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1993

https://hdl.handle.net/10356/91646

https://doi.org/10.1017/CBO9780511526336.029

A local characterization of the graphs of alternating forms and the graphs of quadratic forms over GF(2)

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Abstract

Let Δ be the line graph of $PG(n-1,2)$, $Alt(n,2)$ be the graph of the n-dimensional alternating forms over $GF(2)$, $n \geq 4$. Let Γ be a connected locally Δ graph such that

1. the number of common neighbours of any pair of vertices at distance two is the same as in $Alt(n,2)$.
2. the valency of the subgraph induced on the second neighbourhood of any vertex is the same as in $Alt(n,2)$.

It is shown that Γ is covered either by $Alt(n,2)$ or by the graph of $(n-1)$-dimensional $GF(2)$-quadratic forms $Quad(n-1,2)$.

1. Introduction

In this paper we investigate graphs which are locally the same as the graph $Alt(n, 2)$ of alternating forms on an n-dimensional vector space V over $GF(2)$. An analogous question for $GF(q)$, $q > 2$, is considered in [4]. The local graph of $Alt(n, 2)$ is isomorphic to the Grassmann graph $\begin{bmatrix} V \\ 2 \end{bmatrix}$, i.e. the line graph of $PG(n-1,2)$. Thus, we investigate graphs which are locally $\begin{bmatrix} V \\ 2 \end{bmatrix}$. It turns out that, besides $Alt(n, 2)$, there is another well-known graph which is locally $\begin{bmatrix} V \\ 2 \end{bmatrix}$. It is the graph $Quad(n-1, 2)$ of quadratic forms on an $(n-1)$-dimensional vector space over $GF(2)$. Both $Alt(n, 2)$ and $Quad(n-1, 2)$ are distance regular and have the same parameters, though they are non-isomorphic if $n \geq 5$.

Consider a half dual polar space of type D_n over $GF(2)$. Then the collinearity graph, induced by the complement of a geometric hyperplane,

is always locally $\begin{bmatrix} V \\ 2 \end{bmatrix}$. The number of such hyperplanes, even taken up to the action of the automorphism group of the polar space, increases at least exponentially with n, so that in general, classification of locally $\begin{bmatrix} V \\ 2 \end{bmatrix}$ graphs seems to be a hard problem. In this paper, we restrict ourselves to the case $\mu = 20$, i.e., the number of common neighbours of two vertices at distance 2 is always 20. Both graphs $\mathrm{Alt}(n,2)$ and $\mathrm{Quad}(n-1,2)$ possess this property, as well as another property $a_2 = 15 \cdot 2^{n-1} - 105$, which means that the graph induced by the second neighbourhood of every vertex is regular of the shown valency. This latter condition is rather technical and it is used only once in the proof. Hopefully, in further research it may be shown superfluous.

Under the assumption $\mu = 20$, $\mathrm{Alt}(4,2)$ is the only graph which is locally $\begin{bmatrix} V \\ 2 \end{bmatrix}$ with $n = 4$, while $\mathrm{Alt}(5,2)$ and $\mathrm{Quad}(4,2)$ are the only graphs which are locally $\begin{bmatrix} V \\ 2 \end{bmatrix}$ with $n = 5$. For large n, however, we cannot expect the analogous result, since the quotient graphs of $\mathrm{Alt}(n,2)$ or $\mathrm{Quad}(n-1,2)$ by many subgroups of translations have the same local structure and the same values of μ and a_2. The purpose of this paper is to show that $\mathrm{Alt}(n,2)$ and $\mathrm{Quad}(n-1,2)$ are universal in the following sense.

Main Theorem. *Let Γ be a connected graph which is locally the Grassmann graph $\begin{bmatrix} V \\ 2 \end{bmatrix}$, where V is a vector space of dimension n over $\mathrm{GF}(2)$. Suppose that $\mu(\Gamma) = 20$ and $a_2(\Gamma) = 15 \cdot 2^{n-1} - 105$. Then Γ is covered by either $\mathrm{Alt}(n,2)$ or $\mathrm{Quad}(n-1,2)$.*

The following corollary was a motivation for this work.

Corollary. *If Γ is a distance-regular graph having the same intersection numbers and the same local structure as $\mathrm{Alt}(n,2)$, then Γ is isomorphic to either $\mathrm{Alt}(n,2)$ or $\mathrm{Quad}(n-1,2)$.*

As a non-logical consequence of these investigations, new automorphisms of $\mathrm{Quad}(n,q)$, q even, were found, mixing forms of the same rank, but with different Witt indices [3].

The contents of the paper is as follows. In Section 2 we collect definitions and state the notation. In Section 3 we check that the graph $\mathrm{Quad}(n,2)$ has the prescribed local structure and that it is triangulable, that is, every cycle in it can be decomposed into a product of 3-cycles. Similar results for $\mathrm{Alt}(n,2)$ have been known earlier. In Section 4 we determine the possibilities for the μ-graphs, their intersections, vertex-subgraph relationship, etc. This allows us to determine in Section 5 the second neighbourhood of a vertex. Finally, in Section 6 we apply the covering theorem from [4] to show that every graph under consideration is covered either by $\mathrm{Alt}(n,2)$ or by $\mathrm{Quad}(n-1,2)$.

2. Preliminaries

Let Γ, $\tilde{\Gamma}$ be connected graphs. We say that Γ is a *cover of* $\tilde{\Gamma}$ if there exists a mapping φ from Γ to $\tilde{\Gamma}$ which maps edges to edges and, for every $\gamma \in \Gamma$, induces a bijection from $\Gamma(\gamma)$ to $\tilde{\Gamma}(\varphi(\gamma))$. If the subgraph $\Gamma_2(\alpha)$ is regular for any vertex α of a graph Γ, and its valency is independent on α, then this valency is denoted by $a_2 = a_2(\Gamma)$.

Let V be an n-dimensional vector space over GF(2), $\begin{bmatrix} V \\ 2 \end{bmatrix}$ the Grassmann graph. A *grand clique* of $\begin{bmatrix} V \\ 2 \end{bmatrix}$ is a set of the form $\{\gamma \in \begin{bmatrix} V \\ 2 \end{bmatrix} | x \in \gamma\}$ where $0 \neq x \in V$. A *small clique* of $\begin{bmatrix} V \\ 2 \end{bmatrix}$ is a set of the form $\begin{bmatrix} W \\ 2 \end{bmatrix}$, where $W \in \begin{bmatrix} V \\ 3 \end{bmatrix}$. Any maximal clique of $\begin{bmatrix} V \\ 2 \end{bmatrix}$ is either grand or small. Two distinct grand cliques have exactly one vertex in common, and two distinct small cliques have at most one vertex in common. A grand clique and a small clique can have at most 3 common vertices. Thus, any 4-clique in $\begin{bmatrix} V \\ 2 \end{bmatrix}$ is contained in a unique maximal clique.

Let V be an n-dimensional vector space over GF(q). The *rank* of an alternating form γ on V is defined by rank $\gamma = \dim(V/\text{Rad}\,\gamma)$, where Rad $\gamma = \{u \in V | \gamma(u, v) = 0 \text{ for any } v \in V\}$. Note that rank γ is always even. The *alternating forms graph* Alt(n, q) has as vertices the alternating forms on V. Two alternating forms γ, δ are adjacent whenever rank $(\gamma - \delta) = 2$.

