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#### Abstract

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# A note on the stability number of an orthogonality graph 

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#### Abstract

We consider the orthogonality graph $\Omega(n)$ with $2^{n}$ vertices corresponding to the vectors $\{0,1\}^{n}$, two vertices adjacent if and only if the Hamming distance between them is $n / 2$. We show that, for $n=16$, the stability number of $\Omega(n)$ is $\alpha(\Omega(16))=2304$, thus proving a conjecture by Galliard [7]. The main tool we employ is a recent semidefinite programming relaxation for minimal distance binary codes due to Schrijver [17].

As well, we give a general condition for Delsarte bound on the (co)cliques in graphs of relations of association schemes to coincide with the ratio bound, and use it to show that for $\Omega(n)$ the latter two bounds are equal to $2^{n} / n$.


Keywords: Semidefinite programming, minimal distance codes, stability number, orthogonality graph, Hamming association scheme, Delsarte bound.

AMS subject classification: $90 \mathrm{C} 22,90 \mathrm{C} 27,05 \mathrm{C} 69,05 \mathrm{C} 15$,

## 1 Introduction

## The graph $\Omega(n)$ and its properties

Let $\Omega(n)$ be the graph on $2^{n}$ vertices corresponding to the vectors $\{0,1\}^{n}$, such that two vertices are adjacent if and only if the Hamming distance between them is $n / 2$. Note that $\Omega(n)$ is $k$-regular, where $k=\binom{n}{\frac{1}{2} n}$.
It is known that $\Omega(n)$ is bipartite if $n=2 \bmod 4$, and empty if $n$ is odd. We will therefore assume throughout that $n$ is a multiple of 4 . The graph owns its name to another description, in terms of $\pm 1$-vectors. Then the orthogonality of vectors corresponds to the Hamming distance $n / 2$.

[^0]Moreover, $\Omega(n)$ consists of two isomorphic connected components, $\Omega_{0}(n)$, respectively $\Omega_{1}(n)$, containing all the vertices of even, respectively odd, Hamming weight. For a detailed discussion of the properties of $\Omega(n)$, see Godsil [9], the PhD thesis of Newman [14], and [10].


Figure 1: The connected component of $\Omega(4)$ corresponding to vertices of even Hamming weight.

In this note we study upper bounds on the stability number $\alpha(\Omega(n))$.
Galliard [7] pointed out the following way of constructing maximal stable sets in $\Omega(n)$. Consider the component $\Omega_{\epsilon}(n)$ of $\Omega(n)$, for $1-\epsilon=\frac{n}{4} \bmod 2$, and take all vertices of Hamming weight $\epsilon, \epsilon+2, \ldots, \epsilon+2 \ell, \ldots, n / 4-1$. Obviously, these vertices form a stable set $S$ of $\Omega(n)$ of size

$$
\sum_{i=\epsilon}^{\lfloor n / 8\rfloor}\binom{n}{2 i-\epsilon} .
$$

We can double the size of $S$ by adding the bit-wise complements of the vertices in $S$, and double it again by taking the union with the corresponding stable set in $\Omega_{1-\epsilon}(n)$. Thus we find that

$$
\begin{equation*}
\alpha(\Omega(n)) \geq 4 \sum_{i=\epsilon}^{\lfloor n / 8\rfloor}\binom{n}{2 i-\epsilon}:=\underline{\alpha}(n) \tag{1}
\end{equation*}
$$

For $n=16$ this evaluates to $\alpha(\Omega(n)) \geq 2304$. Galliard et al [8] could show that $\alpha(\Omega(16)) \leq 3912$. In this note we will show that, in fact, $\alpha(\Omega(16))=2304$. This was conjectured by Galliard [7], and Newman [14] has recently conjectured that the value (11) actually equals $\alpha(\Omega(n))$ whenever $n$ is a multiple of 4 .

