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The Universal Covers of the Sporadic Semibiplanes

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We determine the universal covers of the few flag-transitive sporadic semibiplanes. They were already known by computer-aided coset enumeration. The method we are using seems to be new and of interest on its own.

1. INTRODUCTION

This paper is a continuation of [2, 3]. In [2, 3] all known examples of flag-transitive $c \cdot c^*$-geometries, also called *semibiplanes*, were listed and all such geometries satisfying certain extra assumption were classified. The universal covers of certain $c \cdot c^*$-geometries were found using computer-aided coset enumeration. One aim of this paper is to determine the universal covers of these geometries without using a computer. Another aim is to exhibit some of the technique used, which seems to be new and of interest on its own. Roughly speaking, if

(a) there is a good theoretic bound on the number of elements of the (flag-transitive) geometry, and

(b) for any flag-transitive cover $\tilde{\mathcal{G}}$ of the geometry $\mathcal{G}$, there exists a perfect flag-transitive group of automorphisms $\tilde{G}$ of $\tilde{\mathcal{G}}$,

then there is a good chance that the automorphism group of the universal cover contains a flag-transitive subgroup $\tilde{G}$, which is a perfect central extension of G. It turns out that, for the $c \cdot c^*$-geometries considered here, $\tilde{G}$ is such an extension and, moreover, $\tilde{G}/Z(\tilde{G})$ is a simple group. Since the Schur multipliers of the finite simple groups are known, we are able to determine $\tilde{G}$ and thereby the universal cover.

In the Appendix we give the distribution diagrams of the point–circle incidence graphs of the sporadic $c \cdot c^*$-geometries, although only a small part of the information they carry is actually used here.

Our terminology is fairly standard; see the next section. The elementsof $\mathcal{G}$ are called *points, lines* and *circles.* Let G act flag-transitively on $\mathcal{G}$. For a flag $\{p, l, c\}$ and $x \in \{p, l, c\}$, we denote by G_x the stabilizer of x in G. Then $\mathcal{G}$ is isomorphic to its group geometry $\mathcal{G}(G, (G_p, G_l, G_c))$. That is why $\mathcal{G}$ can be reconstructed from G, so we denote $\mathcal{G} = \mathcal{G}(G)$. Also, let n be the number of points incident to a given circle, N being the total number of points in $\mathcal{G}$.

According to [3] (see also Lemma 2.3 of this paper), the stabilizer G_p of a point p is a doubly transitive permutation group. For every doubly transitive permutation group H of degree n there exists a $c \cdot c^*$-geometry with $H \simeq G_p$ and $G \simeq E_{2^{n-1}}{:}\,H$; namely, the two-coloured hypercube $H(n)$—see, for instance, [3] or [4]. We are aware of existence of three other infinite families with G_p an affine doubly transitive permutation group; see [2]. Moreover, there are ten sporadic examples with almost simple point-stabilizer, as follows (here $G = \mathrm{Aut}(\mathcal{G})$):

(i) $\mathcal{G} = \mathcal{G}(L_2(11))$, $n = 5$, $N = 11$, $G \simeq L_2(11)$ and $G_p \simeq A_5$.

(ii) $\mathcal{G} = \mathcal{G}(S_6)$ or $\mathcal{G}(3S_6)$, $n = 6$, $N = 6$ or 18, $G \simeq S_6$ or $3S_6$, respectively, and $G_p \simeq S_5$. Also, $H \leqslant G$, $H \simeq A_6$ or $3A_6$ acts flag-transitively on $\mathcal{G}(S_6)$ or $\mathcal{G}(3S_6)$, respectively.

(iii) $\mathcal{G} = \mathcal{G}(L_3(4))$ or $\mathcal{G}(2L_3(4))$, $n = 10$, $N = 56$ or 112 and $G \simeq L_3(4)2^2$ or $2L_3(4)2^2$,

respectively, and $G_p \simeq P\Gamma L_2(9)$. Also, $H \leqslant G$, $H \simeq L_3(4)$ or $2L_3(4)$ acts flag-transitvely on $\mathscr{G}(L_3(4))$ or $\mathscr{G}(2L_3(4))$, respectively.
(iv) $\mathscr{G} = \mathscr{G}(\mathrm{Aut}(M_{12}))$, $n = 12$, $N = 144$, $G \simeq \mathrm{Aut}(M_{12})$ and $G_p \simeq PGL_2(11)$. Also, $H \leqslant G$, $H \simeq M_{12}$ acts flag-transitively.
(v) $\mathscr{G} = \mathscr{G}(M_{12})$, $n = 11$, $N = 144$, $G \simeq M_{12}$ and $G_p \simeq L_2(11)$.
(vi) $\mathscr{G} = \mathscr{G}(U_3(3))$, $n = 7$, $N = 36$, $G \simeq U_3(3)$ and $G_p \simeq L_3(2)$.
(vii) $\mathscr{G} = \mathscr{G}(M_{22})$ or $\mathscr{G}(2M_{22})$, $n = 15$, $N = 176$ or 352, $G \simeq M_{22}$ or $2M_{22}$, respectively, and $G_p \simeq A_7$.

We suspect that these and the two-coloured hypercube are the only examples having an almost simple non-abelian stabilizer of a point. In [2], under some weak assumptions, this suspicion was confirmed. Here we determine the universal covers of the ten sporadic $c{\cdot}c^*$-geometries.

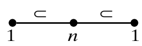
THEOREM A. (*i*) *Both geometries* $\mathscr{G}(L_3(4))$ *and* $\mathscr{G}(M_{22})$ *possess a double cover.*
(ii) *The geometries* $\mathscr{G}(L_2(11))$, $\mathscr{G}(3S_6)$, $\mathscr{G}(\mathrm{Aut}(M_{12}))$, $\mathscr{G}(M_{12})$ *and* $\mathscr{G}(U_3(3))$ *and the double covers* $\mathscr{G}(2L_3(4))$ *and* $\mathscr{G}(2M_{22})$ *are simply connected.*

In fact, Theorem A can be used to eliminate the use of a computer in the proof of the results in [2, 3].