Let V be as before. A map $\gamma : V \to \text{GF}(q)$ is called a *quadratic form* if for any $u, v \in V$ and $a, b \in \text{GF}(q)$, $\gamma(au + bv) = a^2\gamma(u) + b^2\gamma(v) + abB_\gamma(u, v)$ for some bilinear form B_γ. We call B_γ the *bilinear form associated with* γ. In case q is even, B_γ is an alternating form. The *rank* of γ is defined by rank $\gamma = \dim(V/\text{Rad}\,\gamma)$, where Rad $\gamma = \{u \in \text{Rad}\,B_\gamma | \gamma(u) = 0\}$. If q is even and rank $\gamma = \dim V$ odd, then the 1-dimensional subspace Rad B_γ is called the *nucleus* of γ. The *quadratic forms graph* Quad(n, q) has as vertices the quadratic forms on V. Two quadratic forms γ, δ are adjacent whenever rank $(\gamma - \delta) = 1$ or 2. The graph Quad(n, q) is distance-regular and has the same parameters as Alt$(n + 1, q)$ [1].

3. Some properties of Quad(n-1,2)

It is well-known that Alt$(n, 2)$ is locally the Grassmann graph $\begin{bmatrix} V \\ 2 \end{bmatrix}$, where V is an n-dimensional vector space over GF(2).

Proposition 3.1 *The graph* $\Gamma = $ Quad$(n-1, 2)$ *is locally the Grassmann graph* $\begin{bmatrix} V \\ 2 \end{bmatrix}$, *where* V *is a vector space of dimension* n *over* GF(2).

Proof. Let $V = W \oplus \langle e_0 \rangle$, where Γ is the set of all quadratic forms on W. Define the mapping $\varphi : \Gamma(0) \to \begin{bmatrix} V \\ n-2 \end{bmatrix}$ as follows. For $\gamma \in \Gamma(0)$, define $\varphi(\gamma) = \text{Rad}\,\gamma \in \begin{bmatrix} W \\ n-2 \end{bmatrix} \subset \begin{bmatrix} V \\ n-2 \end{bmatrix}$ if rank $\gamma = 1$. If rank $\gamma = 2$, then write $W = \text{Rad}\,\gamma \oplus \langle x_1, x_2 \rangle$. Define $\varphi(\gamma)$ by $\varphi(\gamma) = \text{Rad}\,\gamma \oplus (e_0 + (1 + \gamma(x_2))x_1 +$

$(1 + \gamma(x_1))x_2) \in [\begin{smallmatrix} V \\ n-2 \end{smallmatrix}]$. It is easy to see that φ is well-defined and bijective. Since $\Gamma(0)$ and $[\begin{smallmatrix} V \\ n-2 \end{smallmatrix}]$ have the same valency, φ is an isomorphism if and only if it preserves adjacency.

Let $\gamma, \delta \in \Gamma(0)$ be adjacent. If rank γ = rank δ = 1, then clearly $\varphi(\gamma)$ is adjacent to $\varphi(\delta)$. If rank γ = 1 and rank δ = 2, then we have rank $(\gamma + \delta)$ = 2 and Rad $\delta \subset$ Rad γ. Thus Rad $\delta \subset \varphi(\gamma) \cap \varphi(\delta)$, i.e., $\varphi(\gamma)$ is adjacent to $\varphi(\delta)$. If rank γ = rank δ = 2 and rank $(\gamma + \delta)$ = 1, then Rad γ = Rad $\delta \subset \varphi(\gamma) \cap \varphi(\delta)$, i.e., $\varphi(\gamma)$ is adjacent to $\varphi(\delta)$. Finally suppose that rank γ = rank δ = rank $(\gamma + \delta)$ = 2. Then dim(Rad $\gamma \cap$ Rad δ) = $n - 4$. Choose $x, y, z \in W$ in such a way that $W = ($Rad $\gamma \cap$ Rad $\delta) \oplus \langle x, y, z \rangle$, Rad $\gamma = ($Rad $\gamma \cap$ Rad $\delta) \oplus \langle y \rangle$, and Rad $\delta = ($Rad $\gamma \cap$ Rad $\delta) \oplus \langle z \rangle$. Since $\gamma(y) = \delta(z) = (\gamma + \delta)(y + z) = 0$, we have $\gamma(z) = \delta(y)$, so that the element $e_0 + (1 + \gamma(z))x + (1 + \delta(x))y + (1 + \gamma(x))y$ belongs to $\varphi(\gamma) \cap \varphi(\delta)$. Thus, dim $\varphi(\gamma) \cap \varphi(\delta) \leq n - 3$, i.e., $\varphi(\gamma)$ is adjacent to $\varphi(\delta)$. □

A graph is called *triangulable* if every cycle in it can be decomposed into a product of 3-cycles (cf. [4]).

Proposition 3.2 *If q is a power of 2, then the graph* Quad(n, q) *is triangulable.*

To prove triangulability of a graph Γ it suffices to check the following two conditions (γ a fixed vertex of Γ):

(T1) $\Gamma_{j-1}(\delta) \cap \Gamma(\gamma)$ *is connected for every $\delta \in \Gamma_j(\gamma)$, $j \geq 2$;*

(T2) *if $\delta_0, \delta_1 \in \Gamma_j(\gamma)$, $j \geq 2$ are adjacent, then the subsets $\Gamma_{j-1}(\delta_0) \cap \Gamma(\gamma)$ and $\Gamma_{j-1}(\delta_1) \cap \Gamma(\gamma)$ are at distance at most 1 from each other.*

For a proof of this criterion see the proof of Lemma 6.2 in [4]. From now on we denote by V an n-dimensional vector space over GF(q), q even. Let $\Gamma = $ Quad(n, q), $\bar{\Gamma} = $ Alt(n, q) and let φ be the mapping from Γ to $\bar{\Gamma}$ defined by $\gamma \mapsto B_\gamma$. Clearly, φ takes adjacent vertices to adjacent or equal. Moreover, two quadratic forms are mapped to the same vertex of Alt(n, q) if and only if their difference has rank 1, i.e., such forms are adjacent.

The following technical facts were taken from [1], see 9.5.5(i) for (i), and 9.6.2 for (ii). A proof for (iii) can be found in [1], page 292.

Lemma 3.3 (i) *Let γ and δ be two alternating forms on V. If rank $(\gamma + \delta)$ = rank γ + rank δ, then Rad γ + Rad δ = V and Rad $\gamma \cap$ Rad δ = Rad $(\gamma + \delta)$.*

(ii) *Let γ and δ be quadratic forms on V of rank $2j + 1$ and 2, respectively. If rank $(\gamma + \delta)$ = $2j$ then Rad $\delta \cap$ Rad B_γ = Rad γ.*

(iii) *Let γ be a rank 2 quadratic form and δ a rank 2 alternating form on V, such that rank $(B_\gamma + \delta)$ = 2. Then there is a rank 2 quadratic form δ' with $B_{\delta'} = \delta$, such that $\gamma + \delta'$ also has rank 2.*

Lemma 3.4 *Let γ be a quadratic form on V of even rank $2j$. Then φ establishes an isomorphism between the subgraphs $\Gamma(0) \cap \Gamma_{j-1}(\gamma)$ and $\bar{\Gamma}(0) \cap \bar{\Gamma}_{j-1}(B_\gamma)$. In particular, $\Gamma(0) \cap \Gamma_{j-1}(\gamma)$ is connected.*

