Essays on multi-echelon inventory control

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**ESSAYS ON MULTI-ECHELON INVENTORY
CONTROL**

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Essays on Multi-Echelon Inventory Control

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I would like to dedicate this thesis to my wife. Without her constant love and unselfish support throughout these years, I would not have been able to accomplish this.

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Abstract

Matching demand with supply effectively is a significant objective for supply chain inventory management. Efficient supply chain management requires the effective coordination of shipments from one stage to the next throughout the entire system.

The first essay of this thesis studies fixed-interval ordering policies in a serial system. In particular we consider an echelon-stock fixed-interval order-up-to policy for an N -stage serial supply chain. For reasons of mathematical tractability and historic convention, previous studies on periodic-review inventory control policies have typically accounted system costs at the end of each review period. This accounting method, however, often significantly underestimates system costs when inventory related costs actually accrue in continuous time. The main contribution of the current study is two-fold. First, we provide a simple approach to evaluate the inventory related costs in the system continuously in time. This evaluation approach uses only the demand distribution, is easy to follow and compute and can also be used when end-of-period cost accounting is adopted. Second, we provide an intuitive characterization of an optimal ordering policy by simply equating the marginal echelon inventory related cost to the

inventory holding cost at the upstream stage. This leads to a simple bottom-up procedure to identify an optimal ordering policy for a given replenishment schedule. Using this solution, we develop a top-down search procedure to identify an optimal replenishment schedule subject to the optimization of the ordering policy.

The second essay of this thesis considers a two-level distribution system, in which one central warehouse orders from an external supplier in an interval of a fixed length to supply a group of non-identical retailers. The retailers face independent Poisson processes and order from the warehouse in fixed intervals that are integer-ratio multiples of the warehouse replenishment interval. Optimal inventory control policies for two-level distribution systems are notoriously complicated because of the stock allocation problem when shortage occurs at the warehouse. However, finding warehouse stock allocation policies that are optimal or near-optimal and easy to implement is of practical importance. We assume that the system adopts a base-stock policy, in which the warehouse orders at each review point from the external supplier to raise the echelon-stock inventory position to a fixed base stock level. We investigate different allocation policies for the warehouse to allocate stock to retailers. First, we consider a class of stationary ship-up-to- S policy, in which each retailer's inventory position is raised to a fixed base-stock level at a shipping point. When facing a shortage, we consider two myopic allocation policies: (1) virtual allocation, where the remaining stock is allocated to satisfy the demand

on a first-come first-served basis, (2) myopic optimal allocation, where the remaining stock is allocated to the retailers optimally to minimize system cost. Numerical experiments show that myopic optimal allocation performs better than virtual allocation, particularly in cases with large number of retailers, high service level requirements, large demand rate, large difference between retailers, and short system replenishment cycle. We note that the myopic allocation policies are stationary and myopic in the sense that base-stock levels for retailers are stationary and the warehouse stock will be completely allocated to retailers when facing a shortage. Therefore we construct a heuristic dynamic allocation rule, which explicitly incorporates the benefits of centralizing stock. We demonstrate through numerical experiments that the heuristic allocation rule on average performs at a comparable level with myopic optimal allocation in terms of cost control. However, this heuristic significantly reduces the complexity of system optimization by considering only echelon order-up-to level at the warehouse and can cope with a relatively large number of retailers, a large demand arrival rate and a long replenishment cycle.

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Chapter 1

Introduction

A supply chain consists of all organizations involved directly or indirectly in the provision of a product and/or service required by end customers. Inventory management in a supply chain spans all movement and storage of raw materials, work-in-process inventory, and finished goods from point of origin to point of consumption. Traditionally, inventory decisions are made locally at each stocking point. However, companies have long realized that they can achieve tremendous benefits by integrating their inventory operations with their business partners. The well-known example is the implementation of vendor managed inventory (VMI) in practice. VMI is a streamlined approach to inventory management and order fulfillment. VMI involves collaboration between suppliers and their customers (e.g. distributors, retailers). In VMI, the buyer of a product provides inventory information to a vendor supplier of that product and the supplier takes full responsibility for maintaining the inventory, usually at the buyer's consumption location. Contemporary supply chain management calls for the application of a total systems approach to supply chain inventory management (Jacobs et al.

2009). Although supply chain management has become an important management paradigm, the optimal control of a stochastic multi-echelon supply chain inventory system is still largely an open issue (Wang 2011). The management of multi-echelon inventory systems is a crucial part of supply chain operations. The overall goal is, in general, to minimize the costs for ordering, for capital tied up in the supply chain, and for not providing an adequate customer service (Axsäter 2003).

1.1 Multi-Echelon Inventory Systems

Multi-echelon inventory systems are common in supply chains, in both distribution and production. A distribution system is established when products are distributed over large geographical areas. To provide good service, local stocking points close to the customers in different areas are needed. These local sites may be replenished from a central warehouse close to the production facility. In production, stocks of raw materials, components and finished products are coupled to each other in a similar way. The management of multi-echelon inventory systems is important for both inbound and outbound of companies.

In essence, the management of multi-echelon inventory system is to effectively match demand with supply. How to build such an effective system is an objective for supply chain inventory managers. To ensure timely delivery of the products at the lowest possible cost, they must pay close attention to the coordination of the shipments from one stage to the next throughout the supply chain. The possibilities for efficient control of multi-echelon inventory systems have increased substantially during the last two decades. One reason is the development of

new information technologies, which have dramatically increased the technical possibilities for supply chain coordination. In the following we first introduce typical structures of multi-echelon systems and then we provide a general overview of the literature.

1.1.1 System Structures

There are various supply chain network structures. The simplest is a serial system that has a single facility at each stage. The next simplest structure is an assembly system in which multiple parts are assembled into a single component and, therefore, a single facility can have multiple suppliers. A distribution system supplies a product to many customers and, therefore, can have multiple customers for a supplier. A tree system combines assembly systems and distribution systems, and a general system can include any of the above as part of the system (Zipkin 2000). A prototype network structure for previous studies on multi-echelon inventory systems is a two-level distribution system whereby a central warehouse supplies a product to a group of retailers. This structure includes a serial system as a special case. An assembly system can also be considered as a special case as it can be decomposed into multiple serial systems under certain conditions (Rosling 1989).

The main focus of this thesis is on serial system and distribution inventory systems with stochastic demand. Figure 1.1 illustrates a serial system with N stages. Figure 1.2 depicts a two-level distribution system with a central warehouse and a number of retailers. Figure 1.3 illustrates an assembly system with N suppliers and one manufacturer. In a distribution system each installation has

at most one predecessor. In the opposite case where each installation has at most one successor we have an assembly system. Such systems are common in production. A serial system is obviously a special case both of a distribution system and of an assembly system. More importantly, these typical structures include all the fundamental issues for a multi-echelon inventory control system and yet not have the complexity of a general system for the ease of analysis.

Figure 1.1: N -stage serial system

Figure 1.2: Two-level distribution inventory system

Figure 1.3: Two-level assembly inventory system

In practice we often meet general systems, where some installations have multiple predecessors as well as multiple successors. Such systems are very difficult

to handle by scientific methods. Assembly to order system is such a system that contains not only assembly systems but also distribution systems. For an excellent review on assembly to order system, please refer to Song and Zipkin (2003).

1.1.2 Literature Overview

The study of multi-echelon inventory system begins with deterministic demand. The early literature focuses mainly on stationary-nested policies. Assume that each facility orders at equally-spaced points in time. A nested policy means that each facility orders every time when any of its immediate suppliers does. The seminal work of Roundy (1985, 1986) has typically considered such stationary-nested policy, i.e., power-of-two policy, which requires that products are replenished at constant intervals and these replenishment intervals are all power-of-two multiples of a common base planning period. They showed that this policy can guarantee a 98% worst-case cost bound. The reader is referred to Maxwell and Muckstadt (1985), Roundy (1985), Federgruen and Zheng (1993) and Li and Wang (2007) for more specific review on deterministic demand models.

In reality, the demand is generally stochastic. Therefore later work typically considers stochastic demand for multi-echelon inventory systems. The seminal paper by Scarf and Clark (1960) considered a serial system with stochastic demand. They proved that base-stock policies based on echelon inventory positions are optimal and the optimal base-stock levels are obtained by the minimization of one-dimensional convex cost functions. Subsequent extensions have considered systems with fixed batch sizes or fixed replenishment intervals. These studies

have considered two basic inventory policies: (i) the echelon-stock continuous-review (r, Q) policy, in which a fixed batch of size Q is ordered when the echelon inventory position (inventory on order plus inventory on hand plus inventory in transit to and at its downstream stages minus backorders at the most downstream stage) drops to a reorder point r , and (ii) the echelon-stock periodic-review (S, T) policy, in which the echelon inventory position is reviewed in a fixed interval of length T and stock is ordered at a review point to raise the inventory position to a fixed order-up-to level S .

For a single installation, the (r, Q) policy dominates the (S, T) policy because the former is able to use more updated stock information to optimize system performance (Hadley and Whitin 1963). Probably because of this reason, multi-echelon inventory control models based on the (r, Q) policy have been much more common in the literature. Most of the work on the (r, Q) policy focuses on developing approximate or exact total cost expressions and then optimizing inventory policies. A few notable examples include Axsäter (1990, 1993), Axsäter and Rosling (1993), Cachon (2001), Chen and Zheng (1994). Chen and Zheng (1998) provided an algorithm to optimize the echelon-stock (r, Q) policy for a N -stage serial system. Cachon (2001) studied a distribution system with a batch ordering policy. By assuming a random allocation rule, he provides a method to exactly evaluate the average inventory, backorders and fill rates for each location in the system. Gallego et al. (2007) developed simple approximate methods to analyze a distribution system in which every installation uses a $(S - 1, S)$ policy. There are also studies that provide simple heuristics on the (r, Q) policy, e.g., Shang and Song (2007) and Shang (2008).

Although models with multi-echelon (r, Q) policies are more common in the

literature, periodic-review inventory control policy is much more common in practice and has attracted considerable interest recently, because they provide an efficient mechanism to pool stocks and coordinate inventory activities for multi-echelon inventory systems. The (S, T) policy was first discussed by Hadley and Whitin (1963). Naddor (1975) studied the (S, T) policy for single- and multi-item systems. Cachon (1999) studied the reorder-interval policy in a distribution system. Graves (1996) derived an exact evaluation for base-stock policies in continuous-time, multi-echelon distribution systems with fixed replenishment intervals. Recently, van Houtum et al. (2007) studied a serial system and show that the echelon order-up-to policies are optimal when the reorder intervals are fixed. Chao and Zhou (2009) extended these results to the echelon (r, nQ, T) policies. They provided an algorithm to obtain the optimal reorder points with fixed batch sizes and reorder intervals.

A recent interest is to optimize the length of order intervals for multi-echelon inventory systems. For example, Feng and Rao (2007) considered this problem for a two-stage serial system. Shang and Zhou (2010) studied a N -stage serial inventory system that adopts an echelon-stock (r, nQ, T) policy. Shang et al. (2015) applied the echelon-stock (S, T) policy into a periodical review distribution system with virtual allocation and try to optimize replenishment scheduling. Wang and Axsäter (2013) studied a fixed-interval ordering policy in a distribution system with a myopic stock allocation policy at the warehouse.

For serial systems, optimality of inventory policies has been analyzed by Clark and Scarf (1960), van Houtum (2007) and Chao and Zhou (2009). For distribution systems, the optimal control policy is still unknown. It is generally recognized that, even if it exists, the optimal policy can be very complex (Clark and Scarf

1960). This is because in the presence of multiple retailers, the warehouse must adopt an additional policy to allocate the remaining stock to multiple retailers when facing a shortage. The literature generally focuses on heuristic inventory control and allocation policy.

Jackson (1988) considered a distribution system with a class of ship-up- S policy in one replenishment cycle. The inventory policy specifies that at the start of each interval, the inventory positions of the retailers are raised to their prescribed order-up-to levels as long as sufficient central stock is available. When facing a shortage, the warehouse will allocate all remaining stock among the retailers to minimize their costs over the rest of the cycle, which is referred to as myopic optimal allocation. Graves (1996) first proposed the virtual allocation rule in a periodical-review setting. It assumes that warehouse stock is allocated or reserved to satisfy demand at retailers on a first-come-first-served basis. Axsäter (1990, 1993) provided an exact cost evaluation approach for distribution system with continuous review by assuming the first-come-first-serve allocation rule. Shang et al. (2015) considered a periodic-review distribution system with virtual allocation and fixed-interval order-up-to policy. The literature on virtual allocation generally assume Poisson demand as it provides tractable property for exact evaluation of average system costs. Although virtual allocation is tractable from a mathematical point of view, it is not optimal, since it does not account for differences in needs for stocks at downstream stages as in myopic optimal allocation.

For distribution systems, another stream of literature analyzes the benefits of centralizing inventory at the warehouse, which generally include three categories: (1) external lead time inventory pooling, which can reduce the uncertainty faced by the system by carrying a single inventory during the warehouse lead time rather

than individual retailer inventories (Eppen and Schrage 1981), (2) holding cost saving at the central warehouse (or upstream) as the holding cost at the upstream of a supply chain is usually smaller than that at downstream (Shang and Song 2003, Wang and Wan 2015), (3) rebalancing inventory among retailers during replenishment cycle. That is, the warehouse replenishment is not completely allocated, but a part is kept back to rebalance local stocks at additional shipment opportunities later on in the replenishment cycle. This postponement causes that the out of stock risk is pooled, with the drawback that the central stock is not immediately available to satisfy customer demand (McGavin et al. 1993).

For more information on stochastic multi-echelon inventory systems, interested readers are referred to excellent reviews by Axsäter (2003), van Houtum (2006), and Simchi-Levi and Zhao (2007), and Wang (2011).

1.2 Essays Overview

Although multi-echelon inventory systems that implement (r, Q) policies have been extensively studied, studies on the (S, T) policy are relatively sparse (Shang, 2011). Actually, fixed-interval ordering has been recognized as common practice to facilitate freight consolidations and logistics/production scheduling (Graves, 1996). For example, in production process, it can smooth workloads over time by coordinating significant setup times for heavily loaded machines/production line. Further, it may facilitate coordination with other stages in the supply chain and other functions such as workforce planning, maintenance, and warehousing ((van Houtum et al. 2007)). In distribution networks, regularly scheduled shipments are a common practice to achieve an efficient utilization of transportation resources,

such as container ships and terminals. It facilitates a simple coordination and scheduling of the delivery activities (order picking, loading, unloading, receiving, etc.) at the warehouse and the retailers (Marklund 2011).

Therefore, this thesis will focus on fixed-interval ordering inventory policy. Specifically, we will consider how to optimize (S, T) policy in a serial system with continuous stochastic demand in the first essay and study the allocation problem in a stochastic distribution system in the second essay. We provide an overview for both essays below.

1.2.1 Fixed-interval Order-up-to Policy in Serial Systems

In the first essay, we consider an echelon-stock fixed-interval order-up-to policy in a stochastic serial system. For reasons of mathematical tractability and historic convention, previous studies have typically accounted inventory related costs at the end of each review period (Clark and Scarf 1960, Chen and Zheng 1994, Chen 2000, van Houtum et al. 2007, Chao and Zhou 2009, Shang and Zhou 2010). Inventory related costs, however, often accrue continuously in time in reality. Rudi et al. (2009) studied the effect of end-of-period cost accounting recently. They find that, compared to continuous-time cost accounting, end-of-period cost accounting using continuous-time cost parameters often significantly underestimates inventory related costs.

We first provide an approach to continuously evaluate the expected inventory related costs in the system and subsequently provide a simple way to characterize an optimal solution. We show that under an optimal solution, the marginal cost at an echelon is equal to the inventory holding cost at the upstream stage. This

is intuitive. Consider a unit of stock at stage j at a review point of stage $j - 1$ and the decision between retaining it at stage j or shipping it to the downstream echelon $j - 1$. The marginal cost of the first option is the holding cost h_j . Apparently, this unit of stock should be shipped to the downstream echelon $j - 1$ if the marginal cost at echelon $j - 1$ is lower than h_j . Following this intuition, an optimal inventory position (or order-up-to level) at the downstream stage $j - 1$ is reached when the two marginal costs are equal. This finding provides a simple characterization of an optimal ordering policy for a serial inventory system. Under this decision rule, an optimal base stock level at a stage is determined independently of the base stock levels and reorder intervals at the upstream stages. This leads to a simple bottom-up recursive approach to identify an optimal ordering policy for a given replenishment schedule. Subsequently, given the reorder intervals at the upstream stages, we establish a feasible range for an optimal reorder interval at stage j using the sub-system that consists of stage j and all the upstream stages. This leads to an efficient top-down exhaustive search approach to identify an optimal system replenishment schedule subject to the optimization of the base stock levels.

1.2.2 Stock Allocation in Distribution Systems

The second essay considers a two-level distribution inventory system where a central warehouse supplies a group of non-identical retailers. Customer demands arrive at retailers according to independent Poisson processes. The warehouse orders from an external supplier with ample stock and the retailers order from the warehouse. We assume that the warehouse uses a fixed-interval base-stock

policy, in which the warehouse orders from the external supplier in a fixed replenishment cycle, raising the echelon inventory position to a fixed base stock level at each ordering point. The retailers also adopt fixed-interval base-stock policies, in which each retailer orders in a fixed interval that is an integer-ratio multiple of the warehouse replenishment cycle to raise its inventory position to a fixed order-up-to level at each review point. Our main objective of this study is to investigate stock allocation policies. To focus on this objective, we assume that the replenishment intervals of the warehouse and the retailers are determined exogenously based on operational schedules with other partners. In addition, we assume that material flows through the network are coordinated by a central authority that has full knowledge on the inventory status of the system.

An allocation policy decides on the amount of stock to be kept at the warehouse and to be shipped to each retailer in each shipping period. The literature generally considers virtual allocation and myopic allocation separately. Virtual allocation means that all demands will be satisfied on a first-come-first-served basis. That is, whenever a unit is demanded at one of the retailers, the warehouse assigns one unit in stock or on order to this retailer. At a replenishment period for this retailer, all assigned inventory units at the warehouse are shipped to this retailer. The virtual allocation rule is commonly seen in practice (Shang and Zhou 2015). For example, Wal-Mart's distribution center assigns replenishment stocks to the demands as they occur. The assigned stocks are loaded onto a truck and shipped to the retail stores according to a fixed schedule (Chandran 2003). Readers are referred to Graves (1996) for a detailed discussion of its applications. However, virtual allocation does not account for differences in needs for stocks at downstream stages as in myopic optimal allocation. Myopic optimal

allocation means that when shortage occurs, the available inventory is allocated to minimize the cost incurred at the retailers for the rest of the cycle. A comprehensive comparison between them is in order, which is one of the focuses in our study.

First, we consider stationary ship-up-to- S policy. Under this policy, at the beginning of the warehouse's each ordering cycle, the warehouse places an order to raise the echelon inventory position to S_0 . When the order arrives, the warehouse replenishes retailers' inventory position to their base-stock levels at every shipping point if the warehouse has sufficient inventory on hand. When the warehouse faces a shortage, we consider myopic allocation policies, which means that the available stock at the warehouse is completely allocated to all retailers at the shortage moment. We develop an evaluation approach for evaluating average cost in a single system replenishment cycle for this general myopic allocation policy. Further, we consider two specific myopic allocation rules: virtual allocation and myopic optimal allocation. We also discuss how our approach can be extended to infinite horizon or multiple replenishment cycles. Then we provide an optimization approach to find the optimal stock decisions. Numerical experiments are conducted to compare the two allocation policies. The results show that myopic optimal allocation performs better than virtual allocation, particularly in cases with large number of retailers, high service level requirements, large demand rate, large difference between retailers, and short system replenishment cycle.

Further, we note that there are several concerns for myopic allocation policies. First, although myopic optimal allocation takes relative needs of retailers into account, base-stock levels for the retailers are stationary, which means that they can not be adjusted according to the actual realized demand. Intuitively, there

may exist non-stationary or dynamic policies that could perform better. Second, it requires $N + 1$ control variables. To find their optimal control values, it can be very complex and time-consuming, especially for large number of retailers, high arrive rate and long replenishment cycle. Third, the formulations and the methodologies developed in multi-echelon production and distribution systems are often computationally intractable and difficult to explain to practitioners (Gallego et al. 2004). This is particularly a concern for implementation. Firms may not be willing to implement the optimal solution without understanding what is in the black box. The evaluation and optimization approaches we develop for myopic allocation policies also face this obstacle.

To cope with these challenges, we aim to propose a heuristic dynamic allocation policy, which not only could perform potentially better in terms of cost control but also is easy-to-describe, compute and implement. With this motivation, we establish a computationally efficient heuristic dynamic allocation policy, which explicitly incorporates the three major benefits of stock centralization. For the heuristic allocation policy, it would be extremely difficult to evaluate the long-run average system costs exactly, if not impossible. Therefore we resort to a simulation approach. We demonstrate through an extensive numerical study that the novel heuristic allocation rule on average performs at a comparable level with myopic allocation policy in terms of cost control and sometimes much better. Most importantly, the heuristic policy significantly reduces the complexity of system optimization by considering only system order-up-to level and therefore can cope with a relatively large number of retailers, a large demand arrival rate and a long replenishment cycle. Practitioners should find this policy easy to understand and implement.

1.3 Thesis Structure

The rest of the thesis is organized as follows. In Chapter 2, we present the first essay on stochastic serial system with echelon-stock fixed-interval order-up-to policies. In Chapter 3, we present the second essay on inventory allocation policies in distribution systems. Finally we conclude in Chapter 4 with possible future research directions.

Chapter 2

Fixed-Interval Order-up-to Policies for Serial Systems

2.1 Introduction

Matching demand with supply effectively is an objective for supply chain inventory managers. When companies design and manage their supply chain networks, economy of scale is one of the primary concerns (Shang and Song 2007). To ensure timely delivery of the products at the lowest possible cost, they must pay close attention to the coordination of the shipments from one stage to the next throughout the supply chain.

In this study, we consider an echelon-stock fixed-interval order-up-to policy in a serial system. Fixed-interval ordering has been recognized as common practice to facilitate freight consolidations and logistics/production scheduling (Graves 1996). In particular, we consider an N -stage ($N \geq 2$) serial inventory system. Customer demand arises at only stage 1. Stage 1 orders from stage 2, stage

2 orders from stage 3, etc., and stage N orders from an external supplier. It is assumed that the external supplier has unlimited capacity and hence can meet an order from stage N immediately. This assumption is reasonable because usually in the most upper stage of a supply chain, the product is possibly still in the form of raw materials (or commodity product), which generally can be directly ordered from spot market etc. This is also a common assumption in the literature, such as Axsäter (1993), Graves (1996), Shang et al. (2015). An internal stage j ($j = 1, \dots, N$), however, can meet the demand from the next downstream stage only when stock is available. Demand that is not satisfied by on-hand inventory is fully backlogged. Ordered stock is delivered after a constant leadtime. The system incurs (i) a fixed cost at each review point at each stage, (ii) a linear cost for holding one unit of stock per unit time at each stage, and (iii) a linear cost for backlogging a unit of demand per unit time at stage 1. Note that the review cost is the cost associated with an inventory review. For instance, a manager may have to physically review the inventory status at each inventory order period. Sometimes the review cost may include a shipping cost. An example in Shang and Zhou (2010) is EMC², a leading manufacturer for database servers in the United States. At EMC², the shipping cost per truck is specified in a contract with United Parcel Service (UPS). Consequently, the shipping cost is always incurred in a customer's inventory review period even if there is no material to ship from EMC² (e.g., when there are no orders or insufficient supply).

Figure 2.1: N -stage serial system

Multi-echelon inventory systems have been extensively studied. Control poli-

cies have been developed based on essentially two basic control policies: (i) the continuous-review (r, Q) policy, in which a fixed batch of size Q is ordered when the echelon inventory position (inventory on order plus inventory on hand plus inventory in transit to and at its downstream stages minus backorders at the most downstream stage) drops to a reorder point r , and (ii) the periodic-review (S, T) policy, in which the echelon inventory position is reviewed in a fixed interval of length T and stock is ordered at a review point to raise the inventory position to a fixed order-up-to level S . For a single installation, the (r, Q) policy dominates the (S, T) policy because the former is able to use more updated stock information to optimize system performance. Probably because of this reason, multi-echelon inventory control models based on the (r, Q) policy have been much more common in the literature. Nevertheless, periodic-review inventory control policies have attracted considerable interest recently because they provide an efficient mechanism to pool stocks and coordinate inventory activities for multi-echelon inventory control systems. Wang (2013) demonstrated that a periodic-review inventory control policy can out-perform a continuous-review inventory control policy for a two-level distribution system. In addition, it is well known that periodic-review inventory control policies are easy to implement and common in practice (Graves 1996, van Houtum et al. 2007, Shang and Zhou 2010).

We consider a periodic-review control policy for the serial inventory system. Specifically, we assume that stage 1 orders in a fixed interval. Under this condition, it is easy to prove that an optimal inventory control policy for a serial inventory system is synchronized and nested. This condition requires that a stage order in a fixed interval that is an integer multiple of the reorder interval at the downstream stage and place an order when the upstream stage receives a

shipment from its supplier. In addition, given the reorder intervals, van Houtum et al. (2007) proved that a base stock policy is optimal. These conditions lead to a fixed-interval order-up-to policy for the inventory system, in which each stage follows an echelon-stock (S, T) policy. According to Rao (2003), the (S, T) policy can be applied when the demand process is stationary and has independent increments. This condition holds true for the commonly used Poisson demand processes and the Normal demand model. We assume this condition for the customer demand process at stage 1 and present the analysis for the case of continuous (Normal) demand in this study. It is our understanding that the analysis for the case of discrete (Poisson) demand is similar.

The main contribution of the study is two-fold.

First, we provide a simple approach to evaluate the inventory related costs in the system continuously in time. For reasons of mathematical tractability and historic convention, previous studies have typically accounted inventory related costs at the end of each review period (Arrow et al. 1951, Bellman et al. 1955, Clark and Scarf 1960, Chen and Zheng 1994, Chen 2000, van Houtum et al. 2007, Chao and Zhou 2009, Shang and Zhou 2010). Inventory related costs, however, often accrue continuously in time in reality. Rudi et al. (2009) studied the effect of end-of-period cost accounting recently. They find that, compared to continuous-time cost accounting, end-of-period cost accounting using continuous-time cost parameters often significantly underestimates inventory related costs. This is not surprising because the ending inventory level is always lower than the average inventory level in a review period. They also point out that the end-of-period accounting scheme may inaccurately reflect costs, particularly when stock-outs result in backorders. For example, while the cost of stock-outs may

depend on both the duration and quantity of the stock-out, the end-of-period scheme only considers the quantity. Taking snapshots of inventory at the end of a review period tends to reflect a condition in which the company is least exposed to inventory holding costs and most exposed to stock-outs. This calls for the development of exact continuous-time cost evaluations for periodic-review inventory control models.

Under end-of-period cost accounting, a common method is to use the limiting distribution of the end-of-period echelon inventory level (i.e., inventory on hand or in transit minus backorder at stage 1) to evaluate the expected on-hand inventory and backorder (Clark and Scarf 1960, Chen and Zhen 1994, Chao and Zhao 2009). The application of this method, however, is often difficult when inventory related costs accrue continuously in time. To the best of our knowledge, three studies have considered continuous-time cost evaluations for periodic-review multi-echelon inventory control policies. Axsäter (1993) showed that, when customer demands are satisfied on a first-come first-served basis in a two-level distribution system, a periodic-review policy in which all retailers adopt a common reorder interval can be analyzed in essentially the same way as a continuous-review base stock policy. Feng and Rao (2007) adopted the continuous-time cost evaluation for the single-stage (S, T) policy from Rao (2003) and use simulation to evaluate the inventory related cost for a two-stage serial system. Wang (2013) introduced a method to evaluate the inventory related costs at retailers in a two-level distribution system based on the stocking conditions at the warehouse. We provide a different approach to evaluate a serial inventory system. We first observe that, at echelon 1 (i.e., stage 1), the expected inventory related cost in an ordering cycle can be exactly evaluated given the initial inventory position excluding backorders

at stage 2. Subsequently, for each echelon $1 < j \leq N$, we show that the expected inventory related cost in an ordering cycle can be evaluated by a simple expectation of the inventory related cost at the downstream echelon $j - 1$ with respect to the inventory level at the most upstream stage j . This approach uses only the demand distribution, is easy to follow and compute and can be used under any cost accounting method.

