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Lim, Renon

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**NANYANG
TECHNOLOGICAL
UNIVERSITY**

SINGAPORE

**COMPLEX SYMMETRIC
WEIGHTED COMPOSITION
OPERATORS ON $\mathcal{H}_\gamma(\mathbb{D})$**

RENON LIM

**SCHOOL OF PHYSICAL AND
MATHEMATICAL SCIENCES**

2019

**COMPLEX SYMMETRIC
WEIGHTED COMPOSITION
OPERATORS ON $\mathcal{H}_\gamma(\mathbb{D})$**

RENON LIM

School of Physical and Mathematical Sciences

A thesis submitted to the Nanyang Technological
University in partial fulfilment of the requirement for the
degree of Master of Science

2019

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research done by me except where otherwise stated in this thesis. The thesis work has not been submitted for a degree or professional qualification to any other university or institution. I declare that this thesis is written by myself and is free of plagiarism and of sufficient grammatical clarity to be examined. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

20 April 2019

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Supervisor Declaration

Statement

I have reviewed the content and presentation style of this thesis and declare it of sufficient grammatical clarity to be examined. To the best of my knowledge, the thesis is free of plagiarism and the research and writing are those of the candidate's except as acknowledged in the Author Attribution Statement. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

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Le Hai Khoi

Authorship Attribution

Statement

This thesis contains material from a paper published in the following peer-reviewed journal where I was the first and/or corresponding author.

Chapters 1, 3, 4 and 5 are published as [13].

The contributions of the co-authors are as follows:

- Dr Le Hai Khoi provided the initial project direction and was the corresponding author liaising with the journal's editors.
- I prepared the manuscript drafts and each manuscript was revised and commented on by Dr Le.

20 April 2019
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Renon Lim

Abstract

In this thesis, we investigate the complex symmetric structure of weighted composition operators $W_{\psi,\varphi}$ on the Hilbert space $\mathcal{H}_\gamma(\mathbb{D})$ of holomorphic functions over the open unit disk $\mathbb{D}$ with reproducing kernels $K_w^{(\gamma)}(z) = (1 - \overline{w}z)^{-\gamma}$, where $\gamma \in \mathbb{N}$.

In Chapter 1, we provide some key working definitions and a brief overview of the literature on the topic. Then we proceed to Chapter 2, where we give a sufficient treatment of reproducing kernels, which plays a key role in the proofs of many results presented in this thesis.

In Chapter 3, we consider conjugations on $\mathcal{H}_\gamma(\mathbb{D})$ which takes the form $\mathcal{A}_{u,v}f = u \cdot \overline{f \circ \overline{v}}$ (such conjugations are also known as weighted composition conjugations) and characterize them into two classes, denoted by $\mathcal{C}_1$ and $\mathcal{C}_2$. Thereafter in Chapter 4, we obtain explicit conditions for $W_{\psi,\varphi}$ when it is $\mathcal{C}_1$ -symmetric and $\mathcal{C}_2$ -symmetric respectively. In Chapter 5, we conclude this thesis by discussing some possible future directions on this investigation.

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I would also like to thank the publisher Elsevier and the Journal of Mathematical Analysis and Applications for granting me permission to reproduce my work in [13] as part of my thesis.

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Chapter 1

Introduction

In this thesis, we assume familiarity with the basics of common undergraduate course materials such as (but not limited to): linear algebra, real, complex and functional analysis. Prior to giving a complete introduction to the topic and problem on hand, we first provide some key definitions.

1.1 Key definitions

Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{I} : \mathcal{H} \rightarrow \mathcal{H}$ be the identity operator.

Definition 1.1.1. A mapping $\mathcal{A}$ acting on $\mathcal{H}$ is said to be **anti-linear** if for all $x, y \in \mathcal{H}$ and scalar $\alpha \in \mathbb{C}$,

$$\mathcal{A}(\alpha x + y) = \bar{\alpha}\mathcal{A}x + \mathcal{A}y.$$

The notions of continuity, boundedness and invariance of an anti-linear operator are defined in the same manner as for the case of linear operators. Not surprisingly, boundedness and continuity of an anti-linear operator are also equivalent.

Definition 1.1.2. Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded *linear* operator. The **adjoint**

of T is denoted by T^* and is such that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle$$

for all $x, y \in \mathcal{H}$. If we have instead a bounded *anti-linear* operator $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}$, then its **adjoint** $\mathcal{A}^*$ is such that

$$\langle \mathcal{A}x, y \rangle = \overline{\langle x, \mathcal{A}^*y \rangle} = \langle \mathcal{A}^*y, x \rangle$$

for all $x, y \in \mathcal{H}$.