Proof. Clearly, φ maps the former set to the latter. Hence, it suffices to prove that every alternating form from $\bar{\Gamma}(0) \cap \bar{\Gamma}_{j-1}(B_\gamma)$ has exactly one preimage in $\Gamma(0) \cap \Gamma_{j-1}(\gamma)$, and the preimage of an edge is an edge. Let $\beta \in \bar{\Gamma}(0) \cap \bar{\Gamma}_{j-1}(B_\gamma)$. By Lemma 3.3(i), Rad $\beta \supset$ Rad B_γ. Hence, we may assume for simplicity that Rad $\gamma = 0$. Let us consider V as a symplectic space with respect to the form B_γ. Then $U = \text{Rad}\,(B_\gamma + \beta)$ is orthogonal to Rad β, hence by Lemma 3.3(i), $U = \text{Rad}\,\beta^\perp$. Let a quadratic form δ be defined by Rad $\delta = \text{Rad}\,\beta$ and $\delta|_U = \gamma|_U$. Then $B_\delta = \beta$ and Rad $(\gamma + \delta) = U$, hence $\gamma + \delta$ has rank $2j - 2$. On the other hand, if for a quadratic form δ one has $B_\delta = \beta$ and $\gamma + \delta$ has rank $2j - 2$ then Rad $(\gamma + \delta) = \text{Rad}\,(B_\gamma + \beta) = U$, and hence δ must be defined as above. Hence, φ is indeed a bijection between $\Gamma(0) \cap \Gamma_{j-1}(\gamma)$ and $\bar{\Gamma}(0) \cap \bar{\Gamma}_{j-1}(B_\gamma)$.

Now suppose that δ_1, δ_2 are two rank 2 quadratic forms, such that both $\delta_1' = \gamma + \delta_1$ and $\delta_2' = \gamma + \delta_2$ have rank $2j - 2$. Let $U_i = \text{Rad}\,(B_{\delta_i'})$, $i = 1, 2$. Suppose $B_{\delta_1} + B_{\delta_2}$ has rank 2. By Lemma 3.3(i), it means that $Z = \text{Rad}\,B_{\delta_1} \cap \text{Rad}\,B_{\delta_2}$ has codimension at most 3 in V. In particular, U_1 and U_2, which are both orthogonal to Z, generate at most a 3-dimensional subspace. It follows that $T = U_1 \cap U_2$ is nontrivial. Now the equality $\gamma = \delta_1 + \delta_1' = \delta_2 + \delta_2'$ implies $\delta_1 + \delta_2 = \delta_1' + \delta_2'$. Therefore, both Z and T are in the radical of $\delta_1 + \delta_2$. Since Z has codimension at most 3, and $T \not\subset Z$, we finally have that $\delta_1 + \delta_2$ has rank at most 2. □

Now let us consider the case rank $\gamma = 2j - 1$. Let, as in [1], $R_s(\gamma)$ denotes the set of quadratic forms δ, such that $\gamma + \delta$ has rank s. Then $\Gamma(0) \cap \Gamma_{j-1}(\gamma)$ consists of three parts, namely, $\Omega_1(\gamma) = R_1(0) \cap R_{2j-2}(\gamma)$, $\Omega_2(\gamma) = R_2(0) \cap R_{2j-3}(\gamma)$ and $\Omega_3(\gamma) = R_2(0) \cap R_{2j-2}(\gamma)$. Clearly, $\Omega_1(\gamma)$ is a clique. Furthermore, $\Omega_2(\gamma)$ consists of all rank 2 forms in the preimage of $\bar{\Gamma}(0) \cap \bar{\Gamma}_{j-2}(B_\gamma)$. In particular, $\Omega_2(\gamma)$ induces a connected subgraph.

Lemma 3.5 *If rank $\gamma = 2j - 1$ then $\Gamma(0) \cap \Gamma_{j-1}(\gamma)$ is a connected subgraph.*

Proof. Let $\delta \in \Omega_3(\gamma)$. By Lemma 3.3(ii) Rad $\gamma = \text{Rad}\,B_\gamma \cap \text{Rad}\,\delta$. Hence there is a hyperplane U in V, such that Rad $\delta \subset U$ and Rad $\gamma = \text{Rad}\,B_\gamma \cap U$. Define a rank 1 quadratic form α by Rad $\alpha = U$ and $\alpha|_{\text{Rad}\,B_\gamma} = \gamma|_{\text{Rad}\,B_\gamma}$. Clearly, $\alpha \in \Omega_1(\gamma)$ and δ, α are connected by an edge. Since $\Omega_1(\gamma)$ is a clique, it means that $\Omega_1(\gamma) \cup \Omega_3(\gamma)$ induces a connected subgraph.

Since $\Omega_2(\gamma)$ is connected, it remains to find an edge between $\Omega_2(\gamma)$ and $\Omega_3(\gamma)$. Let δ be as above and let $U = \text{Rad}\,\delta + \text{Rad}\,B_\gamma$. Then U is a hyperplane

in V and, clearly, we can find an alternating form β in $\bar{\Gamma}(0) \cap \bar{\Gamma}_{j-2}(B_\gamma)$, such that $\operatorname{Rad} \beta \subset U$. Then by Lemma 3.3(iii) there is a rank 2 quadratic form α with $B_\alpha = \beta$, such that $\delta + \alpha$ has rank 2. On the other hand, clearly, $\alpha \in \Omega_2(\gamma)$. $\qquad\square$

Lemmas 3.4 and 3.5 give (T1). Next we check (T2).

Lemma 3.6 *If $\gamma, \delta \in \Gamma_j(0)$ are adjacent, then there is an edge between $\Gamma(0) \cap \Gamma_{j-1}(\gamma)$ and $\Gamma(0) \cap \Gamma_{j-1}(\delta)$.*

Proof. Let B_γ and B_δ be of rank $2s$ and $2k$, respectively. We have seen above that $\varphi(\Gamma(0) \cap \Gamma_{j-1}(\gamma))$ contains $\bar{\Gamma}(0) \cap \bar{\Gamma}_{s-1}(B_\gamma)$ and $\varphi(\Gamma(0) \cap \Gamma_{j-1}(\delta))$ contains $\bar{\Gamma}(0) \cap \bar{\Gamma}_{k-1}(B_\delta)$. If $k = s$ then [4], Lemma 6.3(ii) implies that $\bar{\Gamma}(0) \cap \bar{\Gamma}_{s-1}(B_\gamma) \cap \bar{\Gamma}_{k-1}(B_\delta) \neq \emptyset$. If $s < k$ then $\bar{\Gamma}(0) \cap \bar{\Gamma}_{s-1}(B_\gamma) \subset \bar{\Gamma}(0) \cap \bar{\Gamma}_{k-1}(B_\delta)$, and, in particular, we obtain the same conclusion as above. Similarly in the case $s > k$. Hence in any case $\varphi(\Gamma(0) \cap \Gamma_{j-1}(\gamma)) \cap \varphi(\Gamma(0) \cap \Gamma_{j-1}(\delta)) \neq \emptyset$. Since the preimage of any vertex of $\bar{\Gamma}$ is a clique, we obtain the desired edge between $\Gamma(0) \cap \Gamma_{j-1}(\gamma)$ and $\Gamma(0) \cap \Gamma_{j-1}(\delta)$. $\qquad\square$

Since both conditions (T1) and (T2) have been checked, the proof of Proposition 3.2 is complete.

