Two kinds of phase transition in photonic systems with application to optical isolation

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TWO KINDS OF PHASE TRANSITION IN PHOTONIC SYSTEMS WITH APPLICATION TO OPTICAL ISOLATION

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SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

2018

TWO KINDS OF PHASE TRANSITION IN PHOTONIC SYSTEMS WITH APPLICATION TO OPTICAL ISOLATION

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School of Physical and Mathematical Sciences

A thesis submitted to the Nanyang Technological University
in partial fulfillment of the requirement for the degree of
Doctor of Philosophy

2018

For my family and friends

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research done by me except where otherwise stated in this thesis. The thesis work has not been submitted for a degree or professional qualification to any other university or institution. I declare that this thesis is written by myself and is free of plagiarism and of sufficient grammatical clarity to be examined. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

2019-04-04

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Date

Chong Yidong

Authorship Attribution Statement

This thesis contains material from 3 papers published in the following peer-reviewed journals in which I am listed as an author.

Chapter 2 is published as **Xin Zhou, and Yidong Chong. “PT symmetry breaking and nonlinear optical isolation in coupled microcavities.” *Optics Express* 24.7 (2016): 6916-6930.**

The contributions of the co-authors are as follows:

- Yidong Chong conceived the idea.
- I performed the theoretical investigations and simulations, supervised by Yidong Chong.
- I prepared the manuscript drafts. The manuscript was revised by Yidong Chong.

Chapter 3 is published as **Xin Zhou, You Wang, Daniel Leykam, and Yidong Chong. “Optical isolation with nonlinear topological photonics.” *New Journal of Physics* 19.9 (2017): 095002.**

The contributions of the co-authors are as follows:

- Yidong Chong conceived the idea.
- I performed the theoretical investigations and simulations in 1D SSH model and 2D Haldane model sections. You Wang performed the theoretical investigations and simulations in coupled ring resonator section. Daniel Leykam and Yidong Chong supervised the project.
- I prepared the manuscript drafts for 1D SSH model and 2D Haldane model sections. You Wang prepared the manuscript drafts for coupled ring resonator section. The manuscript was revised by Daniel Leykam and Yidong Chong.

Chapter 4 is published as **Xin Zhou, Daniel Leykam, Udvash Chattopadhyay, Alexander Khanikaev, and Yidong Chong. “Realization of a magneto-optical near-zero index medium by an unpaired Dirac point .” *Physical Review B* 98.20 (2018): 205115.**

The contributions of the co-authors are as follows:

- Yidong Chong conceived the idea.

- I and Daniel Leykam performed the main theoretical investigations. I performed the Comsol simulations. Yidong Chong supervised the project.
- I prepared the manuscript drafts. The manuscript was revised by Udvas Chattopadhyay, Alexander Khanikaev, Daniel Leykam and Yidong Chong.

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Abstract

This thesis concerns the study of two kinds of phase transitions, parity-time(PT) symmetric phase transitions and topological phase transitions, using photonic systems.

In the first part of the thesis, we perform a theoretical study of the nonlinear dynamics of a nonlinear optical isolator device based on coupled microcavities with gain and loss. This reveals a correspondence between the boundary of asymptotic stability in the nonlinear regime, where gain saturation is present, and the PT -breaking transition in the underlying linear system. We establish a correspondence between the onset of optical isolation and the PT phase transition of the linear system.

In the second part of the thesis, we study a new class of nonlinear optical isolator based on self-induced topological phase transitions. We show that topological phase transitions can be used to help design nonlinear photonic structures exhibiting power thresholds and discontinuities in their transmittance. This provides a novel route to devising nonlinear optical isolators. We study three representative designs: (i) a waveguide array implementing a nonlinear 1D Su-Schrieffer-Heeger model, (ii) a waveguide array implementing a nonlinear 2D Haldane model, and (iii) a 2D lattice of coupled-ring waveguides. In the first two cases, we find a correspondence between the topological transition of the underlying linear lattice and the power threshold of the transmittance, and show that the transmission behavior is attributable to the emergence of a self-induced topological soliton. In the third case, we show that the topological transition produces a discontinuity in the transmittance curve, which can be exploited to achieve sharp jumps in the power-dependent isolation ratio.

In the third part, we study the realization of an unpaired Dirac cone at the center of the first Brillouin zone, using a gyromagnetic photonic crystal with broken square sub-lattice symmetry and broken time reversal symmetry. The behavior of the Dirac modes can be described by a gyromagnetic effective medium model with near-zero refractive index. When two domains are subjected to opposite magnetic biases, there exist unidirectional edge states along the domain wall. This establishes a novel link between topological edge states and the surface waves of homogenous magneto-optical media.

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Chapter 1

Introduction

1.1 Motivation and overview

States of matter are divided into different phases and the term “phase transition” refers to the changes of states between these different phases[1]. Common examples of phase transition are the melting of snow under the sun and the formation of fog. These so-called classical phase transitions are controlled by parameters like temperature, pressure and magnetic field strength.

In recent years, two unusual kinds of phase transition, parity-time(PT) phase transition[2–6] and topological phase transition[7–9], have drawn lots of attention among physicists. PT phase transitions refer to the changes of energy spectrum from complex to totally real in non-Hermitian systems(which usually do not possess real spectra) when there is a symmetry constraint known as “ PT symmetry”[2]. Topological phase transitions are associated with conceptually new phases of matter, the topological insulators[10, 11]. These transitions happen when a system changes from, e.g., a conventional insulator to a topological insulator. A topological insulator is different from a conventional insulator in the sense that it is insulating in the bulk, but electrons can move in the surface unidirectionally without backscattering even in

the presence of impurities.

In photonics research field, much attention has been focused on realizing phenomena originally predicted in condensed matter or particle physics, but in photonic settings[12–17]. One reason for this is that Maxwell’s equations, under certain conditions, take a form similar to Schrödinger equation[18, 19]. Some of these quantum phenomena require the ability to fine tune the system parameters, and it is easier to set parameters and fabricate samples in photonic systems than in electronic systems in atomic media. Due to their exotic behavior, the realization of parity-time symmetry[17, 20, 21] and topological states[22, 23] in photonics has been two of the most actively-researched areas; devices have been proposed based on different principles, and have been shown to have interesting and possibly useful behaviors. In parity-time symmetric optics, for example, researchers have proposed parity-time symmetric lasers[24, 25], parity-time symmetric optical isolators[26, 27], parity-time symmetric laser absorbers[28]. In topological photonics, researchers have proposed topological waveguides[29], topological lasers[30–33], passive topological mode selection[33] and topological optical isolators [34–36].

Among all of the applications in the field of optics and photonics, this thesis focuses on the realization of optical isolators[37], which have many potential applications in optical communications and computing. Optical isolators are photonic analogues of diodes, which means that they allow light to propagate in one direction and completely block the transmission of light in the opposite direction. Due to the dramatic behavior change associated with parity-time symmetric phase transitions and topological phase transitions, this thesis investigates the possibility of designing optical isolators based upon these phase transitions[26, 27, 34–36]. We will seek to explain the underlying reasons why parity-time symmetric phase transitions and topological phase transitions can help realize efficient optical isolation.

In the rest of this Chapter, we will introduce the photonic platforms referred

to in the rest of this thesis: microring resonators, photonic crystals and photonic lattices (waveguide arrays). These have been the main platforms used to explore parity-time symmetry and topological states in photonics and optics. We will also give a brief introduction to optical isolation, explaining how to realize it in general.

In Chapter 2, we perform a theoretical study of the nonlinear dynamics of nonlinear optical isolator devices based on coupled microcavities with gain and loss. This reveals a correspondence between the boundary of asymptotic stability in the nonlinear (gain-saturated) regime, and the parity-time-breaking transition in the underlying linear system. For zero detuning and weak input intensity, the onset of optical isolation can be rigorously derived, and corresponds to the transition into the parity-time-broken phase of the linear system. When the couplings to the external ports are unequal, the isolation ratio exhibits an abrupt jump at the transition point, which could be exploited to realize an actively controlled nonlinear optical isolator, with strong optical isolation turned on and off by tiny variations in the inter-resonator separation.

In Chapter 3, we investigate how the concept of topological phase transitions can be used to design nonlinear photonic structures exhibiting power thresholds and discontinuities in their transmittance. This provides a novel route to devising nonlinear optical isolators. We study three representative designs: (i) a waveguide array implementing a nonlinear 1D Su-Schrieffer-Heeger model, (ii) a waveguide array implementing a nonlinear 2D Haldane model, and (iii) a 2D lattice of coupled-ring waveguides. In the first two cases, we find a correspondence between the topological transition of the underlying linear lattice and the power threshold of the transmittance, and show that the transmission behavior is attributable to the emergence of a self-induced topological soliton. In the third case, we show that the topological transition produces a discontinuity in the transmittance curve, which can be exploited to achieve sharp jumps in the power-dependent isolation ratio.

In Chapter 4, we turn to a more fundamental topic in topological photonics,

involving the realization of an unpaired Dirac cone at the center of the first Brillouin zone. This is accomplished using a gyromagnetic photonic crystal with broken square sub-lattice symmetry and broken time reversal symmetry. The behavior of the Dirac modes can be described by a gyromagnetic effective medium model with near-zero refractive index, and Voigt parameter near unity. When two domains are subjected to opposite magnetic biases, there exist unidirectional edge states along the domain wall. This work establishes a link between topological edge states and the surface waves of homogenous magneto-optical media.

1.2 Microring resonator

Microring resonators are an important building block in the field of integrated optics, due to their high quality factors and easy integrability[38–40]. The idea of the microring resonator was proposed in the 1960’s by Marcatili[41]. In its simplest form, a microring resonator can be considered as a looped waveguide. It can support multiple resonance modes based on having different integer numbers of wavelengths fitting in the loop. Consider a microring resonator with radius r and refractive index of n_{eff} ; in order for the incoming light with wavelength λ to constructively interfere with itself after multiple rounds, the total phase accumulated should be an integer multiple of 2π :

$$\frac{2\pi}{\lambda} n_{eff} L = 2\pi m, \quad (1.1)$$

where m is an integer, $L = 2\pi r$ is the circumference. This can be reduced to:

$$\lambda = \frac{2\pi r n_{eff}}{m}. \quad (1.2)$$

As we can see, the microring resonator will support multiple discrete wavelengths of incoming light. Light with any other wavelength, incident on the microring resonator,

Figure 1.1: Scanning electron micrograph (SEM) of microring resonators. (a) Single silicon microring resonator (Images: Ken Crozier, Harvard School of Engineering and Applied Sciences) (b) Coupled microring resonators. Inner and outer radius for each ring are $3.5\mu\text{m}$ and $4.5\mu\text{m}$. Figure reproduced from Ref. [33]

will not couple to the resonator.

Microring resonators have found lots of potential applications. To give some examples, they can be used as spectral filters for telecommunications [42, 43], as optical delay line [44, 45], and as sensor devices [46, 47] for detecting small changes in refractive index via shifts in the resonance frequencies.

1.3 Photonic crystal

The field of photonic crystals was started by the publication of two seminal papers by S. John and E. Yablonovitch [48, 49]. In essence, photonic crystals are man-made periodic structures possessing periodic permittivity and permeability, which can be used to control the propagation of light [18]. Photonic crystals can be used to implement photonic bandgap materials, which are marked by photonic band gaps: ranges of frequency where light is forbidden to propagate inside the photonic crystal. This happens when the length-scale of the periods (or lattice constants) of photonic crystal is comparable in size with the wavelength of light propagating inside it. The formation of photonic bandgaps comes from Bragg diffraction; roughly speaking, light does not propagate inside the photonic crystal if the reflected light is in phase, which

Figure 1.2: (a) Schematic of one-, two- and three-dimensional photonic crystals. Different colors stand for different permittivity and permeability distributions. (b) Transverse electric (TE, red line) and transverse magnetic (TM, blue line) band-structures for a 2d triangular array of air drilled in substrate ($\epsilon = 13$). A photonic bandgap is highlighted in yellow. Inset: Brillouin zone (left) and spatial structure, with air columns shown in white and dielectric substrate shown in green (right). Figure reproduced from Ref.[18]

is similar to how the bandgap forms for electrons between the valence band and conduction band of a solid[50]. Fig.1.2 shows the schematic picture of a photonic crystal in 1, 2 and 3 dimensions and a transverse-magnetic and transverse-electric bandstructure with a photonic bandgap.

Numerous optical devices based on photonic crystals have been developed, for example highly-confining waveguides[51, 52], lasers[53, 54], beam splitters[55, 56], slow light waveguides[57, 58], and biosensors[59, 60].

The propagation of light inside a photonic crystal is described by Maxwell's equations:

$$\nabla \cdot \mathbf{D} = \rho \quad (1.3)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (1.4)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1.5)$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}. \quad (1.6)$$

Here, $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, while $\mathbf{D}$ and $\mathbf{B}$ are the displacement and magnetic induction field. When the material is linear and isotropic, we have following constitutive relations:

$$\mathbf{D} = \epsilon_0 \epsilon(\mathbf{r}) \mathbf{E}(\mathbf{r}) \quad (1.7)$$

$$\mathbf{B} = \mu_0 \mu(\mathbf{r}) \mathbf{H}(\mathbf{r}). \quad (1.8)$$

Here, ϵ_0 and μ_0 are the vacuum permittivity and permeability. $\epsilon(\mathbf{r})$ and $\mu(\mathbf{r})$ are the relative permittivity and permeability. Assuming there is no free charge or current($\rho =$

0, $\mathbf{J} = 0$), we get:

$$\nabla \cdot [\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r}, t)] = 0 \quad (1.9)$$

$$\nabla \cdot [\mu(\mathbf{r})\mathbf{H}(\mathbf{r}, t)] = 0 \quad (1.10)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mu_0 \mu(\mathbf{r})\mathbf{H}(\mathbf{r}, t)}{\partial t} \quad (1.11)$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial \epsilon_0 \epsilon(\mathbf{r})\mathbf{E}(\mathbf{r}, t)}{\partial t}. \quad (1.12)$$

By separating the time-dependent part and the spatially dependent part, we take the electric field and magnetic fields to have the following forms:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r})e^{-i\omega t} \quad (1.13)$$

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}(\mathbf{r})e^{-i\omega t}. \quad (1.14)$$

The result is :

$$\nabla \cdot [\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r})] = 0 \quad (1.15)$$

$$\nabla \cdot [\mu(\mathbf{r})\mathbf{H}(\mathbf{r})] = 0 \quad (1.16)$$

$$\nabla \times \mathbf{E}(\mathbf{r}) = i\omega\mu_0\mu(\mathbf{r})\mathbf{H}(\mathbf{r}) \quad (1.17)$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = -i\omega\epsilon_0\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r}). \quad (1.18)$$

Combining Eq.1.17 and Eq.1.18, we arrive at the eigenvalue equation:

$$\frac{1}{\mu(\mathbf{r})}\nabla \times \left[\frac{1}{\epsilon(\mathbf{r})}\nabla \times \mathbf{H}(\mathbf{r}) \right] = \omega^2\epsilon_0\mu_0\mathbf{H}(\mathbf{r}). \quad (1.19)$$

After we obtain $\mathbf{H}(\mathbf{r})$ by solving the eigenvalue equation, we can obtain $\mathbf{E}(\mathbf{r})$ by using

Eq.1.18. At this point, we can define the operator:

$$\hat{\Theta} = \frac{1}{\mu(\mathbf{r})} \nabla \times \frac{1}{\epsilon(\mathbf{r})} \nabla \times \quad (1.20)$$

The eigenvalue equation becomes:

$$\hat{\Theta} \mathbf{H}(\mathbf{r}) = \omega^2 \epsilon_0 \mu_0 \mathbf{H}(\mathbf{r}). \quad (1.21)$$

We can see now that light propagating inside photonic crystal can be thought of an eigenvalue problem and shares a lot of similarities with quantum mechanics. The following table makes a comparison between them:

	Quantum mechanics	Electrodynamics
Field	$\Psi(\mathbf{r}, t) = \Psi(\mathbf{r})e^{-iEt/\hbar}$	$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}(\mathbf{r})e^{-i\omega t}$
Eigenvalue problem	$\hat{H}\Psi = E\Psi$	$\hat{\Theta}\mathbf{H}(\mathbf{r}) = \omega^2 \epsilon_0 \mu_0 \mathbf{H}(\mathbf{r})$
Hermitian operator	$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r})$	$\hat{\Theta} = \frac{1}{\mu(\mathbf{r})} \nabla \times \frac{1}{\epsilon(\mathbf{r})} \nabla \times$

Because the Maxwell equations in periodic media share the same form as Schrödinger's equation in a solid state system, photonic crystals can be used to study some condensed matter phenomena, including those that may be difficult to realize in electronic system, e.g. due to demanding sample fabrication. This includes the realization of photonic topological insulators, which has attracted tremendous attention recently[61–63].

Figure 1.3: (a) Microscope image of femtosecond laser-written waveguides array in fused silica glass. The separation between nearby waveguides is $40\mu m$. Figure reproduced from Ref.[67] (b) A schematic picture shows how light propagate inside a two dimensional photonic lattice. Figure reproduced from Ref.[68]

1.4 Waveguide arrays

Waveguide arrays, or photonic lattices, are systems of evanescently coupled optical waveguides, that are commonly used in the study of discretized light propagation[64]. In these systems, light is confined to weakly-guiding waveguides and can hop to adjacent waveguides by evanescent coupling[64, 65]. The strength of the coupling is determined by the separation between waveguides. Waveguides can be modulated periodically in one dimension or in two transverse dimensions(the x-y plane). They are usually translationally invariant in the longitudinal direction(the z direction, which is also the overall propagation direction of the light).

Under the paraxial approximation[19, 66], the propagation of light in a waveguide array is similar to the dynamics of a quantum system governed by the time-dependent Schrödinger equation. As a result, waveguide arrays have many of the important properties of photonic crystals, like Brillouin zones and photonic bandgaps. But unlike photonic crystals, the period of a waveguide array is usually larger than the wavelength of light, which makes the light confinement nonresonant[64, 65].

Now we will derive the governing equation for light propagating inside a waveguide

array under the paraxial approximation, and show that it shares a form similar to the time-dependent Schrödinger equation. Consider Maxwell's equations in a continuous medium without free charge and current, which have been shown in Eq.1.9-Eq.1.12. Taking the curl of Eq.1.11, use Eq.1.9 and Eq.1.12, we have:

$$\nabla^2 \mathbf{E} + 2\nabla \left(\frac{\nabla n}{n^2} \cdot \mathbf{E} \right) - \frac{n^2}{c^2} \partial_t^2 \mathbf{E} = 0. \quad (1.22)$$

If we assume a weak refractive index modulation ($\nabla n \ll n$), we can discard second term and obtain:

$$\nabla^2 \mathbf{E} - \frac{n^2}{c^2} \partial_t^2 \mathbf{E} = 0. \quad (1.23)$$

The paraxial approximation assumes that the angle between $\mathbf{k}$ vector and the z axis is small, that is $k_z \gg k_{x,y}$. This suggests that we can assume a single frequency of light propagating in the z direction, $\mathbf{E}(x, y, z) = A(x, y, z)e^{-i(k_z z + \omega t)}$, with wave number k_z , frequency ω and amplitude $A(x, y, z)$. Inserting $\mathbf{E}(x, y, z) = A(x, y, z)e^{-i(k_z z + \omega t)}$ into Eq.1.23 gives:

$$2ik_z \partial_z A - \nabla^2 A + \left(k_z^2 - \frac{n^2 \omega^2}{c^2} \right) A - \partial_z^2 A = 0. \quad (1.24)$$

We further assume that amplitude A varies slowly in the z direction, so that $|k_z \partial_z A| \gg |\partial_z^2 A| \approx 0$, which leads us to:

$$2ik_z \partial_z A - \nabla^2 A + \left(k_z^2 - \frac{n^2 \omega^2}{c^2} \right) A = 0. \quad (1.25)$$

We now express the refractive index n as a constant background n_0 plus a linear modulation Δn , $n = n_0 + \Delta n$. Assume the linear modulation is small enough that $\frac{n_0 \omega}{c} \approx k_z$. After making this approximation, we have:

$$2ik_z \partial_z A - \nabla^2 A - 2 \frac{\Delta n}{n_0} k_z^2 A = 0. \quad (1.26)$$

To make Eq.1.26 dimensionless, we define a characteristic transverse length scale a , so that $\tilde{z} = z/(k_z a^2)$, $\tilde{x} = x/a$, $\tilde{y} = y/a$. Now we arrive at:

$$i\partial_{\tilde{z}}A - \frac{1}{2}\nabla^2 A + VA = 0, \quad (1.27)$$

where V is defined as $V = -k_z^2 a^2 \frac{\Delta n}{n}$.

Based on the similarity between the governing equation for light propagation in waveguide arrays and electrons in solid state systems, many quantum phenomena have been studied using waveguide arrays, for example Bloch oscillations[69], Dirac zitterbewegungs[70], and topological phases[71].

1.5 Optical isolation

“Non-reciprocal” photonic media are important to optical communication systems due to their potential application in making optical isolators for modern optical communication systems[72–75]. On-chip optical isolators are devices intended to isolate optical components of different scales, in order to prevent interference between different parts of an optical network. When many different optical components are integrated into a single system, backscattering will cause problems such as parasitic responses and coherent interference. As on-chip optical networks attract more and more attention from researchers, on-chip scale optical isolator is becoming more and more important.

To achieve non-reciprocal light transmission, one needs to break Lorentz reciprocity[37]. Lorentz reciprocity states that under assumptions of a linear, time-independent and symmetric medium, the action of reversing the source and monitor will not change the resulting field measured by the monitor. Violation of either one of these three assumptions can lead to the breakdown of Lorentz reciprocity.