Definition 1.1.3. An *anti-linear* operator $\mathcal{C} : \mathcal{H} \rightarrow \mathcal{H}$ is called a **conjugation** if

- (a) $\mathcal{C}$ is *involutive*, i.e. $\mathcal{C}^2 = \mathcal{I}$ and
- (b) $\mathcal{C}$ is *isometric*, i.e. $\|\mathcal{C}x\| = \|x\|$ for all $x \in \mathcal{H}$.

Since a conjugation is isometric, it is therefore necessarily bounded. We now have all the tools required to properly define the notion of complex symmetry.

Definition 1.1.4. A bounded *linear* operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is said to be **complex symmetric** if there exists a *conjugation* $\mathcal{C} : \mathcal{H} \rightarrow \mathcal{H}$ such that $T = \mathcal{C}T^*\mathcal{C}$ (which is equivalent to either $T\mathcal{C} = \mathcal{C}T^*$ or $T^* = \mathcal{C}T\mathcal{C}$). We also say that T is a $\mathcal{C}$ -symmetric operator if T is complex symmetric with respect to a conjugation $\mathcal{C}$.

We continue to define the linear operator that we will be working with in the sequel. Let $\mathcal{D} \subset \mathbb{C}$ be a domain and let $\mathcal{O}(\mathcal{D})$ be the space of all holomorphic functions on $\mathcal{D}$, endowed with the usual compact-open topology.

Definition 1.1.5. Let $\psi : \mathcal{D} \rightarrow \mathbb{C}$ and $\varphi : \mathcal{D} \rightarrow \mathcal{D}$ be holomorphic functions. For all $f \in \mathcal{O}(\mathcal{D})$ and $z \in \mathcal{D}$, the functions ψ and φ induce a linear operator $W_{\psi, \varphi}$ on $\mathcal{O}(\mathcal{D})$ defined by

$$W_{\psi, \varphi}f(z) = \psi(z) \cdot f(\varphi(z)).$$

The operator $W_{\psi, \varphi}$ is known as a **weighted composition operator** on $\mathcal{O}(\mathcal{D})$. If $\psi \equiv 1$, then we have the usual **composition operator** $C_\varphi = W_{1, \varphi}$.

1.2 Background review

The study of complex symmetric operators was birthed from the intersection of the fields of complex analysis and operator theory, and was pioneered by Garcia and Putinar in their papers [6, 7]. Their initial work sparked interest in the Mathematical community and led to much significant research being done in the area of complex symmetry over the past decade or so (see [5] and the many references therein). It is also notable that the interest stemmed from both practical applications (such as non-Hermitian quantum mechanics) and theoretical aspects (such as spectral analysis of complex symmetric operators). Much of the present results illustrate the diversity of bounded complex symmetric operators, ranging from the likes of Volterra integration operators to compressed Toeplitz operators, and even normal operators.

We now turn our attention to weighted composition operators. Over the better part of the past half a century, weighted composition operators on various function spaces was thoroughly studied and documented (see [3, 15] and the many references therein). A long list of traditional topics studied with respect to weighted composition operators comes to mind, such as boundedness, compactness, compact difference, essential norm, closed range, etc. Branching off from the oft travelled route, we can research on complex symmetry with respect to weighted composition operators.

In recent years, this research was carried out by Jung and colleagues (see [11]) as well as by Garcia and Hammond (see [4]). Independently, they discovered the complex symmetric structure of weighted composition operators on Hardy spaces with respect to what we now know as the **standard conjugation**, i.e. those of the form

$$\mathcal{J}f(z) = \overline{f(\bar{z})},$$

where f belongs to the Hardy space.

Also, Hai and Khoi discovered the complex symmetric structure of weighted

composition operators on both Fock spaces $\mathcal{F}^2(\mathbb{C})$ (see [8]) and $\mathcal{F}^2(\mathbb{C}^n)$ (see [9] and [10]) with respect to a more general conjugation of the form

$$\mathcal{A}_{u,v}f(z) = u(z) \cdot \overline{f(\overline{v(z)})},$$

where u and v are entire functions and f belongs to the respective Fock space.

In the abovementioned papers [4, 8, 9, 10, 11], the crux to discovering the various criterions for the weighted composition operator to be complex symmetric with respect to their respective conjugations and function spaces lies within the reproducing kernels of the respective function spaces in question. We postpone a more detailed explanation on reproducing kernels to the next chapter.