Second, we provide a simple way to characterize an optimal solution. Clark and Scarf (1960), Federgruen and Zipkin (1984), Chen and Zheng (1994), and van Houtum et al. (2007) demonstrated that an echelon-stock base stock policy is optimal for a serial inventory system with periodic review. Under the echelon-stock evaluation approach, a cost charge is defined for each echelon. The optimal solution is found by setting the base stock level at stage 1 to minimize the echelon cost charge. Subsequently, for each stage $j > 1$, a penalty cost is introduced for keeping a stock that is insufficient to meet the demand from the downstream stage and an optimal base stock level is found by minimizing the total of the echelon cost charge and the penalty cost given the base stock levels at the downstream stages. Using the echelon-cost evaluation approach, we derive a simple way to determine an optimal base stock level at each stage. Consider a unit of stock at stage $j + 1$ at a review point for the downstream stage j . Intuitively, this unit of stock should be shipped to stage j if the marginal cost at echelon j is lower than the cost for holding it at stage $j + 1$. Consequently, an optimal base stock level at stage j is reached when the two costs are equal. Indeed we prove that a unique optimal ordering policy is determined by this condition at each echelon given the base stock levels at the downstream stages. This leads to a simple bottom-up recursive approach to identify an optimal ordering policy for

a given replenishment schedule. Subsequently, given the reorder intervals at the upstream stages, we establish a feasible range for an optimal reorder interval at stage j using the sub-system that consists of stage j and all the upstream stages. This leads to an efficient top-down exhaustive search approach to identify an optimal system replenishment schedule subject to the optimization of the base stock levels.

We conduct a numerical study to examine several issues. First, we observe the characteristic of the optimal solution. With the explicit characterization of an optimal ordering policy, we focus on the reorder intervals. The numerical study demonstrates that the optimal reorder intervals retain largely the basic properties of the deterministic economic order quantity model. Second, we examine the impact of the cost accounting methods. We replicate the analysis using the end-of-period cost accounting method in which inventory related costs are evaluated at the end of each period of length one (van Houtum et al. 2007, Chao and Zhou 2009, Shang and Zhou 2010). According to the numerical study, end-of-period cost accounting significantly underestimates system cost as compared to continuous-time cost accounting (ranging from 4.30% to 15.32% with an average of 8.91% and a standard deviation of 2.25%). Consequently, an optimal solution that is developed using this cost accounting method can cause substantial cost inefficiency when inventory related costs actually accrue in continuous time. Finally, we investigate whether the deterministic optimal solution can provide a good approximation. Unfortunately, the numerical study shows that the optimal replenishment schedule is often very different from the deterministic solution. This is reasonable. When demand is stochastic, stock should be placed closer to customers in order to reduce stockout at stage 1. To this end, inventory should be

shipped from upstream stages to downstream stages more quickly and in larger quantities. This leads to longer reorder intervals at the downstream stages as compared to the deterministic case. According to our numerical study, the optimal reorder interval at stage 1 is always longer in the stochastic case than in the deterministic case.

The rest of this chapter is organized as follows. Section 2.2 describes the inventory control system. Section 2.3 evaluates the long-run average system cost. Section 2.4 discusses the optimization of the inventory control system. Section 2.5 presents a numerical study. Section 2.6 concludes the study.

2.2 Literature Review

Multi-echelon serial systems with economies of scale have attracted a lot of attention, for example, Chen and Zheng (1994, 1998), Chen (2000), van Houtum et al. (2007), Chao and Zhou (2009), Shang and Zhou (2010) and the references therein. These literature have been built on essentially two basic inventory policies: continuous review (r, Q) policy and periodic review fixed-interval order-up-to (S, T) policy. Companies usually employ different policies to replenish inventory. For some products, if the responsiveness of inventory replenishment is an important concern, these products are usually replenished according to continuous-review policies. The biggest advantage of the fixed-interval ordering policy is simplicity — it is simpler to coordinate orders, shipments, and labor according to some fixed schedules. Nevertheless, the downside of the periodic-review policy is that it may not be as responsive as the continuous-review policy. For single-stage systems, Hadley and Whitin (1963) compared the (r, Q) and (S, T) policies and show that

the (r, Q) policy is superior to the (S, T) policy. Shang et al. (2010) compared the (r, Q) and the (S, T) policies in a serial system and found (S, T) policies to be cost-effective. Wang (2013) demonstrated that a fixed-interval ordering policy can out-perform a continuous-review inventory policy for a two-level distribution system with numerical experiments.

Models with given echelon-stock (r, Q) policies have been much more common in the literature than those with fixed replenishment intervals. The local and echelon-stock (r, Q) policies have been extensively studied in the literature. Most of the work on the (r, Q) policy focuses on developing approximate or exact total cost expressions, for example, Axsäter(1990,1993), Axsäter and Rosling (1993), Badinelli (1992), Cachon (2001), Chen and Zheng (1994), De Bodt and Graves (1985). Axsäter and Rosling (1993) showed that the local (r, nQ) policy is a special case of the echelon-stock (r, Q) policy. Chen and Zheng (1998) provided an algorithm to find a near-optimal echelon-stock (r, Q) policy for a serial system. Chen (1998) constructed an algorithm to find the optimal reorder points for the local (r, Q) policy. There are studies that provide simple heuristics on the (r, Q) policy, e.g., Shang and Song (2007) and Shang (2008).

Although fixed-interval ordering policy has been proposed very early in the literature, it only starts to attract some attention recently in the multi-echelon area. The (S, T) policy was first discussed by Hadley and Whitin (1963). Nad-dor (1975) studied the (S, T) policy for single- and multi-item systems. Cachon (1999) studied the reorder-interval policy in a distribution system. He showed that the supplier's demand variance will decline as the retailers' reorder interval becomes longer. Graves (1996) derived an exact evaluation for base-stock policies in continuous-time, multi-echelon distribution systems with fixed replenishment

intervals and Poisson demand processes. Axsäter (1993) employed a different cost evaluation approach to study a special case of Graves's model in that the retailers have an identical reorder intervals and the order schedule is nested.

Recently, van Houtum et al. (2007) studied a serial system and show that the echelon-stock (S, T) policies are optimal when the reorder intervals are fixed. They also provided an algorithm to obtain the optimal base-stock levels. Chao and Zhou (2009) extended these results to the echelon-stock (r, nQ, T) policies. Under such a policy, managers review its echelon inventory order position every T periods and orders according to an echelon-stock (r, nQ) policy. That is, if the echelon inventory order position is less than or equal to the echelon reorder point r , the stage orders a quantity of the smallest integer multiple of batch size Q so as to bring the echelon inventory order position above r . They showed the optimality of echelon-stock (r, nQ) policy given fixed reorder intervals and then provide an algorithm to obtain the optimal reorder points with fixed batch sizes and reorder intervals.

However, none of the above papers attempts to find optimal reorder intervals. Feng and Rao (2007) considered a two-stage system with echelon-stock (S, T) policies. They derived the average total cost function and use the golden section search to obtain heuristic reorder intervals. Shang and Zhou (2010) studied a periodic-review, N -stage serial inventory system in which echelon-stock (r, nQ, T) policies are implemented. They provided approaches for system cost evaluation and finding the optimal reorder intervals and other policy parameters such that the average total cost per period is minimized. Shang et al. (2015) applied echelon-stock (S, T) policy into a periodical review distribution system and try to optimize replenishment scheduling and inventory control simultaneously. Wang

and Axsäter (2013) also studied fixed-interval ordering policy in a distribution system with one virtual warehouse and N identical retailers. They provided an approach to develop an exact optimal solution under optimal warehouse stock allocation, in which warehouse stock is allocated in the last minute before delivery with the most updated stock information to minimize the total relevant system cost. They considered optimizing reorder interval and stock decisions for all installations.

The literature has shown that to find a system optimal solution to minimize system cost is often a challenge. In many supply chain systems, the form of the optimal policy is not known (Shang 2011). For serial systems, the echelon base-stock policy has been shown to be optimal for both finite-horizon and infinite-horizon criteria (Clark and Scarf 1960, Federgruen and Zipkin 1984, Chen and Zheng 1994). Chen (2000) applied the echelon-stock (R, nQ) policy in a serial system and showed its optimality. van Houtum et al. (2007) considered a serial system with fixed replenishment interval constraints. They proved that the echelon base-stock policy is optimal for the multi-echelon system and derive newsvendor-type characteristics for the optimal base-stock levels. Chao and Zhou (2009) derived the optimal ordering policy for a multi-echelon system with both batch ordering and fixed replenishment interval.

2.3 Model Formulation

Let stage j order from stage $j + 1$ in a fixed interval of length T_j for $j = 1, \dots, N$, where stage $N + 1$ refers to the external supplier. Under this condition, an optimal ordering policy requires that each stage synchronize the arrival of a shipment from

the upstream stage with an order from the downstream stage (Chao and Zhou 2009). This leads to a synchronized and nested policy. Moreover, given the reorder intervals, a base stock policy is optimal (van Houtum et al. 2007). We assume these conditions and consider a fixed-interval order-up-to policy, in which the reorder intervals follow the integer-ratio requirement, i.e., $T_{j+1} = n_{j+1}T_j$, where n_{j+1} is a positive integer, and stage j orders at each review point to raise the echelon inventory position to a fixed base stock level S_j for $j = 1, \dots, N$.

By the definition above, the ordering policy at echelon j is fully characterized by $\mathbf{S}_j = (S_1, \dots, S_j)$. Alternatively, we can define the ordering policy by installation stock. Let $R_1 = S_1$ and $R_j = S_j - S_{j-1}$ for $j = 2, \dots, N$. The ordering policy at stage j is then to order at a review point to raise the installation inventory position at stage j to R_j and the echelon inventory position at stage $j - 1$ to S_{j-1} . We use the installation-stock definition in the following analysis. Let $\mathbf{R}_j = (R_1, \dots, R_j)$ and $\mathbf{T}_j = (T_1, \dots, T_j)$. We refer to $\mathbf{R}_j$ and $\mathbf{T}_j$ as the ordering policy and the replenishment schedule, respectively, for echelon $j = 1, \dots, N$.

A list of notations is provided in Table 2.1.

Table 2.1: Notations

N	The number of retailers,
l_j	constant leadtime at stage j , $i = 1, 2, \dots, N$,
λ	mean customer demand rate per unit time,
$D[a, a + t)$	The cumulative demand at stage 1 over time interval $[a, a + t)$
h_i	installation cost per unit per unit time at stage j ,
b	cost for backlogging one unit of demand per unit time at stage 1,
T_j	Stage j 's reorder interval,
n_j	The integer ratio of adjacent stages' reorder intervals, i.e., $n_j = T_j/T_{j-1}$,
S_j	Echelon j 's order-up-to level,
R_j	Stage j 's installation order-up-to level, where $R_j = S_j - S_{j-1}$,

We define the interval between two consecutive ordering points as an ordering

cycle for an echelon. Without loss of generality, let stage j place an order at time t_j . Given the reorder interval T_j , the interval: $[t_j, t_j + T_j)$ forms an ordering cycle for echelon j , $j = 1, \dots, N$. The inventory related cost at echelon j for this ordering cycle is evaluated in this interval shifted by the constant leadtime, i.e., in the interval: $[t_j + l_j, t_j + T_j + l_j)$.

We assume that the demand process at stage 1 is stationary and has independent increments. Let $D[a, a + t)$ denote the cumulative demand over the interval $[a, a + t)$ of length t . Then $D[a, a + t)$ is stochastically increasing in t (Rao 2003, Liu and Song 2011). The mean of $D[a, a + t)$ is equal to λt . We let $g(x, t)$ and $G(x, t) = \int_0^x g(y, t) dy$ denote respectively the probability density function and the cumulative distribution function of $D[a, a + t)$ and refer to that $D[a, a + t)$ follows the G-distribution.

To facilitate the analysis below, we also define the following cost functions for $j = 1, \dots, N$:

$\pi_j(R_j, T_j | \cdot)$ = expected inventory holding and backordering cost in an ordering cycle of length T_j at echelon j when the installation inventory position at stage j is raised to R_j and the echelon inventory position at stage $j - 1$ is raised to S_{j-1} at the review point and

$C_j(\mathbf{R}_j, \mathbf{T}_j)$ = average inventory related cost, including inventory holding, backordering and ordering costs, per unit time at echelon j when the echelon adopts a replenishment schedule $\mathbf{T}_j$ and an ordering policy $\mathbf{R}_j$, assuming that stage $j + 1$ is perfectly reliable. The perfectly reliable stage $j + 1$ means that whenever stage j orders, stage $j + 1$ will satisfy the order without shortage.

We note that $\pi_1(R, T_1 | \cdot) = \pi_1(R, T_1)$ for stage 1 and $\pi_j(R_j, T_j | \cdot)$ is actually a function of the ordering policy $\mathbf{R}_j$ and replenishment schedule $\mathbf{T}_j$ at echelon j .

We keep only R_j and T_j in the notation to facilitate the analysis below and omit the others for simplicity. In addition, R_j in this definition represents the actual inventory position excluding backorders at stage $j + 1$. When stage $j + 1$ faces a shortage, the expected inventory holding and backordering cost at echelon j in the ordering cycle can be evaluated by $\pi_j(R, T_j | \cdot)$ where R is equal to R_j minus the shortage.

2.4 Evaluation of Expected Inventory Related Costs in the System

Under the inventory control policy that is defined above, the system regenerates every time when the most upstream stage N orders from the external supplier. To evaluate the long-run average system inventory related cost, we need to consider only one system regenerative cycle. The long-run average inventory related cost per unit time in the system can be obtained by dividing the expected cost in a regenerative cycle by the cycle length.

We consider two types of inventory related costs in the system: (i) review and ordering costs and (ii) inventory holding and backordering costs. Following Shang and Zhou (2010), we assume a fixed review and ordering cost at each review point at each stage and hence the average review and ordering cost at stage j is equal to k_j/T_j per unit time for $j = 1, \dots, N$. Given this cost evaluation, the onus is to evaluate the second type of costs. We provide a simple bottom-up recursive approach to evaluate this type of costs below.

2.4.1 Inventory Holding and Backordering Cost at Echelon 1

We start with echelon 1 or stage 1 and evaluate the expected inventory holding and backordering cost at the echelon in an ordering cycle in the interval: $[t_1 + l_1, t_1 + l_1 + T_1)$.

Assume that the inventory position at stage 1 is raised to R_1 at time t_1 . At any time $t \in [t_1 + l_1, t_1 + l_1 + T_1)$, the inventory level at the stage is equal to $R_1 - D[t_1, t)$ and the instantaneous expected inventory holding and backordering cost at time t is equal to $u_1(t) = E[h_1(R_1 - D[t_1, t))^+ + b(D[t_1, t) - R_1)^+]$, where $(x)^+ = \max\{0, x\}$. Using integration, the cumulative expected inventory holding and backordering cost in the interval $[t_1 + l_1, t_1 + l_1 + T_1)$ is given by $\pi_1(R_1, T_1) = \int_{t_1+l_1}^{t_1+l_1+T_1} u_1(t)dt$, which after simplification yields

$$\pi_1(R_1, T_1) = b[\lambda(T_1/2 + l_1) - R_1]T_1 + (h_1 + b)I_1(R_1, T_1) \quad (2.1)$$

where $I_1(R_1, T_1) = \int_{l_1}^{l_1+T_1} \int_0^{R_1} (R_1 - x)g(x, t)dxdt = \int_{l_1}^{l_1+T_1} \int_0^{R_1} G(x, t)dxdt$ is the expected on-hand inventory in the ordering cycle.

2.4.2 Inventory Holding and Backordering Cost at Echelon $j \geq 2$

We consider now echelon $j = 2, \dots, N$. Since echelon j is comprised of stage j and echelon $j - 1$, $\pi_j(R_j, T_j|.)$ is equal to the sum of the expected inventory holding cost at stage j and the expected inventory holding and backordering cost at echelon $j - 1$ in an ordering cycle. We develop a bottom-up recursive approach

to evaluate $\pi_j(R_j, T_j | \cdot)$.

Assume that the echelon inventory position at stage j is raised to S_j at time t_j . The ordered stock arrives at stage j at time $t_j + l_j$, when stage $j - 1$ orders to raise the echelon inventory position to S_{j-1} . After meeting the order from stage $j - 1$, the inventory level at stage j is equal to $R_j - D[t_j, t_j + l_j]$. Subsequently, stage $j - 1$ is scheduled to order at $t_j + l_j + qT_{j-1}$ and the inventory level at stage j after meeting the order from stage $j - 1$ is equal to $R_j - D[t_j, t_j + l_j + qT_{j-1}]$ at $t_j + l_j + qT_{j-1}$ for $q = 1, \dots, n_j - 1$ in the ordering cycle. Let $\tau_{jq} = l_j + qT_{j-1}$ and $Y_{jq} = D[t_j, t_j + \tau_{jq}]$ denote the cumulative demand from time t_j until time $t_j + \tau_{jq}$. Then Y_{jq} follows the G-distribution with mean $\lambda\tau_{jq}$.

Theorem 1. *Given $\pi_1(R_1, T_1)$, $\pi_j(R_j, T_j | \cdot)$ for $j \geq 2$ can be computed recursively starting from $j = 2$ (bottom-up) according to $\pi_j(R_j, T_j | \cdot) = \sum_{q=0}^{n_j-1} \pi_{jq}(R_j, T_{j-1} | \cdot)$ where*

$$\begin{aligned} \pi_{jq}(R_j, T_{j-1} | \cdot) = & h_j T_{j-1} \int_0^{R_j} G(y, \tau_{jq}) dy + \pi_{j-1}(R_{j-1}, T_{j-1} | \cdot) G(R_j, \tau_{jq}) \\ & + \int_{R_j}^{\infty} \pi_{j-1}(R_{j-1} + R_j - y, T_{j-1} | \cdot) g(y, \tau_{jq}) dy. \end{aligned} \quad (2.2)$$

Proof. According to the discussion above, stage j holds an inventory of $R_j - Y_{jq}$ or zero, whichever is larger, from time $t_j + \tau_{jq}$ to time $t_j + \tau_{j(q+1)}$ for $q = 0, \dots, n_j - 1$. The expected inventory holding cost at stage j is then equal to $h_j T_{j-1} \sum_{q=0}^{n_j-1} \int_0^{R_j} (R_j - y) g(y, \tau_{jq}) dy = h_j T_{j-1} \sum_{q=0}^{n_j-1} \int_0^{R_j} G(y, \tau_{jq}) dy$ in the interval: $[t_j + \tau_{jq}, t_j + \tau_{j(q+1)})$ for $q = 0, \dots, n_j - 1$.

In the meantime, the ordering cycle for stage j contains n_j ordering cycles for echelon $j - 1$. We define $c_q(\cdot | Y_{jq})$ as the expected inventory holding and backordering cost at echelon $j - 1$ for the q^{th} ordering cycle or in the interval:

$[t_j + \tau_{jq} + l_{j-1}, t_j + \tau_{j(q+1)} + l_{j-1})$ given Y_{jq} for $q = 0, \dots, n_j - 1$. Then the expected inventory holding and backordering cost at echelon $j - 1$ in the q^{th} ordering cycle can be evaluated by $\int_0^\infty c_q(\cdot|y)g(y, \tau_{jq})dy$.

We break the integration at $Y_{jq} = R_j$ so that

$$\int_0^\infty c_q(\cdot|y)g(y, \tau_{jq})dy = \int_0^{R_j} c_q(\cdot|y)g(y, \tau_{jq})dy + \int_{R_j}^\infty c_q(\cdot|y)g(y, \tau_{jq})dy. \quad (2.3)$$

Consider the two terms on the right hand side of (2.3). If $Y_{jq} < R_j$, stage j has sufficient stock to raise the inventory position at echelon $j - 1$ to S_{j-1} at time $t_j + \tau_{jq}$ and therefore $c_q(\cdot|Y_{jq}) = \pi_{j-1}(R_{j-1}, T_{j-1}|\cdot)$ in this case. This leads to $\int_0^{R_j} c_q(\cdot|y)g(y, \tau_{jq})dy = \pi_{j-1}(R_{j-1}, T_{j-1}|\cdot)G(R_j, \tau_{jq})$. On the other hand, if $Y_{jq} \geq R_j$, stage j is out of stock at time $t_j + \tau_{jq}$. After the shipment of the remaining stock (if any) at stage j , the actual inventory position at stage $j - 1$ excluding backorders at stage j is equal to $R_{j-1} - (Y_{jq} - R_j) = R_j + R_{j-1} - Y_{jq}$. This leads to $c_q(\cdot|Y_{jq}) = \pi_{j-1}(R_j + R_{j-1} - Y_{jq}, T_{j-1}|\cdot)$ and therefore $\int_{R_j}^\infty c_q(\cdot|y)g(y, \tau_{jq})dy = \int_{R_j}^\infty \pi_{j-1}(R_j + R_{j-1} - y, T_{j-1}|\cdot)g(y, \tau_{jq})dy$.

The summation of the expected costs across $q = 0, \dots, n_j - 1$ yields (2.2). $\square$

The echelon-cost evaluation approach is easy to follow. First, since the ordering cycle for echelon j contains n_j ordering cycles for echelon $j - 1$, the expected inventory related cost at echelon j in an ordering cycle, i.e., $\pi_j(R_j, T_j|\cdot)$, is equal to the sum of the expected inventory related costs in the ordering cycles for echelon $j - 1$, i.e., $\pi_{jq}(R_j, T_{j-1}|\cdot)$ for $q = 0, \dots, n_j - 1$. Second, according to Theorem 1, the inventory related cost in an ordering cycle of echelon $j - 1$ can be exactly evaluated given the cumulative demand in the ordering cycle up to the current ordering point for echelon $j - 1$ and therefore $\pi_{jq}(R_j, T_{j-1}|\cdot)$ can be evaluated by

an expectation of this conditional cost evaluation with respect to the cumulative demand.

The cost evaluation is also easy to compute. First, the computation of $\pi_j(R_j, T_j|\cdot)$ is straightforward because it uses only the demand distribution. Second, we can reduce the infinite integration in (2.2) to a finite one. For this purpose, let us consider an ordering cycle for echelon $j-1$ assuming that the leadtime for stage $j-1$ is $\tau_{jq} + l_{j-1}$ and the echelon inventory position is raised to S_j at time t_j . Using the definition above, the inventory related cost at echelon $j-1$ in the ordering cycle is given by $\pi_{j-1}^q(R_{j-1} + R_j, T_{j-1}|\cdot)$, where the superscript q is used to indicate that the leadtime is $\tau_{jq} + l_{j-1}$ instead of l_{j-1} . We note that, upon the arrival of the ordered stock at time $t_j + \tau_{jq} + l_{j-1}$, the inventory level at stage $j-1$ is equal to $R_{j-1} + R_j - x$, where $x = D[t_j, t_j + \tau_{jq} + l_{j-1}]$ is the leadtime demand. We decompose the leadtime demand into two components according to $D[t_j, t_j + \tau_{jq} + l_{j-1}] = D[t_j, t_j + \tau_{jq}] + D[t_j + \tau_{jq}, t_j + \tau_{jq} + l_{j-1}]$. Given $D[t_j, t_j + \tau_{jq}] = y$, the expected inventory holding and backordering cost at echelon $j-1$ in the interval: $[t_j + \tau_{jq} + l_{j-1}, t_j + \tau_{jq} + l_{j-1} + T_{j-1})$ can be evaluated by $\pi_{j-1}(R_{j-1} + R_j - y, T_{j-1}|\cdot)$. This leads to $\pi_{j-1}^q(R_{j-1} + R_j, T_{j-1}|\cdot) = \int_0^\infty \pi_{j-1}(R_{j-1} + R_j - y, T_{j-1}|\cdot)g(y, \tau_{jq})dy$. Using this evaluation, we obtain $\sum_{q=0}^{n_j-1} \pi_{j-1}^q(R_{j-1} + R_j, T_{j-1}|\cdot) = \tilde{\pi}_{j-1}(R_{j-1} + R_j, T_j|\cdot)$, where $\tilde{\pi}_{j-1}(R_{j-1} + R_j, T_j|\cdot)$ is equal to $\pi_{j-1}(R_{j-1} + R_j, T_j|\cdot)$ with leadtime $l_{j-1} + l_j$. This yields

$$\begin{aligned} \pi_j(R_j, T_j|\cdot) &= h_j T_{j-1} \int_0^{R_j} \sum_{q=0}^{n_j-1} G(y, \tau_{jq}) dy + \tilde{\pi}_{j-1}(R_{j-1} + R_j, T_j|\cdot) \\ &\quad - \sum_{q=0}^{n_j-1} \int_0^{R_j} [\pi_{j-1}(R_{j-1} + R_j - y, T_{j-1}|\cdot) - \pi_{j-1}(R_{j-1}, T_{j-1}|\cdot)] g(y, \tau_{jq}) dy. \end{aligned} \tag{2.4}$$

2.4.3 Long-Run Average Inventory Related Cost in the System

According to Theorem 1, if stage $j + 1$ is perfectly reliable, the average inventory holding and backordering cost per unit time at echelon j is equal to $\pi_j(R_j, T_j | \cdot) / T_j$. Add the average review and ordering cost at stage j and each downstream stage. We obtain the average inventory related cost per unit time at echelon j as

$$C_j(\mathbf{R}_j, \mathbf{T}_j) = \sum_{i=1}^j k_i / T_i + \pi_j(R_j, T_j | \cdot) / T_j \text{ for } j = 1, \dots, N. \quad (2.5)$$

The long-run average inventory related cost per unit time at the system is then given by $C_N(\mathbf{R}_N, \mathbf{T}_N)$. The remaining task is to seek an optimal replenishment schedule $\mathbf{T}_N$ and an optimal ordering policy $\mathbf{R}_N$ to minimize $C_N(\mathbf{R}_N, \mathbf{T}_N)$.

Several points about the system cost evaluation model are worth noting.

First, if end-of-period cost accounting is adopted and costs are accounted discretely at points in an interval of length 1, we can evaluate the expected inventory holding and backordering cost in an ordering cycle at stage 1 according to $\pi_1(R_1, T_1) = \sum_{\tau=1}^{T_1} u_1(t_1 + l_1 + \tau)$, where T_1 is assumed to be an integer. All the analysis in the study still holds.

Second, the expected inventory in-transit from stage $j + 1$ to stage j is equal to λ_j per unit time. The cost for holding this expected inventory is constant and thus omitted.

Finally, it is easy to verify that $\pi_1(R_1, T_1)$ decreases if $R_1 < 0$ and consequently we should consider only $R_1 \geq 0$. Furthermore, if $R_j < 0$ for any $j > 0$, we can

redefine the base stock level at stage j to be zero and the base stock level at the downstream stage $j - 1$ to be $R_{j-1} + R_j$ without affecting the system cost. Because of this property, we consider only $R_j \geq 0$ for the optimization of the inventory control system below.

2.5 System Optimization

We discuss now the optimization of the inventory control system or the solution of the following optimization model:

Minimize: $C_N(\mathbf{R}_N, \mathbf{T}_N)$ w.r.t. $\mathbf{R}_N$ and $\mathbf{T}_N$

Subject to: $T_j = n_j T_{j-1}$ and n_j is a positive integer for $j = 2, \dots, N$.

The inventory control policy is built on the single-stage fixed-interval order-up-to (S, T) policy. Lemma 1 summarizes the basic results for its optimization.

Lemma 1. (a) $C_1(R_1, T_1)$ is coordinate-wise convex but not jointly convex in R_1 and T_1 . (b) Let $R_1^*(T_1)$ denote the optimal R_1 that minimizes $C_1(R_1, T_1)$ for a given $T_1 > 0$. Then $R_1^*(T_1)$ increases in T_1 and is uniquely determined by $\int_{l_1}^{l_1+T_1} G(R_1, t) dt / T_1 = b / (h_1 + b)$. (c) $C_1(R_1^*(T_1), T_1) \geq LB(T_1)$ where $LB(T_1) = k_1 / T_1 + \beta T_1 / 2$ and $\beta = \lambda h_1 b / (b + h_1)$.

Proof. We refer to Liu and Song (2011) for the proof of Lemma 1. □

Because $C_1(R_1, T_1)$ is not jointly convex, the first-order necessary conditions of optimality may not be sufficient. A general approach to identify an optimal solution is to first establish a finite feasible region for an optimal T_1 using the lower bound $LB(T_1)$ and subsequently use exhaustive search to identify an optimal T_1 within the feasible region subject to the optimization of R_1 for a given T_1 . We

refer to the Rao (2003) and Liu and Song (2011) for more discussions on this basic inventory control policy.

2.5.1 Optimal Ordering Policy under a Given Replenishment Schedule

We consider first an optimal ordering policy for a given replenishment schedule.

Lemma 2. *Assume $R_j > 0$ for $j = 1, \dots, N$. Then*

$$\frac{\partial C_N(\cdot)}{\partial R_{j-1}} - \frac{\partial C_N(\cdot)}{\partial R_j} = \left[\frac{\partial C_{j-1}(\cdot)}{\partial R_{j-1}} - h_j \right] P_j^N(\cdot) \quad (2.6)$$

where $P_j^N(\cdot)$ represents a probability as a function of $R_j, \dots, R_N$ for $j = 2, \dots, N$.