1.5.1 Derivation of Lorentz reciprocity

Consider two sets of excitations with field $\mathbf{E}^1(x, y, z)$, $\mathbf{H}^1(x, y, z)$ and $\mathbf{E}^2(x, y, z)$, $\mathbf{H}^2(x, y, z)$. Using the Maxwell's equation without source term for the first excitation:

$$\nabla \times \mathbf{E}^1 = -i\omega\boldsymbol{\mu}\mathbf{H}^1 \quad (1.28)$$

$$\nabla \times \mathbf{H}^1 = i\omega\boldsymbol{\epsilon}\mathbf{E}^1. \quad (1.29)$$

Left mutiplying both sides of Eq.1.28 with $\mathbf{H}^2$ and Eq.1.29 with $\mathbf{E}^2$ yields:

$$\mathbf{H}^2 \nabla \times \mathbf{E}^1 + \mathbf{E}^2 \nabla \times \mathbf{H}^1 = i\omega(\mathbf{E}^2 \boldsymbol{\epsilon} \mathbf{E}^1 - \mathbf{H}^2 \boldsymbol{\mu} \mathbf{H}^1). \quad (1.30)$$

We can get a similar result for the second excitation:

$$\mathbf{H}^1 \nabla \times \mathbf{E}^2 + \mathbf{E}^1 \nabla \times \mathbf{H}^2 = i\omega(\mathbf{E}^1 \boldsymbol{\epsilon} \mathbf{E}^2 - \mathbf{H}^1 \boldsymbol{\mu} \mathbf{H}^2). \quad (1.31)$$

Substracting them we get:

$$\nabla \cdot (\mathbf{E}^1 \times \mathbf{H}^2 - \mathbf{E}^2 \times \mathbf{H}^1) = i\omega(\mathbf{E}^2 \boldsymbol{\epsilon} \mathbf{E}^1 - \mathbf{E}^1 \boldsymbol{\epsilon} \mathbf{E}^2 - \mathbf{H}^2 \boldsymbol{\mu} \mathbf{H}^1 + \mathbf{H}^1 \boldsymbol{\mu} \mathbf{H}^2). \quad (1.32)$$

If the system is linear, static and does not respond to a magnetic field, $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ will be a scalar and symmetric tensor, which means that the right side of Eq.1.32 will be 0, thus we obtain the Lorentz reciprocity condition

$$\nabla \cdot (\mathbf{E}^1 \times \mathbf{H}^2 - \mathbf{E}^2 \times \mathbf{H}^1) = 0. \quad (1.33)$$

1.5.2 From Lorentz reciprocity to optical isolation

Consider a general system with volume V enclosed by a surface Ω . Light can go in and out of this system with the following fields[37, 76]

$$\mathbf{E}_\mu(x, y, z) = (a_\mu e^{-i\beta_\mu z} + b_\mu e^{i\beta_\mu z})\mathbf{e}_\mu(x, y) \quad (1.34)$$

$$\mathbf{H}_\nu(x, y, z) = (a_\nu e^{-i\beta_\nu z} - b_\nu e^{i\beta_\nu z})\mathbf{h}_\nu(x, y). \quad (1.35)$$

Here μ and ν stand for the mode numbers; $\beta_{\mu,\nu}$ stands for the propagation constant of certain mode; $a_{\mu,\nu}$ and $b_{\mu,\nu}$ stand for the mode amplitudes of light going into and out of the system; and $\mathbf{e}_\mu(x, y)$ and $\mathbf{h}_\nu(x, y)$ stand for the unit polarization vectors.

We define the normalization condition as:

$$\int \int_{\Omega} \mathbf{e}_\mu \times \mathbf{h}_\nu^* d\mathbf{S} = 2\delta_{\mu\nu}. \quad (1.36)$$

We focus on the surface of the system ($z = 0$):

$$\mathbf{E}_\mu = (a_\mu + b_\mu)\mathbf{e}_\mu(x, y) \quad (1.37)$$

$$\mathbf{H}_\nu = (a_\nu - b_\nu)\mathbf{h}_\nu(x, y). \quad (1.38)$$

We consider two set of incoming and outgoing waves $\mathbf{A}^1, \mathbf{B}^1$ and $\mathbf{A}^2, \mathbf{B}^2$, where $\mathbf{A}, \mathbf{B}$ are mode amplitude vectors for incoming and outgoing waves. A scattering matrix $\mathbf{S}$ can be defined for incoming and outgoing waves:

$$\mathbf{B} = \mathbf{S}\mathbf{A}. \quad (1.39)$$

Integrating the Lorentz Reciprocity equation Eq.1.33 over the closed volume and

using Eq.1.37 and Eq.1.38; we obtain

$$\sum_{\mu} \sum_{\nu} (b_{\mu}^1 a_{\nu}^2 - a_{\mu}^1 b_{\nu}^2) \int \int_{\Omega} \mathbf{e}_{\mu} \times \mathbf{h}_{\nu} d\mathbf{S} = 0. \quad (1.40)$$

Using the normalization condition Eq.1.36, we have:

$$\sum_{\mu} (b_{\mu}^1 a_{\mu}^2 - a_{\mu}^1 b_{\mu}^2) = 0. \quad (1.41)$$

This give us:

$$(\mathbf{B}^1)^T \mathbf{A}^2 - (\mathbf{A}^1)^T \mathbf{B}^2 = (\mathbf{A}^1)^T \mathbf{S}^T \mathbf{A}^2 - (\mathbf{A}^1)^T \mathbf{S} \mathbf{A}^2 = 0. \quad (1.42)$$

Since the choice of $\mathbf{A}^1$ and $\mathbf{A}^2$ are arbitrary, we have:

$$\mathbf{S}^T = \mathbf{S}. \quad (1.43)$$

Eq.1.43 means that our system is reciprocal, because the scattering maxtrix is symmetric. Thus, a system obeying Lorentz reciprocity can never act as an optical isolator.

1.5.3 Ways to break Lorentz reciprocity

Basically there are three ways to break Lorentz reciprocity and each way corresponds to one kind of optical isolator. The first way is to use a magneto-optical material[77, 78]. In a magneto-optical material, the permittivity is not a symmetric tensor, thus breaking Lorentz reciprocity.

However, there are several limitations facing magneto-optical isolators. One of them is magnetic circular dichroism, which limits the maximum isolation ratio[79]. Moreover, the use of an external magnetic field may also be harmful to nearby devices[80, 81]. Most seriously of all, the magneto-optical effect is weak in the optical

Figure 1.4: (a) Schematic picture of a magneto-optical isolator. (b) Schematic picture of a nonlinear optical isolator. (c) Schematic picture of a spatiotemporal modulated optical isolator. Figure reproduced from Ref.[77, 27, 82]

frequency domain, i.e. the off-diagonal elements in the permittivity matrix are small. To achieve a strong magneto-optical response at optical frequencies, one needs to make the size of the device much larger than the wavelength, which is disadvantageous for many integrated optics applications.

The other way to break Lorentz reciprocity is to use nonlinear materials[83–86]. In a nonlinear material, ϵ is dependent on the electric-field, which allows light coming from different directions to see different effective dielectric permittivity distributions[87]. In recent years, a number of nonlinear optical isolators have been proposed, including using Kerr nonlinearity[88, 89], Brillouin scattering[90] and nonlinearity induced by optomechanical effect[91]. Nonlinearities also tend to be weak in the optical regime, for example the Kerr nonlinear refractive index is around $6 \times 10^{-18} \text{m}^2 \cdot \text{W}^{-1}$ [91]. However, due to advances in making high-quality micro-resonators, nonlinear isolation has become more promising in recent years[26, 27, 90, 92].

The third case where Lorentz reciprocity can be broken is using time-dependent materials, where ϵ and μ depend on time[\[93\]](#). Nonreciprocal transmission arises as a result of modulation of phase for light coming from different directions. However, a practical problem for this type of isolator is its finite bandwidth[\[94\]](#).

Chapter 2

PT symmetry breaking and nonlinear optical isolation in coupled microcavities

Most of the material presented in this chapter was previously published in the journal Optics Express [95]. Wherever figures and materials from other sources have been taken, the authors have been given due credit.

For many years, the implementation of compact optical isolators has been a major research goal in the field of integrated optics [37, 72]. Optical isolation requires the breaking of Lorentz reciprocity; this is traditionally achieved using magneto-optic materials, but such materials are challenging to incorporate into integrated optics devices [96, 97]. The most commonly-pursued alternative method for breaking reciprocity is to exploit optical nonlinearity [26, 27, 72, 83–87].

Two recent demonstrations of nonlinearity-based on-chip optical isolators, by Peng *et al.* [26] and Chang *et al.* [27], have drawn particular attention. These experiments are based on the idea of parity-time(*PT*) symmetry, which is described in Sec.2.1. They featured a pair of coupled whispering-gallery microcavities, one containing loss and the other saturable (nonlinear) gain. Light transmission across the structure was

found to be strongly nonreciprocal, depending on whether it first passed through the gain or loss resonator. Aided by the high Q factors of the resonators, isolation was observed for record-low powers of $\sim 1\mu\text{W}$ [26].

The field of “ PT symmetric optics” concerns optical structures that are invariant under simultaneous parity-flip (P) and time-reversal (T) operations [17, 20, 21, 28, 98–104]. The concept originated from the observation that PT symmetric Hamiltonians, despite being non-Hermitian, can exhibit real eigenvalue spectra [2, 105], as well as “ PT -breaking transitions” between real and complex eigenvalue regimes. We will discuss this phenomenon in detail in Sec.2.1. The PT -breaking transition point is an “exceptional point”, where two eigenstates coalesce and the effective Hamiltonian becomes defective [106, 107]. Near the transition, the dynamical behavior of the optical fields can exhibit highly interesting features [108–111]; for instance, the presence of gain saturation has been found to stabilize PT -symmetric steady states past the usual PT transition point [109, 110].

Despite these intriguing conceptual links, it was not clear from [26, 27] how PT symmetry relates to the working of the nonlinear optical isolators in question. Strictly speaking, PT symmetry holds in the dual-resonator structures only in the linear limit; in the nonlinear regime, the gain saturates and no longer matches the loss, so the structures are not PT symmetric and do not possess distinct “ PT -symmetric” or “ PT -broken” phases. Peng *et al.*, in [26], indicated that optical isolation occurs (in the nonlinear regime) if the system is tuned so that it would be PT -broken *in the linear regime*; however, the actual correspondence was not shown theoretically nor experimentally. The dynamical behavior of the system, including the uniqueness and stability of the steady-state solution(s), was also unexplored.

In this chapter, we perform a theoretical analysis of the dual-resonator structure, aiming to clarify the relationships between the PT phase, the performance of the nonlinear optical isolator, and the uniqueness and stability of the steady-state optical

modes. Using coupled-mode theory [112–115], we study the conditions for steady-state solutions to exist, and the asymptotic stability of those solution(s). We find that stability in the nonlinear system has a close correspondence with the PT transition boundary of the underlying linear system.

In the “weak-input limit”, where the input intensity is low relative to the gain saturation threshold within the amplifying resonator, we show that the nonlinear solutions at non-zero frequency detunings are asymptotically stable in the PT -symmetric phase. In the PT -broken phase, the solutions become unstable at large frequency detunings, and the nonlinear system exhibits limit-cycle oscillations, which might be useful for frequency generation applications (such as frequency combs).

For small frequency detunings, multiple steady-state solutions can exist in the PT -broken phase, but only the highest-intensity solution is asymptotically stable. Specifically at zero detuning, there is always one stable steady-state solution, and the *nonlinear* system exhibits a sharp transition between isolating behavior (corresponding to the PT -broken phase) and reciprocal behavior (corresponding to the PT -symmetric phase). Although this transition coincides exactly with the PT transition point, it is an inherently nonlinear effect, arising from a jump between different solution branches of the transmission intensity equations. However, the performance of the isolator can be significantly limited by the contributions to the nonlinearity caused by a reflected wave [87].

We also show that the performance of the nonlinear optical isolator is modified in a useful way when the two resonator-to-waveguide coupling rates are unequal. In this case, a small shift across the transition point causes the isolation ratio (the ratio between forward and backward transmission intensities) to undergo an abrupt jump, which approaches a discontinuity in the weak-input limit. The magnitude of this jump is given by the ratio of the coupling rates. This phenomenon can be used to realize a nonlinear optical isolator that exhibits very large changes in the isolation

ratio, actively controlled by tiny shifts in (e.g.) the inter-resonator separation.

2.1 Introduction to Parity-time symmetry

2.1.1 Parity-time symmetry in quantum mechanics

One of the postulates of quantum mechanics states that the Hamiltonian which describes the system of interest should be Hermitian, i.e., $H = H^\dagger$ where $\dagger$ stands for Hermitian conjugation (transpose and complex conjugate). This mathematical property ensures the realness of the energy spectrum. However, in 1998, Bender and co-workers[2] discovered a class of non-Hermitian Hamiltonian that can possess real-valued eigenvalues as long as they commute with a parity-time($\hat{P}\hat{T}$) operator. Here, $\hat{P}$ is a parity operator which, for example, flips the sign of position and momentum($x \rightarrow -x, p \rightarrow -p$). $\hat{T}$ is the time reversal operator, which flips the sign of the imaginary unit and the momentum($x \rightarrow x, p \rightarrow -p, i \rightarrow -i$).

To see how a parity-time symmetric Hamiltonian can possess a real energy spectrum, consider the eigenvalue problem:

$$H\psi = E\psi. \quad (2.1)$$

Apply the parity-time($\hat{P}\hat{T}$) operation on both sides of Eq.(2.1):

$$\hat{P}\hat{T}H\psi = \hat{P}\hat{T}E\psi = E^*\hat{P}\hat{T}\psi. \quad (2.2)$$

Since the Hamiltonian H commutes with the parity-time($\hat{P}\hat{T}$) operator, $[H, \hat{P}\hat{T}] = 0$, we have

$$H\hat{P}\hat{T}\psi = \hat{P}\hat{T}E\psi = E^*\hat{P}\hat{T}\psi. \quad (2.3)$$

This implies that there are two possible scenarios. First, if the system is in the “ PT

symmetry unbroken” regime, we have $\hat{P}\hat{T}\psi \propto \psi$, and thus $E = E^*$, which means the spectrum is real. Second, if the system is in the “ PT symmetry broken” regime, $\hat{P}\hat{T}\psi \not\propto \psi$, and the spectrum is complex. The system thus goes through a symmetry-breaking process as the energy spectrum changes from real to complex, which is called a PT transition. The transition point is called an ”exceptional point” because not only the eigenvalues become degenerate but also the eigenstates.

In the context of the Schrödinger wave equation, a necessary condition for the Hamiltonian to be PT symmetric is that the real part of the potential must be an even function, and the imaginary part of the potential must be an odd function. To see this, let us consider Schrödinger wave equation in one dimension:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi, \quad (2.4)$$

where ψ is the single particle’s wave equation, $\hbar$ is the reduced Planck constant defined as $\hbar = \frac{h}{2\pi}$ where h is Planck constant, m is the mass of the particle, and $V(x)$ is the potential felt by the particle.

If the Hamiltonian H commutes with the parity-time($\hat{P}\hat{T}$) operator, we have $[H, \hat{P}\hat{T}] = 0$, which implies that:

$$H\hat{P}\hat{T} = \hat{P}\hat{T}H. \quad (2.5)$$

Hence,

$$\hat{P}\hat{T}H = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V^*(-x) \quad (2.6)$$

$$H\hat{P}\hat{T} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x). \quad (2.7)$$

Thus, $V(x) = V^*(-x)$ is a necessary condition for the Hamiltonian to be PT symmetric.

2.1.2 Parity-time symmetric phase transition and the exceptional point: An example

Consider the following two level Hamiltonian:

$$H = \begin{bmatrix} e^{i\theta} & g \\ g & e^{-i\theta} \end{bmatrix}. \quad (2.8)$$

Here g and θ are real parameters and i is imaginary unit. Let us define the parity operator as [6]:

$$\hat{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (2.9)$$

and let $\hat{T}$ be the complex conjugation operator. With these definitions, the Hamiltonian is indeed PT symmetric:

$$\hat{P}\hat{T}H(\hat{P}\hat{T})^{-1} = H = \begin{bmatrix} e^{i\theta} & g \\ g & e^{-i\theta} \end{bmatrix}. \quad (2.10)$$

The eigenvalues of this Hamiltonian are:

$$E_{1,2} = \cos \theta \pm \sqrt{g^2 - \sin^2 \theta} \quad (2.11)$$

and the corresponding eigenstates are:

$$\psi_{1,2} = \begin{bmatrix} \frac{g}{E_{1,2} - e^{i\theta}} \\ 1 \end{bmatrix}. \quad (2.12)$$

We see there exists two relevant regions of parameters, connected by a critical point:

- $|g| > |\sin \theta|$: in this case, the eigenvalues are real despite the fact that the

Figure 2.1: Eigenvalue spectrum of 2 by 2 parity-time symmetric Hamiltonian. $\theta = \frac{\pi}{2}$.

Hamiltonian is not Hermitian. This is what we call the PT -symmetry-unbroken phase, because:

$$\hat{P}\hat{T}\psi_{1,2} = \psi_{1,2} \quad (2.13)$$

- $|g| < |\sin \theta|$: in this case, the two eigenvalues are complex conjugate of each other and this is what we call the PT -symmetry-broken phase, because:

$$\hat{P}\hat{T}\psi_{1,2} = \psi_{2,1} \quad (2.14)$$

- $|g| = |\sin \theta|$: this corresponds to the exceptional point, where both eigenvalues and eigenstates become degenerate:

$$E_1 = E_2 = \cos \theta \quad (2.15)$$

$$\psi_1 = \psi_2 = \begin{bmatrix} \frac{g}{\cos \theta - e^{i\theta}} \\ 1 \end{bmatrix}. \quad (2.16)$$

If we fix the value of $|\sin \theta|$, the PT phase transition happens as we tune the

value of $|g|$ from larger than $|\sin \theta|$ to smaller than $|\sin \theta|$. The system changes from the PT -symmetry-unbroken phase to the PT -symmetry-broken phase as shown in Fig.2.1. This process is a type of “spontaneous” symmetry breaking, in the sense that the Hamiltonian always commutes with the $\hat{P}\hat{T}$ operator, but in the PT -symmetry-broken phase, the eigenstates do not individually satisfy PT symmetry.

2.1.3 Parity-time symmetry in optics

PT -symmetric quantum mechanics originated as an alternative to the usual laws of quantum mechanics, but it subsequently found many interesting applications in optics. Because of the equivalence between the Schrödinger equation and the paraxial optical diffraction equation described in Sec.1.4, a complex PT -symmetric system can be achieved in an optical waveguide array by spatially modulating the refractive index. To achieve the complex potential mentioned above in optical system, one can judiciously incorporate regions which have gain and loss in a balanced way. Optical realizations of the PT -symmetric systems in the coupled waveguides with modulated potential have been achieved, and PT symmetry and spontaneous PT symmetry breaking have been experimentally observed[20, 21]. Figure.2.2 describes a PT -symmetry breaking process in a PT symmetric optical waveguide. The intensity distribution is nearly even in the PT -symmetry-unbroken phase but it is dominant in the active optical waveguide when the system is in the PT -symmetry-broken phase. Many exotic phenomena can be found in PT -symmetric optics, including unidirectional invisibility[116], simultaneous lasing and coherent perfect absorption[117], unconventional beam refraction[17], and anomalous conical diffraction[107].

Figure 2.2: Observation of the parity-time symmetry breaking process in coupled optical waveguides. Top panel: measured intensity distribution. Down panel: measured relative phase. (a) Parity-time symmetry unbroken case. (b) Parity-time symmetry broken case. Figure reproduced from Ref.[20]

2.2 Coupled-mode equations

A dual-resonator structure is shown schematically in Fig. 2.3(a). The setup is identical to the experiments reported in [26, 27], consisting of two evanescently coupled micro-cavities with resonant frequencies ω_1 and ω_2 . One resonator contains saturable gain, and the other is lossy. The resonators are coupled to separate optical fiber waveguides, which act as input/output ports (labeled 1–4), with couplings κ_1 and κ_2 . The direct inter-resonator coupling rate is μ . In the “forward transmission” configuration, light is injected from port 1 at a fixed operating frequency ω , exiting at ports 2 and 4. Alternatively, in the “backward transmission” configuration, light is injected at port 4 and exit at ports 1 and 3. We are interested in the level of isolation between ports 1 and 4, which serve as the operational input and output ports for the device. The dual-resonator system can be described by coupled-mode equations [26, 27], formulated using the standard framework of coupled-mode theory [112–115]. In this section and the next, we briefly summarize these equations, which have previously been presented

Figure 2.3: (a) Schematic of a resonator with saturable gain coupled to a lossy resonator, with both resonators coupled to optical fiber ports. Solid arrows indicate forward transmission (port 1 $\rightarrow$ 4), and dashed arrows indicate backward transmission (port 4 $\rightarrow$ 1). (b)–(d) Transmission characteristics in the linear (non-gain-saturated) regime, when the gain and loss are PT symmetric ($g = \gamma = 0.4$). Here, we plot the intensity in the gain resonator ($|a_1|^2$) under forward transmission (solid lines), and in the loss resonator ($|a_2|^2$) under backward transmission (dashes), versus the frequency detuning. These resonator intensities are proportional to the forward and backward transmission intensities via Eqs. (2.19) and (2.22). In the PT -symmetric phase $\mu > \gamma$, there are two transmission peaks; in the PT -broken phase $\mu < \gamma$, these merge into a single peak.

in Refs. [26, 27]. For forward transmission, the coupled-mode equations are

$$\frac{da_1}{dt} = (i\Delta\omega_1 + g)a_1 - i\mu a_2 \quad (2.17)$$

$$\frac{da_2}{dt} = (i\Delta\omega_2 - \gamma)a_2 - i\mu a_1 + \sqrt{\kappa_2} s_{\text{in}} \quad (2.18)$$

$$I_F = \kappa_1 |a_1|^2. \quad (2.19)$$

Here, a_1 and a_2 denote the complex amplitudes for the slowly-varying field amplitudes in the gain resonator and loss resonators, respectively; $\Delta\omega_{1,2} \equiv \omega - \omega_{1,2}$ denote the operating frequency detuning from each resonator's resonance frequency; $g > 0$ and $\gamma > 0$ are the net gain rate in resonator 1 and the net loss rate in resonator 2; s_{in} is the amplitude of the incoming light in port 1; and I_F is the power transmitted forward into port 4. For the moment, we assume that there is no reflected wave re-entering the system from port 4; the effects of such a reflected wave will be discussed in Section

2.4.4.