4. μ-graphs

We will say that Γ *has distinct μ-graphs* if $\Gamma(v, u_0) = \Gamma(v, u_1)$ for $u_0, u_1 \in \Gamma_2(v)$ implies $u_0 = u_1$. Assume that the graph Γ is locally $\begin{bmatrix} V \\ 2 \end{bmatrix}$, where V is an n-dimensional vector space over $GF(2)$. Any μ-graph of $\begin{bmatrix} V \\ 2 \end{bmatrix}$ is a 3×3 grid, and $\begin{bmatrix} V \\ 2 \end{bmatrix}$ has distinct μ-graphs. It follows (see [5], Lemma 1) that any μ-graph of Γ is locally a 3×3 grid and that Γ has distinct μ-graphs.

Lemma 4.1 *Suppose $\mu = \mu(\Gamma) = 20$. Then every μ-graph of Γ is isomorphic to $J(6,3)$, and Γ has distinct μ-graphs.*

Proof. It is well-known that there are precisely two connected locally a 3×3 grid graphs, as a reference we can suggest [2], where these graphs appear as a very special case. The Johnson graph $J(6,3)$ is the only locally a 3×3 grid graph with 20 vertices. $\qquad\square$

Now we are going to determine all embeddings of $J(6,3)$ into the Grassmann graph $\begin{bmatrix} V \\ 2 \end{bmatrix}$. We denote by $A(W)$ the set of all nondegenerate alternating forms on W, where W is a 4-dimensional vector space over $GF(2)$. We also denote by $Q(W, x)$ the set of all nondegenerate quadratic forms on W with nucleus $\langle x \rangle$, where W is a 5-dimensional vector space over $GF(2)$, $0 \neq x \in W$. For $\gamma \in A(W)$, we define

$$M_\gamma = \{ U \in \begin{bmatrix} W \\ 2 \end{bmatrix} \,|\, U \text{ is nonisotropic with respect to } \gamma \}.$$

Then $|M_\gamma| = 20$.

For $\gamma \in Q(W, x)$, we define

$$M_\gamma = \{U \in [\textstyle{W \atop 2}] \mid \gamma(u) = 1 \text{ for all } u \in U, u \neq 0\}$$
$$T_\gamma = \{U \in [\textstyle{W \atop 2}] \mid x \notin U, \ \gamma|_U \neq 0, \ \gamma \text{ is linear on } U\}$$
$$T_\gamma^0 = \{U \in [\textstyle{W \atop 2}] \mid \gamma = 0 \text{ on } U\}.$$

Then $|M_\gamma| = 20$, $|T_\gamma| = 45$, $|T_\gamma^0| = 15$.

The subgraph of $[\textstyle{W \atop 2}]$ induced by M_γ, where $\gamma \in A(W)$, or $Q(W, x)$, is isomorphic to $J(6, 3)$. We aim to show that these two are the only subgraphs of $[\textstyle{W \atop 2}]$ isomorphic to $J(6, 3)$ up to automorphisms of $[\textstyle{W \atop 2}]$.

For a finite set Ω, consider the incidence system $\Omega_{2,3} = ((\textstyle{\Omega \atop 2}), (\textstyle{\Omega \atop 3}))$, where $(\textstyle{\Omega \atop i})$ denotes the set of i-subsets of Ω, and the incidence is defined by inclusion. A representation φ of $\Omega_{2,3}$ is an injective mapping $\varphi : (\textstyle{\Omega \atop 2}) \to V - \{0\}$, where V is a vector space over GF(2), satisfying the property

$$\varphi(\{i, j\}) + \varphi(\{j, k\}) + \varphi(\{j, i\}) = 0$$

for any $\{i, j, k\} \in (\textstyle{\Omega \atop 3})$. Since the Johnson graph $J(|\Omega|, 3)$ is the line graph of $\Omega_{2,3}$, we obtain an embedding of the Johnson graph $J(|\Omega|, 3)$ in the Grassmann graph $[\textstyle{W \atop 2}]$ whenever we have a representation of $\Omega_{2,3}$. One can construct the universal representation of $\Omega_{2,3}$ in the sense that any linear relation between the vectors representing pairs from $(\textstyle{\Omega \atop 2})$ follows from the above relations. The dimension of the space generated by the image of $(\textstyle{\Omega \atop 2})$ in the universal representation is the dimension of the left null space of the incidence matrix for $\Omega_{2,3}$ taken over GF(2). If $|\Omega| = 6$, then one checks easily that the dimension of the universal representation is 5.

Lemma 4.2 Let V be an n-dimensional vector space over GF(2) and M a subgraph of $[\textstyle{V \atop 2}]$ isomorphic to $J(6, 3)$. Then one of the following holds.

(i) There exist $W \in [\textstyle{V \atop 6}]$, $\gamma \in A(W)$ such that $M = M_\gamma$.

(ii) There exist $W \in [\textstyle{V \atop 5}]$, $0 \neq x \in W$, $\gamma \in Q(W, x)$ such that $M = M_\gamma$.

Proof. The graph $J(6, 3)$ has 30 maximal cliques. It can be shown that there are 15 maximal cliques, each of which is contained in a grand clique of $[\textstyle{V \atop 2}]$. This establishes a representation of $\Omega_{2,3}$ in V. One checks easily that the embedding (ii) is obtained from the universal representation, while (i) is the only quotient of it. □

In the rest of this section we list some properties of the subgraphs $J(6, 3)$ of $[\textstyle{V \atop 2}]$, mostly found by a computer program. In every case a check is straightforward. Table 1 is self-explanatory. The Petersen graph is the complement of $J(5, 2)$. By the type of a quadratic form, we mean the rank r together with a

description of γ and δ				$M_\gamma \cap M_\delta$	# of δ
$\gamma \in Q(W,x)$	$\delta \in Q(W,x)$ $\delta \neq \gamma$	type of $\gamma + \delta$	1	K_4	15
			2+	N_8	45
			2-	$\emptyset$	15
			3	K_2	180
			4+	3-claw	120
			4-	$\emptyset$	72
	$\delta \in A(W_1)$ $x \in W_1 \in \begin{bmatrix} W \\ 4 \end{bmatrix}$	type of γ on $\mathrm{Rad}\,(B_\gamma\vert_{w_i} + \delta)$	2-	K_4	4
		$\mathrm{Rad}\,(B_\gamma\vert_{w_i} + \delta)$	2+	K_2	12
		$\mathrm{Rad}\,(B_\gamma\vert_{w_i} + \delta) = 0$		K_2	12
$\gamma \in A(W_1)$	$\delta \in Q(W,x)$ $x \in W_1 \in \begin{bmatrix} W \\ 4 \end{bmatrix}$	type of δ on $\mathrm{Rad}\,(B_\delta\vert_{w_i} + \gamma)$	2-	K_4	64
		$\mathrm{Rad}\,(B_\delta\vert_{w_i} + \gamma)$	2+	K_2	192
		$\mathrm{Rad}\,(B_\delta\vert_{w_i} + \gamma) = 0$		K_2	192
	$\delta \in A(W_1)$ $\delta \neq \gamma$	rank$(\gamma + \delta) = 2$		N_{12}	15
		rank$(\gamma + \delta) = 4$		Petersen	12
	$\delta \in A(W_2)$ $\dim W_1 \cap W_2 = 3$	$\gamma\vert_{w_1 \cap w_2} = \delta\vert_{w_1 \cap w_2}$		K_4	4
		$\gamma\vert_{w_1 \cap w_2} \neq \delta\vert_{w_1 \cap w_2}$		K_2	24

Table 1. Intersection of two μ-graphs

sign "+" or "−" when r is even. The sign "+" indicates that the Witt index is $r/2$, "−" indicates that it is $r/2 - 1$.