Definition 1.2.1. Let $U : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded *linear* operator. The operator U is said to be **unitary** if $U^*U = UU^* = \mathcal{I}$. We define an *anti-linear unitary* operator in the same manner.

Definition 1.2.2. Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded *linear* operator. The operator T is said to be **self-adjoint** if $T = T^*$, i.e. $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in \mathcal{H}$.

For a bounded *anti-linear* operator $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}$, $\mathcal{A}$ is **self-adjoint** if $\mathcal{A} = \mathcal{A}^*$, i.e. $\langle \mathcal{A}x, y \rangle = \langle \mathcal{A}y, x \rangle$ for all $x, y \in \mathcal{H}$.

The following observation was mentioned in [9] for Fock space: $\mathcal{A}$ is a conjugation if and only if $\mathcal{A}$ is self-adjoint and unitary. In fact, this observation holds for any separable complex Hilbert space. This will be a useful criterion in characterizing conjugations of the form $\mathcal{A}_{u,v}$.

Proposition 1.2.3. *Let $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded anti-linear operator. Then $\mathcal{A}$ is a conjugation if and only if $\mathcal{A}$ is unitary and self-adjoint.*

Proof. Necessity. Suppose that $\mathcal{A}$ is a conjugation, i.e. $\mathcal{A}$ is involutive and isometric. First, we prove that $\mathcal{A}$ is self-adjoint. By the isometry of $\mathcal{A}$, we have $\|\mathcal{A}(x + y)\| = \|x + y\|$ for any $x, y \in \mathcal{H}$, i.e.

$$\langle \mathcal{A}(x + y), \mathcal{A}(x + y) \rangle = \langle x + y, x + y \rangle. \quad (1.2.1)$$

We apply the properties of an inner product to get

$$\begin{aligned}
\langle \mathcal{A}(x+y), \mathcal{A}(x+y) \rangle &= \langle \mathcal{A}x + \mathcal{A}y, \mathcal{A}x + \mathcal{A}y \rangle \\
&= \langle \mathcal{A}x, \mathcal{A}x + \mathcal{A}y \rangle + \langle \mathcal{A}y, \mathcal{A}x + \mathcal{A}y \rangle \\
&= \langle \mathcal{A}x, \mathcal{A}x \rangle + \langle \mathcal{A}x, \mathcal{A}y \rangle + \langle \mathcal{A}y, \mathcal{A}x \rangle + \langle \mathcal{A}y, \mathcal{A}y \rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle x+y, x+y \rangle &= \langle x, x+y \rangle + \langle y, x+y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle.
\end{aligned}$$

Clearly, $\langle \mathcal{A}x, \mathcal{A}x \rangle = \langle x, x \rangle$ and $\langle \mathcal{A}y, \mathcal{A}y \rangle = \langle y, y \rangle$. Hence (1.2.1) simplifies to

$$\langle \mathcal{A}x, \mathcal{A}y \rangle + \langle \mathcal{A}y, \mathcal{A}x \rangle = \langle x, y \rangle + \langle y, x \rangle. \quad (1.2.2)$$

Doing the same with $\|\mathcal{A}(x+iy)\| = \|x+iy\|$ will give us

$$\begin{aligned}
&\langle \mathcal{A}x, \mathcal{A}(iy) \rangle + \langle \mathcal{A}(iy), \mathcal{A}x \rangle = \langle x, iy \rangle + \langle iy, x \rangle \\
&\iff \langle \mathcal{A}x, i\mathcal{A}y \rangle + \langle i\mathcal{A}y, \mathcal{A}x \rangle = i\langle x, y \rangle + i\langle y, x \rangle
\end{aligned}$$

and hence

$$\langle \mathcal{A}x, \mathcal{A}y \rangle - \langle \mathcal{A}y, \mathcal{A}x \rangle = -\langle x, y \rangle + \langle y, x \rangle. \quad (1.2.3)$$

Adding (1.2.2) and (1.2.3) gives

$$2\langle \mathcal{A}x, \mathcal{A}y \rangle = 2\langle y, x \rangle \iff \langle \mathcal{A}x, \mathcal{A}y \rangle = \langle y, x \rangle$$

and by substituting $y' = \mathcal{A}y$ (equivalent to $\mathcal{A}y' = \mathcal{A}^2y = y$), we will obtain

$$\langle \mathcal{A}x, \mathcal{A}y \rangle = \langle y, x \rangle \iff \langle \mathcal{A}x, y' \rangle = \langle \mathcal{A}y', x \rangle,$$

which tells us that $\mathcal{A}^* = \mathcal{A}$.