Proof. We compute the first partial derivatives of $C_j(\mathbf{R}_j, \mathbf{T}_j)$ for $j = 2, \dots, N$ w.r.t. R_i for $i = 1, \dots, j$ and obtain

$$\frac{\partial C_j(\cdot)}{\partial R_j} = h_j P_j^j(R_j) + \int_{R_j}^{\infty} \frac{\partial C_{j-1}(R_j + R_{j-1} - y, T_{j-1} | \cdot)}{\partial R_j} p_j^j(y) dy \quad \text{and} \quad (2.7)$$

$$\frac{\partial C_j(\cdot)}{\partial R_i} = \frac{\partial C_{j-1}(\cdot)}{\partial R_i} P_j^j(R_j) + \int_{R_j}^{\infty} \frac{\partial C_{j-1}(R_j + R_{j-1} - y, T_{j-1} | \cdot)}{\partial R_i} p_j^j(y) dy \quad \text{for } i < j \quad (2.8)$$

where $P_j^j(y) = \sum_{q=0}^{n_j-1} G(y, \tau_{jq}) / n_j$ is the average of the probabilities for stage j not to run out of stock at stage $j-1$'s review points in the ordering cycle for a given initial inventory position y and $p_j^j(y) = \sum_{q=0}^{n_j-1} g(y, \tau_{jq}) / n_j$ is the first derivative of $P_j^j(y)$ w.r.t. y .

For brevity, we omit the policy variables in the cost functions whenever such an omission causes no confusion. In addition, we note that $\partial\pi_{j-1}(\cdot)/\partial R_i/T_{j-1} = \partial C_{j-1}(\cdot)/\partial R_i$.

We prove the relationship (2.6) using induction.

First, for stage j , we obtain $\partial C_j(\cdot)/\partial R_{j-1}$ by letting $i = j - 1$ in (2.8). Subsequently we subtract (2.7) from (2.8) and obtain

$$\frac{\partial C_j(\cdot)}{\partial R_{j-1}} - \frac{\partial C_j(\cdot)}{\partial R_j} = \left[\frac{\partial C_{j-1}(\cdot)}{\partial R_{j-1}} - h_j \right] P_j^j(R_j) \quad (2.9)$$

since $\partial\pi_{j-1}(R_j + R_{j-1} - y, T_{j-1}|\cdot)/\partial R_j = \partial\pi_{j-1}(R_j + R_{j-1} - y, T_{j-1}|\cdot)/\partial R_{j-1}$ for any $y \geq 0$.

Second, for stage $N - 1$, we assume

$$\frac{\partial C_{N-1}(\cdot)}{\partial R_{j-1}} - \frac{\partial C_{N-1}(\cdot)}{\partial R_j} = \left[\frac{\partial C_{j-1}(\cdot)}{\partial R_{j-1}} - h_j \right] P_j^{N-1}(R_{N-1}|\cdot) \quad (2.10)$$

where $P_j^{N-1}(R_{N-1}|\cdot)$ is a function of $R_j, \dots, R_{N-1}$. We keep the variable R_{N-1} in the function to facilitate the analysis below but omit the others for brevity.

Finally, for stage N , we first obtain $\partial C_N(\cdot)/\partial R_{j-1}$ and $\partial C_N(\cdot)/\partial R_j$ from (2.8) and subsequently subtract the latter from the former. We obtain

$$\begin{aligned} \frac{\partial C_N(\cdot)}{\partial R_{j-1}} - \frac{\partial C_N(\cdot)}{\partial R_j} &= \left[\frac{\partial C_{N-1}(\cdot)}{\partial R_{j-1}} - \frac{\partial C_{N-1}(\cdot)}{\partial R_j} \right] P_N^N(R_N) \\ &\quad + \int_{R_j}^{\infty} \left[\frac{\partial C_{N-1}(R_N + R_{N-1} - y, T_{N-1}|\cdot)}{\partial R_{j-1}} - \frac{\partial C_{N-1}(R_N + R_{N-1} - y, T_{N-1}|\cdot)}{\partial R_j} \right] p_j^N(y) dy. \end{aligned} \quad (2.11)$$

Since (2.10) holds for any given R_{N-1} , we apply (2.10) in (2.11) and obtain (2.6), where $P_j^N(R_N|\cdot) = P_j^{N-1}(R_{N-1}|\cdot)P_N^N(R_N) + \int_{R_j}^{\infty} P_j^{N-1}(R_N + R_{N-1} - y|\cdot)p_j^N(y, \tau_{jq})dy$ is a probability that is obtained as an expected value of $P_j^{N-1}(R_N +$

$R_{N-1} - y|.)$ w.r.t. y . □

The finding in Lemma 2 leads to a simple characterization of an optimal ordering policy.

Theorem 2. *For a given system replenishment schedule $\mathbf{T}_N$, an optimal ordering policy that minimizes $C_N(\cdot)$ is uniquely determined by the following bottom-up procedure.*

(i) *Start with stage 1. Identify R_1^* by $\partial C_1(\cdot)/\partial R_1 = h_2$ and proceed to step (ii) below.*

(ii) *For stage $j > 1$, given $R_1^*, \dots, R_{j-1}^*$ that are previously identified, identify the optimal R_j^* by $\partial C_j(\cdot)/\partial R_j = h_{j+1}$. If the obtained solution is $R_j^* = 0$, then stage j is a virtual stage that does not carry any stock and stage $j - 1$ orders directly from stage $j + 1$. Revise R_{j-1}^* by $\partial C_{j-1}(\cdot)/\partial R_{j-1} = h_{j+1}$ with reorder interval T_j and lead time $l_j + l_{j-1}$. Repeat this step until stage N with $h_{N+1} = 0$.*

Proof. We consider the necessary conditions of optimality, i.e., $\partial C_N(\cdot)/\partial R_j = 0$ for $j = 1, \dots, N$. Using Lemma 2, it is easy to verify that these necessary conditions are equivalent to $\partial C_j(\cdot)/\partial R_j = h_{j+1}$ for $j = 1, \dots, N$ with $h_{N+1} = 0$. A solution of these conditions can be obtained using the bottom-up procedure in the theorem.

We prove below that this solution is unique and provides a minimum optimal solution. To this end, we obtain the second derivatives of $C_j(\cdot)$ w.r.t. R_j and R_i for $i = 1, \dots, j - 1$:

$$\frac{\partial^2 C_j(\cdot)}{\partial R_j^2} = [h_j - \frac{\partial C_{j-1}(\cdot)}{\partial R_{j-1}}] p_j^j(R_j) + \int_{R_j}^{\infty} \frac{\partial^2 C_{j-1}(R_j + R_{j-1} - y, T_{j-1}|\cdot)}{\partial R_{j-1}^2} p_j^j(y) dy \quad (2.12)$$

$$\frac{\partial^2 C_j(\cdot)}{\partial R_j \partial R_i} = \sum_{q=0}^{n_j-1} \int_{R_j}^{\infty} \frac{\partial^2 C_{j-1}(R_j + R_{j-1} - y, T_{j-1}|\cdot)}{\partial R_j \partial R_i} p_j^j(y) dy \quad \text{and} \quad (2.13)$$

$$\frac{\partial^2 C_j(\cdot)}{\partial R_i^2} = \frac{\partial^2 C_{j-1}(\cdot)}{\partial R_i^2} P_j^j(R_j) + \int_{R_j}^{\infty} \frac{\partial^2 C_{j-1}(R_j + R_{j-1} - y, T_{j-1}|\cdot)}{\partial R_i^2} p_j^j(y) dy \quad (2.14)$$

where $p_j^j(y) = \sum_{q=0}^{n_j-1} g(y, \tau_{jq})/n_j$ is the first derivative of $P_j^j(y)$ w.r.t. y .

By Lemma 1, $C_1(\cdot)$ is convex in R_1 . Therefore, R_1^* provides a unique minimum solution.

Given $R_1 = R_1^*$, we apply the conditions: $h_2 - \partial C_1(\cdot)/\partial R_1 = 0$ and $\partial^2 C_1(\cdot)/\partial R_1^2 \geq 0$ in (2.12), (2.13) and (2.14), respectively, for $j = 2$. We obtain $\partial^2 C_2(\cdot)/\partial R_i \partial R_q \geq 0$ for $i, q = 1, 2$. This means that $C_2(\cdot)$ is convex in R_2 and super-modular in R_1 and R_2 . Therefore, if $R_2^* > 0$, R_2^* provides a unique minimum solution. If $R_2^* = 0$, the necessary condition for stage 1 is no longer applicable since stage 1 orders directly from stage 3 with lead time $l_2 + l_1$ and we have to revise R_1^* accordingly.

Similarly, for each stage $j > 2$, given $R_1^*, \dots, R_{j-1}^*$, the same results can be obtained. Consequently, R_j^* provides a unique minimum optimal solution. $\square$

According to Theorem 2, under an optimal solution, the marginal cost at an echelon is equal to the inventory holding cost at the upstream stage. This is intuitive. Consider a unit of stock at stage j at a review point of stage $j - 1$ and the decision between retaining it at stage j or shipping it to the downstream echelon $j - 1$. The marginal cost of the first option is the holding cost h_j . Apparently, this unit of stock should be shipped to the downstream echelon $j - 1$ if the marginal

cost at echelon $j - 1$ is lower than h_j . Following this intuition, an optimal inventory position (or order-up-to level) at the downstream stage $j - 1$ is reached when the two marginal costs are equal. This finding provides a simple characterization of an optimal ordering policy for a serial inventory system. Under this decision rule, an optimal base stock level at a stage is determined independently of the base stock levels and reorder intervals at the upstream stages. This independence condition leads to the bottom-up solution procedure in Theorem 2.

2.5.2 Optimal Replenishment Schedule

We consider now an optimal solution that can fully optimize system performance w.r.t. both ordering policy and replenishment schedule. According to Lemma 1, even for a single stage, the cost function is not jointly convex. As a result, the first-order necessary conditions of optimality may not be sufficient to identify an optimal solution.

We develop an exhaustive search approach, facilitated with bounds on reorder intervals, to identify an optimal solution. The simple characterization of an optimal ordering policy for a given replenishment schedule by Theorem 2 greatly simplifies this task. Let $\mathbf{R}_N^* = (R_1^*, \dots, R_N^*)$ denote the optimal ordering policy for a given replenishment schedule $\mathbf{T}_N$. Then the optimization problem becomes minimizing $C_N(\mathbf{R}_N^*, \mathbf{T}_N)$ w.r.t. $\mathbf{T}_N$.

We first construct a series of lower bounds for $C_N(\mathbf{R}_N^*, \mathbf{T}_N)$ using simplified sub-systems that have original demand and cost parameters. Specifically, for each stage j , we consider the sub-system that consists of stages j to N , in which stage j is the lowest stage with the following parameters: mean customer demand rate per

unit time is λ , lead time is $L_{[1,j]} = \sum_{i=1}^j l_i$, review and ordering cost is $K_{[1,j]} = \sum_{i=1}^j k_i$, inventory holding cost is h_j and backordering cost per unit time is b . Other system parameters remain unchanged. Let $\mathbf{R}_{[j,N]} = (R_j, \dots, R_N)$, $\mathbf{T}_{[j,N]} = (T_j, \dots, T_N)$ and $C_{[j,N]}(\mathbf{R}_{[j,N]}, \mathbf{T}_{[j,N]})$ denote the expected inventory related cost per unit time for the simplified sub-system.

Theorem 3. *For any $1 \leq j \leq N$, $C_N(\mathbf{R}_N, \mathbf{T}_N) \geq C_{[j,N]}(\mathbf{R}_{[j,N]}^*, \mathbf{T}_{[j,N]})$, where $\mathbf{R}_{[j,N]}^*$ denotes an optimal ordering policy for the simplified sub-system given $\mathbf{T}_{[j,N]}$.*

Proof. For any given $\mathbf{T}_N$ and $\mathbf{R}_N$, we reduce the inventory holding cost rate at all the downstream stages 1 to $j-1$ to h_j . It is easy to verify that, under this condition, an optimal ordering policy that is specified by Theorem 2 entails no stock at stages 2 to j . This means that stages 1 to j collapse to a single stage with reorder interval T_j and the system is reduced to the simplified sub-system. Because of the inventory holding cost reduction, the cost for the simplified sub-system is lower (or at least not higher). This proves that the system cost $C_N(\mathbf{R}_N, \mathbf{T}_N)$ is bounded below by $C_{[j,N]}(\mathbf{R}_{[j,N]}^*, \mathbf{T}_{[j,N]})$. $\square$

The lower bound in Theorem 3 loses only the information for the different inventory holding costs at stages 1 to $j-1$ and is generally tight. Using this lower bound, we can establish a feasible range for the reorder interval at each stage given the reorder intervals at the upstream stages. An optimal solution can then be obtained by an exhaustive search in the feasible region subject to the optimization of the ordering policy.

Let $\bar{C}$ denote a system cost that is obtained under a feasible solution. Then $\bar{C}$ provides an upper bound for the minimum system cost. We note that the feasible

solution can be arbitrarily selected. However, a tight upper bound will lead to a narrow feasible region for an optimal reorder interval and hence reduce computation. We use the optimal solution with the deterministic optimal replenishment schedule in our numerical study.

Using the lower bound in Theorem 3, we can establish a feasible region for an optimal T_N or for an optimal n_{j+1} given $\mathbf{T}_{[j+1,N]}$ for $1 \leq j < N$ in two steps. Recall that $T_j = T_{j+1}/n_{j+1}$ for $1 \leq j < N$, where n_{j+1} is a positive integer.

In step 1, we use the lower bound from Lemma 1 and obtain

$$C_{[j,N]}(\mathbf{R}_{[j,N]}^*, \mathbf{T}_{[j,N]}) \geq \sum_{i=j+1}^N k_i/T_i + K_{[1,j]}/T_j + \beta_j T_j/2 \quad (2.15)$$

where $\beta_j = \lambda h_j b / (b + h_j)$. Let $\underline{T}'_N$ and $\overline{T}'_N$ denote the minimum and maximum $T_N \geq 0$ satisfying the condition: $\overline{C} \geq K_{[1,N]}/T_N + \beta_N T_N/2$ and $\underline{n}'_{j+1}$ and $\overline{n}'_{j+1}$ for $j < N$ denote the minimum and maximum positive integer n_{j+1} satisfying the condition: $\overline{C} \geq \sum_{i=j+1}^N k_i/T_i + K_{[1,j]}/T_j + \beta_j T_j/2$, where $T_j = T_{j+1}/n_{j+1}$. Since the right hand side of (2.15) is convex in T_j , such solutions are unique and can be easily obtained.

Subsequently, in step 2, we increase the lower bound (i.e., $\underline{T}'_N$ or $\underline{n}'_{j+1}$) or decrease the upper bound (i.e., $\overline{T}'_N$ or $\overline{n}'_{j+1}$) until any further such change will violate the condition: $C_{[j,N]}(\mathbf{R}_{[j,N]}^*, \mathbf{T}_{[j,N]}) \geq \overline{C}$. Let $\underline{T}_N$ and $\underline{n}_{j+1}$ for $j < N$ denote the lower bound and $\overline{T}_N$ and $\overline{n}_{j+1}$ for $j < N$ denote the upper bound, respectively, that are obtained in this step.

Since $C_N(\mathbf{R}_N, \mathbf{T}_N) \geq C_{[N,N]}(\mathbf{R}_{[N,N]}^*, \mathbf{T}_{[N,N]}) \geq \overline{C}$ for any $T_N < \underline{T}_N$ or $T_N > \overline{T}_N$, an optimal T_N is bounded by $\underline{T}_N \leq T_N \leq \overline{T}_N$. On the other hand, given $\mathbf{T}_{[j+1,N]}$, since $C_N(\mathbf{R}_N, \mathbf{T}_N) \geq C_{[j,N]}(\mathbf{R}_{[j,N]}^*, \mathbf{T}_{[j,N]}) \geq \overline{C}$ for any $n_{j+1} < \underline{n}_{j+1}$ or

$n_{j+1} > \bar{n}_{j+1}$, an optimal n_{j+1} is bounded by $\underline{n}_{j+1} \leq n_{j+1} \leq \bar{n}_{j+1}$ for $j = 1, \dots, N-1$.

We can now use a top-down exhaustive search to identify an optimal solution. We start with stage N and identify a feasible range for T_N : $\underline{T}_N \leq T_N \leq \bar{T}_N$. Subsequently, we proceed to the next downstream stage until stage 1. For each stage $1 \leq j < N$, given $T_N, n_N, \dots, n_{j+1}$, we identify a feasible range for n_{j+1} . An optimal solution is obtained by an exhaustive search. We note that, for any $1 \leq j \leq N$, given $\mathbf{T}_{[j,N]}$, an optimal ordering policy that minimizes $C_{[j,N]}(\mathbf{R}_{[j,N]}, \mathbf{T}_{[j,N]})$ is obtained using Theorem 2.

Example: $N = 3, \lambda = 4, (k_1, k_2, k_3) = (10, 10, 20), (h_1, h_2, h_3) = (2, 1.5, 1), (l_1, l_2, l_3) = (1, 1, 1)$ and $b = 20$. Demand is Normal and has a standard deviation of 0.5.

We use this example to illustrate the solution procedure. We start with the deterministic optimal replenishment schedule: $(n_2, n_3, T_3) = (1, 2, 3.3)$ and, given this schedule, obtain the optimal ordering policy: $(R_1, R_2, R_3) = (11.01, 3.77, 9.74)$. This solution generates a system cost: 28.86, which provides an upper bound for the minimum system cost, i.e., $\bar{C} = 28.86$.

Using the solution procedure above, we first obtain a feasible range for T_3 : $[1.62, 13.16]$. Subsequently, for each feasible T_3 , we identify a feasible range for n_3 and, for each feasible (T_3, n_3) (or T_3 and $T_2 = T_3/n_3$), we identify a feasible range for n_2 . Table 2.2 presents the lower bound and upper bound for (a) n_3 given T_3 (assuming integer values) and (b) n_2 given T_3 and n_3 . We present the bounds for the two steps in each case. For example, given $T_3 = 6$, step 1 identifies a feasible range: $[1, 8]$ and step 2 improves this range to $[1, 4]$ for n_3 . Subsequently, given T_3 and each n_3 in the feasible range, we identify a feasible range for n_2 . For example, given $(T_3, n_3) = (6, 1)$, step 1 identifies a feasible range: $[1, 16]$ and

step 2 improves this range to $[1, 4]$ for n_2 . If no feasible solution exists, the case is indicated by " - ". For example, given $(T_3, n_3) = (6, 4)$, no feasible solution is identified in step 2.

Table 2.2: Bounds for n_3 (given T_3) and n_2 (given T_3 and T_2)

T_3	(a) Bounds for n_3		(b) Bounds for n_2 given n_3							
			$n_3=1$		$n_3=2$		$n_3=3$		$n_3=4$	
	step		step		step		step		step	
	1	2	1	2	1	2	1	2	1	2
1	[1,1]	-								
2	[1,2]	[1,1]	[1,3]	[1,1]						
3	[1,4]	[1,2]	[1,5]	[1,3]	[1,6]	[1,3]				
4	[1,5]	[1,3]	[1,9]	[1,4]	[1,4]	[1,1]	[1,3]	-		
5	[1,7]	[1,3]	[1,12]	[1,4]	[1,6]	[1,2]	[1,4]	[1,1]		
6	[1,8]	[1,4]	[1,16]	[1,4]	[1,8]	[1,2]	[1,5]	[1,1]	[1,3]	-
7	[1,10]	[1,4]	[1,19]	[2,2]	[1,9]	[1,2]	[1,6]	[1,1]	[1,4]	-
8	[1,11]	[1,4]	[1,22]	-	[1,11]	-	[1,7]	-	[1,5]	-
9	[1,13]	[1,3]	[1,24]	-	[1,12]	-	[1,8]	-		
10	[1,14]	[1,2]			[1,13]	-				
11	[1,16]	-								
12	[1,17]	-								
13	[1,19]	-								
14	[1,20]	-								

The numerical example demonstrates interesting features of the solution procedure. First, the lower bounds are rather tight, generating relatively narrow feasible ranges for the reorder intervals. Second, the cascading search procedure improves the feasible range in each step. Consider T_3 . We start with a feasible range $[1.62, 13.16]$ or $[1, 14]$ when only integer values are considered. The identification of a feasible solution for T_2 reduces the feasible range to $[2, 10]$. And this range is further reduced to $[2, 7]$ in the identification of a feasible solution for T_1 . This desirable feature is obtained because of the cascading application of the lower bound $C_{[j,N]}(\mathbf{R}_{[j,N]}^*, \mathbf{T}_{[j,N]})$ for the minimum system cost, which gets tighter

for a lower j . This property can dramatically reduce computation.

2.6 Numerical Study

We present a numerical study in this section. The numerical study contains 128 cases that are formed by the different combinations of the following parameters: $N = 3$, $k_1, k_2 \in \{10, 20\}$, $k_3 \in \{20, 40\}$, $h_2 \in \{1.5, 3\}$, $h_1 = h_2 + \Delta$, $h_3 = h_2 - \Delta$, $\Delta \in \{0.5, 1\}$, $b \in \{10, 20\}$ and $l_1 = l_2 = l_3 = 1$. Customer demand per unit time follows the Normal distribution with mean $\lambda = 4$ and standard deviation $\sigma = 0.5$. The reason that we choose different values for fixed ordering costs, holding costs, and penalty costs is that later on we would like to examine their impact on the optimal solutions, particularly reorder intervals. Since the mean demand is 8 times the standard deviation, the probability of negative demand under the Normal distribution is negligible and hence omitted in the following discussion.

We examine three issues: (i) the optimal solution, (ii) the impact of the cost accounting method (i.e., end-of-period cost accounting v.s. continuous-time cost accounting), and (iii) the deterministic optimal replenishment schedule as an approximation.

2.6.1 The Optimal Solution

We compute the optimal solution and the associated minimum system cost in each case. Note that we could have a closed form solution for stage 1's optimal

order up to level R_1 with the help of equation (2.1) and theorem 2. That is,

$$R_1^* = G^{-1}\left(\frac{h_2 + b}{h_1 + b}\right). \quad (2.16)$$

where G here is the cumulative probability function of normal distribution with mean μT_1 and standard deviation σT_1 .

For illustration, Table 2.3 presents 8 selected cases for $k_1 = 10$, $k_3 = 40$, $h_2 = 1.5$ and $b = 20$ from the numerical experiment. We refer to the optimal solution as Solution 1 and let C^* denote the associated minimum system cost. T_3 is optimized to two decimal places.

Given the complete characterization of the ordering policy in Section 2.4.1, we examine how the optimal reorder intervals relate to the system cost parameters. We conduct a correlation statistical analysis. Table 2.4 provides the summary statistics. According to this correlation analysis, the optimal reorder interval T_j is positively correlated to the ratio k_j/h_j but negatively correlated to the backorder penalty cost b for $j = 1, \dots, N$. This is reasonable. According to the deterministic economic order quantity model, the reorder interval increases with the ordering cost but decreases with the inventory holding cost. The numerical study shows that the optimal solution in the stochastic case still retains largely these basic properties. In addition, the numerical study also shows that a higher backorder penalty cost usually entails a shorter reorder interval. This is because, given a base stock level, a shorter reorder interval can reduce the probability of stockout and hence balance the expected backorder penalty cost.

Moreover, according to Table 2.4, the correlation coefficient of T_j and k_j/h_j is (i) the highest at the current stage, (ii) decreasing monotonically on both

k_2	(h_1, h_3)	Solutions	n_2	n_3	T_3	R_1	R_2	R_3	Cost	Deviation(%)
10	(2.0,1.0)	1	1	1	4.02	20.21	3.42	3.06	30.60	
		2	1	1	4	21.5	3.66	3.32	32.54(29.72)	6.35(-8.65)
		3	1	2	3.82	12.01	3.73	10.69	33.04	7.99
	(2.0,0.5)	1	1	2	5.9	16.07	3.10	15.40	28.49	
		2	1	2	6	17.48	3.46	15.91	30.29(27.40)	6.33(-9.54)
		3	1	4	6.6	11.01	3.44	23.30	30.68	7.69
	(2.5,1.0)	1	2	1	4.46	12.82	12.60	2.91	33.23	
		2	2	1	4	13.17	12.14	3.24	35.57(32.01)	7.04(-10.02)
		3	2	2	4.6	8.77	8.50	12.07	36.31	9.27
	(2.5,0.5)	1	1	3	6.93	13.18	3.39	22.05	30.79	
		2	1	3	6	13.17	3.81	20.05	33.13(29.41)	7.59(-11.24)
		3	1	4	6.32	10.45	3.54	22.58	32.47	5.44
20	(2.0,1.0)	1	1	1	4.39	21.67	3.36	3.00	32.94	
		2	1	1	4	21.5	3.66	3.32	35.04(32.22)	6.38(-8.03)
		3	2	2	5.16	9.62	8.86	13.02	38.07	15.6
	(2.0,0.5)	1	1	2	6.54	17.29	3.05	16.61	31.69	
		2	1	2	6	17.48	3.46	15.91	33.62(30.73)	6.11(-8.60)
		3	2	2	5.84	10.28	8.98	15.23	34.38	8.5
	(2.5,1.0)	1	2	1	4.88	13.57	13.43	2.85	35.42	
		2	2	1	4	13.17	12.14	3.24	38.07(34.51)	7.50(-9.36)
		3	2	2	5.04	9.18	8.93	12.86	39.54	11.64
	(2.5,0.5)	1	1	2	6.1	15.92	3.30	15.84	34.31	
		2	1	2	6	17.11	3.62	15.94	36.97(33.06)	7.74(-10.57)
		3	2	2	5.64	9.73	9.03	14.87	36.10	5.21

	b	k_1/h_1	k_2/h_2	k_3/h_3
T_1^*	-0.213*	0.591**	0.381**	0.096
T_2^*	-0.157	0.484**	0.564**	0.16
T_3^*	-0.068	0.431**	0.496**	0.945**

Note: **, * represent significant at the 0.01, 0.05 level.

(downstream and upstream) sides, and (iii) higher at the downstream stages than at the upstream stages. These patterns have intuitive implications: the optimal reorder intervals are (i) most sensitive to the fixed costs at the current stages, (ii) more sensitive to the fixed costs at more immediate (downstream and upstream) stages, and (iii) more sensitive to the fixed costs at the downstream stages than the upstream stages. In addition, the correlation coefficient of T_j and b decreases on the upstream. This suggests that the impact of the backorder penalty cost decreases along the supply chain on the upstream.

We note that T_3 (or T_N in general) is the only continuous reorder interval in the exhaustive search optimization. To investigate the impact of a search step, we compared the optimal solution for which T_3 is optimized to two decimal places with the optimal solution for which T_3 is assumed to be an integer. The integer assumption increases system cost by an average of 0.59% (ranging from 0.00% to 2.23% with a standard deviation of 0.47%). The numerical study suggests that the minimum system cost is insensitive to T_3 around the optimal solution. This property is similar to the insensitivity of the deterministic economic order quantity formula (Zipkin 2000). Consequently, a relatively large search step for T_3 (e.g., 0.1 or even integer) can generate a reasonably good approximation.

Finally, we would like to examine the impact of σ on the optimal solutions. We let $\sigma = 1$ for the 108 cases we use for the numerical experiment and find the corresponding solution. The increase of σ from 0.5 to 1 indicates higher uncertainty for the arrival demand. According to the numerical results, the system costs increase 7.06% on average when compared to the cases with $\sigma = 0.5$. When the uncertainty increases, the reorder intervals either increase or decrease according to the numerical results. This indicates that the optimal inventory policy copes

with demand uncertainty through two ways: ordering more often and ordering less often but holding more safety stock. The optimal solutions depends in a complex way on all the cost parameters. As Liu and Song (2011) have shown, the cost function of one stage problem with (S, T) policy is even not well behaved, let alone N -stage serial systems. The optimal solutions for all the cases are included in Appendix D.

2.6.2 Impact of Cost Accounting Method

We examine next the impact of the cost accounting method. Following previous studies (von Houtum et al. 2007, Chao and Zhou 2009, Shang and Zhou 2010), we use end-of-period cost accounting and evaluate the inventory related costs in the system at the end of each period of length one. This requires that the continuous-time cost evaluation (1) at stage 1 be modified accordingly (i.e., changing the integration to summation and assuming T_1 to be an integer). The rest of the analysis in our study applies to this case. We replicate the numerical experiment with the end-of-period cost evaluation. The optimal solution is presented and denoted by Solution 2 in Table 2.3. We obtain the continuous-time cost evaluation under this solution. This system cost evaluation is presented in the column under "cost" in the table. We also present the (minimum) discrete-time cost evaluation under this solution in brackets next to the continuous-time cost evaluation.