In Section 2 we give definitions and some basic facts about $c \cdot c^*$-geometries. Section 3 consists of a proof of Theorem A. For each of the examples $\mathscr{G} = \mathscr{G}(\mathrm{G})$ in Theorem A, we proceed as follows. If $\mathscr{G}$ is known to possess a non-trivial cover as a result of a computer-aided coset enumeration, we construct it independently. Namely, we obtain a cover by embedding the amalgam for $\mathscr{G}$ in a covering group of G. Then we show the simple connectedness of (the covers of) $\mathscr{G}$. In some cases this follows immediately from the bound on the number of points. By [14], a $c \cdot c^*$-geometry with maximal parabolic subgroups G_p, G_l and G_c is simply connected iff its automorphism group is the completion of the amalgam of these maximal parabolic subgroups. Hence for the remaining geometries we determine the completion $\tilde{G}$ of the corresponding amalgam.

2. PRELIMINARIES

A geometry $\mathscr{G}$ consisting of points, lines and circles is a $c \cdot c^*$-geometry' or belongs to the diagram



iff:
(1) for every point p, the residue $\mathscr{G}_p$ of p is isomorphic to the geometry of vertices and edges of a complete graph K_{n+2} on $n+2$ vertices, where the circles and the lines in $\mathscr{G}_p$ are the vertices and the edges respectively;
(2) for every line l, the residue $\mathscr{G}_l$ of l is a generalized 2-gon consisting of two points and two circles;
(3) for every circle c, the residue $\mathscr{G}_c$ of c is a complete graph K_{n+2}, where the points and the lines in $\mathscr{G}_c$ are the vertices and the edges respectively.

The following is a condition equivalent to the Intersection Property in [6]:

> (IP) For any two elements x and y, the set of points incident with x and y coincides, if not empty, with the set of points incident with some element z, which is incident with both x and y.

If (IP) holds then the truncation of $\mathscr{G}$ to points and circles (blocks) is a semibiplane—that is, a connected incidence structure satisfying:

(i) any two points are incident with 0 or 2 common blocks;
(ii) any two blocks are incident with 0 or 2 common points. (See, for example, [20]).

On the other hand, each semibiplane yields a $c \cdot c^*$-geometry, the lines of which are the quadruples (P_1, P_2, B_1, B_2) of two different points P_1, P_2, being incident with the two different blocks (circles) B_1, B_2.

A *cover* of a geometry $\mathcal{G}$ is a tuple $(\tilde{\mathcal{G}}, \phi)$ consisting of a geometry $\tilde{\mathcal{G}}$ and an epimorphism ϕ: $\tilde{\mathcal{G}} \to \mathcal{G}$, such that ϕ maps $\mathrm{res}(p)$ isomorphically onto $\mathrm{res}(p^{\phi})$ for any element p of $\mathcal{G}$.

A cover $(\tilde{\mathcal{G}}, \phi)$ is called *universal,* if for each cover $(\bar{\mathcal{G}}, \psi)$, there is a covering $(\tilde{\mathcal{G}}, \theta)$ of $\bar{\mathcal{G}}$, such that $\theta\psi = \phi$. A geometry $\mathcal{G}$ is called *simply connected* if the cover $(\mathcal{G}, id)$ is universal.

Let $(\tilde{\mathcal{G}}, \phi)$ be a finite cover of $\mathcal{G}$. Since ϕ induces an equivalence relation on $\tilde{\mathcal{G}}$, the number of objects of a given type in $\mathcal{G}$ divides the number of objects of this type in $\tilde{\mathcal{G}}$.

If the fundamental group of $\tilde{\mathcal{G}}$ is characteristic in the fundamental group of $\mathcal{G}$, then the automorphism group of $\mathcal{G}$ can be lifted to a group of automorphism of $\tilde{\mathcal{G}}$, i.e. there is a subgroup $\tilde{G}$ of $\mathrm{Aut}(\tilde{\mathcal{G}})$ and a normal subgroup N of $\tilde{G}$, such that $\tilde{G}/N \simeq \mathrm{Aut}(\mathcal{G})$. In particular, since the fundamental group of the universal cover is trivial, $\mathrm{Aut}(\mathcal{G})$ can be lifted to a group of automorphism of the universal cover. These definitions and the last fact can be found in [19] and [18].

An *amalgam* is a collection $\mathcal{A}$ of groups such that any two groups $U, V \in \mathcal{A}$ intersect in some specified element of $\mathcal{A}$. Suppose that there is a group G which is generated by the groups in $\mathcal{A}$. If G is maximal in this respect, then we call G the *completion of the amalgam* $\mathcal{A}$. See [15] for a comprehensive introduction to the amalgam approach in diagram geometry.

LEMMA 2.1. *Let $\mathcal{G}$ be a geometry of rank 3, with $G \leqslant \mathrm{Aut}(\mathcal{G})$ flag-transitive on $\mathcal{G}$. Let $\tilde{\mathcal{G}}$ be the universal cover of $\mathcal{G}$ and $\tilde{G} = N \cdot G$ be the lifting of G into $\mathrm{Aut}(\tilde{\mathcal{G}})$. Assume that G is simple and that, for two different types i, j, the stabilizers in G of the elements of type i and j are perfect groups. Then $\tilde{G}$ is perfect and $C_{\tilde{G}}(N)N/N = 1$ or G.*

PROOF. By residual connectness, $\tilde{G} = \langle \tilde{G}_i, \tilde{G}_j \rangle$. As $\tilde{G}_i$ and $\tilde{G}_j$ are perfect groups, $\tilde{G}' \geqslant \langle \tilde{G}_i', \tilde{G}_j' \rangle = \tilde{G}$, so $\tilde{G}$ is perfect. Since $G = \tilde{G}/N$ is a simple group, the last claim holds as well. □

Let Δ be the distribution diagram of the point–circle incidence graph Γ of a flag-transitive $c \cdot c^*$-geometry $\mathcal{G}$ with respect to some point p. For the definition of the distribution diagram, see [5]. As usual, let $\Gamma_i(p)$ be the vertices of Δ having distance i to p and let $\{p\} = \Gamma_0(p)$ and $\Gamma(p) = \Gamma_1(p)$. If each vertex in $\Gamma_i(p)$ has the same number of neighbours in $\Gamma_{i-1}(p)$, then we denote this number by c_i. By definition, $|\Gamma(p)| = n$.

