A study of Rhaly operators on Dirichlet series with real frequencies

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**NANYANG
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SINGAPORE

**A study of Rhyly operators on
Dirichlet series with real
frequencies**

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School of Physical and Mathematical Sciences

2019

A study of Rhaly operators on Dirichlet series with real frequencies

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A thesis submitted to the Nanyang Technological University
in partial fulfilment of the requirement for the degree of
Master of Science

2019

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

12/11/2018

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Date

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Ronald Koh Joon Wei

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Abstract

In this thesis, we recall several important properties of the general Dirichlet series and characterize the entire Dirichlet series in the first two chapters.

We also recall the Cesàro operator, an operator which was defined based on the Cesàro sum, and is heavily related to the Cesàro limits. A more general version of the Cesàro operator; Rhaly operators, will also be defined and the study of its properties on the Hilbert space of entire Dirichlet series is the bedrock of this thesis.

Criteria will be provided for properties of Rhaly operators on the Hilbert space of entire Dirichlet series like boundedness and invariance. Compactness and Schatten p -class inclusions will also be studied on a smaller class of entire Dirichlet series; the *small weighted* Hilbert space of Dirichlet series. An additional section will also study injectivity and the closed range property of bounded Rhaly operators.

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Chapter 1

Introduction

1.1 Classical Dirichlet series

We begin this thesis with a definition of the classical/ordinary Dirichlet series.

Definition 1.1. A *classical/ordinary Dirichlet series* is the series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z} \tag{1.1}$$

where $(a_n)_{n=1}^{\infty}$ is a complex sequence and z is a complex number.

The ordinary Dirichlet series plays a prominent role in analytic number theory and has been a prime topic of study for many mathematicians ever since its inception [1].

One of the main problems of the current century includes the solving of the non-trivial zeroes of the *Riemann zeta function*, i.e. a specific case of the classical Dirichlet series when $a_n = 1$ for all n ,

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z},$$

the problem is also known as the *Riemann Hypothesis*: are the non-trivial zeroes of the Riemann zeta function on the line $\operatorname{Re}(z) = \frac{1}{2}$?

The convergence of the classical Dirichlet series is an interesting topic. We focus on the specific case where (a_n) and z are real instead. Suppose (1.1) is convergent at some $x_0 \in \mathbb{R}$. Then rewriting (1.1) as

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z} = \sum_{n=1}^{\infty} \frac{a_n}{n^{x_0}} \cdot \frac{1}{n^{z-x_0}},$$

we observe that the sequence

$$b_n := \frac{1}{n^{z-x_0}}$$

is monotonically decreasing and hence (1.1) is convergent for $z > x_0$ by Abel's test of convergence. Note that $z > x_0$ is the "right hand side" of the real number line. The question then is, can we obtain something similar in a more general case?

1.2 General Dirichlet series

We begin this section with the definition of the general Dirichlet series.

Definition 1.2. Let a_n and $z = \sigma + it$ be complex numbers and (λ_n) be a sequence of strictly increasing real positive numbers (the *frequencies* of the series) where $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ (denoted as $0 \leq (\lambda_n) \uparrow \infty$). Then the series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \tag{1.2}$$

is called a *Dirichlet series with real frequencies*.

A point to note is that (1.2) can be clearly seen as a more general case of (1.1), i.e. $\lambda_n = \log n$.

From this point onward, the term "Dirichlet series" shall be referred to as the general Dirichlet series. For Chapter 1 and 2, the proofs are deferred as the focus is mainly on the study of Rhy operators on the Hilbert space of Dirichlet series (Chapter 3). Most of the proofs can be found in [4, 12].

The following result describes the convergence of the Dirichlet series:

Proposition 1.3. *Suppose the series*

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

converges at $z = z_0$. Then it is uniformly convergent in the domain

$$\{z \in \mathbb{C} : |\arg(z - z_0)| \leq \theta < \frac{\pi}{2}\}.$$

The preceding result tells us that the convergence of the Dirichlet series is a half-plane, and it can be reformulated as

Proposition 1.4. *If the series*

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

converges at $z_0 = \sigma_0 + it$, then it is convergent for any z with $\sigma > \sigma_0$.

Similar to the radius of convergence in Taylor's series, we also have the abscissa of convergence of a Dirichlet series:

Definition 1.5. The *abscissa of convergence*, σ_c is defined as

$$\sigma_c = \inf\{\sigma \in \mathbb{R} : (1.2) \text{ converges for any } z \text{ such that } \operatorname{Re}(z) > \sigma\}.$$

Definition 1.6. The *half-plane of convergence* is defined as

$$\mathbb{C}_{\sigma_c} = \{z \in \mathbb{C} : \operatorname{Re}(z) > \sigma_c\}.$$

Remark 1.7. For the sake of consistency, we also consider $\mathbb{C}$ and $\emptyset$ as half-planes, i.e. $\mathbb{C}_{-\infty} = \mathbb{C}$ and $\mathbb{C}_{\infty} = \emptyset$.

The following series are examples of Dirichlet series with their abscissas of convergence.

Example 1.8. By choosing $\lambda_n = \log n$ and $a_n = \frac{1}{n!}$, the series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} = \sum_{n=1}^{\infty} \frac{1}{n! n^z}$$

is convergent for all values of z and hence $\sigma_c = -\infty$.

Example 1.9. By choosing $\lambda_n = \log n$ and $a_n = n!$, the series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} = \sum_{n=1}^{\infty} \frac{n!}{n^z}$$

is convergent for no values of z and hence $\sigma_c = \infty$.

Example 1.10. By choosing $\lambda_n = \log n$ and $a_n = 1$, the series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

is convergent for values of z with $\operatorname{Re}(z) > 1$. Hence $\sigma_c = 1$.

Example 1.11. The series

$$\sum_{n=1}^{\infty} \frac{1}{n} e^{-nz}$$

at $z = -\pi i$ is the alternating harmonic series which is convergent, while on the same line $\sigma = 0$, the series is divergent at $z = 0$. Hence, $\sigma_c = 0$.

Example 1.12. The series

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^2} \cdot \frac{1}{n^z}$$

has an abscissa of convergence $\sigma_c = 1$, while it is convergent on all points on $\sigma = 1$.

In the following two results, we provide two different formulae to calculate the abscissa of convergence.

