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Fundamental Limitation on Achievable Decentralized Performance [1]

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Abstract

A commonly accepted fact is that the diagonal structure of the controller poses fundamental limitations on the achievable performance, but few quantitative results are available for measuring these limitations. This paper provides a lower bound on the achievable quality of disturbance rejection using a decentralized controller for stable discrete time linear systems with time delays, which do not contain any finite zeros on or outside the unit circle. The proposed result is useful for assessing when full multivariable controllers can provide significantly improved performance, as compared to decentralized controllers. The results are also extended to the case, where the individual sub-controllers are restricted to be PID controllers.

Key words: Decentralized control, Minimum variance control, Performance limitations, PID control.

1 Introduction

It is well established that the system itself can pose fundamental limitations on the achievable control performance. These limitations can arise due to the presence of unstable poles and zeros, and also time delay. When these sources of limitations are present, (i) performance gain in some frequency range must be traded-off against performance loss in other frequency ranges [2,6] and, (ii) there exist non-zero lower bounds on the norms of different closed-loop transfer functions, that no controller can improve upon (see *e.g.* [4,12]). An overview of the results on fundamental limitations is available in [16,17].

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In many practical problems of interest, *e.g.* process industries, the use of a decentralized controller is preferred over a full multivariable or centralized controller. In some special cases, the structure of the controller does not impose any additional limitations, than that are encountered with the use of a full multivariable controller. For example, similar to full multivariable controllers, the sensitivity function can be reduced arbitrarily for systems that are diagonal at high frequencies [20] and also for sequentially minimum-phase systems [9]. In general, however, the controller structure can itself act as a source of limitations, which is not taken into account by most of the results available on fundamental limitations.

Cui and Jacobsen [5] have studied the change in location of unstable zeros due to closure of some of the loops of the decentralized controller. Goodwin *et al.* [7] have further established the effect of unstable poles and zeros on achievable performance using decentralized controller. These results are, however, derived under the assumption that the individual loops of the decentralized controller are designed sequentially [5] or independently [7]. Noting that the use of independent or sequential design methods can be conservative, it is better to allow for simultaneous design of decentralized controller to characterize fundamental limitations, as is done in this paper. For decentralized control, it is useful to ensure integrity against loop failures [10]. Unlike the independent design method, however, integrity is difficult to obtain using the simultaneous design method. The issue of integrity is not dealt with in this paper and the characterization of the limitations due to integrity requirements is still an open problem.

We consider stable discrete time linear systems with time delay, which do not contain finite zeros on or outside the unit circle. The performance is measured in terms of the $\mathcal{H}_2$-norm of the closed-loop transfer function between disturbances and outputs. For the resulting cheap or minimum variance (MV) control problem, the achievable performances for single input single output (SISO) systems and for multi-input multi-output (MIMO) systems under full multivariable control are available in [1] and [8], respectively. The exact characterization of the achievable decentralized performance is difficult, as the resulting optimization problem is non-convex [18,3,15]. Clearly, any suboptimal tuning strategy for the decentralized controller provides an upper bound on achievable performance. Some computationally efficient upper bounds for decentralized MV control problem have been previously reported by utilizing the structure of involved optimization problem [19] or explicit characterization of suboptimal solution [11]. In comparison to these results, a lower bound is clearly more useful to analyze the fundamental limitations due to the controller structure.

In this paper, we show that for invertible systems, the controller structure does not impose any additional limitations on the achievable performance

as compared to the full multivariable controller. This result is more general than the corresponding result in [9], as the system is only required to be minimum-phase, but not sequentially minimum-phase. Our main contribution is a lower bound on the achievable performance for the decentralized MV control problem, which indicates a fundamental limitation due to controller structure. The problem formulation inherently allows us to distinguish between the limitations arising due to the time delay and the controller structure.