The following lemma was shown by Wild [Wi] for semibiplanes. In fact, it holds in each $c \cdot c^*$-geometry, i.e. also if (IP) fails. The lemma can be proved using a result of Pasechnik [13], which provides a bound on the number of points of a locally finite $C_2 \cdot L$-geometry. Here we give a direct proof.

LEMMA 2.2. *Let $u \in \Gamma_m(p)$ and $c \in \Gamma_{m+1}(p)$ be neighbours in Γ, $m \geqslant 1$. Then $|\Gamma_m(p) \cap \Gamma(v)| \geqslant |\Gamma_{m-1}(p) \cap \Gamma(u)| + 1$. In particular, $\mathcal{G}$ has at most 2^{n-1} points.*

PROOF. Without loss of generality, we can assume u to be a point and v to be a circle. Let $v_1, \ldots, v_r$ be the neighbours of u in $\Gamma_{m-1}(p)$. Since the residue of u is isomorphic to a complete graph, there exists exactly one line l_i, which is incident to v and v_i, $1 \leqslant i \leqslant r$. As the residue of v is isomorphic to a complete graph as well, in Γ_v

there exists exactly one further point u_i distinct from u incident to l_i, $1 \leqslant i \leqslant r$. This yields the assertion, since $u, u_1, \ldots, u_r \in \Gamma_m(p)$.

Thus, as $c_1 = 1$, we have $|\Gamma_{m-1}(p) \cap \Gamma(u)| \geqslant m$ and $|\Gamma_m(p)| \leqslant \binom{n}{m}$. Hence the number of points is at most $\sum_{i=0}^{[n/2]} \binom{n}{2i} = 2^{n-1}$. $\square$

The following result provides a set of conditions on a group G to be a flag-transitive automorphism group of a $c \cdot c^*$-geometry.

LEMMA 2.3 [3]. *A group G acts flag-transitively on a $c \cdot c^*$-geometry $\mathscr{G}$, iff there are pairwise distinct subgroups $G_1, G_2, G_3 \leqslant G$, satisfying the following conditions*:
(1) *G_i is a doubly transitive permutation group on $\{(G_1 \cap G_3)g, g \in G_i\}$, $i \in \{1, 3\}$;*
(2) *$B \trianglelefteq G_2$, $G_2/B \simeq E_4$, $(G_2 \cap G_i)/B \simeq Z_2$ and $G_i = \langle a_i, G_1 \cap G_3\rangle$, $a_i \in (G_2 \cap G_i)\backslash B$, $i \in \{1, 3\}$ and $B = G_1 \cap G_2 \cap G_3$;*
(3) *$(G_1 \cap G_3) \cap (G_1 \cap G_3)^{a_i} = B$;*
(4) *$G = \langle G_1, G_3\rangle$.*

3. PROOF OF THE THEOREM

3.1. *The geometry with circle size $n = 15$*

Let $\mathscr{G} = \mathscr{G}(M_{22})$. Then $\mathscr{G}$ can be described as follows (see [3]). Let $\mathscr{S} = S(5, 8, 24)$ be a Steiner system on the set $\Omega = \{\alpha_1, \ldots, \alpha_{24}\}$ with set of octads $\mathscr{O}$. The points of $\mathscr{G}$ are the octads, which contain α_1, but not α_{24}; whereas the circles are the octads, which contain α_{24}, but not α_1. The lines of $\mathscr{G}$ are two-coloured sextets $\{L_1, L_2, L_3\}\{L_4, L_5, L_6\}$, where $\{L_1, \ldots, L_6\}$ is a sextet with $\alpha_1 \in L_1$ and $\alpha_{24} \in L_6$. Let a point p be incident to a circle c iff $p \cap c = \varnothing$. A point p (a circle c) is incident to a line $l = \{L_1, L_2, L_3\}\{L_4, L_5, L_6\}$ iff $p = L_1 \cup L_2$ or $p = L_1 \cup L_3$ (respectively, $c = L_4 \cup L_6$ or $c = L_5 \cup L_6$). Finally, $G = \mathrm{Aut}(\mathscr{G}) \simeq M_{22}$ acts flag-transitively on $\mathscr{G}$.

LEMMA 3.1. *There exists a double cover $\bar{\mathscr{G}}$ of $\mathscr{G}(M_{22})$ with automorphism group $H \simeq 2M_{22}$.*

PROOF. We identify $\mathscr{G}$ with the group geometry $\mathscr{G}(G, (G_p, G_l, G_c))$.

Let $H \simeq 2M_{22}$ be the double cover of G and ψ the natural endomorphism from H onto G. We construct an amalgam (H_p, H_l, H_c) in H, such that ψ induces a cover of the group geometry $\mathscr{G}(H, (H_p, H_l, H_c))$ onto $\mathscr{G}(G, (G_p, G_l, G_c))$. Thus we have to find subgroups H_p, H_l and H_c of H, such that for pairwise distinct $x, y, z \in \{p, l, c\}$ the morphism ψ induces an isomorphism of H_x, $H_x \cap H_y$ and $H_x \cap H_y \cap H_z$ onto G_x, $G_x \cap G_y$ and $G_x \cap G_y \cap G_z$, respectively.