We examine the differences between (i) the continuous-time and discrete-time cost evaluations under Solution 2 and (ii) the continuous-time cost evaluations under Solutions 1 and 2. Let C_a and C_e denote the continuous-time cost evaluation and the discrete-time cost evaluation, respectively, under Solution 2. According

to the numerical experiment, the relative deviation: $100(C_e/C_a - 1)\%$ ranges from -15.32% to -4.30% with an average of -8.91% and a standard deviation of 2.25% and the relative deviation: $100(C_a/C^* - 1)\%$ ranges from 3.47% to 13.60% with an average of 7.31% and a standard deviation of 1.78% . The first deviation shows that end-of-period cost accounting indeed significantly underestimates system cost and the second deviation implies that the implementation of the discrete-time optimal solution often leads to a significantly higher system cost if inventory related costs actually accrue in continuous time. The two deviations are displayed in the last column in Table 2.3.

2.6.3 The Deterministic Optimal Replenishment Schedule

Finally, we examine if the deterministic optimal solution provides a good approximation by comparing it with the stochastic optimal solution. This comparison reveals some interesting properties of an optimal system replenishment schedule when demand is stochastic.

We assume that demand is constant and equal to λ per unit time and obtain an optimal deterministic solution for each case of the numerical experiment. We refer to Zipkin (2000) for a simple algorithm to identify an optimal replenishment schedule in this case. Given the replenishment schedule, we identify an optimal ordering policy according to Theorem 2 and evaluate the associated system cost. This heuristic solution is denoted by Solution 3 in Table 2.3. Let C_d denote the system cost under this solution.

According to the numerical experiment, the relative deviation: $100(C_d/C^* - 1)\%$ ranges from 5.03% to 29.16% with an average of 15.44% and a standard

deviation of 6.16%. This suggests that the deterministic solution does not provide a good approximation. A scrutiny of the optimal solutions in the numerical experiment further reveals that the optimal T_1 is higher and the optimal n_j for $j = 2, \dots, N$ is the same or lower than the deterministic optimal solution in every case. This means that stock at an upstream stage is delivered less or at least no more frequently to the downstream stage in the stochastic case as compared to the deterministic case. Given the same (length of the) ordering cycle and base stock level, this implies that stock is shipped to the downstream stage in a larger quantity. This is reasonable. When demand is stochastic, stock should be placed closer to the customers in order to reduce stockout and balance the backorder penalty cost with inventory holding cost. To achieve this objective, downstream stages should carry more stock or equivalently stock should be shipped more quickly in larger quantities from upstream stages to downstream stages. This leads to more stock and a longer reorder interval at stage 1. As a result, the stochastic optimal solution often entails a very different system replenishment schedule as compared to the deterministic optimal solution.

2.7 Concluding Remarks

This study contributes to the literature by providing a different approach to evaluate a serial inventory system. This evaluation approach leads to a simple characterization of an optimal ordering policy, which simply equates the marginal echelon inventory related cost to the inventory holding cost at the upstream stage. Moreover, using this optimal solution for a given system replenishment schedule, we develop a top-down procedure to identify an optimal system re-

plenishment schedule subject to the optimization of the ordering policy. This evaluation method not only provides an alternative to the echelon-stock evaluation approach in the literature. It also has several advantages. First, it uses only the demand distribution and is easy to follow and compute. Second, the characterization of an optimal solution enhances our understanding of optimal control policies for serial inventory systems. Finally, this method can be used to evaluate other periodic-review control policies for serial inventory systems under different cost accounting methods.

Chapter 3

Stock Allocation Policies for Two-Echelon Distribution Systems

3.1 Introduction

We consider a two-level distribution inventory system that consists of a warehouse and a group of non-identical retailers. Customer demand arrives at the retailers according to independent Poisson processes. The warehouse orders from an external supplier with ample stock to replenish the retailers. There are fixed lead times between the external supplier and the warehouse and between the warehouse and the retailers. Unfulfilled demands or orders are fully backlogged. The system is centralized in the sense that the warehouse has full knowledge on the inventory status at the retailers and controls the inventory for the whole system.

We consider a periodic-review inventory control system in which inventory

positions are reviewed periodically in a fixed base planning period (e.g., one day, one week, etc.). We assume that the warehouse orders from the external supplier in an fixed replenishment cycle and replenishes the retailers in a fixed-interval that is an integer-ratio multiple of the warehouse replenishment cycle. This type of time-based inventory policy is very common in practice because it facilitates easy coordination and simple scheduling of inventory activities (order picking, loading, unloading, receiving, etc.) at the warehouse and retailers.

Our main objective of this study is to investigate stock allocation policies. To focus on this objective, we assume that the replenishment intervals of the warehouse and the retailers are determined exogenously based on operational schedules with other partners. Under this condition, no explicit ordering costs will be considered in our model. Economies of scale in ordering can be taken into consideration by the length of the replenishment cycle for the warehouse or a retailer. Material flows through the network are coordinated.

An allocation policy decides on the amount of stock to be kept at the warehouse and to be shipped to each retailer in each shipping period. Optimal inventory control policies for multi-echelon inventory systems are notoriously complicated. In particular, the optimal control of a distribution system with heterogeneous retailers is still largely an open problem. An optimal policy, even if it exists, can be very complex (Clark and Scarf 1960). One reason is that the presence of multiple retailers requires that the warehouse adopt a policy for allocating the remaining stock to retailers when facing a shortage. Because an optimal warehouse stock allocation often depends on the inventory positions at the warehouse and every retailer, the inventory state space for optimal warehouse stock allocation often entails a large dimension and precludes an exact evaluation

of the expected system cost. This problem is commonly referred to as the curse of dimensionality in the literature (Federgruen and Zipkin 1984). As a result, previous studies on distribution system with stochastic demand have developed only approximate solutions to derive the optimal control parameters to minimize system cost or fulfill a given service level. According to Cachon (2001), evaluating the performance of an inventory control policy has been a major challenge, letting alone providing an exact solution for a distribution system. Thus finding effective heuristic policies are of practical importance.

Clark and Scarf (1960) proposed a balance assumption, which allows free transshipments between retailers at a point of warehouse stock allocation, in order to overcome the problem of dimensionality. Later on many studies have adopted this assumption to investigate risk pooling effect in distribution systems (Eppen and Schrage 1981, Macgavin et al. 1993). Nevertheless, these studies have assumed a common replenishment schedule for all retailers in order to focus on the issue of stock pooling and as a result failed to provide a method to evaluate a general periodic-review inventory control policy for a two-level distribution system.

A number of studies have considered general periodic-review inventory control policies for distribution systems (Graves 1996, Axsäter 1990, 1993, Marklund 2011, Shang et al. 2015). These studies, however, have assumed that warehouse stock is allocated or reserved to satisfy demand at retailers on a first-come first-served (FCFS) basis. This allocation method is referred to as virtual allocation or complete reservation in the literature. When stock is reserved to satisfy retailer demand on a FCFS basis, the movement of every unit of stock in the system can be fully tracked. As a result, an exact system cost evaluation can be

produced. Unfortunately, although virtual allocation provides model tractability from a mathematical point of view, it is clearly not optimal because it does not take the relative needs for stocks at the retailers into consideration (Graves 1996). Jackson (1988) proposed a ship-up-to- S policy, under which the warehouse makes shipments to restore a retailer's inventory position to a predetermined level in every period if there is sufficient stock. When the warehouse does not have sufficient stock, all the remaining stock is allocated among the retailers to minimize their costs over the rest of the cycle. We refer this allocation rule as myopic optimal allocation. He developed a function to approximate the expected holding and shortage costs per cycle under the proposed policy.

In view of the literature, the current study aims to provide a comparison of several warehouse stock allocation policies. In our model, we assume that at the beginning each ordering cycle, the warehouse places an order to raise the echelon inventory position to S_0 . Upon the arrival of the warehouse order and subsequently at each shipping point for the retailers, the warehouse decides how much to allocate to reach retailer based on available stock.

First we consider a stationary ship-up-to- S policy with myopic warehouse stock allocation. Under this policy, at each shipping point, the warehouse ships the needed stock to each retailer i to raise its inventory position to a fixed base stock S_i for $i = 1, \dots, N$. When the warehouse faces a shortage, the available stock at the warehouse is completely allocated to the retailers. Two specific myopic allocation rules are considered. Namely, virtual allocation allocates the remaining stock to meet the demand at the retailers on a FCFS basis and myopic optimal allocation allocates the available inventory to minimize the cost incurred at the retailers for the rest of the cycle. We develop an general approach to evaluate

the expected inventory related costs in the system for this class of stationary ship-up-to- S policy with myopic allocation. We provide an approach to optimize base stock levels, i.e., S_0 at the warehouse and $\mathbf{S} = (S_1, \dots, S_N)$ at the retailers.

For cost-efficiency management of distribution systems, important questions are how much total inventory to carry within the system and where to deploy it so as to minimize the total inventory holding and backorder costs (Axsäter et al. 2002). It is important to determine how much central stock to carry at the warehouse. Typically there are three sources of benefits from centralizing inventory at the warehouse: (1) external lead time inventory pooling, which can reduce the uncertainty faced by the system by carrying a single inventory during the warehouse lead time rather than individual retailer inventories (Eppen and Schrage 1981), (2) holding cost saving at the central warehouse (or upstream) as the holding cost at the upstream of supply chains is usually smaller than that at downstream (Shang and Song 2003, Wang and Wan 2015), (3) rebalancing inventory among retailers during a replenishment cycle, which means that the warehouse replenishment is not completely allocated so that a part of the stock is kept back to rebalance local stocks at additional shipment opportunities later on in the replenishment cycle. This postponement causes that the out of stock risk is pooled, with the drawback that the central stock is not immediately available to satisfy customer demand (McGavin et al. 1993).

A myopic allocation policy captures the benefits of stock centralization. However, there are several concerns. First, although myopic optimal allocation takes the retailers' relative needs for inventory into consideration, it does not relate the available stock to future demand since base stock levels for the retailers are not adjusted based on the actual realized demand. Intuitively, a non-stationary or

dynamic policy may perform better. Second, it requires $N + 1$ control variables, i.e., S_0 and $\mathbf{S}$. Finding the optimal values for these control variables can be very complex and time-consuming, especially for large number of retailers, high arrive rate and long replenishment cycle. Third, the formulations and the methodologies developed in multi-echelon production and distribution systems are often computationally intractable and difficult to explain to practitioners (Gallego et al. 2004). This is particularly a concern to implementation. A firm may not be willing or able to implement an inventory control policy without a full understanding. The evaluation and optimization approaches we develop for myopic allocation policies also face this obstacle.

To cope with these challenges, we propose a heuristic dynamic allocation policy, which not only could perform potentially better in terms of cost control but also is easy-to-describe, compute and implement. With this motivation, we establish a computationally efficient heuristic dynamic allocation policy, which explicitly incorporates the three major benefits of stock centralization. In this novel heuristic, we still assume that the warehouse places an order to raise the echelon inventory position to S_0 in a fixed time interval of T_0 . The warehouse will allocate its available inventory to the retailers during the replenishment cycle at each shipping point. The allocation decision at each point includes two parts: (1) how much to reserve at the warehouse, (2) how much to allocate to each retailer. Wang and Wan (2015) studied an N -stage serial system and provides an intuitive characterization of an optimal ordering policy by simply equating the marginal echelon inventory related cost to the inventory holding cost at the upstream stage. Based on this finding, we propose a two-step dynamic heuristic allocation policy. In step 1, we determine how much to allocate to each retailer by simply

equating the marginal echelon inventory related cost to the inventory holding cost at the upstream stage, which is referred as marginal cost allocation. However, this simple allocation may ignore partial benefit of inventory rebalancing during the replenishment cycle. Then, we further make an adjustment to marginal cost allocation according to the needs of retailers for the rest of system cycle, which is referred as refined marginal cost allocation.

With this heuristic allocation, it would be extremely difficult to evaluate the long-run average system costs exactly, if not impossible. Therefore we resort to simulation approach. We demonstrate through an extensive numerical study that the novel heuristic allocation rule can be very effective in terms of cost control. Most importantly, this heuristic policy significantly reduces the complexity of system optimization by considering only system order-up-to level and therefore can cope with a relatively large number of retailers, a large demand arrival rate and a long replenishment cycle. Practitioners should find this policy easy to understand and implement.

The structure of the chapter is organized as follows. In section 3.2, we will provide a review of the relevant literature. In section 3.3, we describe our model setting. In section 3.4, we analyze the myopic allocation policy and provide cost evaluation and optimization methods. In section 3.5, we propose a novel heuristic dynamic allocation policy. Extensive numerical analysis are presented in section 3.6. Finally we conclude in section 3.7.

3.2 Literature Review

There exists a large literature on multi-echelon inventory systems. For serial systems, the optimality of inventory policies are analyzed by Clark and Scarf (1960), van Houtum (2007) and Chao and Zhou (2009). For distribution systems, the optimal control policy is still unknown. It is generally recognized that, even if it exists, the optimal policy can be very complex (Clark and Scarf 1960). A common approach is approximating a distribution system by balance assumption, which allows complete and costless reallocation of stock in the system at a point of inventory review. This assumption was first suggested by Clark and Scarf (1960) to decouple the warehouse stock allocation decision from the inventory positions at individual retailers. Many studies use the balance assumption because of its mathematical tractability. However, the balance assumption does not always provide a good approximation, for example, in situations where the order cycles are long, the demand variability is high, and the retailers are very different in terms of service requirements and demand characteristics (Axsäter 2002, Marklund and Rosling 2012, Doğru et al. 2009). Other literature focuses on specific heuristic allocation policy. In the following we will summarize the literature in two streams. The first stream studies the performance of their proposed heuristic allocation policies. The second stream studies the benefits of stock centralization in a distribution system.

In the first stream, Jackson (1988) considered a distribution system with a class of ship-up- S policy in one replenishment cycle including N equal intervals. The policy specifies that at the start of each interval, the inventory positions of the retailers are raised to their prescribed order-up-to levels as long as sufficient

central stock is available. When facing a shortage, the warehouse will allocate all remaining stock among the retailers to minimize their costs over the rest of the cycle, which is referred to as myopic optimal allocation. His numerical results show significant effects of risk pooling. But his analysis is limited to identical retailer case. Van der Heijden (1999) considered similar setting except when facing a shortage, the warehouse will allocate retailers according to predetermined fixed proportions. He proposed an approximate approach to determine the rationing policy and order-up-to level separately for all stockpoints to satisfy a given fill rate at each retailer. Marklund and Rosling (2012) formulated a stochastic dynamic programming problem to solve how a warehouse should allocate a given amount of stock to N retailers, when m time periods until the next warehouse replenishment arrives. They presented a new lower bound for the system expected cost. Based on the lower bound solution, they suggested a heuristic solution, which outperforms the heuristic by Jackson(1988).

The above literature only focuses on one replenishment cycle and provides only approximate cost evaluation approach. Some other studies assume non-optimal warehouse stock allocation policies under which the expected system cost can be explicitly and exactly evaluated. The exact evaluation of an inventory control policy for a distribution system has been a major challenge (Cachon 2001). Graves (1996) first proposed the virtual allocation rule in a periodical-review setting. He assumed that warehouse stock is allocated or reserved to satisfy demand at retailers on a FCFS basis. Axsäter (1990, 1993) provided an exact cost evaluation approach for distribution system with continuous review by assuming the FCFS allocation rule. Shang et al. (2015) considered a periodic-review distribution system with multiple non-identical retailers and Poisson arrivals. By assuming

virtual allocation and fixed-interval order-up-to policy, they developed an evaluation scheme and provide a method to obtain the optimal base-stock levels and reorder intervals.

Although virtual allocation is tractable from a mathematical point of view, it is not optimal, since it does not account for differences in needs for stocks at downstream stockpoints at allocation times. Axsäter (2007) showed that the FCFS rule can, in the worst case, lead to an asymptotically unbounded relative cost increase as the number of retailers approaches infinity. Wang and Axsäter (2013) studied fixed-interval ordering policy in a distribution system with one virtual warehouse and N non-identical retailers. By assuming Poisson arrivals, they provided an approach to exactly evaluate and optimize the system under the balance assumption. Wang (2013) extends the study to a non-virtual warehouse and relaxes the balance assumption to myopic optimal allocation. But the study is restricted to identical retailers. Our study builds on the work by Wang and Axsäter (2013) and Wang (2013). We extend their research to non-identical retailers case and generalize the cost evaluation approach for ship-up-to- S policy and general myopic allocation.

Besides considering the replenishment cycle of fixed length, other studies consider batch quantity ordering policy. Cachon (2001) studied a distribution system with identical retailers. He proposed a continuous-review batch ordering policy and a random allocation rule and provides a method to exactly evaluate the average inventory, backorders and fill rates for each location in the system. Axsäter et al. (2002) assumed that the warehouse replenishments must be multiples of a given batch quantity and suggest a computationally tractable heuristic for allocating stock from the warehouse to the retailers. They suggested a two-step

allocation heuristic that is used in every period to determine how much to retain at the warehouse for the last period in the order cycle. Their simulation results show that risk pooling becomes very important when the retailers experience large differences in cost and demand characteristics. Gallego et al. (2007) developed simple approximate methods to analyze a distribution system in which every installation uses an $(S - 1, S)$ policy.

The second stream focuses on analyzing the merits of centralizing inventory at the warehouse, which generally include three categories: (1) external lead time inventory pooling, (2) rebalancing inventory among retailers during replenishment cycle, (3) holding cost saving at the central warehouse. Eppen and Schrage (1981) studied the first merit by considering an inventory system with identical retailers and normally distributed demand where the central warehouse is not allowed to carry any stock. They assumed that the retailers place orders directly to the manufacturer, and the manufacturer ships all the stock to the central warehouse. The warehouse receives the inventory after the external leadtime, and then reallocates the warehouse shipments according to inventory consumption during the external leadtime. This reallocation yields a risk-pooling effect during the leadtime. Kumar and Jacobson (1998) built on the Eppen and Schrage model and provide methods for evaluating system costs and the probability that inventory imbalances will occur under different procurement and allocation policies. Schwarz (1989) analyzed the value of risk-pooling over the outside supplier lead time. He concluded that the value is especially high in situations with significant demand variability, long warehouse lead time, and short retailer lead time.

Many literature study the benefits of inventory rebalancing by using two allocation points. Jönsson and Silver (1987) considered a distribution system with a

virtual warehouse and identical retailers by allowing two shipment opportunities in a replenishment cycle, first at the start of the cycle, followed by an opportunity one period before the end of the cycle. The stock is entirely allocated at the first opportunity and reallocated amongst the local stockpoints at the second opportunity using mutual transshipments. They showed that significant safety stock reductions by the second redistribution. Erkip (1984) proposed an alternative allocation scheme, in which a fraction of the warehouse inventory is allocated at the beginning of the order cycle and the rest is retained for an additional allocation opportunity later in the cycle. Jackson and Muckstadt (1989) used a two-period allocation problem to quantify the benefits of keeping stock at the warehouse. McGavin et al. (1993) designed a 50/25 heuristic allocation rule. That is, the replenishment cycle contains two shipment opportunities, one at the start and one after 50% of the replenishment cycle. At the second shipment opportunity, 25% of the mean replenishment cycle demand is reserved. They showed that a second allocation opportunity seems to capture most of the inventory balancing effect of keeping stock at the warehouse. Cao and Silver (2005) assumed that the warehouse is replenished regularly on a fixed cycle. Most of the stock is directly shipped to the retailers but some stock is sent to the central warehouse. At the beginning of any one of the periods during the cycle, the central stock can be completely allocated out to the retailers. They allow only one allocation opportunity to balance inventory among the retailers in one cycle. They proposed a heuristic method to dynamically (as retailer inventory levels change with time) determine the appropriate period in which a allocation is carried out. McGavin et al. (1997) examined the optimality of inventory balancing in a distribution system over T successive time intervals. They showed that for the identical-retailer case, bal-

ancing at the start of the first interval is optimal whenever the T -interval cost function exhibits a general convexity condition but for the non-identical-retailer case, balancing is generally not optimal.

For multi-echelon system, local holding cost at upstage is usually smaller than the holding cost at downstage as the adding value increases, which is also a common assumption in the literature. With this property, Shang and Song (2003) derived effective bounds heuristic solutions for optimal inventory policies in serial systems. Wang and Wan (2015) studied a N -stage serial system with fixed replenishment schedule and echelon order-up-to policy. They find that given a replenishment schedule, under an optimal echelon ordering policy, the marginal cost at an echelon is equal to the inventory holding cost at the upstream stage. This provided an intuitive characterization of an optimal ordering policy by simply equating the marginal echelon inventory related cost to the inventory holding cost at the upstream stage.

3.3 Model Setting

We consider a single-item supply chain that consists of one central warehouse and a group of N non-identical retailers as shown in Figure 3.1. The system adopts a periodic-review inventory control policy in which inventory activities are carried out only at the end or beginning of a fixed base planning period. The length of the base planning period is normalized to be 1 in the analysis. Without further elaboration, all time variables in the analysis are integer multiples of the base planning period or simply integers since the length of the base planning period is normalized to be 1.

Figure 3.1: Structure of the supply chain

The warehouse orders from an external supplier to replenish the retailers. It is assumed that the external supplier has unlimited stock and a shipment from the external supplier arrives at the warehouse after a constant lead time L_0 . This unlimited stock for external supplier assumption is assumed for the modeling purpose and it is reasonable because usually in the most upper stage of a supply chain, the product is possibly still in the form of raw materials (or commodity product), which generally can be directly ordered from spot market etc. This is also a common assumption in the inventory literature, such as Axsäter (1993), Graves (1996), Shang et al. (2015). The warehouse has real-time information on each retailer's demand and inventory level. Lead time from the central warehouse to retailer i is L_i . Demand that is not satisfied is fully backordered at each stocking point and lateral transshipments are not allowed between retailers.

We assume that retailer i faces a Poisson demand with a stationary rate λ_i per period for $i = 1, \dots, N$ and the demand processes at the retailers are independent. Poisson process is a common approach to model demand arrival process in the inventory literature, such as Axsäter (1993), Wang (2013), Shang et al. (2015). There are also some empirical evidence showing that Poisson demand was a good model for the order process, such as in a wholesale company (Ehrenberg 1959, Johnston and Boylan 1996).

The following notations are defined.

(i) $\lambda_0 = \sum_{i=1}^N \lambda_i$,

(ii) $g(x, \lambda)$ and $G(x, \lambda)$ denote the probability mass function and the cumulative probability function, respectively, for the Poisson distribution with mean λ , i.e., $g(x, \lambda) = e^{-\lambda} \lambda^x / x!$ and $G(x, \lambda) = \sum_{y=0}^x g(y, \lambda)$,

(iii) $[t, t+\delta)$ and $[t, t+\delta]$ denote the time intervals over periods $t, t+1, \dots, t+\delta-1$ and periods $t, t+1, \dots, t+\delta$, respectively, and

(iv) $D_i[t, t+\delta)$ and $D_i[t, t+\delta]$ denote the cumulative demand for retailer i over time periods in $[t, t+\delta)$ and $[t, t+\delta]$, respectively.

In addition, we assume that all inventory related costs are accounted at the end of every period and the system faces (i) a local or installation holding cost h_i per unit of on-hand inventory at the warehouse ($i = 0$) or retailer i ($i = 1, 2, \dots, N$) per unit time and (ii) a linear backorder cost b_i per unit of backlogged demand at retailer i ($i = 1, 2, \dots, N$) per unit time. No ordering cost is explicitly considered. A list of major notations used in this study is provided in the appendix.

3.3.1 System Replenishment Schedule

We assume that the central warehouse orders from the outside supplier in a fixed replenishment interval T_0 and replenishes the retailers in a common fixed interval of length τ that is an integer-ratio multiple of the warehouse replenishment interval T_0 , i.e., $\tau = T_0/n$, where n is a positive integer. This type of time-based control is very common in practice because it facilitates easy coordination and simple scheduling of inventory activities (order picking, loading, unloading, receiving, etc.) at the warehouse and retailers. We further assume that T_0 and τ

are determined exogenously, which could be as a result of trade-offs between cycle stock and ordering costs or as a given transport frequency allowing to combine the delivery of multiple products from the same supplier (Van der Heijden 1999). As in Chao and Zhou (2009) and Wang (2013), we further assume that all shipments are synchronized, which means that the warehouse always replenishes the retailers immediately after receiving a shipment from the external supplier. Chao and Zhou (2009) showed that a synchronized shipping policy always dominates a non-synchronized one.

Without loss of generality, we let the warehouse review the inventory position and order from the external supplier at (the beginning of) period 0. The shipment from the external supplier arrives at the warehouse at period L_0 . As $T_0 = n\tau$, there are a total of n shipment opportunities in one replenishment cycle. Specifically, the warehouse allocates and ships stock to the retailers at period $t_j = L_0 + (j - 1)\tau$, for $j = 1, \dots, n$. We refer to periods t_j for $j = 1, \dots, n$ as the shipping points in a warehouse replenishment cycle.

To describe the dynamics of the system in a system replenishment cycle, we illustrate in Figure 3.2 an example of $T_0 = 5$, $\tau = 1$ and $L_0 = 2$. In a replenishment cycle for this example, there are 5 shipping points for the warehouse at time $t_j = 1 + j$ for $j = 1, \dots, 5$. A shipment made by the warehouse to retailer i arrives at the retailer after L_i periods.

Figure 3.2: Replenishment schedule for an example with $\tau = 1$, $L_0 = 2$, $T_0 = 5$

We assume that all events take place at the beginning of a period in the

following sequence:

- (1) The warehouse decides whether to order from the outside supplier.
- (2) Deliveries from the outside supplier arrive at the warehouse.
- (3) All or part of the stock on hand at the warehouse is allocated to the retailers.
- (4) Deliveries from the warehouse arrive at the retailers
- (5) The stochastic demand takes place at the retailers

3.3.2 Warehouse Ordering and Allocation Policies

We assume that the warehouse employs a base-stock policy to order from the external supplier and a ship-up-to- S policy to replenish the retailers (Jackson 1988). Specifically, at each review point, the warehouse orders the needed stock from the outside supplier to raise the echelon inventory position to a base stock level S_0 . Then for the warehouse, at each delivery point, it needs to decide how to allocate its available stock to each retailer.

The main objective of this study is to evaluate different warehouse allocation policies. In view of the literature, we consider two types of warehouse stock allocation policies.

3.3.2.1 Stationary Myopic Allocation Policies

A myopic allocation policy has two requirements. First, the warehouse maintains a stationary base stock or target level S_i for each retailer $i = 1, \dots, N$. Second, no stock is kept at the warehouse when there is unsatisfied demand. According to this definition, a myopic allocation policy requires that the warehouse ship the needed stock to raise the inventory position at retailer i to the same base

stock level S_i at each delivery point if there is sufficient stock. If the warehouse does not have sufficient stock to achieve this objective at a delivery point, all the remaining stock is allocated to the retailers and no delivery can be made to any retailer thereafter until the arrival of the next shipment from the external supplier. We define $\mathbf{S} = \{S_1, S_2, \dots, S_N\}$ to facilitate the discussion below.

In addition, when facing a shortage, the warehouse must adopt a rule to allocate the remaining stock to the retailers. In view of the literature, we consider two allocation policies. First, the unsatisfied demand is met on a FCFS basis according to their occurrence time. This allocation rule is referred to as virtual allocation or complete reservation in the literature (Graves 1996, Shang et al. 2015). Second, the remaining stock is allocated optimally based on the inventory positions at the retailers to minimize system cost. We refer to this allocation rule as myopic optimal allocation (Wang 2013). The former provides modeling tractability but does not consider the relative needs for inventory at the retailers. The latter makes the allocation decision at the moment of shipping based on the real-time inventory position of retailers. However, this allocation rule increases model complexity.

3.3.2.2 Dynamic Allocation Policy

Although myopic optimal allocation takes the retailers' relative needs for inventory into consideration, it fails to relate the current inventory to the future demand. Intuitively, the base stock levels at the retailers, i.e. $\mathbf{S}$, should be determined based on the stock available and the demand to be satisfied. This means that the stationary requirement of the base stock levels at the retailers imposes a limitation on the efficiency of the inventory control system. A more

efficient warehouse stock allocation policy is then to determine S_i for $i = 1, \dots, N$ dynamically at each delivery point based on the inventory status of the system as well as the future demand to be satisfied for the remaining stock. With this motivation, we also consider a dynamic warehouse allocation policy in this study. Nevertheless, because of difficulties in system evaluation and optimization, we restrict the study in this part to a simple heuristic allocation rule.