We also consider the case, where the individual subcontrollers are restricted to be of PID controllers. This additional structural limitation can further limit the achievable performance. Previously, computationally efficient upper bounds for decentralized PID controllers have been presented in [13,11]. Braatz *et al.* [3] have also derived an upper bound on the achievable robust performance evaluated at steady-state for decentralized proportional-integral controllers. The proposed lower bound, however, is more useful for characterizing fundamental limitations, as before. The lower bounds proposed in this paper can be conservative in some cases. The tightening of lower bound, however, requires the solution of a non-convex optimization problem. In this sense, the results of this paper are useful for getting quick insight into the limitations due to controller structure.

2   Preliminaries

For a matrix $A \in \mathbb{R}^{m \times n}$, A_{ij} and A_{*j} represent the ij^{th} element (or block) and j^{th} column of A, respectively. For ease of notation, we define

$$\text{vec}(A) = [A_{11} \, A_{21} \cdots A_{m1} \cdots A_{mn}]^T \tag{1}$$

and for $B \in \mathbb{R}^{n \times p}$

$$\mathcal{M}(A, B) = \begin{bmatrix} A_{*1} \, B_{11} \, A_{*2} \, B_{21} \, \cdots \, A_{*n} \, B_{n1} \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\ A_{*1} \, B_{1p} \, A_{*2} \, B_{2p} \, \cdots \, A_{*n} \, B_{np} \end{bmatrix} \tag{2}$$

A diagonal matrix $C \in \mathbb{R}^{m \times m}$ is vectorized as

$$\text{diag}(C) = [C_{11} \, C_{22} \cdots C_{mm}]^T \tag{3}$$

We denote the system and disturbance models as $G(q^{-1})$ and $H(q^{-1})$, respectively, such that

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$$y = G(q^{-1})\,u + H(q^{-1})\,a \tag{4}$$

Here, y, u and a are controlled outputs, manipulated variables and disturbances, respectively. We make the following simplifying assumptions:

(1) $G(q^{-1})$ and $H(q^{-1})$ are stable, causal transfer matrices, contain no zeros on or outside the unit circle except at infinity (due to time delays), and are square having dimensions $n_y \times n_y$.

(2) $a(t)$ is a random noise sequence with unit variance.

When $H(q^{-1})$ contains zeros outside the unit circle, these zeros can be factored through an all pass factor without affecting the noise spectrum [8]. Further, there is no loss of generality in assuming that the system is affected by noise having unit variance. When $E[a(t)a^T(t)] \neq I$, the disturbance model can always be scaled to satisfy this assumption. The feedback controller $K(q^{-1})$ is assumed to have a diagonal structure.

The $\mathcal{H}_2$ norm of the stable transfer matrix $G(q^{-1})$ is given as

$$\|G(q^{-1})\|_2^2 = \sum_{i=0}^{\infty} \mathrm{tr}(G_i^T\, G_i) \tag{5}$$

where G_i is the ith impulse response coefficient of G. The jk^{th} element (or block) of G_i is represented as $[G_i]_{jk}$. We consider that $G(q^{-1})$ can be factored as

$$G(q^{-1}) = D^{-1}(q^{-1})\tilde{G}(q^{-1}) \tag{6}$$

such that $\tilde{G}(q^{-1})$ and $D^{-1}(q^{-1})$ contain the invertible and non-invertible parts of $G(q^{-1})$, respectively. The interactor matrix generalizes the time delay for SISO systems to the multivariable case [8] and can be written as,

$$D(q) = D_0(q)q^d + D_1(q)q^{d-1} + \cdots D_{d-1}(q)q$$

where d denotes the order of the interactor matrix. We consider that $D(q)$ is a unitary interactor matrix, *i.e.* $D^T(q^{-1})\,D(q) = I$ [8]. In the remaining discussion, we drop the arguments q^{-1} and t for notational simplicity.