Proposition 1.13. *If $\sum_{k=1}^{\infty} a_k$ diverges, then σ_c can be calculated via the following formula*

$$\sigma_c = \limsup_{n \rightarrow \infty} \frac{\log |A_n|}{\lambda_n},$$

where

$$A_n = \sum_{k=1}^n a_k.$$

Proposition 1.14. *If $\sum_{k=1}^{\infty} a_k$ is convergent, then σ_c can be calculated via the following formula*

$$\sigma_c = \limsup_{n \rightarrow \infty} \frac{\log |R_n|}{\lambda_{n+1}},$$

where

$$R_n = \sum_{k=n+1}^{\infty} a_k.$$

We also have a notion of absolute convergence and abscissas of uniform and absolute convergence of Dirichlet series.

Definition 1.15. A Dirichlet series is *absolutely convergent* if the series

$$\sum_{n=1}^{\infty} |a_n e^{-\lambda_n z}| = \sum_{n=1}^{\infty} |a_n| e^{-\lambda_n \sigma} \quad (1.3)$$

is convergent.

Definition 1.16. The *abscissa of absolute convergence*, σ_a is defined as

$$\sigma_a = \inf\{\sigma \in \mathbb{R} : (1.2) \text{ converges absolutely for any } z \text{ such that } \operatorname{Re}(z) > \sigma\}.$$

Definition 1.17. The *abscissa of uniform convergence*, σ_u is defined as

$$\sigma_u = \inf\{\sigma \in \mathbb{R} : (1.2) \text{ converges uniformly for any } z \text{ such that } \operatorname{Re}(z) > \sigma\}.$$

It is clear that absolute convergence implies uniform convergence by the Weierstrass M-test, and also uniform convergence implies convergence. Hence $\sigma_c \leq \sigma_u \leq \sigma_a$.

Chapter 2

Entire Dirichlet series

The aim of this chapter is to characterize some special Dirichlet series which has the abscissa of convergence $\sigma_c = -\infty$, i.e. converges everywhere on the complex plane/is an entire function. We begin this chapter with a result that gives an upper bound to the width of the *strip of conditional convergence*, the strip on the complex plane where the Dirichlet series is conditionally convergent. As mentioned in Chapter 1, the most of the results and proofs in this chapter can be found in [4] and [12].

2.1 Preliminary results

Proposition 2.1. *For the series (1.2), we have the inequality*

$$0 \leq \sigma_a - \sigma_c \leq \limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} =: L.$$

This quantity L is essential in characterizing the entire Dirichlet series, as we will see later.

We also have a lower bound for the abscissa of convergence (which is an important quantity to be seen in this chapter).

Proposition 2.2. *For the series (1.2), we have a lower bound for the abscissa of convergence*

$$\sigma_c \geq \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n}. \quad (2.1)$$

The following result, also known as the Valiron formula [12], is the most important result of this chapter.

Proposition 2.3 (Valiron formula, [12]). *Let L be defined as it was in Proposition 2.1. Suppose L is finite, then this chain of inequalities will always hold for (1.2).*

$$\limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} \leq \sigma_c \leq \sigma_u \leq \sigma_a \leq \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} + L.$$

The Valiron formula is the most important result in helping us characterize a part of the entire Dirichlet series, i.e

Proposition 2.4. *If L , as defined in Proposition 2.1, is finite, then the abscissa of convergence, abscissa of uniform convergence and the abscissa of absolute convergence of (1.2) are $\sigma_c = \sigma_u = \sigma_a = -\infty$, i.e. (1.2) is convergent everywhere on $\mathbb{C}$ if and only if*

$$\limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} = -\infty. \quad (2.2)$$

Example 2.5. Consider the series in Example 1.8, where $a_n = \frac{1}{n!}$ and $\lambda_n = \log n$,

$$\sum_{n=1}^{\infty} \frac{1}{n! n^z}.$$

Note that

$$L = \limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} = \lim_{n \rightarrow \infty} \frac{\log n}{\log n} = 1 < \infty.$$

Now,

$$\limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} = \lim_{n \rightarrow \infty} \frac{\log \frac{1}{n!}}{\log n} = -\infty.$$

So by Proposition 2.4, the series in Example 1.8 represents an entire Dirichlet series.

We also note characterizing entire functions is possible with $L < \infty$; on the other hand, it is also possible to have entire functions when $L = \infty$.

Example 2.6. Consider the series

$$\sum_{n=2}^{\infty} a_n e^{-\lambda_n z}$$

where $\lambda_n = \log \log n$, $a_n = \frac{1}{n!}$. We have

$$L = \limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} = \lim_{n \rightarrow \infty} \frac{\log n}{\log \log n} = \infty.$$

Note that

$$\sum_{n=2}^{\infty} a_n e^{-\lambda_n z} = \sum_{n=2}^{\infty} \frac{1}{n!} \cdot e^{-\log \log n z} = \sum_{n=2}^{\infty} \frac{1}{n!} \cdot \frac{1}{(\log n)^z}.$$

By applying the Ratio Test, we have

$$\sum_{n=2}^{\infty} a_n e^{-\lambda_n z}$$

converges absolutely for all z . That is to say that the series above is an entire function. So when $L = \infty$, it is still possible to have entire functions.

2.2 Characterization of entire Dirichlet series

We continue this chapter by recalling a classical result from complex analysis. This result can be found in any textbook for elementary complex analysis.

Proposition 2.7. *Suppose (f_n) is a sequence of functions, all analytic on an open set U . If (f_n) converges uniformly to f on every compact subset of U , then f is also analytic on U .*

The result above is followed by a result of uniqueness of the Dirichlet series, i.e. if the two Dirichlet series is absolutely convergent for an abscissa large enough and agree on a sequence tending to infinity, then the coefficients of the two series are the same.

Proposition 2.8 ([1]). *Suppose*

$$f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \quad g(z) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n z}$$

are absolutely convergent for all $\sigma > \sigma_a$. Define $(z_k) = (\sigma_k + it_k)$ to be a sequence such that $(\sigma_k) \rightarrow \infty$ as $k \rightarrow \infty$ and let $f(z_k) = g(z_k)$ for all k , then $a_n = b_n$ for all n .

We end of this chapter with a corollary on the unique characterization of the entire Dirichlet series.