Finally we prove that $\mathcal{A}$ is unitary. Since

$$\mathcal{I} = \mathcal{A}^2 = \mathcal{A}\mathcal{A}^* = \mathcal{A}^*\mathcal{A},$$

hence $\mathcal{A}$ is unitary.

Sufficiency. Suppose that $\mathcal{A}$ is unitary and self-adjoint. Then it is easy to see at once that $\mathcal{A}$ is involutive. Since $\mathcal{A} = \mathcal{A}^*$ and $\mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^* = \mathcal{I}$, we must have $\mathcal{A}^2 = \mathcal{I}$.

All that remains is to show that $\mathcal{A}$ is isometric. Since $\mathcal{A}$ is self-adjoint, we can write

$$\langle \mathcal{A}x, y \rangle = \langle \mathcal{A}y, x \rangle$$

for any $x, y \in \mathcal{H}$. In particular, we choose $y = \mathcal{A}x$ and hence

$$\langle \mathcal{A}x, \mathcal{A}x \rangle = \langle \mathcal{A}\mathcal{A}x, x \rangle = \langle x, x \rangle \iff \|\mathcal{A}x\|^2 = \|x\|^2,$$

which means that $\mathcal{A}$ is isometric. $\square$

By Proposition 1.2.3, we can see that the standard conjugation $\mathcal{J}$ is both self-adjoint and unitary.

1.3 Objective of this study

Inspired by the recent development in the study of complex symmetric weighted composition operators, we attempt to investigate the complex symmetric structure of weighted composition operators on $\mathcal{H}_\gamma(\mathbb{D})$, which is the Hilbert space of holomorphic functions over the open unit disk $\mathbb{D}$, equipped with reproducing kernel

$$K_w^{(\gamma)}(z) = (1 - \bar{w}z)^{-\gamma}, \tag{1.3.1}$$

where $\gamma \in \mathbb{N}$. Such a space was studied in [2, 12] for the case of $\gamma \geq 1$. Our choice of restricting γ to be a natural number will be made clear in Chapter 3. By setting $\gamma = 1, 2$, we obtain the reproducing kernels for the classical Hardy space and classical Bergman space respectively.

For any $w \in \mathbb{D}$, the norm of $K_w^{(\gamma)}$ is denoted by

$$\|K_w^{(\gamma)}\| = (1 - |w|^2)^{-\frac{\gamma}{2}}.$$

Hence for any $w, z \in \mathbb{D}$, we can write the normalized reproducing kernel as

$$k_w^{(\gamma)}(z) = \frac{K_w^{(\gamma)}(z)}{\|K_w^{(\gamma)}\|} = \frac{(1 - |w|^2)^{\frac{\gamma}{2}}}{(1 - \bar{w}z)^\gamma}. \quad (1.3.2)$$

Our investigation of this problem can be thought of as a two-step approach. First, we must find all such $\mathcal{A}_{u,v}$ that are conjugations, i.e. find holomorphic functions $u : \mathbb{D} \rightarrow \mathbb{C}$ and $v : \mathbb{D} \rightarrow \mathbb{D}$ such that $\mathcal{A}_{u,v}$ is **(a)** invariant on $\mathcal{H}_\gamma(\mathbb{D})$, **(b)** self-adjoint and **(c)** unitary.

Second, we find all such $W_{\psi,\varphi}$ that are complex symmetric with respect to $\mathcal{A}_{u,v}$, i.e. find holomorphic functions $\psi : \mathbb{D} \rightarrow \mathbb{C}$ and $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ such that **(a)** $W_{\psi,\varphi}$ is invariant on $\mathcal{H}_\gamma(\mathbb{D})$ and **(b)** $W_{\psi,\varphi} = \mathcal{A}_{u,v} W_{\psi,\varphi}^* \mathcal{A}_{u,v}$.

Chapter 2

Reproducing kernels

We devote an entire chapter to detail the necessary knowledge on reproducing kernels for our intended study of complex symmetric weighted composition operators.