## A quantum information game

One motivation for studying the graph $\Omega(n)$ comes from quantum information theory. Consider the following game from [8].
Let $r \geq 1$ and $n=2^{r}$. Two players, A and B , are asked the questions $x_{A}$ and $x_{B}$, coded as $n$-bit strings satisfying

$$
d_{H}\left(x_{A}, x_{B}\right) \in\left\{0, \frac{1}{2} n\right\}
$$

where $d_{H}$ denotes the Hamming distance. A and B win the game if they give answers $y_{A}$ and $y_{B}$, coded as binary strings of length $r$ such that

$$
y_{A}=y_{B} \Longleftrightarrow x_{A}=x_{B} .
$$

A and B are not allowed any communication (except a priori deliberation).
It is known that A and B can always win the game if their $r$ output bits are maximally entangled quantum bits [2] (see also [14]).
For classical bits, it was shown by Galliard et al 8 that the game cannot always be won if $r=4$. The authors proved this by pointing out that whether or not the game can always be won is equivalent to the question

$$
\chi(\Omega(n)) \leq n ?
$$

Indeed, if $\chi(\Omega(n)) \leq n$ then A and B may color $\Omega(n)$ a priori using $n$ colors. The questions $x_{A}$ and $x_{B}$ may then be viewed as two vertices of $\Omega(n)$, and the A and B may answer their respective questions by giving the color of the vertices $x_{A}$ and $x_{B}$ respectively, coded as binary strings of length $\log _{2} n=r$. Galliard et al. [8] showed that $\chi(\Omega(16))>16$, i.e. that the game cannot be won for $n=16$. They proved this by showing that $\alpha(\Omega(16)) \leq 3912$ which implies

$$
\chi(\Omega(16)) \geq\left\lceil\frac{2^{16}}{\alpha(\Omega(16))}\right\rceil \geq\left\lceil\frac{2^{16}}{3912}\right\rceil=17
$$

In this note we sharpen their bound by showing that $\alpha(\Omega(16))=2304$, which implies $\chi(\Omega(16)) \geq 29$.
Our main tool will be a semidefinite programming bound on $\alpha(\Omega(n))$ that is due to Schrijver [17], where it is formulated for minimal distance binary codes.

## 2 Upper bounds on $\alpha(\Omega(n))$

In this section we give a review of known upper bounds on $\alpha(\Omega(n))$ and their relationship.

### 2.1 The ratio bound

The following discussion is condensed from Godsil (9).

Theorem 1. Let $G=(V, E)$ be a $k$-regular graph with adjacency matrix $A(G)$, and let $\lambda_{\min }(A(G))$ denote the smallest eigenvalue of $A(G)$. Then

$$
\begin{equation*}
\alpha(G) \leq \frac{|V|}{1-\frac{k}{\lambda_{\min }(A(G))}} . \tag{2}
\end{equation*}
$$

This bound is called the ratio bound, and was first derived by Delsarte 4 for graphs in association schemes (see Sect. 2.2 for more on the latter).
Recall that $\Omega(n)$ is $k$-regular with $k=\binom{n}{\frac{1}{2} n}$. Ignoring multiplicities, the spectrum of $\Omega(n)$ is given by

$$
\begin{equation*}
\lambda_{m}=\frac{2^{\frac{1}{2} n}}{\left(\frac{1}{2} n\right)!}(m-1)(m-3) \cdots(m-n+1) \quad(m=1, \ldots, n) \tag{3}
\end{equation*}
$$

The minimum is reached at $m=2$, and we get

$$
\begin{equation*}
\lambda_{\min }\left(A(\Omega(n))=\frac{2^{\frac{1}{2} n}}{\left(\frac{1}{2} n\right)!}(1)(-1)(-3) \cdots(-n+3)=-\frac{\left(\frac{1}{2} n\right)}{n-1}\right. \tag{4}
\end{equation*}
$$

The ratio bound therefore becomes

$$
\begin{equation*}
\alpha(\Omega(n)) \leq \frac{2^{n}}{n} \tag{5}
\end{equation*}
$$

This is the best known upper bound on $\alpha(\Omega(n))$, but it is known that this bound is not tight: Frankl and Rödl [6] showed that there exists some $\epsilon>0$ such that $\alpha(\Omega(n)) \leq(2-\epsilon)^{n}$. For specific (small) values of $n$ one can improve on the bound (5), as we will show for $n \leq 32$.

