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A Model of Laminar-Turbulent Transition Based On Viscous Stream Buckling

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Abstract. The model of viscous streams' buckling is used to determine values of the local Reynolds numbers, for which transition to turbulence begins. These values are then used to estimate the global critical Reynolds numbers. The method discussed in this work has been used to determine the critical values of the Reynolds number in some well-known flows, e. g. flow over a flat plate, circular pipe flow and free jets. The values thus found agree well with the known critical Re.

Keywords: transition to turbulence, buckling theory.
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INTRODUCTION

Theoretical analysis of transition has seen a considerable amount of development from the traditional linear normal-mode stability theory [1], to the more advanced nonmodal stability calculations [2], which takes into account the short-term transient growth of perturbations. Even though the latter theory treats the responses to initial condition and external forcing within the same framework [3], and a more complete picture of the instabilities appear, further analysis will be needed to gain a complete picture of transition.

For pipe (Hagen-Poiseuille) flow, the linear stability theory predicts decay of disturbances at all Reynolds number. However, the inclusion of a dynamical system approach to the stability theory has lately given some insights to the presence of coherent states and travelling waves [4,5]. Similar phenomena have been observed in channel (plane Poiseuille) and Couette flow [6]. In the former case the critical Reynolds number from linear stability theory is 5772 [7], while for the latter unconditionally stable flow is again found. Even though the linear theory does provide a critical Reynolds number in the channel flow case, the transition (when performing an experiment) occurs sub-critically.

In controlled experiments, pipe flow has been maintained laminar until Re=100000 [8] while the lowest Re for which turbulent puffs have been observed is 1750 [5], and the lowest Re for which coherent structures have been observed is 773. For channel flow, the experiments by Patel & Head [9] gave a critical Re of 1012. The critical Re for Couette flow has been determined to be 370 from various experimental studies [10].

For boundary layer flow the critical Re from linear stability theory is around 91000, when non-parallel effects are accounted for, which has been confirmed experimentally by excitation of the Tollmien-Schlichting waves into the boundary layer. In uncontrolled experiments where wall roughness and free-stream turbulence affects the stability of the flow, the receptivity becomes important [12], which may lead to by-pass transition [13]. This type of transition is directly non-linear and the linear growth of disturbances is not necessary. The transition in experiments when the free-stream turbulence is minimized, occurs at around Re=2700000 [15]. For boundary layer flow travelling waves have also been observed [11]. The dynamics between the point of instability and transition depend on the secondary instabilities [14] which can be of different forms and eventually leads to turbulent spots before merging to a fully turbulent flow field [16].

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LOCAL CRITICAL RE

In 1996 Bejan [17] formulated his now famous Constructal Law:

For a finite-size system to persist in time (to live), it must evolve in such a way that it provides easier access to the imposed currents that flow through it.

This law implies that among the available modes of energy transport, that one is chosen, which provides an optimal (the fastest) energy delivery at a certain given distance. Applied to viscous fluid flow, where inertia forces compete with viscous forces, this means that there should exist a criterion to quantify this competition. Indeed, the time necessary to transfer

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momentum (energy) by convection at a certain given distance, δ, is

$$t_{\mathrm{conv}} = C\delta / U \tag{1}$$

where C is a proportionality constant that depends on the flow geometry and U denotes a characteristic flow velocity. In turn, the time necessary to transfer momentum (energy) by viscous diffusion at the same distance is

$$t_{\mathrm{diff}} = D\delta^2 / \nu \tag{2}$$

where D is a proportionality constant that depends on the flow geometry and ν denotes the kinematic viscosity of the fluid in question.