The objective of this paper is to characterize the least achievable value of the variance of y, *i.e.* $E[\mathrm{tr}(y\,y^T)]$. The achievable value of $E[\mathrm{tr}(y\,y^T)]$ is limited due to the non-invertible part of G, which cannot be overcome even by a full multivariable controller [8], and possibly also due to the structure of the controller K. To differentiate between these sources of limitations, we denote

$$\min_{K} E[\text{tr}(y\,y^{T})] \;=\; J_{\text{full}} \;+\; J_{\text{decen}}$$

where J_{decen} denotes the "additional" limitation due to the use of decentralized controller. Clearly, a decentralized controller will provide same performance as the full multivariable controller, if $J_{\text{decen}} = 0$.

3 Invertible Systems

First we deal with the case, when G is invertible ($D = I$). For such systems, arbitrarily good performance can be obtained using a full multivariable controller having high gain. We show that such a conclusion also holds for decentralized controllers.

Proposition 1 *Consider a system with stable G and H. If G is invertible, $E[\text{tr}(y\,y^{T})]$ can be reduced arbitrarily using a stable decentralized controller.*

PROOF. For regulatory control, let $u = -K\,y$. Under closed loop conditions,

$$y = (I + G\,K)^{-1}\,H\,a \tag{7}$$

Let K be chosen as $K = (1/\epsilon)\,\hat{K}$, where $\hat{K}$ is a diagonal, stable and invertible transfer matrix. Based on (7),

$$y = \epsilon\,(G\,\hat{K})^{-1}\,(\epsilon\,(G\,\hat{K})^{-1} + I)^{-1}\,H\,a$$

Using small gain theorem (see *e.g.* [17]), the closed-loop system is stable, if

$$\max_{|z|=1}\rho(\epsilon\,(G\,\hat{K}(z^{-1}))^{-1}) < 1$$

$$\text{or }|\epsilon| < \min_{|z|=1}\rho^{-1}((G\,\hat{K}(z^{-1}))^{-1}) = \min_{|z|=1}|\underline{\lambda}(G\,\hat{K}(z^{-1}))|$$

where ρ and $\underline{\lambda}$ are the spectral radius and smallest eigenvalue, respectively. As $G\,\hat{K}$ evaluated at any point on the unit circle is non-singular, $\min_{|z|=1}|\underline{\lambda}(G\,\hat{K}(z^{-1}))|$ is non-zero and $E[\text{tr}(y\,y^{T})]$ remains bounded for $|\epsilon| < \min_{|z|=1}|\underline{\lambda}(G\,\hat{K}(z^{-1}))|$, including $\epsilon = 0$. Thus $\lim_{\epsilon \to 0} E[\text{tr}(y\,y^{T})]$ exists and is given as zero. Now, using continuity arguments, it follows that by choosing ϵ to be sufficiently close to zero, $E[\text{tr}(y\,y^{T})]$ can be reduced arbitrarily with $K = (1/\epsilon)\,\hat{K}$ being stable. $\square$

Note that the result in Proposition 1 is stronger that the corresponding result of Johansson and Rantzer [9], as G is only required to be minimum-phase, but not sequentially minimum-phase. Further, Proposition 1 still holds when norms other than $\mathcal{H}_2$ norm are chosen as performance measure.

Example 2 *Motivated by a similar example in [17], we consider the following system*

$$
G = \begin{bmatrix} 0 & \frac{1}{(1-0.5\,z^{-1})^4} \\ \frac{0.5}{(1-0.25\,z^{-1})^2} & 0 \end{bmatrix}
$$

with $H = 1/(1 - 0.9\,z^{-1})$. When the pairings are selected on the off-diagonal elements of G, the system is sequentially minimum phase and controller structure does not pose any limitations [9]. When diagonal pairings are used, neither the system is sequentially minimum phase nor the independent design method can be applied due to the zero diagonal elements of G. We show that similar to off-diagonal pairings, arbitrarily tight control can be obtained for diagonal pairings, as G is minimum phase.