For backward transmission, a different set of coupled-mode equations holds:

$$\frac{da_1}{dt} = (i\Delta\omega_1 + g)a_1 - i\mu a_2 + \sqrt{\kappa_1} s_{\text{in}} \quad (2.20)$$

$$\frac{da_2}{dt} = (i\Delta\omega_2 - \gamma)a_2 - i\mu a_1 \quad (2.21)$$

$$I_B = \kappa_2 |a_2|^2, \quad (2.22)$$

where I_B is the power transmitted into port 1.

The gain/loss rates g and γ consist of several radiative and non-radiative terms [26]:

$$g = \frac{1}{2} (g' - \gamma_1 - \kappa_1) \quad (2.23)$$

$$\gamma = \frac{1}{2} (\gamma_2 + \kappa_2), \quad (2.24)$$

where g' is the intrinsic amplification rate in resonator 1, and $\gamma_{1,2}$ are the intrinsic loss rates in the resonators. Until stated otherwise, we will impose the following simplifying restrictions:

$$\kappa_1 = \kappa_2 = \gamma_1 = \gamma_2 \quad (2.25)$$

$$\Delta\omega_1 = \Delta\omega_2 \equiv \Delta\omega \quad (2.26)$$

$$g' = \frac{g_0}{1 + |a_1/a_s|^2}. \quad (2.27)$$

Eq. (2.25) corresponds to a “critical coupling” criterion with respect to the individual cavity-waveguide couplings. The intrinsic loss and outcoupling rates are all tuned to the same value; note also that $g = g'/2 - \gamma$. Eq. (2.26) states that the resonators have the same resonance frequency. Eqs. (2.25)–(2.26) serve as simplifying assumptions, to avoid dealing with a proliferation of free parameters; later, we will discuss the impli-

cations of relaxing these assumptions. Another important constraint, PT symmetry, will be imposed in the next section. Eq. (2.27) describes saturable gain, where g_0 is the unsaturated amplification rate, and $a_s \in \mathbb{R}^+$ is a gain saturation threshold. Besides gain saturable nonlinearity, there exist other nonlinearity mechanisms such as Kerr effect[118], two photon absorption(TPA)[119] and thermal nonlinearity[120]. In our study, we focused on gain saturation nonlinearity, as this was the dominant nonlinearity in the reported experiments. Future research could look into extending some of the principles appearing in the present study to photonic systems with other forms of nonlinearity.

The experimentally realized systems reported in [26, 27] operated in the 1550 nm wavelength band, with rate parameters μ , g_0 , γ and $\kappa_{1,2}$ on the order of 10 MHz in [26], and 100 MHz in [27]. The coupling rates μ and $\kappa_{1,2}$ can be tuned via the inter-resonator and resonator-waveguide separations. The input power $|s_{\text{in}}|^2$ ranged from zero to around 10–100 μW [26, 27].

The suitability of the system as an optical isolator is characterized using the “isolation ratio”, which is the ratio of forward to backward transmittance at fixed input power:

$$R \equiv \frac{T_F}{T_B} = \frac{I_F(I_{\text{in}})}{I_B(I_{\text{in}})}, \quad (2.28)$$

where I_F is obtained by solving Eqs. (2.17)–(2.19) with $a_1 = a_2 = 0$ (steady state), and I_B is obtained from Eqs. (2.20)–(2.22). When the system is reciprocal, $R = 1$. The isolation ratio was also used in [26, 27] as the figure of merit for optical isolation. However, it is worth noting that the forward and backward transmissions are being compared under the assumption that, in either case, no reflected wave is present. We will discuss this limitation in greater detail in Section 2.4.4.

2.3 Linear operation

We now impose the important constraint $g = \gamma$. This means that in the linear regime, $a_s \rightarrow \infty$, the gain and loss resonators become PT symmetric. To understand the implications, consider the “closed” system without resonator-fiber couplings. Its detuning eigenfrequencies are (assuming $\Delta\omega_1 = \Delta\omega_2 = \Delta\omega$)

$$\Delta\omega = i \frac{g - \gamma}{2} \pm \sqrt{\mu^2 - \gamma g - \left(\frac{g - \gamma}{2}\right)^2}. \quad (2.29)$$

When $g = \gamma$, these reduce to $\Delta\omega = \pm\sqrt{\mu^2 - \gamma^2}$. As μ and γ are varied while keeping $g = \gamma$, the system has a PT symmetry-breaking transition at $\mu = \gamma$. For $\mu > \gamma$, the detunings are real (PT -symmetric phase), and for $\mu < \gamma$ they are purely imaginary (PT -broken phase).

With the resonator-fiber couplings introduced, the eigenmodes become transmission resonances. In the PT -symmetric phase $\mu > \gamma$, the resonator modes and transmission amplitudes exhibit two intensity peaks, at $\Delta\omega = \pm\sqrt{\mu^2 - \gamma^2}$, corresponding to the (real) detunings of the closed system, as shown in Fig. 2.3(b). In the PT -broken phase $\mu < \gamma$, there is a single peak at zero detuning, as shown in Fig. 2.3(d). As noted by Peng *et al.* [26], the PT -symmetric and PT -broken phases will give very different behaviors once gain saturation is introduced.

In the linear regime, Eqs. (2.17)–(2.19) and Eqs. (2.20)–(2.22) obey optical reciprocity by explicit construction [113]. For fixed s_{in} , the forward and backward transmission amplitudes are exactly equal, $I_F = I_B$; the isolation ratio is $R = 1$, as shown in Fig. 2.3(b)–(d).

2.4 Nonlinear operation

2.4.1 Multiple solutions and stability analysis

We turn now to the nonlinear, gain-saturated regime, setting $g_0 = 4\gamma$, so that $g \rightarrow \gamma$ as $a_s \rightarrow \infty$. This means the system would be PT symmetric in the absence of gain saturation. If we use a_s as the natural intensity scale for the coupled-mode equations, the nonlinear system has four remaining independent parameters: $\Delta\omega$, μ , γ , and $|s_{\text{in}}|^2$.

For finite a_s , optical reciprocity is broken. However, the system is no longer PT symmetric, since $g \neq \gamma$, and thus we can no longer rigorously define “ PT symmetric” or “ PT broken” phases. Still, we can relate the nonlinear system’s behavior to the PT symmetric phases *as defined in the linear limit*.

In the linear regime, the solutions to the coupled-mode equations were unique. With nonlinearity, the coupled-mode equations can have multiple steady-state solutions. For forward transmission, steady-state solutions are determined by combining Eqs. (2.17)–(2.18) into:

$$|\alpha|^2 x^3 + \left(2|\alpha - 1|^2 - 2 - \beta\right)x^2 + \left(|\alpha - 2|^2 - 2\beta\right)x - \beta = 0, \quad (2.30)$$

where α , β , and x are the following dimensionless variables:

$$\alpha = \frac{(-i\Delta\omega + \gamma)^2 + \mu^2}{\gamma(-i\Delta\omega + \gamma)}, \quad \beta = \frac{\mu^2}{\Delta\omega^2 + \gamma^2} \frac{1}{\gamma} \left| \frac{s_{\text{in}}}{a_s} \right|^2, \quad x = \left| \frac{a_1}{a_s} \right|^2. \quad (2.31)$$

Since $x \in \mathbb{R}^+$, there is either one, two, or three physical steady-state solutions. There must be at least one solution, since the polynomial has a positive third-order coefficient and negative zeroth-order coefficient. The backward transmission case is handled similarly, using Eqs. (2.20)–(2.21); it gives the same cubic equation as Eq. (2.30), but

Figure 2.4: (a)–(b) Domains in which the nonlinear coupled-mode equations have multiple steady-state solutions, for forward (a) and backward (b) transmission. Here, we show the parameter space defined by the frequency detuning $\Delta\omega$ and inter-resonator coupling μ , with fixed $\gamma = 0.4$, $s_{\text{in}} = 0.5$, and $a_s = 3$; symbols indicate the points in the parameter space corresponding to the curves in (c) and (d). Within the small- $\Delta\omega$ region bounded by the red curves, the highest-intensity (or only) solution is asymptotically stable. (c)–(d) Plots showing the emergence of multiple solutions at several values of μ , fixing $\Delta\omega = 0$. The horizontal axis is the normalized intensity in the gain resonator, $|a_1/a_s|^2$; the vertical axis is the left-hand side of the cubic Eq. (2.30), and its counterpart for backward transmission; the steady-state coupled-mode equations are satisfied when the curves cross zero.

with the replacement

$$\beta = \frac{1}{\gamma} \left| \frac{s_{\text{in}}}{a_s} \right|^2. \quad (2.32)$$

Solving the polynomial reveals a domain in parameter space where there are three physical steady-state solutions, outside of which the solution is unique. This is shown in Fig. 2.4(a)–(b). The three-solution domain lies within the “*PT*-broken” phase of the linear system, $\mu < \gamma$. Nonlinear dynamical equations Eqs. (2.17)–(2.18) and Eqs. (2.20)–(2.21) display a dynamical symmetry previously studied in [121, 122]. If $(a_1, a_2, \Delta\omega, s_{\text{in}})$ is a solution of these nonlinear equations, then $(a_1^*, -a_2^*, -\Delta\omega, s_{\text{in}}^*)$ will also be an solution. This explains the symmetric features in Fig. 2.4(a)–(b).

The boundaries of the three-solution domain depend on γ and s_{in} , as the choice of forward or backward transmission. It consists of two sets of curves; the black curves in Fig. 2.4(a)–(b) involve a degeneracy of two low-intensity roots of the cubic polynomial [Fig. 2.4(c)–(d)]. Crossing this boundary causes no discontinuity in the intensity of the stable steady-state solution. The red curves in Fig. 2.4(a)–(b) involve the degeneracy of two high-intensity roots of the cubic polynomial (2.30); crossing this boundary destabilizes the steady-state solution.

Next, we discuss the stability of the nonlinear coupled-mode equations. For forward transmission, we combine Eqs. (2.17)–(2.19) and (2.25)–(2.27) with the “*PT* symmetry” condition $g_0 = 4\gamma$, to obtain the time-dependent equations

$$\frac{da_1}{dt} = \left(i\Delta\omega - \frac{\gamma + \kappa_1}{2} + \frac{2\gamma}{1 + |a_1/a_s|^2} \right) a_1(t) - i\mu a_2(t) \quad (2.33)$$

$$\frac{da_2}{dt} = \left(i\Delta\omega - \frac{\gamma + \kappa_2}{2} \right) a_2(t) - i\mu a_1(t) + \sqrt{\kappa_2} s_{\text{in}}. \quad (2.34)$$

We assume a steady-state input s_{in} , and define

$$a_1(t) = \tilde{a}_1 + \rho_1(t) \quad (2.35)$$

$$a_2(t) = \tilde{a}_2 + \rho_2(t), \quad (2.36)$$

where $\tilde{a}_{1,2}$ is the steady-state solution that we wish to analyze and $\rho_{1,2}(t)$ are time-dependent perturbations. We insert this into Eqs. (2.33)–(2.34), omitting terms that are quadratic or higher-order in ρ_1 and ρ_2 . The gain-saturation factor simplifies to:

$$\frac{2\gamma}{1 + |a_s|^{-2}(\tilde{a}_1 + \rho_1)(\tilde{a}_1^* + \rho_1^*)} \approx \frac{2\gamma}{1 + |a_s|^{-2}(|\tilde{a}_1|^2 + \tilde{a}_1\rho_1^* + \tilde{a}_1^*\rho_1)} \quad (2.37)$$

$$\approx \frac{2\gamma}{1 + |\tilde{a}_1/a_s|^2} \left[1 - \frac{\tilde{a}_1\rho_1^*(t) + \tilde{a}_1^*\rho_1(t)}{|a_s|^2 + |\tilde{a}_1|^2} \right]. \quad (2.38)$$

The result is a pair of time-dependent equations,

$$\frac{d\rho_1}{dt} = A\rho_1(t) + B\rho_1^*(t) + C\rho_2(t) \quad (2.39)$$

$$\frac{d\rho_2}{dt} = C\rho_1(t) + D\rho_2(t), \quad (2.40)$$

where

$$A = i\Delta\omega - \frac{\gamma + \kappa_1}{2} + \frac{2\gamma}{1 + |\tilde{a}_1/a_s|^2} - \frac{2\gamma |\tilde{a}_1/a_s|^2}{(1 + |\tilde{a}_1/a_s|^2)^2} \quad (2.41)$$

$$B = -\frac{2\gamma \tilde{a}_1^2/|a_s|^2}{(1 + |\tilde{a}_1/a_s|^2)^2} \quad (2.42)$$

$$C = -i\mu \quad (2.43)$$

$$D = i\Delta\omega - \frac{\gamma + \kappa_2}{2}. \quad (2.44)$$

We then assume that the perturbations have the exponential time-dependence

$$\rho_1(t) = u_1 e^{\lambda t} + v_1^* e^{\lambda^* t} \quad (2.45)$$

$$\rho_2(t) = u_2 e^{\lambda t} + v_2^* e^{\lambda^* t}. \quad (2.46)$$

Plugging these into Eqs. (2.39)–(2.44), we derive the matrix equation

$$\begin{bmatrix} A & B & C & 0 \\ B^* & A^* & 0 & C^* \\ C & 0 & D & 0 \\ 0 & C^* & 0 & D^* \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}. \quad (2.47)$$

A stable state must have Lyapunov exponents $\text{Re}(\lambda) < 0$ for all four eigenvalues. For backward transmission, we can derive equations that have exactly the same form as Eqs. (2.39)–(2.47), except that the steady-state amplitudes $\tilde{a}_1$ and $\tilde{a}_2$ must be computed using Eqs. (2.20)–(2.21).

As discussed in Section 2.4.1, there is a domain in parameter space where the coupled-mode equations admit three solutions. The Lyapunov exponents indicate that the highest-intensity solutions are asymptotically stable. Fig. 2.5(a) plots the Lyapunov exponents for the highest-intensity solutions versus the detuning $\Delta\omega$ (in the parts of this plot that lie outside the three-solution domain, the highest-intensity solution is the only one). The exponents become negative within a frequency band centered around $\Delta\omega = 0$. This agrees with Fig. 2.4(a)–(b). The discontinuity in the $\mu = 0.1$ curve results from crossing into the three-solution domain, whereupon a new branch of asymptotically stable solutions become the highest-intensity solutions. For $\mu > \gamma$, the solution is asymptotically stable for all $\Delta\omega$. As for the lower-intensity solutions, they are partially unstable, with one or more exponents satisfying $\text{Re}(\lambda) > 0$.

Figure 2.5: (a) Lyapunov exponents for the highest-intensity steady-state solution under forward transmission, versus detuning $\Delta\omega$. Results are shown for $\mu \in \{0.1, 0.2, 0.3, 0.4\}$. The other model parameters are fixed at $\gamma = 0.4$ and $s_{\text{in}} = 0.5$. (b) Bounds of the asymptotic stability region under forward transmission, for several values of the amplitude s_{in} , with fixed $\gamma = 0.4$ and $a_s = 3$. The bandwidth of the asymptotic stability region increases with μ , and diverges at $\mu = \gamma$, which is the *PT* transition point of the linear system. (c) Bounds of the asymptotic stability region under backward transmission, with the same model parameters.

Figure 2.6: Time-dependent mode amplitudes under (a) forward transmission for $\Delta\omega = 0$ (three-solution domain), (b) forward transmission for $\Delta\omega = 0.2$ (one-solution domain), (c) backward transmission for $\Delta\omega = 0$ (three-solution domain), and (d) backward transmission for $\Delta\omega = 0.2$ (one-solution domain). The other model parameters are $\mu = 0.1$, $\gamma = 0.4$, $a_s = 3.0$, and $s_{\text{in}} = 0.5$. We start each simulation with initial conditions perturbed from a steady-state solution by $\delta a_1 = \delta a_2 = 0.001$. In the three-solution domain, perturbing the two lower-intensity solutions causes the system to evolve to the highest-intensity steady-state, which is asymptotically stable.

Figure 2.7: Beating amplitudes $\Delta|a_{1,2}|^2/|a_s|^2$, defined as the difference of the maximum and minimum values of $|a_{1,2}(t)|^2/|a_s|^2$ over time t , versus the PT -breaking parameter μ/γ . The amplitudes $a_{1,2}(t)$ are solved numerically using the full time-dependent coupled-mode equations, using $\Delta\omega = 0.5$, $\gamma = 0.4$, $s_{\text{in}} = 0.5$, and $a_s = 3$.

To verify these results, we solve the time-domain coupled-mode equations numerically (using the LSODE solver). Fig. 2.6(a) and (c) show the time-dependent intensities, under forward and backward transmission, within the three-solution domain ($\mu = 0.1$ and $\Delta\omega = 0.0$, with $a_s = 3.0$ $s_{\text{in}} = 0.5$ as before). Perturbations to the lower-intensities steady states cause the system to evolve into the highest-intensity steady state, as expected from the stability analysis.

In the single-solution domain, the steady-state solution loses its stability at large detunings, as indicated in Fig. 2.4(a)–(b). This occurs through a Hopf bifurcation [123]: small perturbations away from the steady state induce a limit cycle, i.e. a self-sustained oscillation in the mode amplitudes, as shown in Fig. 2.6. The oscillation’s mid-point coincides roughly with the real part of the unphysical complex root of Eq. (2.30).

Fig. 2.7 shows the beating amplitude of $\Delta|a_{1,2}|^2/|a_s|^2$, versus μ/γ . The system is detuned so that $\Delta\omega = 0.5$. For small μ/γ , the beating is non-zero, but at $\mu \approx \gamma$ the system crosses the asymptotic stability boundary and reaches a steady state where $\Delta|a_{1,2}|^2/|a_s|^2 = 0$. This is another interesting link between the coupled-mode dynamics and the PT transition.

To summarize, the highest-intensity solution in the three-solution domain is asymp-

totically stable (i.e., the Lyapunov exponents are all negative). The two lower-intensity solutions are *unstable*: small perturbations from these steady states eventually evolve into the highest-intensity state. In the one-solution domain, the solution is asymptotically stable for small detuning $\Delta\omega$, and unstable for large $\Delta\omega$.

Interestingly, the region of asymptotic stability in the nonlinear system is closely connected to the PT symmetry phases of the linear system. For $\mu < \gamma$, which corresponds to the PT -broken phase, the frequency range of asymptotic stability is bounded by the solid and dashed red curves shown in Fig. 2.4(a)–(b). These bounds diverge at $\mu = \gamma$, which corresponds to the transition from the PT -broken to the PT -symmetric phase in the linear system. For $\mu > \gamma$, the steady state solution is asymptotically stable for all $\Delta\omega$.

In the one-solution domain, the onset of asymptotic instability (at large $\Delta\omega$) is associated with the appearance of sustained time-domain beating in both the resonator intensities and the transmittance. This is a Hopf bifurcation [123] from a stable state to limit cycle behavior (see Fig. 2.6).

To my knowledge, this connection between the PT -breaking transition and the asymptotic stability of gain-saturated dynamical equations had not previously been known, prior to the publication of the work described here[95].

2.4.2 Isolation ratios at zero detuning

Let us now focus on zero detuning, $\Delta\omega = 0$. In this case, the analysis of Sec.2.4.1 tells us that there is always an asymptotically stable steady-state solution. We can derive an interesting connection to the PT transition of the linear system. The variables α

and β , defined in Eq. (2.31), simplify to

$$\alpha = 1 + \left(\frac{\mu}{\gamma}\right)^2 \quad (2.48)$$

$$\beta = \frac{|s_{\text{in}}/a_s|^2}{\gamma} \times \begin{cases} \left(\frac{\mu}{\gamma}\right)^2, & \text{(Forward)} \\ 1, & \text{(Backward)}. \end{cases} \quad (2.49)$$

Hence, the cubic polynomial in Eq. (2.30) is entirely determined by two quantities: (i) μ/γ and (ii) $|s_{\text{in}}/a_s|^2/\gamma$. The first quantity is also the tuning parameter for the *PT* transition. The second quantity determines the strength of the input relative to the gain saturation threshold. We are particularly interested in the “weak-input” limit, defined as

$$s_{\text{in}} \ll \sqrt{\gamma} a_s. \quad (2.50)$$

When $\beta \ll 1$, the steady state behavior will be principally determined by the *PT*-tuning parameter μ/γ .

Fig. 2.8 plots the isolation ratio $R \equiv I_F/I_B$ versus μ/γ , for several different values of γ and s_{in} . In the weak-input regime, the isolation ratio curves are almost identical for different γ , which verifies that the system is controlled by the combination μ/γ . For $\mu/\gamma < 1$, corresponding to the *PT*-broken phase of the linear system, we find that $R > 1$, and hence the system functions as a good optical isolator. For $\mu/\gamma > 1$, we find that $R \approx 1$. This agrees with the qualitative behaviors reported in [26]: when the system is operating in *PT* symmetry unbroken regime, no isolation behavior is observed while for *PT* symmetry broken regime, isolation behavior is observed.

Let us examine the vicinity of the transition point in greater detail. Fig. 2.8(c) shows that in the weak input regime, the isolation ratio curve exhibits a kink at

Figure 2.8: Isolation ratio versus μ/γ at zero frequency detuning ($\Delta\omega = 0$), for (a) weak inputs $s_{\text{in}} = 0.15$ and $a_s = 3$, and (b) strong inputs regime $s_{\text{in}} = 9$ and $a_s = 3$, using several choices of γ . In the weak-input regime, the isolation ratio is mainly determined by the PT -breaking parameter μ/γ . The system becomes reciprocal for $\mu/\gamma > 1$, corresponding to the PT -symmetric phase of the linear system. (c) Close-up of the isolation ratio behavior in the weak-input regime, showing the kink in the dependence on μ/γ at the PT transition point $\mu/\gamma = 1$. Circles show exact numerical solutions of the coupled-mode equations, and the solid curve shows the analytic approximations of Eqs. (2.53)–(2.54).