Thus, if t_{conv} is less than t_{diff}, transport by convection is preferred. If, however, the time span required by diffusion is shorter, the diffusion will be the dominant transport phenomenon. Consequently, in the case when t_{conv} and t_{diff} are of the same order of magnitude, both processes will be of equal importance. Equating the right-hand sides of eqns. (1) and (2) cancelling out δ, we obtain a criterion for choosing between the two ways of transport:

$$\delta u / \nu = C / D \tag{3}$$

which is nothing other than the requirement for the local Reynolds number (the Reynolds number based on the diffusion scale δ) to be of the order of C/D. In other words, the local Reynolds number may serve as an excellent criterion for showing the state of competition between the two transport processes: if $\mathrm{Re}_l \gg C / D$, diffusion is negligible and convection fully dominates; on the contrary, if $\mathrm{Re}_l = C / D$, diffusion becomes dominant with convection nonexistent. Hence, it becomes obvious that transition to turbulence occurs when $\mathrm{Re}_l = C / D$.

The constants C and D can be found from the model of buckling viscous streams proposed by Bejan [17].

According to Bejan [17], the transition from laminar flow to turbulent flow is characterized by two scaling laws.

(i) A universal proportionality between longitudinal wavelength and stream thickness, that is, by a meander or buckling phenomenon.

(ii) A critical local Reynolds number, based on the stream velocity scale and the stream thickness scale, provides a quantitative criterion of transition.

If the laminar-turbulent is characterized by the scaling law (i) and (ii), then the challenge to predict it reduces the problem of accounting for these scaling laws theoretically. We now present two alternative theoretical arguments, both capable of predicting the scaling laws (i) and (ii). The first argument is the most direct and is based on the buckling property of inviscid (irrotational) flow [17]. The second approach is based on reviewing the scaling implications of classical results known from the hydrodynamic stability analysis of inviscid (irrotational) flows [17].

An interesting analogy between the buckling of elastic solid columns and the meandering of inviscid (irrotational) streams results from considering the static equilibrium of a finite-size control volume drawn around the stream. If the stream and the control volume thickness is of order δ, and if the stream cross-section is A, then the control volume (or the thin-walled hose surrounding the stream) satisfies the two conditions necessary for sinusoidal infinitesimal buckling in elastic systems [17]:

(a) The control volume is in a state of axial compression subject to the impulse and reaction forces

$$F = \rho U^2 A \tag{4}$$

(b) If subjected to a separate bending test, the control volume develops in its cross-section a resistive bending moment that is directly proportional to the induced curvature

$$M = -\rho U^2 I \frac{d^2 z}{dR^2} \tag{5}$$

In this last expression, I is the area moment of inertia of the stream cross section, $I = \iint_A \varphi^2 dA$, while $(-z'')$ is the local curvature of the infinitesimally deformed control volume. Note also that Eqn. (5) is analogous to the expression $M = -EIz''$ derived from applying a separate bending test of prescribed curvature to a slender elastic beam. This means that in inviscid streams, the product ρU^2 plays the role of modulus of elasticity, a fact confirmed easily by trying to manually bend a thin-walled hose containing a high Reynolds number stream. The stream control volume possesses elasticity, that is, conservative mechanical properties, because in the inviscid flow limit, the material that fills the control volume is incapable of generating entropy.

Conditions (a) and (b) are essential to the static equilibrium of the control volume. The transitional equilibrium is evident, as the two forces F balance each other. However, as in Euler's buckling theory of solid columns, the rotational equilibrium must be preserved even when the two forces F are not perfectly collinear; hence,

$$-M(R)+Fz+M(R=0)=0 \quad (6)$$

or, substituting expressions (4) and (5),

$$(\rho U^2 I)z''+(\rho U^2 I)z+M(R=0)=0 \quad (7)$$

This static rotational equilibrium condition indicates that the static equilibrium shape of the nearly straight stream column is a sinusoid of vanishingly small amplitude and characteristic (*unique*) wavelength,

$$\Lambda_B = 2\pi\left(\frac{I}{A}\right)^{\frac{1}{2}} = \begin{cases} \dfrac{\pi}{2}\delta, & \text{circular cross-section} \\ \dfrac{\pi}{\sqrt{3}}\delta, & \text{rectangular cross-section} \end{cases} \quad (8)$$

i.e., in an order of magnitude sense, $\Lambda_B \sim 2\delta$.