Corollary 2.9. *The series (1.2), with the assumption that $L < \infty$, uniquely represents an entire function if and only if*

$$\limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} = -\infty.$$

Chapter 3

Rhaly operators on the Hilbert space of entire Dirichlet series

Let $0 \leq (\lambda_n) \uparrow \infty$ be a sequence of real numbers. Recall from Chapter 2 (Corollary 2.9) that if this sequence satisfies the condition

$$\limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} < \infty,$$

then the Dirichlet series $\sum_{n=1}^{\infty} c_n e^{-\lambda_n z}$ converges in $\mathbb{C}$ (that is, it uniquely defines an entire function) if and only if

$$\limsup_{n \rightarrow \infty} \frac{\log |c_n|}{\lambda_n} = -\infty.$$

We define a sequence space E as follows

$$E = \left\{ a = (a_n) : \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} = -\infty \right\}.$$

This sequence is associated with the inner product space

$$D = \left\{ f(z) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n z} : \limsup_{n \rightarrow \infty} \frac{\log |c_n|}{\lambda_n} = -\infty \right\}$$

under the inner product

$$\left\langle \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \sum_{n=1}^{\infty} b_n e^{-\lambda_n z} \right\rangle = \sum_{n=1}^{\infty} a_n \overline{b_n}.$$

However, D is not a Hilbert space under the norm generated by this inner product. The inner product can be modified so that we obtain a subspace of D which is a Hilbert space; for each sequence of real positive numbers $\beta = (\beta_n)$ (which we term as the *weights*), we associate it with the following sequence space

$$X_\beta = \left\{ a = (a_n) : \|a\|_{X_\beta} := \left(\sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 \right)^{1/2} < \infty \right\},$$

which is a Hilbert space [5] with the inner product

$$\langle a, b \rangle = \sum_{n=1}^{\infty} a_n \overline{b_n} \beta_n^2. \quad (3.1)$$

Based on the sequence of positive real numbers β , we define a function space of *weighted* entire Dirichlet series

$$H^2(E, \beta) := \left\{ f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \text{ entire} : \|f\|_{H^2(E, \beta)} := \|(a_n)\|_{X_\beta} < \infty \right\}.$$

There is a relationship between X_β and E , as described in the first elementary result below. First, we define the quantities

$$\beta_* := \liminf_{n \rightarrow \infty} \frac{\log \beta_n}{\lambda_n} \quad \text{and} \quad \beta^* = \limsup_{n \rightarrow \infty} \frac{\log \beta_n}{\lambda_n}.$$

Proposition 3.1 ([5]). *There are only three possibilities relating X_β and E :*

1. $X_\beta \subsetneq E \iff \beta_* = \infty$,
2. $E \subsetneq X_\beta \iff \beta^* < \infty$,
3. $E \setminus X_\beta \neq \emptyset$ and $X_\beta \setminus E \neq \emptyset \iff \beta_* < \beta^* = \infty$, which means that the spaces never coincide.

Note that the space $H^2(E, \beta)$ may not necessarily be a Hilbert space, but a result in [5] provides a criterion on β for which $H^2(E, \beta)$ is a Hilbert space.

Proposition 3.2. *Let $0 \leq (\lambda_n) \uparrow \infty$ be a sequence of real numbers. Then the space of entire Dirichlet series $H^2(E, \beta)$ is a Hilbert space if and only if the following condition holds*

$$\beta_* = \liminf_{n \rightarrow \infty} \frac{\log \beta_n}{\lambda_n} = \infty. \quad (3.2)$$

In this thesis, we adopt the convention that $\beta_* = \infty$, so that $H^2(E, \beta)$ is a Hilbert space. Under the convention that $\beta_* = \infty$, Result 3.1 asserts that $X_\beta \subsetneq E$, which means that $H^2(E, \beta) \subsetneq D$. Now, we see that the original definition of $H^2(E, \beta)$ can be reformulated as

$$H^2(E, \beta) := \left\{ f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} : (a_n) \in X_\beta \right\}. \quad (3.3)$$

3.1 Definition of the Rhaly operator R_a

Let $c = (c_n)$ be a complex sequence and let

$$s_n(c) := \sum_{k=1}^n c_k$$

be the n th partial sum of the first n entries of c . The Cesàro operator C was first introduced in [3] and it is defined to be

$$C(c) = \left(\frac{s_n}{n} \right)_{n=1}^{\infty}.$$

C is widely studied in various contexts in mathematics as it is related to the theory of Cesàro limits (see, e.g. [3]).

Let $a = (a_n)$ be a given complex sequence. Rhaly [8] generalized the Cesàro operator by defining a new operator

$$R_a = \begin{bmatrix} a_1 & 0 & 0 & 0 & \cdots \\ a_1 & a_2 & 0 & 0 & \cdots \\ a_1 & a_2 & a_3 & 0 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We note that the Cesàro operator is a specific case of Rhaly operators, namely

$$a_n = \frac{1}{n}, \quad n \in \mathbb{N}.$$

Rhaly operators acting on ℓ^p and other sequence spaces have been studied in various papers (see e.g. [7], [8] and the references therein). However, Rhaly

operators on function spaces have not been studied extensively; so far, only [11] addresses Rhaly operators on small weighted Hardy spaces.

We know from [11] that there are many properties of Rhaly operators on the space of analytic functions on the unit disk $\mathbb{D}$ and small weighted Hardy spaces like invariance, boundedness, etc. The aim of this thesis is to study the properties of Rhaly operators on $H^2(E, \beta)$.

Definition 3.3. For a Dirichlet series

$$f(z) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n z}$$

and a complex sequence $a = (a_n)$, the *Rhaly operator induced by a* is defined as

$$R_a(f)(z) = \sum_{n=1}^{\infty} \left(a_n \sum_{k=1}^n c_k \right) e^{-\lambda_n z}. \quad (3.4)$$

3.2 Preliminary definitions and results

In this section, we cover the required definitions and results required for the next section. These definitions, results as well as the proofs of the results can be found in the references [9, 10, 13].

Definition 3.4. Let X and Y be normed vector spaces. A *linear operator* $T : X \rightarrow Y$ is said to be *bounded* if there exists an $M > 0$ such that for all $x \in X$, we have $\|T(x)\| \leq M \|x\|$. The infimum of such M is the *operator norm* of T , denoted as $\|T\|_{op}$, or simply $\|T\|$.

Definition 3.5. Let V, W be Hilbert spaces. For an operator $T : V \rightarrow W$, the *adjoint* of T , denoted as T^* is the operator $T : W \rightarrow V$ such that

$$\langle T(v), w \rangle_W = \langle v, T^*(w) \rangle_V$$

for all $v \in V$ and $w \in W$.