$\mu/\gamma = 1$. To understand this, we return to the definition of the isolation ratio:

$$R = \frac{I_F}{I_B} = (\mu/\gamma)^{-2} \frac{x_F}{x_B}, \quad (2.51)$$

where x_F and x_B are the solutions to Eq. (2.30) for the forward and backward transmission cases. For $\beta \rightarrow 0$, Eq. (2.30) reduces to

$$x \left(x - \frac{1 - (\mu/\gamma)^2}{1 + (\mu/\gamma)^2} \right)^2 \approx 0. \quad (2.52)$$

For $\mu/\gamma < 1$, the double-root in Eq. (2.52) is positive. Hence, in this approximation, the three-solution domain discussed in Section 2.4.1 extends over the entire range $\mu/\gamma < 1$ along the zero-detuning line. The asymptotically stable solution corresponds to the double-root, which is equal for forward and backward transmission, to lowest order in β . Hence, we can use Eq. (2.51) to show that

$$R \approx (\mu/\gamma)^{-2} \quad \text{for } \mu/\gamma < 1, \quad s_{\text{in}} \ll \sqrt{\gamma} a_s. \quad (2.53)$$

For $\mu/\gamma > 1$, the double-root is negative, so the only valid root in the $\beta \rightarrow 0$ limit is $x = 0$. For non-zero β , this root becomes $\mathcal{O}(\beta)$, so Eq. (2.49) implies that $x_F/x_B \approx (\mu/\gamma)^2$. This yields the isolation ratio

$$R \approx 1 \quad \text{for } \mu/\gamma > 1, \quad s_{\text{in}} \ll \sqrt{\gamma} a_s. \quad (2.54)$$

The limiting expressions (2.53)–(2.54) are plotted in Fig. 2.8(c), and agree well with the numerical solutions. This helps explain why the PT phase of the linear system affects the isolation functionality of the nonlinear system. Both phenomena are determined by the parameter μ/γ , with a critical point at $\mu/\gamma = 1$. The kink in the isolation ratio arises from switching solution branches at the critical point.

2.4.3 Imbalanced input/output couplings

Thus far, we have assumed that the waveguide-resonator couplings, κ_1 and κ_2 , are equal. If the couplings are unequal, the isolation behavior of the system can be quite different. To study this, we replace Eq. (2.25) with

$$\gamma_1 + \kappa_1 = \gamma_2 + \kappa_2 = 2\gamma. \quad (2.55)$$

For $g_0 = 4\gamma$, the gain in resonator 1 is

$$g = \frac{2\gamma}{1 + |a_1/a_s|^2} - \gamma, \quad (2.56)$$

which ensures that the decoupled system remains PT symmetric with critical point $\mu = \gamma$, as before. With this generalization, the steady-state equations (2.30)–(2.32) are altered only by the replacements

$$\begin{aligned} \beta &\rightarrow \frac{\kappa_2}{\gamma}\beta \quad (\text{Forward}) \\ \beta &\rightarrow \frac{\kappa_1}{\gamma}\beta \quad (\text{Backward}). \end{aligned} \quad (2.57)$$

By varying the couplings and losses so that Eq. (2.55) is satisfied, we can access different values of κ_1/κ_2 , subject to the constraint $0 < \kappa_1, \kappa_2 < 2\gamma$.

The discussion of Section 2.4.2 generalizes to this case in a straightforward way. Using the previous zero-detuning and weak-input assumptions, we find that $x_F/x_B \approx 1$ for $\mu/\gamma < 1$, as before; but for $\mu/\gamma > 1$, Eq. (2.57) gives $x_F/x_B \approx \beta_F/\beta_B =$

Figure 2.9: Isolation ratio versus μ/γ for different microcavity-waveguide coupling rates. The system parameters are $\Delta\omega = 0$, $s_{\text{in}} = 0.03$, $a_s = 3$, $\gamma = 1$, and $g_0 = 4\gamma$. Thin solid lines show the analytic approximation in the weak-input limit ($s_{\text{in}} \ll \sqrt{\gamma}a_s$), given by Eq. (2.58).

$(\kappa_2/\kappa_1)(\mu/\gamma)^2$. The isolation ratio now becomes

$$R = (\kappa_1/\kappa_2)(\mu/\gamma)^{-2}x_F/x_B \approx \begin{cases} (\kappa_1/\kappa_2)(\mu/\gamma)^{-2} & \text{for } \mu/\gamma < 1 \\ 1 & \text{for } \mu/\gamma > 1. \end{cases} \quad (2.58)$$

For $\kappa_1 \neq \kappa_2$, this predicts a discontinuity in the isolation ratio at $\mu/\gamma = 1$.

Fig. 2.9 plots dependence of the isolation ratios on μ/γ , at zero detuning, for the cases of (i) $\kappa_1 \ll \kappa_2$, (ii) $\kappa_1 \gg \kappa_2$, and (iii) $\kappa_1 = \kappa_2$. In all three cases, the isolation ratio approaches unity for $\mu/\gamma > 1$. However, for $\mu/\gamma < 1$, the unequal-coupling curves exhibit an abrupt change corresponding to a factor of κ_1/κ_2 (which is two orders of magnitude for these examples). Interestingly, for $\kappa_1 \ll \kappa_2$, the isolation ratio in fact *decreases* below unity, before increasing again as $\mu/\gamma \rightarrow 0$. The numerical results match Eq. (2.58) very well.

This phenomenon may be exploited in device applications for realizing an actively switchable optical isolator. Using a small variation in the μ/γ parameter (e.g., by

varying the inter-cavity separation, which affects μ , we can switch between strong optically isolating and reciprocal regimes.

2.4.4 Effect of a simultaneous reflected wave

We have analyzed the nonlinear system and its isolation ratio under the assumption that light propagates in one direction at a time (i.e., forward or backward). This is a good assumption if the isolator is part of an optical circuit operating with optical pulses, such that any reflected pulse re-entering the isolator (due to scattering from other parts of the circuit) does so at a later time, after the initial pulse has already died away. When forward and backward waves are simultaneously present, however, both contribute to the nonlinearity, causing the isolator to fail. This is a general limitation of optical isolators based on nonlinearity [87].

In order to model simultaneous forward and backward waves, we modify the coupled-mode equations to include resonator modes with the opposite circulation. Similar to the previous “forward” configuration, we suppose the main incident wave enters at port 1 with amplitude s_f . In addition, there is a back-propagating wave, incident at port 4 with amplitude s_b , which eventually exits at port 1. The modified equations are (taking $\Delta\omega = 0$ for simplicity):

$$\frac{da_1}{dt} = ga_1 - i\mu a_2 \quad (2.59)$$

$$\frac{da_2}{dt} = -\gamma a_2 - i\mu a_1 + \sqrt{\kappa_2} s_f \quad (2.60)$$

$$\frac{da'_1}{dt} = ga'_1 - i\mu a'_2 + \sqrt{\kappa_1} s_b \quad (2.61)$$

$$\frac{da'_2}{dt} = -\gamma a'_2 - i\mu a'_1. \quad (2.62)$$

The opposite-circulation mode amplitudes are denoted by a'_1 and a'_2 . The back-propagating wave enters into the final term on the right-hand side of Eq. (2.61),

Figure 2.10: Transmittance T_b of a back-propagating wave, versus the normalized backward incident power $|s_b|^2/\gamma|a_s|^2$. A forward-propagating wave with $|s_f|^2/\gamma|a_s|^2 = 10^{-4}$ is simultaneously present; the other parameters are $a_s = 3$, $\gamma = \kappa_1 = 1$, and $\kappa_2 = 0.01\gamma$. Note that T_b can exceed unity because of the presence of gain in the system; this feature can be suppressed if desired by adding loss to the waveguide leads.

coupling to the a'_1 mode. Both modes in the gain resonator now contribute to the gain saturation, so

$$g = \frac{2\gamma}{1 + |a_1/a_s|^2 + |a'_1/a_s|^2} - \gamma. \quad (2.63)$$

Instead of the isolation ratio, we now consider the transmittance of the backward wave:

$$T_b = \frac{\kappa_2 |a'_2|^2}{|s_b|^2}. \quad (2.64)$$

versus the normalized backward incident power, with fixed forward incident power. The features of this plot can be understood as follows: the functioning of the isolator requires the backward wave to cause gain saturation, but when $|s_b|^2$ is too small, the gain saturation is dominated by the forward wave. Thus, for small $|s_b|^2$ we find that T_b is approximately constant; in fact, it equals the forward transmittance at the power level $|s_f|^2$. As $|s_b|^2$ increases, the backward wave starts to affect the gain saturation, and isolation behavior appears in the form of a decrease in T_b . However, this is only apparent for $\mu < \gamma$ (corresponding to the PT -broken regime of the linear system), since it is in this regime that the isolation ratio deviates from unity.

2.5 Summary

We have analyzed the relationship between the linear and nonlinear behaviors of dual microcavity resonators with gain and loss. Previous analysis had been limited by the fact that the PT transition can only be defined for the linear system. However, we were able to show that the transition corresponds closely with the dynamical and steady-state behaviors of the gain-saturated nonlinear system, for which PT symmetry does not strictly apply.

One connection involves the asymptotic stability of the gain-saturated dynamical system. For $\mu > \gamma$, corresponding to the linear system's " PT -symmetric" phase, the resonances are always asymptotically stable. But for $\mu < \gamma$, corresponding to the linear system's " PT -broken" phase, the coupled-mode dynamics become unstable at sufficiently large frequency detunings, leading to self-sustained oscillations. If steady-state operation is desired, our analysis shows that it is preferable to adopt zero detuning.

We have also been able to connect the PT transition with the performance of the nonlinear system as an optical isolator. Using the "weak-input" approximation, we derived a kink in the isolation ratio at the critical point $\mu = \gamma$, which corresponds to the PT transition point of the linear system. Upon relaxing the constraint of equal waveguide-port couplings, this kink turns into a discontinuity, meaning that the isolation ratios vary extremely quickly with μ/γ in the vicinity of the critical point. This could be useful for using the inter-resonator coupling as an active control parameter.

We assumed that light propagates in one direction at a time, either forward or backwards, which holds for pulsed optical circuits where reflections appear at a later time. If both forward and backward waves are simultaneously present, however, the device will fail to act as a nonlinear isolator if the backward wave is too weak.

This type of PT symmetric optical isolator is technologically promising due to its on-chip design, and the record low input power required for isolation of around $1\mu\text{W}$. The main drawback is that the isolation occurs over a relatively narrow frequency bandwidth, due to the reliance on high-quality-factor resonators.

Following is a comparison of key parameters between nonlinear PT symmetric optical isolator and other types of nonlinear optical isolator:

	PT symmetric isolator	Other types of nonlinear isolator
Isolation ratio	10dB	28dB
Size	$\sim 30\mu\text{m}$	$\sim 5\mu\text{m}$
Operating power	$\sim 1\mu\text{W}$	$\sim 85\mu\text{W}$
Operating wavelength	1550nm	1630nm

Chapter 3

Optical isolation with nonlinear topological photonics

Most of the material presented in this chapter was previously published in the journal New Journal of Physics [35]. Wherever figures and materials from other sources have been taken; the authors have been given due credit.

In this chapter, we explore the possibility of realizing optical isolators using nonlinearity-induced topological phase transitions. The concept of topological phases originated in the field of condensed matter physics [124], and was introduced into photonics some years ago [22, 61–63, 125, 71, 126–130]. Researchers have demonstrated a variety of photonic structures with topologically nontrivial photonic bands, including magneto-optic photonic crystals operating at microwave frequencies [61–63, 125] and non-magneto-optic waveguide structures that can operate at optical frequencies [71, 126, 130]. Such structures possess a distinctive property: when tuned into a “topologically nontrivial” phase, they exhibit topological edge states that are robust against perturbations (and, in some cases, have useful properties such as unidirectionality [125]). Although these topological edge states have mostly been studied in the linear regime, there have been several recent papers exploring how they are affected by optical nonlinearities [131–135]. It appears that photonic topological

transitions—transitions from a conventional or topologically trivial phase to a topologically nontrivial phase—can be “driven” by nonlinearities, so that the light intensity itself determines whether the light can propagate via an edge state. We will study how this behavior might be exploited in nonlinear optical isolators. There have also been a number of papers seeking to implement optical isolators using magneto-optic topological photonics [34, 136], but such schemes lie outside the scope of the present discussion.

We will analyze three representative photonic designs that (i) are known to exhibit topological transitions in the linear regime, and (ii) can feasibly operate in the optical frequency range, where optical isolation is a particularly pressing problem due to the absence of strong magneto-optic effects [72, 34, 136]. Our goal is to obtain a conceptual understanding of the features and limitations of these novel isolation schemes; as such, we will make use of simplified models, based on the coupled-mode theory and transfer matrix frameworks, capturing just the essentials of nonlinearity and bandstructure topology. In particular, we will not attempt to study the actual device geometries and material nonlinearities needed to achieve the nonlinear lattice parameters appearing in our models, nor to optimize our designs to maximize their performance.

3.1 Introduction to topological photonics

Topology is a field of mathematics that studies the property of objects that do not change under continuous deformations, such as bending or stretching. For instance, a physical three dimensional solid object can be described by a topological index, an integer called the “genus”. Thus, a ball has a genus of 0, and a donut has a genus of 1; these two objects can not transform continuously into each other.

The idea of topology was introduced to condensed matter physics with the discovery of a new phase of matter, the topological insulator[9–11]. The motion of

Figure 3.1: (a) TRS broken photonic topological system made of gyromagnetic photonic crystal, unidirectional backscattering-immune topological electromagnetic states were observed. (b) TRS unbroken photonic topological system made of coupled optical ring resonators, topological edge states of light in a two-dimensional system were demonstrated. (c) Floquet photonic topological system made of coupled optical waveguide arrays, topologically protected propagation of visible light is observed. Figure reproduced from Ref.[63, 126, 71].

electrons is forbidden inside the bulk of topological insulator, but they can move on the surface; in certain classes of two dimensional topological insulators, this motion is unidirectional and robust against backscattering, even in the presence of disorder.

Topological insulators arise as a consequence of the wave nature of electrons, and light as a wave can also possess topological states. In 2008, Haldane[61, 62] proposed using a time reversal symmetry(TRS) broken photonic crystal to produce a photonic analogue of quantum Hall effect. Since then, numerous studies of topological states in photonic system have been carried out. In 2009, a group in MIT experimentally demonstrated a photonic analogue of a quantum Hall insulator using a gyromagnetic photonic crystal[63], including the phenomenon of unidirectional propagation of light along the edge. Later, TRS unbroken photonic topological systems were designed using coupled resonators, metacrystals and waveguide arrays. By utilizing spatial modulation along the propagation direction in a waveguide array, Rechtsman *et al.* showed how to generate an effective gauge field for light, leading to the realization of Floquet photonic topological insulator[71] in the optical frequency domain. A group in Maryland designed[126] a lattice of coupled optical ring resonators in which a gauge field is realized via differential optical path lengths in auxiliary coupling rings. Edge states at near-infrared($1.55\mu m$) wavelengths were observed. Another interesting approach involved metacrystals with hexagonal symmetry [129] made of materials with a non-vanishing bianisotropic tensor($\chi \neq 0$). In this design, a pseudo spin degree of freedom is represented by linear combinations of transverse electric and transverse magnetic fields and a quantum anomalous Hall phase can be achieved in each ‘spin’ sector.

3.2 Nonlinear coupled waveguide arrays: 1D SSH model

The first type of structure we will study is an array of coupled optical waveguides, where light is guided (“evolves”) either forward or backward along each waveguide, and can hop between adjacent waveguides via evanescent coupling. We begin with an exemplary waveguide array corresponding to a 1D SSH model [124], with Kerr-like nonlinearities added to the inter-waveguide coupling strengths. Hadad, Khanikaev, and Alù have shown that such a model can exhibit a self-induced topological transition [134], in which the nonlinearity drives a local region of the lattice into a different topological phase, giving rise to self-trapped soliton-like edge states. These nonlinear edge states allow a high-intensity signal injected into a edge waveguide to resist diffraction into the rest of the lattice.

We show that when such a lattice has asymmetric input and output coupling losses, it can function as an efficient optical isolator. Light is injected into one port of an edge waveguide, evolves through a fixed distance, and leaves at the other end of the same waveguide. With appropriately-chosen system parameters, the forward transmittance (via a self-induced topological edge state) is of order unity, while the backward transmittance (without an edge state) is suppressed by several orders of magnitude.

The isolator relies on the fact that the self-induced topological transition has a power threshold—i.e., the soliton-like edge state appears only above a certain power. The asymmetric input and output couplings ensure that the edge state exists under forward transmission, but not backward transmission. However, the existence of a threshold is not unique to the nonlinear SSH model. It has previously been shown that nonlinear lattices with a conventional design can support edge solitons with a power threshold [137–139], distinct from bulk solitons which normally bifurcate from zero

power [140]. In those studies [137–139], the edge soliton thresholds had no apparent connection to the topology of the underlying bandstructure. In the present case, the threshold—and hence the operating power of the isolator—is set by the topological phase transition of the SSH model.

We begin our study with a nonlinear version of the 1D SSH model [134], which is the simplest model to exhibit topological modes. As shown schematically in Fig. 3.2(a), such a model can be implemented with a 1D array of waveguides with nonlinear couplings. Each unit cell consists of two waveguides, with identical wave-guiding characteristics. We take the tight-binding (coupled-mode) approximation, which applies to guided modes moving in one direction along the axial direction z , without backscattering. We let $a_n(z)$, $b_n(z)$ denote the complex wave amplitudes in the two waveguides of unit cell n . With an appropriate gauge choice, these obey the coupled-mode equations [140]

$$i \frac{da_n}{dz} = \kappa_1^n b_n + \kappa_2^{n-1} b_{n-1} \quad (3.1)$$

$$i \frac{db_n}{dz} = \kappa_1^n a_n + \kappa_2^n a_{n+1}, \quad (3.2)$$

where $\kappa_1^n, \kappa_2^n \in \mathbb{R}^+$ are intra-cell and inter-cell coupling coefficients. In the linear regime, the κ parameters are constants independent of n . Chiral symmetry is preserved in the linear SSH model, as can be seen by adding a minus sign to the amplitudes on alternate sublattices.

The linear SSH model has been extensively investigated in photonics, including in femtosecond-laser-written waveguide arrays [141, 142] and plasmonic waveguide arrays [143]. It has a phase transition at $\kappa_1 = \kappa_2$; when $\kappa_1 < \kappa_2$, there is an edge state with zero eigenvalue localized to the left edge of the lattice. We can observe this by exciting the leftmost site (waveguide 0) with input power $I = |a_0(0)|^2$, and letting the state evolve up to a fixed distance Z . As shown in Fig. 3.2(b)–(c), the

excitation diffracts into a superposition of bulk modes for $\kappa_2/\kappa_1 < 1$, but remains confined to the edge for $\kappa_2/\kappa_1 > 1$. In Fig. 3.2(d), we plot the transmittance on the edge waveguide, defined as $\mathcal{T} = |a_0(Z)|^2/I$. This is seen to closely track the edge intensity of the exact edge eigenstate (the normalized eigenstate of the linear Hamiltonian having eigenvalue zero).

We now introduce nonlinearity into the model. In accordance with the goals of this study, we will choose a nonlinearity that is conceptually simple and easy to model, leaving aside the question of how best to physically implement it. The inter-cell coupling coefficient is made dependent on the local intensity, as follows [134]:

$$\kappa_2^n(z) = \kappa_0 + \alpha \left(|a_{n+1}(z)|^2 + |b_n(z)|^2 \right), \quad (3.3)$$

where κ_0 stands for static inter-cell coupling, and α is a Kerr-like coefficient multiplying the sum of the intensities in the two coupled sites. We will take $\kappa_1 = 1.0$, $\kappa_0 = 0.5$, and $\alpha > 0$, i.e. the inter-cell coupling becomes stronger at higher intensities. Thus, the bandstructure is topologically trivial in the linear (zero-intensity) limit, but increasing the intensity will (roughly speaking) drive it into the nontrivial phase. Without loss of generality, we set $\alpha = 1$ (other values are equivalent up to a rescaling of intensities).

The nonlinearity allows for the possibility of a self-induced topological transition [134]. Suppose we prepare an initial state by exciting just the left edge waveguide with input power $I = |a_0(0)|^2$. The light undergoes nonlinear evolution for distance Z , and we compute the transmittance $\mathcal{T}(I) = |a_0(Z)|^2/I$. The results are shown in Fig. 3.3(a). For small I , $\mathcal{T}(I)$ is close to zero (similar to the linear case, the light mostly diffracts into the lattice bulk); but for $I \gtrsim 9$, $\mathcal{T}(I)$ abruptly (but continuously) increases towards unity.

This abrupt change in the nonlinear transmittance is related to a self-induced topological transition. To see this, in Fig. 3.3(b) we plot the averaged values of κ_2/κ_1

Figure 3.2: (a) Schematic of SSH model realized by an array of coupled waveguides. (b,c) Mode intensity distributions in a linear SSH lattice of 29 sites, generated by an excitation on the left-most site (waveguide 1). Results are computed numerically using the z -evolution operator. (b) For a “trivial” lattice with $\kappa_1 = 1$ and $\kappa_2 = 0.2$, the excitation diffracts into the bulk. (c) For a “nontrivial” lattice with $\kappa_1 = 1$ and $\kappa_2 = 2.5$, the excitation is partly confined to the edge. (d) For a “nontrivial” lattice with $\kappa_1 = 1$ and $\kappa_2 = 5$, the excitation is confined to the edge. (e) Transmittance on waveguide 1 (blue line) versus κ_2/κ_1 . The intensity of the edge eigenmode on waveguide 1, $|a_1|^2$, is also shown (green dashes). In (b)–(d), the input/output couplings η_A and η_B are set to unity.