We have $G_p \simeq A_7 \simeq G_c$, $G_l \simeq S_4 \times \mathbb{Z}_2$, $G_p \cap G_c \simeq L_3(2)$, $G_p \cap G_l \simeq S_4 \simeq G_c \cap G_l$ and $B \simeq A_4$ (cf. [3]).

We claim that for x a point or a circle $G_x^{\psi^{-1}} \simeq \mathbb{Z}_2 \times A_7$. Suppose that $G_x^{\psi^{-1}}$ is a non-split extension $2 \cdot A_7$. Then the involutions in G_x are lifted to elements of order 4 in $G_x^{\psi^{-1}}$. As $G \simeq M_{22}$ has only one class of involutions, we obtain $\Omega_1(S) \simeq \mathbb{Z}_2$ for $S \in Syl_2(H)$. Hence S is isomorphic to a quaternion or to a cyclic group, a contradiction with the fact that $S/Z(H) \simeq E_{16}{:}\, D_8$. Thus $G_x^{\psi^{-1}} \simeq \mathbb{Z}_2 \times A_7$.

Let $H_x \leqslant G_x^{\psi^{-1}}$ be such that $H_x \simeq A_7$. Then ψ induces an isomorphism of H_x onto G_x.

It remains to produce the parabolic subgroup H_l. By Lemma 2.3, there exist $a_1 \in G_p \cap G_l \backslash B$ and $a_3 \in G_c \cap G_l \backslash B$, so that $G_p \cap G_l = \langle B, a_1\rangle$, $G_c \cap G_l = \langle B, a_3\rangle$ and $(a_1 a_3)^2 \in B$. Furthermore, a_1 and a_3 may be chosen such that $a_1 a_3$ is an element of order 6 (see [3]). Let $\tilde{a}_1$ and $\tilde{a}_3$ be the preimages of a_1 and a_3 in H_p and H_c, respectively. Define $H_l = \langle B^{\psi^{-1}} \cap H_p, \tilde{a}_1, \tilde{a}_3 \rangle$. As involutions of G are lifted to involutions of H, the order of $\tilde{a}_1\tilde{a}_3$ is 6 and $(\tilde{a}_1\tilde{a}_3)^2$ is an element in $H_p \cap B^{\psi^{-1}}$. Thus $H_l \simeq G_l$ and $H_l^{\psi} = G_l$. As $B^{\psi^{-1}} \cap H_p \leqslant H_p \cap G_c^{\psi^{-1}} = H_p \cap H_c$, we have that ψ also induces an isomorphism of $H_p \cap H_l$, $H_l \cap H_c$, $H_p \cap H_l \cap H_c$ onto $G_p \cap G_l$, $G_l \cap G_c$, $G_p \cap G_l \cap G_c$, respectively. So H_p, H_l and H_c give us the required amalgam. □

Let $\tilde{G}$ be the completion of the amalgam of G_p, G_l and G_c. Hence there exist subgroups $\tilde{G}_p$, $\tilde{G}_l$ and $\tilde{G}_c$ of $\tilde{G}$ forming an amalgam, which is isomorphic to the amalgam of G_p, G_l and G_c. The group geometry $\tilde{\mathscr{G}} = \mathscr{G}(\tilde{G}, (\tilde{G}_p, \tilde{G}_l, \tilde{G}_c))$ is the universal cover of $\mathscr{G}$. Next we show that $\bar{\mathscr{G}}$, which is constructed in Lemma 3.1, is the universal cover.

LEMMA 3.2. *The cover $\bar{\mathscr{G}}$ is simply connected.*

PROOF. By Lemma 3.1 and [18], $\tilde{G}/N \simeq 2M_{22}$ for some normal subgroup N of $\tilde{G}$. Let $N \leqslant M \leqslant \tilde{G}$,such that $G/M \simeq M_{22}$. As each circle is incident to 15 points, the geometry $\tilde{\mathscr{G}}$ has less than 2^{14} points (cf. Lemma 2.2). Hence $|\tilde{G}: G_p| < 2^{14}$, which yields $|N| < 2^{14}/352$. As $352 > 1024/3 = 2^{10}/3$ we obtain $|N| < 2^4 \cdot 3 = 48$.

The completion $\tilde{G}$ is a perfect group, since it is generated by G_p and G_c, which are isomorphic to A_7; see Lemma 2.1.

We claim that $\tilde{G}$ is a perfect central extension of G. Since $|N| \leqslant 48$, the group N is solvable. As $[M: N] = 2$, the group M is solvable as well. Let p be a prime dividing the order of M, such that $O_p(M)$ is non-trivial. Set $Q = O_p(M)/\phi(O_p(M))$. Then $C_{\tilde{G}}(Q)$ is a normal subgroup of $\tilde{G}$ and $\tilde{G}/C_{\tilde{G}}(Q)$ isomorphic to a subgroup of $\mathrm{Aut}(Q)$. As $|M| \leqslant 96$ and $\tilde{G}/M \simeq M_{22}$, we obtain $\tilde{G} = C_{\tilde{G}}(Q)M$ and, as $\tilde{G}$ is a perfect group and M is solvable, $\tilde{G} = C_{\tilde{G}}(Q)$. By the same argumentation we also conclude that $\tilde{G} = C_{\tilde{G}}(\phi(O_p(M)))$. As $\tilde{G}$ is generated by elements the order of which is not divisible by p, a theorem of Burnside, [1, 24.1] yields $[\tilde{G}, O_p(M)] = 1$. Since this argument holds for each prime p with $O_p(M) \neq 1$, we obtain that $\tilde{G}$ acts trivially on the Fitting subgroup $F(M)$ of M. This gives $M \leqslant C_M(F(M))$ and, as M is solvable, $M \leqslant C_M(F(M)) \leqslant F(M)$. Thus $M = Z(G)$ and $\tilde{G}$ is a perfect central extension of G.

According to [16] the Schur multiplier is isomorphic to a cyclic group of order 12. Hence $Z(\tilde{G})$ is a cyclic group, the order of which divides 12 and $|N| \leqslant 6$.