Let N_{12} be the graph with 12 vertices $\{(i,j)|i = 1,2, \; j = 1, \ldots, 6\}$ where two distinct vertices $(i_1, j_1), (i_2, j_2)$ are adjacent if and only if $|j_1 - j_2| = 0, 1$, or 5. Let N_8 be the induced subgraph of N_{12} on the vertices $\{(i,j)|i = 1,2, \; j = 1, \ldots, 4\}$.

Lemma 4.3 Let $\gamma \in A(W)$ with $W \in \begin{bmatrix} V \\ 4 \end{bmatrix}$, or $\gamma \in Q(W,x)$ with $W \in \begin{bmatrix} V \\ 5 \end{bmatrix}$. For $U \in \begin{bmatrix} V \\ 2 \end{bmatrix}$, set $N_{\gamma,U} = \{U_1 \in M_\gamma | U \cap U_1 \neq \emptyset\}$ and regard $N_{\gamma,U}$ as a subgraph of M_γ.

(i) If $\gamma \in A(W)$, $U \in \begin{bmatrix} V \\ 2 \end{bmatrix} - M_\gamma$, and $N_{\gamma,U} \neq \emptyset$, then $N_{\gamma,U}$ is isomorphic to either N_{12} or K_4. Moreover, $N_{\gamma,U} \cong N_{12}$ if and only if $U \in \begin{bmatrix} V \\ 2 \end{bmatrix} - M_\gamma$.

(ii) If $\gamma \in Q(W,x)$, $U \in \begin{bmatrix} V \\ 2 \end{bmatrix} - M_\gamma$, and $N_{\gamma,U} \neq \emptyset$, then $N_{\gamma,U}$ is isomorphic to either N_8 or K_4. Moreover, $N_{\gamma,U} \cong N_8$ if and only if $U \in T_\gamma$.

(iii) The number of $U \in \begin{bmatrix} V \\ 2 \end{bmatrix} - M_\gamma$ with $N_{\gamma,U} \neq \emptyset$ is $15 \cdot 2^{n-1} - 105$ if $\gamma \in A(W)$ and is $15 \cdot 2^{n-1} - 120$ if $\gamma \in Q(W,x)$.

Lemma 4.4 Let $\gamma \in A(W)$ or $Q(W,x)$, $M_\gamma \supset N \cong N_8$. Then N generates W, i.e., there is no proper subspace W_1 of W for which $N \subset \begin{bmatrix} W_1 \\ 2 \end{bmatrix}$ holds.

Lemma 4.5 *Let* $\gamma \in Q(W, x)$, $\delta \in Q(W, y)$. *If* $M_\gamma \cap M_\delta \cong N_8$, *then* $x = y$.

Lemma 4.6 (i) *The graph with vertex set* $A(W)$ *whose edges are* (γ, δ), *where* $M_\gamma \cap M_\delta \cong N_{12}$, *is a connected graph of valency 15.*

(ii) *The graph with vertex set* $Q(W, x)$ *whose edges are* (γ, δ), *where* $M_\gamma \cap M_\delta \cong N_8$, *is a connected graph of valency 45.*

Lemma 4.7 (i) *Let* $U_1, U_2 \in \begin{bmatrix} W \\ 2 \end{bmatrix}$, $\dim W = 4$, $U_1 \cap U_2 = 0$. *Then*

$$|\{\gamma \in A(W) | U_1 \in M_\gamma, \ U_2 \in M_\gamma\}| = 10.$$

(ii) *Let* $U_1, U_2 \in \begin{bmatrix} W \\ 2 \end{bmatrix}$, $\dim W = 5$, $U_1 \oplus U_2 \oplus \langle x \rangle = W$. *Then*

$$|\{\gamma \in Q(W, x) | U_1 \in M_\gamma, \ U_2 \in M_\gamma\}| = 10.$$

Lemma 4.8 *Suppose that* $\gamma, \delta \in Q(W, x)$, *the type of* $\gamma + \delta$ *is* $2-$. *Then*

$$\sum_{W_1 \in \begin{bmatrix} W \\ 4 \end{bmatrix}, \ W_1 \ni x} |\{\alpha \in A(W_1) | M_\alpha \cap M_\gamma \supseteq K_4, \ M_\alpha \cap M_\delta \supseteq K_4\}| = 24.$$

5. Determination of the second neighbourhood

In this section, we assume that Γ is locally $\begin{bmatrix} V \\ 2 \end{bmatrix}$ with $\mu = 20$, and $a_2 = 15 \cdot 2^{n-1} - 105$, where $\begin{bmatrix} V \\ 2 \end{bmatrix}$ is the Grassmann graph on a vector space V of dimension n over $GF(2)$. The assumption on a_2 is, however, unnecessary before Lemma 5.5. Let u, v be vertices of Γ at distance 2. Then u is said to be of type 1 (resp. of type 2) with respect to v if the subgraph $\Gamma(u, v)$ in the Grassmann graph $\Gamma(v)$ satisfies Lemma 4.2(i) (resp. (ii)).

Lemma 5.1 *Let* u, v *be vertices of* Γ *at distance 2. Then* u *is of type 1 with respect to* v *if and only if* v *is of type 1 with respect to* u. *If* u *is of type 1 with respect to* v, *then*

$$|\{w \in \Gamma_2(v) \cap \Gamma(u) | \Gamma(u, v, w) \cong N_{12}\}| = 15.$$

If u *is of type 2 with respect to* v, *then*

$$|\{w \in \Gamma_2(v) \cap \Gamma(u) | \Gamma(u, v, w) \cong N_8\}| = 45.$$

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Proof. If v is of type 2 with respect to u, then by Lemma 4.3 there exists a vertex $w \in \Gamma_2(v) \cap \Gamma(u)$ such that $\Gamma(u, v, w) \cong N_8$. By Lemma 4.4, $\Gamma(u, v)$ and $\Gamma(w, v)$ generate the same subspace in the Grassmann graph $\Gamma(v)$. In particular, u and w are of the same type with respect to v. According to Table 1, N_8 does not occur as the intersection of two μ-graphs of vertices of type 1. Thus, u and w are of type 2 with respect to v, proving the first assertion. The rest of the statements follows from Lemma 4.3. □

For the rest of this section, fix an arbitrary vertex v_0 of Γ and identify $\Gamma(v_0)$ with $\begin{bmatrix} V \\ 2 \end{bmatrix}$. By Lemmas 4.2 and 4.1, one can identify $\Gamma_2(v_0)$ with a subset of $(\bigcup_{W_1} A(W_1)) \cup (\bigcup_{W_2, x} Q(W_2, x))$ in such a way that $U \in \begin{bmatrix} V \\ 2 \end{bmatrix}$ and $\gamma \in \Gamma_2(v_0)$ are adjacent if and only if $U \in M_\gamma$. The goal of this section is to determine $\Gamma_2(v_0)$ as such a subset.

Lemma 5.2 (i) If $A(W) \cap \Gamma_2(v_0) \neq \emptyset$, then $A(W) \subset \Gamma_2(v_0)$.
(ii) If $Q(W, x) \cap \Gamma_2(v_0) \neq \emptyset$, then $Q(W, x) \subset \Gamma_2(v_0)$.