Figure 3.3: Behavior of the nonlinear SSH lattice as an optical isolator. (a) Nonlinear transmittance along waveguide 0, $\mathcal{T}$, versus input power I . The lattice has 29 sites, with $\eta_A = 1$, $\eta_B = 0.5$, $\kappa_1 = 1$, $\kappa_0 = 0.5$, and $\alpha = 1$; the output is at $Z = 20$. Results are computed numerically using evolution operators with step size $\delta z = 10^{-3}$. (b) Averaged values of κ_2/κ_1 versus input power I , where the nonlinear κ_2 is averaged over $18 \leq z \leq 22$ (i.e., around the output Z) and over a few sites (4, 8, or 12) closest to the left lattice edge. The topological transition point of the linear SSH model, $\kappa_2 = \kappa_1$, is indicated by the horizontal dashes. (c) Isolation ratio (IR), forward transmittance T_f , and backward transmittance T_b , versus input power I . Horizontal dashes show the approximate upper bounds $T_{f,b}^{\max} = \eta_A^2 \eta_B^2$ and $\text{IR}^{\max} \approx 1/T(0)$.

as a function of I (since κ_2 varies between sites and also with z , the averages shown here are taken over different numbers of unit cells near the left lattice edge, and over a fixed range of z). We find that the increase in $\mathcal{T}(I)$ starts to occur when $\kappa_2/\kappa_1 \approx 1$, which is precisely the topological transition point of the SSH model. Note that this correspondence is apparent even when we average κ_2 over 12 sites, a relatively far distance from the edge.

The regime of high nonlinear transmittance is due to the self-induced edge soliton described by Hadad *et al.* [134]. This is a nonlinear mode that inherits some properties of the linear SSH model’s edge state, such as leaving the b_n sites unexcited. Its onset also closely matches the topological transition of the SSH model. However, it differs in other ways: its eigenvalue is not pinned to zero (as the SSH model’s “particle-hole” symmetry is broken by the nonlinearity), and the intensity goes to a small constant in the bulk rather than decaying exponentially with distance from the edge [134].

A nonlinear photonic structure with strongly intensity-dependent transmittance can serve as the basis for an optical isolator [26, 27, 95, 144, 145]. To accomplish this, we introduce couplings η_A and η_B , as shown schematically in Fig. 3.2(a). In forward-transmission mode, light is coupled into port A on an edge waveguide, is guided in the $+z$ direction, and is out-coupled at port B of the same edge waveguide. In backward-transmission mode, light enters at B , is guided in the $-z$ direction, and is out-coupled at port A . (Note that this is distinct from “asymmetric light transmission” schemes, such as those studied in [146], where light propagates either left-to-right or right-to-left along the lattice, while being guided in a single axial direction $+z$; those schemes have no direct bearing on the problem of optical isolation, as they do not swap physical input and output ports.) In forward-transmission mode, the input light has intensity I , and the intensity coupled into the edge waveguide is $\eta_A^2 I$; the intensity at the end of the waveguide is $\mathcal{T}(\eta_A^2 I) \eta_B^2 I$, where $\mathcal{T}$ is the nonlinear transmittance of the lattice

itself. The overall forward transmittance is thus

$$T_f = \eta_A^2 \eta_B^2 \mathcal{T}(\eta_A^2 I). \quad (3.4)$$

Similarly, the backward transmittance (in the $-z$ direction) is

$$T_b = \eta_A^2 \eta_B^2 \mathcal{T}(\eta_B^2 I). \quad (3.5)$$

Note that $T_f = T_b$ in the linear regime $I = 0$, in accordance with the reciprocity principle. The isolation ratio is defined by

$$\text{IR} \equiv T_f/T_b = \frac{\mathcal{T}(\eta_A I)}{\mathcal{T}(\eta_B I)}. \quad (3.6)$$

Fig. 3.3(c) shows numerical results for the forward and backward transmission, and the isolation ratio, for $\eta_A = 1$ and $\eta_B = 0.5$. We can understand these results qualitatively by using Eqs. (3.4)–(3.5) and the features of the nonlinear waveguide transmittance $\mathcal{T}(I)$. First, note that the upper bound for the transmittance (in either direction) is $T_{f,b}^{\max} \leq \eta_A^2 \eta_B^2$, which occurs when the light is transmitted predominantly along the edge, with losses only at the input and output ports. As I is increased from zero, the transmittances increase exponentially from the very low value of $\eta_A^2 \eta_B^2 \mathcal{T}(0)$, before saturating near the upper bound. Since $\eta_A \neq \eta_B$, the initial increase of T_f and T_b occur with different exponential factors, and as a result the IR increases exponentially with the input power I . It reaches a maximum of $\text{IR}^{\max} \approx 1/\mathcal{T}(0)$, where $\mathcal{T}(0)$ is the waveguide transmittance in the linear regime. This corresponds to the case where T_f has saturated but not T_b . The results in Fig. 3.3(c) show good agreement with these approximate bounds.

We can compare these results to optical isolation schemes based on "non-topological" edge solitons, such as solitons induced by on-site Kerr nonlinearity [137–139]. Such

solitons also exhibit a power threshold proportional to the coupling κ . Optimizing the isolation ratio in a device of fixed length thus requires a trade-off between minimizing the linear transmittance (larger κ), or minimizing the threshold power (smaller κ). In contrast, in the nonlinear SSH model one can reduce the linear transmittance by increasing $\kappa_{0,1}$ without substantially effecting the threshold power, determined by $\kappa_1 - \kappa_0$. In future studies, the staggered profile of the topological edge soliton could be exploited to enable further optimization of the performance, for example by incorporating lossy elements onto the unexcited b_n sites to further reduce the backward transmission [34].

3.3 Nonlinear coupled waveguide arrays: Haldane model

We generalize the results to a structure exhibiting a 2D topological phase [71]. Unlike the 1D SSH model, the 2D lattice has edge solitons that are mobile, and can travel around defects such as corners [135]. In this case, to achieve strong optical isolation, the input and output must correspond to different (spatially separated) waveguides, chosen according to the nonzero velocity of the edge soliton. Similar to the 1D case, we find that the soliton has a power threshold determined by the underlying linear model’s topological phase transition—in this case, a transition from conventional insulator to Chern insulator. This shows that the soliton-based optical isolation scheme can be extended to more complex 2D lattices. Unlike in 1D topological phases, topologically nontrivial behavior in 2D can occur without requiring any special lattice symmetries, and is thus more robust.

In the linear regime, it has previously been shown that a 2D optical waveguide array can be made to act as a 2D topological insulator with broken “time-reversal” symmetry (a Chern insulator) by adding a helical twist to the waveguides [71]. A

Figure 3.4: (a) Intensities after evolution through a nonlinear Haldane lattice, with parameters $t_1 = 1$, $t_2 = 1/3$, $\phi = \pi/2$, and $m_0 = 2$. A corner site (A) is initially excited with intensity I , and colors indicate site intensities $|a_\mu|^2$ and $|b_\nu|^2$ (with arbitrary normalization) after propagation through $Z = 19.2$. Three cases are shown: $I = 1$, $I = 30$ and $I = 150$. The input/output couplings η_A and η_B are set to unity. (b) Nonlinear transmittance $\mathcal{T}(I)$ from one corner (A) to the opposite corner (B), versus input power I . (c) Mean values of $\langle m_A \rangle - \langle m_B \rangle$, versus I . The averages are taken over sites up to n nearest-neighbor hops from site B, for $n \in \{1, 2, 3, 4\}$. (d) Isolation ratio (IR), forward transmittance T_f , and backward transmittance T_b , versus I .

variant design has been shown to support tunable topological phase transitions [135]. Here, we leave implementation details to one side, and focus instead on the Haldane model [147], the simplest and most well-known 2D model with a topological phase transition between a conventional insulator phase and a Chern insulator phase [147]. This tight-binding model describes a 2D honeycomb lattice with broken time-reversal symmetry; the honeycomb lattice is divided into two sub-lattices, A and B , with on-site mass terms $m_A = m_0$ and $m_B = -m_0$. The other model parameters are the nearest-neighbor hopping $t_1 \in \mathbb{R}$, the next-nearest-neighbor hopping amplitude $t_2 \in \mathbb{R}$, and an Aharonov-Bohm phase $\phi \in [-\pi, \pi]$ which determines the magnetic flux penetrating sub-regions of each unit cell [147]. The bandstructure is in a Chern insulator phase when

$$|m_0/t_2| < 3\sqrt{3}|\sin \phi|, \quad (3.7)$$

with Chern numbers $C = \text{sgn}(\phi)$. The other phase is a conventional insulator with Chern number $C = 0$.

We now introduce a nonlinearity designed to drive the system through a topological transition. We make the on-site mass terms on the A and B sublattices nonlinear, depending on the local intensity, as follows:

$$\begin{aligned} m_A^\mu &= \frac{m_0}{1 + |a_\mu|^2} \\ m_B^\nu &= \frac{-m_0}{1 + |b_\nu|^2}, \end{aligned} \quad (3.8)$$

where $m_0 \in \mathbb{R}$ is the Haldane model's mass parameter in the linear limit; μ, ν are site indices on the A and B sublattices respectively; and $a_\mu, b_\nu \in \mathbb{C}$ denote the optical wave amplitudes (wavefunction) at those sites. Similar saturable nonlinearities have previously been studied in the context of non-topological photonic lattices with mobile discrete solitons [148]. We will take $\phi = \pi/2$, $t_2 = 1/3$, and $m_0 = 2$, so that according to Eq. (3.7) the linear system is in the trivial insulator phase. With

increasing intensity, the saturable nonlinearity decreases the on-site mass parameters, “driving” the system towards the Chern insulator phase.

The resulting nonlinear propagation is shown in Fig. 3.4(a). Input light is injected onto a single site at a corner of the lattice, with intensity $I = |a_0|^2$, and undergoes nonlinear evolution over a fixed distance Z ; as discussed in Section 3.3, the z axis plays the role of time. For small I , the light diffracts into the bulk of the lattice, in accordance with the fact that the bandstructure of the linear system is topologically trivial. For large I , the light remains tightly confined to the edge, and propagates in one direction along the edge, including around corners, in accordance with the existence of a unidirectional edge state in the Chern insulator phase.

The nonlinear transmittance $\mathcal{T}(I)$ is plotted in Fig. 3.4(b), and it exhibits a power threshold similar to what we observed in the nonlinear SSH model. Here, $\mathcal{T}(I)$ is defined as the transmittance from the input site (A) at $z = 0$, to the output site (B) on the opposite lattice corner at $z = Z$. Again, we can demonstrate a close correspondence between the power threshold and the topological transition of the linear lattice. In Fig. 3.4(c), we plot $\langle m_A \rangle - \langle m_B \rangle$ versus I , where $\langle m_A \rangle$ and $\langle m_B \rangle$ are the nonlinear on-site mass terms averaged over sites closest to the output waveguide at $z = Z$. Based on Eq. (3.7), the linear lattice exhibits a topological transition at $m_A - m_B = |6\sqrt{3}t_2 \sin \phi| = 2\sqrt{3}$; in Fig. 3.4(c), we indeed observe that the nonlinear transmittance’s threshold power occurs as $\langle m_A \rangle - \langle m_B \rangle$ drops below this value. In Fig. 3.4(d), we plot the isolation ratio and forward and backward transmittances. The behavior is similar to the nonlinear SSH results in Fig. 3.3(c). One difference is that T_f and T_b are dissimilar even though the input/output couplings are symmetric (here we set $\eta_A = \eta_B = 1$); this is because the inputs and outputs are on different sites, so the lattice itself provides the asymmetry.

Optical isolation based on traveling edge solitons has important qualitative differences compared to schemes based on immobile solitons, such as the nonlinear SSH

model discussed in Section 3.2. The device length Z and/or the choice of output waveguide must be matched to the edge soliton velocity, so as to ensure a high forward transmittance. This kind of traveling discrete soliton is not easily achievable with “conventional” nonlinear lattice designs not tied to a topological transition; in those cases, traveling solitons require excitation of several waveguides [148], and discrete solitons are typically immobile and/or suffer from strong radiative losses [149].

3.4 Nonlinear coupled ring lattices

The final type of nonlinear isolator that we will study is based on a periodic 2D array of coupled ring-like waveguides [126, 128, 130, 150–152]. Unlike the waveguide array case, which had a separate “evolution” axis z (with light injected at $z = 0$, and undergoing nonlinear evolution up to $z = Z$), here the system is entirely on-chip. Light is in-coupled and out-coupled at two different positions on the lattice edge, and the steady-state solution within the nonlinear lattice is determined self-consistently. Roughly speaking, this is like solving a nonlinear steady-state scattering problem at a fixed “energy”, rather than a nonlinear evolution problem over a fixed “time interval”.

We show that the coupled-ring lattice can also function as an efficient optical isolator, but with substantially different characteristics due to its steady-state nature. The isolation is again based on a topological phase transition in the lattice bandstructure [128, 150], driven by a nonlinearity-induced variation in the effective coupling between rings. Unlike in the waveguide array, the transition manifests as a discontinuity in the nonlinear transmittance: above a critical input power, light propagation switches abruptly from very low transmittance (via bulk states) to very high transmittance (via edge states). Thus, with varying input power, the structure is able to exhibiting a discontinuous jump from very low to very high isolation ratios.

The structure is shown schematically in Fig. 3.5(a), and consists of ring-shaped

waveguides arranged in a 2D square lattice, with adjacent “site rings” connected by auxiliary “coupler rings”. The structure is assumed to be engineered so that there is negligible back-scattering at the coupling regions where neighboring waveguides approach one another; in other words, the circulation of light in the site rings—clockwise or anti-clockwise—is preserved under inter-site hopping [126, 130].

Suppose the entire lattice, including the auxiliary rings, is periodic. In this case, the lattice’s bandstructure is known to exhibit a topological transition [128]: as we increase the effective coupling between site rings, the bandstructure goes from a conventional phase to a topologically nontrivial phase. In the latter, there exists (for each circulation) a family of topological edge states that move unidirectionally along the lattice edge [128, 150–152].

For the waveguide arrays discussed in Section 3.2–3.3, the “forward” and “backward” modes of the optical isolator corresponded to $+z$ and $-z$ propagation. By contrast, the present coupled-ring lattice structure is “on-chip”, i.e. purely 2D. In forward-transmission mode, light is coupled into one ring, propagates through the lattice in a given circulation direction (say, clockwise), and is subsequently out-coupled. In backward-transmission mode, the input and output ports are switched, and hence the in-lattice propagation takes place via the *opposite* circulation (anti-clockwise). It is important to note that this switches the directionality of the topological edge states. For instance, in Fig. 3.5(a) we show a right-moving edge state on the upper edge, with clockwise circulation; the reciprocal partner is a left-moving edge state on the upper edge, with anti-clockwise circulation.

The transmission of light through the lattice can be calculated using transfer matrices [126, 128]. Fig. 3.5(b) shows a coupler ring joining two site rings; the complex

Figure 3.5: (a) Schematic of a lattice of coupled rings, consisting of “site rings” (large circles) arranged in a square lattice, separated by auxiliary “coupler rings” (small circles). Light is coupled in/out of the lattice on the left and right, with coupling parameters θ_A and θ_B . (b) Close-up of an inter-site coupling, showing the definitions of the complex wave amplitudes $\{a, b, c, d\}$ (in the site rings) and $\{\alpha, \beta, \gamma, \delta\}$ (in the coupler ring).

wave amplitudes labeled in this figure are related by

$$\begin{bmatrix} c \\ \gamma \end{bmatrix} = S_c(\theta_0) \begin{bmatrix} a \\ \alpha \end{bmatrix}, \quad \begin{bmatrix} d \\ \delta \end{bmatrix} = S_c(\theta_0) \begin{bmatrix} b \\ \beta \end{bmatrix}, \quad (3.9)$$

where

$$S_c(\theta_0) = \begin{bmatrix} \sin \theta_0 & i \cos \theta_0 \\ i \cos \theta_0 & \sin \theta_0 \end{bmatrix} \quad (3.10)$$

is a unitary 2×2 scattering matrix describing evanescent coupling with energy conservation and negligible back-scattering, as well as 180-degree rotational symmetry [126, 128]. The strength of the evanescent coupling is described by θ_0 . Moreover, wave amplitudes acquire a phase $\xi_{1,2}$ on traversing each arm of the coupler ring:

$$\alpha = e^{i\xi_1} \delta, \quad \beta = e^{i\xi_2} \gamma. \quad (3.11)$$

Figure 3.6: (a)–(b) Band diagrams for a semi-infinite lattice of coupled rings, with width 10 unit cells in the y direction. For the coupler ring parameters, we choose $\theta_0 = \pi/6$ and two different choices of the phase shift ξ ; the resulting bandstructures are (a) topologically trivial for $\xi_1 = \xi_2 = \xi = 0.55\pi$, and (b) topologically nontrivial for $\xi = 0.85\pi$. (c) Numerically-obtained phase diagram of the system for $\theta_0 = \pi/6$. Grey regions show the ungapped parts of the bandstructure, white regions show trivial (T) bandgaps, and pink regions show nontrivial (NT) bandgaps. For fixed $\phi = \pi/4$, the system can move from a trivial to a nontrivial gap by varying $\xi_{1,2}$ as indicated by the arrow. The gaps depend only on ξ_1 and ξ_2 through the combination $\xi_1 + \xi_2$.

The effective coupling between adjacent site rings can be determined [128, 150] from the parameters θ_0 and $\xi_{1,2}$ (which depend, in turn, on the waveguide geometry and operating frequency). We will assume that the couplings in the x and y direction are identical.

The bandstructure is defined in terms of the phase shift ϕ over each quarter of a site ring, as indicated in Fig. 3.5(b). This plays the role of a “quasi-energy” [128, 150], whose value is fixed by the waveguide geometry and operating frequency; we can regard ϕ as being analogous to the chemical potential in a band insulator. (Note

that the other model parameters, θ_0 and $\xi_{1,2}$, will also simultaneously depend on the operating frequency; in designing a real device, these model parameters must be mapped onto physical quantities including the frequency and geometrical parameters.) In Fig. 3.6(a)–(b), we plot band diagrams of ϕ versus k for a semi-infinite lattice with transverse width of 10 unit cells; here, k is the usual Bloch wavenumber, defined as the phase shift produced by translating one unit cell along the strip [128]. We fix $\theta_0 = \pi/6$, and show results for two choices of $\xi = \xi_1 = \xi_2$. Fig. 3.6(a) shows a conventional bandstructure, while Fig. 3.6(b) shows a nontrivial bandstructure with gaps spanned by topological edge states. Evidently, we can switch between these two distinct cases by tuning only ξ , with all other model parameters kept constant. Fig. 3.6(c) shows a phase diagram indicating the parameter choices for observing trivial and nontrivial gaps.

Fig. 3.7(a) shows the transmittance across the linear lattice for fixed $\phi = \pi/4$ and varying coupler ring phase shifts $\xi = \xi_1 = \xi_2$. The transmittance is close to zero for $\xi \approx \pi/2$, when the system is in a trivial gap, and approaches unity when ξ increases and the system enters a topologically nontrivial gap. For intermediate ξ , the system lies in a band, and the transmittance exhibits numerous resonances.

We now introduce a nonlinearity designed to drive the system through the topological transition. Let the phase shift parameter on each arm of a coupler ring be intensity-dependent:

$$\xi = \xi_0 + \kappa \mathcal{I}, \quad (3.12)$$

where ξ_0 is the phase shift in the linear limit, κ is a Kerr coefficient, and $\mathcal{I}$ is the local intensity in the arm of the coupler ring, defined as $\mathcal{I} = |\psi|^2$ where ψ is the local complex amplitude. Physically, this may be accomplished by fabricating the coupler rings out of a nonlinear material (note that the intensity within the coupler rings will be strongly enhanced if they are close to resonance with the operating frequency [128]).

Figure 3.7: (a) Transmittance T through a linear lattice with phase shift ξ in all arms of all coupler rings. The input/output couplings are taken to be ideal (the coupling matrices have the form of Eq. (3.10) with $\theta_A = \theta_B = \pi/2$), and the lattice parameters are $\phi = \pi/4$, and $\theta_0 = \pi/6$. For $\xi \approx \pi/2$, ϕ lies in a trivial gap, and T is close to zero; with increasing $\Delta\xi$, ϕ enters a band (grey region) and T exhibits bulk transmission resonances; for still larger ξ , ϕ enters a nontrivial gap (pink region) and T approaches unity. Vertical dashes indicate the boundaries between the band and gap regions shown in Fig. 3.6(c). (b) Transmittance through the nonlinear lattice versus input power I , using nonlinearity parameters $\xi_0 = \pi/2$ and $\kappa = 1$, with all other parameters kept the same as in (a). This and subsequent results are obtained by solving the nonlinear transfer matrix relations self-consistently via a numerical nonlinear solver. (c) Normalized field intensity distribution at $I = 3.21$, just above the large discontinuity in (b). For clarity, the coupler rings are omitted and only the site ring intensities are shown. (d) Mean phase shifts $\langle \xi \rangle = \langle (\xi_1 + \xi_2)/2 \rangle$ versus I , with averages taken over both arms of the eight coupler rings on the upper lattice edge [see Fig. 3.5(a)]. Horizontal dashes indicate the boundaries from Fig. 3.6(c).

We choose $\phi = \pi/4$, $\theta_0 = \pi/6$ (both assumed to be intensity-independent), $\xi_0 = \pi/2$, and $\kappa = 1$. Referring to the phase diagram in Fig. 3.6(c), we see that in the linear limit the bands are topologically trivial, with ϕ lying within a trivial gap. Increasing $\mathcal{I}$, and hence ξ , drives the system (locally) into a nontrivial gap. In interpreting this phase diagram, note that although the nonlinearity changes ξ independently in the two arms of a coupler ring, the band gaps of the linear system depend only on the *sum* of ξ in the two arms.

Fig. 3.7(b) plots the transmittance across the lattice, $\mathcal{T}(I)$, versus input power I . Here, we assume the input and output couplings to be perfect (the coupling matrices have the form of Eq. (3.10), with coupling angle $\pi/2$). These results are obtained by using a standard numerical nonlinear solver to find self-consistent solutions to the entire set of transfer matrix relations within the lattice, including the nonlinear phase shifts described by Eq. (3.12). From Fig. 3.7(b), we observe that the transmittance in the linear limit, $\mathcal{T}(0)$, is negligible, in accordance with the fact that the system is in a trivial gap. Moreover, with increasing I , $\mathcal{T}(I)$ initially remains low, but at a critical intensity it jumps discontinuously to $\mathcal{T} \approx 0.8$. Above this discontinuity, transmission takes place along the upper lattice edge, as shown in Fig. 3.7(c).