3.4 System Cost Evaluation and Optimization under Myopic Allocation

In this section, we consider the inventory control system under myopic warehouse stock allocation. We first develop a general cost evaluation for the distribution system under myopic warehouse stock allocation. The features of two specific myopic warehouse stock allocation policies, namely, virtual allocation and optimal allocation, are subsequently, incorporated. We derive the optimal base-stock levels for the warehouse and retailers.

3.4.1 Expected Inventory Related Cost in a System Replenishment Cycle

Because the considered inventory control system is built on the single-stage fixed-interval order-up-to policy (Rao 2003), we first introduce the cost function of this generic inventory control policy. Subsequently, we define a system replenishment cycle, analyze the probabilities for the warehouse to run out of stock at a shipping point, and define a general allocation function when a shortage occurs. Finally,

we aggregate the cost evaluations at the warehouse and retailers to derive an evaluation of the expected inventory related cost in a system replenishment cycle.

3.4.1.1 The Single-stage Fixed-interval Order-up-to Policy

Suppose that the warehouse always has ample stock. Then each retailer i operates under the single-stage fixed-interval order-up-to policy, in which the inventory position is raised to a fixed order-up-to level in a fixed interval. To facilitate the analysis below, we first present the evaluation of a single-stage fixed-interval order-up-to policy for retailer i , in which the inventory position at the retailer is raised to an order-up-to level S_i in a fixed interval of length T_i .

Suppose that retailer i places an order to raise its inventory position to S_i at period t_0 . The shipment will arrive L_i periods later. Let $\pi_i(S_i, T_i)$ denote the expected inventory related cost at the retailer in the interval $[t_0 + L_i + 1, t_0 + T_i + L_i]$. At the end of any period $[t_0, t)$ for $t_0 + L_i < t \leq t_0 + T_i + L_i$, the inventory level (i.e., inventory on-hand minus backorders) at the retailer is equal to $S_i - D_i[t_0, t]$ and the expected inventory holding and backordering cost at the retailer in the interval $[t_0, t)$ is then equal to $h_i E[(S_i - D_i[t_0, t])^+] + b_i E[(D_i[t_0, t] - S_i)^+]$. Then $\pi_i(S_i, T_i)$ is equal to the sum of the expected inventory holding and backordering costs in the interval $[t_0 + L_i + 1, t_0 + L_i + T_i]$,

$$\pi_i(S_i, T_i) = \sum_{t=t_0+L_i+1}^{t_0+L_i+T_i} \left\{ h_i E[(S_i - D_i[t_0, t])^+] + b_i E[(D_i[t_0, t] - S_i)^+] \right\}.$$

We obtain after simplification

$$\pi_i(S_i, T_i) = b_i \left[\lambda_i \left(L_i + \frac{T_i + 1}{2} \right) - S_i \right] T_i + (h_i + b_i) \sum_{t=L_i+1}^{L_i+T_i} \sum_{x=0}^{S_i-1} G(x, \lambda_i t). \quad (3.1)$$

3.4.1.2 A System Replenishment Cycle

Without loss of generality, we initiate the system with a warehouse order to the external supplier at time 0, raising the system or echelon-stock inventory position to S_0 . Subsequently, the warehouse repeats this action in a fixed interval of length T_0 at time $T_0, 2T_0$, etc. We assume that the system adopts the same inventory control policy in each of these intervals and refer to such an interval as a system replenishment cycle. Under this condition, we consider only a single system replenishment cycle for the cost evaluation analysis below. More specifically, we consider the system replenishment cycle $[0, T_0)$. Since the shipment from the external supplier arrives at the warehouse at time L_0 , we evaluate the inventory related cost at the warehouse in the interval $[L_0, L_0 + T_0)$. Accordingly, the warehouse makes the first delivery to retailer i at time L_0 immediately after receiving the shipment from the external supplier at time L_0 . The first shipment from the warehouse arrives at the retailer at time $L_0 + L_i$. We evaluate the inventory related cost at retailer i in the interval $[L_0 + L_i, L_0 + L_i + T_0)$ for $i = 1, \dots, N$.

In addition, let $R = S_0 - \sum_{i=1}^N S_i$. We can define the warehouse ordering policy alternatively and equivalently as to order the needed stock to raise the (installation) inventory position at the warehouse to R and the inventory position at each retailer $i = 1, \dots, N$ to S_i . Since this definition is simple and easy to follow, it is used in the analysis below.

3.4.1.3 Analysis at Warehouse Shipping Points

Consider a shipping point at the warehouse at time t_j . If there is sufficient stock, the warehouse will ship the needed stock to raise retailer i 's inventory position to S_i . Otherwise, the remaining stock will be completely allocated and shipped to the retailers at this point and no more shipment will be made to any retailer thereafter until the arrival of the next shipment from the external supplier.

More specifically, we consider the warehouse stocking conditions at the shipping points $t_j = L_0 + (j - 1)\tau$ for $j = 1, \dots, n$. Let $Y_j = \sum_{i=1}^N D_i[0, t_j)$ denote the total demand at the retailers in the interval $[0, t_j)$ for $j = 1, \dots, n$ and $X_j = Y_j - Y_{j-1}$ for $j = 2, \dots, n$, where $Y_0 = 0$ and $t_{n+1} = L_0 + T_0$. Then Y_{j-1} and X_j are independent and follow Poisson distributions with mean $\lambda_0 t_{j-1}$ and $\lambda_0 \tau$, respectively, for $j > 0$. Let $Y_0 = 0$ and $X_1 = Y_1$ to accommodate the case at t_1 . Given X_j and Y_j , we will characterize whether the warehouse is out of stock or not at each shipping point in the following.

(1) Out of stock at t_j

Suppose the warehouse is out of stock at t_j for $j = 1, \dots, n$. This means that for shipping points t_k ($k < j$) before t_j , the warehouse has sufficient stock to bring the inventory position of each retailer back to its target level. For shipping points t_k ($k > j$) after t_j , the warehouse will be out of stock and there will be no shipment to retailers after t_j in the current system cycle. At t_j , the warehouse only has $R - Y_{j-1} > 0$ stock left for allocation. Now the warehouse needs x_j amount of stock to bring each retailer's inventory position back to the target level. Therefore the warehouse out of stock at t_j implies that $X_j \geq R - Y_{j-1}$.

Let the probability of out of stock at t_j be P_j . Then we have

$$P_j = P(R - Y_{j-1} > 0 \quad \& \quad X_j \geq R - Y_{j-1}), \quad \text{where} \quad P_1 = P(Y_1 \geq R).$$

For example, consider that the shortage occurs at shipping point t_2 in Figure 3.2. Thus when evaluating the inventory-related cost at retailers, we need to consider two intervals: $[t_1 + L_i, t_2 + L_i)$ and $[t_2 + L_i, t_1 + T_0 + L_i)$, since the inventory positions of the retailers will not be raised up to $\mathbf{S}$ at period t_2 . The evaluation for other shipping points applies similarly.

(2) No out of stock in the system replenishment cycle

It is noted that t_n is the last shipping point in the current cycle and, therefore, the warehouse has no shortage in the current system replenishment cycle if $Y_n < R$.

Let the probability of no stock-out be P_0 and we have

$$P_0 = P(Y_n < R) = G(R - 1, \lambda_0 t_n).$$

In this case, at each shipping point, the inventory position of retailer i is raised back to S_i . There will be n deliveries and each delivery will raise the inventory position of each retailer back to its target level. Thus when evaluating the expected inventory-related cost at the retailers in the replenishment cycle, we need to consider n intervals of length τ .

3.4.1.4 Inventory Allocation

If a shortage occurs at shipping point t_j for $j = 1, 2, \dots, n$, then the warehouse needs to allocate the available but insufficient amount of stock $R - Y_{j-1}$ to the retailers, where $Y_0 = 0$ for notation convenience.

Let $D_{ji} = D_i[t_{j-1}, t_j)$ denote the demand at retailer i in the interval $[t_{j-1}, t_j)$ for $j = 1, \dots, n$, where $t_0 = 0$. Note $X_j = \sum_{i=1}^N D_{ji}$. Let IP_{ji} be the inventory position of retailer i before allocation at t_j and let S_{ji} be the inventory position of retailer i after allocation at t_j . Then we have $IP_{ji} = S_{ji} - D_{ji}$. Define vectors $\mathbf{D}_j = \{D_{j1}, D_{j2}, \dots, D_{jN}\}$, $\mathbf{IP}_j = \{IP_{j1}, IP_{j2}, \dots, IP_{jN}\}$, and $\mathbf{S}_j = \{S_{j1}, S_{j2}, \dots, S_{jN}\}$.

After this allocation, there will be no further replenishment for the rest of the current cycle, i.e., the interval $[t_j, L_0 + T_0)$. The length of the rest of cycle after allocation is denoted by $T'_j = (n - j + 1)\tau$.

Now we define a general allocation function $\mathbf{S}_j = \Psi(Y_{j-1}, X_j, \mathbf{IP}_j, T'_j)$. Variables X_j , Y_{j-1} , and $\mathbf{IP}_j$ represent the inventory status of the system before allocation at t_j . The distribution of X_j and Y_{j-1} have been given in the analysis of the previous section. The function $\Psi(\cdot)$ determines how to allocate the available inventory $R - Y_{j-1}$ and the output is the inventory positions of all the retailers after allocation. Of course we have $S_{ji} \geq IP_{ji}$. Later we will provide details on function $\Psi(\cdot)$ for specific allocation rules we consider.

Define $\phi_j(\mathbf{S}_j, T'_j | Y_{j-1}, X_j, \mathbf{D}_j) = \sum_{i=1}^N \pi_i(S_{ji}, T'_j)$. Then $\phi_j(\cdot)$ represents all the retailers' expected inventory related cost in the rest of the cycle after allocation, where the value of $\mathbf{S}_j$ depends on the allocation function $\Psi(\cdot)$ with the inventory status Y_{j-1} , X_j , and $\mathbf{D}_j$.

Further define $\varphi_j(\cdot | Y_{j-1}, X_j) = \frac{E}{\mathbf{D}_j}[\phi_j(\mathbf{S}_j, T'_j | Y_{j-1}, X_j, \mathbf{D}_j)]$, which is the ex-

pected value of $\phi_j(\cdot)$ on stochastic vector $\mathbf{D}_j$.

3.4.1.5 Cost Aggregation

In this part, we will first aggregate the retailers' inventory related cost in the replenishment cycle and then derive the holding cost incurred at the warehouse. For the retailers, we have derived the expected inventory related cost for the case that each retailer's inventory position is raised back to its target level, which is $\pi_i(S_i, \tau)$. When shortage occurs, the retailers' expected inventory related cost for the rest of the cycle after allocation is $\varphi_j(\cdot | Y_{j-1}, X_j)$. In the following we will add up all the installation' cost together for one replenishment cycle and derive the system average cost in one cycle for all possible values of Y_{j-1} and X_j , which characterize when and how much the shortage occurs. Let y_{j-1} and x_j represent the value of random variables Y_{j-1} and X_j , respectively.

(1) Out of stock at t_j

If the warehouse is out of stock after allocation at shipping point t_j , there are $(j - 1)$ shipping points before t_j that each retailer's inventory position is raised back to its target level. Let C_{rj} be the retailers' cost in one replenishment cycle if out of stock occurs at t_j .

Then if $j = 1$, $Y_0 = 0$ and $X_1 \geq R$. As the warehouse is out of stock at the first shipping point, we only need to consider one interval of length T'_1 to evaluate the retailers' expected costs. By aggregating over all possible values of $X_1(\geq R)$, we have $C_{r1} = c_1$, where

$$c_1 = \sum_{x_1=R}^{\infty} \varphi_1(\cdot | y_0, x_1) g(x_1, \lambda_0 L_0).$$

If $j \geq 2$, $Y_{j-1} > R$ and $X_j \geq R - Y_{j-1}$. Then we need to consider $j - 1$ intervals of length τ before t_j when the retailers are raised back to $\mathbf{S}$ and one interval of length T'_j after the warehouse is out of stock. Then we have $C_{rj} = P_j \sum_{k=1}^{j-1} \sum_{i=1}^N \pi_i(S_i, \tau) + c_j$, where

$$c_j = \sum_{y_{j-1}=0}^{R-1} \sum_{x_j=R-y_{j-1}}^{\infty} \varphi_j(\cdot | y_{j-1}, x_j) g(x_j, \lambda_0 \tau) g(y_{j-1}, \lambda_0 t_{j-1}).$$

In summary, without loss of accuracy, we drop the subscript of x_j and y_{j-1} for the ease of exposition, we have

$$C_{r1} = \sum_{x=R}^{\infty} \varphi_1(\cdot | y, x) g(x, \lambda_0 L_0) \quad (3.2)$$

$$C_{rj} = P_j \sum_{k=1}^{j-1} \sum_{i=1}^N \pi_i(S_i, \tau) + \sum_{y=0}^{R-1} \sum_{x=R-y}^{\infty} \varphi_j(\cdot | y, x) g(x, \lambda_0 \tau) g(y, \lambda_0 t_{j-1}), \quad \text{for } j = 2, \dots, n. \quad (3.3)$$

(2) No out of stock

Since there is no shortage in the system replenishment cycle, the retailers' cost could be easily evaluated as follows, which is denoted by C_{r0} ,

$$C_{r0} = P_0 \sum_{k=1}^n \sum_{i=1}^N \pi_i(S_i, \tau). \quad (3.4)$$

By summarizing $n + 1$ cases at the above, we can have the retailers' expected inventory related cost in one system cycle, denoted by C_r , for all possible values of X_j and Y_{j-1} . Then $C_r = \sum_{j=1}^n C_{rj}$. Further, note that $P_0 + \sum_{j=1}^n P_j$ means that out of stock occurs after shipping point t_{j-1} , which equals $G(R - 1, \lambda_0 t_{j-1})$.

With this and by some mathematical manipulations, we have

$$\begin{aligned}
C_r &= P_0 \sum_{k=1}^n \sum_{i=1}^N \pi_i(S_i, \tau) + \sum_{j=2}^n P_j \sum_{k=1}^{j-1} \sum_{i=1}^N \pi_i(S_i, \tau) + \sum_{j=1}^n c_j \\
&= \sum_{i=1}^N \pi_i(S_i, \tau) P_0 + \sum_{j=1}^{n-1} \sum_{i=1}^N \pi_i(S_i, \tau) G(R-1, \lambda_0 t_j) + \sum_{j=1}^n c_j \\
&= \sum_{j=1}^n \sum_{i=1}^N \pi_i(S_i, \tau) G(R-1, \lambda_0 t_j) + \sum_{j=1}^n c_j. \tag{3.5}
\end{aligned}$$

(3) Warehouse's cost

Now we investigate the holding cost of warehouse incurred in one system cycle. As we know, the holding cost incurs only when there is left stock carried over to next shipping point, then we will look at it at each shipping point. After shipping at t_j , the holding cost of left stock will be incurred in next τ periods, i.e., during the interval $[t_j, t_{j+1})$. The average left stock after shipping at t_j can be obtained by $\sum_{y_j=0}^R (R - y_j) g(y_j, \lambda_0 t_j)$. Then the holding cost incurred in $[t_j, t_{j+1})$ is $h_0 \sum_{y_j=0}^R (R - y_j) g(y_j, \lambda_0 t_j) \tau$. We summarize the holding cost incurred in one replenishment cycle. We can have the warehouse's cost incurred during one system cycle, denoted by C_w .

$$\begin{aligned}
C_w &= h_0 \sum_{j=1}^n \sum_{y_j=0}^R (R - y_j) g(y_j, \lambda_0 t_j) \tau \\
&= h_0 \sum_{j=1}^n \sum_{y=0}^{R-1} G(y, \lambda_0 t_j) \tau. \tag{3.6}
\end{aligned}$$

Finally, we obtain the average system cost for single replenishment cycle by

the following function, denoted by $C(R, \mathbf{S})$.

$$\begin{aligned}
C(R, \mathbf{S}) &= \frac{1}{T_0} (C_w + C_r) \\
&= \frac{1}{T_0} \left\{ h_0 \sum_{j=1}^n \sum_{y=0}^{R-1} G(y, \lambda_0 t_j) \tau + \sum_{j=1}^n \sum_{i=1}^N \pi_i(S_i, \tau) G(R-1, \lambda_0 t_j) + \sum_{j=1}^n c_j \right\}.
\end{aligned} \tag{3.7}$$

3.4.2 Allocation Rules

In this section, we analyze specific allocation rules, mainly virtual allocation and myopic optimal allocation. We will show that the balance condition can be incorporated by simply relaxing a constraint in myopic optimal allocation.

3.4.2.1 Virtual Allocation

Virtual allocation means that the orders from the retailers are fulfilled by FCFS rule. Then at shipping point t_j , after allocation, there are still $Y_j - R$ amount of the orders not fulfilled by the warehouse, which basically are the backorders at the warehouse after allocation, denoted by B_0 . Since the orders are fulfilled on a FCFS basis, we could derive the distribution of the amount of backorders for retailer i , denoted by B_{0i} . We can apply binomial disaggregation on $Y_j - R$ to obtain the distribution of B_{0i} because demand arrival at each retailer follows an independent Poisson process (Shang et al., 2015). Then we have

$$p(B_{0i} = x) = \binom{B_0}{x} \left(\frac{\lambda_i}{\lambda_0} \right)^x \left(1 - \frac{\lambda_i}{\lambda_0} \right)^{B_0 - x}, \quad B_0 = Y_j - R$$

Then we obtain the inventory position of each retailer after allocation, i.e.,

$S_{ji} = S_i - B_{0i}$. Thus the expected retailer i 's inventory related cost incurred in $[t_j + L_i, t_1 + T_0 + L_i)$ after allocation can be derived and denoted by $E_{B_{0i}}[\pi_i(S_{ji}, T'_j)]$. Further, summarizing over N retailers achieves $\varphi_j(\cdot | Y_{j-1}, X_j)$ under virtual allocation, that is,

$$\begin{aligned}\varphi_j(\cdot | Y_{j-1}, X_j) &= \sum_{i=1}^N E_{B_{0i}}[\pi_i(S_{ji}, T'_j)] \\ &= \sum_{i=1}^N \sum_{B_{0i}} \pi_i(S_{ji}, T'_j) p(B_{0i}).\end{aligned}$$

3.4.2.2 Myopic Optimal Allocation

We first make a commitment assumption to facilitate the cost evaluation. Under this assumption, the warehouse's order still raises the echelon inventory position to S_0 except that the order will commit to raise the warehouse's and the retailers' inventory positions to target levels R and $\mathbf{S}$, respectively. In other words, the order amount placed to raise retailer i 's inventory position to S_i will be committed to be delivered to retailer i when the order arrives. Only the rest amount of the order (i.e., R) will subject to further allocation. This commitment assumption corresponds to a practice in industry called direct shipping, which is used for avoiding double-handling cost. The distribution of imported pharmaceutical products in Japan is an important example of the application of the policy of direct shipments to retailer locations (Cao and Silver 2005). Another reason for this commitment assumption is that it enables us to exactly evaluate average system costs for single warehouse order cycle. More specifically, at shipping point t_1 , the warehouse will first deliver the committed amount to each retailer and then we could characterize the inventory position at retailer i before the allocation at

the first shipping point t_1 by $S_i - D_{1i}$. This could facilitate characterizing the myopic optimal allocation decisions. Later on we will discuss when this commitment assumption can be relaxed and how it will affect long-run average system cost.

Myopic optimal allocation assumes that the warehouse optimally allocates the remaining stock right before making a shipment with more updated stock information to minimize system cost. Under this condition, an optimal warehouse stock allocation at t_j is determined with the information of $\mathbf{D}_j = (D_{j1}, D_{j2}, \dots, D_{jN})$, Y_{j-1} and X_j , according to the following optimization model :

$$\phi_j(\cdot | Y_{j-1}, X_j, \mathbf{D}_j) = \text{Min}_{S_{j1}, \dots, S_{jN}} \sum_{i=1}^N \pi_i(S_{ji}, T'_j) \quad (3.8)$$

$$\text{s.t.} \quad \sum_{i=1}^N (S_i - S_{ji}) = Y_j - R \quad (3.9)$$

$$S_{ji} \geq S_i - D_{ji} \quad \text{for } i = 1, \dots, N. \quad (3.10)$$

Retailer i 's inventory position before stock allocation at t_j is then equal to $S_i - D_{ji}$. Since we assume that the shortage occurs at t_j , we have $Y_{j-1} \leq R$ and $Y_{j-1} + X_j > R$. Then before allocation, the warehouse still has $R - Y_{j-1}$ stock on-hand. Then the decision will allocate $R - Y_{j-1}$ stock completely among the retailers. After the allocation we denote the inventory position of retailer i by S_{ji} . Then we could have $\sum_{i=1}^N S_{ji} - (\sum_{i=1}^N S_i - X_j) = R - Y_{j-1}$, which leads to constraint (3.9), because the amount of flow-in should be equal to that of flow-out.

Constraint (3.10) means we assume that no transshipments between retailers are allowed. Then the inventory position at retailer i after stock allocation must

not be lower than $S_i - D_{ji}$, i.e., $S_{ji} \geq S_i - D_{ji}$. Note that when this constraint is ignored, the model above will turn into the case under commonly a balance assumption, which means the free transshipments among retailers are allowed.

To minimize (3.8), a unit of warehouse stock must be allocated to the retailer with the lowest cost increase (or the highest cost decrease). Therefore, we can start with $S_{ji} = S_i - D_{ji}$ for $i = 1, 2, \dots, N$ and allocate the available stock one unit by one unit to the retailer with the lowest $\Delta_{S_{ji}} \Pi_i(S_{ji}, T'_j)$ until $\sum_{i=1}^N (S_i - S_{ji}) = Y_j - R$. This leads to an optimal stock and an exact evaluation of $\phi_j(\cdot | Y_{j-1}, X_j, \mathbf{D}_j)$.

$$\Delta_{S_{ji}} \Pi_i(S_{ji}, T'_j) = -b_i T'_j + (b_i + h_i) \sum_{t=L_i+1}^{L_i+T'_j} G(S_{ji}, \lambda_i t). \quad (3.11)$$

We now derive the distribution of $\mathbf{D}_j$. It is easy to see that D_{ji} follows an independent Poisson distribution with mean $\lambda_i L_0$ for $j = 1$ and $\lambda_i \tau$ for $j > 1$ and $\sum_{i=1}^N D_{ji} = X_j$. Under this conditions, given X_j , $\mathbf{D}_j$ follows a multinomial distribution as follows,

$$p(\mathbf{D}_j) = (X_j!) \prod_{i=1}^N \frac{1}{D_{ji}!} \left(\frac{\lambda_i}{\lambda_0} \right)^{d_{ji}}.$$

Then taking the expected value over $\mathbf{D}_j$ achieves $\varphi_j(\cdot | Y_{j-1}, X_j)$,

$$\varphi_j(\cdot | Y_{j-1}, X_j) = \sum_{\mathbf{D}_j} \phi_j(\cdot | Y_{j-1}, X_j, \mathbf{D}_j) p(\mathbf{D}_j).$$

3.4.3 Optimization

In this section, we will provide an approach to optimize ordering policy. It is known that, even for a serial system with fixed-interval replenishment schedule, the system cost function is not jointly convex (Shang et al. 2010). Consequently, there is no standard method to find an optimal solution. We will characterize the structural properties of the system cost function and develop a procedure to identify an optimal solution. Let $S_0^v, R^v, \mathbf{S}^v$ represent optimal solution for virtual allocation. Let $S_0^{mo}, R^{mo}, \mathbf{S}^{mo}$ represent optimal solutions for myopic optimal allocation.

In this part, with given T_0 , we describe the approach for finding corresponding optimal stock decisions at each installation, i.e., R and $\mathbf{S}$.

First we have the following theorem.

Theorem 4. *When replenishment schedule is given, (a) Under virtual allocation and for given R , $C(R, \mathbf{S})$ is separable convex in $\mathbf{S}$, (b) $\Delta_R C(R, \mathbf{S}) > 0$ when $R > \bar{R}$, where $\bar{R}$ is the minimum R satisfying the following condition*

$$h_0 \sum_{j=1}^n G(R, \lambda_0 t_j) \tau \geq b \sum_{j=1}^n [1 - G(R, \lambda_0 t_j)] T'_j,$$

where $b = \max\{b_1, b_2, \dots, b_N\}$.

The proof of Theorem 1(a) is straightforward. According to the system cost function $C(R, \mathbf{S})$, under virtual allocation, the backorders for each retailer will be independent of $\mathbf{S}$. Then given R , $C(R, \mathbf{S})$ is separable convex in S_i as it is a typical newsboy problem for each retailer. This result will facilitate finding the optimal $\mathbf{S}$ under virtual allocation policy when R is given.

The proof of Theorem 1(b) is a bit complex, which is placed in the appendix for the ease of exposition. According to Theorem 1(b), we can derive an upper bound $\bar{R}$ for R .

To find the optimal $\mathbf{S}$, we provide an approach to find upper bound $\bar{S}_i$ and lower bound $\underline{S}_i$ for each retailer. First we assume that there is infinite stock at the warehouse, which means there will never be shortage at the warehouse, i.e., $R = \infty$. Then any retailer i will act as one stage problem with (S_i, τ) policy. Then finding optimal $S_i^*(\tau)$ for one stage problem with no shortage provides a lower bound $\underline{S}_i$ for optimal S_i in our problem. On the other hand, we assume that there is no stock at the warehouse, i.e., $R = 0$. Then any retailer i will act as if ordering directly from the outside supplier. Then the problem for retailer i is one stage problem with (S_i, T_0) policy and lead time $L_0 + L_i$. Because of no risk pooling, finding optimal $S_i^*(T_0)$ for one stage problem provides an upper bound $\bar{S}_i$ for optimal S_i in our problem. Let $\underline{\mathbf{S}} = \{\underline{S}_1, \dots, \underline{S}_N\}$, $\bar{\mathbf{S}} = \{\bar{S}_1, \dots, \bar{S}_N\}$.

When R is given, under virtual allocation, optimal $\mathbf{S}$ can be easily found by bisection search according to Theorem 1(a). The optimal solution under virtual allocation is denoted by $\mathbf{S}^v(R)$. Then given balance assumption, we can find optimal $\mathbf{S}$ by enumerating all possible combinations of $\mathbf{S}$ by using the lower $\underline{\mathbf{S}}$ and upper bounds $\bar{\mathbf{S}}$. The optimal solution under balance condition is denoted by $\mathbf{S}^b(R)$.

Now we consider how to find myopic optimal $\mathbf{S}$ for optimal allocation rule with given R . The optimal solution under optimal condition is denoted by $\mathbf{S}^{mo}(R)$.

We borrow the following theorem from Wang (2015) to refine our optimization process.

Theorem 5. (Wang, 2015) *The expected cost $C(R, \mathbf{S})$ is (a) coordinate-wise convex in S_i , for $i = 1, 2, \dots, N$; (b) super-modular in S_i and R , for $i = 1, 2, \dots, N$.*

Theorem 2 (b) basically implies that the stock at the warehouse and the retailers are complements. That is, under an optimal solution, if more stock is kept at one location, then no more stock is needed at the other locations. Given this property, we could easily have $\mathbf{S}^{mo}(R+1) \leq \mathbf{S}^{mo}(R)$, which could further reduce our search space on $\mathbf{S}$ when R is increasing.

To summarize, for any given T_0 , we can find a corresponding upper bound $\bar{R}$ and the lower bound $\underline{R} = 0$. Then we enumerate all possible $R \in [\underline{R}, \bar{R}]$, starting with $\underline{R}$. For each R , find the corresponding optimal $\mathbf{S}^{mo}(R)$. With next step $R+1$, we could further refine the upper bounds for $\mathbf{S}^{mo}(R+1)$, which could be found by $\min[\mathbf{S}^{mo}(R), \bar{\mathbf{S}}]$.

3.4.4 Remarks on Cost Evaluation

The core idea of the evaluation approach is based on Wang and Axsäter (2013) and Wang (2013). We have extended their approach to an exact cost evaluation framework for a class of ship-up-to- S policy with a general myopic allocation policy. We specifically consider virtual allocation and myopic optimal allocation. It is worth to note that balance assumption can be incorporated by simply relaxing a constraint in myopic optimal allocation.

With equation (3.7) we are able to exactly evaluate the expected average system costs for one replenishment cycle. Now we discuss how to extend our evaluation approach to infinite horizon and when the commitment assumption can be relaxed for myopic optimal allocation.

Proposition 1. *Given our model setting and stationary ship-up-to- S policy, the system replenishment cycle is regenerative for virtual allocation. For myopic optimal allocation, the system cycle may not be regenerative. However, for any myopic allocation rule, if the probability of shortage occurs at the first shipping point is relatively small, the average system cost for single cycle evaluated by the approach we proposed can be good approximation for the long-run horizon.*

3.5 Dynamic Allocation: A Heuristic Policy

In the section of introduction, we have discussed the benefits of centralizing inventory at the warehouse, which may come from three potential sources: (1) external lead time inventory pooling, (2) holding cost saving at the central warehouse, (3) rebalancing inventory among retailers during a replenishment cycle. The first benefit could be captured by an order-up-to policy based on the echelon stock at the warehouse. The second and third benefits, however, depend on how the warehouse allocates the stock to the retailers.