The buckling wavelength is a geometric property of the finite-size control volume, a length about twice the transversal dimension δ. The $\Lambda_B \sim 2\delta$ scaling, predicted by the buckling theory of inviscid stream, accounts for the scaling law (i).

Before showing how the buckling property also accounts for the scaling law (ii), it is worth making the following observations:

1. The buckling wavelength of an inviscid stream is unique (and of order δ) because the compressive load $\rho U^2 A$ is always proportional to the elasticity modulus ρU^2. This feature sharply distinguishes the buckling of inviscid streams from that of elastic solid columns where F and E are independent. This is why in solid columns we encounter an infinity of Λ_B's (an additional degree of freedom) and, out of these, we must determine a discrete sequence of special Λ_B's that satisfy end-clamping conditions. In the case of inviscid streams, the buckling wavelength is unique, and end-boundary conditions are not an issue (where along the jet the first meander appears depends on the scaling law (ii), as is shown later in this section).

2. The buckling theory of inviscid streams invokes the static equilibrium of a finite-size region of the flow field and, as such, represents a dramatic departure from the methodology that prevails in contemporary fluid mechanics. Routine fluid mechanics analysis has as its starting point the Navier-Stokes equations, which account for mechanical equilibrium among infinitesimally small fluid packets $dR\ d\varphi\ dz$. However, the invocation of mechanical equilibrium in finite-size systems is no mystery to those familiar with engineering thermodynamics, especially with the thermodynamics of flow systems such as rotating machines and rockets.

3. Although the proportionality $\Lambda_B \sim 2\delta$ is universal, the control volume of transversal dimension δ has been selected arbitrarily. Any fluid fiber, that is, any control volume of thickness $\delta' \neq \delta$, satisfies conditions (a) and (b) for infinitesimal buckling. Out of this infinity of fibers, however, only a special class is in a state of unstable equilibrium. The instability of inviscid flow, the discovery that certain fluid fibers are unstable, is an entirely different flow property and the contribution of an entirely different theory (hydrodynamic stability).

4. The buckling property or the scaling law $\Lambda_B \sim 2\delta$ is widely observed in natural flows and can also be visualized in the laboratory. An extensive photographic record of such observations can be found in literature: among these, we note the river meandering phenomenon, the waving of flags and the meandering fall of paper ribbons, the buckling of fast liquid jets shot through the air, the wrinkling of two-dimensional fluid layers being pushed from one end, and the sinuous structure of all turbulent plumes.

Practically anyone can visualize the buckling scaling $\Lambda_B \sim 2\delta$ by placing an obstacle under the capillary water column falling from a faucet.

We now return to the scaling law (ii) armed with the idea that a stream has the property if it is inviscid. The inviscidity (or viscidity) of the stream, and hence, its vorticity, is a flow property, not a fluid property. It is inappropriate to refer to fluids such as honey and lava as viscous when, if the respective streams are wide and fast enough, they buckle (meander) in a way similar to rivers and water columns.

Since buckling is a property of inviscid stream, it can be argued that the laminar-turbulent transition is related to the plume's transition from the state of viscid stream to that of inviscid stream. In time, viscous (vorticity) diffusion penetrates in the direction normal to the stream-ambient interface so that in time of order

$$t_\nu \sim \frac{\delta^2}{16\nu} \tag{9}$$

the steam is fully viscous. The above time-scale follows from the error-function solution to the problem of viscous (*vorticity*) diffusion normal to an impulsively started wall; according to this solution, the time of viscous penetration to the stream centerline (to a depth $\delta/2$) obeys the scaling law

$$\frac{\delta/2}{2\sqrt{\nu t_\nu}} \sim 1 \tag{10}$$

Whether or not the stream becomes viscous depends on how fast it can buckle as an inviscid stream. The end-result of the incipient buckling analyzed early in this section is the birth of eddies, as the crests of the Λ_B waves roll at the stream-ambient interface. From symmetry, the Λ_B wave moves along the stream with a velocity of order $U/2$; hence, the buckling time or the time of eddy formation is

$$t_B \sim \frac{\Lambda_B}{U/2} \tag{11}$$

so that the frequency of eddy formation is $\omega \sim \frac{\pi U}{\Lambda_B} \sim \frac{\pi U}{2\delta}$. Note, that as δ increases, the frequency of eddy formation decreases.