For each $w \in W$, note that by defining the linear functional $l_w(v) = \langle T(v), w \rangle_W$, we can obtain a unique $u \in V$ such that

$$l_w(v) = \langle T(v), w \rangle_W = \langle v, u \rangle_V.$$

By defining $T^*(w)$ as u , the adjoint operator T^* can be defined by using the Riesz Representation Theorem for each $w \in W$.

Definition 3.6. Let H be a Hilbert space. A bounded linear operator $T : H \rightarrow H$ is called a *Hilbert-Schmidt* operator if for any orthonormal basis $\{e_j\}_{j \in J}$ of H ,

$$\|T\|_{HS} := \left(\sum_j \|T(e_j)\|^2 \right)^{1/2} < \infty.$$

$\|T\|_{HS}$ is known as the *Hilbert-Schmidt* norm.

Example 3.7. Let $(e_j)_{j=1}^\infty$ be the canonical basis for the Hilbert space of square summable sequences (see Example 3.7), i.e.

$$e_j = (0, \dots, 0, 1, 0, \dots),$$

where 1 is in the j th coordinate of the sequence.

Define an operator $T : l^2(\mathbb{R}) \rightarrow l^2(\mathbb{R})$ such that

$$T(x) = \sum_{n=1}^{\infty} \frac{1}{n} \langle x, e_n \rangle e_n,$$

where $\langle, \rangle$ denotes the standard dot product. T is clearly linear and for any $i \in \mathbb{N}$,

$$T(e_i) = \frac{1}{i} e_i$$

and hence

$$\|T\|_{HS} = \sum_{i=1}^{\infty} \|T(e_i)\|^2 = \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty.$$

Hence, T is Hilbert-Schmidt.

Example 3.8. On the other hand, consider another operator T_1 on $l^2(\mathbb{R})$ such that for all $x \in l^2(\mathbb{R})$,

$$T_1(x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \langle x, e_n \rangle e_n$$

where $(e_j)_{j=1}^\infty$ is the standard basis of $l^2(\mathbb{R})$ and $\langle, \rangle$ denotes the standard dot product as in Example 3.7. Using techniques similar to Example 3.7, we can see that T_1 is not Hilbert-Schmidt as

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is the divergent harmonic series.

Definition 3.9. Let H be a Banach space. A bounded linear operator $T : H \rightarrow H$ is said to be *compact* if T maps the unit ball of H to a relatively compact subset of H , i.e. a subset of H which has compact closure.

Example 3.10. A Hilbert-Schmidt operator is compact (see the next proposition).

Proposition 3.11. Let H be a Hilbert space and $T : H \rightarrow H$ be a Hilbert-Schmidt operator. Then T is compact.

It is also well-known that a Hilbert space is separable if and only if it has a countable orthonormal basis. Henceforth, we define the *Schatten p -class* operators on separable Hilbert spaces.

Definition 3.12. Suppose H is a separable Hilbert space. For $1 \leq p < \infty$, the *Schatten p -class* consists of all bounded linear operators $T : H \rightarrow H$ such that the Schatten p -norm $\|T\|_p$ is finite, i.e.

$$\|T\|_p := \left(\sum_{n=1}^{\infty} s_n^p(T) \right)^{\frac{1}{p}} < \infty.$$

Here, $s_n(T)$ refers to the n th eigenvalue of the Hermitian operator $|T| := \sqrt{T^*T}$.

Example 3.13. The Hilbert-Schmidt class of operators is a particular case of the Schatten p -class operators, precisely when $p = 2$:

Let $\{s_i(T)\}_{i=1}^{\infty}$ be the eigenvalues of $|T|$, then $\{s_i^2(T)\}_{i=1}^{\infty}$ are eigenvalues for T^*T . Let $\{e_i\}_{i=1}^{\infty}$ be the set of corresponding orthonormal eigenvectors of T^*T which forms an orthonormal basis of H . Then

$$\begin{aligned} \|T\|_{HS} &= \left(\sum_{j=1}^{\infty} \|T(e_j)\|^2 \right)^{1/2} = \left(\sum_{j=1}^{\infty} \langle T(e_j), T(e_j) \rangle \right)^{1/2} \\ &= \left(\sum_{j=1}^{\infty} \langle T^*T(e_j), e_j \rangle \right)^{1/2} = \left(\sum_{j=1}^{\infty} s_j^2(T) \right)^{\frac{1}{2}} = \|T\|_2. \end{aligned}$$

There is an important property of the Schatten p -class of operators, the decreasing nature of its norm as exemplified by the result below which we will not prove as it is a standard textbook exercise.

Proposition 3.14. *For any sequence of real numbers $x = (x_n)_{n=1}^{\infty}$ and p, q are real numbers such that $1 \leq q < p < \infty$,*

$$\|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |x_n|^q \right)^{\frac{1}{q}} =: \|x\|_q.$$

We now have all we need for the next section, where we study properties of the Rhaly operators R_a on the Hilbert space of entire Dirichlet series like invariance, boundedness, compactness and closed range.

3.3 Properties of the Rhaly operator R_a

The first question we should ask is: under what conditions does $R_a(f)(z)$ converge on the whole complex plane $\mathbb{C}$ (i.e. represents an entire function)?

Theorem 3.15. *$R_a(f)$ is invariant on D if and only if $a \in E$.*

Proof. First, suppose that $R_a(f)$ is invariant on D . Let

$$f(z) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n z}$$

such that $c_1 = 1$ and $c_k = 0$ for all $k \geq 2$, i.e. $f(z) = c_1 e^{-\lambda_1 z}$. Clearly f is in D . We now note that for every $n \geq 1$,

$$a_n \sum_{k=1}^n c_k = a_n,$$

hence

$$R_a(f)(z) = \sum_{n=1}^{\infty} \left(a_n \sum_{k=1}^n c_k \right) e^{-\lambda_n z} = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

is in D by invariance assumption. Hence $a \in E$.

Conversely, suppose that $a \in E$. Let

$$f(z) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n z} \in D.$$

Define $b_n = a_n \sum_{k=1}^n c_k$. It suffices to show that

$$\limsup_{n \rightarrow \infty} \frac{\log |b_n|}{\lambda_n} = -\infty.$$

Since $f \in D$, this implies that $f(0) = \sum_{n=1}^{\infty} c_n$ is convergent.