2.1 Basic definition and results

A **reproducing kernel Hilbert space (RKHS)** is a Hilbert space of functions such that for each point in the domain of the functions, the linear evaluation functionals, acting from the said Hilbert space to its underlying field, are bounded. For this chapter, we let $\mathcal{H}$ be a complex RKHS instead and let $\mathcal{D}$ be the domain of $f \in \mathcal{H}$. The linear evaluation functional at a point $w \in \mathcal{D}$ is the map $L_w : \mathcal{H} \rightarrow \mathbb{C}$, defined as $L_w(f) = f(w)$.

Recall the following famous theorem in the study of Hilbert spaces (see Chapter 1 of [1] for more details).

Theorem 2.1.1 (Riesz Representation Theorem). *Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{F}$. If $L : \mathcal{H} \rightarrow \mathbb{F}$ is a bounded linear functional, then there is a unique vector $h_0 \in \mathcal{H}$ such that for all $h \in \mathcal{H}$, $L(h) = \langle h, h_0 \rangle$.*

Since we are interested in $\mathcal{H}$ as a complex RKHS, the Riesz Representation Theorem tells us that for all $w \in \mathcal{D}$, there exists a unique function $K_w \in \mathcal{H}$ such

that for all $f \in \mathcal{H}$, $L_w(f) = \langle f, K_w \rangle$. This unique function K_w is known as the **reproducing kernel** for $\mathcal{H}$.

We refer the interested reader to [14] for more details and examples of RKHS. We will only provide the necessary results on RKHS required for our treatment of this thesis.

The first notable result is that the linear span of the reproducing kernels K_w is dense in $\mathcal{H}$. A function $f \in \mathcal{H}$ is orthogonal to the linear span of K_w if and only if $\langle f, K_w \rangle = 0$ for all $w \in \mathcal{D}$. But this is exactly $f \equiv 0$. Hence the orthogonal complement of the linear span of K_w is $\{0\}$, that is to say the linear span of K_w is dense in $\mathcal{H}$.

The second notable result is that for any sequence of functions from $\mathcal{H}$, the convergence in norm on $\mathcal{H}$ is always stronger than pointwise convergence of the sequence of functions. The proof of this result makes use of the Cauchy-Schwarz inequality. Let (f_n) be a sequence from $\mathcal{H}$ such that $\|f_n\| \rightarrow 0$. Let $w \in \mathcal{D}$ and $\varepsilon > 0$ be given. Then there is a $N \in \mathbb{N}$ such that for all $n \geq N$, $\|f_n\| < \frac{\varepsilon}{\|K_w\|}$. Clearly,

$$|f_n(w)| = |L_w(f_n)| = |\langle f_n, K_w \rangle| \leq \|f_n\| \|K_w\| < \varepsilon.$$

Both of these basic results actually play a key role in obtaining important criteria in the next section.

2.2 Useful propositions

Let V and W be Banach spaces. We define the graph of a linear transformation $T : V \rightarrow W$ to be the set $\{(x, Tx) : x \in V\}$. The following is a famous theorem in functional analysis and operator theory (see Chapter 4 of [1] for more details).

Theorem 2.2.1 (Closed Graph Theorem). *Let V and W be Banach spaces and $T : V \rightarrow W$ be a linear transformation. If the graph of T is closed, then T is continuous.*

We state Proposition 12.7 of Chapter 4 in [1], which characterizes the closed graph of a linear transformation sequentially.

Proposition 2.2.2. *Let V and W be Banach spaces and $T : V \rightarrow W$ be a linear transformation. The graph of T is closed if and only if for all sequences (x_n) from V such that $x_n \rightarrow 0$ and $T(x_n) \rightarrow y$, we have $y = 0$.*

We are now able to prove the equivalence of boundedness and invariance of $W_{\psi,\varphi}$ on $\mathcal{H}$, which is our first useful proposition. This result is very well known amongst mathematicians working with weighted composition operators on RKHS and is often implicitly accepted without mention in research papers.

Proposition 2.2.3. *$W_{\psi,\varphi}$ is bounded on $\mathcal{H}$ if and only if $W_{\psi,\varphi}$ is invariant on $\mathcal{H}$, i.e. $W_{\psi,\varphi} : \mathcal{H} \rightarrow \mathcal{H}$.*

Proof. Necessity. Suppose that $W_{\psi,\varphi}$ is bounded on $\mathcal{H}$. For all $f \in \mathcal{H}$, there exist an $M > 0$ such that $\|W_{\psi,\varphi}f\| \leq M \|f\|$. Since $\|f\| < \infty$, we also have $\|W_{\psi,\varphi}f\| < \infty$ and thus $W_{\psi,\varphi}f \in \mathcal{H}$ too.