We have analyzed a class of myopic allocation policies, which implicitly incorporate the benefits of centralizing stock at the warehouse. Virtual allocation offers mathematical tractability but it does not take the relative needs of the retailers into account when making the allocation decision. Although myopic optimal allocation takes the retailers' relative needs for inventory into consideration, it fails to relate the current inventory to the future demand. Intuitively, the base stock levels at the retailers, i.e. $\mathbf{S}$, should be determined based on the stock available and the demand to be satisfied. This means that the stationary requirement of the base stock levels at the retailers imposes a limitation on the

efficiency of the inventory control system. A more efficient warehouse stock allocation policy is then to determine S_i for $i = 1, \dots, N$ dynamically at each delivery point based on the inventory status of the system as well as the future demand to be satisfied for the remaining stock. Dynamic allocation sounds intriguing. However, the dynamic programming model is a challenge and computationally unrealistic to solve especially when the number of retailers is large and/or the warehouse replenishment cycle is long (Axsäter et al. 2002). Therefore, with these motivations we try to find a simple heuristic allocation rule that might not only perform better in terms of inventory costs, but also should require much less effort for the optimization and could cope with large parameters.

3.5.1 Allocation Rule

In this heuristics, the warehouse still places an order to an echelon order-up-to level S_0 in a fixed time interval of T_0 . The echelon order-up-to level can exploit the external lead time risk pooling effect. Then the warehouse needs to determine how to allocate its available inventory to retailers during the replenishment cycle at each shipping point. The allocation decision at each point includes two parts: (1) how much to reserve at the warehouse, (2) how much to allocate to each retailer.

The first decision is to exploit the benefits of system holding cost saving and rebalancing inventory in the future. In the following we will describe a two-step heuristic allocation rule, in which the first step is to exploit the benefits of holding cost saving and the second step refines the first step's decision to exploit the benefits of rebalancing among retailers.

First, we introduce a finding in Wang and Wan (2015). They study a N -stage serial system with fixed replenishment schedule and echelon order-up-to policy. They find that given a replenishment schedule, under an optimal echelon ordering policy, the marginal cost at an echelon is equal to the inventory holding cost at the upper stage. This is intuitive. Consider a unit of stock at the upper stage, the decision between retaining it at upper stage or shipping it to the downstream echelon. The marginal cost of the first option is the installation holding cost h at upper stage. Apparently, this unit of stock should be shipped to the downstream echelon if the marginal cost at down stage is lower than h . Following this intuition, an optimal inventory position (or order-up-to level) at the downstream stage is reached when the two marginal costs are equal. This finding provides a simple characterization of an optimal ordering policy for a serial inventory system. This provides an intuitive characterization of an optimal ordering policy by simply equating the marginal echelon inventory related cost to the inventory holding cost at the upstream stage.

Now we start to describe the two-step heuristic allocation rule.

Step 1 With the same intuition in the serial system, we try to determine how much to allocate to each retailer by simply equating the marginal echelon inventory related cost to the inventory holding cost at the upstream stage. As we know, the single-stage cost function for each retailers is convex in the order-up-to level. Thus we can easily determine the order-up-to level for retailer i by the following.

Start with initial value $S_i = 0$, then increase S_i unit by unit until the equation

below is satisfied,

$$\Delta\pi_i(S_i, \tau) = \pi_i(S_i + 1, \tau) - \pi_i(S_i, \tau) \geq h_0\tau. \quad (3.12)$$

Suppose at shipping point t_j , the available inventory at the warehouse is $I_0(t_j)$ and the inventory position of retailer i is IP_{ji} before the allocation. After obtaining S_i by equation (3.12), the amount we should allocate to retailer i should be $\max(0, S_i - IP_{ji})$. We call this step of allocation “marginal cost allocation”.

However, this simple allocation in step 1 will possibly ignore partial benefit of inventory rebalancing for the rest of system cycle. Consider one example, where there are retailer 1 and retailer 2 with same holding costs but quite different demand rate and penalty cost. Suppose $\lambda_1 = 1, b_1 = 20$ and $\lambda_2 = 5, b_2 = 5$. Let S_1 and S_2 be base-stock levels. Consider a cycle length 5 and at period 2. Suppose in period 2 only one demand arrives at retailer 1 but 3 demands arrive at retailer 2. Suppose there are still 4 units of central stock at the warehouse. With myopic allocation policy, we should deliver 1 unit of stock to retailer 1 and 3 units of stock to retailer 2 to raise back their base-stock levels. Is this optimal? Probably not. As the penalty cost at retailer 2 is quite small compared to p_1 . Thus it is probably better to reserve some stock for future replenishment to retailer 1 in the rest of replenishment cycle while keeping retailer 2 under shortage.

Motivated by the example above we would like to make an refinement in step 2 to the preliminary allocation decision that has been made in step 1.

Step 2 The basic idea in this step is that we treat the rest of the current system cycle as a whole starting from the current shipping point and consider the current period as the last shipping point. By using the similar method in step 1

, we determine how much to allocate to each retailer. If the available warehouse inventory is not enough to cover the rest of cycle, we will use myopic optimal allocation to refine the decision in step 1. The refinement could reserve partial inventory to those retailers who can save more system costs.

Specifically, consider the rest of the cycle as one whole time interval. The length of the rest of the cycle is T'_j . Then we determine how much we should allocate to each retailer for this interval. Then if the available inventory at the warehouse is enough to cover the total allocation amount, we can allocate each retailer up to S_i . Otherwise, myopic optimal allocation is used to determine how much we should allocate to the retailers with available inventory on hand for the rest of the cycle. We denote the optimal allocation amount for retailer i is $\tilde{S}_i$. If $\tilde{S}_i < S_i$, we will only allocate $\tilde{S}_i$ to retailer i , rather than S_i . This refinement enables us to capture more benefits of inventory rebalancing. We refer the allocation adjustment based on marginal cost allocation as “refined marginal cost allocation”.

To sum up, the specific procedure of this two-step heuristic allocation rule is described as follows.

- (1) Compute S_i by using equation (3.12);
- (2) Start with $S = 0$, then increase S unit by unit until the following equation is satisfied and we obtain S'_i ,

$$\Delta\pi_i(S, T'_j) = \pi_i(S + 1, T'_j) - \pi_i(S, T'_j) \geq 0.$$

Then we denote the total requirement amount from the warehouse to fulfill the retailers for the rest of current replenishment cycle by $x = \sum_{i=1}^N \max(S'_i - IP_{ji}, 0)$

(3) If $x < I_0(t_j)$, the amount allocated to retailer i is $\max(0, S_i - IP_{ji})$, which means we allocate retailer i 's inventory position up to S_i , for $i = 1, \dots, N$. Then the allocation finished. Otherwise, if $x \geq I_0(t_j)$, move to next step;

(4) In this step, we consider the rest of cycle as a whole and determine how much we should allocate to each retailer with the available inventory I_0 by using myopic optimal allocation as follows.

$$\begin{aligned} \varphi_j(\cdot | I_0(t_j), IP_{ji}) &= \text{Min}_{\tilde{S}_1, \dots, \tilde{S}_N} \sum_{i=1}^N \pi_i(\tilde{S}_i, T'_j) \\ \text{s.t.} \quad &\sum_{i=1}^N (\tilde{S}_i - IP_{ji}) = I_0(t_j) \\ &\tilde{S}_i \geq IP_{ji} \quad \text{for } i = 1, \dots, N. \end{aligned}$$

After solving the optimization problem above, we obtain $\tilde{S}_i$. Now we decide how much to allocate for retailer i . If $S_i > \tilde{S}_i$, we allocate retailer i 's inventory position up to $\tilde{S}_i$, otherwise allocate up to S_i , for $i = 1, \dots, N$. Then the allocation is finished.

Note that this procedure will be executed at every shipping point. However, it would be difficult to exactly estimate average system cost. Therefore we will use simulation to test its performance comparing to myopic allocation policies.

3.5.2 Cost Evaluation

With two-step heuristic allocation rule, it would be extremely difficult to exactly evaluate the long-run average system costs, if not impossible. Therefore we resort to simulation approach. For the reason of comparison, we also will use simulation

to estimate the average cost for myopic allocation policy. We make a commitment assumption to exactly evaluate system costs. In the simulation, we could easily relax this assumption.

According to the heuristic allocation rule, the only decision variable for the inventory control policy is the echelon order-up-to level (S_0), which reduces the whole problem into one-dimensional problem. This will significantly reduce the complexity of optimization and the classic “the curse of dimensionality” problem can be avoided.

Further, this heuristic allocation approach has several other advantages: (1) it is based on managerial intuitions, therefore the practitioners can find it easy to understand and implement, (2) can cope with a relatively large number of retailers, a demand arrival rate and a long replenishment cycle.

3.5.3 Heuristic Solution for S_0

As we have discussed before, the optimization problem for the new allocation heuristic is reduced to one-dimension problem, i.e., system order-up-to level S_0 . To find its optimal value, the easiest way is to search a large range on S_0 with average system cost evaluation based on simulation. However, this could still be time-consuming if the range is too large. Therefore, we would like to introduce a heuristic approach to provide a close-to-optimal initial value for S_0 . First we turn the distribution system into N independent two-stage serial systems. Let serial system i consist of the warehouse as the upper stage and retailer i as the downstream stage. Then we try to find an optimal system order-up-to level, denoted by S_{0i} , for two-stage serial system i . Now there are no pooling effect for

there N serial systems, we can use the summation $\sum_{i=1}^N S_{0i}$ as an upper bound for the original distribution system's echelon order-up-to level. A lower bound can be easily established by considering N single-stage problem when all the retailers can directly order from the external suppliers. A search between the lower bound and upper bound enables us to find the optimal S_0 for the distribution system by simulation approach for cost evaluation.

3.6 Numerical Experiments

In this section, we will conduct numerical experiments to investigate the performance of the allocation policies. First, we will discuss the values of the parameters we will use for the experiments. Then we start to compare the performance of virtual allocation and myopic optimal allocation. Further, we investigate the effectiveness of the heuristic dynamic allocation policy (i.e., refined marginal cost heuristics). For the ease of exposition, we use V , MO , and MC_r to represent virtual allocation, myopic optimal allocation, and the heuristic dynamic allocation, respectively. Sometimes we let the lower case of these abbreviations be the superscripts of variables $S_0, \mathbf{S}, C$ to indicate the corresponding optimal solutions and minimum system average costs. Part of the numerical results are presented in Table 9-15 in Appendix D.

The following values of parameters will be used to conduct the experiment for $N = 2$.

$T_0 \in \{3, 4, 5, 6, 7\}$, $\tau = 1$, $L_0 = 1$, $h_0 = 1$, $L_i = L_j = 1$, $h_i = h_j = 1.1$, $SL_i \in \{80\%, 90\%, 99\%\}$, $\lambda_i = \lambda_j \in \{1.5, 2, 2.5, 3\}$, for $i, j = 1, 2$.

The retailers are identical in unit holding cost and demand rate but can be

different in service levels, as we would like to examine the impact of different services levels requirements at the retailers. We let SL_i represent the service level of retailer i and SL_i is in $\{80\%, 90\%, 99\% \}$. As we need the unit shortage cost to evaluate the system cost, we find the corresponding b_i by using the critical ration result from the classic newsvendor problem, i.e., $SL_i = \frac{b_i}{h_i + b_i}$. For instance, if $h = 1.1$, then according to the equation b_i is set to be 108.9 to achieve 99% service level. Note that these service levels are used to help set the values of the corresponding unit shortage costs and the actual system may not be able to achieve such service levels.

There are 120 cases for all combination of parameters for $N = 2$. Then we extend each of the 108 cases to $N = 4$ by adding 2 retailers that are identical to the original two retailers. Then in total we have 240 cases. For every case, we will find the optimal solution (i.e., $S_0^{mo}, \mathbf{S}^{mo}$) for myopic optimal allocation and the optimal solution (i.e., $S_0^v, \mathbf{S}^v$) for virtual allocation. We also obtain the corresponding minimum system average costs C^{mo} and C^v .

As we know, in the exact cost evaluation for myopic optimal allocation, we have a commitment assumption when the warehouse places an order. In the simulation programming, we can easily relax this assumption and then compare it with the costs with commitment assumption. With the optimal solution $S_0^{mo}, \mathbf{S}^{mo}$, we use the simulation to compute the system average costs to test the differences between the cases with commitment assumption and non-commitment assumption. By using the above cases, we test the differences. The results show that the average difference is only 0.003% with a standard deviation of 0.022%. This shows that the cost evaluation approach we developed for single system cycle can be a very effective approximate method for infinite-horizon. This confirms

the conclusion in Proposition 1. Later on, when we try to evaluate system costs by simulation approach, the commitment assumptions will always be relaxed.

3.6.1 Myopic Allocation Policies

We have provided an evaluation approach for myopic allocation policy, in which virtual allocation and myopic optimal allocation are specifically incorporated. Now we would like to comprehensively compare their differences with respect to several parameters, i.e., the number of retailers, the length of the replenishment cycle, retailer's service levels, and demand arrival rates. For the convenience of illustration, we define the relative (or percentage) difference between policy 1 and policy 2 by $\frac{policy1-policy2}{policy2} \times 100\%$, which is labeled by "policy 1 vs policy 2". Table 3.1 presents the statistics of the minimum cost under each myopic allocation policy and their relative differences for the 240 cases.

The last row of Table 3.1 indicates the differences between virtual allocation and myopic optimal allocation for the 240 cases. The result shows that compared with virtual allocation, myopic optimal allocation saves an average of 1.49% with a standard deviation of 0.69%. Then to examine the impact of N, T_0, λ , we divide all cases into groups according to the values of the three parameters. The detailed results are also presented in Table 3.1. According to the results, the relative differences between the two allocation policies increase in N and λ and decrease in T_0 .

We then investigate the impact of the retailers' service level requirements. We divide the cases into groups according to service level requirements. For $N = 4$, we add two retailers that are same with the original two in cases with $N = 2$.

Table 3.1: Comparison between V and MO

Parameters		C^{mo}	C^v	V vs MO (%)
N	2	19.10	19.34	1.21 (0.61)
	4	35.72	36.34	1.77(0.67)
λ	1.5	21.16	21.46	1.37(0.57)
	2.0	25.47	25.88	1.51(0.72)
	2.5	29.54	30.01	1.55(0.74)
	3.0	33.47	34.01	1.62(0.73)
T_0	3	21.49	21.94	1.93(0.64)
	4	24.40	24.83	1.62(0.64)
	5	27.37	27.80	1.44(0.66)
	6	30.37	30.80	1.29(0.67)
	7	33.41	33.83	1.17(0.64)
Total		27.41	27.84	1.49(0.69)

Note: Numbers in the parentheses represent the corresponding standard deviation. Similarly hereinafter.

Then all cases are divided into six groups as in Table 3.2. The groups can be differentiated by service level at retailer 1 and service level at retailer 2. In Table 3.2, group 1-3 consider cases with identical retailers. Group 4-6 consider non-identical retailers. The results in Table 3.2 indicate that the differences are larger in cases with non-identical retailers than in cases with identical retailers. As the service level requirements increase, the differences also increase. As service levels differences between retailers increase, the performance difference between the two allocation policy also increase.

In summary, compared to virtual allocation, myopic optimal allocation performs particularly better in cases with large number of retailers, high service level requirements, large demand rate, large differences between retailers, and short system replenishment cycle.

More precisely, we investigate how the relative differences between V and MO are affected by number of retailers, length of warehouse order cycle, and

Table 3.2: Comparison between MO and V for different service levels

(SL_1, SL_2)	C^{mo}	C^v	V vs MO (%)
(80%,80%)	20.14	20.33	0.96 (0.42)
(90%,90%)	24.96	25.22	1.01(0.40)
(99%,99%)	36.88	37.32	1.14(0.40)
(80%,90%)	22.57	22.92	1.47(0.57)
(90%,99%)	31.04	31.65	1.91(0.46)
(80%,99%)	28.87	29.60	2.45(0.41)

other relevant system parameters by performing a correlation analysis. We let $E(SL)$ measure the average service level of the retailers and $Std(SL)$ measure the standard deviation of the retailers' service level. We use $E(SL)$ and $Std(SL)$ for analyzing the impact of the service levels requirement and the extent that the retailers are different from each other. Then we performs a correlation analysis by using *SPSS* software. The results are presented in Table 3.3. The correlation results are consistent with what we have discussed above and show high statistical significance levels.

Table 3.3: Correlation results

	N	T_0	λ	$E(SL)$	$Std(SL)$
MO vs V	0.402**	-0.375**	0.088*	0.142*	0.739**

Note: **, * represent significant at the 0.01, 0.05 level, respectively.

3.6.2 Heuristic Dynamic Allocation Policy

In this section we investigate the performance of the heuristic dynamic allocation policy. Now we will use simulation approach to evaluate average system costs for all allocation policies. We try to consider experiments with more retailers and longer replenishment cycles. Then first we consider the following system parameters for cases with $N = 2$.

$T_0 \in \{3, 4, 5, 6, 7, 8, 9\}$, $\tau = 1$, $N = 2$, $L_0 = 1$, $h_0 = 1$, $L_i = L_j = 1$, $h_i = h_j = 1.1$, $SL_i \in \{80\%, 90\%, 99\%\}$, $\lambda_i = \lambda_j \in \{2, 4, 6\}$, for $i, j = 1, 2$.

We further consider more retailers. Specifically, we extend each of the 126 cases to $N = 4$ by adding 2 retailers that are identical to the original two retailers. By further adding the original two retailers we expand to cases with $N = 6$. Therefore now we have 378 cases ($N = 2, 4, 6$) in total for numerical experiments. Similar to the previous section, our objective is to more comprehensively compare the allocation policies, i.e., virtual allocation, myopic optimal allocation, and refined marginal cost allocation. We are also interested in how their differences are affected by the number of retailers, the length of warehouse order cycle, demand arrival rate, and service levels.

For each of the 378 cases, we first use the heuristic dynamic allocation policy we proposed and find a reasonable good solution S_0^h . For a fair comparison of allocation policies, we use the same S_0^h for myopic optimal allocation and virtual allocation (Marklund and Rosling 2012). As there are only two types of retailers, we are able to find the optimal $\mathbf{S}$ for myopic optimal allocation and virtual allocation with given S_0^h by enumeration approach using lower bounds and upper bounds. We use simulation approach to evaluate system cost for all allocation policies. The comparison results are presented in Table 3.4 and 3.5. Similarly, we divide the cases into groups to examine the impact of the parameters.

The results in Table 3.4 show that MC_r performs better than MO on average by reducing system costs 0.23%. It also indicates that MC_r performs well particularly for large demand arrival rate, long replenishment cycle, and small number of retailers. Table 3.4 presents the results categorized by service levels. It shows

that MC_r performs better than MC for cases with non-identical retailers. When the retailers are identical, MC_r performs better when the service levels are low. Note that even when MO performs better than MC_r in group (95%, 95%), the average difference is relatively small under 1%. Further, the results show that MC_r outperforms V , particularly in cases with long replenishment cycle, large demand arrival rate, and large service level differences between retailers.

Table 3.4: Comparison among MO , V , and MC_r

Parameters		C^{mc_r}	C^{mo}	C^v	MO vs MC_r (%)	V vs MC_r (%)
N	2	31.97	32.21	32.64	0.54(1.57)	1.62(3.05)
	4	61.04	61.30	62.61	0.21(1.40)	2.03(2.81)
	6	90.06	90.21	92.15	-0.06(1.31)	1.85(2.36)
λ	2.0	37.25	37.06	37.52	-0.35(1.12)	0.73(1.70)
	4.0	61.35	61.64	62.81	0.44(1.24)	2.06(2.44)
	6.0	84.48	85.02	87.08	0.58(1.74)	2.72(3.45)
T_0	3	40.45	40.24	40.92	-0.36(1.21)	0.98(1.99)
	4	47.19	47.11	47.83	-0.19(0.92)	0.98(1.89)
	5	54.03	54.03	55.01	-0.02(0.93)	1.56(1.97)
	6	60.87	61.01	62.26	0.18(1.08)	1.88(2.36)
	7	67.87	68.19	69.59	0.42(1.30)	2.15(2.84)
	8	74.91	75.46	77.05	0.67(1.83)	2.38(3.32)
	9	81.85	82.62	84.62	0.87(2.07)	2.91(3.83)
Total		61.02	61.24	62.47	0.23(1.45)	1.84(2.75)

Table 3.5: Comparison among MO , V , and MC_r for different service levels

(SL_1, SL_2)	C^{mc_r}	C^{mo}	C^v	MO vs MC_r	V vs MC_r (%)
(80%,80%)	47.66	47.60	47.94	0.01(0.75)	0.69(1.19)
(90%,90%)	57.06	56.90	57.42	-0.21(0.39)	0.62(0.58)
(99%,99%)	78.92	78.37	79.87	-0.78(0.72)	0.61 (1.68)
(80%,90%)	52.40	52.34	52.88	-0.21(0.58)	0.71(0.73)
(90%,99%)	67.43	67.84	69.69	0.47(0.79)	2.78(1.83)
(80%,99%)	62.69	64.39	67.00	2.10(2.33)	5.59(4.09)

Similarly, we perform a correlation analysis. The correlation results are presented in Table 3.6, which shows that compared to MO , MC_r performs better in

cases with large arrival rate, low service levels, and large difference on service levels among retailers. The correlation results are consistent with what we observed in Table 3.4 and 3.5.

Table 3.6: Correlation results

	N	T_0	λ	$E(SL)$	$Std(SL)$
MO vs MC_r	-0.170**	0.289**	0.261**	-0.343**	0.199**
V vs MC_r	0.035	0.238**	0.295**	-0.220**	0.369**

Note: **, * represent significant at the 0.01, 0.05 level, respectively.

We have shown that refined marginal cost allocation policy performs at a comparable level with myopic optimal allocation on average in terms of cost control. However, in terms of finding a reasonable S_0 , heuristic dynamic allocation policy can save much more computation time as it only considers single variable S_0 , while myopic allocation policy has to consider $N + 1$ inventory control parameters. For our experiments, finding solution S_0^h for MC_r allocation policy takes an average computation time of 322.91 seconds. However, finding optimal solution for myopic allocation policy takes an average computation time of 916.86 seconds. Note that currently we only consider cases with $N = 2, 4$. According to the optimization approach we have proposed for myopic optimal allocation, the computation time could increase exponentially as N increases, because a search of $N + 1$ dimensional space is required.

3.7 Concluding Remarks

We consider a periodic-review distribution system, in which one central warehouse places orders on a fixed replenishment cycle and then in turn replenishes N non-identical retailers. First, we consider a class of stationary ship-up-to- S policy

with myopic allocation policy, namely, virtual allocation and myopic optimal allocation. We provide an general approach for exactly evaluating the expected system cost in single system order cycle. An approach for optimizing inventory control parameters is also provided. Numerical Experiments show that myopic optimal allocation performs better than virtual allocation, particularly in cases with large number of retailers, high service level requirements, large demand rate, large difference between retailers, and short system replenishment cycle.

We have discussed that myopic allocation policy can not be adjusted according to the actual realized demand. It can be very complex and time-consuming for finding their optimal inventory control parameters, especially for large number of retailers, high arrive rate and long replenishment cycle. Further, with myopic allocation policy as a performance benchmark, we construct a computationally efficiency dynamic inventory allocation policy. We propose a simple heuristic dynamic allocation policy based on managerial insights, called refined marginal cost allocation. We demonstrate through an extensive numerical study that the heuristic dynamic allocation policy can outperform myopic optimal allocation on average, and in particular in cases with large arrival rate and large difference among retailers. Most importantly, this allocation policy significantly reduces the complexity of system optimization by considering only system order-up-to level and therefore can cope with a relatively large number of retailers, a large demand arrival rate and a long replenishment cycle. Practitioners should find this policy easy to understand and implement.

However, the heuristic dynamic allocation policy we have proposed is based on managerial insights. Although extensive numerical results have shown its effectiveness, it still lack of rigorous mathematical analysis. One of the focuses

in our future work is to derive more analytical results or developing worst-case bounds on the performance of the heuristic allocation policy. Another focus is to compare it to other heuristic allocation policy in the literature.

Chapter 4

Conclusions and Future Directions

4.1 Conclusions

In the first essay, we study a serial system with continuous stochastic demand. For reasons of mathematical tractability and historic convention, previous studies have typically accounted inventory related costs at the end of each review period. We provide an approach to continuously evaluate system inventory-related costs. Further, we provide a simple and intuitive characterization of an optimal ordering policy, which simply equates the marginal echelon inventory related cost to the inventory holding cost at the upstream stage. Moreover, using this optimal solution for a given system replenishment schedule, we develop a top-down procedure to identify an optimal system replenishment schedule subject to the optimization of the ordering policy. This study contributes to the understanding of optimal inventory policy of a stochastic serial system. Our numerical experiments show that end-of-period cost accounting can underestimate the system costs significantly if the costs actually incur continuously in time, which suggests that practitioners

should pay attention to their cost accounting methods when implementing an inventory management system.

In the second essay, we consider a periodic-review distribution system and aim to find effective heuristic allocation policies of practical importance. First, we consider myopic allocation policies, namely, virtual allocation and myopic optimal allocation. We provide an approach for evaluating the expected system cost. An approach for optimizing this class of policies is also provided. This contributes to the literature on the evaluation of a stochastic distribution system. Then numerical experiments are conducted to compare the two allocation policies. The results show that myopic optimal allocation performs better than virtual allocation, particularly in cases with large number of retailers, high service level requirements, large demand rate, large difference between retailers, and short system replenishment cycle. The results provide better understanding for practitioners on when to use the two allocation policies. Further, we provide a heuristic dynamic allocation policy, which on average performs at a comparable level with myopic allocation policy in terms of cost control. However, the heuristic policy significantly reduces the complexity of system optimization by considering only system order-up-to level and therefore can cope with a relatively large number of retailers, a large demand arrival rate and a long replenishment cycle. Practitioners should find this policy easy to understand and implement.

4.2 Future Directions

In this thesis, we have focused on centralized multi-echelon inventory systems where a central planner makes all decisions to optimize system performance.

However, in reality, many supply chains involve many independent members who make decisions independently to maximize their own benefits. Although a centralized solution can optimize system performance, it is not always in the best interest of every individual member. Therefore, the implementation of a centralized solution or the coordination of a decentralized supply chain that aims to optimize system performance must have a mechanism to align the objectives of individual supply chain members (Wang 2011). Although, many studies have considered decentralized multi-echelon inventory control systems (see, e.g., Cachon and Zipkin 1999; Chen 1999; Axsäter 2001, 2005; Shang et al. 2009), the literature on decentralized systems is still limited as compared to the literature on centralized systems. Consequently, the coordination of decentralized multi-echelon inventory systems with stochastic demand and/or lead time represents great opportunities and challenges for future research.

Second, penalty cost for shortage has been commonly used in the inventory literature. However, it is a variable that is very difficult to measure in practice. Instead, service level is a good proxy for penalty cost. But there is only a limited literature considering service level in decentralized supply chain coordination in a relative simple setting (Katok et al. 2008, Sieke et al. 2012). Therefore, decentralized system coordination with service levels for multi-echelon systems, especially for distribution systems, also represents a good future direction.

Third, we have only considered a single class of demand for every retailer. However, in reality retailers often face multiple classes of demand. The literature often considers this problem for just one retailer or supplier (Arslan et al. 2007, Kleijn et al. 1998). How to optimize multi-echelon inventory system with multiple classes of demands can also be an interesting future direction.