Assume that $3 \mid |Z(\tilde{G})|$. For $N = \langle n \rangle$, we then have $O_3(N) = \langle n^2 \rangle \simeq \mathbb{Z}_3$. As $O_3(N)\tilde{G}_p$ splits over $O_3(N)$ and as 3 does not divide $|\tilde{G} : O_3(N)\tilde{G}_p| = |O_2(N)| \cdot 176$, it follows from Gaschütz's theorem [1, 10.4], that $\tilde{G}$ splits over $O_3(N)$. As $\tilde{G}$ is a perfect group, $O_3(N) = 1$, in contradiction to our assumption. So $Z(\tilde{G}) \simeq \mathbb{Z}_2$ or $\mathbb{Z}_4$.

Assume that $Z(\tilde{G}) \simeq \mathbb{Z}_4$. Hence G_p lifts to $\tilde{G}_p \times Z(\tilde{G}) \simeq \mathbb{Z}_4 \times A_7$ in $\tilde{G}$. The group $G \simeq M_{22}$ acts not only on the Steiner system $\mathscr{S} = S(5, 8, 24)$, (see the construction of $\mathscr{G}(M_{22})$ above) but also on the Steiner system $\mathscr{T} = T(3, 6, 22)$ on the set $\Omega' = \{\alpha_2, \ldots, \alpha_{23}\}$, where the hexads are the octads of $\mathscr{S}$ containing both α_1 and α_{24}. Without loss of generality we can sssume that $p = \{\alpha_1, \ldots, \alpha_8\}$ and $Z = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}\}$ are octads. Then p is a point of our geometry $\mathscr{G}$ and Z an hexad in $\mathscr{T}$. Moreover, $T = \mathrm{stab}_{G_p}(Z) \simeq (A_4 \times \mathbb{Z}_3)2$ and the stabilizer of Z in G is a split extension $S_Z\colon A_Z$, $S_Z \simeq E_{16}$, $A_Z \simeq A_6$. Moreover, $T = (S_Z \cap T)\colon (A_Z \cap T)$, $S_Z \cap T \simeq E_4$ and $A_Z \cap T \simeq E_9\mathbb{Z}_2$.

In order to derive a contradiction to our assumption, we determine the preimage of $S_2 \cap T$ in $\tilde{G}$. In [16] a construction of the preimage of $S_Z A_Z$ in $4M_{22}$ is given. From this construction we derive that S_Z lifts in G to $(S_1 * S_2) * Z(\tilde{G})$, where $S_1 \simeq S_2 \simeq \mathbb{Q}_8$.

Furthermore the groups $S_1Z(\tilde{G})/Z(\tilde{G})$ and $S_2Z(\tilde{G})/Z(\tilde{G})$ are fixed by some Sylow-3-subgroup, say X, of A_Z. As $\mathrm{Syl}_3(T) \subseteq \mathrm{Syl}_3(S_Z : A_Z)$ we may choose the point p such that $X \leqslant T$. Hence X normalizes $S_Z \cap T$ and the preimage $\tilde{X}$ of X in $\tilde{G}$ normalizes the preimage of $S_Z \cap T$ in $\tilde{G}$. As each Sylow-3-subgroup of A_Z fixes exactly two subgroups of order 4 in S_Z, only the two groups $S_1Z(\tilde{G})/Z(\tilde{G})$ and $S_2Z(\tilde{G})/Z(\tilde{G})$ are fixed by X. Thus $S_Z \cap T$ lifts to $Z(\tilde{G}) * S_i$ for $i = 1$ or 2. This gives us a contradiction, since by our assumption the preimage of $S_Z \cap T$ is isomorphic to $\mathbb{Z}_4 \times E_4$.

Hence $\tilde{\mathscr{G}} = \mathscr{G}(2M_{22})$ is simply connected. □

Lemmas 3.1 and 3.2 prove Theorem A for $\mathscr{G} = \mathscr{G}(M_{22})$.

3.2. *The geometries with circle sizes $n = 11$ or 12*

Let $\mathscr{G} = \mathscr{G}(M_{12})$ or $\mathscr{G}(\mathrm{Aut}(M_{12}))$. The geometry $\mathscr{G}(M_{12})$ was constructed by Buekenhout [7]. Here the stabilizer of a point and the stabilizer of a circle are conjugated maximal subgroups in M_{12}.

The geometry $\mathscr{G}(\mathrm{Aut}(M_{12}))$ was found by Leonard [12] and a construction is given in [5, p. 371]. Take the Steiner system $\mathscr{S} = S(5, 8, 24)$ and two complementary dodecads D_1 and D_2. Then, $\mathrm{stab}_{M_{24}}(D_1) \simeq M_{12}$. Define a graph Δ with vertex set $D_1 \times D_2$, where two pairs (d_1, d_2), (e_1, e_2) are non-adjacent either if $d_1 = e_1$ or $d_2 = e_2$ or if there is an octad B in $\mathscr{S}$ with $B \cap D_1 = \{d_1, e_1\}$ and $\{d_2, e_2\} \subset B \cap D_2$. Then Δ has exactly 144 12-cliques. The points are the vertices of Δ and the circles the 12-cliques. Thus the stabilizer of a point is contained in a maximal subgroup of G which is isomorphic to M_{11}, and the stabilizer of a circle is a maximal subgroup in M_{12}.