Proof. (i) Let $\gamma \in A(W) \cap \Gamma_2(v_0)$. Then by Lemma 5.1, we have

$$|\{\delta \in \Gamma_2(v_0) \cap \Gamma(\gamma) | M_\delta \cap M_\gamma \cong N_{12}\}| = 15.$$

Lemma 4.4 implies that if $M_\delta \cap M_\gamma \cong N_{12}$, then $\delta \in A(W)$. By Lemma 4.6(i) we find $A(W) \subset \Gamma_2(v_0)$.

(ii) The proof is analogous, using Lemmas 4.5, 4.6(ii) and Table 1. □

Lemma 5.3 For any $W \in \begin{bmatrix} V \\ 4 \end{bmatrix}$, one and only one of the following holds.
(i) $A(W) \subset \Gamma_2(v_0)$,
(ii) $Q(\tilde{W}, x) \subset \Gamma_2(v_0)$ for exactly one $\tilde{W} \in \begin{bmatrix} V \\ 5 \end{bmatrix}$ and exactly one $x \in \tilde{W}$ with $\tilde{W} = W \oplus \langle x \rangle$.

Proof. Choose $U_1, U_2 \in \begin{bmatrix} W \\ 2 \end{bmatrix}$ such that $U_1 \cap U_2 = 0$. Since $|\Gamma(U_1, U_2) \cap v_0^\perp| = 10 < \mu(\Gamma)$, we can find a vertex γ in $\Gamma_2(v_0) \cap \Gamma(U_1, U_2)$. If γ is of type 1 with respect to v_0, then $\gamma \in A(W)$, so that (i) holds by virtue of Lemma 5.2(i). If γ is of type 2 with respect to v_0, then $\gamma \in Q(\tilde{W}, x)$ for some $\tilde{W} \in \begin{bmatrix} V \\ 5 \end{bmatrix}$, $x \in \tilde{W}$ with $W \subset \tilde{W}$. It follows easily that $x \notin W$. By Lemma 5.2(ii), $Q(\tilde{W}, x) \subset \Gamma_2(v_0)$. If (i) and (ii) hold simultaneously (resp. if the pair $(\tilde{W}, x)$ is not unique), then $\Gamma(U_1, U_2)$ contains subsets $\Gamma(U_1, U_2) \cap v_0^\perp$, $\Gamma(U_1, U_2) \cap Q(\tilde{W}, x)$, $\Gamma(U_1, U_2) \cap A(W)$, (resp. $\Gamma(U_1, U_2) \cap Q(\tilde{W}', x')$ for another pair $(\tilde{W}', x')$), each having cardinality 10 by Lemma 4.7. Since the intersections are clearly trivial, we obtain a contradiction with $\mu = 20$. □

Lemma 5.4 Let $\gamma, \delta \in \Gamma_2(v_0)$, $M_\gamma \cap M_\delta \neq \emptyset$. Then γ is adjacent to δ if and only if $M_\gamma \cap M_\delta$ contains a 4-clique.

Proof. By Lemma 4.3(i–ii), if γ is adjacent to δ, then $M_\gamma \cap M_\delta$ is one of N_{12}, N_8, or K_4, all of which contain K_4. The converse is a special case of [4], Lemma 2.2. $\square$

Lemma 5.5 *If $Q(W, x) \subset \Gamma_2(v_0)$, $W_1 \in [{}^V_4]$, $\dim(W \cap W_1) = 3$ and $x \notin W_1$, then $A(W_1) \not\subset \Gamma_2(v_0)$.*

Proof. Suppose $A(W_1) \subset \Gamma_2(v_0)$. Let $\gamma \in A(W_1)$, and choose $\delta \in Q(W, x)$ in such a way that $M_\gamma \cap [{}^{W \cap W_1}_2] = M_\delta \cap [{}^{W \cap W_1}_2]$. Then $M_\gamma \cap M_\delta \cong K_4$, hence γ and δ are adjacent by Lemma 5.4. It is easy to see that there exists an $U_0 \in M_\delta$ such that $U_0 \cap W_1 = \emptyset$. On the other hand, Lemma 4.3(iii) together with the assumption on a_2 implies

$$\begin{aligned}
\Gamma_2(\gamma) \cap \Gamma(v_0) &= \{U \in [{}^V_2] - M_\gamma | N_{\gamma, U} \neq \emptyset\} \\
&= \{U \in [{}^V_2] - M_\gamma | U \cap W_1 \neq \emptyset\}.
\end{aligned}$$

This is a contradiction since $U_0 \in \Gamma_2(\gamma)$. $\square$

For brevity, let us call a 4-space W of type A (resp. of type Q) if Lemma 5.3 (i) (resp. (ii)) holds.

Lemma 5.6 *Let $W_0, W_1 \in [{}^V_4]$, $\dim(W_0 \cap W_1) = 3$. Suppose that W_0 is of type A, and W_1 is of type Q, i.e., $Q(W_1', x) \subset \Gamma_2(v_0)$ with $W_1' = W_1 \oplus \langle x \rangle$. Then*

(i) $x \in W_0 \subset W_1'$,

(ii) *every $W \in [{}^{W_1'}_4]$ with $x \in W$ is of type A.*

(iii) *every $W \in [{}^V_4]$ with $x \in W$ and $\dim(W \cap W_1') = 3$, is of type A.*

Proof. (i) If $W_0 \not\subset W_1'$, then $W_0 \cap W_1' = W_0 \cap W_1$, which is impossible by Lemma 5.5. Thus, $W_0 \subset W_1'$. If $x \notin W_0$, then $W_1' = W_0 \oplus \langle x \rangle$, which would imply, by Lemma 5.3, that W_0 is of type Q.

(ii) Since W_0 is of type A, we may assume $W \neq W_0$. If W is of type Q, then $Q(W', y) \subset \Gamma_2(v_0)$ for some $y \notin W$, $W' = W \oplus \langle y \rangle$. By (i), we have $y \in W_0 \subset W'$. This implies $W' = W_1'$, and by Lemma 5.3, $y = x \in W$, a contradiction. Thus, W is of type A.

(iii) If W is of type Q, then $Q(W', y) \subset \Gamma_2(v_0)$ for some $y \notin W$, $W' = W \oplus \langle y \rangle$. Let U_1, U_2, U_3 be the 4-spaces of W_1' containing $W \cap W_1'$. Then U_1, U_2, U_3 are of type A by (ii), and hence by (i), we have $U_i \subset W'$ for $i = 1, 2, 3$. This forces $W' = W_1'$, which is a contradiction. $\square$

Let $\tilde{Q}(W, x) = Q(W, x) \cup (\bigcup_{x \in W_1 \in [{}^W_4]} A(W_1))$.

Lemma 5.7 *We have either*

$$\text{(i)} \qquad \Gamma_2(v_0) = \bigcup_{W \in \left[\begin{smallmatrix} V \\ 4 \end{smallmatrix}\right]} A(W),$$

or

$$\text{(ii)} \qquad \Gamma_2(v_0) = \bigcup_{W \in \left[\begin{smallmatrix} V \\ 5 \end{smallmatrix}\right], W \ni x} \hat{Q}(W, x)$$

for some nonzero element $x \in V$.

Proof. Suppose first that every vertex $u \in \Gamma_2(v_0)$ is of type 1 with respect to v_0. Then $\Gamma_2(v_0)$ is a union of $A(W)$'s, so the first equality holds.