Since $\sum_{n=1}^{\infty} c_n$ is convergent, let

$$\gamma := \sum_{n=1}^{\infty} c_n.$$

Now, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left| \sum_{k=1}^n c_k - \gamma \right| < 1.$$

Then for all $n \geq N$,

$$\begin{aligned} \frac{\log |b_n|}{\lambda_n} &= \frac{\log |a_n|}{\lambda_n} + \frac{\log \left| \sum_{k=1}^n c_k - \gamma + \gamma \right|}{\lambda_n} \\ &\leq \frac{\log |a_n|}{\lambda_n} + \frac{\log(1 + |\gamma|)}{\lambda_n}. \end{aligned}$$

Clearly, the second term of the right hand side of the inequality goes to 0, hence we also have the required result by taking the limit superior on both sides of the inequality. Therefore,

$$\limsup_{n \rightarrow \infty} \frac{\log |b_n|}{\lambda_n} = -\infty,$$

and hence $R_a(f) \in D$, proving invariance of the Rhaly operator R_a . $\square$

As mentioned earlier, we will focus our study of Rhaly operators on the Hilbert space $H^2(E, \beta)$. First, the following theorem encapsulates invariance of R_a on $H^2(E, \beta)$.

Theorem 3.16. *The Rhaly operator R_a is invariant on $H^2(E, \beta)$ if and only if $a \in X_\beta$.*

Proof. Suppose R_a is invariant on $H^2(E, \beta)$. Then $R_a(f)(z) \in H^2(E, \beta)$ for any $f \in H^2(E, \beta)$.

Define

$$f(z) = \sum_{k=1}^{\infty} c_k e^{-\lambda_k z}$$

where $c_1 = 1$ and $c_k = 0$ for all $k \geq 2$. Clearly f is in $H^2(E, \beta)$, since

$$\|(c_n)\|^2 = \sum |c_n|^2 \beta_n^2 = \beta_1^2 < \infty.$$

Then for this particular f ,

$$R_a(f)(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \in H^2(E, \beta),$$

so $a \in X_\beta$.

Conversely, suppose $a \in X_\beta$. Let

$$f(z) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n z} \in H^2(E, \beta).$$

Clearly $f(0) = \sum_{n=1}^{\infty} c_n$ is convergent, since f is in $H^2(E, \beta)$. Hence for any $n \in \mathbb{N}$, the sequence of partial sums

$$s_n := \sum_{k=1}^n c_k$$

are bounded. Therefore, there exists an $M > 0$ such that for all $n \in \mathbb{N}$, we have

$$|s_n|^2 = \left| \sum_{k=1}^n c_k \right|^2 \leq M.$$

As a result,

$$\|R_a(f)\|^2 = \sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 \left| \sum_{k=1}^n c_k \right|^2 \leq M \cdot \sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 < \infty.$$

Hence $R_a(f)$ is invariant on $H^2(E, \beta)$. □

Theorem 3.17. R_a is bounded on $H^2(E, \beta)$ if and only if $a \in X_\beta$.

Proof. Suppose $a \in X_\beta$. We can see from the proof of Theorem 3.16 that R_a is bounded on $H^2(E, \beta)$.

Now suppose that R_a is bounded on $H^2(E, \beta)$. Then there exists an $M > 0$ such that $\|R_a(f)\| \leq M \|f\|$ for all $f \in H^2(E, \beta)$. This implies that R_a is invariant, and hence by Theorem 3.16, $a \in X_\beta$. □

Now, let us suppose that the sequence β satisfies the condition

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n^2} < \infty. \tag{3.5}$$

Under this condition, β_n is assumed to be “large”. Since we also require

$$\sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 < \infty,$$

this loosely means that $|a_n|$ has to be small in order to compensate for the sum being finite. To add on to the above, under this condition, we are only looking at a relatively small subset of the Hilbert space of entire Dirichlet series, which we are going to term as the *small weighted* Hilbert space of entire Dirichlet series. From now on, we will refer the *small weighted* Hilbert space of entire Dirichlet series as $H^2(E, \beta)$ satisfying condition (3.5).

Also, $H^2(E, \beta)$ has a countable orthonormal basis (i.e. $\{\frac{1}{\beta_k} e^{-\lambda_k z}\}_{k=1}^{\infty}$), it follows that $H^2(E, \beta)$ is separable. Hence it makes sense to define the Schatten p -class inclusions for operators on $H^2(E, \beta)$.

Theorem 3.18. *The following statements for the operator R_a on the small weighted Hilbert space of entire Dirichlet series are equivalent:*

1. R_a belongs to Schatten p -class for all $p \geq 2$,
2. R_a is Hilbert-Schmidt,
3. R_a is compact,
4. $a \in X_{\beta}$.

Proof. (1) $\iff$ (2) The Schatten 2-class is the Hilbert-Schmidt class of operators (see Example 3.13) and also the Schatten norms are decreasing in nature (see Proposition 3.14); i.e if T is a compact linear operator on a Hilbert space H and $1 \leq p \leq q \leq \infty$, then $\|T\|_p \geq \|T\|_q$.

(2) $\implies$ (3) Hilbert-Schmidt operators are compact operators. (Proposition 3.11)

(3) $\implies$ (4) Compact operators are assumed to be bounded. This result then follows from Theorem 3.17.

(4) $\implies$ (2) Suppose $a \in X_{\beta}$. Consider the orthonormal basis $\{e_k\}_{k=1}^{\infty}$ for $H^2(E, \beta)$ where $e_k = \beta_k^{-1} e^{-\lambda_k z}$. It is easy to see that for each $k \geq 1$,

$$e_k = \sum_{j=1}^{\infty} c_j e^{-\lambda_j z}$$

where $c_k = \beta_k^{-1}$ and $c_j = 0$ whenever $j \neq k$. Therefore, for each k ,

$$R_a(e_k)(z) = \sum_{n=k}^{\infty} a_n \frac{1}{\beta_k} e^{-\lambda_n z}.$$

Now,

$$\|R_a(e_k)\|^2 = \sum_{n=k}^{\infty} |a_n|^2 \frac{1}{\beta_k^2} \beta_n^2 = \frac{1}{\beta_k^2} \sum_{n=k}^{\infty} |a_n|^2 \beta_n^2.$$

which gives

$$\|R_a\|_{HS}^2 = \sum_{k=1}^{\infty} \|R_a(e_k)\|^2 = \underbrace{\sum_{k=1}^{\infty} \frac{1}{\beta_k^2}}_{< \infty, \text{ due to (3.5)}} \cdot \sum_{n=k}^{\infty} |a_n|^2 \beta_n^2 < \infty.$$

Hence R_a is Hilbert-Schmidt. □

We summarize the above results in the following theorem.