In Fig. 3.7(d), we plot $\langle \xi \rangle$ versus I , averaging over the coupler rings on the upper lattice edge. This shows that the large discontinuity in $\mathcal{T}(I)$ found in Fig. 3.7(b) occurs at values of $\langle \xi \rangle$ corresponding to the boundary between the trivial gap and the band, and between the band and the nontrivial gap. By comparison with the linear system's phase diagram from Fig. 3.6(c), it appears that the nonlinear lattice is “jumping” past the in-band regime.

Fig. 3.7(b) and (d) also shows a “secondary” discontinuity, where $\langle \xi \rangle$ jumps to a larger value while remaining in the topological gap region. This seems to be triggered when the nonlinearity causes the value of ξ in a few individual coupler rings to exceed the upper boundary of the topological gap region, entering into another in-band region at $\xi \sim 1.1\pi$. This destabilizes the solution, and the system compensates by jumping to a different field distribution, with larger $\langle \xi \rangle$, that keeps all coupling rings in the topological gap region. On both sides of this secondary discontinuity, the field distribution remains confined to the upper lattice edge.

Unlike the nonlinear Haldane model studied in Section 3.3, there is no “time-of-flight” limitation on the propagation of light from the input to the output port, due to the steady-state nature of the model. As such, optical isolation in the nonlinear

Figure 3.8: Isolation ratio (IR), forward transmittance T_f , and backward transmittance T_b , versus input power I , in disordered nonlinear coupled ring lattices. Thin lines show results for 100 different disorder realizations, and thick black lines show the mean values. The disorder consists of shifts in individual coupler ring phase shifts ξ_j , normally distributed with mean 0 and standard deviation 0.01π . Input/output loss is applied at port B by multiplying amplitudes by $e^{0.1}$ (with no loss at port A); all other parameters are the same as in Fig. 3.7.

coupled-ring lattice can exhibit topological protection against lattice defects. To test this idea, we apply an input/output asymmetry by adding a small amount of loss (coupling factor of $e^{-0.1}$) to one of the ports (B). We then compute the self-consistent forward transmittance (port A to B), backward transmittance (B to A), and isolation ratio. Disorder is introduced by adding normally distributed shifts to each ξ_i parameter (with standard deviation 0.01π). The results are shown in Fig. 3.8. For a range of input powers near $I \gtrsim 3$, there is a sharp jump in the mean forward transmittance $\langle T_f \rangle$, and correspondingly a jump in the mean isolation ratio. These jumps come from ensemble averages of transmittance discontinuities shifted by the disorder; note that the secondary discontinuities, being much smaller in magnitude, are “smeared out” and are thus not visible in the plotted mean values.

3.5 Summary

In this chapter, we have studied how optical isolation can be accomplished in three different models of topological photonics. A few basic ingredients are common to all three models. Firstly, the model must be nonlinear, so as to break optical reciprocity [37]. Secondly, the structure must contain an asymmetry (e.g., asymmetric input/output couplings) that distinguishes between “forward” and “backward” directions. The third ingredient is the use of a topological phase transition to associate forward transmission and backward transmission with different topological phases, whose physical properties are qualitatively different from each other. This last design principle is reminiscent of recently-proposed optical isolation schemes based on parity/time-reversal (PT) symmetric structures, which rely on a non-Hermitian transition (between “PT symmetric” and “PT broken” phases) rather than a topological phase transition [26, 27, 95, 153].

In the presence of nonlinearity, topological phase transitions manifest as “self-induced” topological solitons: local regions of the lattice where the optical field self-sustains its own edge state-like behavior, even when the lattice is topologically trivial in the zero-intensity limit. Such solitons were previously discovered in the 1D nonlinear SSH model [134], as well as 2D continuum “Floquet topological insulator” models [132, 135]. Here, we have shown that the nonlinear SSH soliton is useful for optical isolation, and moreover that topological solitons also occur in two new 2D models—nonlinear versions of the Haldane model [147] and the coupled-ring lattice [126, 128, 130, 150]. The soliton in the coupled-ring lattice model is notable for being a static (steady-state) solution, whereas the solitons in the nonlinear Haldane model and other previously-studied 2D lattices [132, 135] are dynamical. As we have seen, this gives rise to the distinctive feature: a discontinuity (not just a threshold or kink) in the power-dependent optical transmittance.

To realize a self-induced topological soliton, nonlinearity must be applied to a model in a non-arbitrary way: as exemplified by Eqs. (3.3), (3.8), and (3.12), the nonlinearity needs to act on parameters that drive the system towards a topological phase transition. Interestingly, homogeneous nonlinearities such as local Kerr effects, which are the most commonly-studied nonlinearities in lattice models [137–139], may *not* be suited to inducing topological phase transitions. Roughly speaking, such uniform nonlinearities play the role of altering the scalar potential, which is an inefficient way to induce topological band inversions. In real experiments, nonlinearities may be inhomogeneous and present in both on-site terms and “off-diagonal” (inter-site coupling) terms [154]. Our results demonstrate that the latter, though frequently ignored, can lead to soliton behaviors that are both distinctive and useful. It should also be noted that our study has omitted the temporal effects of optical nonlinearity, such as frequency generation and pulse dispersion; these may be important in certain materials, or for ultrashort pulses [154].

In future studies, it would be interesting to introduce additional features to the models that could further improve their performance as optical isolators. For instance, loss can be selectively added to the lattice to suppress the transmittance from bulk modes and/or diffusive non-topological edge modes, while leaving the topological edge modes relatively unaffected. It would also be interesting to make a comparison with various non-topological nonlinear isolator designs that are fine-tuned to achieve high isolation ratios and/or high transmittance [145, 155–157]; one might be able to show that isolation schemes based on topological solitons can achieve similarly high performance, while being less sensitive to random defects due to the intrinsic robustness of the topological edge modes.

Dynamical effects in the coupled-ring lattice are another avenue for further study. For example, the transmittance jumps in Fig. 3.7(b) may form hysteresis loops under an additional slow modulation of the input intensity. Such designs may also be useful

for limiting unwanted dynamical reciprocity [87], as the nonlinear isolation is provided by topological edge states localized in both frequency and space. Small amplitude signals with sufficiently large frequency detuning from the input could propagate via the qualitatively different bulk modes, and might thus be efficiently filtered or suppressed.

The study described in this Chapter should be regarded as a set of theoretical proofs of concept. Further research is needed to develop experimentally realistic designs. However, even at this stage it is highly promising to find that combining nonlinearity with topological features can give rise to robust optical isolation. Future designs may include more specific design elements, such as selectively adding losses to the bulk to attenuate the unwanted backscattered light and hence increase isolation ratios. One major limitation, which was identified in this work and should be further investigated in the future, is the need to tailor the time of flight in the nonlinear SSH and Haldane topological isolators; i.e., the need to choose the input/output ports and propagation distances carefully, which reduces design flexibility. Another problem is that in the nonlinear SSH and coupled ring resonator isolators, asymmetry is introduced by adding input and output losses, which reduces the efficiency of the device.

Chapter 4

Realization of magneto-optical near-zero index medium by an unpaired Dirac point

Most of the material presented in this chapter was previously published in the journal Physical Review B[158]. Wherever figures and materials from other sources have been taken; the authors have been given due credit.

4.1 Introduction

In two-dimensional (2D) lattices, Dirac points are linear band-crossing points associated with emergent relativistic two-component particles. Since the work of Haldane [147], they have been recognized as the elementary “building blocks” for topological phases, occurring at the boundaries separating parametric domains of different band topology [10, 11].

If time-reversal symmetry ($\mathcal{T}$) is unbroken, Dirac points come in pairs [159, 160]; in triangular and honeycomb lattices such as graphene, the pairs are pinned to high-

Figure 4.1: (a) Atomic structure of graphene. (b) Bandstructure of honeycomb lattice. Inset: Dirac cone dispersion near a corner of Brillouin zone. Figure reproduced from Ref.[162, 163]

symmetry points at the corners of the Brillouin zone [147, 161], see Fig.4.1. By breaking $\mathcal{T}$ and lattice symmetries, however, these constraints can be relaxed.

In this Chapter, we describe an experimentally realizable $\mathcal{T}$ -broken photonic crystal of gyromagnetic cylinders, whose bandstructure exhibits an unpaired Dirac point at the center of the Brillouin zone (Γ). We show that the unpaired Dirac point causes the photonic crystal to act as an effective electromagnetic medium with two interesting properties: near-zero (NZ) refractive index [164] and strongly enhanced magneto-optical activity. Using this effective medium description, we establish a link between topological edge states and surface waves in magneto-optical NZ media, two phenomena that had previously been regarded as conceptually distinct. Unlike the more familiar case of paired Dirac points in triangular or honeycomb photonic crystals (Fig.4.1), the Dirac modes in our system can be said to act as an effective medium because they interact isotropically with current sources, and there does not exist another valley for them to scatter into.

A number of earlier studies have already established the remarkable fact that photonic crystals with conical band-crossing points can act as NZ index effective media [165–169] (Fig. 4.2). Intuitively, this is due to the vanishing of the phase velocity in the

Figure 4.2: (a) Bandstructure of a two dimensional square photonic crystal with a Dirac-like cone. (b) Cloaking effect demonstrated in a microwave experiment. Upper panel: experimental set-up made of photonic crystal formed of alumina rods and a Dirac-like point occurs at frequency 10.3 GHz . A metallic obstacle (blue) is placed inside the sample. Lower panel: The distribution of out-of-plane electric field E_z outside the sample. Figure reproduced from Ref.[165].

vicinity of a band-crossing point at Γ . This provides a practical method for using all-dielectric photonic structures to realize NZ index media, which have numerous exotic and desirable capabilities such as geometry-insensitive wave-guiding [164]. Although the type of band-crossing point utilized in these schemes [165–170] is sometimes called “Dirac-like”, due to the conical dispersion, it is not actually described by a Dirac Hamiltonian, as evidenced by the fact that the cone is attached to an inseparable flat band [171, 172]. The underlying photonic crystals in these studies are $\mathcal{T}$ -symmetric, and so are the effective media, which have NZ refractive index but no magneto-optical activity.

As we have noted, an unpaired Dirac point only exists if $\mathcal{T}$ is broken. If the cone is centered at Γ , it may be described by a NZ index effective medium; but unlike previously studied cases [165–170], the medium exhibits magneto-optical activity (non-vanishing imaginary off-diagonal terms in the permeability tensor).

Intriguingly, the enhancement of magneto-optical effects has previously been iden-

tified as a promising application of NZ index media. Engheta and co-workers [164, 173–175] have noted that the strength of magneto-optical effects is characterized by a Voigt parameter—the ratio of (imaginary) off-diagonal to (real) diagonal components of the permeability or permittivity tensor—which becomes large in NZ index media as the denominator becomes small. We indeed find extremely strong magneto-optical activities in our effective medium, with a near-unity effective Voigt parameter.

Another striking feature of magneto-optical NZ index media is that they can support surface magneto-plasmon-like [176] waves that move unidirectionally along a sample edge without being back-scattered by edge deformations and other imperfections [173–175]. The unidirectional and robust nature of these modes is strongly reminiscent of the photonic topological edge states [22] of 2D photonic crystals with topologically nontrivial bandstructures, which have their existence guaranteed by topological principles. To our knowledge, no definite connection between these two phenomena has previously been identified. The present system sheds light on the issue: the unpaired Dirac point occurs at a topological transition, and we show that the remnants of the topological edge states on one side of the transition correspond to the edge states of the magneto-optical NZ index effective medium. (For other recent explorations of the links between surface states of continuous media and topology, see Refs. [170, 177, 178].)

The unpaired Dirac point at Γ , like the non-Dirac points studied in Refs. [165–169], is “accidental” in the sense that it must be realized by fine-tuning the lattice. (By contrast, paired Dirac points like those in graphene (Fig. 4.1) can exist without fine-tuning, if they are protected by a lattice symmetry.) Recently, a photonic system containing an unpaired Dirac point has been proposed [179] and observed [180], based on an array of helical optical waveguides; however, this Dirac point exists in a “Floquet” bandstructure defined via the paraxial equations of waveguide mode evolution, which has no easy effective medium interpretation. In Ref. [181], it was

shown that a $\mathcal{T}$ -broken photonic crystal can host an unpaired Dirac point possessing many interesting behaviors, such as one-way Klein tunneling; however, this Dirac point occurs at a corner of the Brillouin zone (similar to the Haldane model [147]), rather than the Γ point.

The unpaired photonic Dirac cone that we will study is obtained by breaking optical reciprocity through the magneto-optical effect, similar to previously-studied optical isolators based on photonic topological insulators, which operated at microwave frequencies. Because magneto-optical effects are weak at optical frequencies, this phenomenon cannot be scaled up to optical frequencies. In this respect, our interest in the topic is primarily fundamental.

4.2 Gyromagnetic photonic crystal with unpaired Dirac point

We will study a photonic crystal based on gyromagnetic rods in a 2D lattice. Similar photonic crystals have previously been used to realize photonic topological insulators [61–63, 125, 182, 183]; in those cases, the $\mathcal{T}$ -symmetric bandstructures contain band-crossing points (either paired Dirac points, or quadratic band-crossing points [184]), and topologically nontrivial gaps are opened as soon as $\mathcal{T}$ is broken by a biasing magnetic field $H \neq 0$ pointing in the out-of-plane ($\hat{z}$) direction. By contrast, we seek a topological transition occurring at a *nonzero* value of H . To achieve this, we search a two-dimensional space spanned by H and a sublattice symmetry breaking parameter.

As shown in Fig. 4.3(a), we take the unit cell to be a square of lattice constant a , with two ferrite rods lying along the diagonal, surrounded by air. The rods are separated by $a/\sqrt{2}$, and form two square sublattices. They have radii r_1 and r_2 respectively; to further break the sublattice symmetry, we enclose the second rod in a

Figure 4.3: (a) Schematic of a 2D photonic crystal unit cell: a square of side a , containing two ferrite rods with radii r_1 and r_2 , which form square sublattices. One rod has a dielectric cladding with refractive index n_2 and outer radius r_s . We set $a = 4$ cm, $r_1 = 0.088a$, $r_2 = 1.21 r_1$, and $r_s = 1.46 r_1$. The ferrite has $\epsilon_1 = 15$ and $\mu_0 M_s = 0.191$ T. (b) TM photonic bandstructure for $n_2 = 5$ and $\mu_0 H = 0.125$ T. Bands 4 and 5 meet at an unpaired Dirac point at Γ . (c) Bandstructure near the Dirac frequency ($\omega_D = 25.7$ GHz). (d) Heat map (in arbitrary units) showing the minimum frequency separation between bands 4 and 5, versus n_2 and $\mu_0 H$. The Dirac point occurs at a topological band transition; the Chern numbers for the 4th band and 5th band are -1 and 1 respectively in the nontrivial phase, and both 0 in the trivial phase. The star indicates the parameters used in subplots (b) and (c).

dielectric sheath with refractive index n_2 and outer radius r_s . The ferrite material has dielectric function ϵ_1 ; when biased by an out-of-plane magnetic field H , its magnetic permeability tensor is [185]

$$\overset{\leftrightarrow}{\mu} = \begin{bmatrix} \mu & i\alpha & 0 \\ -i\alpha & \mu & 0 \\ 0 & 0 & \mu_0 \end{bmatrix}, \quad (4.1)$$

where $\mu = 1 + \omega_m \omega_0 / (\omega_0^2 - \omega^2)$, $\alpha = \omega_m \omega / (\omega_0^2 - \omega^2)$, $\omega_m = g\mu_0 M_s$, $\omega_0 = g\mu_0 H$, $g = 1.76 \times 10^{11} \text{ C/kg}^{-1}$ is the gyromagnetic ratio, M_s is the saturation magnetization, and μ_0 is the permeability of free space. Our specific parameter choices are listed in the caption of Fig. 4.3.

By tuning H and other system parameters, we can generate a Dirac point between bands 4 and 5, as shown in Fig. 4.3(b)–(d). The Dirac point appears at Γ (i.e., $k_x = k_y = 0$), with angular frequency $\omega_D \approx 25.7 \text{ GHz}$, and is unpaired. (These and subsequent numerical results were obtained with Comsol Multiphysics.)

Fig. 4.3(d) plots part of the phase diagram, showing that the Dirac point occurs along a topological phase boundary where $H \neq 0$. The gap is topologically nontrivial; the calculated Chern numbers for the two bands are ± 1 below the critical H , and 0 above. As H decreases further below the depicted range, these two bands undergo another topological transition (featuring a Dirac point at X) and re-open a trivial gap, consistent with the principle that all gaps are topologically trivial for $H = 0$.

The unpaired Dirac point is associated with topological edge states. Fig. 4.4(a)–(c) show bandstructures computed for a strip that is periodic along y , and divided along x into two domains with opposite magnetic bias. Below a critical field strength $|H_c|$, there are two branches of gap-spanning edge states on the domain wall, consistent with the Chern number difference of 2 [Fig. 4.4(a)]. At the transition, the edge states merge continuously into a single branch which passes through the Dirac point

Figure 4.4: (a)–(c) Bandstructures for semi-infinite strip with two domains of opposite magnetic biases $\pm H$. Results are shown for (a) $\mu_0|H| = 0.12$ T (nontrivial gap), (b) $\mu_0|H| = 0.125$ T (unpaired Dirac point), and (c) $\mu_0|H| = 0.13$ T (trivial gap). The angular frequencies are relative to $\omega_D = 25.7$ GHz. The red curves indicate edge states localized at the interface between the domains; grey dashes are spurious edge states localized at the edges of the computational cell. (d) Spatial distributions of field intensity $|E_z|^2$ for the edge states indicated by corresponding symbols in (b). Only the 8 unit cells nearest to the domain wall are shown; the actual computational cell is 100 unit cells wide along x .

[Fig. 4.4(b)]. Above $|H_c|$, the edge states no longer span the band gap [Fig. 4.4(c)]. At each frequency, the edge states co-exist with bulk Dirac modes, but are always centered at the domain wall, as shown in Fig. 4.4(d). Right at the Dirac point, the edge state delocalizes while the bulk density of states goes to zero.

4.3 Effective medium approximation and one-way edge propagation

4.3.1 Derivation of effective medium parameter

We seek an effective medium which can be mapped to the gyromagnetic photonic crystal. Maxwell's equations in a gyromagnetic medium, with no free charges or currents, are

$$\nabla \times \mathbf{E} = i\omega \overset{\leftrightarrow}{\mu} \mathbf{H} \quad (4.2)$$

$$\nabla \times \mathbf{H} = -i\omega \epsilon \mathbf{E} \quad (4.3)$$

$$\nabla \cdot (\epsilon \mathbf{E}) = 0 \quad (4.4)$$

$$\nabla \cdot (\mu \mathbf{H}) = 0. \quad (4.5)$$

Here, ϵ is the scalar permittivity, and the permeability has the form

$$\overset{\leftrightarrow}{\mu} = \begin{bmatrix} \mu & i\alpha & 0 \\ -i\alpha & \mu & 0 \\ 0 & 0 & \mu_0 \end{bmatrix}. \quad (4.6)$$

Assume a uniform medium, so that both ϵ and $\overset{\leftrightarrow}{\mu}$ are position-independent. We consider TM modes with $E_x = E_y = H_z = 0$. For this polarization, Eq. (4.4) is

automatically satisfied. Combining Eq. (4.2) with (4.6) gives

$$-\nabla_{2D}^2 E_z = i\omega [\mu(\partial_x H_y - \partial_y H_x) - i\alpha(\partial_x H_x + \partial_y H_y)], \quad (4.7)$$

where $\nabla_{2D}^2 \equiv \partial_x^2 + \partial_y^2$ is the 2D Laplacian. Next, combining Eq. (4.5) with (4.6) gives

$$\mu(\partial_x H_x + \partial_y H_y) = -i\alpha(\partial_x H_y - \partial_y H_x). \quad (4.8)$$

Combining Eqs. (4.7)–(4.8) and Eq. (4.3) yields

$$\left[\nabla_{2D}^2 + \epsilon\mu \left(1 - \frac{\alpha^2}{\mu^2} \right) \omega^2 \right] E_z = 0. \quad (4.9)$$

Here $n^2 = \epsilon\mu$ where n is effective refractive index. So far, we have not made any assumptions about the frequency dispersion of the medium. We now assume that the bulk dispersion obeys the Dirac dispersion relation

$$\omega = \omega_D + v_D |\mathbf{k}|, \quad (4.10)$$

where ω_D is the Dirac frequency, v_D is the effective speed of the Dirac modes, and $\mathbf{k} = (k_x, k_y)$ is the 2D in-plane wavevector. Comparing Eqs. (4.9)–(4.10), we deduce that

$$\epsilon\mu \left(1 - \frac{\alpha^2}{\mu^2} \right) = \left[\frac{1}{v_D} \frac{\delta\omega}{\omega_D} \right]^2, \quad (4.11)$$

where $\delta\omega \equiv \omega - \omega_D$ (we assume throughout that $|\delta\omega| \ll \omega_D$). This means that the effective refractive index vanishes for $\delta\omega \rightarrow 0$; this is a NZ index medium.