Next suppose that there is a vertex $u \in \Gamma_2(v_0)$ of type 2 with respect to v_0. This means that there exists a $W_1 \in \left[\begin{smallmatrix} V \\ 4 \end{smallmatrix}\right]$ of type Q. Since $|Q(W, x)|$ does not divide $|\Gamma_2(v_0)|$ ($|\Gamma_2(v_0)|$ is known since μ is known), there exists a $W_0 \in \left[\begin{smallmatrix} V \\ 4 \end{smallmatrix}\right]$ of type A. Since the Grassmann graph $\left[\begin{smallmatrix} V \\ 4 \end{smallmatrix}\right]$ is connected, we may assume without loss of generality that $\dim(W_0 \cap W_1) = 3$. By Lemma 5.6(i), we see that $\hat{Q}(W_1', x) \subset \Gamma_2(v_0)$ for some $x \in W_0 - W_1$, $W_1' = W_1 \oplus \langle x \rangle$. Let $W_2' \in \left[\begin{smallmatrix} V \\ 5 \end{smallmatrix}\right]$, $x \in W_2'$, $\dim(W_1' \cap W_2') = 4$. We want to show that $\hat{Q}(W_2', x) \subset \Gamma_2(v_0)$. In order to do so, it suffices to prove that $Q(W_2', x) \subset \Gamma_2(v_0)$, by virtue of from Lemma 5.6(ii). Let $W_2 \in \left[\begin{smallmatrix} W_2' \\ 4 \end{smallmatrix}\right]$ with $x \notin W_2$ and $W_1' \in \left[\begin{smallmatrix} V \\ 3 \end{smallmatrix}\right]$. By Lemma 5.5, W_2 must be of type Q. Applying Lemma 5.6(i) for $(W_1' \cap W_2', W_2)$, we see that $\hat{Q}(W_2', y) \subset \Gamma_2(v_0)$ for some $y \in W_1' \cap W_2' - W_2$. If $y \neq x$, then there exists a $W \in \left[\begin{smallmatrix} W_2' \\ 4 \end{smallmatrix}\right]$ such that $x \in W$, $y \notin W$. Since $W_2' = W \oplus \langle y \rangle$, W is of type A, while W is of type A by Lemma 5.6(iii). This contradiction proves $x = y$, so that $Q(W_2', x) \subset \Gamma_2(v_0)$. We have shown $\hat{Q}(W_2', x) \subset \Gamma_2(v_0)$ for any $W_2' \in \left[\begin{smallmatrix} V \\ 5 \end{smallmatrix}\right]$ with $x \in W_2'$, $\dim(W_1' \cap W_2') = 4$. Now it follows from connectivity of the Grassmann graph $\left[\begin{smallmatrix} V/\langle x \rangle \\ 4 \end{smallmatrix}\right]$ that $\hat{Q}(W_2', x) \subset \Gamma_2(v_0)$ for every $W_2' \in \left[\begin{smallmatrix} V \\ 5 \end{smallmatrix}\right]$ with $x \in W_2'$. Finally, a simple counting shows that the sets $\hat{Q}(W, x)$ cover the whole of $\Gamma_2(v_0)$. $\qquad\square$

It is straightforward to check that Lemma 5.7(i) holds if $\Gamma = \text{Alt}(n, 2)$, and Lemma 5.7(ii) holds if $\Gamma = \text{Quad}(n - 1, 2)$. Moreover, two vertices γ, δ of $\text{Quad}(n - 1, 2)$ at distance 2 are of type 1 to each other if and only if rank $(\gamma + \delta) = 3$. If Γ satisfies Lemma 5.7(ii), then the grand clique $C = \{U \in \left[\begin{smallmatrix} V \\ 2 \end{smallmatrix}\right] | U \ni x\}$ of the Grassmann graph $\left[\begin{smallmatrix} V \\ 2 \end{smallmatrix}\right] \cong \Gamma(v_0)$ will be called the *nucleus* with respect to v_0. If $u \in \Gamma_2(v_0)$, then $\Gamma(v_0, u) \cap C = K_4$ or $\emptyset$ depending on whether u is of type 1 or 2 with respect to v_0, and the nucleus is the only such grand clique. The following lemma gives a more convenient characterization of the nucleus.

Lemma 5.8 *Let* γ *be a type 2 vertex with respect to* v_0, *and* $W \in \left[\begin{smallmatrix} V \\ 5 \end{smallmatrix}\right]$ *be the subspace generated by* M_γ. *Then the nucleus* C *with respect to* v_0 *is*

characterized by the properties $\left[\begin{smallmatrix} W \\ 2 \end{smallmatrix}\right] \cap C \neq \emptyset$ and

$$\{U \in M_\gamma | U \cap U_0 \neq \emptyset\} \cong K_4 \quad \text{for any } U_0 \in \left[\begin{smallmatrix} W \\ 2 \end{smallmatrix}\right] \cap C.$$

Proof. Note that γ can be regarded as a nondegenerate quadratic form on W, and its nucleus x is characterized by the property: for any $U_0 \in \left[\begin{smallmatrix} W \\ 2 \end{smallmatrix}\right]$ with $U_0 \ni x$, U_0 contains a unique nonsingular 1-space distinct from $\langle x \rangle$. There are exactly four elements in M_γ containing a given nonsingular 1-space. □

6. Proof of the main theorem

Let Γ and $\tilde{\Gamma}$ be graphs with the same local structure, let v be a vertex of Γ and let $\tilde{v}$ be a vertex of $\tilde{\Gamma}$. An isomorphism $\sigma : \tilde{v}^\perp \to v^\perp$ with $\sigma(\tilde{v}) = v$ is called *extendable* if there exists a bijection $\sigma' : \tilde{v}^\perp \cup \Gamma_2(\tilde{v}) \to v^\perp \cup \Gamma_2(v)$, mapping edges to edges and satisfying $\sigma'|_{\tilde{v}^\perp} = \sigma$. In this case the mapping σ' is called the *extension* of σ.

Let Γ be locally $\left[\begin{smallmatrix} V \\ 2 \end{smallmatrix}\right]$ with $\mu = 20$ and $a_2 = 15 \cdot 2^{n-1} - 105$, where V is a vector space of dimension n over $GF(2)$.

Lemma 6.1 *Suppose v_0 is a vertex of Γ, such that Lemma 5.7(i) holds. Take $\tilde{\Gamma} = \text{Alt}(n,2)$ and let $\tilde{v} \in \tilde{\Gamma}$. Then every isomorphism $\sigma : \tilde{v}^\perp \to v^\perp$ is extendable.*

Proof. Clearly, every μ-graph $\tilde{\Gamma}(\tilde{v},\tilde{u})$ in $\tilde{v}^\perp$ is mapped onto a μ-graph $\Gamma(v,u)$ in $v^\perp$. By the second part of Lemma 4.1, such a vertex u is defined uniquely and, hence, we can define the extension σ' by $\sigma'(\tilde{u}) = u$. It remains to check that σ' maps an edge to an edge. Unless both ends of the edge in $\tilde{v}^\perp \cup \tilde{\Gamma}_2(\tilde{v})$ belong to $\tilde{\Gamma}_2(\tilde{v})$, the claim follows by definition. Let $u \in \tilde{\Gamma}_2(\tilde{v})$. Identifying $\tilde{\Gamma}(u)$ with $\left[\begin{smallmatrix} V \\ 2 \end{smallmatrix}\right]$, it follows from Lemma 4.3(iii) that the set

$$S = \{w \in \tilde{\Gamma}(u) - \tilde{\Gamma}(u,\tilde{v}) | \tilde{\Gamma}(u,\tilde{v},w) \supset K_4\}$$

has cardinality $a_2 = 15 \cdot 2^{n-1} - 105$. By Lemma 5.4, S is contained in $\tilde{\Gamma}(u) \cap \tilde{\Gamma}_2(\tilde{v})$, whose cardinality is also a_2. This implies $S = \tilde{\Gamma}(u) \cap \tilde{\Gamma}_2(\tilde{v})$. Now if (u,w) is an edge in $\tilde{\Gamma}_2(\tilde{v})$, then $\tilde{\Gamma}(u,\tilde{v},w) \supset K_4$, so that $\Gamma(\sigma'(u),v,\sigma'(w)) \supset K_4$. Again by Lemma 5.4, we see that $(\sigma'(u),\sigma'(w))$ is an edge. □