### 2.2 The Delsarte bound and $\vartheta^{\prime}$

Here we are going to use more linear algebra that naturally arise around $\Omega(n)$. We recall the following definitions, cf. e.g. Bannai and Ito [1].

Association schemes. An association scheme $\mathcal{A}$ is a commutative subalgebra of the full $v \times v$-matrix algebra with a distinguished basis $\left(A_{0}=I, A_{1}, \ldots, A_{n}\right)$ of 0-1 matrices, with an extra property that $\sum_{i} A_{i}$ equals the all-ones matrix. One often views $A_{j}, j \geq 1$, as the adjacency matrix of a graph on $v$ vertices; $A_{j}$ is often referred to as the $j$-th relation of $\mathcal{A}$. As the $A_{j}$ 's commute, they have $n+1$ common eigenspaces $V_{i}$. Then $\mathcal{A}$ is isomorphic, as an algebra, to the algebra of diagonal matrices $\operatorname{diag}\left(P_{0, j}, \ldots, P_{n, j}\right)$, where $P_{i j}$ denotes the eigenvalue of $A_{j}$ on $V_{i}$. The matrix $P=\left(P_{i j}\right)$ is called first eigenvalue matrix of $\mathcal{A}$. The set of $A_{j}$ 's is closed under taking transpositions: for each $0 \leq j \leq n$ there exists $j^{\prime}$ so that $A_{j}=A_{j^{\prime}}^{T}$. In particular, $P_{i j}=\overline{P_{i j^{\prime}}}$. An association scheme with all $A_{j}$ symmetric is called symmetric, and here we shall consider such schemes only. There is a matrix $Q$ (called second eigenvalue matrix) satisfying $P Q=Q P=v I$. In what follows it is assumed (as is customary in the literature)
that the eigenspace $V_{0}$ corresponds to the eigenvector $(1, \ldots, 1)$; then the 0 -th row of $P$ consists of the degrees $v_{j}$ of the graphs $A_{j}$. It is remarkable that the 0 -th row of $Q$ consists of dimensions of $V_{i}$.

Let $\vartheta^{\prime}$ denote the Schrijver $\vartheta^{\prime}$-function [16]:

$$
\vartheta^{\prime}(G)=\max \{\operatorname{Tr}(J X): \operatorname{Tr}(A X)=0, \operatorname{Tr}(X)=1, X \succeq 0, X \geq 0\}
$$

For any graph $G$ one has $\alpha(G) \leq \vartheta^{\prime}(G)$. Moreover, $\vartheta^{\prime}(G)$ is smaller than or equal to the ratio bound (2) for regular graphs, as noted by Godsil [9] Sect. 3.7]. For graphs with adjacency matrices of the form $\sum_{j \in \mathcal{M}} A_{j}$, with $\mathcal{M} \subset\{1, \ldots, n\}$ and $A_{j}$ 's from the $0-1$ basis of an association scheme $\mathcal{A}$, the bound $\vartheta^{\prime}$ coincides, as was proved by Schrijver [16], with the following bound due to Delsarte [3, 4]

$$
\begin{equation*}
\max 1^{T} w \text { subject to } w \geq 0, Q^{T} w \geq 0, w_{0}=1, w_{j}=0 \text { for } j \in \mathcal{M} \tag{6}
\end{equation*}
$$