In both cases $G \leqslant \mathrm{Aut}(\mathscr{G})$, $G \simeq M_{12}$, acts flag-transitively on $\mathscr{G}$ with $G_p \simeq G_c \simeq L_1(11)$. Let $\tilde{\mathscr{G}}$ be the universal cover of $\mathscr{G}$. Then there is a subgroup $\tilde{G}$ in $\mathrm{Aut}(\tilde{\mathscr{G}})$, such that $\tilde{G}/N \simeq G$ for some normal subgroup N of $\tilde{G}$. By Lemma 2.2, $[\tilde{G} : \tilde{G}_p]$ is at most 2^{10} for $\mathscr{G}(M_{12})$ and at most 2^{11} for $\mathscr{G}(\mathrm{Aut}(M_{12}))$. Thus $|M| \leqslant 7$ or 14, respectively. Hence, as in Section 3.1, we obtain that $\tilde{G}$ is a perfect central extension of G. This gives $|N| \leqslant 2$ (see [18]). The Mathieu group M_{12} has three classes of subgroups isomorphic to $L_2(11)$. Two of them consist of non-maximal subgroups and they fuse in $\mathrm{Aut}(M_{12})$. The third class consists of maximal subgroups. Suppose that $|N| = 2$. By [10], the maximal subgroups isomorphic to $L_2(11)$ in M_{12} are lifted to $SL_2(11)$, which is a contradiction with the fact that $\tilde{G}_p \simeq \tilde{G}_c \simeq L_2(11)$.

3.3. *The geometry with circle size $n = 10$*

Let $\mathscr{G} = \mathscr{G}(L_3(4))$ and $G \leqslant \mathrm{Aut}(\mathscr{G})$, $G \simeq L_3(4)$. The geometry $\mathscr{G}$ can be described as follows; see [2]. Let $\mathscr{S} = S(3, 6, 22)$ be a Steiner system on the set $\Omega = \{\alpha_1, \ldots, \alpha_{22}\}$. Then the points and the circles of $\mathscr{G}$ are the hexads, which do not contain α_{22}. A point p is incident to a circle c iff their intersection is empty. Let $\{p, c\}$ be an incident point–circle pair. Then the stabilizer of p in G and the stabilizer of c in G are subgroups isomorphic to A_6 and their intersection is isomorphic to $E_9 : \mathbb{Z}_4$. Moreover, we have $G_l \simeq \mathbb{Z}_4 * D_8$ and $B \simeq \mathbb{Z}_4$.

Let $H \simeq 2L_3(4)$ be the double cover of G and ψ the natural endomorphism from H onto G.

Lemma 3.3. *The group H is isomorphic to a subgroup of the double cover of M_{22}.*

Proof. Assume the contrary. Then there is a subgroup U in $2M_{22}$ with $U \simeq \mathbb{Z}_2 \times L_3(4)$. Let $S \in \mathrm{Syl}_2(U)$. Then $S = (Z \times S_1)S_2$, $Z = Z(U)$, $S_1 \simeq E_{16}$ and $S_2 \simeq E_4$.

Since $L_3(4)$ has only one class of involutions, by the same argument as for M_{22} involutions of $L_3(4)$ lift to involutions in $2L_3(4)$. Hence S_1 lifts to a group isomorphic to E_{32} in $2L_3(4)$. Let $K \simeq 4M_{22}$. Then the preimage of U in K is isomorphic either to $\mathbb{Z}_4 \times L_3(4)$ or to $\mathbb{Z}_4 * 2L_3(4)$. Thus S_1 lifts in K to a group isomorphic to $\mathbb{Z}_4 \times E_{16}$ or $\mathbb{Z}_4 * E_{32}$, respectively, which is in both cases an abelian group. This gives us a contradiction since in K the group S_1 lifts to $\mathbb{Z}_4 * \mathbb{Q}_8 * \mathbb{Q}_8$—see the proof of Lemma 3.2, which is not an abelian group. Thus H is isomorphic to a subgroup of $2M_{22}$. □

LEMMA 3.4. *There exists a double cover* $(\bar{\mathcal{G}}, \psi)$ *of* $\mathcal{G}(L_3(4))$ *which admits as group of automorphisms* $H \simeq 2L_3(4)$.

PROOF. In the same manner as in Lemma 3.1 we identify $\mathcal{G}$ with the group geometry $\mathcal{G}(G, (G_p, G_l, G_c))$ and we construct an amalgam (H_p, H_l, H_c) in H, such that ψ induces a cover of the group geometry $\mathcal{G}(H, (H_p, H_l, H_c))$ onto $\mathcal{G}(G, (G_p, G_l, G_c))$.

According to Lemma 3.3 we may assume that $H \leqslant 2M_{22}$.

As in $2M_{22}$ the subgroups isomorphic to A_6 are lifted to subgroups isomorphic to $\mathbb{Z}_2 \times A_6$ (see the proof of Lemma 3.1), we have $G_x^{\psi^{-1}} \simeq \mathbb{Z}_2 \times A_6$ for $x \in \{p, c\}$. Let H_x be the subgroup of $G_x^{\psi^{-1}}$ isomorphic to A_6. Set $H_l = \langle N_{H_p}(B^{\psi^{-1}} \cap H_p),\ N_{H_c}(B^{\psi^{-1}} \cap H_c)\rangle$. We claim that (H_p, H_l, H_c) gives us the desired amalgam. Hence it remains to show $H_p \cap H_c \simeq E_9{:}\,\mathbb{Z}_4$, $H_t \simeq G_l \simeq \mathbb{Z}_4 * D_8$, $H_p \cap H_l \simeq H_l \cap H_c \simeq D_8$ and $H_p \cap H_l \cap H_c \simeq \mathbb{Z}_4$.

First we show that $H_p \cap H_c \simeq E_9{:}\,\mathbb{Z}_4$. Set $T = O_3(H_p \cap H_c)$. As the stabilizers G_p and G_c are conjugated in G, the groups H_p and H_c are also conjugated in H. Since $\mathrm{Syl}_3(H_p \cap H_c) \subseteq \mathrm{Syl}_3(H_c)$ we have $H_p^g = H_c$ for some $g \in N_H(T)$.