We express the controller as $K = (1/\epsilon)\,I$. Based on the proof of Proposition 1, this parameterization of K implies that the closed loop system remains stable for $|\epsilon| < \min_{|z|=1} |\underline{\lambda}(G(z^{-1}))| = 0.251$. For $\epsilon = 0.25$, $E[\mathrm{tr}(y\,y^T)] = 30.15$. When ϵ is chosen as $0.1, 0.01$ and 0.001, $E[\mathrm{tr}(y\,y^T)]$ reduces to 0.12, 1.09×10^{-3} and 1.08×10^{-5}, respectively, demonstrating that arbitrarily good performance can be obtained using a high-gain decentralized controller.

4 Limitations due to Controller Structure

Unlike invertible systems, the controller structure can pose additional limitations on the achievable performance of non-invertible systems ($D \neq I$). In this section, we derive an explicit lower bound on the achievable decentralized performance for non-invertible systems. This result is further extended to include the case, when the individual sub-controllers are restricted to be of PID type.

4.1 Problem Formulation

Using (4) and (6),

$$
y = D^{-1}\tilde{G}\,u + H\,a \tag{8}
$$

Define $y_1 = q^{-d} D y$ and $\bar{H} = q^{-d} D H$. By multiplying both sides of (8) by $q^{-d} D$, we have

$$y_1 = q^{-d} \tilde{G} u + \bar{H} a \tag{9}$$

For regulatory control, $u = -K y_1$ and

$$y_1 = (I + q^{-d} \tilde{G} K)^{-1} \bar{H} a \tag{10}$$

Using Diophantine's identity,

$$\bar{H} = \bar{F} + q^{-d} \bar{R}$$

When the closed loop system is stable, we can expand $(I + q^{-d} \tilde{G} K)^{-1}$ to get,

$$\begin{aligned} y_1 &= (I - q^{-d} \tilde{G} K + q^{-2d} (\tilde{G} K)^2 + \cdots) (\bar{F} + q^{-d} \bar{R}) a \\ &= \left(\bar{F} + q^{-d} L - q^{-2d} \tilde{G} K L + \cdots \right) a \end{aligned} \tag{11}$$

where

$$L = \bar{R} - \tilde{G} K \bar{F} \tag{12}$$

Since $E[\mathrm{tr}(y y^T)] = E[\mathrm{tr}(y_1 y_1^T)]$ and $\bar{F}$ is controller invariant, L can be set to zero to obtain the full multivariable minimum variance (MV) control law, where $J_{\mathrm{full}} = \|\bar{F}\|_2^2$ [8]. When the controller has structural constraints, this may not be possible since K has fewer degrees of freedom than the full multivariable controller. To characterize the achievable decentralized performance, one can minimize the contribution of controller dependent terms on $E[\mathrm{tr}(y y^T)]$. In general, this problem tends to be non-linear and a lower bound on the achievable value of $E[\mathrm{tr}(y y^T)]$ is discussed next.

4.2   Decentralized Controller

Let the closed loop system be $S = (I + q^{-d} \tilde{G} K)^{-1} \bar{H}$. Based on (5)

$$E[\mathrm{tr}(y y^T)] = \sum_{i=0}^{\infty} \mathrm{tr} \left(S_i^T S_i \right)$$

where S_i denote the ith impulse response matrix of S. Thus $E[\mathrm{tr}(y y^T)] \geq \sum_{i=0}^{n} \mathrm{tr} \left(S_i^T S_i \right)$ for any finite n. Then, the non-linear nature of the involved

minimization problem can be overcome by selecting n such that S_i, $0 \leq i \leq n$, depend linearly on K and a lower bound on $E[\text{tr}(y\,y^T)]$ is obtained by minimizing $\sum_{i=0}^{n} \text{tr}\left(S_i^T S_i\right)$. Here, we select $n = (2d-1)$, as in addition to L_i, $i \geq d$, the impulse response coefficients of the non-linear term $\tilde{G}\,K\,(\bar{R} - \tilde{G}\,K\,\bar{F})$ also contribute to S_j, $j \geq 2d$. Then,

$$E[\text{tr}(y\,y^T)] \geq \|\bar{F}\|_2^2 + \sum_{i=0}^{d-1} \text{tr}\left(L_i^T L_i\right) \tag{13}$$

where L is given by (12). Since $\bar{F}$ is controller independent, a lower bound on $E[\text{tr}(y\,y^T)]$ can derived by minimizing the contribution of the second term in (13).