where $Q$ is the second eigenvalue matrix of $\mathcal{A}$.
The bound (6) is often stated for (and was originally developed for) bounding the maximal size of a $q$-ary code of length $n$ and minimal distance $d$; then the association scheme $\mathcal{A}$ becomes the Hamming distance association scheme $H(n, q)$ and $\mathcal{M}=\{1, \ldots, d-1\}$. The relations of $H(n, q)$ can be viewed as graphs on the vertex set of $n$-strings on $\{0, \ldots, q-1\}$ : the $j$-th graph of $H(n, q)$ is given by

$$
\left(A_{j}\right)_{X Y}= \begin{cases}1 & \text { if } d_{H}(X, Y)=j \\ 0 & \text { otherwise }\end{cases}
$$

For $H(n, q)$ the first and the second eigenvalue matrices $P$ and $Q$ coincide, and are given by $P_{i j}=K_{j}(i)$, where $K_{k}$ is the Krawtchouk polynomial

$$
K_{k}(x):=\sum_{j=0}^{k}(-1)^{j}(q-1)^{k-j}\binom{x}{j}\binom{n-x}{k-j}
$$

For $\Omega(n)$, the bound (6) is as above with $\mathcal{A}=H(n, 2)$ and $\mathcal{M}=\left\{\frac{n}{2}\right\}$. Newman [14] has shown computationally that $\vartheta^{\prime}(\Omega(n))=2^{n} / n$ if $n \leq 64$, i.e. the ratio and $\vartheta^{\prime}$ bounds coincide for $\Omega(n)$ if $n \leq 64$. We show that it is the case for all $n$, as an easy consequence of the following.

Proposition 1. Let $\mathcal{A}$ be an association scheme with the $0-1$ basis $\left(A_{0}, \ldots, A_{n}\right)$ and eigenvalue matrices $P$ and $Q$. Let $A_{r}$ have the least eigenvalue $\tau=P_{\ell r}$ and assume

$$
v_{r} P_{\ell i} \geq v_{i} \tau, \quad 0 \leq i \leq n
$$

Then the Delsarte bound (6), with $\mathcal{M}=\{r\}$, and the ratio bound (2) for $A_{r}$ coincide.

Proof. Let $P_{j}$ denote the $j$-th row of $P$.
As we already mentioned, the bound (2) for regular graphs always majorates (6). Thus it suffices to present a feasible vector for the LP in (6) that gives the objective value the same as (2).

We claim that

$$
a=\frac{-\tau}{v_{r}-\tau} P_{0}^{T}+\frac{v_{r}}{v_{r}-\tau} P_{\ell}^{T}
$$

is such a vector. It is straightforward to check that $a_{0}=1$ and $a_{r}=0$, as required. By the assumption of the proposition, $a \geq 0$. As $P Q=v I$, any nonnegative linear combination $z$ of the rows of $P$ satisfies $Q^{T} z^{T} \geq 0$. As $a^{T}$ is such a combination, we obtain $Q^{T} a \geq 0$.
Finally, to compute $1^{T} a$, note that $1^{T} P_{0}^{T}=v$ and $1^{T} P_{\ell}^{T}=0$.
Corollary 1. The bounds (6) and (2) coincide for $\Omega(n)$.
Proof. We apply Proposition to $\mathcal{A}=H(n, 2)$ and $r=\frac{n}{2}$. Then the eigenvalues of $A_{r}=\Omega(n)$ given in (3) comprise the $r$-th column on $P$, in particular the least eigenvalue $\tau$ equals $P_{2, r}$, by (4) above. The assumption of the proposition translates into ${ }^{1}$

$$
\binom{n}{\frac{n}{2}} K_{i}(2)-\binom{n}{i} K_{\frac{n}{2}}(2)=\frac{2^{\frac{n}{2}+2}(n-2)!(n-1)!!\left(\frac{n}{2}-i\right)^{2}}{i!\left(\frac{n}{2}\right)!(n-i)!} \geq 0
$$

as claimed.