Let us calculate $N_H(T)$. We have $N_G(T^{\psi}) = N_{M_{22}}(T^{\psi}) = N_M(T^{\psi}) \simeq E_9{:}\,\mathbb{Q}_8$, where $M \leqslant M_{22}$, $M \simeq M_{10}$. Furthermore, $N_H(T) = N_G(T^{\psi})^{\psi^{-1}} = N_M(T^{\psi})^{\psi^{-1}}$. As $(M')^{\psi^{-1}} \simeq \mathbb{Z}_2 \times A_6$ and for any $x \in M^{\psi^{-1}} \backslash (M')^{\psi^{-1}}$ one has $1 \neq x^2 \in (M')^{\psi^{-1}}$, we obtain $M^{\psi^{-1}} \simeq \mathbb{Z}_2 \times M_{10}$. Thus $N_M(T^{\psi})^{\psi^{-1}}$ splits over $Z(H)$. Hence $N_H(T) \simeq \mathbb{Z}_2 \times E_9{:}\,\mathbb{Q}_8$.

For $x = p$ or c, we have $B^{\psi^{-1}} \cap H_x \simeq \mathbb{Z}_4$ and $N_{H_x}(T) = T(B^{\psi^{-1}} \cap H_x)$. Without loss of generality, we may assume that $o(g) = 4$. Then, as $N_H(T) \simeq \mathbb{Z}_2 \times E_9{:}\,\mathbb{Q}_8$, we obtain $[B^{\psi^{-1}} \cap H_p, g] \leqslant B^{\psi^{-1}} \cap H_p$, which gives $N_{H_p}(T) = N_{H_c}(T)$ and $H_p \cap H_c = N_{H_p}(T) \simeq E_9{:}\,\mathbb{Z}_4$.

Let $\langle n \rangle = B^{\psi^{-1}} \cap H_p$. Then $H_l \leqslant N_H(\langle n \rangle)$. We have $N_{H_p}(\langle n \rangle) \simeq D_8$. Let $N_{H_p}(\langle n \rangle) = \langle a_1, n \rangle$ and $N_{H_c}(\langle a_3, n \rangle) = \langle a_3, n \rangle$. Due to Lemma 2.3, we have $(a_1 a_3)^2 \in B^{\psi^{-1}}$. As $G_l \simeq \mathbb{Z}_4 * D_8$, we obtain $(a_1 a_3)^3 \in \{n^2, zn^2\}$, where $\langle z \rangle = Z(H)$. Assume that $(a_1 a_3)^2 = n^2 z$. Then $(a_1 a_3 n)^2 = z$, which contradicts the fact that involutions of G are lifted to involutions in H. Hence $H_l \cap H_x = N_{H_x}(\langle n \rangle) \simeq D_8$ for $x \in \{p, c\}$, $H_l \simeq \mathbb{Z}_4 * D_8$ and $H_p \cap H_l \cap H_c \simeq \mathbb{Z}_4$, which shows the assertion. □

LEMMA 3.5. *The geometry* $\bar{\mathcal{G}}$ *constructed in Lemma* 3.4 *is simply connected and* $\mathrm{Aut}(\bar{\mathcal{G}}) \simeq 2L_3(4)2^2$.

PROOF. Let $(\tilde{\mathcal{G}}, \phi)$ be the universal cover of $\mathcal{G}$ and set $G = \mathrm{Aut}(\mathcal{G})$ and $\tilde{G} = \mathrm{Aut}(\tilde{\mathcal{G}})$. By [2], the stabilizer $\tilde{G}_p$ of a point p in $\tilde{G}$ is not isomorphic to A_{10} or S_{10}. So, as G_p is a doubly transitive permutation group of degree 10 (cf. Lemma 2.3) $\tilde{G}_p \simeq G_{p^{\phi}} \simeq \mathrm{Aut}(A_6)$ and $\tilde{G}$ acts o the fibres of ϕ, i.e. $\tilde{G}/K \simeq G$, where K is the kernel of ϕ.

Let Δ and $\tilde{\Delta}$ be the distribution diagrams of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ with respect to the points p and p^{ϕ}, respectively. As in G the stabilizer of a point and the stabilizer of a circle are conjugated subgroups, G_p fixes a circle c. As (IP) holds in $\mathcal{G}$ and there are 56 points and 56 circles, we have $|\Gamma(p)| = 10$, $|\Gamma_2(p)| = 45 = |\Gamma_3(p)|$, $|\Gamma_4(p)| = 10$ and $|\Gamma_5(p)| = 1$, and $c_2 = 2$, $c_3 = 8$, $c_4 = 9$ and $c_5 = 10$.

Since ϕ is a covering, ϕ maps bijectively the circles and the lines in $\mathscr{G}_p$ onto the circles and the lines in $\mathrm{res}(p^\phi)$, respectively. Moreover, by definition of the point–circle incidence graph, $\Gamma(p)$ represents the circles which are in $\mathscr{G}_p$ and, as each line is incident with exactly two points, $\Gamma_2(p)$ represents the lines in $\mathscr{G}_p$. Hence, as ϕ is a covering, ϕ maps $\Gamma(p) \cup \Gamma_2(p)$ bijectively onto $\Gamma(p^\phi) \cup \Gamma_2(p^\phi)$.

Let $u \in \Gamma_3(p)$ and $v \in \Gamma_2(p) \cap \Gamma(u)$. Then, as the restriction of ϕ on $\mathrm{res}(v)$ is an isomorphism between $\mathrm{res}(v)$ and $\mathrm{res}(v^\phi)$, we obtain $u^\phi \in \Gamma_3(p^\phi)$. Let $w \in \Gamma_3(p^\phi)$. Then there is some $x \in \Gamma_2(p)$, such that $x^\phi \in \Gamma_2(p^\phi) \cap \Gamma(w)$. Hence, as ϕ maps $\Gamma(p) \cup \Gamma_2(p)$ bijectively onto $\Gamma(p^\phi) \cup \Gamma_2(p^\phi)$, we obtain $w = y^\phi$ for some $y \in \Gamma_3(p) \cap \Gamma(x)$. Thus ϕ maps $\Gamma_3(p)$ onto $\Gamma_3(p^\phi)$.

