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Equilibrium-based Valuation of Option Prices in Jump-diffusion Models

Huang Huamei

NANYANG TECHNOLOGICAL UNIVERSITY
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School of Humanities and Social Sciences

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Doctor of Philosophy

2008

To my parents...

with unbounded love ...

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Huang Huamei

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Summary

This thesis studies the use of general equilibrium approach in valuing contingent financial claims under jump-diffusion settings.

Jump-diffusion process is a mathematical technique aiming to account for discontinuity in asset prices by introducing jumps. Equilibrium-based valuation is a modeling perspective aiming to give an economic explanation for how asset prices are determined. We model the basic variables as jump-diffusion processes and apply the equilibrium-based approach to discover pricing relationships. From there we manage to derive a compact-structure jump-enhanced Black-Scholes formula for a standard European-style call option. It is shown to aggregate all existing relevant formulas into one and proven to hold a sensible interpretation from economic standpoint. Some numerical examples and computation results are presented in support of the formula's performance.

Following the theme, two application are examined. One is a mean-reversion jump-diffusion short rate and bond option pricing model, the other is a two-country two-good jump-diffusion model and foreign currency option pricing. These extensions consolidate the argument that the equilibrium-based approach is effective.

Natura non facit saltum -Nature does not jump.

Alfred Marshall
Principle of Economics
1890

Notations

elements of options:

t, u, s	time
T	maturity date
τ	time to maturity
S	stock price
μ	drift
σ	volatility
q	dividend yield
D	dividend payout
K	strike price
C	call option price
P	put option price
H	bond option price
A	risk-free savings account
$\mathfrak{B}$	risk-free pure discount bond

r	real interest rate
R	nominal interest rate
X_t	spot exchange rate

general mathematical symbols:

Ω	state space
$\mathcal{F}$	filtration
$\mathbb{P}, \mathbb{Q}, \mathbb{R}$	probability measure, probability
E	expectation
Var	variance
Cov	covariance
$\mathbf{I}_A$	indicator function, taking on 1 when in the set A
$\lVert \cdot \rVert$	a n-dimensional Euclidean norm
diag	diagonal
$\Phi(\cdot)$	cumulative standard normal distribution
$\Psi(a, b)$	Poisson distribution
$\sim N(\mu, \sigma^2)$	normally distributed with expectation μ and variance σ^2
dt	small increment in t
T	transposed
$o(x)$	equal-order infinitesimal, i.e. $\lim_{x \to 0}\{o(x)/x\} = 0$
$\equiv$	defined to be

various variables and parameters:

B	standard Brownian motion
N	Poisson process
λ	intensity of a Poisson process
Y, X, L, D, G, H	jump amplitude
U	utility
W	wealth

c	consumption
M	money supply
δ	endowment
ε, ξ	state price deflator
η	Radon-Nikodym derivative
θ, φ	discount rate
γ	coefficient of relative risk aversion
ϖ, ρ, ϱ	coefficient of correlation
V	portfolio
ϕ	market price of risk

abbreviations:

GBM	geometric Brownian motion
ODE	ordinary differential equation
i.i.d.	identically, independently distributed
SDE	stochastic differential equation
PDE	partial differential equation
EMM	equivalent martingale measure
BS	Black-Scholes
CRRA	Constant Relative Risk Aversion
HARA	Hyperbolic Absolute Risk Aversion
CAPM	Capital Asset Pricing Model

Chapter 1

Introduction

Option valuation is a major accomplishment of modern finance. It spurred the development and widespread use of familiar financial options, such as calls and puts in common assets, and exotic options. It also sparked a firestorm of research by economic theorists and empiricists. The research in this area aims at a better understanding of the stochastic evolution of financial markets through the formulation of appropriate mathematical models, as well as at the development of efficient new methodologies for pricing and hedging financial derivatives.

A precise description of the stochastic process followed by the underlying asset price is a prerequisite to attain the option evaluation. The Brownian motion model for stock price movements was incorporated into the option pricing theory by Black, Scholes in their highly acclaimed 1973 paper [18], and hence named the Black-Scholes model. Basically it assumes that the stock price is log-normally distributed, while the stock returns in non-overlapping periods, defined as the changes in the logarithm of stock prices, are identically, independently and normally distributed. Over the decades, despite the popularity and longevity of the Black-Scholes model, empirical biases and systematic mispricing have been widely recognized.

The current research is actively evolving in two directions. The first one involves the relaxation of some of the underlying assumptions with a view to developing a theory that can account for realistic phenomena, e.g., with transaction costs. The second direction focuses on the generalization of the price dynamics of the traded assets to include broader classes of processes, such as the so-called Lévy processes and their extensions. The purpose of this dissertation is not to study the general case of a Lévy driving process but rather to concentrate on a specific subclass of jump-diffusions. A jump-diffusion is a mixture of a continuous diffusion process and a discrete jump process and the most widely used jump-diffusion is the independent sum of a Brownian motion and a compensated Poisson process (also called Poisson-Gaussian processes).

The condition of no-arbitrage among the various assets, namely the assumption that an investor cannot make excess returns without suffering any risk, has an elegant and useful characterization in the language of martingale theory. Under "market completeness" assumptions, martingale techniques offer a powerful tool for pricing and hedging derivatives, and provide a framework for the analysis of numerous other financial applications. Assuming complete markets, an asset can generally be priced by breaking into a sequence of history-contingent claims, evaluating each component of that sequence with the relevant state discounter and then adding up those values. This allows us to price any asset whose payoff could be synthesized as a measurable function of the economy's state but is untenable in an incomplete market context.

There is yet another distinct approach, one that does not require the assumption that there are complete markets — the equilibrium-based approach. It spells out fewer aspects of the economy and assumes fewer markets, but nevertheless derives testable inter-temporal restrictions on prices and returns of different assets, and also across those prices and returns and consumption allocations. In fact, it can rely on the Euler Equations for a utility-maximizing agent and supply stringent restrictions without specifying a concrete general equilibrium model.

The goal of the dissertation is to examine the implications of the equilibrium-based valuation on a jump-diffusion setting. In our model, the exogenous endowment or the technology variable is specified to evolve as a jump-diffusion process, which renders a market portfolio price following a jump-diffusion process and endogenously determines the interest rate in the economy. An asset is valued in market equilibrium on the basis of its characteristics as well as on the basis of preferences and endowment of an agent/investor representing the capital market. Under some canonical assumptions regarding the distribution of jump magnitude and the structure of the economy, we obtain a compact-structure jump-enhanced Black-Scholes formula for standard European call options. It is shown to include all related existing formulas as special cases. More importantly, we substantiate the systematic nature of jump risks and explore its relevance to option pricing. Some data and simulation results are presented in support of the performance of the formula.

Jumps are also expected to be a satisfactory modeling device for interest rate as stylized facts from the bond markets suggest that jump behavior is ubiquitous. We thus extend the approach to bond option pricing to show that the general equilibrium framework can accommodate stochastic interest rate. A feature of mean-reversion is added to the basic jump-diffusion process, which entails a mean-reverting jump-diffusion short rate process. This is quite interesting as both jumps and mean-reversion have been perceived to be present and accounted for in interest rate behavior. Jumps in the interest rate may result from information surprises such as economic announcements that are different from expectations; mean-reversion reflect a central tendency or government control. For the purpose of analysis, Fourier inversion technique is used.

Given the ample empirical evidence that jumps are present in foreign exchange rates, we carry forward the theory from a single economy to an international economy. We use the general equilibrium synthesis where real, financial and monetary aspects are integrated. Our two-country two-good jump-diffusion model and modified valuation formula of foreign currency option are to our knowledge new to the literature.

The detailed content of the dissertation comprises six chapters. Chapter 1 briefly outlines the problem. Chapter 2 presents an overview of the benchmark option pricing methodologies and an exposition of the general equilibrium analytics. Chapter 3 introduces the jump processes and explains how jumps make the market incomplete. Chapter 4 provides a menu of option price formulas in jump-diffusion models and demonstrates ours along with numerical assessment. Extension to stochastic interest rate and resulting bond option pricing formulas are also offered. Chapter 5 deals with an international equilibrium model and develops a currency option price formula. The last chapter concludes this treatise and mentions a few future research directions.

Chapter 2

General Pricing Mechanism

A *derivative security* (also called *contingent claim*) is a financial asset whose payoff depends on the value of some underlying variables. The underlying assets may be stocks, stock indices, foreign currencies, debt instruments, commodities and future contracts. A (so-called *plain vanilla*) *option* gives the holder the right to buy or sell the underlying asset for a strike or exercise price K (fixed when the option is written) at a later time. If the holder has the right to buy the asset, the option is a *call*, while a *put* gives the holder the right to sell the asset. The option is called *European* if it can only be exercised at a certain date (the expiration or maturity date T). An *American* option can be exercised at any time before the expiration date.

A European call option offers the purchaser limited downside loss as given by the premium paid combined with unlimited upside potential. The value of the option at expiry is the payoff of the option. It is zero if exercising the option doesn't provide a profit and positive otherwise. For a long position in a European call option it can be written as $C_T = \max(S_T - K, 0)$, where K is the strike price and S_T is the spot price of the asset at maturity of the contract.

If the spot price is below the strike price for a call option then the option is said to be *out-of-the-money.* On the contrary, if the spot price is above the strike price then the option is said to be *in-the-money.* If the strike price is equal to the spot price, the option is *at-the-money.*

An option that has no chance of ever being exercised would be worthless; however, if an option has a high probability of being exercised then one should expect to pay more for it. A fundamental principle behind the option price prior to maturity is that the higher the probability of an option being exercised, the higher will be its price other things being equal. Therefore those factors influencing the likelihood of an option being exercised will be determinants of the price to be paid for the option.

2.1 No-arbitrage Pricing

There are basically two approaches to option pricing.

The first approach, following the path laid out by Black and Scholes (1973) [18], is to infer from the replicating argument that the price of the derivative must follow a certain equation of motion. This fundamental equation is either a partial difference equation or a partial differential equation (PDE), depending on whether the underlying asset price is modeled in discrete time or in continuous time. In either case it describes how the arbitrage-free price of the derivative changes with respect to the various "state variables"—typically, time and the prices of assets that comprise the replicating portfolio. The solution gives price as a weighted sum of the prices of the primary assets and thereby indicates precisely what the replicating portfolio is and how it can be adjusted as the state variables evolve.

The second way to find a derivative's arbitrage-free price relies on the "no-something-for-nothing" property of the market. Adopting Arrow-Debreu characterization of assets as bundles of state-contingent receipts, we can view the market value of an asset in an arbitrage-free market as simply the discounted sum of the values of products of the cash amounts received in various states and times. Specifically, this method makes use of equivalent martingale measures (EMM). The probability distribution of the underlying stock is adjusted so that the discounted value of the stock price becomes a martingale. A *martingale* has a remarkable property that its expectation function is constant. One can then value the option C_t as the discounted value of its terminal payoff C_T: $C_t = e^{-r\tau}E^{\mathbb{Q}}[C_T|\mathcal{F}_t]$, where $\tau \equiv T - t$ is the time to expiration, r is the risk-free interest rate, and $\mathcal{F}_t$ is the filtration describing the history of the underlying process up to time t and $\mathbb{Q}$ is the adjusted probability measure.

2.1.1 PDE

Consider first the PDE approach. Assume a simple continuous process for the underlying stock price S_t:

$$\frac{dS_t}{S_t} = (\mu_S - q)dt + \sigma_S dB_t \tag{2.1}$$

where μ_S is the relative drift, or the instantaneous expected return on the stock q is the dividend rate and σ_S is the instantaneous volatility of the return. For ease of notation, we assume that μ_S, σ_S and q neither depend on calendar time t nor on the current stock price S_t. $B = \{B_t\}$ is a standard Brownian motion under the physical probability measure $\mathbb{P}$ which captures the underlying uncertainty in this market. Trading in this asset is unrestricted, i.e., no taxes, transaction costs or other frictions. Likewise, investors can invest without restrictions at the constant risk-free asset r. The above differential equation admits a solution for the stock price in the form of a geometric Brownian motion(GBM).

$$S_t = S_0 \exp\left[(\mu_S - q - \frac{1}{2}\sigma_S^2)t + \sigma_S B_t\right] \tag{2.2}$$

The same source of uncertainty, i.e., the Brownian motion B, will affect both stock and option prices. Suppose that we seek to value a European call option that pays off C_T at T. It would be possible to construct a hedge portfolio, which must earn the risk-free rate of return in the absence of arbitrage opportunities. That is, we can form a portfolio consisting of one call option and short n_t shares of the stock. The value of this portfolio at t is $V_t = C_t - n_t S_t - nqS_t t$.

This portfolio is also self-financing, which means that for each period the accession of a new asset has to be financed through the sale of some other assets. In this case, the number of units of the stock n_t has to be adjusted in each period in order to maintain the portfolio V_t risk-free and self-financing. Assuming that the current price $C_t \equiv C(S_t, \tau)$ of the security has suitable differentiability properties, one can apply Itô's lemma [44] to get

$$\begin{aligned} dV_t &= dC_t - n_t dS_t - n_t q S_t dt \\ &= [\frac{\partial C_t}{\partial t} + (\mu_S - q)S_t C_S + \frac{1}{2}\sigma_S^2 S_t^2 C_{SS}]dt + \sigma_S S_t C_S dB_t - n_t S_t(\mu_S dt + \sigma_S dB_t) \end{aligned}$$

where C_S denotes the partial derivative of C_t with respect to S_t and C_{SS} denotes the second order partial derivative.

We can choose $n_t = C_S$ to knock out the dB_t term so that the portfolio grows at a deterministic rate.

$$\frac{dV_t}{V_t} = rdt$$

Therefore we have the following partial differential equation

$$\frac{\partial C_t}{\partial t} + \frac{1}{2}\sigma_S^2 S_t^2 C_{SS} + (r - q)S_t C_S = rC_t \tag{2.3}$$

Append the above PDE to the boundary conditions and we can obtain the unique solution $C(S_t, \tau)$. There are two boundary conditions for a European call option. The first one, $C(S_T, 0) = \max(S_T - K, 0)$ comes from the fact that a call gives its holder the right but not the obligation to buy the underlying stock at time T. The second one, $C(0,\tau)=0$ holds because zero is an absorbing barrier of S_t as defined in equation (2.1). Once S_t becomes zero, it stays there so that the price of the option contract must be zero.

2.1.2 EMM

It is noteworthy that the instantaneous expected return of the underlying stock μ_S does not appear in (2.3). In other words, the risk-preferences of individual investors are never relevant for solving the partial differential equation. Therefore all people, risk-avoiders and risk-lovers, should place the same value on the derivative security, and thus the derivative can be priced as if all agents were risk-neutral. So a derivative can be valued by taking the expectation of the option payoff at maturity using the risk-neutral probability density function of the underlying asset, and then discounting this expected value back to the present at the risk-free rate.

$$C_t = e^{-r\tau} E^{\mathbb{Q}}[C_T|\mathcal{F}_t] \tag{2.4}$$

The risk-neutral measure $\mathbb{Q}$ is also termed the *equivalent martingale measure*(EMM). Accordingly, this pricing method is known as martingale pricing technique. In a more formal setting, Harrison and Kreps (1979) [31] show that for a security market model to be sensible from an economic standpoint, the existence of a risk-neutral measure is equal to the absence of arbitrage. Moreover, Harrison and Pliska (1981) [32] recognize that the martingale measure is unique if and only if the market is complete.

To apply this approach, we need to find the risk-neutral measure $\mathbb{Q}$ which makes the discounted stock prices $\widehat{S_t} = \left(e^{-rt}S_t + \int_0^t e^{-ru}S_u q du\right)$ (plus accumulated dividends if any) a martingale

$$E^{\mathbb{Q}}[\widehat{S_T}|\mathcal{F}_t] = \widehat{S_t}.$$

The Girsanov theorem provides the general method to convert a probability measure $\mathbb{P}$ into its equivalent martingale measure [54].

$$\mathbb{Q}(A) = E^{\mathbb{P}}[\exp{(-\phi B_T - \frac{1}{2}\phi^2 T)}\mathbf{I}_A] \quad \forall A \in \mathcal{F}_T \tag{2.5}$$

The new measure $\mathbb{Q}$ is called "equivalent" since it assigns positive probabilities to the same sets as the original probability measure $\mathbb{P}$. Although the two measures are different, with Radon-Nikodym derivative η_t we can recover one measure from the other.

$$\eta_t \equiv \frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t = \exp(-\phi B_t - \frac{1}{2}\phi^2 t) \quad t \in [0, T] \tag{2.6}$$

η is a $\mathbb{P}$-martingale with initial value $\eta_0 = 1$. Combining this with the fact that $\eta_T > 0(\mathbb{P} - a.s.)$ enables us to conclude that $\mathbb{Q}$ is a probability measure and equivalent to $\mathbb{P}$. Under $\mathbb{Q}$ measure $\tilde{B}_t = B_t + \phi t; \tilde{B}_0 = 0$ is a new standard Brownian motion. $\phi = \frac{\mu_S - r}{\sigma_S}$ is termed the *market price of risk* (or the Sharpe ratio) as it measures the reward (i.e., the risk premium) per unit risk. The stock process under measure $\mathbb{Q}$ has a drift rate equal to r but the diffusion term is unaffected.

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma_S d\tilde{B}_t \tag{2.7}$$

Because η_T depends on the market price of risk, the measure $\mathbb{Q}$ can be interpreted as a risk-adjusted probability measure.

$$S_t = E_t^{\mathbb{Q}}\left[e^{-r\tau}S_T + \int_t^T e^{-r(u-t)}S_u q du\right] \tag{2.8}$$

This expression shows that the asset price equals the expected value of the discounted dividends augmented by the expected value of the discounted terminal price under $\mathbb{Q}$. This formula can be restated in terms of expectations with respect to $\mathbb{P}$. Indeed, changing the measure yields the alternative representation

$$S_t = E_t\left[\xi_{t,T}S_T + \int_t^T \xi_{t,u}S_u q du\right] \tag{2.9}$$

where $\xi_{t,T} \equiv e^{-r\tau}\eta_T/\eta_t$ and $E[\quad]$ without superscript means it is under the original market measure $\mathbb{P}$. The quantity $\xi_u \equiv \xi_{0,u}$ is known as the *state price deflator*, which captures the market wide pricing information. (The terms state price deflator, state price density, state discount factor, pricing kernel are synonymous.) The Arrow-Debreu price at time 0 of a dollar received at date u in state s equals $\xi_u d\mathbb{P}(s)$. Conditional Arrow-Debreu price at time t for cash flows received at time u are given by $\xi_{t,u}d\mathbb{P}(s)$. Arrow-Debreu prices are also known as state prices. The present value of the underlying asset is the sum of cash flows multiplied by the state prices.

With complete market, the PDE and the EMM approaches give the same results. The following is the celebrated Black-Scholes formula solved for the price at time t of a European call option with strike price K and maturity date T written on a stock S_t paying a constant dividend at rate q.

$$C_{BS}\left(S_t,\tau;r,q,\sigma_S,K\right) = S_t e^{-q\tau}\Phi(h) - Ke^{-r\tau}\Phi(h-\sigma_S\sqrt{\tau}) \tag{2.10}$$

where

$$\begin{aligned} h &= \frac{\ln(S_t/K)+(r-q)\tau}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2} \\ \Phi(h) &= \int_{-\infty}^{h} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)dx \end{aligned}$$

$\Phi(\cdot)$ is a cumulative standard normal distribution. It has been agreed upon that $(S_t,\tau;r,q,\sigma_S,K)$ are the six fundamental direct determinants of option value. The same set of variables matter for both calls and puts, albeit not always in the same way. Similar arguments lead to the value of a European put option,

$$P_{BS}\left(S_t,\tau;r,q,\sigma_S^2,K\right) = Ke^{-r\tau}\Phi(-h+\sigma_S\sqrt{\tau}) - S_t e^{-q\tau}\Phi(-h) \tag{2.11}$$

with h as defined above.

In some versions of the formula, it is assumed that stock does not pay dividend and the BS formula is the same as the above one except for $q = 0$. Allowing cash dividend payment requires only a minor modification which is to subtract the present value of dividends from stock price or to use $e^{-q\tau}S_t$ instead of S_t. Here we present a general formula but in later chapters q may or may not be included depending on the context.

2.2 Equilibrium Framework

A critical aspect of the models described in the first section is the ability to hedge all risks with the existing menu of assets. This property is called market completeness. When the underlying source of risk consists of a d-dimensional Brownian motion, market completeness can be ensured if $d + 1$ securities, namely d risky assets and one locally riskless assets, are freely traded. However when additional risks are incorporated, markets are often incomplete.

Incomplete markets have generally more than one equivalent martingale measure. It is necessary to select an appropriate one among several possible measures for option pricing. The equilibrium approach advocates selecting the equivalent martingale measure by giving a description of a general equilibrium model. Pioneered by Lucas [47] and Cox et al.[23] this approach rests on a notion that the economy, in the aggregate, behaves like a single well-defined agent. Analysis of this representative agent's behavior combined with market-clearing conditions lead to equilibrium values for the interest rate, the market price of risk, and the prices of primary and derivative assets.

2.2.1 Structure of the Economy

Let us sketch a continuous time version of the Lucas pure exchange economy model. The economy has a finite time horizon $[0, T]$: all processes described below are understood to "live" on that interval. The fundamental quantity, the single infinitely-divisible good, like crops falling from the productive "tree", is endowed according to an Itô process

$$\frac{d\delta_t}{\delta_t} = \mu_t^\delta + (\sigma_t^\delta)^\intercal d\mathbf{B}_t \tag{2.12}$$

where $\mathbf{B} = \{\mathbf{B}_t\}$ is a n-dimensional Brownian motion representing n exogenous shocks to the economy at time t. The infinite state space Ω is in this case the set of all paths of the Brownian motion $\mathbf{B}$. μ^δ is a progressively measurable process representing the expected consumption growth rate and σ^δ is a d-dimension vector of progressively measurable processes capturing the volatility exposures of the consumption growth rate with respect to the various sources of risk.

Financial markets are composed of primary and derivative assets. Primary assets include n risky stocks and one riskless asset. The riskless asset is in zero net supply and pays annually interest rate at the rate r. We may refer to this asset as the bank account. Let $\mathbf{A} = \{A_t\}$ denote the price process of the bank account. The increment to the balance of the account over an infinitesimal interval $[t, t + dt]$ is known at time t to be

$$\frac{dA_t}{A_t} = rdt. \tag{2.13}$$

A time zero deposit of A_0 will grow to

$$A_t = A_0 e^{rt} \tag{2.14}$$

at t.

Each stock is in unit supply (one share outstanding) and pays dividends. Stock i pays D_t^i per unit time, $\mathbf{D}_t = (D_t^1, \dots, D_t^n)^\intercal$ and

$$\frac{dD_t}{D_t} = \underline{\mu_t^D} dt + \underline{\sigma_t^D} d\mathbf{B}_t \tag{2.15}$$

with (μ_t^D, σ_t^D) progressively measurable. Assuming that dividends are the only source of consumption mandates the consistency condition $\sum_{i=1}^n D_t^i = \delta_t$.

The price of n type of stocks are modeled as general Itô processes. The price process $S^i = \{S_t^i\}$ of the i-th risky asset is assumed to be of the form

$$\frac{dS_t^i + D_t^i dt}{S_t^i} = \mu_t^i dt + (\sigma_t^i)^\intercal d\mathbf{B}_t. \tag{2.16}$$

The price dynamics of all the n risky assets can be written compactly in vector notation as

$$d\mathbf{S}_t + \mathbf{D}_t dt = \operatorname{diag}(\mathbf{S}_t)(\underline{\mu_t^S} dt + \underline{\sigma_t^S} d\mathbf{B}_t)$$

where

$$\mathbf{S}_t = \begin{pmatrix} S_t^1 \\ S_t^2 \\ \vdots \\ S_t^n \end{pmatrix}, \mathbf{D}_t = \begin{pmatrix} D_t^1 \\ D_t^2 \\ \vdots \\ D_t^n \end{pmatrix}, \operatorname{diag}(\mathbf{S}_t) = \begin{bmatrix} S_t^1 & 0 & \cdots & 0 \\ 0 & S_t^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_t^n \end{bmatrix}$$

$$\underline{\mu_t^S} = \begin{pmatrix} \mu_t^1 \\ \mu_t^2 \\ \vdots \\ \mu_t^n \end{pmatrix}, \underline{\sigma_t^S} = \begin{bmatrix} \sigma_{1t}^1 & \sigma_{2t}^1 & \cdots & \sigma_{dt}^1 \\ \sigma_{1t}^2 & \sigma_{2t}^2 & \cdots & \sigma_{dt}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1t}^n & \sigma_{2t}^n & \cdots & \sigma_{dt}^n \end{bmatrix}$$

We assume that the process $\{\mu^i : i = 1, \dots, n\}$ and $\{\sigma_z^i : z = 1, \dots, d; i = 1, \dots, n\}$ are "well-behaved", e.g. generating prices with finite variances. The economic interpretation of $\underline{\mu_t^S}$ is the expected rate of return per time period over the next instant. The matrix $\underline{\sigma_t^S}$ captures the sensitivity of the prices to the exogenous shocks.

Derivative assets are in zero net supply and consist of k securities with payoff $(f^j, \mathcal{Y}_T^j) : j = 1, \ldots, k$, where f^j is a continuous, progressively measurable payment and $\mathcal{Y}_T^j$ is a terminal, measurable cash flow. The price dynamics of all the k derivative assets $\mathbf{V}_t = (V_t^1, V_t^2, \ldots, V_t^k)^\intercal$ are conjectured to satisfy Itô processes.

$$\begin{aligned} d\mathbf{V}_t + \mathbf{f}_t dt &= \operatorname{diag}(\mathbf{V}_t)(\underline{\mu_t^V} dt + \underline{\sigma_t^V} d\mathbf{B}_t) \qquad (2.17) \\ \mathbf{V_T} &= \mathcal{Y}_\mathbf{T} \end{aligned}$$

with progressively measurable coefficient matrix $\{(\underline{\mu_t^V}, \underline{\sigma_t^V}) : j = 1, \ldots, k\}$.