### 2.3 Schrijver's improved SDP-based bound

Recently, Schrijver [17] has suggested a new SDP-based bound for minimal distance codes, that is at least as good as the $\vartheta^{\prime}$ bound, and still of size polynomial in $n$. It is given as the optimal value of a semidefinite programming (SDP) problem.
In order to introduce this bound (as applied to $\alpha(\Omega(n))$ ) we require some notation.
For $i, j, t \in\{0,1, \ldots, n\}$, and $X, Y \in\{0,1\}^{n}$ define the matrices

$$
\left(M_{i, j}^{t}\right)_{X, Y}= \begin{cases}1 & \text { if }|X|=i,|Y|=j, d_{H}(X, Y)=i+j-2 t \\ 0 & \text { otherwise }\end{cases}
$$

The upper bound is given as the optimal value of the following semidefinite program:

$$
\bar{\alpha}(n):=\max \sum_{i=0}^{n}\binom{n}{i} x_{i, 0}^{0}
$$

subject to

$$
\begin{aligned}
x_{0,0}^{0} & =1 \\
0 & \leq x_{i, j}^{t} \leq x_{i, 0}^{0} \text { for all } i, j, t \in\{0, \ldots, n\} \\
x_{i, j}^{t} & =x_{i^{\prime}, j^{\prime}}^{t^{\prime}} \text { if }\left\{i^{\prime}, j^{\prime}, i^{\prime}+j^{\prime}-2 t^{\prime}\right\} \text { is a permutation of }\{i, j, i+j-2 t\} \\
x_{i, j}^{t} & =0 \text { if }\{i, j, i+j-2 t\} \cap\left\{\frac{1}{2} n\right\} \neq \emptyset,
\end{aligned}
$$

[^1]as well as
$$
\sum_{i, j, t} x_{i, j}^{t} M_{i, j}^{t} \succeq 0, \quad \sum_{i, j, t}\left(x_{i+j-2 t, 0}^{0}-x_{i, j}^{t}\right) M_{i, j}^{t} \succeq 0 .
$$

The matrices $M_{i, j}^{t}$ are of order $2^{n}$ and therefore too large to compute with in general. Schrijver pointed out that these matrices form a basis of the Terwilliger algebra of the Hamming scheme, and worked out the details for computing the irreducible block diagonalization of this (non-commutative) matrix algebra of dimension $O\left(n^{3}\right)$.
Thus, analogously to the $\vartheta^{\prime}$-case, the constraint $\sum_{i, j, t} x_{i, j}^{t} M_{i, j}^{t} \succeq 0$ is replaced by

$$
\sum_{i, j, t} x_{i, j}^{t} Q^{T} M_{i, j}^{t} Q \succeq 0
$$

where $Q$ is an orthogonal matrix that gives the irreducible block diagonalization. For details the reader is referred to Schrijver [17]. Since SDP solvers can exploit block diagonal structure, this reduces the sizes of the matrices in question to the extent that computation is possible in the range $n \leq 32$.

### 2.4 Laurent's improvement

In Laurent [13] one finds a study placing the relaxation 17] into the framework of moment sequences of 11 12. This study also explains the relationship with known lift-and-project methods for obtaining hierarchies of upper bounds on $\alpha(G)$.
Moreover, Laurent 13 suggests a refinement of the Schrijver relaxation that takes the following form:

$$
l_{+}(n):=\max 2^{n} x_{0,0}^{0}
$$

subject to

$$
\begin{aligned}
0 & \leq x_{i, j}^{t} \leq x_{i, 0}^{0} \text { for all } i, j, t \in\{0, \ldots, n\} \\
x_{i, j}^{t} & =x_{i^{\prime}, j^{\prime}}^{t^{\prime}} \text { if }\left\{i^{\prime}, j^{\prime}, i^{\prime}+j^{\prime}-2 t^{\prime}\right\} \text { is a permutation of }\{i, j, i+j-2 t\} \\
x_{i, j}^{t} & =0 \text { if }\{i, j, i+j-2 t\} \cap\left\{\frac{1}{2} n\right\} \neq \emptyset,
\end{aligned}
$$