Appendix A: List of Notations in Chapter 3

- N The number of retailer,
- L_0 The lead time for the warehouse,
- L_i The lead time for retailer i from the warehouse, $i = 1, 2, \dots, N$,
- λ_i The demand rate of retailer i per period, and $\lambda_0 = \sum_{i=1}^N \lambda_i$,
- $D_i[t, t + \delta)$ The cumulative demand for retailer i over time periods in $[t, t + \delta)$,
- h_i The local or installation holding cost per unit of on-hand inventory at the warehouse ($i = 0$) or retailer i ($i = 1, 2, \dots, N$) per period,
- b_i The backorder cost b_i per unit of backlogged demand at retailer i ($i = 1, 2, \dots, N$) per period,
- T_0 The length of the order interval of the warehouse,
- τ The length of the replenishment interval for the retailers and $\tau = T_0/n$, where n is a positive integer,

-
- t_j The shipping point (or period) in one replenishment cycle, $j = 1, 2, \dots, n$,
- S_0 The echelon order-up-to level at the warehouse,
- S_i The base-stock level of retailer i , $i = 1, 2, \dots, N$,
- Y_j The total demand at the retailers in the interval $[0, t_j)$ and $X_j = Y_j - Y_{j-1}$,
- $D_{ji}[t_{j-1}, t_j)$ The demand at retailer i in the interval $[t_{j-1}, t_j)$ and $\mathbf{D}_j = \{D_{j1}, D_{j2}, \dots, D_{jN}\}$,
- IP_{ji} The inventory position of retailer i before allocation at t_j and $\mathbf{IP}_j = \{IP_{j1}, IP_{j2}, \dots, IP_{jN}\}$,
- S_{ji} The the inventory position of retailer i after allocation at t_j and $\mathbf{S}_j = \{S_{j1}, S_{j2}, \dots, S_{jN}\}$,
- $\pi_i(S, T)$ The total inventory related cost for retailer i with fixed-interval (T) base-stock (S) policy in one replenishment cycle,
- T'_j The length of the interval from shipping point t_j to the start of the next replenishment cycle $t_0 + L_0 + T_0$ and $T'_j = (n - j + 1)\tau$.

Appendix B: Proof of Theorem 4(b) in Chapter 3

In the following we only show the proof under optimal allocation at the moment of delivery. For other allocation policies, similar argument in the proof can be applied with small modifications.

If the warehouse base stock level is $R > 0$ and $y_{j-1} + x_j = R$, the warehouse has the exact stock to raise the inventory position at each retailer to S_i and $\varphi_j(\cdot | y_{j-1}, x_j) = \sum_{i=1}^N \pi_i(S_{ji}, T'_j)$. Let

$$f(R) = \sum_{j=1}^n c_j = \sum_{x=R}^{\infty} \varphi_1(\cdot | y, x) g(x, \lambda_0 t_1) + \sum_{j=2}^n \sum_{y=0}^{R-1} \sum_{x=R-y}^{\infty} \varphi_j(\cdot | y, x) g(x, \lambda_0 \tau) g(y, \lambda_0 t_{j-1}).$$

Consider $\varphi_j(R, \cdot | y, x)$ as a function of R , then

$$\begin{aligned}
\Delta f(R) &= f(R+1) - f(R) = \sum_{x=R+1}^{\infty} \varphi_1(R+1, \cdot | y, x) g(x, \lambda_0 t_1) - \sum_{x=R}^{\infty} \varphi_1(R, \cdot | y, x) g(x, \lambda_0 t_1) \\
&+ \sum_{j=2}^n \sum_{y=0}^R \sum_{x=R+1-y}^{\infty} \varphi_j(R+1, \cdot | y, x) g(x, \lambda_0 \tau) g(y, \lambda_0 t_{j-1}) \\
&- \sum_{j=2}^n \sum_{y=0}^{R-1} \sum_{x=R-y}^{\infty} \varphi_j(R, \cdot | y, x) g(x, \lambda_0 \tau) g(y, \lambda_0 t_{j-1}) \\
&= \sum_{x=R+1}^{\infty} \Delta_R \varphi_1(\cdot | y, x) g(x, \lambda_0 t_1) - \varphi_1(R, \cdot | y, x=R) g(R, \lambda_0 t_1) \\
&+ \sum_{j=2}^n \sum_{y=0}^R \sum_{x=R+1-y}^{\infty} \Delta_R \varphi_j(\cdot | y, x) g(x, \lambda_0 \tau) g(y, \lambda_0 t_{j-1}) \\
&+ \sum_{j=2}^n \sum_{x=R+1-y}^{\infty} \varphi_j(R, \cdot | y=R, x) g(x, \lambda_0 \tau) g(R, \lambda_0 t_{j-1}) \\
&- \sum_{j=2}^n \sum_{y=0}^{R-1} \varphi_j(R, \cdot | y, x=R-y) g(R-y, \lambda_0 \tau) g(y, \lambda_0 t_{j-1}) \\
&= \sum_{x=R+1}^{\infty} \Delta_R \varphi_1(\cdot | y, x) g(x, \lambda_0 t_1) - \sum_{i=1}^N \pi_i(S_i, T_0) g(R, \lambda_0 t_1) \\
&+ \sum_{j=2}^n \sum_{y=0}^R \sum_{x=R+1-y}^{\infty} \Delta_R \varphi_j(\cdot | y, x) g(x, \lambda_0 \tau) g(y, \lambda_0 t_{j-1}) \\
&+ \sum_{j=2}^n \left\{ \left[\sum_{i=1}^N \pi_i(S_i, T'_{j-1}) - \sum_{i=1}^N \pi_i(S_i, \tau) \right] g(R, \lambda_0 t_{j-1}) \right. \\
&\quad \left. - \sum_{i=1}^N \pi_i(S_i, T'_j) g(R, \lambda_0 t_j) \right\}.
\end{aligned}$$

Let

$$\begin{aligned}
H &= \sum_{j=1}^n \sum_{i=1}^N \pi_i(S_i, \tau) g(R, \lambda_0 t_j) - \sum_{i=1}^N \pi_i(S_i, T_0) g(R, \lambda_0 t_1) \\
&+ \sum_{j=2}^n \left\{ \left[\sum_{i=1}^N \pi_i(S_i, T'_{j-1}) - \sum_{i=1}^N \pi_i(S_i, \tau) \right] g(R, \lambda_0 t_{j-1}) - \sum_{i=1}^N \pi_i(S_i, T'_j) g(R, \lambda_0 t_j) \right\} \\
&= \sum_{j=1}^n \sum_{i=1}^N \pi_i(S_i, \tau) g(R, \lambda_0 t_j) + \sum_{j=1}^{n-1} \left\{ \left[\sum_{i=1}^N \pi_i(S_i, T'_j) - \sum_{i=1}^N \pi_i(S_i, \tau) \right] g(R, \lambda_0 t_j) \right\} \\
&- \sum_{i=1}^N \pi_i(S_i, T_0) g(R, \lambda_0 t_1) - \sum_{j=2}^n \sum_{i=1}^N \pi_i(S_i, T'_j) g(R, \lambda_0 t_j) \\
&= 0.
\end{aligned}$$

With $H = 0$, we could further simplify $\Delta_R C(\cdot)$ and have the following

$$\begin{aligned}
\Delta_R C(\cdot) &= C(R+1, \mathbf{S}) - C(R, \mathbf{S}) \\
&= \frac{1}{T_0} \left\{ h_0 \sum_{j=1}^n G(R, \lambda_0 t_j) \tau + \Delta f(R) + \sum_{j=1}^n \sum_{i=1}^N \pi_i(S_i, \tau) g(R, \lambda_0 t_j) \right\} \\
&= \frac{1}{T_0} \left\{ h_0 \sum_{j=1}^n G(R, \lambda_0 t_j) \tau + \sum_{x_1=R+1}^{\infty} \Delta_R \varphi_1(\cdot | y, x) g(x, \lambda_0 t_1) \right. \\
&\quad \left. + \sum_{j=2}^n \sum_{y=0}^R \sum_{x=R+1-y}^{\infty} \Delta_R \varphi_j(\cdot | y, x) g(x, \lambda_0 \tau) g(y, \lambda_0 t_{j-1}) \right\}. \tag{B1}
\end{aligned}$$

Given y_{j-1} and x_j ($y_{j-1} < R$ and $y_{j-1} + x_j \geq R$), if the warehouse base stock level increases from R to $R+1$, this is equivalent to increasing the warehouse stock by one unit. Under the warehouse allocation, this unit of stock is allocated to retailer. This implies $\Delta_R \varphi_j(\cdot | y_{j-1}, x_j) = \Delta_R \sum_{i=1}^N \pi_i(S_{ji}, T'_j)$. According to equation (3.9), $\Delta_R \varphi_j(\cdot | y_{j-1}, x_j) \geq -b_j T'_j \geq -b T'_j$, where $b = \max\{b_1, b_2, \dots, b_N\}$, and consequently. Apply this inequality to (B1) We have

$$\Delta_R C(.) \geq \frac{1}{T_0} \left\{ h_0 \sum_{j=1}^n G(R, \lambda_0 t_j) \tau - b \sum_{j=1}^n [1 - G(R, \lambda_0 t_j)] T'_j \right\}.$$

Let $\Delta_R C(.) \geq 0$, we have

$$h_0 \sum_{j=1}^n G(R, \lambda_0 t_j) \tau \geq b \sum_{j=1}^n \left[1 - G(R, \lambda_0 t_j) \right] T'_j.$$

Appendix C: Proof of Proposition 1 in Chapter 3

If the system replenishment cycle is regenerative under infinite horizon, the analysis for single replenishment cycle can be carried into infinite horizon, and then we are able to exactly evaluate long-run average system cost. If it is not, we will discuss when this approach can be good approximation for long-run average system cost.

We are interested in the inventory level at the warehouse and the inventory positions at all retailers for system cost evaluation, which as a whole we refer as system inventory status. Basically, to show whether the replenishment cycle is regenerative, we need to show whether the inventory status of the current cycle is independent of inventory status at the end of last cycle.

To illustrate replenishment cycle, we assume that the current cycle begins at period t_0 , i.e., the warehouse places an order and the order arrives at the beginning of period t_1 . We will focus two periods t_0 and t_1 as shown in Figure 3. If the inventory status t_1 is independent of last cycle, the following periods in the current cycle are also independent of last cycle.

Let $IP_i^-(t_0)$ be the inventory positions of all installations ($i = 0$ for the

Figure 1: An example of system replenishment cycle

warehouse, $i = 1, \dots, N$ for the retailers) right before the warehouse places orders at period t_0 . Now we look at the inventory status of the system at the beginning of period t_1 after the warehouse receives the order but before making any allocation decision.

If no shortage occurs at shipping point t_1 , the system inventory status is independent between replenishment cycles. No shortage means that after shipping at point t_1 , retailer i 's inventory position is raised up to S_i for $i = 1, \dots, N$ and the warehouse's inventory position should be greater than or equal to 0, i.e., $IP_0(t_1) = R - D[t_0, t_1] \geq 0, IP_i(t_1) = S_i$, which indicates that the inventory status of every installation is independent of last cycle. Under this case, the commitment assumption for myopic optimal allocation will not be necessary as it will be automatically satisfied. This implies that if the probability of shortage at the first shipping point (i.e., t_1) is relatively small, our evaluation in one single cycle can be good approximation for long-run average system cost. This should be reasonable in most cases, especially for long warehouse replenishment cycle.

Now we discuss what happens if shortage occurs at shipping point t_1 .

First, we consider virtual allocation. At shipping point t_1 , the backorders at the warehouse after allocation $B_0 = Y_1 - R$. The distribution of backorders for retailer i , B_{0i} , follows a binomial distribution. Then we obtain the inventory position of each retailer after allocation, i.e., $S_{ji} = S_i - B_{0i}$, which is independent of last replenishment cycle. Then $IP_i^-(t_1) = S_i - D[t_0, t_1], IP_0^-(t_1) = R$, which

means that each warehouse replenishment forms a regenerative cycle for the system. Basically, the cost evaluation approach for virtual allocation is only based on echelon stock at the warehouse.

Then, for myopic optimal allocation, the commitment assumption assumes that the warehouse orders from the external supplier to raise the inventory position at retailer i to S_i and the installation inventory position at the warehouse to R . Then we consider two cases:

(1) if $IP_0^-(t_0) \leq 0$, then the warehouse places an order to raise its installation inventory position to R . When the order arrives at t_1 , the warehouse first fulfills the backorders from last cycle and then still has R for fulfilling the lead time demand. If under shortage, R amount of stock is used to fulfill lead time demand according to allocation rule, which is also independent of last cycle. Thus for this case, the system cycle also can be regenerative.

(2) if $IP_0^-(t_0) > 0$, although the warehouse raise its the inventory position to R , the warehouse would make allocations during the lead time interval $[t_0, t_1)$ by using the available stock amount $IP_0^-(t_0) > 0$ from last cycle. If this happens, when the warehouse order arrives at t_1 , the available stock at the warehouse will be less than R and the amount depends on value of $IP_0^-(t_0)$ from last cycle. Therefore, for this case, the system cycle can not be regenerative.

In summary, the system replenishment cycle is regenerative for virtual allocation. But for myopic optimal allocation, system cycle may not be always regenerative. For any allocation rule, if the probability of shortage occurs at the first shipping point is relatively small, the average system cost for single cycle can be good approximation for the infinite horizon.

Appendix D: Numerical Results of Chapter 2 and Chapter 3

In the following, Table 1-8 are numerical results for Chapter 2. Table 9-15 are numerical results for Chapter 3.

Table 1: Optimal solutions for cases with $k_1 = 10$, $k_2 = 10$, and $k_3 = 20$

h_1	h_2	h_3	b	σ	T_1^*	T_2^*	T_3^*	R_1^*	R_2^*	R_3^*	$Cost$
2	1.5	0.5	10	0.5	2.7	2.7	5.4	14.63	2.85	13.95	23.52
			10	1	3.40	3.40	3.40	18.09	2.13	3.45	25.47
			20	0.5	2.45	2.45	4.9	14.11	3.24	13.5	24.89
			20	1	3.40	3.40	3.40	18.89	2.61	3.78	27.63
2	1.5	1	10	0.5	3.49	3.49	3.49	17.63	3.27	2.73	23.72
			10	1	3.44	3.44	3.44	18.24	2.99	2.45	25.40
			20	0.5	3.4	3.4	3.4	17.82	3.47	3.18	25.24
			20	1	3.43	3.43	3.43	19.08	3.30	2.84	27.69
2.5	1.5	0.5	10	0.5	2.51	2.51	5.02	13.33	3.11	13.32	25.29
			10	1	2.36	2.36	4.72	13.27	2.97	13.07	27.51
			20	0.5	2.39	2.39	4.78	13.42	3.43	13.32	27.00
			20	1	2.34	2.34	4.68	13.98	3.32	13.50	30.05
2.5	1.5	1	10	0.5	3.25	3.25	3.25	16.03	3.45	2.91	25.98
			10	1	3.32	3.32	3.32	16.72	3.39	2.65	27.82
			20	0.5	3.11	3.11	3.11	16.14	3.69	3.31	27.93
			20	1	2.97	2.97	2.97	16.38	3.77	3.10	30.73
3.5	3	2	10	0.5	2.82	2.82	2.82	15.2	2.88	2.25	29.57
			10	1	2.91	2.91	2.91	16.45	2.37	1.84	31.68
			20	0.5	2.63	2.63	2.63	14.85	3.22	2.91	32.28
			20	1	2.69	2.69	2.69	16.11	2.87	2.55	35.60
3.5	3	2.5	10	0.5	2.82	2.82	2.82	15.2	3.4	1.73	29.57
			10	1	2.91	2.91	2.91	16.42	3.24	0.99	31.69
			20	0.5	2.64	2.64	2.64	14.89	3.64	2.48	32.28
			20	1	2.78	2.78	2.78	16.53	3.47	1.83	35.61
4	3	2	10	0.5	2.69	2.69	2.69	14.11	3.13	2.42	31.04
			10	1	2.87	2.87	2.87	15.31	2.84	2.06	33.32
			20	0.5	2.48	2.48	2.48	13.81	3.45	3.03	34.14
			20	1	2.53	2.53	2.53	14.74	3.31	2.72	37.80
4	3	2.5	10	0.5	2.69	2.69	2.69	14.11	3.58	1.97	31.04
			10	1	2.75	2.75	2.75	14.87	3.59	1.41	33.28
			20	0.5	2.49	2.49	2.49	13.84	3.78	2.7	34.14
			20	1	2.69	2.69	2.69	15.39	3.85	2.04	37.71

Table 2: Optimal solutions for cases with $k_1 = 10$, $k_2 = 10$, and $k_3 = 40$

h_1	h_2	h_3	b	σ	T_1^*	T_2^*	T_3^*	R_1^*	R_2^*	R_3^*	$Cost$
2	1.5	0.5	10	0.5	3.06	3.06	6.12	16.02	2.70	15.30	26.97
			10	1	3.06	3.06	6.11	16.76	2.26	15.50	28.79
			20	0.5	2.95	2.95	5.9	16.07	3.10	15.40	28.49
			20	1	2.80	2.80	5.59	16.45	2.83	15.26	31.03
2	1.5	1	10	0.5	4.31	4.31	4.31	20.80	3.06	2.48	28.86
			10	1	4.26	4.26	4.26	21.33	2.88	2.12	30.40
			20	0.5	4.02	4.02	4.02	20.21	3.42	3.06	30.60
			20	1	4.11	4.11	4.11	21.71	3.18	2.70	32.91
2.5	1.5	0.5	10	0.5	2.43	2.43	7.29	12.99	3.18	22.35	29.01
			10	1	3.03	3.03	6.05	15.65	2.71	15.55	31.10
			20	0.5	2.31	2.31	6.93	13.18	3.39	22.05	30.79
			20	1	2.26	2.26	6.77	13.65	3.36	22.24	33.62
2.5	1.5	1	10	0.5	2.31	4.62	4.62	12.61	12.52	2.38	31.30
			10	1	2.29	4.57	4.57	12.99	12.64	1.89	33.25
			20	0.5	2.23	4.46	4.46	12.82	12.60	2.91	33.23
			20	1	2.26	4.51	4.51	13.63	13.10	2.42	35.99
3.5	3	2	10	0.5	3.44	3.44	3.44	17.56	2.69	1.92	35.96
			10	1	3.48	3.48	3.48	18.59	2.18	1.53	37.90
			20	0.5	3.2	3.2	3.2	17.08	3.09	2.73	39.10
			20	1	3.28	3.28	3.28	18.55	2.62	2.30	42.24
3.5	3	2.5	10	0.5	3.44	3.44	3.44	17.56	3.31	1.3	35.96
			10	1	3.51	3.51	3.51	18.74	3.02	0.62	37.90
			20	0.5	3.2	3.2	3.2	17.08	3.51	2.31	39.10
			20	1	3.25	3.25	3.25	18.47	3.32	1.59	42.23
4	3	2	10	0.5	3.27	3.27	3.27	16.24	2.96	2.12	37.75
			10	1	3.33	3.33	3.33	16.99	2.70	1.80	39.80
			20	0.5	3.07	3.07	3.07	16.05	3.33	2.85	41.34
			20	1	3.12	3.12	3.12	17.04	3.12	2.50	44.68
4	3	2.5	10	0.5	3.28	3.28	3.28	16.28	3.47	1.59	37.75
			10	1	3.41	3.41	3.41	17.29	3.39	1.02	39.81
			20	0.5	3.07	3.07	3.07	16.05	3.72	2.46	41.35
			20	1	3.09	3.09	3.09	16.95	3.73	1.88	44.69

Table 3: Optimal solutions for cases with $k_1 = 10$, $k_2 = 20$, and $k_3 = 20$

h_1	h_2	h_3	b	σ	T_1^*	T_2^*	T_3^*	R_1^*	R_2^*	R_3^*	$Cost$
2	1.5	0.5	10	0.5	3.88	3.88	3.88	19.12	2.47	3.36	26.41
			10	1	4.07	4.07	4.07	20.64	1.84	3.36	27.96
			20	0.5	3.67	3.67	3.67	18.85	2.97	3.68	28.03
			20	1	3.87	3.87	3.87	20.76	2.43	3.71	30.35
2	1.5	1	10	0.5	3.9	3.9	3.9	19.2	3.16	2.61	26.42
			10	1	3.92	3.92	3.92	20.11	2.88	2.24	28.00
			20	0.5	3.7	3.7	3.7	18.97	3.46	3.12	28.05
			20	1	3.79	3.79	3.79	20.45	3.24	2.76	30.38
2.5	1.5	0.5	10	0.5	3.58	3.58	3.58	17.23	2.79	3.45	28.92
			10	1	4.09	4.09	4.09	19.50	2.33	3.43	30.79
			20	0.5	2.15	4.3	4.3	12.51	11.7	3.6	30.89
			20	1	2.10	4.20	4.20	13.08	11.43	3.64	33.80
2.5	1.5	1	10	0.5	3.62	3.62	3.62	17.37	3.4	2.79	28.93
			10	1	3.83	3.83	3.83	18.59	3.29	2.47	30.72
			20	0.5	2.05	4.1	4.1	12.14	11.9	3	30.89
			20	1	3.40	3.40	3.40	17.98	3.68	3.01	33.86
3.5	3	2	10	0.5	3.14	3.14	3.14	16.42	2.77	2.08	32.92
			10	1	3.18	3.18	3.18	17.44	2.31	1.67	34.95
			20	0.5	2.9	2.9	2.9	15.92	3.15	2.82	35.86
			20	1	2.97	2.97	2.97	17.25	2.78	2.42	39.10
3.5	3	2.5	10	0.5	3.14	3.14	3.14	16.42	3.34	1.51	32.92
			10	1	3.22	3.22	3.22	17.52	3.14	0.85	34.95
			20	0.5	2.96	2.96	2.96	16.15	3.57	2.37	35.86
			20	1	2.94	2.94	2.94	17.25	3.43	1.69	39.07
4	3	2	10	0.5	2.99	2.99	2.99	15.22	3.03	2.26	34.56
			10	1	3.06	3.06	3.06	16.00	2.77	1.96	36.71
			20	0.5	2.81	2.81	2.81	15.06	3.37	2.94	37.93
			20	1	2.88	2.88	2.88	16.09	3.19	2.60	41.39
4	3	2.5	10	0.5	3	3	3	15.25	3.53	1.76	34.56
			10	1	3.07	3.07	3.07	16.00	3.42	1.31	36.73
			20	0.5	2.82	2.82	2.82	15.1	3.74	2.56	37.92
			20	1	2.84	2.84	2.84	16.00	3.78	1.99	41.39

Table 4: Optimal solutions for cases with $k_1 = 10$, $k_2 = 20$, and $k_3 = 40$

h_1	h_2	h_3	b	σ	T_1^*	T_2^*	T_3^*	R_1^*	R_2^*	R_3^*	$Cost$
2	1.5	0.5	10	0.5	3.48	3.48	6.96	17.60	2.58	16.83	30.05
			10	1	3.47	3.47	6.93	18.28	2.14	17.00	31.86
			20	0.5	3.27	3.27	6.54	17.29	3.05	16.61	31.69
			20	1	3.39	3.39	6.77	18.81	2.59	17.50	34.17
2	1.5	1	10	0.5	4.61	4.61	4.61	21.90	3.07	2.38	31.11
			10	1	4.78	4.78	4.78	23.31	2.73	1.96	32.63
			20	0.5	4.39	4.39	4.39	21.67	3.36	3.00	32.94
			20	1	4.52	4.52	4.52	23.31	3.12	2.60	35.24
2.5	1.5	0.5	10	0.5	3.23	3.23	6.46	15.92	2.92	15.95	32.29
			10	1	3.16	3.16	6.31	16.11	2.67	16.00	34.33
			20	0.5	3.05	3.05	6.1	15.92	3.30	15.84	34.31
			20	1	2.97	2.97	5.93	16.32	3.14	16.00	37.16
2.5	1.5	1	10	0.5	2.46	4.92	4.92	13.18	13.03	2.29	33.39
			10	1	2.53	5.05	5.05	13.86	13.44	1.81	35.33
			20	0.5	2.44	4.88	4.88	13.57	13.43	2.85	35.42
			20	1	2.34	4.67	4.67	13.94	13.39	2.36	38.20
3.5	3	2	10	0.5	3.71	3.71	3.71	18.59	2.61	1.76	38.76
			10	1	3.78	3.78	3.78	19.73	2.08	1.35	40.64
			20	0.5	3.47	3.47	3.47	18.13	3.04	2.64	42.08
			20	1	3.53	3.53	3.53	19.46	2.60	2.21	45.20
3.5	3	2.5	10	0.5	3.71	3.71	3.71	18.59	3.25	1.12	38.76
			10	1	3.78	3.78	3.78	19.73	2.98	0.46	40.64
			20	0.5	3.44	3.44	3.44	18.01	3.52	2.18	42.08
			20	1	3.54	3.54	3.54	19.50	3.32	1.48	45.18
4	3	2	10	0.5	3.53	3.53	3.53	17.21	2.88	1.97	40.69
			10	1	3.60	3.60	3.60	17.98	2.60	1.65	42.67
			20	0.5	3.29	3.29	3.29	16.89	3.28	2.79	44.50
			20	1	3.29	3.29	3.29	17.65	3.08	2.45	47.84
4	3	2.5	10	0.5	3.53	3.53	3.53	17.21	3.43	1.41	40.69
			10	1	3.61	3.61	3.61	18.01	3.33	0.92	42.68
			20	0.5	3.29	3.29	3.29	16.89	3.66	2.41	44.51
			20	1	3.29	3.29	3.29	17.67	3.67	1.84	47.82

Table 5: Optimal solutions for cases with $k_1 = 20$, $k_2 = 10$, and $k_3 = 20$

h_1	h_2	h_3	b	σ	T_1^*	T_2^*	T_3^*	R_1^*	R_2^*	R_3^*	$Cost$
2	1.5	0.5	10	0.5	3.88	3.88	3.88	19.12	2.47	3.36	26.41
			10	1	4.08	4.08	4.08	20.68	1.85	3.30	28.04
			20	0.5	3.67	3.67	3.67	18.85	2.97	3.68	28.03
			20	1	3.87	3.87	3.87	20.76	2.43	3.71	30.35
2	1.5	1	10	0.5	3.9	3.9	3.9	19.2	3.16	2.61	26.42
			10	1	3.96	3.96	3.96	20.26	2.86	2.26	28.02
			20	0.5	3.7	3.7	3.7	18.97	3.46	3.12	28.05
			20	1	3.80	3.80	3.80	20.49	3.22	2.78	30.47
2.5	1.5	0.5	10	0.5	3.58	3.58	3.58	17.23	2.79	3.45	28.92
			10	1	4.09	4.09	4.09	19.50	2.33	3.43	30.79
			20	0.5	2.83	2.83	5.66	15.08	3.35	15.01	30.93
			20	1	3.66	3.66	3.66	18.97	2.93	3.85	33.88
2.5	1.5	1	10	0.5	3.62	3.62	3.62	17.37	3.4	2.79	28.93
			10	1	3.57	3.57	3.57	17.67	3.32	2.56	30.67
			20	0.5	3.37	3.37	3.37	17.12	3.66	3.27	30.99
			20	1	3.45	3.45	3.45	18.21	3.66	3.00	33.69
3.5	3	2	10	0.5	3.14	3.14	3.14	16.42	2.77	2.08	32.92
			10	1	3.25	3.25	3.25	17.71	2.27	1.65	34.94
			20	0.5	2.9	2.9	2.9	15.92	3.15	2.82	35.86
			20	1	2.97	2.97	2.97	17.25	2.78	2.42	39.10
3.5	3	2.5	10	0.5	3.14	3.14	3.14	16.42	3.34	1.51	32.92
			10	1	3.21	3.21	3.21	17.48	3.15	0.86	34.94
			20	0.5	2.96	2.96	2.96	16.15	3.57	2.37	35.86
			20	1	2.96	2.96	2.96	17.33	3.39	1.71	39.04
4	3	2	10	0.5	2.99	2.99	2.99	15.22	3.03	2.26	34.56
			10	1	3.06	3.06	3.06	16.00	2.77	1.96	36.71
			20	0.5	2.81	2.81	2.81	15.06	3.37	2.94	37.93
			20	1	2.88	2.88	2.88	16.09	3.19	2.60	41.39
4	3	2.5	10	0.5	3	3	3	15.25	3.53	1.76	34.56
			10	1	3.07	3.07	3.07	16.00	3.42	1.31	36.73
			20	0.5	2.82	2.82	2.82	15.1	3.74	2.56	37.92
			20	1	2.84	2.84	2.84	16.00	3.78	1.99	41.39