All assets, primary and derivatives, are freely traded at the relevant prices. These prices, as well as the rate of interest, are endogenous in equilibrium. No specific restrictions are placed on n, d and k. Hence $n + k < d$ is an admissible market structure.

The economy has a representative agent who maximizes welfare by consuming and investing in the assets available. The consumption space is the space of progressively measurable, nonnegative processes. Preferences over consumption processes, denoted by $\mathbf{c} \equiv \{c_t : t \in [0, T]\}$, are represented by the von Neumann-Morgensten utility

$$\mathbf{U}(\mathbf{c}) \equiv E[\int_0^T e^{-\theta t} U(c_t) dt] \qquad (2.18)$$

where $U(c_t)$ is the utility of consumption at time t. θ is the time preference rate or subjective discount rate of the agent. The utility function is assumed to be strictly increasing and concave in consumption and to satisfy the limiting (Inada) conditions $\lim_{c \to 0} U_c(c_t) = \infty$, $\lim_{c \to \infty} U_c(c_t) = 0$ for all $t \in [0, T]$, where U_c is the derivative with respect to consumption.

Investment policies are progressively measurable processes denoted by $\pi \equiv (\pi^S, \pi^V)$ where π^S represents the vector of fractions of wealth invested in stocks and π^V the vector of fractions invested derivatives. The complement $1 - (\pi^S + \pi^V)^\intercal \mathbf{1}$ is the fraction of wealth invested in the risk-free asset ($\mathbf{1}$ is a vector of ones with suitable dimensions). For a given consumption-portfolio policy $(\mathbf{c}, \pi)$ wealth W evolves according to the dynamic budget constraint.

$$\begin{aligned} dW_t = (rW_t - c_t)dt + W_t(\pi_t^S)^\intercal[(\underline{\mu_t}^S - r\mathbf{1})dt + \underline{\sigma_t}^S d\mathbf{B}_t] \\ + W_t(\pi_t^V)^\intercal[(\underline{\mu_t}^V dt - r\mathbf{1})dt + \underline{\sigma_t}^V d\mathbf{B}_t] \end{aligned} \tag{2.19}$$

subject to some initial condition W_0 and nonnegativity condition $W_t \geq 0$. The representative agent maximize over policies $(\mathbf{c}, \pi)$ satisfying the constraint. Policies that solve this maximization problem are said to be optimal for $\mathbf{U}$ at the given price process $(\mathbf{S}, \mathbf{V}, r)$.

2.2.2 Equilibrium Conditions

A competitive rational-expectation equilibrium for an economy is a collection of price processes $\{(\mathbf{S} : \underline{\mu}^S, \underline{\sigma}^S); (\mathbf{V} : \underline{\mu}^V, \underline{\sigma}^V), r\}$ and consumption-portfolio policies $(\mathbf{c}, \pi)$ such that

- given the asset prices, an agent has chosen an optimal trading strategy according to her preferences and endowments, i.e. $(\mathbf{c}, \pi)$ is optimal for $\mathbf{U}$ at the price process $(\mathbf{S}, \mathbf{V}, r)$.
- markets clear, i.e., total consumption equal to total production, $c_t = \delta_t$, and total demand equal total supply for each asset, $W\pi^S = 1$, $W\pi^V = 0$ and $W[1 - (\pi^S + \pi^V)^\intercal \mathbf{1}] = 0$.

Under suitable conditions, standard arguments can be invoked to show that the optimal consumption satisfies the first-order condition $e^{-\theta t} U_c(c_t) = a\xi_t$ where $U_c(c_t)$ denotes the first order derivative of $U(c_t)$ with respect to c_t, a is a constant and ξ_t is the relevant state price density implied by the given price process $(\mathbf{S}, \mathbf{V}, r)$.

Equilibrium in the goods market then gives the condition $e^{-\theta t}U_c(\delta_t) = a\xi_t$. Since $\xi_0 = 1$ it must be that $a = U_c(\delta_0)$ and therefore

$$\xi_t = e^{-\theta t}\frac{U_c(\delta_t)}{U_c(\delta_0)} \tag{2.20}$$

The equilibrium state price density is therefore equal to the marginal rate of substitution between consumptions at time t and 0. The same arguments that were used in prior sections can be applied in this context to establish the price representations

$$S_t^j = E_t[\int_t^T \xi_{t,u}D_u^j du] \tag{2.21}$$

$$V_t^j(f,\mathcal{Y}) = E_t[\xi_{t,T}\mathcal{Y}_T^j + \int_t^T \xi_{t,u}f_u^j du] \tag{2.22}$$

Let us write the dynamics of a state price deflator as

$$\frac{d\xi_t}{\xi_t} = \mu_t^\xi dt + (\sigma_t^\xi)^\intercal d\mathbf{B}_t$$

for some relative drift μ_t^ξ and some volatility vector σ_t^ξ. Define $\zeta_t = \xi_t A_t = \xi_t e^{rt}$. By Itô's lemma,

$$\frac{d\zeta_t}{\zeta_t} = (\mu_t^\xi + r)dt + (\sigma_t^\xi)^\intercal d\mathbf{B}_t$$

Since ζ is a martingale by the property of a state price deflator, we must have $\mu_t^\xi = -r$, i.e., the relative drift of a state price deflator process is equal to the negative of the interest rate.

If $\mathbb{Q}$ is a risk-neutral measure, the discounted stock prices plus accumulated dividends, if any, are $\mathbb{Q}$-martingales.

$$\widehat{\mathbf{S}}_t = \mathbf{S}_t e^{-rt} + \int_0^t e^{-ru}\mathbf{D}_u du$$

An application of Itô's lemma shows that the dynamics of the discounted prices is

$$d\widehat{\mathbf{S}}_t = \mathrm{diag}(\widehat{\mathbf{S}}_t)[(\underline{\mu_t^S} - r\mathbf{1})dt + \underline{\sigma_t^S}d\mathbf{B}_t] \tag{2.23}$$

The change of measure from $\mathbb{P}$ to $\mathbb{Q}$ is captured by the random variable, $\eta_t = E_t[d\mathbb{Q}/d\mathbb{P}]$. Then it follows from the *Martingale Representation Theorem* [54], that a d-dimensional process $\underline{\phi_t}$ exists such that

$$d\eta_t = -\eta_t(\underline{\phi_t})^\intercal d\mathbf{B}_t$$

or, equivalently (using $\eta_0 = E[d\mathbb{Q}/d\mathbb{P}] = 1$),

$$\eta_t = \exp\left(-\frac{1}{2}\int_0^t \|\underline{\phi_u}\|^2 du - \int_0^t (\underline{\phi_u})^\intercal dB_u\right) \tag{2.24}$$

where $\|\cdot\|$ represents a n-dimensional Euclidean norm.

According to Girsanov's Theorem [54], the process $\tilde{\mathbf{B}}$ defined by

$$d\tilde{\mathbf{B}}_t = d\mathbf{B}_t + \underline{\phi_t}dt, \quad \tilde{\mathbf{B}}_0 = 0 \tag{2.25}$$

is then a standard Brownian motion under the $\mathbb{Q}$ measure. Substituting $d\mathbf{B}_t = d\tilde{\mathbf{B}}_t - \underline{\phi_t}dt$ into (2.23), we obtain

$$\widehat{d\mathbf{S}}_t = \text{diag}(\widehat{\mathbf{S}}_t)[(\underline{\mu_t^S} - r\mathbf{1} - \underline{\sigma_t^S} \times \underline{\phi_t})dt + \underline{\sigma_t^S}d\tilde{\mathbf{B}}_t]. \tag{2.26}$$

If discounted prices are to be $\mathbb{Q}$-martingales, the drift must be zero, so

$$\underline{\mu_t^S} - r\mathbf{1} = \underline{\sigma_t^S} \times \underline{\phi_t} \tag{2.27}$$

From these arguments it follows that the existence of a solution $\underline{\phi}$ to this system of equations is a necessary condition for the existence of a risk-neutral measure. Note that the system has n equations (one for each asset) in d unknowns, $\phi_1, \ldots, \phi_d$ (one for each exogenous shock). The process $\underline{\phi} = \{\underline{\phi}_t\}$ is called a *market price of risk* process.

Let us now look at the relation between the market price of risk and the state price deflator. Suppose ϕ is a market price of risk and η_t defines the associated risk-neutral measure. We know that under a regularity condition, the process $\xi = \{\xi_t\}$ defined by

$$\xi_t = \eta_t e^{-rt} = \exp\left(-rt - \frac{1}{2}\int_0^t \|\underline{\phi_u}\|^2 du - \int_0^t (\underline{\phi_u})^\intercal dB_u\right) \tag{2.28}$$

is a state price deflator. Since $d\eta_t = -\eta_t(\underline{\phi_t})^\intercal dB_t$, an application of Itô's lemma implies that

$$\frac{d\xi_t}{\xi_t} = -[rdt + (\underline{\phi_t})^\intercal d\mathbf{B}_t] \tag{2.29}$$

As we have just shown, the relative drift of a state price deflator equals the negative of the short term interest rate. Now we see that the volatility vector of a state price deflator equals the negative of a market price of risk.

Usually we have implicitly assumed that S_t is the real value of the asset, i.e., deflated by consumer prices. However when analyzing foreign exchange rate or currency derivatives, monetary factors cannot be omitted. In that case the nominal state price deflator is needed. Consider the nominal pricing equation for an asset with a payoff that depends on a future nominal asset value. Inflation enters the pricing equation exogenously as a numeraire. Suppose we are working with a nominal price P_t and that I_t is the level of consumer price. Then the nominal state price deflator:

$$\xi_t^* \equiv \xi_t \frac{I_t}{I_T}. \tag{2.30}$$

We can then equally easily work with the pricing

$$P_t = E_t[\xi_t^* P_T] = E_t[\xi_t I_t S_T]. \tag{2.31}$$

The nominal riskfree bond price and nominal interest rate are obtained by evaluating equation (2.31) with a payoff function of unity. It can also be shown that the nominal state price deflator is characterized by the pair of the nominal interest rate $R(t)$ and the market price for nominal risk $\Theta(t)$.

$$\frac{d\xi_t^*}{\xi_t^*} = -[R(t)dt + \Theta(t)^\intercal d\mathbf{B}_t] \tag{2.32}$$

Back to the real economy described above, the equilibrium (real) interest rate $r = \{r_t\}$ and the (real) market price of risk $\phi = \{\phi_t\}$ can be obtained by applying

Itô's lemma to both sides of equation (2.20). This yields

$$\begin{aligned} r_t &= \theta + \gamma(\delta_t)\mu_t^\delta - \frac{1}{2}\gamma(\delta_t)\varsigma(\delta_t)\|\sigma_t^\delta\|^2 && (2.33) \\ \phi_t &= \gamma(\delta_t)\sigma_t^\delta && (2.34) \end{aligned}$$

where

$$\gamma(c_t) = -\frac{U_{cc}(c_t)}{U_c(c_t)}c_t$$

is the *relative risk aversion coefficient* and

$$\varsigma(c_t) = -\frac{U_{ccc}(c_t)}{U_{cc}(c_t)}c_t$$

is the *relative prudence coefficient.*

We can observe the following relations:

First, there is a positive relation between the discount rate θ and the equilibrium interest rate r. The intuition behind this is that when the agent of the economy is impatient and has a high demand for current consumption, the equilibrium interest rate must be high in order to encourage the agent to save now and postpone consumption.

Second, the multiplier of μ_δ is γ, the relative risk aversion of the representative agent, which is positive. Hence, there is a positive relation between the expected growth in aggregate consumption and the equilibrium interest rate. This can be explained as follows: we expect higher future consumption and hence lower future marginal utility, so postponed payments due to saving have lower value. Consequently, a higher return on saving is needed to maintain market clearing.

Finally, there is a negative relation between the variance of aggregate consumption $\sigma_\delta(\sigma_\delta)^\intercal$ and interest rate r. Indeed this is true as long as the representative agent has decreasing absolute risk aversion which makes $U_{ccc}(c_t)$ positive. The intuition is that the greater the uncertainty about future consumption, the more will the agent appreciate the sure payments from the risk-free asset and hence the lower a return is necessary to clear the market for borrowing and lending.

Explicit expressions for the interest rate and security prices are impossible without specifying the utility function. In the literature, the momentary utility function for a representative agent often takes the following iso-elastic form.

$$U(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} \tag{2.35}$$

where $\gamma > 0$ and $\gamma \neq 1$. When $\gamma = 1$ it becomes the logarithmic utility. γ is the Arrow-Pratte coefficient measuring the relative risk aversion. This utility function, also called power law utility, exhibits constant relative risk aversion (CRRA). It exhibits several interesting properties that deserve examination. First, it is compatible with risk neutrality $\gamma = 0$ and also includes $\gamma = 1$. Second, with this function form, the risk premiums predicted by the model are resistant to changes in wealth levels and in the size of the economy. Third, to the extent that economic agents share the same utility function, we can aggregate individual choices, even if agents have different levels of wealth.

With CRRA, the economy has a flat term structure if the drift and variance rates of endowment are constant.

$$\begin{aligned} r &= \theta + \gamma\mu^{\delta} - \frac{1}{2}\gamma(\gamma+1)(\sigma^{\delta})^2 \\ \phi &= \gamma\sigma^{\delta} \end{aligned}$$

2.2.3 Consumption CAPM

To facilitate the exposition of the consequent Consumption Capital Asset Pricing Model, let us first consider a discrete-time version of the above equilibrium pricing model.

The representative wants to maximize $E[\sum_0^T \varphi^t U(c_t)]$, where φ is the discount rate in discrete setting. The Euler equation for the price of an asset is

$$S_t^i = E_t\left[\varepsilon_{t+1}(S_{t+1}^i + D_{t+1}^i)\right]. \tag{2.36}$$

$\varepsilon_{t+1} = \varphi U_c(c_{t+1})/U_c(c_t)$ is a one-period state price deflator for converting a time $t+1$ payoff into time t value. By repeated substitution we can get

$$S_t^i = E_t[\sum_{j=1}^{T-t}(\prod_{s=1}^{j}\varepsilon_{t+s})D_{t+j}^i].$$

For a T-period risk-free pure discount bond which has only a terminal payoff $D_T = 1$, $D_s = 0$ for $s \neq T$, then its price at time t is

$$\mathfrak{B}_t^T \equiv \frac{1}{R_{\mathfrak{B}}} = E_t[\varepsilon_T] \tag{2.37}$$

where $R_{\mathfrak{B}} \equiv 1 + r$ represents the gross rate of return on the risk-free bond. Let $R_{t+1}^i \equiv (S_{t+1}^i + D_{t+1}^i)/S_t^i$ denote the gross rate of return on the asset between t and $t+1$. Substitute ε_t and R_{t+1}^i into (2.36), we get

$$1 = E_t[\varepsilon_{t+1}R_{t+1}^i]. \tag{2.38}$$

Making use of $E[xy] = E[x]E[y] + \text{Cov}[x, y]$ to decompose the product, this becomes

$$\begin{aligned} 1 &= E_t[\varepsilon_{t+1}]E_t[R_{t+1}^i] + \text{Cov}[\varepsilon_{t+1}, R_{t+1}^i] \\ &= \frac{E_t[R_{t+1}^i]}{R_{\mathfrak{B}}} + \text{Cov}\left[\varphi\frac{U_c(c_{t+1})}{U_c(c_t)}, R_{t+1}^i\right] \end{aligned}$$

Rearrange it into

$$E_t[R_{t+1}^i] - R_{\mathfrak{B}} = -\frac{\text{Cov}[U_c(c_{t+1}), R_{t+1}^i]}{E_t[U_c(c_t)]}. \tag{2.39}$$

This equation is known as *Consumption-CAPM*. It indicates that all assets have an expected return to the risk-free return plus a risk adjustment which depends on the covariance of its return with marginal rate of consumption. Since $U_c(c_t)$ is high when c_t is low, stock with a high return when c_t is low has a low expected return: it offers insurance against consumption fluctuations.

Apply this to a market portfolio which is perfectly correlated with the state price deflator, we have

$$E_t[\varepsilon_{t+1}] = R_{\mathfrak{B}} - R_{\mathfrak{B}}\mathrm{Var}[\varepsilon_{t+1}].$$

Use this to replace the right hand side of (2.39) we have

$$E_t[R^i_{t+1}] - R_{\mathfrak{B}} = \beta^i_t(E_t[\varepsilon_{t+1}] - R_{\mathfrak{B}}) \tag{2.40}$$

where

$$\beta^i_t = \frac{\mathrm{Cov}[\varepsilon_{t+1}, R^i_{t+1}]}{\mathrm{Var}[\varepsilon_{t+1}]}$$

This is known as the *security market line*, which highlights the general measure of the systematic risk of a risky asset S^i in a consumption model.

For the continuous time, the instantaneous return on a risky asset

$$\begin{aligned}
\xi_t S^i_t &= E_t[\int_t^T \xi_u D^i_u du] + E_t[\int_t^T \xi_u dS^i_u] \\
&= E_t[\int_{u=t}^{\Delta} \xi_u D^i_u du] + E_t[\xi_{t+\Delta} S^i_{t+\Delta}] \\
&\approx \xi_t D^i_t \Delta + E_t[\xi_t S^i_t + (\xi_{t+\Delta} S^i_{t+\Delta} - \xi_t S^i_t)]
\end{aligned}$$

which is exact as $\Delta \to 0$ and hence

$$0 = \xi_t D^i_t dt + E_t[d(\xi_t S^i_t)]$$

Note that $rdt = -E_t[d\xi_t/\xi_t]$. The expected instantaneous return on a risky asset is the sum of capital gain and the dividend.

$$\mu^i_S = E_t\left[\frac{dS^i_t}{S^i_t}\right] + \frac{D^i_t}{S^i_t}dt$$

We eventually get the important relationship

$$E_t\left[\frac{dS_t}{S_t}\right] = rdt - \mathrm{Cov}\left[\frac{d\xi_t}{\xi_t}\frac{dS^i_t}{S^i_t}\right]. \tag{2.41}$$

which is the continuous time analogue of (2.39).

The reason for presenting the CAPM/C-CAPM here is to link them to our analysis. On one hand, our representative agent paradigm can lead to the same security market line as the financial portfolio theory does. On the other hand, the CAPM/C-CAPM is an equilibrium model to price financial assets of any kind even if standard implementation is usually limited to common stocks. CAPM and Black-Scholes option pricing are often presented in so different contexts that they seem to belong to disparate sections of finance or economics. However, we have to bear in mind that CAPM is very general, concerning stocks, bonds and derivative assets, including puts and calls. Option pricing is less special than it appears if we perceive that common stocks are call options written on the assets of the firm and corporate bonds are equivalent to default-free bonds plus a short position in puts. Moreover, in such instances as jump-diffusion models, Black-Scholes option pricing loses its predominance and CAPM/C-CAPM provides a plausible mechanism.

The valuation formulas (2.31) issued from this general equilibrium analysis have the same structure as those prevailing in the complete market settings of earlier sections. In light of this structural property, the representative agent model is said to have *effectively complete markets.* We will soon see that the formulas are especially useful in settings with systematic jump risks where not all the risks can be hedged away. In those instances pricing based on the representative paradigm provides a simple and often attractive approach to valuation.

Chapter 3

Jump-Diffusion Models

Documentation from various empirical studies shows that the Black-Scholes-type geometric Brownian motion (GBM) models are inadequate, both in relation to their description power as well as for the systematic mispricing that they may induce. The contributions of the literature to the present volume deal with various generalizations of the basic GBM. Here we focus on the fact that returns of various asset prices and interest rates may exhibit a jumping behavior. We thus study possible superpositions of jump and diffusion processes. Jump-diffusion models have some intuitive appeal in that they let prices change continuously most of the time, but they also take into account the fact that from time to time larger jumps may occur that cannot be adequately modeled by pure diffusion processes.

3.1 Literature Survey

Among the empirical studies documenting discontinuity in equity prices and interest rates, one may quote Ball and Torus (1985) [9], Jorion (1985) [43], Andersen et al. (1998) [6] and Jiang (1998) [40] etc. Jump effects tend to be prevalent in regulated intervention environments such as the interest rate and foreign exchange markets. Energy prices and electricity prices is another common case, which are known for abrupt and unanticipated large changes caused by supply shocks such as outage, generating or transmission constraints. As Merton [49] emphasizes, routine trading information releases are well depicted by smooth changes yet bursts of information are often reflected in price behavior by jumps.

Jumps play two important related roles in modeling: one is to provide time series data a better fit and the other is to afford greater flexibility in matching derivative prices—i.e. in modeling dynamics under an equivalent martingale measure. Equivalence of physical and martingale measures require that both admit jumps if either does yet their frequency and magnitudes can be quite different under the two measures. Jump-diffusion processes can flexibly accommodate a wide range of skewness and kurtosis, of which raw statistical evidence is strongly suggestive [28]. Kurtosis can substantially affect the pricing of derivative securities.

Press (1976) [55] noted long ago that the analytical characteristics of a Poisson mixture of normal distributions agree with the properties of the empirical distributions of security prices. To augment the Brownian motion, Press introduces the first jump-diffusion model. On the other hand, a first approach extending the Black-Scholes option pricing formula with inclusion of jumps is that of Merton (1976) [49]. However when the underlying stock follows a jump-diffusion process and transactions are solely possible in the stock and the risk-free asset, options will have to elude arbitrage-oriented pricing. That stock weights eliminating the linear diffusion risk cannot simultaneously remove the non-linear jump risk, vice versa, because the option price is a convex function of the stock price. A non-hedgeable residual risk remains, which one is only able to eliminate via portfolio strategies under very restrictive assumptions. See, [24], [37], [42], [38], [48].

Merton [49] makes an assumption that the jump component describes the evolution of returns for an individual asset and is uncorrelated with returns on the market portfolio. In another word, the jump risk is non-systematic and hence will not be priced in equilibrium. So one can follow Black and Scholes [18] and equate the option value to the expected value of its payoff discounted at the risk-free rate.

However, this hypothesis was later criticized by showing that such an assumption is equivalent to the existence of a non-jump market portfolio containing the stock [17]. Contradictorily, empirical evidence of discontinuities in the daily and weekly price changes for diversified equity portfolios can be found in [37], [9], [43] and [58]. Kim (1994) [45] also documented that jumps in stock returns are indeed systematic. Moreover, Merton's assumption is obviously violated if the asset under consideration is the market portfolio itself [7],

An alternative to circumvent the problem is to assume that the market, if not already complete, can be made complete by the introduction of additional contracts. The non-uniqueness in the pricing problem then translates into the freedom that one has in the specification of these new contracts. After completion of the market, prices and hedging strategies can be determined uniquely. See e.g., [59], [11], [50], [39],

[56], [25], and [61]. These models have the advantage over Merton's model that the absence of arbitrage ensures a unique price of the option without making any assumptions about a non-priced jump risk. The drawbacks of these models are:

1. The jump amplitude has to fulfill certain requirements: either it has to be a finite discrete random variable or it has to be a predictable process. In

this way the jumps may be controlled to some extent. In fact, if the jump amplitude is an independent absolute continuous random variable, the market will still be incomplete.

2. The stochastic evolution of a continuum of other assets has to be specified, which complicates the whole analysis.

In a more general case, the risk associated with the jump component is systematic and non-diversifiable and then general equilibrium arguments is called for. See [35], [7], [51], [14], [1], [4], and [52]. Like in the Arrow-Debreu tradition, the security prices and returns are shown to arise from the interactions of profit maximizing firms and utility maximizing households. The advantage of this approach is that one can derive the Arrow-Debreu price of "jump insurance" and construct the asset price process under the adjusted probability measure, which is fundamental to option valuation.

When it comes to solving the continuous-time portfolio problem in an equilibrium context, there are two parallel methods. One is the stochastic control approach [21] [23], based on standard results of stochastic control theory. The optimal solution is computed by solving the so-called Hamilton-Jacobi-Bellman Equation in two steps. The first step consists of searching for the optimal portfolio strategy as a function of the (unknown) optimal expected utility. Inserting this portfolio and consumption strategy into the Hamilton-Jacobi-Bellman Equation results in a non-linear partial differential equation, whose solution forms the second step. With additional assumption like a HARA (hyperbolic absolute risk aversion) utility function, there are closed form solutions for this problem. In general, however, it is very hard to get explicit solutions to the Hamilton-Jacobi-Bellman Equation. Bates (1988) [13] and Ahn (1992) [1] etc. have used this method in a jump-diffusion setting and obtained option pricing results relying on the indirect utility of wealth function. The other method, the Euler equation approach, is much simpler but comparably useful and essentially accordable with the stochastic control theory. This method makes it possible to derive the fundamental valuation equation for contingent claims, interest rate sensitive or foreign exchange sensitive or otherwise, with just first-order conditions [47].

3.2 Jump Processes

In this section we review basic definitions and results needed for the study of jump-diffusion models limiting ourselves to univariate point processes.

3.2.1 Poisson Process

A point process is intended to describe events that occur randomly over time. It can be represented as a sequence of nonnegative random variables $0 = T_0 < T_1 < T_2 < \infty$ where the generic T_n is the n-th instant of occurrence of an event. The process may equivalently be represented via its associated counting process

$$N_t = n \quad \text{if} \quad t \in [T_n, T_{n+1}) \quad n \geq 0, \tag{3.1}$$

which counts the number of events up to and including time t. The non-explosion condition is $N_t < \infty$ for $t \geq 0$.

Definition 1. A point process N_t adapted to a filtration $\mathcal{F}_t$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *Poisson process* if

1. $N_0 = 0$;
2. N_t is a process with independent increments;
3. $N_t - N_s$ is a Poisson random variable with mean $\Lambda_{s,t}$.

Usually one assume $\Lambda_{s,t} = \int_s^t \lambda_u du$ for a deterministic function λ_t, called the *intensity* of the Poisson process N_t. It can be seen that if λ_t is a constant λ, then $T_{n+1} - T_n$ are i.i.d., exponential random variables with parameter λ. A natural interpretation of the intensity is:

- the probability of one jump in an interval of length $\triangle$ is $\lambda\triangle + o(\triangle)$;
- the probability of two or more jumps in an interval of length $\triangle$ is $o(\triangle)$;
- the number of jumps in non-overlapping intervals are independent.