The stream can buckle only if $t_B < t_\nu$; in other words, if the buckling frequency number $N_B = t_\nu/t_B$ is greater than one.

In conclusion, the time-scale argument presented above recommends the following criterion for transition:

$$N_B = \frac{t_\nu}{t_B}\begin{cases} <1 & \text{laminar flow} \\ =1 & \text{transition} \\ >1 & \text{turbulent flow} \end{cases} \tag{12}$$

The fascinating aspect of this criterion is that after replacing t_B and t_ν by $\frac{\Lambda_B}{U/2}$ and $\frac{\delta^2}{16\nu}$ respectively, it reads

$$\mathrm{Re}_l = \frac{U\delta}{\nu}\begin{cases} <100 & \text{laminar flow} \\ =100 & \text{transition} \\ >100 & \text{turbulent flow} \end{cases} \tag{13}$$

for circular streams. The group $(U\delta)/\nu$ is the local Reynolds number Re_l, which is based on the longitudinal velocity scale U and the transverse dimension of the stream δ.

Therefore, the local Reynolds number criterion (13) correctly predicts the scaling law (ii) for laminar-turbulent transition.

Note also that Re_l = 116 for the rectangular stream [see Eqn. (8)] and is of the same order of magnitude for most of other shapes. Nevertheless, the smallest value of the critical Re is achieved for circular streams.

GLOBAL CRITICAL RE

The task is now to provide a general expression that relates the local critical Reynolds number and the corresponding global Reynolds number, i.e. the Reynolds number based on a characteristic size of the flow domain, where the transition process begins, x_{cr}.

It follows from Eqn. (2) that $\delta_{\mathrm{cr}} = k\sqrt{x_{\mathrm{cr}}}$, where the value of the parameter k depends on the domain geometry.

Hence,

$$x_{\mathrm{cr}} = \left(\frac{16\nu\,\mathrm{Re}_{l,\mathrm{cr}}}{kU}\right)^2 \tag{14}$$

where the coefficient 16 is due to Eqn. (9). On the other hand,

$$x_{\mathrm{cr}} = \frac{\nu\,\mathrm{Re}_{\mathrm{cr}}}{U} \tag{15}$$

Combining (14) and (15), we obtain

$$\mathrm{Re}_{\mathrm{cr}} = \frac{128\nu}{U}\left(\frac{\mathrm{Re}_{l,\mathrm{cr}}}{k}\right)^2 \qquad (16)$$

Equation (16) represents a relationship between the local and global critical values of the Reynolds number; this relationship is universal.

MODEL VALIDATION

Consider now a flow over a flat plate, for which the boundary layer thickness is given by

$$\delta = \left(1.83\sqrt{\frac{\nu}{U}}\right)\sqrt{x} \qquad (17)$$

Hence, the parameter k in (16) is $\left(1.83\sqrt{\frac{\nu}{U}}\right)$ for the flow over a flat plate. Hence, for $\mathrm{Re}_{l,\mathrm{cr}} = 116$ (buckling of rectangular viscous streams), it follows from (16) that the critical value of the Reynolds number is 514309, which is in excellent accord with the experimental value.

CONCLUSIONS

The model of viscous streams' buckling has been used to determine values of the local Reynolds numbers, for which transition to turbulence begins. A universal relationship that relates critical values of the local Re with the critical values of the global Re has been derived. Then, the method discussed in this work has been used to determine the critical values of the Reynolds number in some well-known flows, e. g. flow over a flat plate, circular pipe flow and free jets. The values thus found agree well with the known critical Re.

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