The impulse response coefficients of L are given as

$$
\begin{bmatrix} L_0 \\ L_1 \\ \vdots \\ L_{d-1} \end{bmatrix} = \begin{bmatrix} \bar{R}_0 \\ \bar{R}_1 \\ \vdots \\ \bar{R}_{d-1} \end{bmatrix} - \begin{bmatrix} \tilde{G}_0 & 0 & \cdots & 0 \\ \tilde{G}_1 & \tilde{G}_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ \tilde{G}_{d-1} & \cdots & \cdots & \tilde{G}_0 \end{bmatrix} \begin{bmatrix} K_0 & 0 & \cdots & 0 \\ K_1 & K_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ K_{d-1} & \cdots & \cdots & K_0 \end{bmatrix} \begin{bmatrix} \bar{F}_0 \\ \bar{F}_1 \\ \vdots \\ \bar{F}_{d-1} \end{bmatrix}
$$

By vectorizing L_i, $i = 0, \cdots, (d-1)$, we have

$$L_v = \bar{R}_v - \tilde{G}_H K_v$$

where

$$
\begin{aligned}
L_v &= \left[\text{vec}(L_0)^T \ \text{vec}(L_1)^T \ \cdots \ \text{vec}(L_{d-1})^T \right]^T \\
\bar{R}_v &= \left[\text{vec}(\bar{R}_0)^T \ \text{vec}(\bar{R}_1)^T \ \cdots \ \text{vec}(\bar{R}_{d-1})^T \right]^T \\
K_v &= \left[\text{diag}(K_0)^T \ \text{diag}(K_1)^T \ \cdots \ \text{diag}(K_{d-1})^T \right]^T
\end{aligned} \tag{14}
$$

$$
\tilde{G}_H = \begin{bmatrix} \mathcal{M}(\tilde{G}_0, \bar{F}_0) & 0 & \cdots & 0 \\ \mathcal{M}(\tilde{G}_0, \bar{F}_1) + \mathcal{M}(\tilde{G}_1, \bar{F}_0) & \mathcal{M}(\tilde{G}_0, \bar{F}_0) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ \sum_{i=0}^{d-1} \mathcal{M}(\tilde{G}_i, \bar{F}_{(d-1)-i}) & \cdots & \cdots & \mathcal{M}(\tilde{G}_0, \bar{F}_0) \end{bmatrix} \tag{15}
$$

where $\text{vec}(\cdot)$, $\mathcal{M}(\cdot, \cdot)$ and $(\cdot)_v$ are defined by (1), (2) and (3), respectively. Now, the contribution of the second term in (13) can be minimized by minimizing

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$L_v^T L_v$ using regression. Based on these developments, we present the main contribution of this paper.

Proposition 3 *A lower bound on achievable decentralized performance is given as*

$$\min_K E[\text{tr}(y\, y^T)] \geq \|\bar{F}\|_2^2 + \left\| \left(I - \tilde{G}_H\, \tilde{G}_H^\dagger \right) \bar{R}_v \right\|_2^2 \tag{16}$$

where $\bar{R}_v$ and $\tilde{G}_H$ are given by (14) and (15), respectively.

As $\|\bar{F}\|_2^2$ is the achievable value for full multivariable controller, the second term in (16) provides a lower bound on the "additional" limitations due to the use of a diagonal controller, *i.e.* J_{decen}.