Notice that these parallel the characterizations of a Brownian motion process: both are processes with independent increments; the increments of a Brownian motion are normally distributed while those of a Poisson process are Poisson distributed. The Brownian motion is the natural prototype for the continuous component of asset price changes whereas the Poisson process is the basic building block for the discontinuous component. On the other hand, while the Brownian motion is itself a martingale, a Poisson process process as such is not. It becomes a martingale if one subtracts from N_t the process given by its mean. Indeed, since $E[N_t - N_s|\mathcal{F}_s] = E[\int_s^t \lambda_u du|\mathcal{F}_s]$,

$$M_t \equiv N_t - \int_0^t \lambda ds \tag{3.2}$$

is a $\mathcal{F}_t$-martingale and sometimes called a compensated Poisson process.

Let N_t be a Poisson process with constant intensity λ. The infinitesimal change in N over dt is:

$$dN_t = \begin{cases} 0, & \text{with probability } 1 - \lambda dt; \\ 1, & \text{with probability } \lambda dt\ . \end{cases} \tag{3.3}$$

Given that the Poisson event occurs (i.e., some important information on the stock arrives), there is a "drawing" from a distribution to determine the impact of this information on the stock price. In other words, if S_t is the stock price at time t and Y is the random variable description of this drawing, then, neglecting the continuous part, S_{t+dt} will be S_tY, given that one such jump occurs. The addition of jump process will increase the security price by a proportional amount $k \equiv E[Y-1]$. Thus a so-called jump-diffusion model is in this thesis always encapsulated by the equation:

$$\frac{dS_t}{S_{t-}} = (\mu_S - q - \lambda k)dt + \sigma_S dB_t + (Y-1)dN_t \tag{3.4}$$

where μ_S is the instantaneous expected return on the stock, σ_S^2 is the instantaneous variance of the return, conditional on no occurrence of price jumps. The B, N and Y are independent processes. From this stochastic differential equation, S_t can be explicitly expressed as

$$S_t = S_0 e^{(\mu_S - q - \frac{1}{2}\sigma_S^2 - \lambda k)t + \sigma_S B_t} Y^{n_t} \tag{3.5}$$

$Y^n = 1$ if $n = 0$; $Y^{n_t} = \prod_{j=1}^{n_t} Y_j$ for $n \geq 1$ where Y_j are i.i.d. and n is the number of jumps, Poisson distributed with parameter λdt.

It is standard to assume Y_js have independent lognormal distribution, which involves a generic jump whose magnitude fluctuates between minus one and infinity, thus allowing the generation of both downward and upward jumps. There are a number of other candidate distributions: e.g., the uniform, truncated lognormals, double exponential, a mixture of Beta and Pareto distributions. The choice of the distribution for the jump size has important implication for the kurtosis and skewness of the return process. The lognormal distribution is able to reproduce the leptokurtic feature of the stock return distribution and the "volatility smile" of option prices. It also leads to closed-form analytical solutions for option prices and has many desirable properties.

When the jump amplitude is lognormally distributed, i.e. $y_j = \ln Y_j \sim N(\mu_y, \sigma_y^2)$.

Consequently, the mean percentage jump in the endowment flow is $k = e^{\mu_y + \frac{1}{2}\sigma_y^2} - 1$.

$$S_T = S_t \exp\left[(\mu_S - q - \frac{1}{2}\sigma_S^2 - \lambda k)(T-t) + \sigma_S(B_T - B_t) + \sum_{j=n_t+1}^{n_T} y_j\right] \tag{3.6}$$

3.2.2 Change of Measure for Jumps

Consider a generic jump-diffusion model for a stock price.

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dB_t + X_t dN_t - \lambda_t E[X]dt \tag{3.7}$$

where N is a Poisson process with an intensity under $\mathbb{P}$ denoted dy λ_t. The sizes X of the different jumps are assumed to be independent and identically distributed. The mean jump size under $\mathbb{P}$ is equal to $E[X] \equiv E^{\mathbb{P}}[X]$ and -1 is a lower bound. Under the "risk-neutral" measure $\mathbb{Q}$ the dynamics of the stock price are

$$\frac{dS_t}{S_{t-}} = rdt + \sigma_S d\tilde{B}_t + X_t dN_t - \lambda_t^{\mathbb{Q}} E^{\mathbb{Q}}[X]dt \tag{3.8}$$

where changing the measure from $\mathbb{P}$ to $\mathbb{Q}$ we introduce a new standard Brownian motion $\tilde{B}_t$ according to

$$d\tilde{B}_t = dB_t + \phi_t dt$$

with ϕ_t as the market price of diffusion risk. For the jump risk component, the intensity changes from λ_t to $\lambda_t^{\mathbb{Q}}$, and the distribution of the jump size changes from $\mathbb{P}(dx)$ to $\mathbb{Q}(dx)$. By the Girsanov theorem for point processes [27], the change of measure from $\mathbb{P}$ to $\mathbb{Q}$ is given by the Radon-Nikodym derivative

$$\eta_T = \frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t, \quad \eta_t = E^{\mathbb{P}}[\eta_T|\mathcal{F}_t]. \tag{3.9}$$

It can be represented by

$$\begin{aligned}\eta_t &= \exp\left[-\int_0^t \int_{-1}^{\infty} [h(u,x) - 1]\mathbb{P}(u,dx)\lambda_u du + \int_0^t \ln h(u, X_u) dN_u\right] \\ &= \exp\left[-\int_0^t E^{\mathbb{P}}[h(u,X_u) - 1|\mathcal{F}_u]\lambda_u du + \int_0^t \ln h(u,X_u)dN_u\right]\end{aligned}$$

where h is some function of time t and of the jump size X that describes the measure transform, similar to the term ϕ in case of a diffusion process which captures the change in the drift. The stochastic differential equation for η_t is

$$\frac{d\eta_t}{\eta_{t-}} = -\int_{-1}^{\infty} [h(t,x) - 1]\mathbb{P}(t,dx)\lambda_t dt + [h(t,X_t) - 1]dN_t \tag{3.10}$$

From this expression we see that η_t is indeed a $\mathbb{P}$-martingale. Given this measure transform, the new intensity of the Poisson process under $\mathbb{Q}$ is given by

$$\lambda_t^{\mathbb{Q}} = \lambda_t E[h(t,X_t)|\mathcal{F}_t], \tag{3.11}$$

and the distribution of the jump size under $\mathbb{Q}$ is given by

$$\mathbb{Q}(t,dx) = \frac{h(t,x)}{E[h(t,X_t)|\mathcal{F}_t]}\mathbb{P}(t,dx). \tag{3.12}$$

For instance, if the jump amplitude is lognormally distributed with mean μ_y and standard deviation σ_y and so the mean percentage jump in the stock price is $k = \exp{(\mu_y + \sigma_y^2/2)} - 1$. That is,

$$\frac{dS_t}{S_{t-}} = (\mu_S - k\lambda)dt + \sigma_S dB_t + (e^y - 1)dN_t$$

We can have one such Radon-Nikodym derivative:

$$\frac{d\eta_t}{\eta_{t-}} = (e^{a+by_t-b\mu_y-\frac{1}{2}b^2\sigma_y^2} - 1)dN_t - (e^a - 1)\lambda dt$$

where a and b are predictable processes and where ξ_0=1. By construction, the process $\eta_t, 0 \leq t \leq T$ is a martingale of mean one so that the measure $\mathbb{Q}$ thus defined is indeed a probability measure. Under $\mathbb{Q}$ the jump arrival intensity $\lambda^{\mathbb{Q}}$ and the mean jump size $k^{\mathbb{Q}}$ change from their counterparts λ and k in the reference measure $\mathbb{P}$ to

$$\begin{aligned} \lambda^{\mathbb{Q}} &= \lambda e^a \\ 1 + k^{\mathbb{Q}} &= (1+k)e^{b\sigma_y^2} \end{aligned}$$

3.3 Pricing Relations in Presence of Jumps

3.3.1 Incomplete Market

The expected return on the stock is given by

$$\begin{aligned}\mu_S &= r + \sigma_S\phi + E^{\mathbb{P}}[X]\lambda^{\mathbb{P}} - E^{\mathbb{Q}}[X]\lambda^{\mathbb{Q}} \\ &= r + \sigma_S\phi + \int_{-1}^{\infty} x[\lambda^{\mathbb{P}}\mathbb{P}(dx) - \lambda^{\mathbb{Q}}\mathbb{Q}(dx)]\end{aligned} \tag{3.13}$$

Note that the stock is exposed to a diffusion risk B and to jump risk relating to N and X. The exposure to diffusion risk is measured by the volatility coefficient σ_S. To be hedged against this type of risk, we need one instrument with non-zero diffusion sensitivity, and consequently, there is one market price ϕ of diffusion risk.

On the other hand there is one market price of risk for each possible jump size. This follows from the fact that to be hedged against jump risk, we need one instrument for each possible jump size. To see this consider the following simple example where the stock is only exposed to jump risk. There are two possible jump sizes x_1 and x_2, and no diffusion risk. The change in the stock price is given by

$$\frac{dS_t}{dS_{t-}} = \mu_S - \lambda E[X]dt + x_1 dN_t^1 + x_2 dN_t^2 \tag{3.14}$$

where $dN_t^i = 1$ $(i = 1, 2)$ if a jump of size x_i occurs and zero, otherwise. The price of a derivative contract C is a function of the stock price and the change in its value is given by

$$\frac{dC_t}{C_{t-}} = \mu_C dt + f^C(x_1)dN_t^1 + f^C(x_2)dN_t^2 \tag{3.15}$$

where $f^C(x)$ is the percentage change in the claim price if the stock price jumps by x and μ_C is the drift of the claim price. At the moment, we are not concerned with the exact functional form of f^C and μ_C. To create a risk-free portfolio, we form a portfolio of the claim and the hedge instruments which is no longer exposed to any of the risk factors. With two risk factors, namely a jump of size x_1 and a jump of size x_2, the situation is similar to a trinomial model (where we implicitly assume that the two Poisson processes do not jump simultaneously), and we need in general two instruments exposed to these risk factors to eliminate jump risk.

The example can be easily generalized to more than two jump sizes. We need one hedge instrument for every possible jump size, and thus there is not one market price of jump risk, but there is one for each possible jump size. Consequently, for a continuous jump size distribution there are infinitely many market prices of jump risk. The total jump premium of the stock can be represented as the integral over the compensations for each individual jump size x. Obviously, the exposure of the stock to a jump of size x is just equal to x, and its contribution to the jump risk premium is equal to this exposure times the difference $\lambda\mathbb{P}(dx) - \lambda^{\mathbb{Q}}\mathbb{Q}(dx)$ between the intensity under the physical and under the risk-neutral measure. This difference thus includes all the information about the pricing of jump of size x.

If the intensities of a jump size x are the same under $\mathbb{P}$ and $\mathbb{Q}$, a jump of size x in the stock is not priced. To see what happens if a jump is not priced, consider a negative jump size $x < 0$ first. If the risk-neutral intensity is greater than the physical intensity, the difference $\lambda^{\mathbb{P}}\mathbb{P}(dx) - \lambda^{\mathbb{Q}}\mathbb{Q}(dx)$ is negative and the contribution to the total risk premium is positive. Intuitively, when it comes to pricing, the investor over-estimates the probability of this negative jump. For a positive jump the argument goes the other way around.

3.3.2 CAPM in a Jump-diffusion Economy

Recall the pure exchange economy with one representative agent and one perishable consumption good. As usual, the economy is endowed with a stochastic flow of the consumption good except that its rate evolves as a jump-diffusion process.

$$\frac{d\delta_t}{\delta_{t-}} = [\mu_\delta - \lambda E(X-1)]dt + \sigma_\delta dB_t^\delta + (X-1)dN_t \tag{3.16}$$

where μ_δ and σ_δ are the mean and standard deviation of the diffusion component. B^δ is the standard Brownian motion, N is a separated Poisson process with an intensity parameter λ. $X-1$ is the jump amplitude with $\ln X \sim N(\mu_x, \sigma_x^2)$.

Let the price dynamics of a security be the following jump-diffusion process

$$\frac{dS_t}{S_{t-}} = (\mu_S - \lambda k)dt + \sigma_S dB_t^S + (Y-1)dN_t \tag{3.17}$$

where μ_S and σ_S are the mean and standard deviation of the diffusion component, another Brownian motion B^S, N_t is the same Poisson process as above but the jump size is $Y-1$. $\ln Y \sim N(\mu_y, \sigma_y^2)$. $k \equiv \lambda E(Y-1)$.

It can be seen that the jump process increases the dimension of the basic uncertainty source by one. It is thus possible to set up a market structure where the same equilibrium argument is unaffected. So we can still use the state price deflator

$$\xi_t = e^{-\theta t}\frac{U_c(\delta_t)}{U_c(\delta_0)},$$

whose dynamics follow Itô's lemma pertaining to a Poisson process applied to the function $\xi_t = f(\delta_t)$.

Lemma 3.1 (Itô's lemma for jump-diffusion processes). *Given a process*

$$dx_t = \mu(x_{t-}, t-)dt + \sigma(x_{t-}, t-)dB_t + JdN_t,$$

let $f(x_t, t)$ be a continuously differentiable function of x and t, then,

$$df(x_t,t) = \left(\frac{\partial f}{\partial t} + \mu(x_t,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(xt,t)\frac{\partial^2 f}{2\partial x^2}\right)dt + \frac{\partial f}{\partial x}\sigma(x_t,t)dB_t + [f(x_{t-}+J) - f(x_t,t-)]$$

With CRRA utility function,

$$\xi_t = e^{-\theta t}\left(\frac{\delta_t}{\delta_0}\right)^{-\gamma} \tag{3.18}$$

the dynamics of ξ_t can be represented as

$$\frac{d\xi_t}{\xi_{t-}} = \left[-\theta - \gamma[\mu_\delta - \lambda E(X-1)] + \frac{1}{2}(\gamma^2+\gamma)\sigma_\delta^2\right] dt - \gamma\sigma_\delta dB_t^\delta + \left[\frac{\xi_{(}X\delta_{t-})}{\xi(\delta_{t-})} - 1\right] dN_t.$$

We can also substitute the differential equation for $d\delta_t/\delta_t$ to get

$$\frac{d\xi_t}{\xi_t} = -\gamma\frac{d\delta_t}{\delta_t} + [-\theta + \frac{1}{2}(\gamma^2+\gamma)\sigma_\delta^2]dt + [\frac{\xi_{(}X\delta_{t-})}{\xi(\delta_{t-})} - 1 + \gamma(X-1)]dN_t$$

which suggests that $d\delta_t/\delta_t$ and $d\xi_t/\xi_t$ are not perfectly correlated. Rewrite the process for the state price deflator as

$$\frac{d\xi_t}{\xi_t} = [\mu_\xi - \lambda E(L-1)]dt + \sigma_\xi dB_t^\delta + (L-1)dN_t \tag{3.19}$$

where

$$\begin{aligned} \mu_\xi &= -\theta - \gamma[\mu_\delta - \lambda E(X-1)] + \frac{1}{2}(\gamma^2+\gamma)\sigma_\delta^2 + \lambda E(L-1) \\ \sigma_\xi &= -\gamma\sigma_\delta \\ L &= \frac{\xi(X\delta_t)}{\xi(\delta_t)} = X^{-\gamma} \end{aligned}$$

Using the property $\mu_\xi = -r$ we can obtain the equilibrium interest rate.

$$r = \theta + \gamma[\mu_\delta - \lambda E(X-1)] - \frac{1}{2}(\gamma^2+\gamma)\sigma_\delta^2 - \lambda E(L-1) \tag{3.20}$$

To derive the pricing relations, we need to invoke the product rule of generalized Itô's Lemma.

Lemma 3.2 (Itô's lemma for two correlated processes). *Given*

$$\begin{aligned} dx_1 &= \mu_1 dt + \sigma_1 dB_t^1 + J_1 dN_t \\ dx_2 &= \mu_2 dt + \sigma_2 dB_t^2 + J_2 dN_t \end{aligned}$$

where B^1 and B^2 are two Brownian motion processes correlated through correlation coefficient ϖ and N_t is a Poisson jump process. Then the dynamics of an asset whose payoff is $f = f(x_1, x_2, t)$ is

$$\begin{aligned} df = &\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}\mu_1 + \frac{\partial f}{\partial x_2}\mu_2 + \frac{\partial^2 f}{2\partial x_1^2}\sigma_1^2 + \frac{\partial^2 f}{\partial x_1 \partial x_2}\varpi\sigma_1\sigma_2 + \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2}\sigma_2^2\right) dt \\ &+ \frac{\partial f}{\partial x_1}\sigma_1 dB_t^1 + \frac{\partial f}{\partial x_2}\sigma_2 dB_t^2 + [f(x_1+J_1, x_2+J_2, t) - f(x_1, x_2, t)]dN_t. \end{aligned}$$

Therefore we have,

$$\begin{aligned}\frac{d\xi_t S_t}{\xi_t S_t} &= [\mu_\xi - \lambda E(L-1) + \mu_S - \lambda E(Y-1) + \varpi\sigma_\xi\sigma_S]dt \\ &\quad + \sigma_\xi dB_t^\delta + \sigma_S dB_t^S + (YL-1)dN_t\end{aligned}$$

where ϖ denotes the correlation coefficient between B^δ and B^S. By the martingale property, it must hold that

$$0 = \mu_\xi - \lambda E(L-1) + \mu_S - \lambda E(Y-1) + \varpi\sigma_\xi\sigma_S + \lambda E[YL-1]$$

and since $\mu_\xi = -r$, we have

$$\begin{aligned}\mu_S &= r + \lambda E(L-1) + \lambda E(Y-1) - \varpi\sigma_\xi\sigma_S - \lambda E[YL-1] \\ &= r - \varpi\sigma_\xi\sigma_S - \lambda E[(L-1)(Y-1)] \qquad (3.21) \\ &= r - \varpi\sigma_\xi\sigma_S - \lambda E(L-1)E(Y-1) - \lambda \mathrm{Cov}[L,Y]\end{aligned}$$

where $\mathrm{Cov}[L, Y]$ is the covariance between the state-price deflator jump size and the asset price jump size.

This result (3.21) corresponds to the security market line equation in a jump-diffusion economy. The risk premium of a security can be separated into two parts: the premium for the continuous part of the return and the premium for the jump part. Each premium is proportional to its covariance with the corresponding part of the state price deflator. Because of the jump component, the asset's expected return earns an additional jump risk premium. This jump risk has two items. The first component is the negative of the product of the expected net asset jump size and the state-price deflator jump size times the jump frequency. The second component is the negative of the covariance between the asset and the pricing standard jump size times the jump arrival frequency.

The covariance between the jump parts of the security price and state price deflator is zero unless jumps occur simultaneously. Thus, if the state price deflator does not jump simultaneously with the security, then the jump risk of the security will not be priced. This is a very intuitive result. Ordinarily an upward jump in a security price should produce a downward jump in state price deflators, since investors holding the security will be wealthier and hence value dollars less highly at the margin. This negative covariance between the jump parts implies that the jump risk will earn a premium. However, if the risk is diversified, then state price deflators will not jump. In this case, the jump risk will not earn a premium.

Since

$$\mathrm{Cov}\left[\frac{d\xi_t}{\xi_t}, \frac{dS_t}{S_t}\right]$$

$$= E\{\big([-\lambda E(L-1)]dt + \sigma_\xi dB_t^\delta + (L-1)dN_t\big)\,\big([\mu_S - \lambda k]dt + \sigma_S dB_t^S + (Y-1)dN_t\big)\}$$

$$= \varpi\sigma_\xi\sigma_S - \lambda E(L-1)E(Y-1)dt - \lambda\mathrm{Cov}[(L-1),(Y-1)],$$

the security market line can be rewritten as

$$E_t\left[\frac{dS_t}{S_t}\right] = rdt - \mathrm{Cov}\left[\frac{d\xi_t}{\xi_t}, \frac{dS_t}{S_t}\right]. \tag{3.22}$$

Again, the negative covariance between the price change and the state price deflator change determines the security risk premium. So like the diffusion economy, the fundamental valuation equation can be generalized by endogenizing the state price deflator to versions of the capital asset pricing model (CAPM), the inter-temporal CAPM and the consumption CAPM. In another word, the security market line is valid whenever it can be formulated, i.e., whenever the state price deflator exists, risk premia exist and their local covariances exist. The intuition for this result is very strong. For example, a security which tends to have its high returns when state prices are low will not be a desirable asset and will trade at a low price, relative to its expected payoff. Thus a large negative correlation with state prices will translate into a high risk premium.

An important perception is that asset pricing theories are largely developed based on continuous price processes, they also shed insightful light on discontinuous processes. Restrictions such as systematic jump risk [37] are sufficient but not necessary at all. If the state price deflator process is continuous, then all the jumps represent idiosyncratic risk and the pricing relations look exactly the same as in existing models. On the other hand, if the state price deflator can jump, we generally retain most of the economic content of existing models, but we lose some of the local linearity, and subsequently the pricing formulas are less simple.

Chapter 4

Option Pricing Formulas

Last chapter has justified the economic content of equilibrium-based valuation. This chapter will derive formally the option pricing results alongside with all other authors' formulas that feature jumps in stock returns. We draw inspiration from our formula and try to derive others in an identic format. That is, the call premium is a weighted average of the conditional Black-Scholes premiums, each conditioned on the number of jumps. The weights are the Poisson probability of n jumps with the "adjusted" jump arrival frequency. The Black-Scholes values when there are n jumps are computed using the "adjusted" risk-free interest rate and volatility. Moreover, external parameters like the discount rate are made absent from the formulas and the dividend rate is condensed so that the option price is a function of variables mostly observable and universal.

In addition to the formalization, we compare them in terms of the theoretical significance as well as practical value. Some numerical analysis is conducted to verify our inferences and qualitative conjecture.

Furthermore, an extension is attempted to a mean-reversion jump-diffusion short rate process. The equilibrium approach is still applicable to interest rate derivatives thereof.

4.1 Cox & Ross Pure Jump Model

The literature addressing the discontinuity in stock processes can be tracked back to Cox and Ross (1976) [24] who put forward a pure jump process

$$\frac{dS_t}{S_{t-}} = \mu_S dt + (Y-1)dN_t. \tag{4.1}$$

As the name implied, there is no diffusion component. The percentage change in the value of the stock on the interval from to is composed of a drift term and a jump term. In contrast to the continuous process, it follows a deterministic movement upon which are superimposed discrete jumps. Most of the time the stock price grows at rate μ_S. However occasionally it exhibits jumps equal to $Y-1$ times the current price. They used a constant Y. Jumps occur to a Poisson process at the rate of λ. The terminal stock price distribution is log-Poisson. By assuming constant jump amplitude, the market is complete, a risk-free hedge could be formed and used to value options.

The risk-neutral measure corresponding to jump process $\mathbb{Q}$ also exists, under which $\Psi(a,b) = \sum_{n=a}^{\infty} \frac{e^{-b}b^n}{n!}$ is the complementary Poisson distribution function for S_t with argument a and b. $b = (r-\mu_S)Y\tau/(Y-1)$. a stands for the minimum number of jumps that the stock must make for the call to finish in the money. Thus a is the smallest non-negative integer such that $S_t e^{\mu_S \tau} Y^a > K$. The final payoff of an option written on a stock with such a pure jump is:

$$\max(S_T - K, 0) = \begin{cases} 0, & \text{for all } n < a; \\ S_T - K, & \text{for all } n \geq a. \end{cases}$$

A European call option has a price formula

$$\begin{aligned} C_{CR} &= E_t^{\mathbb{Q}}[e^{-r\tau}\max(S_T - K, 0)] \\ &= S_t\Psi(a,b) - Ke^{-r\tau}\Psi(a, b/K). \end{aligned} \tag{4.2}$$

This formula looks a lot like the Black-Scholes formula but with Poisson instead of Gaussian distribution function. A pure jump model leads to the situation where the implied distribution has a fat right tail and a thin left tail (the opposite to that observed for equities). Arguably the model is unrealistic in that jumps can only be positive.

4.2 Merton's Formula

Merton (1976) [49] proposes the first jump-diffusion model for stock prices:

$$\frac{dS_t}{S_{t-}} = (\mu_S - \lambda k)dt + \sigma_S dB_t + (Y-1)dN_t$$

with parameters as noted before.