Let $v \in \Gamma_2(p)$, then $\tilde{G}_{p,v} \simeq \mathbb{Z}_8\colon E_4$ and $\tilde{G}_{p,v}$ has two orbits on $\Gamma(v)$ of lengths 2 and 8, respectivley. Hence $\tilde{G}_p$ acts transitively on $\Gamma_3(p)$, which yields that each vertex in $\Gamma_3(p)$ has the same number of neighbours in $\Gamma_2(p)$, say $c_3(\tilde{\Delta})$.

We claim that $c_3(\tilde{\Delta})$ divides c_3. The covering ϕ induces an equivalence relation on $\Gamma_3(p)$. As $\tilde{G}$ acts on the fibres of ϕ, the stabilizer $\tilde{G}_p$ acts transitively on the classes of the equivalence relation and so each class has the same number of points. This gives that $|\Gamma_3(p^\phi)|$ divides $|\Gamma_3(p)|$. Hence, as a vertex in $\Gamma_3(p)$ ($\Gamma_3(p^\phi)$) has $c_3(\tilde{\Delta}) = 8$ neighbours in $\Gamma_2(p)$ ($\Gamma_2(p^\phi)$), we have that

$$c_3(\tilde{\Delta}) = \frac{8\,|\Gamma_2(p)|}{|\Gamma_3(p)|} = \frac{8\,|\Gamma_2(p^\phi)|}{|\Gamma_3(p)|} \quad \text{divides} \quad \frac{8\,|\Gamma_2(p^\phi)|}{|\Gamma_3(p^\phi)|} = c_3.$$

By Lemma 2.2, $c_3(\tilde{\Delta})$ is at least 3, so $c_3(\tilde{\Delta}) = 4$ or 8.

Assume that $c_3(\tilde{\Delta}) = 8$. By Lemma 2.2, a straightforward counting argument shows that $\tilde{\Delta} = \Delta$, in contradiction to Lemma 3.4. Hence $c_3(\tilde{\Delta}) = 4$ and, again by Lemma 2.2, the number of points of $\tilde{\mathscr{G}}$ is less of equal to $1 + 45 + 108 + 51 + 4 = 209$. Since $\tilde{\mathscr{G}}$ already has 112 points and the number of points of $\tilde{\mathscr{G}}$ divides the number of points of $\mathscr{G}$, we obtain $\tilde{\mathscr{G}} = \mathscr{G}$ as claimed. Moreover, as $\tilde{G}_p \simeq \mathrm{Aut}(A_6)$, the second part of the statement follows. □

3.4. *The geometry with circle size* $n = 7$

Let $\mathscr{G} = \mathscr{G}(U_3(3))$. The geometry can be seen as follows. The group $G \simeq U_3(3)$ has a rank 4 representation on 36 points on the coset of its subgroup $H \simeq L_3(2)$ with orbitals of lengths 1, 21, 7 and 7. Define a graph Δ, the vertices of which are the conjugates of H in G, two vertices being adjacent iff the corresponding subgroups intersect in a subgroup isomorphic to D_8. Then G has two orbits of 7-cliques, each of length 36. The group $\mathrm{Aut}(G)$, also acting on Δ, interchanges these two orbits. The points of Γ are the vertices and the circles are the 7-cliques in one of these two orbits. This example is due to [17] (see also [9]). Due to Lemma 2.2, and as the number of points in $\mathscr{G}$ divides the number of points in the universal cover, $\mathscr{G}$ is simply connected.

3.5. *The geometries with circle sizes* $n = 6$ *or* 5

Let $\mathscr{G} = \mathscr{G}(3S_6)$ or $\mathscr{G}(L_2(11))$. In [11] the first geometry is explicitly given. The second is a biplane on 11 points, i.e. any two points are incident with exactly two circles and any two circles with exactly two points.

Again by Lemma 2.2, the universal cover has at most 2^5 or 2^4 points, respectively. Hence in both cases $\mathscr{G}$ is simply connected.

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Appendix: Distribution Diagrams

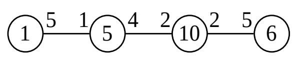

Figure A1. $L_2(11)$.

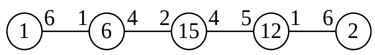

Figure A2. $3S_6$.

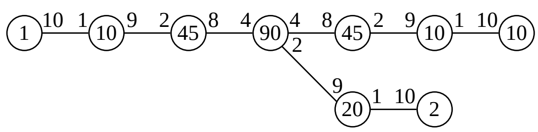

Figure A3. $2L_3(4)$.

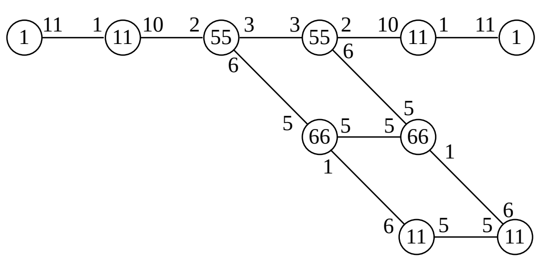

Figure A4. M_{12} of degree 11.

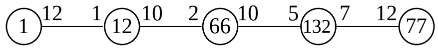

FIGURE A5. M_{12} of degree 12.

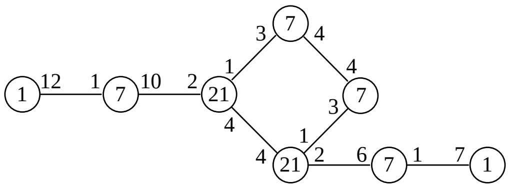

FIGURE A6. $U_3(3)$.

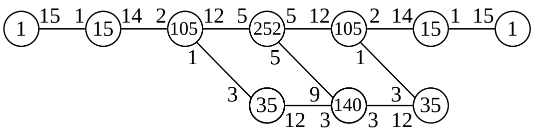

FIGURE A7. $2M_{22}$.

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