Next, consider two adjacent domains, separated by a domain wall at $x = 0$. The magnetic bias has opposite signs for $x > 0$ and $x < 0$. In other words, for $x < 0$ we

replace α with $-\alpha$ in Eq. (4.6). We seek solutions of the form

$$E_z = E_0 e^{-\gamma|x|+ik_y y}. \quad (4.12)$$

These modes are localized around $x = 0$, with wavenumber k_y parallel to the domain wall and penetration depth $1/\gamma$. Note that E_z is necessarily continuous across the domain wall. Substituting into Eq. (4.9) yields the condition

$$k_y^2 - \gamma^2 = \epsilon\mu \left(1 - \frac{\alpha^2}{\mu^2}\right) \omega^2. \quad (4.13)$$

Moreover, knowing E_z we can use Eq. (4.2) to retrieve the magnetic fields:

$$\mathbf{H} = \frac{1}{i\omega} \overset{\leftrightarrow}{\mu}^{-1} \nabla \times \mathbf{E} \quad (4.14)$$

$$\begin{bmatrix} H_x \\ H_y \end{bmatrix} = \frac{E_0}{i\omega} \overset{\leftrightarrow}{\mu}^{-1} \begin{bmatrix} ik_y \\ \pm\gamma \end{bmatrix} e^{\mp\gamma x + ik_y y} \quad (4.15)$$

$$= \frac{E_0}{i\omega(\mu^2 - \alpha^2)} \begin{bmatrix} i(\mu k_y - \alpha\gamma) \\ \pm(\mu\gamma - \alpha k_y) \end{bmatrix} e^{\mp\gamma x + ik_y y}, \quad (4.16)$$

where the $\pm$ signs denote the $x > 0$ and $x < 0$ domains respectively. Due to the absence of surface currents, H_y must also be continuous across the domain wall, and hence

$$\gamma = \frac{\alpha k_y}{\mu}. \quad (4.17)$$

The parameter γ must be positive. Eq. (4.17) implies that if a domain wall state exists for $k_y > 0$, then $\alpha/\mu > 0$ for $k_y > 0$. Conversely, if a domain wall state exists for $k_y < 0$, then $\alpha/\mu < 0$ for $k_y < 0$. In Fig. 4.4(b) and Fig. 4.5, we observe that domain wall states of the photonic crystal exist for both positive and negative k_y . The domain wall state's relative frequency $\delta\omega \equiv \omega - \omega_D$ switches sign as k_y switches

sign. This implies that α/μ switches sign across the Dirac frequency.

Using Eq. (4.17), we can simplify Eq. (4.13) to $\epsilon\mu\omega^2 = k_y^2$. Applying the Dirac medium condition (4.11) gives the dispersion relation for the domain wall states:

$$\delta\omega^2 = v_D^2 \left[1 - \left(\frac{\alpha}{\mu} \right)^2 \right] k_y^2. \quad (4.18)$$

4.3.2 One way edge state

Consider two uniform domains with equal and opposite Voigt parameters $\tilde{\alpha}/\tilde{\mu}$, separated by a straight domain wall parallel to y . Maxwell's equations support TM modes confined to the domain wall by the mismatch in $\tilde{\alpha}/\tilde{\mu}$. If Eq. (4.11) is obeyed, these satisfy

$$|\delta\omega| = v_D \sqrt{1 - (\tilde{\alpha}/\tilde{\mu})^2} |k_y| \quad (4.19)$$

$$\gamma = (\tilde{\alpha}/\tilde{\mu}) k_y, \quad (4.20)$$

where k_y is the wavenumber along the domain wall and $1/\gamma$ is the penetration depth. In order for the modes to decay away from the domain wall, we require $\gamma > 0$.

It should be noted that similar edge states can exist if $\tilde{\epsilon}$, $\tilde{\mu}$, and $\tilde{\alpha}$ are frequency-independent (so that Eq. (4.11) does *not* hold). In that case, the edge dispersion relation (4.19) is replaced by $\omega = v_0 \sqrt{1 - (\tilde{\alpha}/\tilde{\mu})^2} |k_y|$, where v_0 is the speed of light in the bulk. Then $k_y \rightarrow 0$ at zero frequency, rather than at a finite Dirac frequency ω_D .

Eqs. (4.19)–(4.20) can be systematically fitted to photonic crystal simulations. From the bulk band diagram, we estimate $v_D \approx 1.26 \times 10^7 \text{ ms}^{-1}$; the Dirac cone is nearly isotropic, with fits along different crystal axes giving less than 0.5% variation in v_D . Next, we extract γ by calculating the intensity profile $\langle |E_z|^2 \rangle$ for the photonic crystal edge states, averaging over each unit cell, and fitting to an envelope

Figure 4.5: (a) Edge state decay parameter γ versus k_y . These data are extracted from two-domain photonic crystal simulations by fitting $\langle |E_z|^2 \rangle$ (averaged over each unit cell) to an $\exp(-2\gamma|x|)$ envelope. (b) Close-up of the bandstructure near the Dirac point. The solid curves show the numerically-calculated dispersion for the edge states (red) and bulk states (blue); the dashes are spurious edge states localized at edges of the computational cell, which is 300 unit cells wide with all other parameters the same as in Fig. 4.4(b). The green dots show the effective medium prediction, calculated using the fitted value of v_D and the decay parameters from (a).

$I_0 \exp(-2\gamma|x|)$. The results are shown in Fig. 4.5(a).

Comparing this to Eq. (4.20), we deduce that the Voigt parameter $\tilde{\alpha}/\tilde{\mu}$ switches sign across ω_D ; very near ω_D , it has almost constant magnitude $|\tilde{\alpha}/\tilde{\mu}| \approx 0.94$. From Eq. (4.19), this also implies that $\tilde{\epsilon}\tilde{\mu} \rightarrow 0$ as $\omega \rightarrow \omega_D$.

Finally, we use the fitted values of v_D and γ together with Eqs. (4.19)–(4.20) to plot the edge state dispersion curve $\delta\omega = \pm v_D \sqrt{k_y^2 - \gamma^2}$. As shown in Fig. 4.5(b), near the Dirac frequency the dispersion curve predicted by the effective medium theory agrees with the edge state dispersion curve from the photonic crystal simulations.

The effective Voigt parameter of $|\tilde{\alpha}/\tilde{\mu}| \approx 0.94$ is close to the maximum allowed value; it cannot exceed unity, as such a medium would lie in the “Hall opacity” regime [173] where no propagating bulk modes exist. The actual Voigt parameter within the underlying ferrite rods is $\alpha/\mu \approx 1.53$ at the operating frequency. However, the rods occupy only 6% of the photonic crystal’s area. An area-weighted homogenization

of the photonic crystal yields an effective Voigt parameter of only ≈ 0.09 , while a different homogenization scheme for magneto-optical media [186] yields ≈ 0.2 (see Sec. 4.3.3). Thus, the Voigt parameter of the Dirac point-induced effective medium is substantially enhanced relative to the homogenized photonic crystal, in agreement with previous arguments that NZ index media can enhance magneto-optical activity [164, 173–175].

Although the edge states delocalize ($\gamma \rightarrow 0$) at the Dirac frequency, transmission along the edge appears to be quite robust *near* the Dirac frequency. This is due to (i) the reduction in the density of bulk states available to scatter into, and (ii) the fact that the edge states are unidirectional and thus unable to back-scatter. Fig. 4.6 shows the transmission along a domain wall with a perfect electrical conductor obstacle. The simulations for both the photonic crystal and the effective medium show a transmittance dip at the Dirac frequency, as shown in Fig. 4.6(c). At small detunings, however, the surface waves are able to partially bypass the obstacle, as shown by the field intensity plots in Fig. 4.6(a) for the photonic crystal, and Fig. 4.6(b) for the effective medium.

4.3.3 Effective Voigt parameter under crude homogenization

In the previous section, we noted that the effective medium’s effective Voigt parameter of $|\tilde{\alpha}/\tilde{\mu}| \approx 0.94$ is remarkably large, given that (i) the actual Voigt parameters in the ferrite rods is $\alpha/\mu \approx 1.5$ at the operating frequency, and (ii) the rods occupy only 6% of the photonic crystal’s area (see Fig. 4.3). Crudely weighting α/μ by area yields an effective Voigt parameter of approximately 0.092.

A slightly more sophisticated scheme for homogenizing magneto-optic media was derived in Ref. [186]. That paper considered cylinders of gyroelectric material, with

Figure 4.6: (a)–(b) Spatial distribution of the field intensity $|E_z|^2$, for surface waves moving along a domain wall with a perfect electrical conductor obstacle. (a) Photonic crystal simulation, including an expanded view of the region near the obstacle. (b) Effective medium simulation, using the effective medium fitting parameters from Fig. 4.5(b). The source positions are indicated by yellow stars. In both cases, the detuning is $\delta\omega \approx 0.07$ GHz. However, for the effective medium the Dirac frequency is taken to be $\omega_D \approx 25.23$ GHz, a reduction of 1.8% relative to the bulk bandstructure calculation; this value is determined from the transmittance dip shown in (c), and appears to be a discretization error. (c) Normalized transmittances versus frequency detuning $\delta\omega$, for the photonic crystal and the effective medium model. The maximum transmittance over the frequency range is normalized to unity.

dielectric tensors of the form

$$\overset{\leftrightarrow}{\epsilon}_2 = \begin{bmatrix} \epsilon_2 & i\gamma & 0 \\ -i\gamma & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_2 \end{bmatrix}. \quad (4.21)$$

The background medium has permittivity ϵ_1 , and the magnetic permeability is unity throughout. It was found that the homogenized effective dielectric parameters are

$$\tilde{\epsilon}_x = \tilde{\epsilon}_y = \epsilon_1 + f(\epsilon_2 - \epsilon_1) \left[1 + \frac{(1-f)(\epsilon_2 - \epsilon_1)}{2\epsilon_1} \right]^{-1} \quad (4.22)$$

$$\tilde{\epsilon}_z = (1-f)\epsilon_1 + f\epsilon_2 \quad (4.23)$$

$$\tilde{\gamma} = \gamma f \left[1 + \frac{(1-f)(\epsilon_2 - \epsilon_1)}{2\epsilon_1} \right]^{-1}, \quad (4.24)$$

where $\tilde{\epsilon}_x$, $\tilde{\epsilon}_y$, and $\tilde{\epsilon}_z$ are the on-diagonal components, and $\tilde{\gamma}$ is the magneto-optical off-diagonal component of the effective permittivity tensor, and f is the volume fraction of the gyroelectric component.

We adapt this calculation to our gyromagnetic photonic crystal by the mapping

$$\left\{ \begin{array}{l} \nabla \times \mathbf{E} = i\omega \mathbf{B} \\ \nabla \times \mathbf{B} = -i\omega \overset{\leftrightarrow}{\epsilon} \mathbf{E} \\ \mu = 1 \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \nabla \times \mathbf{B} = -i\omega \mathbf{E} \\ \nabla \times \mathbf{E} = i\omega \overset{\leftrightarrow}{\mu} \mathbf{H} \\ \epsilon = 1 \end{array} \right\} \quad (4.25)$$

As this is only intended as a rough comparison, we ignore the non-unity permittivities of the ferrite rods. At the operating frequency of $\omega \approx \omega_D$, the ferrite permeability is

$$\overset{\leftrightarrow}{\mu} = \begin{bmatrix} \mu & i\alpha & 0 \\ -i\alpha & \mu & 0 \\ 0 & 0 & \mu_0 \end{bmatrix}, \quad \mu \approx -3.2, \quad \alpha \approx -4.9. \quad (4.26)$$

The volume fraction is $f \approx 0.06$. Thus,

$$\tilde{\mu} = 1.2, \quad \tilde{\alpha} = 0.3. \quad (4.27)$$

This corresponds to a Voigt parameter of $\tilde{\alpha}/\tilde{\mu} \approx 0.23$.

4.4 Summary

We have designed a realistic gyromagnetic photonic crystal with an unpaired Dirac point at Γ . The Dirac medium serves as a magneto-optical NZ index effective medium [164–169, 173–175], and we have identified the “remnant” of the topological edge state with the magneto-plasmon-like surface states of the effective medium. The effective Voigt parameter is ≈ 0.94 , which is very near the “Hall opacity” regime [173], and enhanced by approximately five-fold relative to the homogenized value.

We have focused on the behavior of domain walls in the bulk of an otherwise homogeneous photonic crystal. For finite arbitrarily-shaped domains, however, the effective medium approximation to a photonic crystal is known to be problematic, as it assumes that the Bloch modes of interest have a dominant Fourier component, which is typically not the case for higher bands. This limits the accuracy with which quantities such as reflection and transmission coefficients can be reproduced [187]. Moreover, if domain boundaries do not preserve the bulk symmetries of the underlying crystal, the effective medium must be complemented by additional mode-matching boundary layers [188]. In the future, it would be interesting to identify models exhibiting unpaired Dirac cones in lower bands, which might mitigate these limitations [165].

Chapter 5

Conclusion

In this thesis, we have studied parity-time phase transitions and topological phase transitions in various photonics system ranging from coupled ring resonators to photonic crystals and lattices. Based on the observation of dramatic behavior change associated with the phase transitions, optical isolation assisted by parity-time phase transitions and topological phase transitions were studied and explored.

In the first part of this thesis, we analyzed the relationship between the linear and nonlinear behaviors of dual microcavity resonators with gain and loss. We clarified the relationships between the PT phase, the performance of nonlinear optical isolation and the stability of the steady-state optical modes. The enhancement of optical isolation in PT -broken phase observed in [26] was explained by the switch of solution branches at the critical point of the nonlinear system.

In the second part of this thesis, we studied the possibility of realizing nonlinear optical isolators using the concept of topological phase transitions. We studied three representative designs: (1) a waveguide array implementing a nonlinear 1D Su-Schrieffer-Heeger model, (2) a waveguide array implementing a nonlinear 2D Haldane model, and (3) a 2D lattice of coupled-ring waveguides. Optical isolation behavior was demonstrated to be associated with self-induced topological phase transitions in these

systems. The isolation behavior relies on the fact that the self-induced topological transition has a power threshold i.e., the soliton-like edge state appears only above a certain power. Asymmetric input and output couplings can be used to ensure that the edge state exists under forward transmission, but not backward transmission.

In the last part of the thesis, we focused on the topological transition point itself, and showed that it is associated with a new type of photonic crystal which can host an unpaired Dirac point at the center(Γ point) of the first Brillouin zone. The unpaired Dirac point in a gyromagnetic photonic crystal can be mapped, through effective medium theory, to a magneto-optical near-zero index medium. The effective Voigt parameter is greatly enhanced, consistent with predictions. We then focused on the domain wall states in both photonic crystal systems and the effective medium system it maps into, and found unidirectional one way edge states that are robust against backscattering. This work has shed some light on the relationship between two seemingly unrelated phenomena: topological edge state in photonic crystals and surface magnetoplasmon-like waves in uniform magneto-optical near-zero index media.

In the future, we aim to collaborate with experimentalists to realize some of the phenomena predicted in this thesis, including the discontinuities in isolation factor near the PT transition point described in Chapter 2, topologically robust nonlinear optical isolation following one of the designs in Chapter 3, and evidence for the the unpaired Dirac point described in Chapter 4. Further theoretical research is also required into certain issues; for example, in the topic of nonlinear topological isolation (Chapter 3), it would be interesting to introduce more features to increase the robustness of the edge states, such as selectively adding loss to attenuate bulk modes.

Bibliography

- [1] Hidetoshi Nishimori and Gerardo Ortiz. *Elements of phase transitions and critical phenomena*. OUP Oxford, 2010.
- [2] Carl M Bender and Stefan Boettcher. *Physical Review Letters*, 80(24):5243, 1998.
- [3] Carl M Bender, Stefan Boettcher, and Peter N Meisinger. *Journal of Mathematical Physics*, 40(5):2201–2229, 1999.
- [4] Carl M Bender, Dorje C Brody, and Hugh F Jones. *Physical Review Letters*, 89(27):270401, 2002.
- [5] Carl M Bender. *Contemporary Physics*, 46(4):277–292, 2005.
- [6] Carl M Bender. *Reports on Progress in Physics*, 70(6):947, 2007.
- [7] Haijun Zhang, Chao-Xing Liu, Xiao-Liang Qi, Xi Dai, Zhong Fang, and Shou-Cheng Zhang. *Nature Physics*, 5(6):438, 2009.
- [8] YL Chen, James G Analytis, J-H Chu, ZK Liu, S-K Mo, Xiao-Liang Qi, HJ Zhang, DH Lu, Xi Dai, Zhong Fang, et al. *Science*, 325(5937):178–181, 2009.
- [9] Joel E Moore. *Nature*, 464(7286):194, 2010.

-
- [10] M Zahid Hasan and Charles L Kane. *Reviews of Modern Physics*, 82(4):3045, 2010.
 - [11] Xiao-Liang Qi and Shou-Cheng Zhang. *Reviews of Modern Physics*, 83(4):1057, 2011.
 - [12] Ilya Vorobeichik, Edvardas Narevicius, Gershon Rosenblum, Meir Orenstein, and Nimrod Moiseyev. *Physical Review Letters*, 90(17):176806, 2003.
 - [13] G Della Valle, M Ornigotti, E Cianci, V Foglietti, P Laporta, and S Longhi. *Physical Review Letters*, 98(26):263601, 2007.
 - [14] Stefano Longhi. *Physical Review Letters*, 97(11):110402, 2006.
 - [15] Masaru Onoda, Shuichi Murakami, and Naoto Nagaosa. *Physical Review Letters*, 93(8):083901, 2004.
 - [16] Omur Hosten and Paul Kwiat. *Science*, 319(5864):787–790, 2008.
 - [17] Konstantinos G Makris, R El-Ganainy, DN Christodoulides, and Ziad H Musslimani. *Physical Review Letters*, 100(10):103904, 2008.
 - [18] John D Joannopoulos, Steven G Johnson, Joshua N Winn, and Robert D Meade. *Photonic crystals*. Princeton University Press, 2011.
 - [19] Daniel Leykam. Wave and spectral singularities in photonic lattices. 2015.
 - [20] Christian E Rüter, Konstantinos G Makris, Ramy El-Ganainy, Demetrios N Christodoulides, Mordechai Segev, and Detlef Kip. *Nature Physics*, 6(3):192, 2010.
 - [21] Alois Regensburger, Christoph Bersch, Mohammad-Ali Miri, Georgy Onishchukov, Demetrios N Christodoulides, and Ulf Peschel. *Nature*, 488(7410):167, 2012.

-
- [22] Ling Lu, John D Joannopoulos, and Marin Soljačić. *Nature Photonics*, 8(11): 821, 2014.
- [23] Tomoki Ozawa, Hannah M Price, Alberto Amo, Nathan Goldman, Mohammad Hafezi, Ling Lu, Mikael Rechtsman, David Schuster, Jonathan Simon, Oded Zilberberg, et al. *arXiv preprint arXiv:1802.04173*, 2018.
- [24] Liang Feng, Zi Jing Wong, Ren-Min Ma, Yuan Wang, and Xiang Zhang. *Science*, 346(6212):972–975, 2014.
- [25] Hossein Hodaei, Mohammad-Ali Miri, Matthias Heinrich, Demetrios N Christodoulides, and Mercedeh Khajavikhan. *Science*, 346(6212):975–978, 2014.
- [26] Bo Peng, Şahin Kaya Özdemir, Fuchuan Lei, Faraz Monifi, Mariagiovanna Gianfreda, Gui Lu Long, Shanhui Fan, Franco Nori, Carl M Bender, and Lan Yang. *Nature Physics*, 10(5):394, 2014.
- [27] Long Chang, Xiaoshun Jiang, Shiyue Hua, Chao Yang, Jianming Wen, Liang Jiang, Guanyu Li, Guanzhong Wang, and Min Xiao. *Nature Photonics*, 8(7): 524, 2014.
- [28] Stefano Longhi. *Physical Review A*, 82(3):031801, 2010.
- [29] Ling Lu, John D Joannopoulos, and Marin Soljačić. *Nature Physics*, 12(7):626, 2016.
- [30] Gal Harari, Miguel A Bandres, Yaakov Lumer, Mikael C Rechtsman, YD Chong, Mercedeh Khajavikhan, Demetrios N Christodoulides, and Mordechai Segev. *Science*, 359(6381):eaar4003, 2018.
- [31] Miguel A Bandres, Steffen Wittek, Gal Harari, Midya Parto, Jinhan Ren, Mordechai Segev, Demetrios N Christodoulides, and Mercedeh Khajavikhan. *Science*, 359(6381):eaar4005, 2018.

-
- [32] Henning Schomerus. *Optics Letters*, 38(11):1912–1914, 2013.
 - [33] Han Zhao, Pei Miao, Mohammad H Teimourpour, Simon Malzard, Ramy El-Ganainy, Henning Schomerus, and Liang Feng. *Nature Communications*, 9(1):981, 2018.
 - [34] Ramy El-Ganainy and Miguel Levy. *Optics Letters*, 40(22):5275–5278, 2015.
 - [35] Xin Zhou, You Wang, Daniel Leykam, and Yi Dong Chong. *New Journal of Physics*, 19(9):095002, 2017.
 - [36] DD Solnyshkov, O Bleu, and G Malpuech. *Applied Physics Letters*, 112(3):031106, 2018.
 - [37] Dirk Jalas, Alexander Petrov, Manfred Eich, Wolfgang Freude, Shanhui Fan, Zongfu Yu, Roel Baets, Miloš Popović, Andrea Melloni, John D Joannopoulos, et al. *Nature Photonics*, 7(8):579, 2013.
 - [38] Kerry J Vahala. *Nature*, 424(6950):839, 2003.
 - [39] DK Armani, TJ Kippenberg, SM Spillane, and KJ Vahala. *Nature*, 421(6926):925, 2003.
 - [40] Li Ge, Liang Feng, and Harald GL Schwefel. *Photonics Research*, 5(6):OM1–OM3, 2017.
 - [41] Enrique AJ Marcatili. *Bell Labs Technical Journal*, 48(7):2071–2102, 1969.
 - [42] Wim Bogaerts, Pieter Dumon, Dries Van Thourhout, Dirk Taillaert, Patrick Jaenen, Johan Wouters, Stephan Beckx, Vincent Wiaux, and Roel G Baets. *IEEE Journal of Selected Topics in Quantum Electronics*, 12(6):1394–1401, 2006.