4.2.1 Replicating Portfolio

To derive an equation for an option price contingent on a stock following a jump-diffusion process described above, let us attempt to set up a perfect hedge with one option and n_t shares of the stock like Black and Scholes (1973) [18]. The value of this portfolio at time t is $V_t = C_t - n_t S_t$. The marginal change in the value of the portfolio is

$$\begin{aligned} dV_t &= dC_t - ndS_t \\ &= \left(\frac{\partial C_t}{\partial t} + C_S(\mu_S - \lambda k)S_t + \frac{1}{2}C_{SS}S_t^2\sigma_S^2\right)dt + C_S\sigma_S S_t dB_t + [C(YS_{t-}) - C(S_t)]dN_t \\ &- n_t S_t[(\mu_S - \lambda k)dt + \sigma_S dB_t + (Y-1)dN_t]. \end{aligned}$$

There are two random terms dB_t and dN_t. With only one choice parameter n_t, it would not be possible (in general) to form a perfect hedge between the stock and the option since they are not perfectly correlated. With the hedge ratio $n_t = \partial C_t / \partial S_t$ substituted, we get

$$dV_t = \left(\frac{\partial C_t}{\partial t} + \frac{1}{2}C_{SS}S_t^2\sigma_S^2\right)dt + \{-C_S(Y-1) + [C(YS_{t-}) - C(S_t)]\}dN_t.$$

This portfolio is still risky due to the jump component. Merton (1976) [49] invokes an assumption that the jump is firm-specific or non-systematic. In other words, the rare events that cause sudden discontinuous changes in the price of a stock affect only that stock, or, at most the stock of a few other firms (such as the other party in a litigation or in a merger). The risk of these sudden changes will be diversifiable and the market will consequently pay no risk premium over the risk-free rate for bearing this risk. Thus the equilibrium expected rate of return on the partially hedged portfolio is the risk-free rate r. We can have:

$$\frac{\partial C_t}{\partial t} + \frac{1}{2} C_{SS} S_t^2 \sigma_S^2 S_t + (r - \lambda k) C_S + \lambda [C(Y S_{t-}) - C(S_{t-})] = r C_t. \tag{4.3}$$

Along with boundary conditions this is a mixed difference-differential equation. When $\lambda = 0$, it reduces to the Black-Scholes PDE. The solution also satisfies the Feynman-Kac Theorem:

$$C_t = E_t^{\mathbb{Q}}[e^{-r\tau} \max{(S_T - K, 0)}]$$

where

$$\frac{dS_t}{S_{t-}} = (r - \lambda k)dt + \sigma_S d\tilde{B}_t + (Y - 1)dN_t.$$

$\tilde{B}_t = B_t + \phi t; \tilde{B}_0 = 0$ is a standard Brownian motion under $\mathbb{Q}$ measure. Let us work out the explicit expression for this option price. Define $\alpha_S^{\mathbb{Q}} = r - \frac{\sigma_S^2}{2} - \lambda k$ and $Z_S^{\mathbb{Q}} = \sigma_S(\tilde{B}_T - \tilde{B}_t) + \sum_{j=n_t+1}^{n_T} y_j$, $y_j = \ln Y_j$ as described in section 3.2, then

$$C_t = E_t^{\mathbb{Q}} \left[e^{-r\tau} \max(S_t e^{\alpha_S^{\mathbb{Q}} + Z_S^{\mathbb{Q}}} - K, 0) \right]$$

Taking expectation conditional on knowing that exactly n Poisson jumps occur during the life of the option

$$\begin{aligned} C_t &= E_t^{\mathbb{Q}}[e^{-r\tau} \max{(S_T - K, 0)}] \\ &= E_t^{\mathbb{Q}} \left[E_t^{\mathbb{Q}}[e^{-r\tau} \max{(S_T - K, 0)}] | N_T - N_t = n \right] \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} E_t^{\mathbb{Q}} \left[e^{-r\tau} \max(S_t e^{\alpha_S^{\mathbb{Q}}(n) + Z_S^{\mathbb{Q}}} - K, 0) \right] \end{aligned} \tag{4.4}$$

where $Z_S^Q(n) = \sigma_S(\tilde{B}_T - \tilde{B}_t) + \sum_{j=1}^{n} y_j$. The mean and variance of $Z_S^Q(n)$ is $E[Z_S^Q(n)] = n\mu_y$ and $\mathrm{Var}[Z_S^Q(n)] = \sigma_S^2\tau + n\sigma_y^2$. Therefore $\ln(S_T/S_t)$ follows the normal distribution

$$\ln(S_T/S_t) \sim N(\alpha_S^Q\tau + n\mu_y, \sigma_S^2\tau + n\sigma_y^2)$$

Note that $E_t^Q[e^{-r\tau}\max(S_t e^{\alpha_S^Q + Z_S^Q} - K, 0)]$ mirrors the form of the Black-Scholes model, so we can borrow from the Black-Scholes formula.

$$\begin{aligned} C_t &= E_t^Q[e^{-r\tau}\max(S_T - K, 0)] \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} e^{-\lambda k\tau + n(\mu_y + \frac{1}{2}\sigma_y^2)} C_{BS}(S_t, \tau; r_M, \sigma_M, K) \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda_M\tau}(\lambda_M\tau)^n}{n!} C_{BS}(S_t, \tau; r_M, \sigma_M, K) \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} \lambda_M &= \lambda E[Y] \\ r_M &= r + \lambda E[1 - Y] + \frac{n}{\tau}(\mu_y + \frac{1}{2}\sigma_y^2) \\ \sigma_M^2 &= \sigma_S^2 + \frac{n}{\tau}\sigma_y^2 \end{aligned}$$

It can be seen that the Merton formula is essentially a weighted sum of Black-Scholes values (C_{BS}). λ_M may be viewed as the "adjusted" jump arrival intensity, r_M the "conditional" risk-free interest rate and σ_M the "transformed" volatility. Actually λ_M can be regarded as the jump intensity under the risk-neutral measure as well.

This type of adjustment is intuitively appealing. Consider the jump frequency, for positive shocks to the economy (good state), the risk-adjusted jump arrival rate is lower than the true arrival rate, for negative shocks (bad state) the risk-adjusted jump arrival rate is higher than the true arrival rate. This shifting of "probability" mass from the good states to the bad states is how the risk premium is extracted in this economy.

4.2.2 Fourier Inversion

In the following we shall produce the option pricing formula in Merton's model through another method, *Fourier inversion.* The intention is to illustrate the technique of Fourier inversion, which will be necessitated when the basic variable's density function is complicated.

The risk-neutralized model has the same form as in the Black-Scholes model, but with the drift $\mu_S - q$ replaced by $r - \lambda k - q$. The SDE under the risk neutral $\mathbb{Q}$ measure has the explicit solution

$$S_t = S_0 e^{(r-q-\lambda k-\frac{1}{2}\sigma_S^2)t+\sigma\tilde{B}_t}\prod_{n=1}^{n_t} Y_n$$

Let $S_t^* = \ln S_t$. Then conditional on S_t^* and $n_T - n_t = n$, the distribution of S_T^* is normal with mean $(r - q - \lambda k - \frac{1}{2}\sigma_S^2)\tau$ and variance $\sigma_S^2\tau + n\sigma_y^2$. The time t price of a call option $C_t(S_t, \tau)$ can be written as

$$C_t = E_t^{\mathbb{Q}}[e^{-r\tau}\max(S_T - K, 0)] = e^{-r\tau}E_t^{\mathbb{Q}}[(S_T - K)\mathbf{I}_{S_T>K}].$$

The term $E_t^{\mathbb{Q}}[\mathbf{I}_{S_T>K}] = E_t^{\mathbb{Q}}[\mathbf{I}_{S_T^*>\ln K}] = \mathbb{Q}(S_T > K)$ can be computed as follows. Let $\wp(a) = E_t^{\mathbb{Q}}[e^{aiS_T^*}]$ be the characteristic function of S_T^* where a is an arbitrary real number and $i = \sqrt{-1}$. If this characteristic function is known analytically, we can use the Fourier inversion formula [34]

$$\mathbb{Q}(S_T^* > X) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\wp(a)\frac{e^{-iaX}}{ia}\right) da \tag{4.6}$$

where Re refers to the real part.

The characteristic function in Merton's [49] model can be computed explicitly.

$$\begin{aligned}
\wp(a) = E_t^{\mathbb{Q}}[e^{iaS_T^*}] &= \sum_{n=0}^{\infty}\frac{(\lambda\tau)^n e^{-\lambda\tau}}{n!}E_t^{\mathbb{Q}}[e^{iaS_T^*}|S_t^*, n_T - n_t = n] \\
&= \sum_{n=0}^{\infty}\frac{(\lambda\tau)^n e^{-\lambda\tau}}{n!}e^{ia[S_t^*+(r-q-\lambda k-\frac{1}{2}\sigma_S^2)\tau]-\frac{1}{2}a^2\sigma^2\tau} \times \left[\exp(ia\mu_y - \frac{1}{2}a^2\sigma_y^2)\right]^n \qquad (4.7) \\
&= \exp\left\{ia[S_t^* + (r - q - \lambda k - \frac{1}{2}\sigma_S^2)\tau] - \frac{1}{2}a^2\sigma^2\tau - \lambda\tau + \lambda\tau e^{ia\mu_y-\frac{1}{2}a^2\sigma_y^2}\right\}
\end{aligned}$$

The term $E_t^{\mathbb{Q}}[S_T \mathbf{I}_{S_T > K}]$ can be transformed to a probability computation by using the change of measure. Let $\eta_t = e^{-(r-q)t} S_t / S_0$ and call the measure $d\mathbb{R} = \eta_T d\mathbb{Q}$. Then $E_t^{\mathbb{Q}}[\eta_T \wp(S_T^*)] = E_t^{\mathbb{R}}[\wp(S_T^*)]$ and so $E_t^{\mathbb{Q}}[S_T \mathbf{I}_{S_T > K}] = e^{(r-q)\tau} S_t \mathbb{R}(S_T^* > \ln K)$. The characteristic function under the new measure can be computed explicitly as

$$\begin{aligned} g(a) = E_t^{\mathbb{R}}[e^{iaS_T^*}] &= E_t^{\mathbb{Q}}[\frac{\eta_T}{\eta_t} e^{iaS_T^*}] \\ &= e^{-(r-q)\tau - S_t^*} E_t^{\mathbb{Q}}[e^{(1+ia)S_T^*}] \\ &= e^{-(r-q)\tau - S_t^*} \wp(-i+a) \end{aligned} \tag{4.8}$$

where $\wp(\cdot)$ is defined above. The probability $\mathbb{R}(S_T^* > \ln K)$ can be computed using the Fourier inversion with $\wp$ replaced by g.

To summarize, the price of a European call in Merton's jump- diffusion model is

$$C_M(S_T, \tau) = S_t e^{-q\tau} \mathbb{R}(S_T > K) - e^{-r\tau} \mathbb{Q}(S_T > K) \tag{4.9}$$

where the two probabilities can be computed using the Fourier transform inversion and the explicit formulas. The result is equivalent to Merton's original formula that expresses the option price as an infinite weighted sum of Black-Scholes values. Although the expression of the characteristic functions are lengthy, the final equation involves two one-dimensional integrals that can be computed numerically.

4.3 Naik & Lee Formula on Market Portfolio

Naik and Lee (1990) [51] consider a continuous-time infinite-horizon variation of the Lucas model. They limit the analysis to a single firm completely financed by equity with one share outstanding. Therefore they interpret the equity share of the firm as the market portfolio and the dividends of the firm as the aggregate dividends in the economy. The equity share is perfectly divisible and competitively traded at instant t for a price S_t (in terms of the consumption good). Also available for trading are other claims in zero net supply. Aggregate dividends are assumed to follow a compound diffusion-Poisson process

$$\frac{d\delta_t}{\delta_t} = [\mu_\delta - \lambda E(X-1)]dt + \sigma_\delta dB_t^\delta + (X-1)dN_t \tag{4.10}$$

where μ_δ is the instantaneous expected rate of change of aggregate dividends, σ_δ is the instantaneous variance of rate of change in aggregate dividends conditional on the Poisson event not occurring, B^δ is a standard Brownian motion, N is a Poisson process with parameter λ, $X-1$ is the jump size and $\ln X \sim N(\mu_x, \sigma_x^2)$.

The representative agent's inter-temporal utility (assuming a CRRA form) maximization problem gives rise to the Euler equation.

$$S_t = \frac{E_t\left[\int_t^\infty e^{-\theta(u-t)}U_c(c_u)\delta_u du\right]}{U_c(c_t)} \tag{4.11}$$

At this price, the investor will never change her current holdings of the security even though he is given the opportunity to do so. The authors point out that the equilibrium real price at time t of the market portfolio in the economy is a constant.

$$\begin{aligned}
S_t &= \frac{E_t\left[\int_t^\infty e^{-\theta(u-t)}U_c(\delta_u)\delta_u du\right]}{U_c(\delta_t)} \\
&= \int_t^\infty E_t\left[e^{-\theta(u-t)}(\frac{\delta_u}{\delta_t})^{-\gamma}\delta_u\right]du \\
&= \delta_t\int_t^\infty E_t\{\,\exp[-\theta(u-t)+(1-\gamma)[\mu_\delta-\frac{\sigma_\delta^2}{2}-\lambda E(X-1)](u-t) \\
&\qquad\qquad +\sigma_\delta(B_u-B_t)+\sum_{j=n_t+1}^{n_u}x_j]\}du \\
&= \delta_t\int_t^\infty e^{\{\theta+(1-\gamma)[\mu_\delta-\frac{\sigma_\delta^2}{2}-\lambda E(X-1)]\}(u-t)}E_t\left(e^{(1-\gamma)\sigma_\delta(B_u-B_t)}\right)\times E_t\left(e^{(1-\gamma)\sum_{j=n_t+1}^{n_u}x_j}\right)du \\
&= \delta_t\int_t^\infty e^{\{\theta+(1-\gamma)[\mu_\delta-\frac{\sigma_\delta^2}{2}-\lambda E(X-1)]}(u-t)\}E_t\left(e^{(1-\gamma)^2\sigma_\delta^2(u-t)}\right)\times E_t\left(e^{\lambda E[X^{1-\gamma}](u-t)}\right)du \\
&= \delta_t\int_t^\infty \exp\{-[\theta-g(\gamma)](u-t)\}du
\end{aligned}$$

where $g(\gamma) \equiv (1-\gamma)[\mu_\delta - \lambda E(X-1)] - \frac{1}{2}\gamma(1-\gamma)\sigma_\delta^2 + \lambda[E(X^{1-\gamma})-1]$. Finally,

$$S_t = \frac{\delta_t}{\theta - g(\gamma)}. \tag{4.12}$$

So it is clear that the equilibrium stochastic evolution of S_t is given by

$$\frac{dS_t}{S_t} = [\mu_\delta - \lambda E(X-1)]dt + \sigma_\delta dB_t^\delta + (X-1)dN_t \tag{4.13}$$

This, then, endogenizes a mixed jump-diffusion process for the price of the market portfolio. It also shows that the endogenously derived dividend yield on the market portfolio in the above economy q is constant.

$$q = \theta - g(\gamma) \tag{4.14}$$

Likewise, Naik and Lee show that the term structure in the present economy is flat with the instantaneous risk-free interest rate given by

$$\begin{aligned} r &= \theta - g(\gamma+1) \\ &= \theta + \gamma[\mu_\delta - \lambda E(X-1)] - \frac{1}{2}\gamma(\gamma+1)\sigma_\delta^2 - \lambda[E(X^{-\gamma})-1] \end{aligned} \tag{4.15}$$

The call option price formula is obtained by using (4.11) for a standard European option.

$$C_N = \sum_{n=0}^{\infty} \frac{e^{-\lambda_N \tau}(\lambda_N \tau)^n}{n!} C_{BS}(S_t, \tau; r_N, q, \sigma_N, K) \tag{4.16}$$

where

$$\begin{aligned} \lambda_N &= \lambda E[X^{1-\gamma}] \\ r_N &= r + \frac{n}{\tau}[\mu_x + (1-2\gamma)\frac{\sigma_x^2}{2}] + \lambda\{E[X^{-\gamma}] - E[X^{1-\gamma}]\} \\ \sigma_N^2 &= \sigma_\delta^2 + \frac{n}{\tau}\sigma_x^2 \end{aligned}$$

Our expressions do not appear the same as those in Naik and Lee's original paper. Besides notational differences, there are two major modifications for the convenience of comparison and investigation. First, it is arranged in a way that the dividend rate in computing C_{BS}, the Black Scholes value corresponding to each possible jump is a constant and the jump intensity differs. Secondly, expressions for the interest rate and dividend yield have been incorporated so that variables such as discount rate θ and the expected rate of endowment μ_δ do not show in the final formula.

4.4 Ahn's Formula

Ahn (1992) [1]'s setting is a production economy: there are a single good and n production processes whose return follows jump-diffusion processes. Given her budget constraint, each consumer chooses c the consumption, a the vector of the proportion of wealth W to be invested in each of the n production processes, the number of contingent claims and the amount of risk-free borrowing or lending, in order to maximize expected utility given by $E[\int_0^T \ln c_t dt]$. The indirect utility function J is determined by solving the maximization problem. Since consumers are identical, all wealth is invested in the production processes in equilibrium.

The dynamics of primary assets are assumed to follow a jump-diffusion process as defined in prior sections:

$$\frac{dS_t}{S_t} = [\mu_S - q - \lambda E(Y-1)]dt + \sigma_S B_t^S + (Y-1)dN_t$$

Ahn and Thompson 1988 [2] proves that the optimal consumption is equal to $c^* = \theta W$ in this economy and hence the level of wealth also follows a jump-diffusion process given by

$$\frac{dW_t}{W_t} = [\mu_W - \theta - \lambda_W E(D-1)] + \sigma_W dB_t^W + (D-1)dN_t \qquad (4.17)$$

where μ_W is expected growth rate of wealth and σ_W is the volatility. B^W is a standard Brownian motion different from B^S associated with the stock price movement. Let ϖ denote the correlation coefficient between B^S and B^W. But the jump process N_t is assumed to have the same intensity λ as the one above due to simultaneous information arrivals. D is the jump size, lognormally distributed with μ_d and σ_d^2 and ρ denote the correlation coefficient between y and d, the logarithm of the stock price jump size and the wealth jump size.

The value at time t of any contingent claim paying C_T at time T is given by

$$C_t = E_t \left[e^{-\theta(T-t)} \frac{J_W(W_T)}{J_W(W_t)} C_T \right] \tag{4.18}$$

where $J(W_t)$ is the value function, which is just the utility of the optimal consumption at time t. Subscripts refers to first-order derivative so $J_W(W_t) = U_c(c_t) = 1/(\theta W_t)$. Applying the valuation equation for a default-free bond yielding 1 at time T yields the risk-free interest rate

$$r = \mu_W - \sigma_W^2 - \lambda E(D-1) - \lambda\{E[D^{-1}] - 1\}. \tag{4.19}$$

Notice that Ahn also has a flat term structure but the risk free interest rate does not include the discount rate as in ours. With log utility, the Naik Lee is similar except for the existence of discount rate, too. This difference comes from the fact that Ahn's is expressed with production parameters (in a production economy) but ours is expressed with consumption parameters in an exchange economy.

The price at time t of a European call option is

$$C_A = \sum_{n=0}^{\infty} \frac{e^{-\lambda_A \tau}(\lambda_A \tau)^n}{n!} C_{BS}(S_t, \tau; r_A, q, \sigma_A, K) \tag{4.20}$$

where

$$\begin{aligned} \lambda_A &= \lambda E[YD^{-1}] \\ r_A &= r + \frac{n}{\tau}(\mu_y - \rho\sigma_d\sigma_y + \frac{\sigma_y^2}{2}) + \lambda\{E[D^{-1}] - E[YD^{-1}]\} \\ \sigma_A^2 &= \sigma_S^2 + \frac{n}{\tau}\sigma_y^2 \end{aligned}$$

4.5 Our Framework

4.5.1 Option Pricing Formula

We make use of the following conditional expectation formulas [57] to obtain our option pricing results.

Lemma 4.1. *If random variables x and y are bivariate normally distributed, then*

$$\begin{aligned} E[e^{y} I_{x \geq a}] &= e^{E(y)+\frac{1}{2}Var(y)} \Phi\left(\frac{-a+E(x)+Cov(x,y)}{\sqrt{Var(x)}}\right) \\ E[e^{x+y} I_{x \geq a}] &= e^{E(x)+E(y)+\frac{1}{2}Var(x+y)} \Phi\left(\frac{-a+E(x)+Var(x)+Cov(x,y)}{\sqrt{Var(y)}}\right) \end{aligned}$$

Now let us use the equilibrium pricing equation to value a standard European call option.

$$C_t = E_t[\xi_{t,T} C_T] \tag{4.21}$$

where

$$\begin{aligned} C_T &= \max\left(S_T - K, 0\right) \\ S_T &= S_t \exp\left[(\mu_S - \frac{1}{2}\sigma_S^2 - \lambda k)\tau + \sigma_S(B_T^S - B_t^S) + \sum_{j=n_t+1}^{n_T} y_j\right] \\ \xi_{t,T} &= \exp\left[[\mu_\xi - \frac{1}{2}\sigma_\xi^2 - \lambda E(L-1)]\tau + \sigma_\xi(B_T^\delta - B_t^\delta) + \sum_{j=n_t+1}^{n_T} l_j\right] \end{aligned}$$

Let $\alpha_S = \mu_S - \frac{\sigma_S^2}{2} - \lambda k$ and $Z_S = \sigma_S(B_T^S - B_t^S) + \sum_{j=n_t+1}^{n_T} y_j$, $\alpha_\xi = \mu_\xi - \frac{\sigma_\xi^2}{2} - \lambda E(L-1)$ and $Z_\xi = \sigma_\xi(B_T^\xi - B_t^\xi) + \sum_{j=n_t+1}^{n_T} l_j$, then the option valuation equation can be written as

$$C_t = E_t[e^{\alpha_\xi \tau + Z_\xi} \max\left(S_t e^{\alpha_S \tau + Z_S} - K, 0\right)]$$

Take expectations conditional on knowing that exactly n Poisson jumps will occur during the life of the option.

$$\begin{aligned}
C_t &= E_t[e^{\alpha_\xi \tau + Z_\xi} \max\left(S_t e^{\alpha_S \tau + Z_S} - K, 0\right)] \\
&= E\left[E[e^{\alpha_\xi \tau + Z_\xi} \max\left(S_t e^{\alpha_S \tau + Z_S} - K, 0\right) | n_T - n_t = n]\right] \\
&= \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} E\left[e^{\alpha_\xi \tau + Z_\xi(n)} \max(S_t e^{\alpha_S \tau + Z_S(n)} - K, 0)\right] \\
&= \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \left[S_t E_t[e^{\alpha_\xi \tau + Z_\xi(n) + \alpha_S \tau + Z_S(n)} \mathbf{I}_{S_T \geq K}] - K E_t[e^{\alpha_\xi \tau + Z_\xi(n)} \mathbf{I}_{S_T \geq K}]\right]
\end{aligned} \tag{4.22}$$

where $Z_\xi(n) = \sigma_\xi(B_T^\delta - B_t^\delta) + \sum_{j=1}^n l_j$ and $Z_S(n) = \sigma_S(B_T^S - B_t^S) + \sum_{j=1}^n y_j$. The mean and variance of $Z_\xi(n)$ is $E[Z_\xi(n)] = n\mu_l$ and $\text{Var}[Z_\xi(n)] = \sigma_\xi^2 \tau + n\sigma_l^2$. Similarly, the mean and variance of $Z_S(n)$ is $E[Z_S(n)] = n\mu_y$ and $\text{Var}[Z_S(n)] = \sigma_S^2 \tau + n\sigma_y^2$. Let$\varpi$ denote the correlation coefficient between B^δ and B^S, ρ denote the correlation coefficient between $l = \ln L$ and $y = \ln Y$, then $\text{Cov}[Z_\xi(n), Z_S(n)] = \varpi\sigma_\xi\sigma_S\tau + n\rho\sigma_l\sigma_y$. Now focus on the conditional expectation. For the first part,

$$\begin{aligned}
& E_t[\exp\left(\alpha_\xi \tau + Z_\xi(n) + \alpha_S \tau + Z_S(n)\right) \mathbf{I}_{S_T \geq K}] \\
&= \exp\left(\alpha_\xi \tau + E[Z_\xi(n)] + \alpha_S \tau + E[Z_S(n)] + \frac{1}{2}\text{Var}[Z_S(n)] + \frac{1}{2}\text{Var}[Z_\xi(n)] + \text{Cov}[Z_\xi(n), Z_S(\right. \\
&\times \Phi\left(\frac{\ln(S_t/K) + \alpha_S \tau + E[Z_S(n)] + \text{Var}[Z_S(n)] + \text{Cov}[Z_S(n), Z_\xi(n)]}{\sqrt{\text{Var}[Z_S(n)]}}\right) \\
&= \exp\{[\mu_\xi - \lambda E(L-1) + \mu_S - \lambda E(Y-1) + \varpi\sigma_\xi\sigma_S]\tau + n(\mu_l + \mu_y + \frac{\sigma_l^2}{2} + \rho\sigma_l\sigma_y + \frac{\sigma_y^2}{2})\} \\
&\times \Phi\left(\frac{\ln(S_t/K) + r_H \tau}{\sqrt{\sigma_H^2 \tau}} + \frac{1}{2}\sqrt{\sigma_H^2 \tau}\right)
\end{aligned}$$

where

$$\begin{aligned}
r_H &= \mu_S - \lambda E(Y-1) + \varpi\sigma_\xi\sigma_S + \frac{n}{\tau}(\mu_l + \frac{\sigma_y^2}{2} + \rho\sigma_l\sigma_y) \\
\sigma_H^2 \tau &= \sigma_S^2 \tau + n\sigma_y^2.
\end{aligned}$$

The second part becomes

$$
\begin{aligned}
E_t\{\exp[\alpha_\xi\tau + Z_\xi(n)]\mathbf{I}_{S_T\geq K}\} &= \exp\{\alpha_\xi\tau + E[Z_\xi(n)] + \frac{1}{2}\mathrm{Var}[Z_\xi(n)]\}\\
&\quad\times\Phi\left(\frac{\ln(S_t/K) + \alpha_S\tau + E[Z_S(n)] + \mathrm{Cov}[Z_S(n), Z_\xi(n)]}{\sqrt{\mathrm{Var}[Z_S(n)]}}\right)\\
= \exp\{[\mu_\xi - \lambda E(L-1)\tau] + n\mu_l + n\sigma_l^2/2\} &\times \Phi\left(\frac{\ln(S_t/K) + r_H\tau}{\sqrt{\sigma_H^2\tau}} - \frac{1}{2}\sqrt{\sigma_H^2\tau}\right).
\end{aligned}
$$

Substitute the expressions for the two parts into (4.22) we have

$$
\begin{aligned}
& S_tE_t[e^{\alpha_\xi\tau + Z_\xi(n) + \alpha_S\tau + Z_S(n)}\mathbf{I}_{S_T\geq K}] - KE_t[e^{\alpha_\xi\tau + Z_\xi(n)}\mathbf{I}_{S_T\geq K}]\\
&= \exp\{[\mu_\xi - \lambda E(L-1) + \mu_S - \lambda E(Y-1) + \varpi\sigma_\xi\sigma_S]\tau + n(\mu_l + \mu_y + \frac{\sigma_l^2}{2} + \rho\sigma_l\sigma_y + \frac{\sigma_y^2}{2})\}\\
&\times [S_t\Phi(h_H) - Ke^{-r_H\tau}\Phi(h_H - \sigma_H\sqrt{\tau})],
\end{aligned}
$$

where

$$
h_H = \frac{\ln(S_t/K) + r_H\tau}{\sqrt{\sigma_H^2\tau}} - \frac{1}{2}\sqrt{\sigma_H^2\tau}.
$$

Note that $E[YL] = \exp(\mu_l + \mu_y + \frac{\sigma_l^2}{2} + \rho\sigma_l\sigma_y + \frac{\sigma_y^2}{2})$, therefore we have

$$
C_H = \sum_{n=0}^{\infty}\frac{e^{(v-\lambda)\tau}[\lambda E(YL)\tau]^n}{n!}C_{BS}(S_t, \tau; r_H, \sigma_H, K) \tag{4.23}
$$

where

$$
v = \mu_S + \mu_\xi - \lambda E(L-1) - r + \lambda E(Y-1) + \varpi\sigma_\xi\sigma_S = -\lambda E(LY-1).
$$

We have substituted $\mu_\xi = -r$ and the security market line (3.21). The results can be rearranged to a simple form:

$$
C_H = \sum_{n=0}^{\infty}\frac{e^{\lambda_H\tau}(\lambda_H\tau)^n}{n!}C_{BS}(S_t, \tau; r_H, \sigma_H, K) \tag{4.24}
$$

where

$$
\begin{aligned}
\lambda_H &= \lambda E(YL)\\
r_H &= r - \lambda E[(Y-1)L] + \frac{n}{\tau}(\mu_y + \frac{\sigma_y^2}{2} + \rho\sigma_l\sigma_y)\\
\sigma_H^2 &= \sigma_S^2 + \frac{n}{\tau}\sigma_y^2
\end{aligned}
$$

Although this is already an applicable formula, the state price deflator process is admittedly unobservable. We'd better endogenize the state price deflator process in the equilibrium setting using its definition by incorporating the investor's marginal rate of substitution. Given CRRA and a jump-diffusion consumption flow, we have the following relations (refer to (3.17)):

$$L = X^{-\gamma} \tag{4.25}$$

$$\sigma_\xi = -\gamma\sigma_\delta \tag{4.26}$$

Substitute these relations into the above formulas and using the properties of log-normal distribution, allowing for a constant dividend yield q (This is an inessential modification.), we complete our option valuation formulas as below.

$$C_H = \sum_{n=0}^{\infty} \frac{e^{\lambda_H \tau}(\lambda_H \tau)^n}{n!} C_{BS}(S_t, \tau; r_H, q, \sigma_H, K) \tag{4.27}$$

where

$$\begin{aligned} \lambda_H &= \lambda E[YX^{-\gamma}] \\ r_H &= r - \lambda E[(Y-1)X^{-\gamma}] + \frac{n}{\tau}(\mu_y + \frac{\sigma_y^2}{2} - \gamma\varrho\sigma_x\sigma_y) \\ \sigma_H^2 &= \sigma_S^2 + \frac{n}{\tau}\sigma_y^2 \end{aligned}$$

where ϱ is the correlation coefficient between the logarithm of the asset price jump size $\ln Y$ and the endowment (aggregate consumption) jump size $\ln X$.