Theorem 3.19. *Let R_a be the Rhaly operator induced by a complex sequence $a = (a_n)$. Then the following assertions are equivalent:*

1. $a \in X_{\beta}$,
2. R_a is invariant on $H^2(E, \beta)$,
3. R_a is bounded on $H^2(E, \beta)$,

Moreover, for the small weighted Hilbert space of entire Dirichlet series, (1)-(3) are equivalent to the following statements

4. R_a is compact,
5. R_a is in the Schatten p -class for $p \geq 2$.

3.4 Closed range characterization of R_a

In this section, we characterize the closed range property of bounded Rhaly operators. Unless otherwise stated, the Rhaly operator's domain in this section is assumed to be $H^2(E, \beta)$. We begin this section with two definitions regarding *finite* and *strictly nonzero* sequences.

Definition 3.20. Let $a = (a_n)_{n=1}^{\infty}$ be a complex sequence. We say that a is an *finite* sequence if there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n = 0$.

In this definition, we adopt the convention that the zero sequence is a finite sequence.

Definition 3.21. Let $a = (a_n)_{n=1}^{\infty}$ be a complex sequence. We say that a is a *strictly nonzero* sequence if $a_n \neq 0$ for all $n \in \mathbb{N}$.

Suppose we have a sequence $a = (a_n)_{n=1}^{\infty}$ that is not strictly nonzero. Then there exists an $n_0 \in \mathbb{N}$ such that $a_{n_0} = 0$. Looking at the subsequent indices of the sequence after n_0 , we have that either all the subsequent terms after a_{n_0} are zero, that is, for all $n > n_0$, $a_n = 0$, i.e. a is a finite sequence, or that there exists a subsequent term after a_{n_0} that is nonzero; there exists an $n_1 \in \mathbb{N}$ such that $n_1 \geq n_0$ and $a_{n_1+1} \neq 0$, which in this case, we say that a has a *finite gap* of zeroes. We also note that it is also possible for sequences that are *not* strictly nonzero to have the two properties mentioned above.

Using strict nonzero-ness of the sequence a , we can characterize the injectivity of R_a as we will see in the following theorems.

Theorem 3.22. *Suppose R_a is bounded. If a is a strictly nonzero sequence, then R_a is injective.*

Proof. Assume that a is a strictly nonzero sequence and let

$$f(z) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n z}, \quad g(z) = \sum_{n=1}^{\infty} d_n e^{-\lambda_n z}$$

be in $H^2(E, \beta)$ such that $R_a(f) = R_a(g)$. Since R_a is bounded, it follows from Theorem 3.19 that R_a is invariant on $H^2(E, \beta)$. Therefore $R_a(f) \in H^2(E, \beta)$ and by uniqueness of entire Dirichlet series, we have

$$a_n \sum_{k=1}^n c_k = a_n \sum_{k=1}^n d_k$$

for all $n \in \mathbb{N}$. Since $a_n \neq 0$ for all $n \in \mathbb{N}$, this yields

$$\sum_{k=1}^n c_k = \sum_{k=1}^n d_k$$

for all $n \in \mathbb{N}$. By induction, $c_k = d_k$ for all k and hence $f = g$. Therefore R_a is injective. $\square$

It also turns out that strict nonzero-ness of a is a necessary condition for R_a to be injective, as exemplified by the following theorem.

Theorem 3.23. *If R_a is injective, then a is a strictly nonzero sequence.*

Proof. From the comments directly after Definition 3.21, we show that if a is either a finite sequence or has a finite gap of zeroes, then R_a is not injective, i.e. we can find two Dirichlet series in $H^2(E, \beta)$

$$f(z) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n z}, \quad g(z) = \sum_{n=1}^{\infty} d_n e^{-\lambda_n z}$$

such that $R_a(f) = R_a(g)$ but $f \neq g$.

Assume first that a is not a strictly nonzero sequence, that is, there exists an $n_0 \in \mathbb{N}$ such that $a_{n_0} = 0$. Then at least one of these two cases will occur:

Case 1: For all $n \geq n_0$, $a_n = 0$.

In this case, consider two Dirichlet series

$$f(z) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n z}, \quad g(z) = \sum_{n=1}^{\infty} d_n e^{-\lambda_n z}$$

such that $c_k = d_k = 0$ for $k \neq n_0, n_0 + 1$, and $c_{n_0} = d_{n_0+1} = 0$, $c_{n_0+1} = d_{n_0} = 1$. It is clear that f, g are both in $H^2(E, \beta)$ and $f \neq g$. However, for all $n \in \mathbb{N}$, we have

$$a_n \sum_{k=1}^n c_k = 0 = a_n \sum_{k=1}^n d_k,$$

hence $R_a(f) = R_a(g)$. This shows that R_a is not injective.

Case 2: $\exists n_1 \in \mathbb{N}$, $n_1 \geq n_0$ such that $a_k = 0$ for all $n_0 \leq k \leq n_1$ and $a_{n_1+1} \neq 0$.

In this case, we also consider two Dirichlet series

$$f(z) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n z}, \quad g \equiv 0$$

where $c_{n_0} = 1$, $c_{n_1+1} = -1$ and $c_k = 0$ otherwise. Then it is also clear that $f, g \in H^2(E, \beta)$ and $f \neq g$. Similarly to Case 1, $R_a(f) = R_a(g) = 0$, and hence R_a is not injective. $\square$

We note that in Theorem 3.23, boundedness of R_a is not needed as the range of R_a is not important; we only need to show that R_a maps two different functions to the zero function, which is obviously in the range of R_a since R_a is linear.

Summarizing Theorem 3.22 and 3.23, we can characterize injectivity for bounded Rhaly operators in the following theorem.

Corollary 3.24. *Suppose R_a is bounded. Then R_a is injective if and only if a is a strictly nonzero sequence.*

We now begin to characterize the closed range property of Rhaly operators.