as well as

$$
\sum_{i, j, t} x_{i, j}^{t} M_{i, j}^{t} \succeq 0
$$

and

$$
\left(\begin{array}{cc}
1-x_{0,0}^{0} & c^{T} \\
c & \sum_{i, j, t}\left(x_{i+j-2 t, 0}^{0}-x_{i, j}^{t}\right) M_{i, j}^{t}
\end{array}\right) \succeq 0
$$

where $c:=\sum_{i=0}^{n}\left(x_{0,0}^{0}-x_{0, i}^{0}\right) \chi_{i}$, and $\chi_{i}$ is defined by

$$
\left(\chi_{i}\right)_{X}:= \begin{cases}1 & \text { if }|X|=i \\ 0 & \text { else }\end{cases}
$$

This SDP problem may be block-diagonalised as before to obtain an SDP of size $O\left(n^{3}\right)$.

## 3 Computational results

To summarize, the bounds we have mentioned satisfy:

$$
\underline{\alpha}(n) \leq \alpha(\Omega(n)) \leq l^{+}(n) \leq \bar{\alpha}(n) \leq \vartheta^{\prime}(\Omega(n))=2^{n} / n .
$$

In Table we show the numerical values for $\bar{\alpha}(n)$ and $l_{+}(n)$ that were obtained using the SDP solver SeDuMi by Sturm [18, with Matlab 7 on a Pentium IV machine with 1GB of memory. Maltab routines that we have written to generate the corresponding SeDuMi input are available online [15.

| $n$ | $\underline{\alpha}(n)$ | $l_{+}(n)$ | $\bar{\alpha}(n)$ | $\vartheta^{\prime}(\Omega(n))=\left\lfloor 2^{n} / n\right\rfloor$ |
| :---: | :---: | :---: | :---: | :---: |
| 16 | 2304 | 2304 | 2304 | 4096 |
| 20 | 20,144 | $20,166.62$ | $20,166.98$ | 52,428 |
| 24 | 178,208 | 183,373 | 184,194 | 699,050 |
| 28 | $1,590,376$ | $1,848,580$ | $1,883,009$ | $9,586,980$ |
| 32 | $14,288,896$ | $21,103,609$ | $21,723,404$ | $134,217,728$ |

Table 1: Lower and upper bounds on $\alpha(\Omega(n))$.
Note that the lower and upper bounds coincide for $n=16$, proving that $\alpha(\Omega(16))=2304$. The best previously known upper bound, obtained by an ad hoc method, was $\alpha(\Omega(16)) \leq 3912$ [8].
The value $\bar{\alpha}(20)=20,166.98$ implies that

$$
\alpha(\Omega(20)) \in\{20144,20148,20152,20156,20160,20164\}
$$

since $\alpha(\Omega(n))$ is always a multiple of 4 . Another implication is that $n=20$ is the smallest value of $n$ where the upper bounds $\bar{\alpha}(n)$ and $l_{+}(n)$ are not tight. It is worth noticing that the Schrijver and Laurent bounds $\left(\bar{\alpha}(n)\right.$ and $l_{+}(n)$ respectively) give relatively big improvements over the Delsarte bound $\frac{2^{n}}{n}$. This is in contrast to the relatively small improvements that these bounds give for binary codes, cf. 17, 13. We also note that these relaxations are numerically ill-conditioned for $n \geq 24$. This makes it difficult to solve the corresponding SDP problems to high accuracy. The recent study by De Klerk, Pasechnik, and Schrijver [5] suggests a different way to solve such SDP problems, leading to larger SDP instances, but which may avoid the numerical ill-conditioning caused by performing the irreducible block factorization.

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[^1]:    ${ }^{1}$ Here $m!!=m(m-2)(m-4) \ldots$, the double factorial.