Here a fully stated economic equilibrium is used to price the options. Both the jump risk and the diffusion risk are priced in this equilibrium. In contrast with Merton's, the formula for the price of a call option on the market portfolio is equivalent to

$$\begin{aligned} C_t &= E_t[\xi_{t,T} \max(S_T - K, 0)] \\ &= E_t[\xi_{t,T}]E_t[\max(S_T - K, 0)] + \mathrm{Cov}[\xi_{t,T}, \max(S_T - K, 0)] \\ &= e^{-r\tau}E_t[\max(S_T - K, 0)] + \mathrm{Cov}[e^{-\theta\tau}\frac{U_c(c_T)}{U_c(c_t)}, \max(S_T - K, 0)]. \end{aligned} \tag{4.28}$$

Thus, the call price is a sum of two terms. The first term is similar to the standard EMM approach: it is the expected cash flow on the option (discounted at the risk-free interest rate which is shown to be constant in equilibrium) when the underlying asset earns its equilibrium instantaneous expected return of μ_S. Without no-arbitrage argument, we do not conclude that the option is priced as if the underlying asset earns an expected return equal to the risk-free interest rate. The second term is the equilibrium price of the jump and diffusion risks implicit in the option's price: it equals the covariance of the option's payoff with the change in the marginal utility of equilibrium aggregate consumption. Under this parametrization, they can explicitly evaluate that term and do not need to assume that the jump correlation with returns on aggregate consumption is zero.

4.5.2 Corresponding PDE

By using the security market line (3.21), one can retrieve the corresponding partial differential-difference equations (PDE) of the above option pricing results. Recall that the asset price change follows a jump-diffusion process. The price change of a call option written on the asset can be derived by Itô's lemma. As shown, the call value is a function of the asset price and time only; that is, $C_t = C(S_t, \tau)$, so

$$dC_t = \left[\frac{\partial C_t}{\partial t} + C_S[\mu_S - \lambda E(Y-1)]S_t + \frac{1}{2}C_{SS}S_t^2\sigma_S^2\right]dt + C_S\sigma_S S_t dB_t^S + [C(YS_t) - C_t]dN_t.$$

This can be written into the following standard form:

$$\frac{dC_t}{C_t} = \left[\mu_C - \lambda E[\frac{C(YS_t)}{C_t} - 1]\right]dt + \sigma_C dB_t^S + [\frac{C(YS_t)}{C_t} - 1]dN_t \tag{4.29}$$

where

$$\begin{aligned} \mu_C &= \frac{\frac{\partial C_t}{\partial t} + C_S[\mu_S - \lambda E(Y-1)]S_t + \frac{1}{2}C_{SS}S_t^2\sigma_S^2}{C_t} + \lambda E[\frac{C(YS_t)}{C_t} - 1] \\ \sigma_C &= C_S\frac{S_t}{C_t}\sigma_S \end{aligned}$$

Now recall that all traded assets including a call option must satisfy the security market line.

$$\mu_C = r - \varpi\sigma_\xi\sigma_C - \lambda E\left[[\frac{C(YS_t)}{C_t} - 1](L-1)\right] \tag{4.30}$$

Substituting μ_C and σ_C (4.27) in gives us

$$\frac{\frac{\partial C_t}{\partial t} + C_S[\mu_S - \lambda E(Y-1)]S_t + \frac{1}{2}C_{SS}S_t^2\sigma_S^2}{C_t} + \lambda E\left[\frac{C(YS_t)}{C_t} - 1\right]$$
$$= r - \varpi\sigma_\xi C_S\frac{S_t}{C_t}\sigma_S - \lambda E\left[[\frac{C(YS_t)}{C_t} - 1](L-1)\right].$$

Simplifying, we obtain the following equation that an option price must satisfy

$$\frac{1}{2}C_{SS}S_t^2\sigma_S^2 + \frac{\partial C_t}{\partial t} + C_S[\mu_S + \varpi\sigma_\xi\sigma_S - \lambda E(Y-1)]S_t + \lambda E\{[C(YS_t) - C_t]L\} = rC_t$$

It can be proved that the option pricing formula (4.23) solves the above equation subject to the appropriate boundary conditions.

As the option's underlying stock is traded, it also satisfies the security market line:

$$\mu_S = r - \varpi\sigma_\xi\sigma_S - \lambda E[(L-1)(Y-1)]$$

therefore the PDE can be rewritten as

$$\frac{1}{2}C_{SS}S_t^2\sigma_S^2 + \frac{\partial C_t}{\partial t} + C_S\{r - \lambda E[L(Y-1)]\}S_t + \lambda E\{[C(YS_t) - C_t]L\} = rC_t. \tag{4.31}$$

Notice in case that $L = 1$, this will return to the PDE in [49]. This will be interpreted in the next section.

To enrich the economic content, we can substitute (4.25) and (4.26) to deliver the equilibrium results.

$$\frac{1}{2}C_{SS}S_t^2\sigma_S^2 + \frac{\partial C_t}{\partial t} + C_S\{r - \lambda E[X^{-\gamma}(Y-1)]\}S_t + \lambda E\{[C(YS_t) - C_t]X^{-\gamma}\} = rC_t \tag{4.32}$$

This fundamental valuation equation applies to any claim satisfying the smoothness condition and written in the economy. It resembles the equation of [21] and [13] but differs from them. In their valuation equation, the PDE is expressed in terms of the indirect utility function or value function that is a solution for the investor's consumption-portfolio choice problem. However a closed-form solution for the indirect utility is unobtainable in most cases unless very restrictive simplifications are assumed. In contrast, (4.32) involves only the consumption.

The presentation of this equation is rather meaningful. Although we've concerned ourselves with only plain vanilla options, the variety of new option contracts has increased enormously in recent years. Many types of so-called *exotic options* are now popular items in the over-the-counter markets. Their payoff involve various pattern of cash flows and payment can be spread evenly through time or occur at unspecific dates. With a little more effort, we will be able to value many other types of options by the essential pricing equation. Sometimes we may need resort to numerical methods to work out the solutions and there may not be neat analytical closed-form results.

4.6 Comparison and Numerical Experiments

So far we have presented the above option pricing formulas and found a way to homogenize them in a meaningful format. It can be seen that our results encompass other authors' as special cases.

As $\lambda \to 0$, the process S_t converges to a lognormal diffusion and the pricing formula for the call option degenerates to the conventional Black-Scholes formula. The other three jump-diffusion formulas also collapse to Black-Scholes formula.

As $\sigma_S \to 0, \sigma_y \to 0$, the jump diffusion process converges to a pure jump process with a constant jump size. This becomes the Cox & Ross model and the full-fledged option price formula simplifies to theirs. Transform (4.28) into

$$C_t = e^{-r\tau} \sum_{n=0}^{\infty} \frac{e^{-b} b^n}{n!} E_t[\max\left(S_t e^{\mu_S \tau - \lambda(Y-1)\tau} Y^n - K, 0\right)]$$

with

$$\begin{aligned} b &= \lambda\tau \exp\left(\mu_l + \mu_y\right) \\ &= \frac{\lambda\tau(e^{\mu_l+\mu_y} - e^{\mu_l})}{e^{\mu_y} - 1} Y \\ &= -\frac{\mu_S - \lambda E(Y-1) - r}{e^{\mu_y} - 1} \tau E(Y), \end{aligned}$$

where the last line follows from (3.21)

$$\begin{aligned} \mu_S - r &= -\lambda E(L-1)E(Y-1) \\ &= -\lambda(e^{\mu_l} - 1)(e^{\mu_y} - 1) \\ &= -\lambda(e^{\mu_l+\mu_y} - e^{\mu_l} - e^{\mu_y} + 1). \end{aligned}$$

This is consistent with equation (4.2).

Merton's [49] formula is a special case of our results with $L = 1$. In another word, Merton's result is justified only if the state price deflator does not contain a jump component. However in light of the empirical evidence that the market portfolio does experience occasional jumps, Merton's option prices permits arbitrage opportunities as they deviate from their true values.

In a general equilibrium framework, $L = X^{-\gamma}$ where γ is the measure of agents' risk aversion. $L = 1$ implies either $\gamma = 0$, i.e. the investor is risk-neutral or $X = 1$, that is, aggregate consumption contains no jumps and therefore asset jump risk does not command a jump risk premium. This suggests that major changes in the market do not affect consumption. In fact, consumers cut down on major purchases in economic downturns. Thus, Merton's formula, which assumes away the jump risk premium, leads to questionable economic implications.

Index options cannot be properly valued using the Merton's formula since in this case the jump is, by definition, completely non-diversifiable. But our option formula can be applied to index option.

Under CRRA and constant investment opportunity set, a largely possible conclusion is that the return on the market portfolio is perfectly correlated with the aggregate consumption, i.e., $S_t = a + bc_t$ for some constants a, b. Then we have $c_t = \delta_t$, $\mu_S = \mu_\delta$ and $\mu_y = \mu_x$. If we let $a = 0, b = 1$ as in Naik and Lee 1990 [51].

$$C_H = \sum_{n=0}^{\infty} \frac{e^{(v-r)\tau}[\lambda E(YL)\tau]^n}{n!} C_{BS}(S_t, \tau; r_H, q, \sigma_H, K),$$

where

$$\begin{aligned} v &= \mu_\delta - r - \lambda E(Y-1) - \lambda E(L-1) - \gamma\sigma_\delta^2 \\ r_H &= \mu_S - \lambda E(Y-1) - \gamma\sigma_\delta^2 + \frac{n}{\tau}(\mu_y + \frac{1}{2}\sigma_y^2 - \gamma\sigma_y^2) \end{aligned}$$

we end up like (4.16).

Ahn's [1] formula may also be specialized to value index options by assuming perfect correlation between the index and the total wealth. In such instances, $\varpi = 1$, $\rho = 1$, $q = \theta$ and $\mu_S = \mu_W, \sigma_S = \sigma_W, \mu_y = \mu_d, \sigma_d = \sigma_y$. With these restrictions, the formula simplifies to

$$C_A = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} C_{BS}(S_t, \tau; r_A, \theta, \sigma_A, K)$$

where

$$r_A = r + \lambda E(Y^{-1} - 1) + \frac{n}{\tau}(\mu_y - \frac{1}{2}\sigma_y^2).$$

It is the same as both our formula and Naik & Lee when restricting to the log utility function. Therefore ours is more general in this sense. On the other hand, our model and Ahn's include valuation of options on assets not perfectly correlated with the market but Naik & Lee's does not. Obviously if the market jumps some stocks must jump with it but the responsiveness may vary.

Albeit similar and conformable, the jump-diffusion formulas are still distinct. Examination of the adjusted jump arrival frequency reveals the following relation. λ_M, λ_N, λ_A and λ_H differ in the expectation term multiplying λ, which is a function of the asset price jump distribution, the consumption jump distribution, the correlation between them and the degree of risk aversion. For assets stemming against the market, (i.e. negatively correlated with the market portfolio), $\lambda_H > \lambda_M$. As a larger jump arrival frequency results in higher option values, we can infer that Merton's formula is likely to undervalue such options. On the contrary, Merton's tends to overvalue options positively correlated with the market. Examination of the adjusted risk-free interest rate r_M, r_N, r_A, r_H, gives us no clear-cut relations because of the offsetting two covariance terms.

Although we can't make conclusive remarks so far about the effects of jump risk premiums on option values, we can test the above inferences and intuition using numerical experiments. In the subsequent part we select some representative parameter values and calculate our pricing formula values as well as other author's formula values. We use Matlab programming (Codes available upon request.) and all the data is summarized in the four tables in the Appendix. In the bracket beneath those prices is the percentages by which the value exceeds the corresponding Black-Scholes values. According to our analysis (4.28), the difference between the price of an option and the expected value of its cashflow discounted at the risk-free rate is the jump risk premium. This kind of numerical trial follows [57], [51], [Fuh, 2000] and [Navas, 2002]; the purpose is mainly to validate the qualitative conjecture and to lay a groundwork for empirical investigation.

Let us price a standard European call option on a hypothetical aggregate consumption index (i.e. $\varrho = 1$ and $\mu_x = \mu_y, \sigma_x = \sigma_y$) with five possible expiration dates: one month, three months, six months and one year, two years to maturity. The exercise price of the index is 50 and the risk free interest rate is 10% per annum. Suppose that the true process describing the dynamics of stock returns is a jump-diffusion process. The volatility for diffusion part is 20% per annum. As to the jump frequency, we conduct a two-fold numerical examination: $\lambda = 2$ and $\lambda = 7$ which translate into 2 and 7 jumps in one year on average respectively. Assume that the logarithmic jump size follow a normal distribution with $\mu_y = -0.0032, \sigma_y = 0.08$. Thus the expected jump amplitude is zero.

If an investor incorrectly believes that the asset returns follow a diffusion process, she will use the Black Scholes formulas to price options. the volatility she estimates from time series data will be total volatility instead of the diffusion part only. Thus the Black-Schole value should be computed using the volatility-adjustment procedure suggested by Merton: $\sigma^2_{\text{total}} = \sigma^2_{\text{diffusion}} + \lambda\sigma^2_{\text{jump}}$. Continuous dividend yields 2%. We shall first use a relative risk aversion γ of degree 1, i.e., the log-utility function. The number of jumps in the Poisson sum is cut off at 100 to ensure a precision of 10^{-6}.

The first table reports these prices. Our formula produces exactly the same results as Naik & Lee, Ahn. But Merton's model, which should not be used to price index option, overvalues in-the-money options but undervalues out-of-the-money options. In most cases, the jump risk premia that the Black-Scholes ignores are insignificant. But larger errors occur for out-of-the-money options than for in-the-money options. This is because the effect of the jump risk on call option pricing is significant when the price is very low. Perhaps one jump can be sufficient to bring the option back into the money. The bias percentage diminishes as the time to expiration increases. The effect of the jump arrival frequency on option pricing depends on whether the option is in or out of the money. For in-the-money options, the discrepancy seems greater when the frequency is higher but the opposite is true for out-of-the-money options.

The second table shows how jump-diffusion values depend on the importance of the jump component of the underlying asset price movement. We now consider an asset that is correlated to the market portfolio with $\varrho = 0.8$. The jump amplitude distribution parameters are $\mu_y = 0, \mu_x = -09.0018, \sigma_x = 0.06$. We still use $\gamma = 1$ and consequently our results are the same as Ahn's.

The jump component is measured by $\kappa = \lambda\sigma^2_{\text{jump}}/\sigma^2_{\text{total}}$: the percentage of the total volatility explained by the jump part and λ: the expected number of jumps per year. We subject our formulas to combinations of low and high jump λ and σ_y.

- For a given λ and σ_{total}, as the variance of a typical jump increases, κ increases and the call option value departs further from its corresponding Black-Scholes value.
- But for a given κ, as the intensity λ increases, the jump component becomes more like a diffusion part. T time diversification reduces the effect of the jump risk on option pricing. As a result the jump diffusion values approximate their corresponding Black-sholes values.

The third table looks at the effect of systematic jump risk, an indicator of which is the correlation coefficients between the asset prices and the consumption flow or market portfolio. From (4.27), they enter both the "adjusted" jump intensity and the "conditional" interest rate expressions. As the correlation of the underlying asset's logarithmic jump with the market portfolio's logarithmic jump increases, out-of-the-money options become less valuable but in-the-money options become more valuable. Merton's values are closest to ours when the correlation coefficient is 0, that is, when the asset jump risk is uncorrelated with the consumption jump risk. Merton's values still differ from ours because Merton assumes total jump risk is diversifiable while zero correlation implies only the systematic jump risk is zero. Therefore, the effect of the jump risk premium on option pricing is minimized when the jump size of the underlying asset is uncorrelated with that of the market.

When the correlation between consumption and asset price jumps is more negative, out-of-the-money option values are higher since the option's hedging services for jump risk becomes more valuable. This makes strong sense: if an asset is negatively correlated with the market portfolio, that is, counter-cyclical, it delivers a high cash flow when the state of the economy is bad. Therefore, people tend to view this asset as hedging against "bad times" and are ready to pay high for it. In case the jump risk cannot be hedged, options written on these assets supply a valuable service and thus should have higher values. Merton undervalues such options and this mispricing is more severe when the market is more volatile and investors are more risk averse.

The last table shows how the different degrees of the representative agent's risk aversion might influence our computation results. A look suggests that the jump risk premia decrease slightly when γ increases. When $\gamma = 0$, that is, investors are risk neutral, our formula and Naik & produce the same results as Merton. A more volatile jump will cause larger price differences.

To conclude, given that prices of an asset follow a jump-diffusion process, equilibrium option pricing formulas are superior to Merton's and others. The presence of jumps particularly benefits the pricing of short-term out-of-the-money options while the equilibrium approach "effectively" completes the market despite additional jump risks. The combination is not only economically reasonable but also potentially able to lessen the Black-Scholes pricing bias. Having said these, it is not perfect and concerns arise with it.

To begin with, one can never be absolutely sure that the dynamics of an asset price is a jump-diffusion process. As mentioned in prior sections, the actual underlying asset price distributions often depart significantly from the Black-Sholes lognormal. A corollary of this is the volatility smile – the way in which at-the-money options often have a lower volatility than out-of-the-money options or in-the-money options; or sometimes a skew. Volatility is the most critical parameter for option pricing – option prices are very sensitive to changes in volatility. Nevertheless it cannot be directly observed and has to be estimated. Volatility smile or skew complicates the tasks of pricing and hedging options.

Consider the task of calculating an option's delta which measures the sensitivity of an option's theoretical value to a change in the price of the underlying asset. If we assume the sticky delta model, this will affect how we calculate the option's delta, i.e., the rate of change of the option price with the underlying asset price. Changes in implied volatilities that are expected to accompany changes in the value of the underlying will impact the option's value. Deltas need to adjusted to reflect this. This is more than a theoretical consideration. If a trader is dynamically hedging an option's position and fails to incorporate volatility smile or skew into her delta calculations, her hedge ratio will be off. This justifies the practical value of more sophisticated models like jump-diffusions. Besides its ability to explain the volatility smile, it is advantageous for hedging not simply pricing options, especially exotic options which are less liquid, (i.e. their prices are not so readily visible in the market place).

Secondly, jump-diffusion models or equilibrium pricing models do pose challenges to empirical research. If we use Merton's formula, three more parameters – jump intensity, jump distribution need to be estimated. A number of authors have tried maximum likelihood, efficient methods of moments and indirect inference. They either estimate from historical equity price data or try to find suitable parameters that minimizes the difference between the proposed model and the synchronous market option prices. For estimation purposes using daily data, it is convenient to approximate the Poisson jump process by a binomial jump process. As for equilibrium approach, it has been suggested to select an index as a proxy for the aggregate consumption. An alternative possibility is based on estimating/filtering the state price deflator or market price of risk using market data.

The objective of such empirical studies should not be the comparison of model and market prices bur rather to the extant differences between competing models. Actual market prices do differ from model prices. However, by contrasting these model prices, we are able to ascertain whether such discrepancies should be attributed to the mis-specification of the underlying asset price process. A lot of authors have documented that jump-diffusion models can in principle eliminate the systematic biases of the Black-Scholes model. The gain from this added feature is worth the additional complexity.

Finally, a natural question is whether the deviation of the model option prices from market prices really significant. This will depend on what one plans to do with options. For a market maker who trades frequently with high turnover, the answer may be yes whereas for a positioner holding for the long term then the answer may be no. So further extensions and improvement of jump-diffusion models are still needed.

4.7 Stochastic Interest Rate

Up to this point, we have been examining the pricing theory without paying attention to the term structure of interest rates. Under certain assumptions, the short term interest rate can be constant. However, bonds with various maturities certainly differ in their interest rates and even the short rate is unlikely to be always constant either. There is now a burgeoning literature that develops bond pricing models using jump-diffusion processes. In this section we are going to add a feature of mean-reversion to the basic jump-diffusion processes, which will be entailed to the short rate process, and to look at some of the consequences this has for the interest rate derivatives. With mean-reversion introduced, the terminal density of the underlying asset is dependent on the specific time at which jumps occur. The mathematics will be a bit complicated but still tractable.