Theorem 3.25. *If a is a finite sequence, then R_a has a closed range, i.e.*

$$R_a(H^2(E, \beta)) = \overline{R_a(H^2(E, \beta))}.$$

Proof. Suppose a is a finite sequence. If a is the zero sequence, the result follows trivially. Otherwise, there exists an $N \in \mathbb{N}$ such that for all $n > N$, $a_n = 0$ and $a_k \neq 0$ for some $1 \leq k \leq N$. We define the set A as the set of indices of a_n such that $a_n \neq 0$, i.e.

$$A := \{n \in \mathbb{N} : a_n \neq 0\}.$$

Since a is a finite sequence and is not the zero sequence, A is a nonempty finite set. Hence, we write A as

$$A = \{n_1, n_2, \dots, n_i\}$$

for some natural number $1 \leq i \leq N$. Note that each $n_j \in \mathbb{N}$ and we will also assume these numbers are in increasing order.

In order to show that R_a has a closed range, let $(y_m)_{m=1}^\infty$ be a sequence in $R_a(H^2(E, \beta))$ such that $y_m \rightarrow y$ in norm. Since $y_m \in R_a(H^2(E, \beta))$, there exists $x_m \in H^2(E, \beta)$ such that $y_m = R_a(x_m)$ for each m . Our aim is to show that $y \in R_a(H^2(E, \beta))$.

Since the sequences a and λ_n are already determined, the sequence terms y_m are only determined by the coefficients of the Dirichlet series x_m , i.e.

$$y_m = \sum_{n=1}^N \left(a_n \sum_{k=1}^n c_{k,m} \right) e^{-\lambda_n z}$$

where $x_m = \sum_{k=1}^{\infty} c_{k,m} e^{-\lambda_k z} \in H^2(E, \beta)$. Since $y_m \rightarrow y$ in norm, y_m is Cauchy in $R_a(H^2(E, \beta))$. Let $\varepsilon > 0$ be given. Then there exists a $K \in \mathbb{N}$ such that for any $a, b \geq K$,

$$\begin{aligned} \|y_a - y_b\| &= \sum_{n=1}^N |a_n|^2 \left| \sum_{k=1}^n (c_{k,a} - c_{k,b}) \right|^2 \beta_n^2 \\ &= \sum_{n \in A} |a_n|^2 \left| \sum_{k=1}^n (c_{k,a} - c_{k,b}) \right|^2 \beta_n^2 < \varepsilon. \end{aligned}$$

Claim: For all natural numbers $1 \leq r \leq i$,

$$\left(\sum_{k=n_{r-1}+1}^{n_r} c_{k,m} \right)_{m=1}^{\infty}$$

is Cauchy in $\mathbb{C}$. (Here we define $n_0 = 0$.) We do this by strong induction. When $r = 1$, we have

$$|a_{n_1}|^2 \left| \sum_{k=1}^{n_1} (c_{k,a} - c_{k,b}) \right|^2 \beta_{n_1}^2 \leq \sum_{n \in A} |a_n|^2 \left| \sum_{k=1}^n (c_{k,a} - c_{k,b}) \right|^2 \beta_n^2 < \varepsilon.$$

Since $a_{n_1}, \beta_{n_1} \neq 0$, the claim is proven for $r = 1$. Suppose for some $1 \leq j \leq i-1$, the claim is true for $r = 1, \dots, j$. As $a_{n_{j+1}}, \beta_{n_{j+1}} \neq 0$, note that

$$|a_{n_{j+1}}|^2 \left| \sum_{k=1}^{n_{j+1}} (c_{k,a} - c_{k,b}) \right|^2 \beta_{n_{j+1}}^2 \leq \sum_{n \in A} |a_n|^2 \left| \sum_{k=1}^n (c_{k,a} - c_{k,b}) \right|^2 \beta_n^2 < \varepsilon,$$

so that $\left(\sum_{k=1}^{n_{j+1}} c_{k,m} \right)_{m=1}^{\infty}$ is Cauchy in $\mathbb{C}$. Then

$$\begin{aligned} \left| \sum_{k=n_{j-1}+1}^{n_{j+1}} c_{k,m} \right| &= \left| \sum_{k=1}^{n_{j+1}} c_{k,m} - \sum_{k=1}^{n_1} c_{k,m} - \dots - \sum_{k=n_{j-1}+1}^{n_j} c_{k,m} \right| \\ &\leq \left| \sum_{k=1}^{n_{j+1}} c_{k,m} \right| + \left| \sum_{k=1}^{n_1} c_{k,m} \right| + \dots + \left| \sum_{k=n_{j-1}+1}^{n_j} c_{k,m} \right|. \end{aligned}$$

By the induction hypothesis and $\left(\sum_{k=1}^{n_{j+1}} c_{k,m} \right)_{m=1}^{\infty}$ is Cauchy in $\mathbb{C}$, we can see that the claim is proven for $r = j + 1$. Hence the claim is true.

Since $\mathbb{C}$ is complete, for each $1 \leq r \leq i$, there exists $h_{n_r} \in \mathbb{C}$ such that

$$\sum_{k=n_{r-1}+1}^{n_r} c_{k,m} \rightarrow h_{n_r}$$

as $m \rightarrow \infty$. By the earlier claim, since there are finitely many such sequences

$$\left(\sum_{k=n_{r-1}+1}^{n_r} c_{k,m} \right)_{m=1}^{\infty},$$

we may choose a $K_1 \in \mathbb{N}$ such that for any $1 \leq r \leq i$ and $m \geq K_1$,

$$\left| \sum_{k=n_{r-1}+1}^{n_r} c_{k,m} - h_{n_r} \right| < \frac{\varepsilon^{1/2}}{\sqrt{2} \cdot N^{3/2} \cdot \max_{1 \leq r \leq i} \{|a_{n_r}|\} \cdot \max_{1 \leq i \leq N} \{\beta_{n_r}\}}.$$

Define a complex sequence $(d_k)_{k=1}^{\infty}$ such that

$$d_k = \begin{cases} 0, & \text{if } k \notin A, k \leq N \\ h_{n_r}, & \text{if } k = n_r \text{ for some } r \in \{1, \dots, i\} \\ c_{k,1}, & \text{if } k > N. \end{cases}$$

Based on this sequence d_k , define a new function

$$f(z) = \sum_{n=1}^N \left(a_n \sum_{k=1}^n d_k \right) e^{-\lambda_n z}.$$

We can see that $f \in R_a(H^2(E, \beta))$ as

$$\begin{aligned} \sum_{k=1}^{\infty} |d_k|^2 \beta_k^2 &= \sum_{k=1}^N |d_k|^2 \beta_k^2 + \sum_{k=N+1}^{\infty} |d_k|^2 \beta_k^2 \\ &= \sum_{r=1}^i |h_{n_r}|^2 \beta_k^2 + \sum_{k=N+1}^{\infty} |c_{k,1}|^2 \beta_k^2 < \infty, \end{aligned}$$

as $x_1 = \sum_{k=1}^{\infty} c_{k,1} e^{-\lambda_k z}$ is in $H^2(E, \beta)$.