Table 6: Optimal solutions for cases with $k_1 = 20$, $k_2 = 10$, and $k_3 = 40$

h_1	h_2	h_3	b	σ	T_1^*	T_2^*	T_3^*	R_1^*	R_2^*	R_3^*	$Cost$
2	1.5	0.5	10	0.5	3.48	3.48	6.96	17.60	2.58	16.83	30.05
			10	1	3.47	3.47	6.93	18.28	2.14	17.00	31.86
			20	0.5	3.27	3.27	6.54	17.29	3.05	16.61	31.69
			20	1	3.67	3.67	7.34	19.96	2.51	18.50	34.37
2	1.5	1	10	0.5	4.61	4.61	4.61	21.90	3.07	2.38	31.11
			10	1	4.78	4.78	4.78	23.31	2.73	1.96	32.63
			20	0.5	4.39	4.39	4.39	21.67	3.36	3.00	32.94
			20	1	4.52	4.52	4.52	23.31	3.12	2.60	35.24
2.5	1.5	0.5	10	0.5	3.31	3.31	6.62	16.26	2.85	16.28	32.30
			10	1	3.31	3.31	6.62	16.68	2.61	16.56	34.49
			20	0.5	3.05	3.05	6.1	15.92	3.30	15.84	34.31
			20	1	3.25	3.25	6.50	17.44	3.03	17.03	37.40
2.5	1.5	1	10	0.5	4.23	4.23	4.23	19.63	3.26	2.62	34.06
			10	1	4.36	4.36	4.36	20.49	3.19	2.31	35.72
			20	0.5	4.08	4.08	4.08	19.82	3.57	3.15	36.38
			20	1	4.18	4.18	4.18	20.95	3.57	2.83	39.09
3.5	3	2	10	0.5	3.71	3.71	3.71	18.59	2.61	1.76	38.76
			10	1	3.82	3.82	3.82	19.88	2.06	1.34	40.65
			20	0.5	3.47	3.47	3.47	18.13	3.04	2.64	42.09
			20	1	3.54	3.54	3.54	19.50	2.58	2.22	45.22
3.5	3	2.5	10	0.5	3.71	3.71	3.71	18.59	3.25	1.12	38.76
			10	1	3.78	3.78	3.78	19.73	2.98	0.46	40.64
			20	0.5	3.44	3.44	3.44	18.01	3.52	2.18	42.09
			20	1	3.52	3.52	3.52	19.42	3.37	1.45	45.19
4	3	2	10	0.5	3.53	3.53	3.53	17.21	2.88	1.97	40.69
			10	1	3.68	3.68	3.68	18.26	2.57	1.63	42.69
			20	0.5	3.29	3.29	3.29	16.89	3.28	2.79	44.50
			20	1	3.33	3.33	3.33	17.79	3.05	2.47	47.90
4	3	2.5	10	0.5	3.53	3.53	3.53	17.21	3.43	1.41	40.69
			10	1	3.60	3.60	3.60	17.98	3.34	0.91	42.68
			20	0.5	3.29	3.29	3.29	16.89	3.66	2.41	44.51
			20	1	3.30	3.30	3.30	17.69	3.72	1.80	47.84

Table 7: Optimal solutions for cases with $k_1 = 20$, $k_2 = 20$, and $k_3 = 20$

h_1	h_2	h_3	b	σ	T_1^*	T_2^*	T_3^*	R_1^*	R_2^*	R_3^*	$Cost$
2	1.5	0.5	10	0.5	4.31	4.31	4.31	20.76	2.32	3.27	28.86
			10	1	4.21	4.21	4.21	21.18	1.79	3.29	30.42
			20	0.5	4.06	4.06	4.06	20.38	2.87	3.64	30.57
			20	1	3.96	3.96	3.96	21.10	2.41	3.71	32.85
2	1.5	1	10	0.5	4.31	4.31	4.31	20.76	3.1	2.48	28.86
			10	1	4.31	4.31	4.31	21.56	2.78	2.13	30.41
			20	0.5	4	4	4	20.15	3.42	3.06	30.60
			20	1	4.30	4.30	4.30	22.47	3.11	2.67	33.05
2.5	1.5	0.5	10	0.5	3.92	3.92	3.92	18.47	2.67	3.39	31.59
			10	1	3.82	3.82	3.82	18.55	2.40	3.50	33.26
			20	0.5	3.8	3.8	3.8	18.76	3.15	3.72	33.76
			20	1	3.60	3.60	3.60	18.74	2.96	3.84	36.62
2.5	1.5	1	10	0.5	3.84	3.84	3.84	18.19	3.36	2.7	31.60
			10	1	3.71	3.71	3.71	18.17	3.30	2.50	33.39
			20	0.5	3.81	3.81	3.81	18.8	3.61	3.18	33.80
			20	1	4.21	4.21	4.21	21.06	3.57	2.82	36.64
3.5	3	2	10	0.5	3.44	3.44	3.44	17.56	2.69	1.92	35.96
			10	1	3.56	3.56	3.56	18.89	2.14	1.50	37.91
			20	0.5	3.2	3.2	3.2	17.08	3.09	2.73	39.10
			20	1	3.40	3.40	3.40	19.04	2.58	2.25	42.27
3.5	3	2.5	10	0.5	3.44	3.44	3.44	17.56	3.31	1.3	35.96
			10	1	3.51	3.51	3.51	18.74	3.02	0.62	37.90
			20	0.5	3.2	3.2	3.2	17.08	3.51	2.31	39.10
			20	1	3.38	3.38	3.38	18.97	3.36	1.47	42.26
4	3	2	10	0.5	3.27	3.27	3.27	16.24	2.96	2.12	37.75
			10	1	3.33	3.33	3.33	16.99	2.70	1.80	39.80
			20	0.5	3.07	3.07	3.07	16.05	3.33	2.85	41.34
			20	1	3.10	3.10	3.10	16.97	3.12	2.51	44.74
4	3	2.5	10	0.5	3.28	3.28	3.28	16.28	3.47	1.59	37.75
			10	1	3.44	3.44	3.44	17.37	3.45	0.96	39.81
			20	0.5	3.07	3.07	3.07	16.05	3.72	2.46	41.35
			20	1	3.07	3.07	3.07	16.87	3.73	1.90	44.75

Table 8: Optimal solutions for cases with $k_1 = 20$, $k_2 = 20$, and $k_3 = 40$

h_1	h_2	h_3	b	σ	T_1^*	T_2^*	T_3^*	R_1^*	R_2^*	R_3^*	$Cost$
2	1.5	0.5	10	0.5	3.8	3.8	7.6	18.81	2.49	18.00	32.83
			10	1	3.90	3.90	7.80	20.00	1.92	18.50	34.81
			20	0.5	3.86	3.86	7.72	19.61	2.91	18.85	34.59
			20	1	3.56	3.56	7.12	19.54	2.52	18.07	37.32
2	1.5	1	10	0.5	4.93	4.93	4.93	23.12	3.00	2.28	33.20
			10	1	5.13	5.13	5.13	24.68	2.63	1.87	34.73
			20	0.5	4.75	4.75	4.75	23.08	3.31	2.94	35.13
			20	1	5.05	5.05	5.05	25.37	3.02	2.50	37.52
2.5	1.5	0.5	10	0.5	3.62	3.62	7.24	17.37	2.78	17.40	35.30
			10	1	3.57	3.57	7.14	17.63	2.52	17.52	37.29
			20	0.5	3.4	3.4	6.8	17.23	3.25	17.19	37.50
			20	1	3.25	3.25	6.50	17.44	3.03	17.03	40.48
2.5	1.5	1	10	0.5	4.51	4.51	4.51	20.64	3.25	2.51	36.35
			10	1	4.51	4.51	4.51	21.02	3.15	2.30	38.03
			20	0.5	4.35	4.35	4.35	20.83	3.57	3.09	38.81
			20	1	4.35	4.35	4.35	21.59	3.51	2.82	41.61
3.5	3	2	10	0.5	3.96	3.96	3.96	19.54	2.54	1.62	41.37
			10	1	4.04	4.04	4.04	20.72	1.96	1.22	43.20
			20	0.5	3.72	3.72	3.72	19.12	2.96	2.58	44.87
			20	1	3.80	3.80	3.80	20.57	2.48	2.12	47.92
3.5	3	2.5	10	0.5	3.96	3.96	3.96	19.54	3.23	0.94	41.37
			10	1	4.04	4.04	4.04	20.72	2.92	0.27	43.20
			20	0.5	3.71	3.71	3.71	19.08	3.46	2.08	44.87
			20	1	3.74	3.74	3.74	20.38	3.22	1.38	47.92
4	3	2	10	0.5	3.77	3.77	3.77	18.09	2.81	1.84	43.42
			10	1	3.83	3.83	3.83	18.78	2.52	1.58	45.37
			20	0.5	3.56	3.56	3.56	17.9	3.25	2.7	47.45
			20	1	3.63	3.63	3.63	18.97	2.97	2.35	50.74
4	3	2.5	10	0.5	3.77	3.77	3.77	18.09	3.42	1.24	43.42
			10	1	3.81	3.81	3.81	18.74	3.36	0.72	45.36
			20	0.5	3.54	3.54	3.54	17.84	3.65	2.29	47.45
			20	1	3.67	3.67	3.67	19.14	3.60	1.68	50.70

Table 9: Comparison results between V and MO for $N = 2$ and $\lambda = 2$

N	λ	b_1	b_2	T_0	S_0^{mo}	S_0^v	C^{mo}	C^v	V vs MO (%)
2	2	4.4	4.4	3	21	21	9.92	10.04	1.13
2	2	9.9	9.9	3	24	24	12.61	12.77	1.25
2	2	108.9	108.9	3	32	32	19.70	19.95	1.27
2	2	4.4	9.9	3	23	23	11.26	11.43	1.49
2	2	4.4	108.9	3	27	27	14.89	15.18	1.99
2	2	9.9	108.9	3	28	28	16.16	16.52	2.23
2	2	4.4	4.4	4	24	24	11.35	11.45	0.92
2	2	9.9	9.9	4	28	28	14.33	14.45	0.83
2	2	108.9	108.9	4	36	36	21.94	22.16	1.00
2	2	4.4	9.9	4	26	26	12.84	12.97	1.07
2	2	4.4	108.9	4	30	31	16.76	17.12	2.18
2	2	9.9	108.9	4	32	32	18.16	18.46	1.65
2	2	4.4	4.4	5	27	27	12.81	12.90	0.71
2	2	9.9	9.9	5	31	31	16.05	16.16	0.71
2	2	108.9	108.9	5	40	40	24.20	24.40	0.84
2	2	4.4	9.9	5	29	29	14.43	14.56	0.88
2	2	4.4	108.9	5	34	34	18.69	19.12	2.33
2	2	9.9	108.9	5	36	36	20.21	20.47	1.29
2	2	4.4	4.4	6	30	30	14.30	14.38	0.56
2	2	9.9	9.9	6	35	35	17.82	17.92	0.57
2	2	108.9	108.9	6	44	44	26.45	26.64	0.72
2	2	4.4	9.9	6	32	32	16.07	16.19	0.79
2	2	4.4	108.9	6	38	38	20.68	21.12	2.15
2	2	9.9	108.9	6	39	40	22.23	22.53	1.34
2	2	4.4	4.4	7	33	33	15.82	15.89	0.46
2	2	9.9	9.9	7	38	38	19.56	19.66	0.50
2	2	108.9	108.9	7	48	48	28.70	28.88	0.63
2	2	4.4	9.9	7	36	36	17.72	17.84	0.64
2	2	4.4	108.9	7	42	41	22.70	23.15	1.97
2	2	9.9	108.9	7	43	44	24.25	24.60	1.44

Table 10: Comparison results between V and MO for $N = 3$ and $\lambda = 3$

N	λ	b_1	b_2	T_0	S_0^{mo}	S_0^v	C^{mo}	C^v	V vs MO (%)
2	3	4.4	4.4	3	31	31	12.81	12.96	1.17
2	3	9.9	9.9	3	35	35	16.13	16.33	1.19
2	3	108.9	108.9	3	44	44	24.60	24.93	1.34
2	3	4.4	9.9	3	33	33	14.47	14.67	1.41
2	3	4.4	108.9	3	38	38	18.81	19.24	2.28
2	3	9.9	108.9	3	40	40	20.42	20.79	1.81
2	3	4.4	4.4	4	36	36	15.00	15.12	0.84
2	3	9.9	9.9	4	40	40	18.68	18.85	0.89
2	3	108.9	108.9	4	50	50	27.89	28.19	1.07
2	3	4.4	9.9	4	38	38	16.82	17.00	1.06
2	3	4.4	108.9	4	43	44	21.62	22.16	2.47
2	3	9.9	108.9	4	45	45	23.34	23.78	1.92
2	3	4.4	4.4	5	40	40	17.20	17.32	0.66
2	3	9.9	9.9	5	45	45	21.28	21.43	0.74
2	3	108.9	108.9	5	56	56	31.19	31.47	0.88
2	3	4.4	9.9	5	43	43	19.25	19.42	0.89
2	3	4.4	108.9	5	48	49	24.52	25.11	2.38
2	3	9.9	108.9	5	51	51	26.32	26.78	1.74
2	3	4.4	4.4	6	45	45	19.48	19.58	0.51
2	3	9.9	9.9	6	50	50	23.90	24.04	0.60
2	3	108.9	108.9	6	62	62	34.49	34.74	0.74
2	3	4.4	9.9	6	48	48	21.71	21.89	0.79
2	3	4.4	108.9	6	54	55	27.44	28.12	2.45
2	3	9.9	108.9	6	57	57	29.35	29.82	1.59
2	3	4.4	4.4	7	49	50	21.80	21.89	0.41
2	3	9.9	9.9	7	56	56	26.55	26.67	0.46
2	3	108.9	108.9	7	68	68	37.77	38.01	0.64
2	3	4.4	9.9	7	53	53	24.20	24.38	0.74
2	3	4.4	108.9	7	60	61	30.42	31.16	2.43
2	3	9.9	108.9	7	62	62	32.37	32.88	1.56

Table 11: Comparison results between V and MO for $N = 4$ and $\lambda = 2$

N	λ	b_1	b_2	T_0	S_0^{mo}	S_0^v	C^{mo}	C^v	V vs MO (%)
4	2	4.4	4.4	3	42	42	18.64	18.99	1.87
4	2	9.9	9.9	3	48	48	23.61	24.05	1.87
4	2	108.9	108.9	3	61	62	36.56	37.25	1.88
4	2	4.4	9.9	3	45	46	21.11	21.61	2.41
4	2	4.4	108.9	3	52	54	27.79	28.72	3.34
4	2	9.9	108.9	3	55	56	30.19	30.97	2.56
4	2	4.4	4.4	4	48	48	21.43	21.71	1.33
4	2	9.9	9.9	4	54	55	26.82	27.21	1.45
4	2	108.9	108.9	4	69	69	40.64	41.30	1.62
4	2	4.4	9.9	4	51	53	24.12	24.68	2.32
4	2	4.4	108.9	4	60	61	31.36	32.37	3.20
4	2	9.9	108.9	4	62	64	33.86	34.70	2.50
4	2	4.4	4.4	5	54	54	24.27	24.52	1.04
4	2	9.9	9.9	5	61	61	30.09	30.44	1.18
4	2	108.9	108.9	5	77	77	44.77	45.35	1.31
4	2	4.4	9.9	5	57	60	27.21	27.84	2.30
4	2	4.4	108.9	5	67	68	34.99	35.95	2.74
4	2	9.9	108.9	5	70	72	37.59	38.59	2.64
4	2	4.4	4.4	6	60	60	27.19	27.43	0.88
4	2	9.9	9.9	6	68	68	33.43	33.75	0.95
4	2	108.9	108.9	6	84	85	48.90	49.45	1.12
4	2	4.4	9.9	6	64	66	30.35	30.82	1.55
4	2	4.4	108.9	6	74	75	38.60	39.46	2.24
4	2	9.9	108.9	6	77	79	41.36	42.30	2.28
4	2	4.4	4.4	7	66	67	30.12	30.34	0.73
4	2	9.9	9.9	7	75	75	36.82	37.11	0.79
4	2	108.9	108.9	7	92	93	53.03	53.57	1.01
4	2	4.4	9.9	7	71	73	33.53	34.04	1.52
4	2	4.4	108.9	7	81	82	42.28	43.20	2.19
4	2	9.9	108.9	7	84	86	45.20	45.99	1.76

Table 12: Comparison results between V and MO for $N = 4$ and $\lambda = 3$

N	λ	b_1	b_2	T_0	S_0^{mo}	S_0^v	C^{mo}	C^v	V vs MO (%)
4	3	4.4	4.4	3	61	61	24.00	24.47	1.97
4	3	9.9	9.9	3	69	69	30.20	30.75	1.82
4	3	108.9	108.9	3	86	86	45.57	46.53	2.11
4	3	4.4	9.9	3	65	67	27.11	27.84	2.69
4	3	4.4	108.9	3	74	76	34.97	36.23	3.61
4	3	9.9	108.9	3	77	79	37.91	39.03	2.95
4	3	4.4	4.4	4	70	70	28.23	28.62	1.41
4	3	9.9	9.9	4	78	79	34.96	35.49	1.50
4	3	108.9	108.9	4	97	98	51.61	52.49	1.71
4	3	4.4	9.9	4	75	76	31.63	32.19	1.78
4	3	4.4	108.9	4	84	86	40.26	41.40	2.84
4	3	9.9	108.9	4	88	90	43.43	44.52	2.49
4	3	4.4	4.4	5	79	80	32.63	32.97	1.02
4	3	9.9	9.9	5	89	89	39.95	40.42	1.18
4	3	108.9	108.9	5	109	109	57.67	58.49	1.42
4	3	4.4	9.9	5	84	87	36.32	37.00	1.88
4	3	4.4	108.9	5	95	97	45.70	46.86	2.55
4	3	9.9	108.9	5	99	100	49.04	50.06	2.09
4	3	4.4	4.4	6	89	89	37.06	37.37	0.84
4	3	9.9	9.9	6	99	99	45.02	45.45	0.96
4	3	108.9	108.9	6	120	121	63.74	64.52	1.23
4	3	4.4	9.9	6	94	97	41.07	41.75	1.67
4	3	4.4	108.9	6	106	107	51.17	52.40	2.42
4	3	9.9	108.9	6	110	112	54.67	55.76	2.00
4	3	4.4	4.4	7	98	98	41.57	41.85	0.67
4	3	9.9	9.9	7	109	110	50.15	50.55	0.80
4	3	108.9	108.9	7	132	133	69.84	70.59	1.08
4	3	4.4	9.9	7	104	107	45.90	46.56	1.45
4	3	4.4	108.9	7	117	119	56.71	58.08	2.41
4	3	9.9	108.9	7	122	123	60.35	61.50	1.90

Table 13: Comparison results among V , MO and MC_r for cases with $T_0 = 4$

N	T_0	λ	b_1	b_2	S_0^h	C^{mc_r}	C^{mo}	C^v	MO vs MC_r (%)	V vs MC_r (%)
2	4	2	4.4	9.9	26	12.78	12.74	12.87	-0.33	0.65
2	4	2	4.4	108.9	31	16.90	16.80	16.95	-0.59	0.30
2	4	4	4.4	9.9	49	20.73	20.71	20.92	-0.06	0.91
2	4	4	4.4	108.9	56	26.06	26.41	27.10	1.34	3.97
2	4	6	4.4	108.9	80	34.20	34.75	35.76	1.61	4.57
2	4	6	9.9	108.9	85	37.27	37.30	37.62	0.07	0.95
2	4	2	4.4	4.4	24	11.25	11.37	11.30	1.00	0.43
2	4	2	9.9	9.9	28	14.44	14.51	14.48	0.48	0.21
2	4	4	4.4	4.4	48	18.52	18.62	18.76	0.54	1.29
2	4	4	9.9	9.9	52	22.66	22.65	23.02	-0.06	1.59
2	4	6	4.4	4.4	70	25.09	25.10	25.37	0.04	1.10
2	4	6	108.9	108.9	91	44.22	44.15	44.47	-0.18	0.55
4	4	2	4.4	108.9	60	32.06	31.67	32.16	-1.23	0.30
4	4	4	4.4	9.9	98	39.72	39.69	40.25	-0.07	1.34
4	4	4	4.4	108.9	110	49.50	49.79	51.24	0.58	3.51
4	4	4	9.9	108.9	113	52.56	52.60	55.05	0.08	4.75
4	4	6	4.4	9.9	144	54.12	53.94	54.84	-0.33	1.33
4	4	6	4.4	108.9	157	65.09	66.18	69.00	1.68	6.00
4	4	6	9.9	108.9	165	70.70	71.07	72.50	0.52	2.54
4	4	4	4.4	4.4	94	35.54	35.59	35.92	0.14	1.05
4	4	4	9.9	9.9	104	43.55	43.29	44.03	-0.58	1.11
4	4	6	4.4	4.4	139	48.93	48.83	49.53	-0.21	1.22
4	4	6	9.9	9.9	152	59.29	58.99	59.84	-0.49	0.93
4	4	6	108.9	108.9	179	83.66	83.39	83.74	-0.32	0.10
6	4	2	4.4	108.9	90	47.18	46.34	47.38	-1.77	0.44
6	4	2	9.9	108.9	93	50.83	50.15	51.11	-1.33	0.55
6	4	4	4.4	9.9	147	58.64	58.58	59.17	-0.09	0.90
6	4	4	4.4	108.9	163	72.65	73.42	75.22	1.07	3.55
6	4	4	9.9	108.9	170	78.86	78.91	80.40	0.07	1.95
6	4	6	4.4	9.9	217	80.10	79.75	81.10	-0.44	1.24
6	4	6	4.4	108.9	236	96.00	98.23	102.44	2.32	6.71
6	4	6	9.9	108.9	246	103.95	104.54	107.19	0.58	3.12
6	4	2	4.4	4.4	72	32.15	31.96	32.47	-0.60	1.00
6	4	2	9.9	9.9	83	40.05	40.09	40.67	0.09	1.55
6	4	4	4.4	4.4	142	52.72	52.77	53.33	0.09	1.15
6	4	4	9.9	9.9	153	64.27	63.94	65.59	-0.51	2.06
6	4	6	4.4	4.4	209	72.61	72.52	73.29	-0.13	0.94
6	4	6	9.9	9.9	227	87.67	87.06	88.39	-0.70	0.82
6	4	6	108.9	108.9	267	122.07	121.26	123.05	-0.67	0.80

Table 14: Comparison results among V , MO and MC_r for cases with $T_0 = 6$

N	T_0	λ	b_1	b_2	S_0^h	C^{mc_r}	C^{mo}	C^v	MO vs MC_r (%)	V vs MC_r (%)
2	6	2	4.4	9.9	32	15.99	15.97	16.13	-0.17	0.83
2	6	2	4.4	108.9	38	20.60	20.75	21.12	0.74	2.55
2	6	4	4.4	9.9	63	27.31	27.35	27.41	0.15	0.36
2	6	4	4.4	108.9	70	33.38	34.18	34.53	2.41	3.43
2	6	4	9.9	108.9	73	36.22	36.48	37.18	0.72	2.66
2	6	6	4.4	108.9	102	44.54	45.87	48.22	2.99	8.28
2	6	6	9.9	108.9	105	48.38	48.97	50.69	1.21	4.78
2	6	2	4.4	4.4	30	14.24	14.24	14.31	0.00	0.51
2	6	2	9.9	9.9	34	17.76	17.82	17.99	0.35	1.34
2	6	4	4.4	4.4	59	24.44	24.45	24.58	0.05	0.56
2	6	4	9.9	9.9	66	29.74	29.74	29.80	0.00	0.18
2	6	4	108.9	108.9	80	42.05	41.99	42.51	-0.16	1.09
2	6	6	4.4	4.4	88	34.46	34.44	34.54	-0.07	0.23
2	6	6	9.9	9.9	98	41.34	41.27	41.45	-0.16	0.28
2	6	6	108.9	108.9	115	56.60	56.51	57.64	-0.16	1.85
4	6	2	4.4	9.9	65	30.77	30.58	30.91	-0.59	0.47
4	6	2	4.4	108.9	74	39.05	39.36	39.65	0.79	1.54
4	6	2	9.9	108.9	76	41.30	41.16	42.72	-0.34	3.44
4	6	4	4.4	9.9	124	52.56	52.64	53.23	0.15	1.26
4	6	4	4.4	108.9	135	62.47	64.37	67.24	3.04	7.64
4	6	4	9.9	108.9	142	67.53	68.28	70.53	1.12	4.44
4	6	6	4.4	9.9	185	73.31	73.38	74.13	0.10	1.12
4	6	6	4.4	108.9	199	86.00	88.63	93.95	3.05	9.24
4	6	6	9.9	108.9	209	91.74	92.05	96.37	0.33	5.04
4	6	2	108.9	108.9	86	50.03	48.93	50.11	-2.20	0.16
4	6	4	4.4	4.4	118	47.46	47.42	48.02	-0.08	1.18
4	6	4	9.9	9.9	131	57.37	57.34	57.50	-0.06	0.22
4	6	4	108.9	108.9	157	79.72	79.43	80.05	-0.37	0.41
4	6	6	4.4	4.4	176	67.39	67.24	67.81	-0.21	0.64
4	6	6	9.9	9.9	194	79.59	79.43	80.18	-0.20	0.73
4	6	6	108.9	108.9	224	106.31	106.06	109.50	-0.24	3.00
6	6	2	4.4	9.9	96	45.51	45.00	45.74	-1.14	0.51
6	6	2	4.4	108.9	109	57.03	57.10	58.00	0.12	1.70
6	6	2	9.9	108.9	115	61.08	60.81	62.17	-0.45	1.79
6	6	4	4.4	9.9	188	77.79	77.84	78.35	0.06	0.71
6	6	4	4.4	108.9	204	92.09	94.03	97.99	2.11	6.40
6	6	4	9.9	108.9	215	100.65	101.22	102.74	0.57	2.08
6	6	6	4.4	9.9	277	109.05	109.06	110.66	0.01	1.48
6	6	6	4.4	108.9	299	126.93	130.14	136.12	2.53	7.24
6	6	6	9.9	108.9	312	135.71	136.46	142.34	0.55	4.89

Table 15: Comparison results among V , MO and MC_r for cases with $T_0 = 8$

N	T_0	λ	b_1	b_2	S_0^h	C^{mc_r}	C^{mo}	C^v	MO vs MC_r (%)	V vs MC_r (%)
2	8	2	4.4	108.9	45	24.29	24.65	24.87	1.52	2.42
2	8	2	9.9	108.9	46	25.90	26.12	26.31	0.86	1.57
2	8	4	4.4	9.9	76	33.57	33.68	34.29	0.32	2.14
2	8	4	4.4	108.9	85	40.58	42.19	43.70	3.97	7.69
2	8	4	9.9	108.9	88	43.88	44.63	45.06	1.72	2.70
2	8	6	4.4	9.9	113	47.82	47.91	48.28	0.17	0.95
2	8	6	4.4	108.9	122	55.48	60.31	63.52	8.71	14.48
2	8	6	9.9	108.9	128	60.04	61.26	62.70	2.04	4.43
2	8	2	108.9	108.9	52	30.94	30.83	31.22	-0.37	0.92
2	8	4	9.9	9.9	80	36.72	36.72	37.02	0.00	0.81
2	8	4	108.9	108.9	96	51.04	50.97	51.46	-0.13	0.83
2	8	6	4.4	4.4	108	43.95	43.92	44.02	-0.07	0.17
2	8	6	9.9	9.9	119	51.93	51.93	52.26	-0.02	0.63
4	8	2	4.4	9.9	78	36.99	36.95	37.38	-0.12	1.05
4	8	2	4.4	108.9	88	45.69	46.34	47.93	1.42	4.91
4	8	2	9.9	108.9	91	49.18	49.35	50.40	0.35	2.48
4	8	4	4.4	9.9	151	65.33	65.47	66.56	0.21	1.87
4	8	4	4.4	108.9	166	76.85	80.18	82.65	4.33	7.54
4	8	4	9.9	108.9	172	82.01	83.28	85.97	1.55	4.83
4	8	6	4.4	9.9	227	93.76	93.94	94.22	0.19	0.48
4	8	6	4.4	108.9	242	107.56	113.78	121.56	5.78	13.02
4	8	6	9.9	108.9	253	114.38	115.47	120.09	0.95	4.99
4	8	2	9.9	9.9	81	40.36	40.31	40.93	-0.14	1.41
4	8	4	4.4	4.4	144	59.76	59.77	60.25	0.01	0.82
4	8	4	9.9	9.9	158	70.57	70.49	71.48	-0.11	1.28
4	8	4	108.9	108.9	187	95.33	95.05	97.47	-0.30	2.24
4	8	6	4.4	4.4	216	86.19	86.07	86.42	-0.14	0.26
4	8	6	9.9	9.9	236	100.84	100.70	101.31	-0.14	0.47
4	8	6	108.9	108.9	273	130.74	130.65	133.88	-0.07	2.40
6	8	2	4.4	9.9	116	54.46	54.23	54.90	-0.43	0.81
6	8	2	4.4	108.9	129	66.78	67.46	69.84	1.03	4.59
6	8	2	9.9	108.9	138	72.34	72.34	73.53	-0.01	1.64
6	8	4	4.4	9.9	227	96.72	96.90	97.96	0.18	1.29
6	8	4	4.4	108.9	246	112.95	116.80	122.11	3.41	8.11
6	8	4	9.9	108.9	259	121.86	122.65	126.14	0.65	3.51
6	8	6	4.4	9.9	340	138.56	138.79	140.21	0.17	1.19
6	8	6	4.4	108.9	363	158.95	166.14	174.94	4.52	10.06
6	8	6	9.9	108.9	378	168.43	169.72	177.65	0.77	5.47

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