4.7.1 The Basic Stochastic Process

Suppose the endowment (the same as equilibrium aggregate consumption) follows an exponential mean-reverting jump-diffusion process

$$\frac{dc_t}{c_t} = \{\ell[\mu_c - \lambda E(X-1) - \ln c_t]\}dt + \sigma_c dB_t^c + (X-1)dN_t \tag{4.33}$$

with ℓ denoting the speed of adjustment, $\mu_c - \lambda E(X-1)$ the long run mean rate, σ_c^2 the instantaneous variance, and B^c a standard Brownian motion. N is a Poisson process with a jump intensity of λ. X denotes the gross jump size of consumption triggered with a log-normal distribution $\ln X \sim N(\mu_x, \sigma_x^2)$. B^c, N and X are assumed to be independent. Let $c_t^* = \ln c_t$, then by Itô's lemma,

$$dc_t^* = \ell(\mu_c^* - c_t^*)dt + \sigma_c dB_t^c + \ln X dN_t. \tag{4.34}$$

where $\mu_c^* = \mu_c - \lambda E(X-1) - \frac{1}{2\ell}\sigma_c^2$. Therefore the solution to equation is

$$c_T = \exp\left(\mu_c^* + (\ln c_t - \mu_c^*)e^{-\ell\tau} + \int_t^T \sigma_c e^{-\ell(T-u)}dB_u^c + \int_t^T e^{-\ell(T-u)}\ln X dN_u\right) \tag{4.35}$$

Since $\int_t^T \sigma_c e^{-\ell(T-u)}dB_u^c$ is normally distributed with mean zero and variance $\int_t^T \sigma_c^2 e^{-2\ell(T-u)}d$ $\int_t^T e^{-\ell(T-u)}\ln X dN_u$ is a product of a random number of independent normal distribution variables, we can derive the moments of c_T^* conditional on c_t^*.

$$\begin{aligned} E[c_T^*|c_t^*] &= \mu_c^* + (\ln c_t - \mu_c^*)e^{-\ell\tau} + \frac{\lambda\mu_x}{\ell}(1-e^{-\ell\tau}) \\ \mathrm{Var}[c_T^*|c_t^*] &= \frac{1}{2\ell}(1-e^{-2\ell\tau})[\sigma_c^2 + \lambda(\mu_x^2 + \sigma_x^2)] \end{aligned}$$

In the below we will derive the moment generating function $E_t[e^{bc_T^*}]$ for c_T^* conditional on c_t^* for later use. Substituting for c_T^* gives

$$\begin{aligned} E_t[e^{bc_T^*}] &= E_t\Big[\exp\Big(b\mu_c^* + b(\ln c_t - \mu_c^*)e^{-\ell\tau} + \int_t^T b\sigma_c e^{-\ell(T-u)}dB_u^c + \int_t^T be^{-\ell(T-u)}\ln X dN_u\Big) \\ &= e^{b\mu_c^*+b(\mu_c^*-\ln c_t)e^{-\ell\tau}+\frac{1}{2}\frac{1}{2\ell}b^2\sigma_c^2(1-e^{-2\ell\tau})} \times E_t[e^{\int_t^T be^{-\ell(T-u)}\ln XdN_u}] \end{aligned}$$

Focus on the jump part. The Riemann-Stigels integral in the exponent can be computed by taking expectation conditional on the number n of jumps of size x_j and the timing t_{ij} of the i-th jump of size x_j-th size.

$$E_t[e^{\int_t^T be^{-\ell(T-u)}\ln XdN_u}] = E\left[E\left(\exp[b\int_t^T e^{-\ell(T-u)}\ln XdN_u]|n, \{t_{ij}\}_{i=1,\ldots,n}\right)\right].$$

The timing of the jumps t_{ij} is uniformly distributed on $(t, T]$, allowing

$$E_t[e^{\int_t^T be^{-\ell(T-u)}\ln XdN_u}] = E\left[E\left(\prod_{j=1}^{n}\frac{1}{\tau}\int_t^T \exp\left(bx_je^{-\ell(T-u)}\right)du|n\right)\right].$$

Using the Poisson distribution for n and normal distribution for x_j gives

$$\begin{aligned} &E_t\left[\exp\left(\int_t^T be^{-\ell(T-u)}\ln XdN_u\right)\right] \\ &= E_t\left[\sum_{n=1}^{\infty}\frac{(\lambda\tau)^n e^{-\lambda\tau}}{n!}\left(\frac{1}{\tau}\int_t^T \exp\left(bx_je^{-\ell(T-u)}\right)du\right)^n\right] \\ &= E_t\left[\exp\left(-\tau\lambda\left(1-\frac{1}{\tau}\int_t^T \exp\left(bx_je^{-\ell(T-u)}\right)du\right)\right)\right] \\ &= E_t\left[\exp\left(-\tau\lambda+\lambda\int_t^T \exp\left(b\mu_xe^{-\ell(T-u)}+\frac{1}{2}b^2\sigma_x^2e^{-2\ell(T-u)}\right)du\right)\right] \end{aligned}$$

Therefore the moment generating function is

$$\begin{aligned} E_t[e^{bc_T^*}] = \exp[&b\mu_c^* + b(\ln c_t - \mu_c^*)e^{-\ell\tau} + \frac{1}{4\ell}b^2\sigma_c^2(1-e^{-2\ell\tau}) \\ &- \tau\lambda + \lambda\int_t^T \exp[b\mu_xe^{-\ell(T-u)}+\frac{1}{2}b^2\sigma_x^2e^{-2\ell(T-u)}]du]. \end{aligned} \tag{4.36}$$

4.7.2 Interest Rate Process

Pure discount bonds pay a single dividend 1 at maturity. Using this fact the Euler equation gives us the price of a discount bond which matures at time T.

$$\begin{aligned}\mathfrak{B}_t^T &= E_t[\xi_{t,T}] \\ &= \frac{E_t[e^{-\theta\tau}c_T^{-\gamma}]}{c_t^{-\gamma}} \\ &= E_t[\exp\left(-\theta\tau - \gamma\ln c_T + \gamma\ln c_t\right)]\end{aligned}$$

where the second step follows from (3.18). Making use of (4.36) and simplifying gives the price of a discount bond.

$$\begin{aligned}\mathfrak{B}_t^T = \exp\big\{ &-\tau\theta + \gamma(\ln c_t - \mu_c^*)(1-e^{-\ell\tau}) + \frac{1}{4\ell}\gamma^2\sigma_c^2(1-e^{-2\ell\tau}) \\ &-\tau\lambda + \lambda\int_t^T \exp[-\gamma\mu_x e^{-\ell(T-u)} + \frac{1}{2}\gamma^2\sigma_x^2 e^{-2\ell(T-u)}]du\big\}\end{aligned} \tag{4.37}$$

In the equation above, the price of bond is a function of the current level of the random process that drives the output, the parameters of the underlying jump processes, the level of agent risk aversion, and the time to maturity. Long term bonds are more sensitive to the current level and to jump shocks than short term bonds because of the factor $1-e^{-\ell\tau}$.

From the price of discount bonds, expressions for the yields of bonds of different maturity (the yield curve) and the short-term interest rate can be obtained. The yield-to-maturity is given as

$$\begin{aligned}ytm_t^T &= -\frac{\ln\mathfrak{B}_t^T}{\tau} \\ &= \theta - \frac{1}{\tau}\gamma(\ln c_t - \mu_c^*)(1-e^{-\ell\tau}) - \frac{1}{4\ell\tau}\gamma^2\sigma_c^2(1-e^{-2\ell\tau}) \\ &+ \lambda - \lambda\frac{1}{\tau}\int_t^T \exp[-\gamma\mu_x e^{-\ell(T-u)} + \frac{1}{2}\gamma^2\sigma_x^2 e^{-2\ell(T-u)}]du.\end{aligned} \tag{4.38}$$

The short term interest rate is the yield-to-maturity on an instantaneously maturing bond and is given as

$$\begin{aligned} r_t &= \lim_{T\to t} ytm_t^T \\ &= \theta - \gamma\ell(\ln c_t - \mu_c^*) - \frac{1}{2}\gamma^2\sigma_c^2 + \lambda[1 - \exp(\gamma\mu_x + \frac{1}{2}\gamma^2\sigma_x^2)]. \end{aligned} \tag{4.39}$$

The stochastic processes governing the short-term interest rate is

$$\begin{aligned} dr_t &= -\gamma\ell d\ln c_t \\ &= -\gamma\ell[\ell(\mu_c^* - \ln c_t)dt + \sigma_c dB_t^c + \ln X dN_t] \\ &= \ell(r_t^* - r_t)dt - \gamma\ell\sigma_c dB_t^c - \gamma\ell\ln X dN_t \end{aligned} \tag{4.40}$$

where $r_t^* = \theta - \frac{1}{2}\gamma^2\sigma_c^2 + \lambda[1 - \exp(\gamma\mu_x + \frac{1}{2}\gamma^2\sigma_x^2)]$ is a central tendency parameter, i.e., the long-run mean of the interest rate. It depends on the discount rate, the volatility of growth in the economy, the jump sizes and associated arrival rate and the risk aversion of agents. The short term interest rate mimics the technology process and follows a mean reversion jump-diffusion process.

4.7.3 Bond Option Price

Let H_t denote the price at time t of a contingent claim that has a payoff H_s at time s. Using the Euler equation, H_t is given as

$$H_t = \frac{E_t[e^{-\theta(s-t)}c_T^{-\gamma}H_s]}{c_t^{-\gamma}}.$$

Bond option, an option to buy or sell a type of bond by a certain date for a particular price, is one of the most popular over-the-counter interest rate products. A European call option expiring at s written on a discount bond which matures at T, is a contingent claim with $H_s = \max(\mathfrak{B}_s^T - K, 0)$. Its price at time $t \in (0, s)$ is

$$H_t(s,T,K) = \frac{E_t\left[e^{-\theta(s-t)}c_s^{-\gamma}\max(\mathfrak{B}_s^T - K, 0)\right]}{c_t^{-\gamma}} \tag{4.41}$$

The option pricing formula can be derived.

$$
\begin{aligned}
H_t(s,T,K) &= E_t\left[e^{-\theta(s-t)}\frac{c_s^{-\gamma}}{c_t^{-\gamma}}(\mathfrak{B}_s^T - K)|\mathfrak{B}_s^T > K\right] \\
&= E_t\left[e^{-\theta(s-t)}\frac{c_s^{-\gamma}}{c_t^{-\gamma}}\mathfrak{B}_s^T|\mathfrak{B}_s^T > K\right] - E_t\left[e^{-\theta(s-t)}\frac{c_s^{-\gamma}}{c_t^{-\gamma}}K|\mathfrak{B}_s^T > K\right]
\end{aligned}
$$

Using the expression for $\mathfrak{B}_s^T$, one can write $\mathfrak{B}_s^T > K$ as

$$
\ln c_s > \mu_c^* + \frac{\ln K + (T-s)\theta - \frac{1}{2\ell}\gamma^2\sigma_c^2(1-e^{-2\ell(T-s)}) + \lambda(\tau - \int_s^T e^{-\gamma\mu_x e^{-\ell(T-u)}+\frac{1}{2}\gamma^2\sigma_x^2 e^{-2\ell(T-u)}}}{\gamma(1-e^{-\ell(T-s)})}
$$

Use $\mathfrak{d}$ to denote the right hand side. Then

$$
H_t(s,T,K) = E_t\left[e^{-\theta(s-t)}\frac{c_s^{-\gamma}}{c_t^{-\gamma}}\mathfrak{B}_s^T|c_s > e^{\mathfrak{d}}\right] - E_t\left[e^{-\theta(s-t)}\frac{c_s^{-\gamma}}{c_t^{-\gamma}}K|c_s > e^{\mathfrak{d}}\right].
$$

Using the expressions for $\mathfrak{B}_t^T$ and $\mathfrak{B}_s^t$ we can get

$$
\begin{aligned}
H_t(s,T,K) &= \mathfrak{B}_t^T\frac{E_t\left[e^{-\theta(s-t)}\frac{c_s^{-\gamma}}{c_t^{-\gamma}}\mathfrak{B}_s^T|c_s > e^{\mathfrak{d}}\right]}{E_t\left[e^{-\theta(T-t)}\frac{c_T^{-\gamma}}{c_t^{-\gamma}}\right]} - K\mathfrak{B}_t^s\frac{E_t\left[e^{-\theta(s-t)}\frac{c_s^{-\gamma}}{c_t^{-\gamma}}|c_s > e^{\mathfrak{d}}\right]}{E_t\left[e^{-\theta(s-t)}\frac{c_s^{-\gamma}}{c_t^{-\gamma}}\right]} \\
&= \mathfrak{B}_t^T\frac{E_t\left[e^{-\theta(s-t)}\frac{c_s^{-\gamma}}{c_t^{-\gamma}}E_s[e^{-\theta(T-s)}\frac{c_T^{-\gamma}}{c_s^{-\gamma}}]|c_s > e^{\mathfrak{d}}\right]}{E_t\left[e^{-\theta(T-t)}\frac{c_T^{-\gamma}}{c_t^{-\gamma}}\right]} - K\mathfrak{B}_t^s\frac{E_t\left[c_s^{-\gamma}|c_s > e^{\mathfrak{d}}\right]}{E_t\left[c_s^{-\gamma}\right]} \\
&= \mathfrak{B}_t^T\frac{E_t[c_T^{-\gamma}|c_s > e^{\mathfrak{d}}]}{E_t(c_T^{-\gamma})} - K\mathfrak{B}_t^s\frac{E_t\left[c_s^{-\gamma}|c_s > e^{\mathfrak{d}}\right]}{E_t\left[c_s^{-\gamma}\right]}
\end{aligned}
$$

Note that

$$
c_T = \exp\left(\mu_c^* + (\ln c_s - \mu_c^*)e^{-\ell(T-s)} + \int_s^T \sigma_c e^{-\ell(T-u)}dB_u^c + \int_s^T e^{-\ell(T-u)}\ln X dN_u\right),
$$

we can rewrite the above as

$$
\begin{aligned}
H_t(s,T,K) &= \mathfrak{B}_t^T\frac{E_t[\exp(-\gamma\ln c_s e^{-\ell(T-s)})|c_s > e^{\mathfrak{d}}]}{E_t[\exp(-\gamma\ln c_s e^{-\ell(T-s)})]} - K\mathfrak{B}_t^s\frac{E_t\left(c_s^{-\gamma}|c_s > e^{\mathfrak{d}}\right)}{E_t\left(c_s^{-\gamma}\right)} \\
&= \mathfrak{B}_t^T\frac{E_t[\exp(-\gamma c_s^* e^{-\ell(T-s)})|c_s^* > \mathfrak{d}]}{E_t[\exp(-\gamma c_s^* e^{-\ell(T-s)})]} - K\mathfrak{B}_t^s\frac{E_t[\exp(-\gamma c_s^*|c_s^* > \mathfrak{d})]}{E_t[\exp(-\gamma c_s^*)]}.
\end{aligned}
\tag{4.42}
$$

We then choose a new probability measure that is equivalent to the market probability measure, under which the option price can be written in a form similar to the Black-Scholes formula. The prices are obtained using the characteristic function inversion technique [34].

Assume that we are at time t, and that we are looking ahead to time s. We are interested in the distribution of c_s^* given the current value c_t^*. In order to get the $(s-t)$-interval characteristic function $\wp(c_t^*, s; a) = E_t[e^{iac_s^*}]$, we solve its Kolmogorov backward equation.

$$\frac{\partial \wp(c_t^*, s; a)}{\partial c_t^*} \ell(\mu_c^* - c_t^*) + \frac{1}{2} \frac{\partial \wp(c_t^*, s; a)^2}{\partial c_t^{*2}} \sigma_c^2 - \frac{\partial \wp(c_t^*, s; a)}{\partial s} + \lambda E[\wp(c_t^* + \ln X, s; a) - \wp(c_t^*, s; a)] = 0 \tag{4.43}$$

subject to the boundary condition that $\wp(c_t^*, 0; a) = e^{iac_t^*}$.

$$\begin{aligned} \wp(c_t^*, s; a) = \exp \big\{ & ia\mu_c^* + ia(\ln c_t - \mu_c^*)e^{-\ell(s-t)} - \frac{1}{4\ell} a^2 \sigma_c^2 (1 - e^{-2\ell(s-t)}) \\ & - (s-t)\lambda + \lambda \int_t^s \exp[ia\mu_x e^{-\ell(s-u)} - \frac{1}{2} a^2 \sigma_x^2 e^{-2\ell(s-u)}] du \big\} \end{aligned} \tag{4.44}$$

Let $\mathbb{P}(c_s^*)$ be the probability density function of c_s^* under the market measure. Define a new equivalent probability measure

$$\mathbb{R}(c_s^*, b) = \frac{e^{bc_s^*}}{E(e^{bc_s^*})} \mathbb{P}(c_s^*) = \frac{c_s^b}{E(c_s^b)} \mathbb{P}(c_s). \tag{4.45}$$

The characteristic function $\wp(c_t^*, s; a, b)$ of $\mathbb{R}(c_s^*, b)$ is given as

$$\begin{aligned} \wp(a, b) = E_t^{\mathbb{R}}[e^{iac_s^*}] = E_t \left[e^{iac_s^*} \frac{e^{bc_s^*}}{E_t[e^{bc_s^*}]} \right] &= \frac{E_t[e^{(ia+b)c_s^*}]}{E_t[e^{bc_s^*}]} \\ = \exp[ia\mu_c^* + ia(c_t^* - \mu_c^*)e^{-\ell(s-t)} + \frac{1}{2\ell}(-a + 2bi)\sigma_c^2(1 - e^{-2\ell(s-t)}) \\ + \lambda \int_t^s \exp[(ai + b)\mu_x e^{-\ell(s-u)} + \frac{1}{2}(ai + b)^2 \sigma_x^2 e^{-2\ell(s-u)}] du \\ - \lambda \int_t^s \exp[b\mu_x e^{-\ell(s-u)} + \frac{1}{2} b^2 \sigma_x^2 e^{-2\ell(s-u)}] du] \end{aligned}$$

Using the change of probability measure,

$$H_t(s, T, K) = \mathfrak{B}_t^T \Upsilon_1 - K \mathfrak{B}_t^s \Upsilon_2 \tag{4.46}$$

where

$$\begin{aligned}\Upsilon_1 &= \text{prob}\left[c_s^* > \eth | \mathbb{R}(c_s^*, b = -\gamma e^{-\ell})\right] = \int_{\eth}^{\infty} \mathbb{R}(c_s^*, -\gamma e^{-\ell(T-s)}) dc_s^* \\ \Upsilon_2 &= \text{prob}\left[c_s^* > \eth | \mathbb{R}(c_s^*, b = -\gamma)\right] = \int_{\eth}^{\infty} \mathbb{R}(c_s^*, -\gamma) dc_s^*.\end{aligned}$$

Υ_1 and Υ_2 can be efficiently computed by Fourier inversion and the characteristic function of the probability density function $\mathbb{R}(c_s^*, b)$. The above expression for the price of the call option is similar in form to those obtained by Merton (1976) [49] and Naik and Lee (1990) [51] for options on stocks. There is one major difference: in their option pricing models, the timing of the jumps does not matter. In the option price we provide, the timing of the jumps is important because of the mean reversion in the output process. Mean reversion causes jumps far from the expiration date to be damped more; the impact is smaller on expiration date bond price. Thus the timing of the jumps determines the end of period short-term interest rate and bond price. In this extended model, jumps affect the output and consumption and are still systematic, causing the jump risk to be priced.

Chapter 5

Foreign Currency Option Pricing

With the increasing globalization of world financial markets, derivative products linked to exchange rates are assuming a new importance. The correct computation of the derivative prices calls for a good model for the stochastic processes that generate the time paths of foreign exchange rates.

Let X_t be the spot exchange rate at time t, the value of one unit of the foreign currency measured in the domestic currency. Consider a European call option from the point of view of domestic country, i.e., a contract which gives its owner the right to buy certain amount of foreign currency at a pre-specified exchange rate K at the date T. The size of the contract varies but we can normalize to one. Its payoff at T is $\max(X_T - K, 0)$. Note that from the point of view of foreign country, it is a European put option, i.e., an option to sell one unit of their (foreign) currency, which will be worth X_T domestic currency at time T, to obtain K units of domestic currency. Thus its payoff in foreign currency at T is $\max\left(\frac{X_T - K}{X_T}, 0\right)$.

5.1 A Brief Review

Jump-diffusion processes have also been shown to be statistically superior to simple diffusions for foreign exchange rates, see for example Akgiray and Booth (1988) [3]. They are more accurate than pure diffusions in valuing and hedging derivatives on the exchange rate, such as currency option.

The model of Garman and Kohlhagen (1983) [29], which is a modification of the Black-Scholes model, provides an analytical formula for currency option values. Like the original Black-Scholes formula, its strength lies in its simplicity but a number of authors have reported empirical biases and mispricing. Bodurtha and Courtadon (1987) [19], Jiang (1998) [40] and Tucker (1991) [60] suggest using Merton (1976)'s [49] jump-diffusions model where jump risk is assumed diversifiable. The problem with this assumption brings the same difficulty as mentioned above in equity markets. Dumas et al. (1995) [26] notices an inconsistency when using Merton's approach with currency options: the value of an option as seen in one country does not match the value of the same option in the other country. Bardhan (1995) [10] also admits this problem and shows further that investors in home and foreign countries demand different premiums for the same source of risk. We will show later that in an international general equilibrium setting, the inconsistency problem will disappear.

Existing models for the valuation of foreign currency options are mostly based on the arbitrage-free approach. See for a partial list, [53], [16], [29], [22], [5], [34], [41], [26], [30], [10], [15], [20], [33]. Typically, these authors first specify some exogenous process for the spot exchange rate, the factor risk premium, and the domestic and foreign term structure of interest rates and then follow a variation of the Black and Scholes argument to derive a partial valuation equation. While this approach has generated many practically useful foreign exchange claims valuation formulas, there is no guarantee that the arbitrary choice of the exogenous processes will be consistent with any international general equilibrium in which the factor risk premium , the spot exchange rate and domestic and foreign interest rates are endogenously determined.

In addition, it is expected that money supply should play a role in the model given that the existence of foreign exchange claims is precisely due to the existence of different monies. Possible sources of policy shocks include inflation pressure reactions and some discretionary actions taken by central banks may cause jumps in exchange rates [62].

Among the international general equilibrium models in the literature, several methods of incorporating nominal pricing have been used including transaction cost technologies such as a cash-in-advance constraint [46], overlapping generations formulations and money-in-the-utility-function formulations
Basak and Gallmeyer (1999) [12], Bakshi and Chen (1997) [8] have extended Lucas (1982)'s [46] discrete-time two country monetary model and value foreign currency option in an international equilibrium context. With the cash-in-advance constraint, the nominal exchange rates and international term structure of interest rates are endogenously determined. However they still use the diffusion-type process.

Our treatment of foreign currency option part builds on Bakshi and Chen (1997) [8] and is an attempt to bridge the above mentioned gap. In another word, our consideration of jump risks establish a link with the jump-diffusion formulation of equity options. Exchange rate dynamics are characterized and closed-form formulas for a standard European currency option are obtained.

5.2 Garman & Kohlhagen Formula

Garman and Kohlhagen (1983) [29] for the first time extend the Black-Scholes model from equity options to the realm of currency options. Freely floating exchange rate regimes are usually modeled by letting the exchange rate evolve according to a geometric diffusion process with constant drift and variance.

$$\frac{dX_t}{X_t} = \mu_F dt + \sigma_X dB_t^X \tag{5.1}$$

Risk-free assets, that is, the savings accounts in domestic currency and foreign currency evolve respectively as:

$$\begin{aligned} dA_t &= A_t R dt & (5.2) \\ d\bar{A}_t &= \bar{A}_t \bar{R} dt. & (5.3) \end{aligned}$$

R is domestic country risk-free nominal interest rate and $\bar{R}$ is foreign country risk-free nominal interest rate. From now on a bar on a variable stands for the foreign counterpart of the corresponding domestic variable.

The price of foreign savings account denoted in domestic currency is $F_t = X_t \bar{A}_t$. By Itô's lemma,

$$\frac{dF_t}{F_t} = (\bar{R} + \mu_X)dt + \sigma_X dB_t^X$$

Use the Radon-Nikodym derivative to define the equivalent martingale measure $\mathbb{Q}$,

$$\eta_t = \frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t = \exp\left(-\psi B_t^X - \frac{1}{2}\psi^2 t\right)$$

where $\psi = (\bar{R} - R + \mu_X)/\sigma_X$ is the exchange rate risk premium in domestic market. By Girsanov's Theorem, $\check{B}_t^X = \psi t + B_t^X$ is a standard Brownian motion under $\mathbb{Q}$.

$$\frac{dX_t}{X_t} = (R - \bar{R})dt + \sigma_X d\check{B}_t^X \tag{5.4}$$

$\mathbb{Q}$ is just the risk neutral measure for domestic investors. This equation reveals that a foreign currency is analogous to a stock providing a known dividend yield. The owner of foreign currency receives a "dividend yield" equal to the risk-free interest rate in the foreign currency. To valuate the call option, we apply the standard Black-Scholes technique.

$$\begin{aligned} C_{GK} &= E_t^{\mathbb{Q}}[e^{-R\tau}\max\left(X_T - K\right), 0] \\ &= X_t e^{-\bar{R}\tau}\Phi(\hbar) - Ke^{-R\tau}\Phi(\hbar - \sigma_X\sqrt{\tau}) \end{aligned} \tag{5.5}$$

where

$$\hbar = \frac{\ln(X_t/K) + (R - \bar{R})\tau}{\sigma_X\sqrt{\tau}} + \frac{\sigma_X\sqrt{\tau}}{2}$$

That is only the domestic investor's outlook. Let us now transport ourselves to foreign country where investors observe the inverse exchange rate $\bar{X}_t = 1/X_t$. (that is, foreign country currency units per unit of domestic currency). Of course, domestic and foreign here still refers to our original domestic and foreign. $\bar{X}_t = 1/X_t$ is the inverse function of X.

$$\frac{d\bar{X}}{\bar{X}} = (-\mu_X + \sigma_X^2)dt - \sigma_X dB_t^X \tag{5.6}$$

where the source of risk is exactly the same Brownian motion except for an opposite direction. In the same manner, we know that the risk neutral drift is $\bar{R} - R$ and the risk premium demanded by foreign investors is $\bar{\psi} = -(\bar{R} - R - \mu_X + \sigma_X^2)/\sigma_X = \psi - \sigma_X$. The pricing formula for a European put option is

$$\bar{P}_{GK} = Ke^{-R\tau}\Phi(\sigma_X\sqrt{\tau} - \hbar) - X_t e^{-\bar{R}\tau}\Phi(-\hbar). \tag{5.7}$$

According to the law of one price, $\bar{P}_t$ converted into the domestic currency at the spot exchange rate should be the same as C_t. It is verified that

$$C_{GK}(X_t, \tau; R, \bar{R}, \sigma_X, K) = X_t\bar{P}_{GK}(K/X_t, \tau; R, \bar{R}, \sigma_X, 1), \tag{5.8}$$

which means that a currency option hence has the same domestic currency value, whether viewed from domestic country or foreign country.

5.3 Merton-style Currency Option Formula

On the ground of studies showing that jump-diffusion models perform well in modeling the foreign exchange rate process, Bodurtha and Courtadon (1987) [19], Jiang (1998) [40] and Tucker (1991) [60] suggest using Merton's jump-diffusions model for currency option valuation. In Merton's model, investors are not paid for jump risks since the jumps are assumed to be diversifiable.

Dumas et al. (1995) [26] point out that if both the domestic and foreign investors assume their own risk neutral processes, even in the case where the jump component in the exchange rate is uncorrelated with the consumption, applying Merton's formula generates an analog to Siegel's paradox. The paradox refers to the violation of the parity conditions between domestic and foreign investors' valuations.

Assuming an exchange rate dynamics as follows,

$$\frac{dX_t}{X_t} = [\mu_X - \lambda(Y-1)]dt + \sigma_X dB_t^X + (Y-1)dN_t \tag{5.9}$$

where μ_X is the instantaneous drift and σ_X is the volatility of the diffusion part B^X, N is an independent Poisson process with intensity λ and jump size $Y-1$. $\ln Y \sim N(\mu_y, \sigma_y)$.