Though the expression for lower bound on $E[\text{tr}(y\, y^T)]$ is messy, insights can be drawn by considering special cases. We consider G with interactor matrix of the form $D = q\, I$, *i.e.* unit delay in all elements of G and $H = 1/(1 - a\, q^{-1})\, I$, $|a| \leq 1$. Then, $F = I$ and the achievable multivariable performance is $\|F\|_2^2 = n_y$. Through straightforward algebraic manipulations, the lower bound in Proposition 3 can be simplified as

$$J_{\text{decen}} \geq a^2 \sum_{j=1}^{n_y} \frac{\sum_{i \neq j}[\tilde{G}_0]_{ij}^2}{\sum_j [\tilde{G}_0]_{ij}^2} = a^2 \sum_{j=1}^{n_y} \frac{1}{1 + \frac{[\tilde{G}_0]_{jj}^2}{\sum_{i \neq j}[\tilde{G}_0]_{ij}^2}} \tag{17}$$

Note that as $[\tilde{G}_0]_{jj} \to 0$ for all i, the above expression approaches its maximum value, which is $a^2\, n_y$. When $a = 1$ (step-type disturbances), the relative difference between achievable multivariable and decentralized performances is 100% showing fundamental difficulty in the use of a decentralized controller.

Remark 4 *Based on the above discussion, it follows that to avoid significant performance loss, associated with the use of decentralized controller for rejection of disturbances passing through a first-order filter (including step-type disturbances), it should be ensured that $[\tilde{G}_0]_{jj}^2 \gg \sum_{i \neq j}[\tilde{G}_0]_{ij}^2$ for all j. This condition is similar to the concept of diagonal dominance, whose implications for achieving closed-loop stability through independent design of decentralized controller are well-known [14,17].*

4.3 Decentralized PID controller

In many practical problems, the individual subcontrollers are fixed to be of PID type. In this section, we show how Proposition 3 can be modified to find a

lower bound on the achievable performance for decentralized PID controllers. We note that in Proposition 3, the bound is derived by solving the optimization problem with respect to the impulse response coefficients of the controller K. Then, by setting $K_k = 0$ for all $k > p$, a lower bound with only p non-zero impulse response coefficients can be obtained.

We consider that the decentralized PID controller is expressed as,

$$K_{\text{PID}} = \frac{1}{\Delta} \sum_{i=0}^{2} C_i q^{-i} = \frac{1}{\Delta} C$$

where $\Delta = 1 - q^{-1}$ is the integrator and C has the same diagonal structure as the controller K. By considering $1/\Delta$ as a part of $\tilde{G}$ (or $\bar{F}$) and minimizing $\sum_{i=0}^{d-1} \text{tr}\left(L_i^T L_i\right)$ with respect to C, a lower bound on the achievable PID performance can be derived. Then Propositions 3 can be used by considering only the first 3 block columns of $\tilde{G}_H$ in (15). To ensure that the assumption of stability of G is satisfied, the integrator can be moved just inside the unit circle without affecting the result significantly. Note that these results provide information about limitations due to the PID structure of sub-controllers, as compared to unrestricted decentralized control, only when the order of the interactor matrix is greater than 3.

5 Examples and Discussion

The results of the previous section provide a lower bound on the achievable decentralized performance and are useful in quantifying the fundamental limitations due to the controller structure. In general, however, the proposed results can be conservative in the sense that there may not exist a decentralized controller that closely matches the lower bound. In this section, we demonstrate the usefulness and conservatism of the proposed results using simple examples.

Example 5 *We consider the system with*

$$G = \begin{bmatrix} \frac{-0.1q^{-2}}{(1-0.1q^{-1})(1-0.2q^{-1})} & \frac{-0.25q^{-1}(1-0.3q^{-1})}{(1-0.1q^{-1})(1-0.2q^{-1})} \\ \frac{0.5q^{-1}(1+0.9q^{-1})}{(1-0.1q^{-1})(1-0.2q^{-1})} & \frac{-0.1q^{-2}}{(1-0.1q^{-1})(1-0.2q^{-1})} \end{bmatrix} \tag{18}$$

and $H = 1/(1 - 0.5q^{-1}) I$. Then, $D = q I$, $F = I$ and the achievable multivariable performance is $J_{\text{full}} = \|F\|_2^2 = 2$.