Note that if $m \geq K_1$, and any natural number $1 \leq j \leq i$,

$$\begin{aligned}
\left| \sum_{k=1}^{n_j} c_{k,m} - \sum_{k=1}^{n_j} d_k \right| &= \left| \sum_{k=1}^{n_j} c_{k,m} - \sum_{r=1}^i h_{n_r} \right| \\
&= \left| \sum_{r=1}^j \left(\sum_{k=n_{r-1}+1}^{n_r} c_{k,m} - h_{n_r} \right) \right| \\
&\leq \sum_{r=1}^j \left| \sum_{k=n_{r-1}+1}^{n_r} c_{k,m} - h_{n_r} \right| \\
&< \sum_{r=1}^j \frac{\varepsilon^{1/2}}{\sqrt{2} \cdot N^{3/2} \cdot \max_{1 \leq r \leq i} \{|a_{n_r}|\} \cdot \max_{1 \leq r \leq i} \{\beta_{n_r}\}} \\
&\leq \frac{\varepsilon^{1/2}}{\sqrt{2} \cdot N^{1/2} \cdot \max_{1 \leq r \leq i} \{|a_{n_r}|\} \cdot \max_{1 \leq r \leq i} \{\beta_{n_r}\}}.
\end{aligned}$$

We now claim that $y_m \rightarrow f$ in norm. For $m \geq K_1$, we have

$$\begin{aligned}
\|y_m - f\| &= \left\| \sum_{n=1}^N \left(a_n \sum_{k=1}^n c_{k,m} \right) e^{-\lambda_n z} - \sum_{n=1}^N \left(a_n \sum_{k=1}^n d_k \right) e^{-\lambda_n z} \right\| \\
&= \sum_{r=1}^i |a_{n_r}|^2 \left| \sum_{k=1}^{n_r} c_{k,m} - \sum_{k=1}^{n_r} d_k \right|^2 \beta_{n_r}^2 \\
&< \frac{\varepsilon}{2} \cdot \sum_{r=1}^i \frac{|a_{n_r}|^2}{\max_{1 \leq r \leq i} \{|a_{n_r}|\}^2} \cdot \frac{\beta_{n_r}^2}{\max_{1 \leq r \leq i} \{\beta_{n_r}\}^2} \frac{1}{N} \leq \frac{\varepsilon}{2}.
\end{aligned}$$

Now, since $y_m \rightarrow y$ in norm, there exists $K_2 \in \mathbb{N}$ such that for all $m \geq K_2$,

$$\|y_m - y\| < \frac{\varepsilon}{2}.$$

Choosing $M = \max\{K_1, K_2\}$, for all $m \geq M$, we have

$$\|y - f\| \leq \|y - y_m\| + \|y_m - f\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and this implies that $y \equiv f \in R_a(H^2(E, \beta))$ and the proof is complete. $\square$

Under the additional assumptions that R_a is bounded on the small weighted Hilbert space of entire Dirichlet series, we can obtain a converse statement of Theorem 3.25. Compactness is needed in this case, which in turn is true if the domain of R_a is the small weighted Hilbert space of entire Dirichlet series (see Theorem 3.19).

Theorem 3.26. *Suppose R_a is bounded on the small weighted Hilbert space of entire Dirichlet series. If R_a has a closed range, then a is a finite sequence.*

Proof. Suppose R_a has a closed range. Since R_a is bounded, Theorem 3.19 tells us that R_a is a compact on the small weighted Hilbert space of entire Dirichlet series and hence it must be finite-dimensional (see [10, Theorem 4.18b]). Assume on the contrary that a is not a finite sequence. Choosing an orthonormal basis $\{e_k\}$ as the orthonormal basis in Theorem 3.18, it follows that

$$\dim(R_a(H^2(E, \beta))) \geq \dim(\text{span}\{R_a(e_k) : k \in \mathbb{N}\}) = \infty,$$

which is a contradiction. Therefore a is a finite sequence. $\square$

Remark 3.27. Let X, Y be Banach spaces. There is actually a well-known approach to show that a bounded operator $T : X \rightarrow Y$ has a closed range; an injective operator T has a closed range if and only if it is *bounded below*, that is, there exists an $M > 0$ such that for all $x \in X$, $M \|x\| \leq \|T(x)\|$. However, notice that we are unable to use this theorem to show that R_a has a closed range in Theorem 3.25. The reason is that if a is a finite sequence, then by Theorem 3.24, R_a is not injective.

Summarizing Theorems 3.25 and 3.26, we obtain a characterization for the closed range property of bounded Rhaly operators.

Corollary 3.28. *Suppose R_a is bounded on the small weighted Hilbert space of entire Dirichlet series. Then R_a has a closed range if and only if a is a finite sequence.*

Chapter 4

Conclusion and Future Work

In this thesis, foundations were laid for Rhaly operators on the Hilbert space of entire Dirichlet series and certain important properties like invariance, boundedness were shown. Compactness, Hilbert-Schmidtness and Schatten p -class inclusions were proved on a smaller subset of the Hilbert space of entire functions, namely when the condition (3.5) is true.

We would also like to point out that the results in Theorem 3.19 are analogous to that of the results in [11], except the fact that the invariance and boundedness properties of R_a are independent of condition (3.5), which in [11], the boundedness and invariance properties are dependent on the condition (3.5). In other words, the boundedness and invariance properties hold for not just the small weighted Hilbert space of entire Dirichlet series, but also the more general space $H^2(E, \beta)$.

That being said, there are possibilities yet to be explored; namely

- In Theorem 3.19, the compactness property is proven using Hilbert-Schmidtness of the operator R_a , which was in turn dependent on condition (3.5). However, not all compact operators are Hilbert-Schmidt, hence compactness of R_a may be independent of (3.5); is it true that R_a is compact if (3.5) is not satisfied, i.e.

$$\sum_{k=1}^{\infty} \frac{1}{\beta_k^2} = +\infty?$$

- Other properties of R_a can also be investigated, such as surjectivity, compact difference, cyclicity and perturbation, etc.
- What is the numerical range of R_a ? (see [6, Section 2.3])

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