As a mimic of Merton (1976) [49] for stock options, the formula for a European currency option from the point of view of domestic country is

$$\acute{C}_M = \sum_{n=0}^{\infty} \frac{e^{-\acute{\lambda}_M}(\acute{\lambda}_M \tau)^n}{n!} C_{GK}(X_t, \tau; R_M, \bar{R}, \sigma_M, K) \tag{5.10}$$

where

$$\begin{aligned}
\acute{\lambda}_M &= \lambda E(Y) \\
R_M &= R - \lambda E(Y-1) + \frac{n}{\tau}(\mu_y + \frac{1}{2}\sigma_y^2) \\
\sigma_M^2 &= \sigma_X^2 + \frac{n}{\tau}\sigma_y^2.
\end{aligned}$$

C_{GK} is defined in (5.5). From the foreign country's point of view, the corresponding put option formula is

$$\bar{P}_M = \sum_{n=0}^{\infty} \frac{e^{-\bar{\lambda}_M}(\bar{\lambda}_M \tau)^n}{n!} P_{GK}(\bar{X}_t, \tau; \bar{R}_M, R, \sigma_M, \bar{K}) \tag{5.11}$$

where

$$\begin{aligned} \bar{\lambda}_M &= \lambda E(Y^{-1}) \\ \bar{R}_M &= R - \lambda E(Y^{-1} - 1) - \frac{n}{\tau}(\mu_y - \frac{1}{2}\sigma_y^2) \\ \sigma_M^2 &= \sigma_X^2 + \frac{n}{\tau}\sigma_y^2 \end{aligned}$$

P_{GK} is defined in (5.7). According to the law of one price, $\bar{P}_M$ converted into the domestic currency at the spot exchange rate should be the same as C_M. However in presence of jumps, it does not hold that

$$C_M(X_t, \tau; \ldots, K) = X_t \bar{P}_M(K/X_t, \tau; \ldots, 1)$$

Therefore a domestic and a foreign investor do not assign the same value to that same security, if both make the assumption that the jump risk is non-priced. In fact, a zero price for jump risk is an untenable assumption in the international financial market, when the investors look at returns from different currency points of view.

In order to eliminate this paradox, Bardhan (1995) [10] has advocated a "directional adjustment", i.e., the extra term $\sigma_X^2 + \lambda E(\frac{1}{Y} - 1)$ appearing in the drift for the inverse exchange rate $\bar{\mu}_X$.

$$\begin{aligned} \frac{d\bar{X}_t}{\bar{X}_t} &= [\mu_X + \lambda E(Y-1) + \sigma_X^2]dt - \sigma_X dB_t^X + (\frac{1}{Y} - 1)dN_t \\ &= [\bar{\mu}_X - \lambda E(\frac{1}{Y} - 1)]dt - \sigma_X dB_t^X + (\frac{1}{Y} - 1)dN_t \end{aligned}$$

where $\bar{\mu}_X = -\mu_X + \lambda E(Y-1) + \sigma_X^2 + \lambda E(\frac{1}{Y} - 1)$. He suggests that the foreign investor makes a corresponding correction if using the domestic observations or directly use to get her risk-neutral expectation for $d\bar{X}_t/\bar{X}_t$. Her adjustment is essentially the same as what Hull and White [36] suggest about using the foreign country money account as numeraire for the inverse exchange rate process.

The ultimate problem in the application of Merton's stock option model to exchange rate option is its assumption that the jump risk is non-systematic or uncorrelated with the market. This assumption might have some justification in stock market but is problematic for currency market. Since the exchange rate reflects one nations's purchasing power relative to another nation, the exchange rate is inherently correlated with aggregate fundamental forces that affect the market.

5.4 Equilibrium Model

In what follows, we will develop an international general equilibrium framework. Specifically, we will extend Lucas (1982) [46] discrete-time two country monetary model to continuous-time version in a jump-diffusion setting.

5.4.1 Structure of the Economy

The economy has a finite horizon $[0, T]$. There are two countries named domestic and foreign country and two goods freely traded between them. Each country has a stochastic non-storable endowment or production of its unique goods. Denote the domestic dividend as δ, which are exogenously given as

$$\frac{d\delta_t}{\delta_{t-}} = [\mu_\delta - \lambda_\delta E(G-1)]dt + \sigma_\delta dB_t^\delta + (G-1)dN_t^\delta \tag{5.12}$$

where B^δ is a Brownian motion process and N^δ is an independent Poisson process with intensity parameter λ_δ and an amplitude $G-1$. $\ln G \sim N(\mu_g, \sigma_g^2)$.

There is one risky stock, which represents the ownership of the productive technology for the domestic good. The total supply of this risky stock is normalized to one. Denote its nominal price at time t as S_t. Then its real price in terms of domestic goods is S_t/p_t where p_t is the price of the domestic good at time t.

Since in either country monetary authorities control the monetary aggregate only imperfectly through intermediate instrument, we model the evolution of the monetary aggregate as a stochastic process. The monetary authority sets the money supply on the basis of long term target for nominal money growth. Assume that the monetary policy is exogenous, real productivity shocks or inflation shocks do not feed back into the nominal side of the economy.

$$\frac{dM_t}{M_{t-}} = [\mu_M - \lambda_M E(H-1)]dt + \sigma_M dB_t^M + (H-1)dN_t^M \tag{5.13}$$

where B^M is a Brownian motion process and N^M is an independent Poisson process with intensity parameter λ_M and an amplitude $H-1$. $\ln H \sim N(\mu_h, \sigma_h^2)$. This money supply process incorporates both frequent fluctuations and infrequent large shocks to the money supply. Possible sources of policy shocks include inflation pressure reactions and some discretionary actions taken by central banks.

The foreign country also has

$$\frac{d\bar{\delta}_t}{\bar{\delta}_{t-}} = [\mu_{\bar{\delta}} - \lambda_{\bar{\delta}} E(\bar{G}-1)]dt + \sigma_{\bar{\delta}} dB_t^{\bar{\delta}} + (\bar{G}-1)dN_t^{\bar{\delta}} \tag{5.14}$$

where $B^{\bar{\delta}}$ is a Brownian motion process and $N^{\bar{\delta}}$ is an independent Poisson process with intensity parameter $\lambda_{\bar{\delta}}$ and an amplitude $\bar{G}-1$. $\ln \bar{G} \sim N(\mu_{\bar{g}}, \sigma_{\bar{g}}^2)$.

$$\frac{d\bar{M}_t}{\bar{M}_{t-}} = [\mu_{\bar{M}} - \lambda_{\bar{M}} E(\bar{H}-1)]dt + \sigma_{\bar{M}} dB_t^{\bar{M}} + (\bar{H}-1)dN_t^{\bar{M}} \tag{5.15}$$

where $B^{\bar{M}}$ is a Brownian motion process and $N^{\bar{M}}$ is an independent Poisson process with intensity parameter $\lambda_{\bar{M}}$ and an amplitude $\bar{H}-1$. $\ln \bar{H} \sim N(\mu_{\bar{h}}, \sigma_{\bar{h}}^2)$.

The four processes described above form the basis of what is referred to as the primitives of the economy. Call $s_t = (\delta_t, \bar{\delta}_t, M_t, \bar{M}_t)$ the state in period t. It follows a Markov process with the distribution of s_{t+1} given by the distribution functions

$\mathbb{P}(s_{t+1}|s_t)$. That is, the probability distribution of s_{t+1} depends on the realization of s_t only. They are assumed to be independent, measured with respect to a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Together with the specification of the utility function for the representative agent, they induce equilibrium prices for other assets. Among these assets, there are domestic and foreign risk-free bonds, domestic and foreign risky stocks and other contingent claims on the stocks or the spot exchange rate or the money transfer. As discussed in [46], there is a certain arbitrariness in which assets are assumed traded. The important aspect is that there are enough assets to allow for a stationary equilibrium.

The two countries each have an identical representative agent who maximizes expected utility and have the same preferences over the two goods, as given below

$$E[\int_0^T e^{-\theta t} U(c_t, \bar{c}_t) dt] \tag{5.16}$$

for a consumption bundle of c_t domestic goods and $\bar{c}_t$ foreign goods. θ is the discount rate. For simplicity, we use a log-utility Cobb-Douglas form function.

$$U(c_t, \bar{c}_t) = \vartheta \ln c_t + (1 - \vartheta) \ln \bar{c}_t \tag{5.17}$$

where $\vartheta \in [0, 1]$ is the expenditure share on the domestic good.

5.4.2 Agents' Decision Problem

We shall first look at the decision problem of the home agent. Initially, the agent is endowed with one share of the domestic risky stock, one unit of domestic money holdings and one share of the equity claims for domestic monetary transfer. Her consumption over time is financed by a continuous trading strategy $\{m_t, \breve{m}_t, \pi_t, \forall t \geq 0\}$, where m_t is the domestic money holding by the domestic agent at time t and $\breve{m}_t$ is the foreign money holding by the domestic agent at time t and $\pi_t = (\pi_t^S, \pi_t^{\breve{S}}, \pi_t^A, \pi_t^V, \pi_t^{\breve{V}})$ represents the portfolio holdings consists of all the financial assets held by the domestic agent at time t. The nominal prices of all financial assets are denoted by $\mathbf{P}_t = (S_t, \breve{S}_t, A_t, V_t, \breve{V}_t)^\intercal$ and corresponding vector of dividends $\mathbf{q}_t$. Note that the nominal prices of foreign equity claims $\breve{S}_t$ and $\breve{V}_t$ are prices at which they are traded in domestic financial market and therefore measured in domestic currency. The cumulative dividends up to t is $\mathfrak{q} = \int_0^t q_u du$. To make the budget manageable, assume for now that the agent does not invest in the foreign bond market. At time t, the domestic agent's real wealth is

$$\mathbf{w}_t = \frac{m_t + \breve{m}_t X_t + \pi_t \mathbf{P}_t}{p_t}. \tag{5.18}$$

The flow budget constraint is

$$(c_t + \breve{c}_t)dt + d\mathbf{w}_t = m_t d\left(\frac{1}{p_t}\right) + \pi_t(d\mathbf{P}_t + d\mathbf{q}_t)d\left(\frac{1}{p_t}\right) \tag{5.19}$$

where c_t and $\breve{c}_t$ are the domestic agent's consumption of domestic and foreign goods. This constraint intuitively states that the sum of the wealth increase $d\mathbf{w}_t$ and consumption flow $(c_t + \breve{c}_t)$ is bounded by the dividend and capital gain from the portfolio.

The cash-in-advance constraint states that domestic currency is used to buy domestic goods at the price p_t, whereas foreign currency is used to pay for foreign goods at the price $\bar{p}_t$. The purchases must obey the liquidity constraint

$$p_t c_t \leq m_t; \quad \bar{p}_t \breve{c}_t \leq \breve{m}_t. \tag{5.20}$$

With these constraints, one can apply the same technique to derive the necessary first-order conditions. In equilibrium the security trading will not affect the holdings or the consumption flow of the representative agent. As the two goods are traded in the respective producing countries and only in the country's currency, the domestic country's output δ_t can only be purchased and consumed by paying a total of $\delta_t p_t$. The foreign country's output $\bar{\delta}_t$ can only be purchased and consumed by paying a total of $\bar{\delta}_t \bar{p}_t$.

The representative agent in foreign country is identical with the domestic one. She maximizes expected utility given by

$$E\left[\int_0^T e^{-\theta t}[\vartheta \ln c_t + (1-\vartheta)\ln \bar{c}_t]dt\right]$$

for a consumption bundle of c_t domestic goods and $\bar{c}_t$ foreign goods. The foreign agent is endowed with one share of the foreign risky stock, one unit of domestic money holdings and one share of the equity claims for domestic monetary transfer. Her consumption over time is financed by a continuous trading strategy $\{\bar{m}_t, m'_t, \bar{\pi}_t\}, \forall t \geq 0$, where $\bar{m}_t$ is the foreign money holding by the foreign agent at time t and m'_t is her domestic money holding and $\bar{\pi}_t = (\bar{\pi}_t^{\bar{S}}, \bar{\pi}_t^{S'}, \bar{\pi}_t^{\bar{A}}, \bar{\pi}_t^{\bar{V}}, \bar{\pi}_t^{V'})$ represents the portfolio holdings consists of all the financial assets held by the foreign agent at time t. The foreign-currency-denominated prices of all financial assets is $\bar{\mathbf{P}}_t = (\bar{S}_t, S'_t, \bar{A}_t, \bar{V}_t, V'_t)^\intercal$ and corresponding vector of dividends $\bar{\mathbf{q}}_t$. The cumulative dividends up to t is $\bar{\mathfrak{q}} = \int_0^t \bar{q}_u du$. At time t, the foreign agent's real wealth is

$$\bar{\mathbf{w}}_t = \frac{\bar{m}_t + m'_t\bar{X}_t + \bar{\pi}_t\bar{\mathbf{P}}_t}{\bar{p}_t}.$$

The flow budget constraint is

$$(\bar{c}_t + c'_t)dt + d\bar{\mathbf{w}}_t = \bar{m}_t d\left(\frac{1}{\bar{p}_t}\right) + \bar{\pi}_t(d\bar{\mathbf{P}}_t + d\bar{\mathfrak{q}}_t)d\left(\frac{1}{\bar{p}_t}\right)$$

where $\bar{c}_t$ and c'_t are the domestic agent's consumption of domestic and foreign goods. Her cash-in-advance constraint dictates that

$$\bar{p}_t\bar{c}_t \leq \bar{m}_t; \quad p_t c'_t \leq m'_t$$

The first-order conditions arise from the foreign agent's problem are just the same as those from the domestic side.

5.4.3 Equilibrium and Characterization

Given the above setup, a perfect pooling equilibrium exists in which the domestic and foreign representative agent each consume half of each of the two goods.

$$c_t = c'_t = \frac{1}{2}\delta_t \tag{5.21}$$

$$\breve{c}_t = \bar{c}_t = \frac{1}{2}\bar{\delta}_t \tag{5.22}$$

and in which each agent holds half of the domestic and half of the foreign equity shares for every time t.

$$\pi_t^S = \bar{\pi}_t^{S'} = \frac{1}{2}; \quad \pi_t^{\breve{S}} = \bar{\pi}_t^{\bar{S}} = \frac{1}{2} \tag{5.23}$$

$$\pi_t^V = \bar{\pi}_t^{V'} = \frac{1}{2}; \quad \pi_t^{\breve{V}} = \bar{\pi}_t^{\bar{S}} = \frac{1}{2} \tag{5.24}$$

Then the equilibrium demand for the domestic currency is $m_t = \frac{1}{2}p_t\delta_t$ for the domestic agent and $m'_t = \frac{1}{2}p_t\delta_t$ for the foreign agent, implying a total demand of $p_t\delta_t$ domestic currency. Similarly, the total demand for foreign currency is $\breve{m}_t + \bar{m}_t = \bar{p}_t\bar{\delta}_t$. In the perfect pooling equilibrium, the monetary policy for each country should be such that its money supply equals demand.

$$M_t = p_t\delta_t; \quad \bar{M}_t = \bar{p}_t\bar{\delta}_t \tag{5.25}$$

A look at the equations tells us that holding δ_t, $\bar{\delta}_t$ and $\bar{M}_t$ fixed, an increase in domestic money supply M_t will lead to a proportional increase in domestic price level p_t , which in turn causes a proportional increase in exchange rate X_t . Thus a rise in money supply alone will only proportionally depreciate the value of the domestic currency, whereas a rise in foreign money supply will have the opposite effect.

The equilibrium nominal exchange rate must satisfy the equation below via the purchasing power parity.

$$X_t = \frac{U_{\bar{c}}(c_t, \bar{c}_t)}{U_c(c_t, \bar{c}_t)} \frac{p_t}{\bar{p}_t} \tag{5.26}$$

where subscripts on U denote the respective partial derivatives. Other assets' pricing equations may be obtained via the substitution argument that in equilibrium the investor should be indifferent between consuming now and holding a nominal bond at the margin and then spending the return on consumption goods later.

$$\mathbf{P}_t = E_t[\int_t^T \xi^*_{t,T}\mathbf{q}_u du] \tag{5.27}$$

where

$$\xi^*_{t,T} = e^{-\theta}\frac{U_c(c_T, \bar{c}_T)}{U_c(c_t, \bar{c}_t)}\frac{p_t}{p_T}$$

is the nominal state price deflator. (2.30)

Under the utility function specified and the market clearing conditions, (5.26) becomes

$$\begin{aligned} X_t &= \frac{(1-\vartheta)c_t}{\vartheta\breve{c}_t}\frac{p_t}{\bar{p}_t} \\ &= \frac{(1-\vartheta)\frac{1}{2}\delta_t}{\vartheta\frac{1}{2}\bar{\delta}_t}\frac{p_t}{\bar{p}_t} \\ &= \frac{1-\vartheta}{\vartheta}\frac{M_t}{\bar{M}_t}. \end{aligned} \tag{5.28}$$

The nominal exchange rate is decreasing in the expenditure share of the domestic goods and proportional to the ratio of domestic to foreign money supplies. More surprisingly, it is independent of production output in either country, which means foreign exchange levels are driven by monetary policies. This detachment of exchange rate from the real side of the economy is due to the perfect-pooling assumption. Under this assumption, the cross-border trade in goods occurs irrespective of the exchange rates. Consequently the cash-in-advance constraint and monetary policies become the sole determination of exchange rate. (There is however one indirect way in which real factors can still influence the determination of the exchange rate, that is, the monetary authorities can index monetary growth to real shocks and make M_t and $\bar{M}_t$ dependent on the real sector.)

Itô's lemma yields the dynamics of the exchange rate.

$$
\begin{aligned}
\frac{dX_t}{X_t} &= [\mu_M - \lambda_M E(H-1) - \mu_{\bar{M}} + \lambda_{\bar{M}} E(\bar{H}-1)]dt + \sigma_M dB_t^M - \sigma_{\bar{M}} dB_t^{\bar{M}} \\
&\quad + (H-1)dN_t^M + (\bar{H}^{-1}-1)dN_t^{\bar{M}}
\end{aligned}
\tag{5.29}
$$

Under the log-utility function, the nominal asset pricing kernel now becomes

$$
\begin{aligned}
\xi^*_{t,T} &= e^{-\theta\tau}\left(\frac{c_T}{c_t}\right)^{-1}\frac{p_t}{p_T} \\
&= e^{-\theta\tau}\left(\frac{\frac{1}{2}\delta_T}{\frac{1}{2}\delta_t}\right)^{-1}\frac{p_t}{p_T} \\
&= e^{-\theta\tau}\frac{M_t}{M_T}.
\end{aligned}
$$

The foreign agent's nominal state price deflator, $\bar{\xi}^*{}_t = e^{-\theta\tau}\bar{M}_t/\bar{M}_T$, can be obtained likewise. Then we can apply Itô's lemma and the property that the drift of a nominal state price deflator equals the negative of the nominal interest rate.

$$
R = \theta + \mu_M - \lambda_M E(H-1) - \sigma_M^2 - \lambda_M E(H^{-1}-1) \tag{5.30}
$$

$$
\bar{R} = \theta + \mu_{\bar{M}} - \lambda_{\bar{M}} E(\bar{H}-1) - \sigma_{\bar{M}}^2 - \lambda_{\bar{M}} E(\bar{H}^{-1}-1) \tag{5.31}
$$

We can see that in this special case of log-utility function, the nominal interest rates turn out to be independent of the dividends' processes and solely determined by the corresponding country's monetary policy. With a more general class of utility functions such as CRRA, this may not necessarily the case. Knowing both the domestic and the foreign term structure of interest rates allows us to rewrite (5.29) as

$$
\begin{aligned}
\frac{dX_t}{X_t} &= [R - \bar{R} + \sigma_M^2 + \lambda_M E(H^{-1}-1) - \lambda_{\bar{M}} E(\bar{H}^{-1}-1)]dt + \sigma_M dB_t^M - \sigma_{\bar{M}} dB_t^M \\
&\quad + (H-1)dN_t^M + (\bar{H}^{-1}-1)dN_t^{\bar{M}}
\end{aligned}
\tag{5.32}
$$

The key feature of the above exchange rate is that it is derived endogenously from the underlying processes for the money supply. This endogeneity is in contrast with the arbitrariness in partial equilibrium currency option models. The exchange rate and the interest rates are all endogenously determined as part of a general international equilibrium. This guarantees that no internal inconsistency will arise among the processes assumed for them independently. It also makes it possible to gain some insights into how derivative prices may respond to a change in any underlying variable or structural parameter. The exchange rate dynamics incorporate the two independent jump components from the two countries' money supply. Obviously, the jump in the exchange rate must be priced.

5.4.4 Option Valuation

Let us now value a European call option on the exchange rate.

$$C_t = E_t[\xi^*_{t,T} C_T] = E_t\left[e^{-\theta\tau}\frac{M_t}{M_T}\max\left(X_T - K, 0\right)\right] \tag{5.33}$$

Since the exchange rate is a function $X_t = \frac{1-\vartheta}{\vartheta}\frac{M_t}{\bar{M}_t}$, then

$$\begin{aligned} C_t &= E_t\left[e^{-\theta\tau}\frac{M_t}{M_T}\max\left(\frac{1-\vartheta}{\vartheta}\frac{M_T}{\bar{M}_T} - K, 0\right)\right] \\ &= E_t\left[e^{-\theta\tau}\max\left(\frac{1-\vartheta}{\vartheta}\frac{M_t}{\bar{M}_T} - K\frac{M_t}{M_T}, 0\right)\right] \\ &= E_t\left[e^{-\theta\tau}\max\left(X_t\frac{\bar{M}_t}{\bar{M}_T} - K\frac{M_t}{M_T}, 0\right)\right] \end{aligned}$$

Define $\alpha_M = \mu_M - \lambda_M E(H-1) - \frac{1}{2}\sigma_M^2$ and $Z_M = \sigma_M(B_T^M - B_t^M) + \sum_{j=n_t+1}^{n_T} h_j$ for domestic money supply process M_t. Likewise, for foreign country money supply process $\alpha_{\bar{M}} = \mu_{\bar{M}} - \lambda_{\bar{M}} E(\bar{H}-1) - \frac{1}{2}\sigma_{\bar{M}}^2$ and $Z_{\bar{M}} = \sigma_{\bar{M}}(B_T^{\bar{M}} - B_t^{\bar{M}}) + \sum_{j=n_t+1}^{n_T} \bar{h}_j$. Then we can rewrite the valuation equation.

$$C_t = E_t\left[e^{-\theta\tau}\max\left(X_t e^{-\alpha_{\bar{M}}\tau - Z_{\bar{M}}} - K e^{-\alpha_M\tau - Z_M}, 0\right)\right]$$

Taking expectation conditional on n_M jumps occurring in domestic money supply and $n_{\bar{M}}$ jumps occurring in foreign money supply.

$$
\begin{aligned}
C_t = & \sum_{n_{\bar{M}}=0}^{\infty}\sum_{n_M=0}^{\infty}\frac{e^{-\lambda_M\tau}(\lambda_M\tau)^{n_M}}{n_M!}\frac{e^{-\lambda_{\bar{M}}\tau}(\lambda_{\bar{M}}\tau)^{n_{\bar{M}}}}{n_{\bar{M}}!} \\
& \times E_t\left[e^{-\theta\tau}\left(X_t e^{-\alpha_{\bar{M}}\tau - Z_{\bar{M}}(n_{\bar{M}})}\mathbf{I}_{X_T>K} - K e^{-\alpha_M\tau - Z_M(n_M)}\mathbf{I}_{X_T>K}\right)\right]
\end{aligned}
$$

where $Z_M(n_M) = \sigma_M(B_T^M - B_t^M) + \sum_{j=1}^{n_M} h_j$ and $Z_{\bar{M}}(n_{\bar{M}}) = \sigma_{\bar{M}}(B_T^{\bar{M}} - B_t^{\bar{M}}) + \sum_{j=1}^{n_{\bar{M}}} \bar{h}_j$. Now let us focus on the conditional expectation.

$$
\begin{aligned}
E_t\left[e^{-\theta\tau-\alpha_{\bar{M}}\tau - Z_{\bar{M}}(n_{\bar{M}})}\mathbf{I}_{X_T>K}\right] = & \exp\left(-\theta\tau - \alpha_{\bar{M}}\tau - E[Z_{\bar{M}}(n_{\bar{M}})] + \frac{1}{2}\mathrm{Var}[Z_{\bar{M}}(n_{\bar{M}})]\right) \\
& \times \Phi\left(\frac{\ln(X_t/K) - \alpha_{\bar{M}}\tau - E[Z_{\bar{M}}(n_{\bar{M}})] + \alpha_M\tau + E[Z_M(n_M)] + \mathrm{Var}[Z_{\bar{M}}(n_{\bar{M}})]}{\sqrt{\mathrm{Var}[Z_M(n_M)] + \mathrm{Var}[Z_{\bar{M}}(n_{\bar{M}})]}}\right) \\
= & \, e^{-r_f\tau}\Phi\left(\frac{\ln(X_t/K) + (r_d - r_f)\tau}{\sqrt{\sigma_F^2\tau}} + \frac{1}{2}\sqrt{\sigma_F^2\tau}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
r_f &= \theta + \mu_{\bar{M}} - \lambda_{\bar{M}}E(\bar{H} - 1) + \frac{n_{\bar{M}}}{\tau}(\mu_{\bar{h}} - \frac{1}{2}\sigma_{\bar{h}}^2) - \sigma_{\bar{M}}^2 \\
r_d &= \theta + \mu_M - \lambda_M E(H - 1) + \frac{n_M}{\tau}(\mu_h - \frac{1}{2}\sigma_h^2) - \sigma_M^2 \\
\sigma_F^2 &= (\sigma_M^2 + \sigma_{\bar{M}}^2)\tau + n_M\sigma_h^2 + n_{\bar{M}}\sigma_{\bar{h}}^2.
\end{aligned}
$$

In the same way, the second part conditional expectation can be expressed as

$$
E_t\left[e^{-\theta\tau-\alpha_M\tau - Z_M(n_M)}\mathbf{I}_{X_T>K}\right] = e^{-r_d\tau}\Phi\left(\frac{\ln(X_t/K) + (r_d - r_f)\tau}{\sqrt{\sigma_F^2\tau}} - \frac{1}{2}\sqrt{\sigma_F^2\tau}\right).
$$

These lead to our final formula

$$
C_F = \sum_{n_{\bar{M}}=0}^{\infty}\sum_{n_M=0}^{\infty}\frac{e^{-\lambda_M\tau}(\lambda_M\tau)^{n_M}}{n_M!}\frac{e^{-\lambda_{\bar{M}}\tau}(\lambda_{\bar{M}}\tau)^{n_{\bar{M}}}}{n_{\bar{M}}!}C_{GK}(X_t, \tau; r_d, r_f, \sigma_F, K). \quad (5.34)
$$

We can also express the conditional interest rate by substituting in the domestic and foreign nominal interest rate.

$$
\begin{aligned}
r_d &= R + \lambda_M E(H^{-1} - 1) + \frac{n_M}{\tau}(\mu_h - \frac{1}{2}\sigma_h^2) \\
r_f &= \bar{R} + \lambda_{\bar{M}} E(\bar{H}^{-1} - 1) + \frac{n_{\bar{M}}}{\tau}(\mu_{\bar{h}} - \frac{1}{2}\sigma_{\bar{h}}^2)
\end{aligned}
$$

Valuation equations can be produced by the same argument as in the closed pure exchange economy.