-
- [43] Wim Bogaerts, Shankar Kumar Selvaraja, Pieter Dumon, Joost Brouckaert, Katrien De Vos, Dries Van Thourhout, and Roel Baets. *IEEE Journal of Selected Topics in Quantum Electronics*, 16(1):33–44, 2010.
- [44] Joyce KS Poon, Jacob Scheuer, Yong Xu, and Amnon Yariv. *JOSA B*, 21(9):1665–1673, 2004.
- [45] Andrea Melloni, A Canciamilla, C Ferrari, F Morichetti, L O’Faolain, TF Krauss, Re De La Rue, A Samarelli, and M Sorel. *IEEE Photonics Journal*, 2(2):181–194, 2010.
- [46] Katrien De Vos, Irene Bartolozzi, Etienne Schacht, Peter Bienstman, and Roel Baets. *Optics Express*, 15(12):7610–7615, 2007.
- [47] Frank Vollmer and Stephen Arnold. *Nature Methods*, 5(7):591, 2008.
- [48] Sajeev John. *Physical Review Letters*, 58(23):2486, 1987.
- [49] Eli Yablonovitch. *Physical Review Letters*, 58(20):2059, 1987.
- [50] Charles Kittel and Paul McEuen. *Introduction to solid state physics*, volume 8. Wiley New York, 1996.
- [51] Attila Mekis, JC Chen, I Kurland, Shanhui Fan, Pierre R Villeneuve, and JD Joannopoulos. *Physical Review Letters*, 77(18):3787, 1996.
- [52] Steven G Johnson, Shanhui Fan, Pierre R Villeneuve, John D Joannopoulos, and LA Kolodziejski. *Physical Review B*, 60(8):5751, 1999.
- [53] Marko Lončar, Tomoyuki Yoshie, Axel Scherer, Pawan Gogna, and Yueming Qiu. *Applied Physics Letters*, 81(15):2680–2682, 2002.
- [54] Hatice Altug, Dirk Englund, and Jelena Vučković. *Nature Physics*, 2(7):484, 2006.

-
- [55] Mehmet Bayindir, B Temelkuran, and E Ozbay. *Applied Physics Letters*, 77(24):3902–3904, 2000.
- [56] Xiaofang Yu and Shanhui Fan. *Applied Physics Letters*, 83(16):3251–3253, 2003.
- [57] Toshihiko Baba. *Nature Photonics*, 2(8):465, 2008.
- [58] Thomas F Krauss. *Journal of Physics D: Applied Physics*, 40(9):2666, 2007.
- [59] Nina Skivesen, Amølie Têtu, Martin Kristensen, Jørgen Kjems, Lars H Frandsen, and Peter I Borel. *Optics Express*, 15(6):3169–3176, 2007.
- [60] Elewout Hallynck and Peter Bienstman. *Optics Express*, 18(17):18164–18170, 2010.
- [61] S Raghu and FDM Haldane. *Physical Review A*, 78(3):033834, 2008.
- [62] FDM Haldane and S Raghu. *Physical Review Letters*, 100(1):013904, 2008.
- [63] Zheng Wang, Yidong Chong, John D Joannopoulos, and Marin Soljačić. *Nature*, 461(7265):772, 2009.
- [64] Ivan L Garanovich, Stefano Longhi, Andrey A Sukhorukov, and Yuri S Kivshar. *Physics Reports*, 518(1-2):1–79, 2012.
- [65] Demetrios N Christodoulides, Falk Lederer, and Yaron Silberberg. *Nature*, 424(6950):817, 2003.
- [66] Stefano Longhi. *Laser & Photonics Reviews*, 3(3):243–261, 2009.
- [67] Alexander Szameit, Jonas Burghoff, Thomas Pertsch, Stefan Nolte, Andreas Tünnermann, and Falk Lederer. *Optics Express*, 14(13):6055–6062, 2006.
- [68] Yidong Chong. *Nature*, 496(7444):173, 2013.
- [69] Stefano Longhi. *EPL (Europhysics Letters)*, 76(3):416, 2006.

-
- [70] Felix Dreisow, Matthias Heinrich, Robert Keil, Andreas Tünnermann, Stefan Nolte, Stefano Longhi, and Alexander Szameit. *Physical Review Letters*, 105(14):143902, 2010.
- [71] Mikael C Rechtsman, Julia M Zeuner, Yonatan Plotnik, Yaakov Lumer, Daniel Podolsky, Felix Dreisow, Stefan Nolte, Mordechai Segev, and Alexander Szameit. *Nature*, 496(7444):196, 2013.
- [72] Marin Soljačić and John D Joannopoulos. *Nature Materials*, 3(4):211, 2004.
- [73] Lorenzo Pavesi. In *Advances in Optical Science and Engineering*, pages 7–10. Springer, 2015.
- [74] Vilson R Almeida, Carlos A Barrios, Roberto R Panepucci, and Michal Lipson. *Nature*, 431(7012):1081, 2004.
- [75] David AB Miller. *IEEE Journal of Selected Topics in Quantum Electronics*, 6(6):1312–1317, 2000.
- [76] Charles Vassallo. *Optical Wave Sciences and Technology*, 1:Sec–5, 1991.
- [77] Yuya Shoji, Tetsuya Mizumoto, Hideki Yokoi, I-Wei Hsieh, and Richard M Osgood Jr. *Applied Physics Letters*, 92(7):071117, 2008.
- [78] Ming-Chun Tien, Tetsuya Mizumoto, Paolo Pintus, Herbert Kromer, and John E Bowers. *Optics Express*, 19(12):11740–11745, 2011.
- [79] Enrico Milani. *Applied Optics*, 32(27):5217–5221, 1993.
- [80] Lei Bi, Juejun Hu, Peng Jiang, Dong Hun Kim, Gerald F Dionne, Lionel C Kimerling, and CA Ross. *Nature Photonics*, 5(12):758, 2011.
- [81] Richard L Espinola, Tomoyuki Izuhara, Ming-Chun Tsai, Richard M Osgood, and Horst Dötsch. *Optics Letters*, 29(9):941–943, 2004.

-
- [82] Hugo Lira, Zongfu Yu, Shanhui Fan, and Michal Lipson. *Physical Review Letters*, 109(3):033901, 2012.
- [83] Katia Gallo, Gaetano Assanto, Krishnan R Parameswaran, and Martin M Fejer. *Applied Physics Letters*, 79(3):314–316, 2001.
- [84] Andrey E Miroshnichenko, Etienne Brasselet, and Yuri S Kivshar. *Applied Physics Letters*, 96(6):26, 2010.
- [85] M Krause, H Renner, and E Brinkmeyer. *Electronics Letters*, 44(11):691–693, 2008.
- [86] Christopher G Poulton, Ravi Pant, Adam Byrnes, Shanhui Fan, MJ Steel, and Benjamin J Eggleton. *Optics Express*, 20(19):21235–21246, 2012.
- [87] Yu Shi, Zongfu Yu, and Shanhui Fan. *Nature Photonics*, 9(6):388, 2015.
- [88] Pankaj K Sahoo and Joby Joseph. *Applied Optics*, 52(34):8252–8257, 2013.
- [89] Li Fan, Leo T Varghese, Jian Wang, Yi Xuan, Andrew M Weiner, and Minghao Qi. *Optics Letters*, 38(8):1259–1261, 2013.
- [90] Xinpeng Huang and Shanhui Fan. *Journal of Lightwave Technology*, 29(15):2267–2275, 2011.
- [91] Zheqi Wang, Lei Shi, Yi Liu, Xinbiao Xu, and Xinliang Zhang. *Scientific Reports*, 5:8657, 2015.
- [92] Nicholas Bender, Samuel Factor, Joshua D Bodyfelt, Hamidreza Ramezani, Demetrios N Christodoulides, Fred M Ellis, and Tsampikos Kottos. *Physical Review Letters*, 110(23):234101, 2013.
- [93] Zongfu Yu and Shanhui Fan. *Nature Photonics*, 3(2):91, 2009.

-
- [94] Negar Reiskarimian and Harish Krishnaswamy. *Nature Communications*, 7: 11217, 2016.
- [95] Xin Zhou and YD Chong. *Optics Express*, 24(7):6916–6930, 2016.
- [96] Horst Dötsch, Norbert Bahlmann, Oleksandr Zhuromskyy, Manfred Hammer, Ludger Wilkens, Reinald Gerhardt, Peter Hertel, and Anatoly F Popkov. *JOSA B*, 22(1):240–253, 2005.
- [97] Miguel Levy. *JOSA B*, 22(1):254–260, 2005.
- [98] YD Chong, Li Ge, and A Douglas Stone. *Physical Review Letters*, 106(9):093902, 2011.
- [99] Martin Wimmer, Alois Regensburger, Mohammad-Ali Miri, Christoph Bersch, Demetrios N Christodoulides, and Ulf Peschel. *Nature Communications*, 6:7782, 2015.
- [100] R El-Ganainy, KG Makris, DN Christodoulides, and Ziad H Musslimani. *Optics Letters*, 32(17):2632–2634, 2007.
- [101] Konstantinos G Makris, Ramy El-Ganainy, Demetrios N Christodoulides, and Ziad H Musslimani. *Physical Review A*, 81(6):063807, 2010.
- [102] ZH Musslimani, Konstantinos G Makris, Ramy El-Ganainy, and Demetrios N Christodoulides. *Physical Review Letters*, 100(3):030402, 2008.
- [103] A Guo, GJ Salamo, D Duchesne, R Morandotti, M Volatier-Ravat, V Aimez, GA Siviloglou, and DN Christodoulides. *Physical Review Letters*, 103(9):093902, 2009.
- [104] Liang Feng, Ye-Long Xu, William S Fegadolli, Ming-Hui Lu, José EB Oliveira, Vilson R Almeida, Yan-Feng Chen, and Axel Scherer. *Nature Materials*, 12(2): 108, 2013.

-
- [105] Carl M Bender, MV Berry, and Aikaterini Mandilara. *Journal of Physics A: Mathematical and General*, 35(31):L467, 2002.
- [106] WD Heiss. *Journal of Physics A: Mathematical and Theoretical*, 45(44):444016, 2012.
- [107] Hamidreza Ramezani, Tsampikos Kottos, Vassilios Kovanis, and Demetrios N Christodoulides. *Physical Review A*, 85(1):013818, 2012.
- [108] Xin-You Lü, Hui Jing, Jin-Yong Ma, and Ying Wu. *Physical Review Letters*, 114(25):253601, 2015.
- [109] Kosmas V Kepesidis, Thomas J Milburn, Julian Huber, Konstantinos G Makris, Stefan Rotter, and Peter Rabl. *New Journal of Physics*, 18(9):095003, 2016.
- [110] Absar U Hassan, Hossein Hodaei, Mohammad-Ali Miri, Mercedeh Khajavikhan, and Demetrios N Christodoulides. *Physical Review A*, 92(6):063807, 2015.
- [111] B Peng, ŞK Özdemir, S Rotter, H Yilmaz, M Liertzer, F Monifi, CM Bender, F Nori, and L Yang. *Science*, 346(6207):328–332, 2014.
- [112] Hermann A Haus. *Waves and fields in optoelectronics*. Prentice-Hall,, 1984.
- [113] Wonjoo Suh, Zheng Wang, and Shanhui Fan. *IEEE Journal of Quantum Electronics*, 40(10):1511–1518, 2004.
- [114] Rafif E Hamam, Aristeidis Karalis, JD Joannopoulos, and Marin Soljačić. *Physical Review A*, 75(5):053801, 2007.
- [115] Shuang Zhang, Dentcho A Genov, Yuan Wang, Ming Liu, and Xiang Zhang. *Physical Review Letters*, 101(4):047401, 2008.

-
- [116] Zin Lin, Hamidreza Ramezani, Toni Eichelkraut, Tsampikos Kottos, Hui Cao, and Demetrios N Christodoulides. *Physical Review Letters*, 106(21):213901, 2011.
- [117] Yong Sun, Wei Tan, Hong-qiang Li, Jensen Li, and Hong Chen. *Physical Review Letters*, 112(14):143903, 2014.
- [118] Xiaogang Chen, Nicolae C Panoiu, Iwei Hsieh, Jerry I Dadap, and Richard M Osgood. *IEEE Photonics Technology Letters*, 18(24):2617–2619, 2006.
- [119] M Dinu, Francesco Quochi, and H Garcia. *Applied Physics Letters*, 82(18):2954–2956, 2003.
- [120] Vilson R Almeida and Michal Lipson. *Optics Letters*, 29(20):2387–2389, 2004.
- [121] Simon Malzard, Emiliano Cancellieri, and Henning Schomerus. *Optics Express*, 26(17):22506–22518, 2018.
- [122] Simon Malzard and Henning Schomerus. *New Journal of Physics*, 20(6):063044, 2018.
- [123] Steven Strogatz, Mark Friedman, A John Mallinckrodt, and Susan McKay. *Computers in Physics*, 8(5):532–532, 1994.
- [124] B Andrei Bernevig and Taylor L Hughes. *Topological insulators and topological superconductors*. Princeton University Press, 2013.
- [125] Zheng Wang, YD Chong, John D Joannopoulos, and Marin Soljačić. *Physical Review Letters*, 100(1):013905, 2008.
- [126] Mohammad Hafezi, S Mittal, J Fan, A Migdall, and JM Taylor. *Nature Photonics*, 7(12):1001, 2013.
- [127] Kejie Fang, Zongfu Yu, and Shanhui Fan. *Nature Photonics*, 6(11):782, 2012.

-
- [128] GQ Liang and YD Chong. *Physical Review Letters*, 110(20):203904, 2013.
- [129] Alexander B Khanikaev, S Hossein Mousavi, Wang-Kong Tse, Mehdi Kargarian, Allan H MacDonald, and Gennady Shvets. *Nature Materials*, 12(3):233, 2013.
- [130] S Mittal, J Fan, S Faez, A Migdall, JM Taylor, and M Hafezi. *Physical Review Letters*, 113(8):087403, 2014.
- [131] Chong Li, Xiaoyong Hu, Hong Yang, and Qihuang Gong. *AIP Advances*, 7(2):025203, 2017.
- [132] Yaakov Lumer, Yonatan Plotnik, Mikael C Rechtsman, and Mordechai Segev. *Physical Review Letters*, 111(24):243905, 2013.
- [133] Mark J Ablowitz, Christopher W Curtis, and Yi-Ping Ma. *Physical Review A*, 90(2):023813, 2014.
- [134] Yakir Hadad, Alexander B Khanikaev, and Andrea Alu. *Physical Review B*, 93(15):155112, 2016.
- [135] Daniel Leykam and Yi Dong Chong. *Physical Review Letters*, 117(14):143901, 2016.
- [136] Ramy El-Ganainy, Alexander Eisfeld, Miguel Levy, and Demetrios N Christodoulides. *Applied Physics Letters*, 103(16):161105, 2013.
- [137] S Suntsov, KG Makris, DN Christodoulides, GI Stegeman, A Hache, R Morandotti, H Yang, G Salamo, and M Sorel. *Physical Review Letters*, 96(6):063901, 2006.
- [138] S Suntsov, KG Makris, DN Christodoulides, GI Stegeman, R Morandotti, H Yang, G Salamo, and M Sorel. *Optics Letters*, 32(21):3098–3100, 2007.

-
- [139] Falk Lederer, George I Stegeman, Demetri N Christodoulides, Gaetano Assanto, Moti Segev, and Yaron Silberberg. *Physics Reports*, 463(1-3):1–126, 2008.
- [140] DN Christodoulides and RI Joseph. *Optics Letters*, 13(9):794–796, 1988.
- [141] Julia M Zeuner, Mikael C Rechtsman, Yonatan Plotnik, Yaakov Lumer, Stefan Nolte, Mark S Rudner, Mordechai Segev, and Alexander Szameit. *Physical Review Letters*, 115(4):040402, 2015.
- [142] S Weimann, M Kremer, Y Plotnik, Y Lumer, S Nolte, KG Makris, M Segev, MC Rechtsman, and A Szameit. *Nature Materials*, 16(4):433, 2017.
- [143] Qingqing Cheng, Yiming Pan, Qianjin Wang, Tao Li, and Shining Zhu. *Laser & Photonics Reviews*, 9(4):392–398, 2015.
- [144] Reji Philip, M Anija, Chandra S Yelleswarapu, and DVGLN Rao. *Applied Physics Letters*, 91(14):141118, 2007.
- [145] Benoy Anand, Ramakrishna Podila, Kiran Lingam, SR Krishnan, S Siva Sankara Sai, Reji Philip, and Apparao M Rao. *Nano Letters*, 13(12):5771–5776, 2013.
- [146] Stefano Longhi. *Optics Letters*, 40(7):1278–1281, 2015.
- [147] F Duncan M Haldane. *Physical Review Letters*, 61(18):2015, 1988.
- [148] R Morandotti, U Peschel, JS Aitchison, HS Eisenberg, and Y Silberberg. *Physical Review Letters*, 83(14):2726, 1999.
- [149] Thomas RO Melvin, AR Champneys, Panayotis G Kevrekidis, and J Cuevas. *Physical Review Letters*, 97(12):124101, 2006.
- [150] Michael Pasek and YD Chong. *Physical Review B*, 89(7):075113, 2014.

-
- [151] Wenchao Hu, Jason C Pillay, Kan Wu, Michael Pasek, Perry Ping Shum, and YD Chong. *Physical Review X*, 5(1):011012, 2015.
- [152] Fei Gao, Zhen Gao, Xihang Shi, Zhaoju Yang, Xiao Lin, Hongyi Xu, John D Joannopoulos, Marin Soljačić, Hongsheng Chen, Ling Lu, et al. *Nature Communications*, 7:11619, 2016.
- [153] Sergey V Suchkov, Andrey A Sukhorukov, Jiahao Huang, Sergey V Dmitriev, Chaohong Lee, and Yuri S Kivshar. *Laser & Photonics Reviews*, 10(2):177–213, 2016.
- [154] Robert W Boyd. *Nonlinear optics*. Academic press, 2003.
- [155] Stefano Lepri and Giulio Casati. *Physical Review Letters*, 106(16):164101, 2011.
- [156] Stefano Lepri and Boris A Malomed. *Physical Review E*, 87(4):042903, 2013.
- [157] Yi Xu and Andrey E Miroshnichenko. *Physical Review B*, 89(13):134306, 2014.
- [158] Xin Zhou, Daniel Leykam, Udas Chattopadhyay, AB Khanikaev, and Yi Dong Chong. *Physical Review B*, 98(20):205115, 2018.
- [159] Holger Bech Nielsen and Masao Ninomiya. *Nuclear Physics B*, 185(1):20–40, 1981.
- [160] Holger Bech Nielsen and Masao Ninomiya. *Nuclear Physics B*, 193(1):173–194, 1981.
- [161] Gordon W Semenoff. *Physical Review Letters*, 53(26):2449, 1984.
- [162] Antonio Castro Neto, Francisco Guinea, and Nuno Miguel Peres. *Physics World*, 19(11):33, 2006.
- [163] AH Castro Neto, Francisco Guinea, Nuno MR Peres, Kostya S Novoselov, and Andre K Geim. *Reviews of Modern Physics*, 81(1):109, 2009.

-
- [164] Iñigo Liberal and Nader Engheta. *Nature Photonics*, 11(3):149, 2017.
- [165] Xueqin Huang, Yun Lai, Zhi Hong Hang, Huihuo Zheng, and CT Chan. *Nature Materials*, 10(8):582, 2011.
- [166] CT Chan, Zhi Hong Hang, and Xueqin Huang. *Advances in OptoElectronics*, 2012, 2012.
- [167] Jian-Wen Dong, Ming-Li Chang, Xue-Qin Huang, Zhi Hong Hang, Zhi-Chao Zhong, Wen-Jie Chen, Zhan-Yun Huang, and Che Ting Chan. *Physical Review Letters*, 114(16):163901, 2015.
- [168] Muhammad Waseem Ashraf and Muhammad Faryad. *JOSA B*, 33(6):1008–1013, 2016.
- [169] Saman Jahani and Zubin Jacob. *Nature Nanotechnology*, 11(1):23, 2016.
- [170] Meng Xiao and Shanhui Fan. *Physical Review B*, 96(10):100202, 2017.
- [171] Dmitry Green, Luiz Santos, and Claudio Chamon. *Physical Review B*, 82(7):075104, 2010.
- [172] Daniel Leykam and Anton S Desyatnikov. *Advances in Physics: X*, 1(1):101–113, 2016.
- [173] Arthur R Davoyan and Nader Engheta. *Physical Review Letters*, 111(25):257401, 2013.
- [174] Arthur R Davoyan, Ahmed M Mahmoud, and Nader Engheta. *Optics Express*, 21(3):3279–3286, 2013.
- [175] Fereshteh Abbasi, Arthur R Davoyan, and Nader Engheta. *New Journal of Physics*, 17(6):063014, 2015.

-
- [176] Bin Hu, Ying Zhang, and Qi Jie Wang. *Nanophotonics*, 4(1):383–396.
- [177] Mário G Silveirinha. *Physical Review B*, 92(12):125153, 2015.
- [178] Mário G Silveirinha. *Physical Review B*, 94(20):205105, 2016.
- [179] Daniel Leykam, MC Rechtsman, and YD Chong. *Physical Review Letters*, 117(1):013902, 2016.
- [180] Jiho Noh, Sheng Huang, Daniel Leykam, Yi Dong Chong, Kevin P Chen, and Mikael C Rechtsman. *Nature Physics*, 13(6):611, 2017.
- [181] Xiang Ni, David Purtseladze, Daria A Smirnova, Alexey Slobozhanyuk, Andrea Alù, and Alexander B Khanikaev. *Science Advances*, 4(5):eaap8802, 2018.
- [182] Jin-Xin Fu, Rong-Juan Liu, and Zhi-Yuan Li. *Applied Physics Letters*, 97(4):041112, 2010.
- [183] Yin Poo, Rui-xin Wu, Zhifang Lin, Yan Yang, and CT Chan. *Physical Review Letters*, 106(9):093903, 2011.
- [184] YD Chong, Xiao-Gang Wen, and Marin Soljačić. *Physical Review B*, 77(23):235125, 2008.
- [185] David M Pozar. *Microwave engineering*. John Wiley & Sons, 2009.
- [186] Masanori Abe and Manabu Gomi. *Japanese Journal of Applied Physics*, 23(12R):1580, 1984.
- [187] Wojciech Śmigaj and Boris Gralak. *Physical Review B*, 77(23):235445, 2008.
- [188] Constantin R Simovski. *Journal of Optics*, 13(1):013001, 2010.