Lemma 6.2 *Suppose v_0 is a vertex of Γ, such that Lemma 5.7(ii) holds. Take $\tilde{\Gamma} = \text{Quad}(n-1,2)$ and let $\tilde{v} \in \tilde{\Gamma}$. Then an isomorphism $\sigma : \tilde{v}^\perp \to v^\perp$ is extendable if and only if it maps the nucleus with respect to $\tilde{v}$ to the nucleus with respect to v. In particular, if there exist type 2 vertices $u \in \Gamma_2(v)$ and $\tilde{u} \in \tilde{\Gamma}_2(\tilde{v})$, such that $\sigma(\tilde{\Gamma}(\tilde{v},\tilde{u})) = \Gamma(v,u)$, then σ is extendable.*

Proof. The "only if" part is obvious, so let us consider an isomorphism $\sigma : \tilde{v}^\perp \to v^\perp$, which maps the nucleus with respect to $\tilde{v}$ to the nucleus with respect to v. As in Lemma 6.1, the μ-graphs in $\tilde{v}^\perp$ are mapped onto μ-graphs in $v^\perp$ and this gives us the extension σ'. We only need to check that σ' maps edges from $\tilde{\Gamma}_2(\tilde{v})$ onto edges.

Let $u \in \tilde{\Gamma}_2(\tilde{v})$. If u is of type 1 with respect to $\tilde{v}$ then, as in the previous lemma, we show that for every edge (u, w) in $\tilde{\Gamma}_2(\tilde{v})$ we have $\tilde{\Gamma}(u, \tilde{v}, w) \supset K_4$ and hence $(\sigma'(u), \sigma'(w))$ is an edge by Lemma 5.4. Suppose u is of type 2 with respect to $\tilde{v}$. Consider once again the set $S = \{w \in \tilde{\Gamma}(u) - \tilde{\Gamma}(u, \tilde{v}) | \tilde{\Gamma}(u, \tilde{v}, w) \supset K_4\} \subset \tilde{\Gamma}(u) \cap \tilde{\Gamma}_2(\tilde{v})$. This time, $|S| = a_2 - 15$ by Lemma 4.3 (iii). Identify now $\tilde{\Gamma}(\tilde{v})$ with $\binom{V}{2}$ and suppose $u = \gamma \in Q(W, x) \subset \tilde{\Gamma}_2(\tilde{v})$, where $0 \neq x \in W \in \binom{V}{5}$. From Table 1, there are 15 elements $\delta \in Q(W, x)$ such that the type of $\gamma + \delta$ is $2-$, and for such a vertex δ, $M_\gamma \cap M_\delta = \emptyset$ holds. By Lemma 4.8, there are 24 vertices in $\tilde{Q}(W, x)$ which are of type 1 with respect to $\tilde{v}$ and are adjacent to both γ and δ. Since $\mu = 20 < 24$, γ and δ must be adjacent. Therefore, $\tilde{\Gamma}(u) \cap \tilde{\Gamma}_2(\tilde{v})$ consists of the disjoint union of S and the set

$$S_0 = \{\delta \in Q(W, x) | \text{type of } \gamma + \delta \text{ is } 2-\}.$$

Clearly, $\sigma'(w)$ is adjacent to $\sigma'(u)$ if $w \in S$. If $\delta \in S_0$, then by the first part of the proof, the 24 type 1 common neighbours of $\gamma = u$ and δ in $\tilde{Q}(W, x)$ are mapped to common neighbours of $\sigma'(u)$ and $\sigma'(\delta)$, so that $\sigma'(u)$ and $\sigma'(\delta)$ are adjacent since $\mu < 24$.

The last statement of the lemma follows from Lemma 5.8. $\qquad \square$

Proof of Main Theorem. First suppose that for any vertices u, v of Γ at distance 2, u is of type 1 with respect to v. Let $\tilde{\Gamma} = \text{Alt}(n, 2)$. For any vertex v of Γ and for any vertex $\tilde{v}$ of $\tilde{\Gamma}$ and for any isomorphism $\sigma : \tilde{v}^\perp \to v^\perp$, σ is extendable by Lemma 6.1. Let σ' be the extension of σ. The mapping σ' maps edges to edges. Hence, if $w \in \tilde{\Gamma}(\tilde{v})$, then $\sigma'|_{w^\perp}$ is also extendable by Lemma 6.1. Now, by [4], Proposition 6.1, all the hypotheses of [4], Theorem 7.1 are satisfied, so Γ is covered by $\text{Alt}(n, 2)$.

Next suppose that there exist vertices u_0, v_0 of Γ which are of type 2 with respect to each other. Again we want to use [4], Theorem 7.1, this time with $\tilde{\Gamma} = \text{Quad}(n-1, 2)$. By Lemma 6.2, there exists an extendable isomorphism $\tilde{v}_0^\perp \to v_0^\perp$, where $\tilde{v}_0$ is a vertex of $\tilde{\Gamma}$. Suppose that $\tilde{v}$ is a vertex of $\tilde{\Gamma}$ and that v is a vertex of Γ and $\sigma : \tilde{v}^\perp \to v^\perp$ is an extendable isomorphism with the extension σ'. Then σ' maps edges to edges. Let $w \in \tilde{\Gamma}(\tilde{v})$. We want to show that $\sigma'|_{w^\perp}$ is extendable. If $u \in \tilde{v}^\perp \cup \tilde{\Gamma}_2(\tilde{v})$, then clearly $\sigma'(\tilde{\Gamma}(w, u)) = \Gamma(\sigma'(w), \sigma'(u))$. If, moreover, u is of type 2 with respect to w, then, as well, $\sigma'(u)$ is of type 2 with respect to $\sigma'(w)$. By Lemma 6.2, it implies that the nucleus with respect to w is mapped to the nucleus with respect to $\sigma'(w)$. Again by Lemma 6.2 we obtain that $\sigma'|_{w^\perp}$ is extendable.

It remains to find a vertex $u \in \tilde{v}^{\perp} \cup \tilde{\Gamma}_2(\tilde{v})$, which is of type 2 with respect to w. Since $\tilde{\Gamma} = \text{Quad}(n-1, 2)$, it means that we must find, for every quadratic form γ of rank ≤ 2, a quadratic form δ of rank at most 4, such that $\text{rank}(\gamma + \delta) = 4$. This is easy to check, so that $\sigma'|_{w^{\perp}}$ is indeed extendable. Triangulability of $\text{Quad}(n-1, 2)$ has been shown in Proposition 3.2. Now all the hypotheses of [4], Theorem 7.1 are satisfied, so Γ is covered by $\text{Quad}(n-1, 2)$.

$\square$

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