The currency option prices depend intuitively on the fundamental parameters. First, an increase in the domestic money supply volatility σ_M, or the volatility of jump size σ_h induces a lower r_d and a higher σ_F: the joint consequence is not clear since C_{GK} is an increasing function of the conditional domestic interest rate r_d, and the conditional exchange rate volatility σ_F. Second, a higher volatility of foreign money supply $\sigma_{\bar{M}}$, or higher volatility of the corresponding jump $\sigma_{\bar{h}}$, imply a higher call price because it reduces r_d and increases σ_F simultaneously. Further, the call value is positively related to the instantaneous expected growth rate of the domestic money supply μ_M and negatively related to the instantaneous expected growth rate of foreign money supply $\mu_{\bar{M}}$. The effects of parameters $(\lambda_M, \lambda_{\bar{M}}, \mu_h, \mu_{\bar{h}})$ on currency call prices are ambiguous. Note that if there were no jump component in foreign money supply, the currency call prices would reduce to Merton style price equations with jump amplitude changing from Y to H^{-1}. In this case, the only jump uncertainty underlying the exchange rate would be from the domestic money supply and this jump uncertainty is not priced.

5.4.5 Pricing the Foreign Side

Let us now examine from the foreign country's perspective, that is, we try to price from the other side of the trade.

$$\begin{aligned}\frac{d\bar{X}_t}{X_t} &= [\mu_{\bar{M}} - \lambda_{\bar{M}}E(\bar{H}-1) - \mu_M + \lambda_M E(H-1)]dt + \sigma_{\bar{M}}dB_t^{\bar{M}} - \sigma_M dB_t^M \\ &\quad + (\bar{H}-1)dN_t^{\bar{M}} + (H^{-1}-1)dN_t^M\end{aligned} \tag{5.35}$$

We shall first value a European call option on the inverse exchange rate.

$$\begin{aligned}
\bar{C}_t &= E_t[\bar{\xi}^*_{t,T}\bar{C}_T] = E_t\left[e^{-\theta\tau}\frac{\bar{M}_t}{\bar{M}_T}\max\left(\bar{X}_T - \bar{K}, 0\right)\right] \\
\bar{C}_t &= E_t\left[e^{-\theta\tau}\frac{\bar{M}_t}{\bar{M}_T}\max\left(\frac{\vartheta}{1-\vartheta}\frac{\bar{M}_T}{M_T} - \bar{K}, 0\right)\right] \\
&= E_t\left[e^{-\theta\tau}\max\left(\bar{X}_t\frac{M_t}{M_T} - \bar{K}\frac{\bar{M}_t}{\bar{M}_T}, 0\right)\right] \\
&= E_t\left[e^{-\theta\tau}\max\left(\bar{X}_t e^{-\alpha_M\tau - Z_M} - \bar{K}e^{-\alpha_{\bar{M}}\tau - Z_{\bar{M}}}, 0\right)\right]
\end{aligned}$$

Taking expectation conditional on n_M jumps occurring in domestic money supply and $n_{\bar{M}}$ jumps occurring in foreign money supply.

$$\begin{aligned}
\bar{C}_t = \sum_{n_{\bar{M}}=0}^{\infty}\sum_{n_M=0}^{\infty} & \frac{e^{-\lambda_M\tau}(\lambda_M\tau)^{n_M}}{n_M!}\frac{e^{-\lambda_{\bar{M}}\tau}(\lambda_{\bar{M}}\tau)^{n_{\bar{M}}}}{n_{\bar{M}}!} \\
& \times E_t\left[e^{-\theta\tau}\left(\bar{X}_t e^{-\alpha_M\tau - Z_M(n_M)}\mathbf{I}_{\bar{X}_T > \bar{K}} - \bar{K}e^{-\alpha_{\bar{M}}\tau - Z_{\bar{M}}(n_{\bar{M}})}\mathbf{I}_{\bar{X}_T > \bar{K}}\right)\right]
\end{aligned}$$

The same tedious derivation yields.

$$\bar{C}_F = \sum_{n_{\bar{M}}=0}^{\infty}\sum_{n_M=0}^{\infty}\frac{e^{-\lambda_M\tau}(\lambda_M\tau)^{n_M}}{n_M!}\frac{e^{-\lambda_{\bar{M}}\tau}(\lambda_{\bar{M}}\tau)^{n_{\bar{M}}}}{n_{\bar{M}}!}\bar{C}_{GK}(\bar{X}_t, \tau; r_f, r_d, \sigma_F, \bar{K}) \quad (5.36)$$

The corresponding put option formula is

$$\bar{P}_F = \sum_{n_{\bar{M}}=0}^{\infty}\sum_{n_M=0}^{\infty}\frac{e^{-\lambda_M\tau}(\lambda_M\tau)^{n_M}}{n_M!}\frac{e^{-\lambda_{\bar{M}}\tau}(\lambda_{\bar{M}}\tau)^{n_{\bar{M}}}}{n_{\bar{M}}!}\bar{P}_{GK}(\bar{X}_t, \tau; \bar{r}_f, \bar{r}_d, \sigma_F, \bar{K}) \quad (5.37)$$

According to the law of one price, $\bar{P}_F$ converted into the domestic currency at the spot exchange rate should be the same as C_F. Since $C_{GK}(X_t, \tau; r_d, r_f, \sigma_F, K) = X_t\bar{P}_{GK}(K/X_t, \tau; r_f, r_d, \sigma_F, 1)$, thus

$$C_F(X_t, \tau; \ldots, K) = X_t\bar{P}_F(K/X_t, \tau; \ldots, 1). \quad (5.38)$$

It is self-evident that our currency option formula satisfy the property that its value does not depend on whose point of view is taken, that of the domestic or foreign investor without assuming away the jump risk. There is no instance of Siegle's paradox in our model. Actually, as Siegle and Dumas surmises, information about investor risk-preferences or more primitive economic variables would help to pin down the fair price.

We have examined a continuous-time two country dynamic monetary equilibrium. With mild assumptions on all exogenous distributions on all exogenous distributions, we provide pricing results for the nominal exchange rate and currency options. We can also easily obtain the equity option prices and modified security market line within this international economy context. With incorporation of money, the real quantities in the economy are identical to a benchmark economy with no money since the representative agent's marginal utility is not dependent on money balances. However the nominal interest rate and nominal prices will be influenced by aggregate money balances in each country.

Chapter 6

Concluding Remarks

The contribution of this dissertation is two-fold. Firstly, the classic Lucas (1978) [47] general equilibrium model is extended and applied to price stock options and bond options in the presence of jumps in the process that determines output in the economy. We have not only abandoned the assumption by Merton of unpriced jumps [49], but also explored the consequences of systematic jump risks. We go beyond other authors who have also made similar attempts in that our results are more generalized, meaningful yet clear. To an extent, this is a unified treatment of jump-diffusion option pricing models and a link between the CAPM and option pricing. The other half of this dissertation employs an international general equilibrium model originating with Lucas (1982) [46] and develops a foreign currency option pricing formula in a jump-diffusion setting. In contrast with the existing partial equilibrium models, the exchange rate process is endogenously determined and there is no inconsistency for exchange rate option prices from two sides (domestic and foreign) of the trade. All the formulas have been feasibly studied and analytically solved, lending themselves to application and/or empirical tests.

Our model has exclusively priced ordinary call and put options. They can be used as "building blocks" for constructing much more general and complex options. It is expected that jump-diffusion models may also prove useful for exotic options. Especially for those path-dependent options, a rigorous consideration of jumps in the sample paths of the underlying asset prices seem more important than for the standard non-path-dependent options.

It is worthwhile suggesting further avenues of research, which could benefit from the framework of this thesis. First, an extensive examination of which type of information surprises causes jumps is an open question. Locating jumps in the data and associating them with market events is one possible way. Secondly, parallelling the equity option models, jump-diffusion processes have provoked the attention in credit risk models [63] too. With firm value evolving as a jump-diffusion process, a firm can default instantaneously because of a sudden drop in its value. However, the treatment of jump risks thereinto still follows Merton (1976) [49]. Therefore models that integrate market and credit risk in a general equilibrium await development.

Appendix A

Tables

Here is a list of all parameters and variable values used in each table.

Table 1: Pricing a call option on consumption index with different maturity.

$$K = 50, \quad r = 0.1, \quad q = 0.02, \quad \sigma_S = 0.2,$$
$$\mu_x = \mu_y = -0.0032, \quad \sigma_x = \sigma_y = 0.08, \quad \varrho = 1, \quad \gamma = 1$$

Table 2: Importance of jump component.

$$K = 50, \quad r = 0.1, \quad q = 0.02, \quad \tau = 0.25,$$
$$\mu_x = -0.0018, \quad \sigma_x = 0.06, \quad \mu_y = 0, \quad \varrho = 0.8, \quad \gamma = 1$$

Table 3: The effect of systematic risk.

$$K = 50, \quad r = 0.1, \quad q = 0.02, \quad \tau = 0.25, \quad \sigma_S = 0.2,$$
$$\mu_x = -0.0018, \quad \sigma_x = 0.06, \quad \mu_y = -0.0032, \quad \sigma_y = 0.08, \quad \lambda = 2, \quad \gamma = 1$$

Table 4: The effect of different utility functions.

$$K = 50, \quad r = 0.1, \quad q = 0.02, \quad \tau = 0.25, \quad \sigma_S = 0.2,$$
$$\mu_x = -0.0018, \sigma_x = 0.06, \quad \mu_y = 0, \quad \varrho = 1, \quad \lambda = 2$$

$\lambda = 2$ $\sigma_{total} = 0.2298$				S_t	τ		$\lambda = 7$ $\sigma_{total} = 0.2912$		
BS	Merton	Huang					BS	Merton	Huang
0.00043	0.0045	0.0038		40	**1/12**		0.0059	0.0221	0.019
	(938%)	(774%)						(273%)	(220%)
0.0943	0.1054	0.1006		45			0.2346	0.2488	0.2351
	(11.80%)	(6.71%)						(6.07%)	(0.24%)
1.4889	1.4506	1.4526		50			1.839	1.753	1.7592
	(-2.56%)	(-2.43%)						(-4.68%)	(-4.34%)
5.4164	5.4297	5.4361		55			5.5627	5.5814	5.6002
	(0.25%)	(-0.36%)						(0.34%)	(0.67%)
0.0782	0.0891	0.085		40	**0.25**		0.2373	0.2545	0.2424
	(14.06%)	(8.70%)						(7.25%)	(2.16%)
0.7122	0.702	0.6968		45			1.1718	1.1435	1.1336
	(-1.42%)	(-2.14%)						(-2.41%)	(-3.26%)
2.7833	2.7556	2.7601		50			3.3781	3.3238	3.3356
	(-0.99%)	(-0.83%)						(-1.61%)	(-1.26%)
6.4509	6.4496	6.4595		55			6.8758	6.8612	6.8861
	(-0.01%)	(-0.14%)						(-0.21%)	(0.15%)
0.4576	0.4598	0.454		40	**0.5**		0.912	0.9052	0.8961
	(0.50%)	(-0.75%)						(-0.75%)	(-1.75%)
1.7152	1.699	1.6972		45			2.46	2.4238	2.4276
	(-0.93%)	(-1.04%)						(-1.48%)	(-1.32%)
4.2077	4.1877	4.1944		50			5.0248	4.9803	5.0036
	(-0.47%)	(-0.31%)						(-0.89%)	(-0.42%)
7.8274	7.8219	7.8331		55			8.4955	8.4666	8.5034
	(-0.07%)	(-0.07%)						(-0.34%)	(0.09%)
1.5669	1.5596	1.5568		40	**1**		2.4448	2.1126	2.4268
	(-0.45%)	(-0.63%)						(-13.6%)	(-0.74%)
3.545	3.5303	3.5335		45			4.6247	4.0557	4.6094
	(-0.41%)	(-0.32%)						(-12.3%)	(-0.33%)
6.4522	6.439	6.4483		50			7.5425	6.6852	7.5397
	(-0.2%)	(-0.05%)						(-11.4%)	(-0.04%)
10.1193	10.1137	10.1259		55			11.073	9.8834	11.0864
	(-0.05%)	(-0.06%)						(-10.7%)	(-0.12%)
3.9655	3.9421	3.9597		40	**2**		5.2964	0.8133	5.2925
	(-0.58%)	(-0.14%)						(-84.6%)	(-0.07%)
6.658	6.626	6.6562		45			8.0748	1.2939	8.0785
	(-0.47%)	(-0.02%)						(-84%)	(-0.04%)
9.9686	9.9298	9.9726		50			11.3411	1.8705	11.354
	(-0.39%)	(-0.04%)						(-83.5%)	(-0.11%)
13.756	13.7107	13.7656		55			14.9982	2.5229	15.0196
	(-0.33%)	(-0.07%)						(-83.2%)	(0.14%)

	$\sigma_{total} = 0.2296 \quad \kappa = 50\% \quad \sigma_S = 0.1625$					λ	$\sigma_{total} = 0.2296 \quad \kappa = 20\% \quad \sigma_S = 0.2054$				
S_t	Huang	Merton	NaikLee	BS	σ_y		Huang	Merton	NaikLee	BS	σ_y
40	0.1876	0.2012	0.1317	0.0782	0.2966	0.3	0.1072	0.112	0.0953	0.0782	0.1879
	(140%)	(157%)	(69%)				(37.30%)	(43.40%)	(22.10%)		
45	0.5876	0.6028	0.5311	0.7121			0.6812	0.6863	0.6696	0.7121	
	(-17.5%)	(-15.4%)	(-25.4%)				(-4.3%)	(-3.58%)	(-5.93%)		
50	2.4612	2.4617	2.4838	2.7831			2.7083	2.708	2.713	2.7831	
	(-11.6%)	(-11.6%)	(-10.8%)				(-2.68%)	(-2.69%)	(-2.51%)		
55	6.3531	6.3419	6.4317	6.4508			6.4306	6.426	6.4475	6.4508	
	(-1.51%)	(-1.69%)	(-0.29%)				(-0.31%)	(-0.38%)	(-0.08%)		
40	0.1498	0.1603	0.1274	0.0782	0.1625	1	0.0919	0.095	0.0886	0.0782	0.103
	(91.50%)	(105%)	(62.90%)				(17.60%)	(21.50%)	(13.40%)		
45	0.6535	0.6683	0.6239	0.7121			0.6994	0.7035	0.6952	0.7121	
	(-8.26%)	(-6.17%)	(-12.4%)				(-1.78%)	(-1.21%)	(-2.37%)		
50	2.601	2.5992	2.6135	2.7831			2.7497	2.7488	2.7513	2.7831	
	(-6.55%)	(-6.62%)	(-6.1%)				(-1.2%)	(-1.23%)	(-1.14%)		
55	6.4277	6.4136	6.4668	6.4508			6.4481	6.4435	6.4537	6.4508	
	(-0.36%)	(-0.58%)	(-0.25%)				(-0.04%)	(-0.11%)	(-0.05%)		
40	0.1221	0.13	0.1118	0.0782	0.1149	2	0.0855	0.0876	0.0843	0.0782	0.0728
	(56.14%)	(66.2%)	(43%)				(9.42%)	(12.20%)	(7.93%)		
45	0.6777	0.6904	0.6618	0.7121			0.7046	0.7077	0.7029	0.7121	
	(-4.86%)	(-3.07%)	(-7.08%)				(-1.03%))	(-0.6%)	(-1.27%)		
50	2.6715	2.6687	2.6782	2.7831			2.7647	2.7635	2.765	2.7831	
	(-4.02%)	(-4.12%)	(-3.78%)				(-0.66%)	(-0.7%)	(-0.64%)		
55	6.4479	6.4346	6.468	6.4508			6.4512	6.4473	6.4531	6.4508	
	(-0.05%)	(-0.25%)	(0.27%)				(-0.01%)	(-0.05%)	(-0.04%)		
40	0.1016	0.1072	0.0977	0.0782	0.0812	4	0.0818	0.0832	0.0815	0.0782	0.0515
	(30.11%)	(37.30%)	(25.10%)				(4.61%)	(6.51%)	(4.34%)		
45	0.6915	0.701	0.6848	0.7121			0.7079	0.71	0.7074	0.7121	
	(-2.86%)	(-1.52%)	(-3.79%)				(-0.58%))	(-0.29%)	(-0.66%)		
50	2.7219	2.7187	2.7241	2.7831			2.774	2.7727	2.7736	2.7831	
	(-2.19%)	-2.30%	(-2.11%)				(-0.33%)	(-0.37%)	(-0.34%)		
55	6.4543	6.4432	6.4619	6.4508			6.4524	6.4492	6.4523	6.4508	
	(-0.06%)	(-0.11%)	(-0.18%)				(-0.03%)	(-0.02%)	(-0.02%)		
40	0.0915	0.0958	0.0901	0.0782	0.0614	7	0.08	0.0811	0.0801	0.0782	0.0389
	(17.10%)	(22.60%)	(15.30%)				(2.45%)	(3.82%)	(2.57%)		
45	0.6987	0.7056	0.696	0.7121			0.7092	0.7106	0.7091	0.7121	
	(-1.88%))	(-0.9%)	(2.26%)				(-0.36%)	(-0.17%)	(-0.38%)		
50	2.7483	2.745	2.7482	2.7831			2.778	2.7767	2.7772	2.7831	
	(-1.25%)	(-1.37%)	(-1.25%)				(-0.17%)	(0.22%)	(0.20%)		
55	6.4557	6.4465	6.4577	6.4508			6.4523	6.4496	6.4515	6.4508	
	(-0.07%)	(-0.07%)	(0.11%)				(-0.03%)	(-0.01%)	(-0.02%)		

$\lambda = 2 \quad \kappa = 24.24\%$					S_t	ϱ		$\lambda = 7 \quad \kappa = 52.83\%$			
BS	Huang	Merton	NaikLee					BS	Huang	Merton	NaikLee
0.0782	0.0859	0.0891	0.085		40	**1**		0.2373	0.2449	0.2545	0.2424
	(9.86%)	(14.10%)	(8.70%)						(3.20%)	(7.25%)	(-2.16%)
0.7122	0.6977	0.702	0.6968		45			1.1718	1.1347	1.1435	1.1336
	(-2.02%)	(-1.42%)	(-2.14%)						(-3.17%)	(-2.41%)	(-3.26%)
2.7833	2.7584	2.7556	2.7601		50			3.3781	3.3308	3.3238	3.3356
	(-0.89%)	(-1%)	(-0.83%)						(-1.4%)	(-1.61%)	(-1.26%)
6.4509	6.4566	6.4496	6.4595		55			6.8758	6.8786	6.8612	6.8861
	(-0.09%)	(-0.02%)	(-0.14%)						(-0.04%)	(-0.21%)	(-0.15%)
0.0782	0.0876	0.0891			40	**0.5**		0.2373	0.2499	0.2545	
	(12.00%)	(14.10%)							(5.33%)	(7.25%)	
0.7122	0.7	0.702			45			1.1718	1.1396	1.1435	
	(-1.7%)	(-1.42%)							(-2.74%)	(-2.41%)	
2.7833	2.7572	2.7556			50			3.3781	3.328	3.3238	
	(-0.93%)	(-1%)							(-1.49%)	(-1.61%)	
6.4509	6.4533	6.4496			55			6.8758	6.8704	6.8612	
	(-0.04%)	(-0.02%)							(-0.07%)	(-0.21%)	
0.0782	0.0894	0.0891			40	**0**		0.2373	0.2554	0.2545	
	(14.30%)	(14.10%)							(7.64%)	(7.25%)	
0.7122	0.7026	0.702			45			1.1718	1.1456	1.1435	
	(-1.33%)	(-1.42%)							(-2.23%)	(-2.41%)	
2.7833	2.7565	2.7556			50			3.3781	3.3264	3.3238	
	(-0.96%)	(-1%)							(-1.53%)	(-1.61%)	
6.4509	6.4502	6.4496			55			6.8758	6.8632	6.8612	
	(-0.01%)	(-0.02%)							(-0.18)	(-0.21%)	
0.0782	0.0913	0.0891			40	**-0.5**		0.2373	0.2613	0.2545	
	(16.80%)	(14.10%)							(10.10%)	(7.25%)	
0.7122	0.7056	0.702			45			1.1718	1.1527	1.1435	
	(-0.91%)	(-1.42%)							(-1.63%)	(-2.41%)	
2.7833	2.7562	2.7556			50			3.3781	3.3262	3.3238	
	(-0.97%)	(-1%)							(-1.54%)	(-1.61%)	
6.4509	6.4474	6.4496			55			6.8758	6.857	6.8612	
	(-0.05%)	(-0.02%)							(-0.27%)	(-0.21%)	
0.0782	0.0933	0.0891			40	**-1**		0.2373	0.2676	0.2545	
	(19.40%)	(14.10%)							(12.80%)	(7.25%)	
0.7122	0.7089	0.702			45			1.1718	1.1608	1.1435	
	(-0.45%)	(-1.42%)							(-0.94%)	(-2.41%)	
2.7833	2.7563	2.7556			50			3.3781	3.3274	3.3238	
	(-0.96%)	(-1%)							(-1.5%)	(-1.61%)	
6.4509	6.445	6.4496			55			6.8758	6.8517	6.8612	
	(-0.09%)	(-0.02%)							(-0.35%)	(-0.21%)	

$\sigma_y = 0.08$ $\sigma_{total} = 0.2298$ $\kappa = 24.24\%$				S_t	γ	$\sigma_y = 0.2$ $\sigma_{total} = 0.3464$ $\kappa = 33.33\%$			
BS	Huang	Merton	NaikLee			BS	Huang	Merton	NaikLee
0.0782		0.0917		40	**0**	0.4513		0.5671	
		(17.40%)						(25.70%)	
0.7122		0.706		45		1.6157		1.4844	
		(-0.86%)						(-8.13%)	
2.7833		2.7553		50		3.9145		3.5986	
		(-1%)						(-8.07%)	
6.4509		6.446		55		7.3115		7.1256	
		(-0.07%)						(-2.54%)	
0.0782	0.0899	0.0917	0.0892	40	**0.5**	0.4513	0.5483	0.5671	0.5074
	(15.00%)	(17.40%)	(14.10%)				(21.50%)	(25.70%)	(12.40%)
0.7122	0.7032	0.706	0.7021	45		1.6157	1.4639	1.4844	1.4222
	(-1.25%)	(-0.86%)	(-1.4%)				(-9.4%)	(-8.13%)	(-12%)
2.7833	2.7558	2.7553	2.7558	50		3.9145	3.5959	3.5986	3.5976
	(-0.98%)	(-1%)	(-0.98%)				(-8.14%)	(-8.07%)	(-8.1%)
6.4509	6.4489	6.446	6.4497	55		7.3115	7.1396	7.1256	7.1781
	(-0.03%)	(-0.07%)	(-0.02%)				(-2.35%)	(-2.54%)	(-1.83%)
0.0782	0.0881	0.0917	0.087	40	1	0.4513	0.5306	0.5671	0.4601
	(12.80%)	(17.40%)	(11.30%)				(17.60%)	(25.70%)	(1.94%)
0.7122	0.7008	0.706	0.6991	45		1.6157	1.4456	1.4844	1.3848
	(-1.58)	(-0.86%)	(-1.82%)				(-10.5%)	(-8.13%)	(-14.3%)
2.7833	2.7569	2.7553	2.7575	50		3.9145	3.596	3.5986	3.6278
	(-0.94%)	(-1%)	(-0.92%)				(-8.14%)	(-8.07%)	(-7.3%)
6.4509	6.4522	6.446	6.4543	55		7.3115	7.1558	7.1256	7.2547
	(-0.02%)	(-0.07%)	(-0.05%)				(-2.13%)	(-2.54%)	(-0.78%)
0.0782	0.0852	0.0917	0.0833	40	**2**	0.4513	0.4986	0.5671	0.3998
	(9.01%)	(17.40%)	(6.59%)				(10.50%)	(25.70%)	(-11.4%)
0.7122	0.6976	0.706	0.6957	45		1.6157	1.4158	1.4844	1.3864
	(-2.03%)	(-0.86%)	(-2.3%)				(-12.4%)	(-8.13%)	(-14.2%)
2.7833	2.7612	2.7553	2.7643	50		3.9145	3.6047	3.5986	3.7862
	(-0.79%)	(-1%)	(-0.68%)				(-7.91%)	(-8.07%)	(-3.28%)
6.4509	6.4603	6.446	6.4659	55		7.3115	7.1946	7.1256	7.4857
	(-0.15%)	(-0.07%)	(0.23%)				(-1.6%)	(-2.54%)	-(2.38%)
0.0782	0.096	0.0917	0.0977	40	**-1**	0.4513	0.6086	0.5671	0.7309
	(22.80%)	(17.40%)	(25.10%)				(34.80%)	(25.70%)	(61.90%)
0.7122	0.7131	0.706	0.7164	45		1.6157	1.5324	1.4844	1.6886
	(0.14%)	(-0.86%)	-(0.61%)				(-5.16%)	(-8.13%)	(4.50%)
2.7833	2.7562	2.7553	2.7578	50		3.9145	3.6125	3.5986	3.6983
	(-0.97%)	(-1%)	(-0.91%)				(-7.72%)	(-8.07%)	(-5.53%)
6.4509	6.4415	6.446	6.4408	55		7.3115	7.104	7.1256	7.0927
	(-0.14%)	(-0.07%)	(-0.15%)				(-2.84%)	(-2.54%)	(-2.99%)

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