Using Proposition 3, we find that when a decentralized controller with diagonal

pairings is used, $J_{\text{decen}} \geq 0.5$. Thus $E[\text{tr}(y\,y^T)] \geq 2.5$ implying additional limitations due to controller structure. For these pairings, additional limitation due to controller structure primarily arises, as the diagonal elements of $\tilde{G}_0$ are zero; see also Remark 4. The static controller

$$K = \begin{bmatrix} -0.066 & 0 \\ 0 & -0.47 \end{bmatrix}$$

is designed using trial and error, which provides $E[\text{tr}(y\,y^T)] = 2.65$.

When the off-diagonal pairings are used, the lower bound on J_{decen} reduces to zero indicating that the diagonal structure of the decentralized controller is not always limiting, as one may expect. The following sub-optimal controller designed using trial and error

$$K = \begin{bmatrix} \frac{-1.69}{1-0.47q^{-1}} & 0 \\ 0 & \frac{0.86}{1+0.71q^{-1}} \end{bmatrix}$$

provides $E[\text{tr}(y\,y^T)] = 2.08$. For both pairings, the achieved performance is close to the proposed lower bound showing the tightness. Though usually difficult to obtain using simultaneous design method, the sub-optimal controllers also provide integrity against loop failures for both pairings.

In Example 5, the lower bound proposed in Proposition 3 can be matched closely for the system considered, but this bound is loose in general. This issue is illustrated by the following example.

Example 6 *We revisit Example 5 with the disturbance model being $H = (1/\Delta)\,I$ i.e. step-type disturbances. As before, we have $D = q\,I$, $F = I$ and $J_{\text{full}} = 2$. Based on Proposition 3, we find that with the use of diagonal pairings, $J_{\text{decen}} \geq 2$ and thus $E[\text{tr}(y\,y^T)] \geq 4$. We, however, could not find a diagonal controller that matches the lower bound indicated by Proposition 3 closely. With the use of the diagonal controller*

$$K = \begin{bmatrix} \frac{-1.39\,(1-0.73q^{-1})}{1-q^{-2}} & 0 \\ 0 & \frac{-4.15\,(1-0.32q^{-1})}{1-q^{-2}} \end{bmatrix}$$

$E[\text{tr}(y\,y^T)] = 8.34$, which could not be reduced substantially even by using a more sophisticated controller. This indicates the conservatism of the proposed lower bound.

To confirm this finding, we note that the lower bound in Proposition 3 is derived by considering only the first 2 impulse response matrices of the closed-loop transfer matrix. The primary difficulty in improving the lower bound by considering additional impulse response matrices is their nonlinear dependence on the controller. We increase the number of terms sequentially and use Matlab function fminunc with multiple randomized initial guesses to solve the resulting non-convex optimization problems. When $5, 10, 15$ and 20 terms of S are considered, the lower bound on $E[\mathrm{tr}(y\,y^T)]$ is found to be $6.85, 8.00, 8.14$ and 8.16, respectively, confirming the conservatism of Proposition 3.

Example 6 demonstrates that Proposition 3 may grossly underestimate the extent of limitations due to controller structure. Nevertheless, the proposed lower bound provides quick insight into the limitations and its' improvement without solving computationally expensive non-convex optimization problems is an issue for further research.

6   Conclusions

We derived a lower bound on the achievable performance for the decentralized minimum variance control problem. This result is useful for analyzing fundamental limitations due to controller structure, which are in addition to the limitations encountered with a full multivariable controller. For many systems, there exists a controller such that the lower bound is closely matched, but the bound is loose in general. Future research will focus on tightening the bound and also extending it to unstable, non-minimum phase systems. The main challenge in deriving improved bounds is the non-convexity of the resulting optimization problem, when using decentralized controllers.

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