Sufficiency. Suppose that $W_{\psi,\varphi} : \mathcal{H} \rightarrow \mathcal{H}$. Consider the graph of $W_{\psi,\varphi}$, which is the set $\{(f, W_{\psi,\varphi}f) : f \in \mathcal{H}\}$ and the identity map

$$Id : (\mathcal{H}, \|\cdot\|) \rightarrow (\mathcal{H}, |\cdot|).$$

We claim that $Id \circ W_{\psi,\varphi}$ is continuous. Let (f_n) be a sequence of functions from $(\mathcal{H}, \|\cdot\|)$ such that $f_n \rightarrow 0$ and $W_{\psi,\varphi}f_n \rightarrow g$. Since $\|f_n\| \rightarrow 0$, we must also have $|f_n(z)| \rightarrow 0$ for all $z \in \mathcal{D}$, which gives $|f_n(\varphi(z))| \rightarrow 0$ for all $z \in \mathcal{D}$. Let $z \in \mathcal{D}$ and $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(\varphi(z))| < \frac{\varepsilon}{|\psi(z)|}$. Thus

$$|Id \circ W_{\psi,\varphi}(f_n(z))| = |W_{\psi,\varphi}f_n(z)| = |\psi(z)| |f_n(\varphi(z))| < \varepsilon.$$

Thus $Id \circ W_{\psi,\varphi}$ is continuous. We can now write

$$\lim_{n \rightarrow \infty} Id \circ W_{\psi,\varphi}(f_n(z)) = Id \circ W_{\psi,\varphi}(0) = 0.$$

The continuity of Id allows us to write

$$0 = \lim_{n \rightarrow \infty} Id \circ W_{\psi, \varphi}(f_n(z)) = Id \left(\lim_{n \rightarrow \infty} W_{\psi, \varphi}(f_n(z)) \right) = g(z).$$

Hence, Proposition 2.2.2 tells us that the graph of $W_{\psi, \varphi}$ is closed. As a consequence of the Closed Graph Theorem, $W_{\psi, \varphi}$ is continuous, therefore bounded.

□

Our next useful proposition is a direct corollary of the fact that the linear span of K_w is dense in $\mathcal{H}$.

Proposition 2.2.4. *Let $W_{\psi, \varphi} : \mathcal{H} \rightarrow \mathcal{H}$ and $\mathcal{C}$ be a conjugation on $\mathcal{H}$. Then $W_{\psi, \varphi}$ is $\mathcal{C}$ -symmetric if and only if for all $w \in \mathcal{D}$,*

$$W_{\psi, \varphi} \mathcal{C} K_w = \mathcal{C} W_{\psi, \varphi}^* K_w.$$

The above Proposition 2.2.4 is remarkable as it tells us that it suffices to work with the reproducing kernels instead of an arbitrary $f \in \mathcal{H}$ when dealing with complex symmetry. If $W_{\psi, \varphi}$ is bounded on $\mathcal{H}$, then for all $w \in \mathcal{D}$, we have

$$W_{\psi, \varphi}^* K_w = \overline{\psi(w)} K_{\varphi(w)}. \quad (2.2.1)$$

We prove it by utilizing the definitions of an adjoint and reproducing kernel.

$$\begin{aligned} \langle f, W_{\psi, \varphi}^* K_w \rangle &= \langle W_{\psi, \varphi} f, K_w \rangle = W_{\psi, \varphi} f(w) \\ &= \psi(w) f(\varphi(w)) = \psi(w) \langle f, K_{\varphi(w)} \rangle \\ &= \left\langle f, \overline{\psi(w)} K_{\varphi(w)} \right\rangle. \end{aligned}$$

This formula (2.2.1) will be useful in many of the proofs in the later chapters. The next proposition is our last useful result for this chapter and is similar to one of the many results in [4, 11].

Proposition 2.2.5. *Let $\psi : \mathcal{D} \rightarrow \mathbb{C}$ and $\varphi : \mathcal{D} \rightarrow \mathcal{D}$ be holomorphic functions such that $W_{\psi, \varphi} : \mathcal{H} \rightarrow \mathcal{H}$. If $W_{\psi, \varphi}$ is complex symmetric, then $\psi(z)$ is either identically zero or non-zero for all $z \in \mathcal{D}$.*

Proof. Suppose not. That is, we suppose that $W_{\psi,\varphi}$ is complex symmetric but $\psi(z)$ is not identically zero and there is some $z_0 \in \mathcal{D}$ such that $\psi(z_0) = 0$. Then we can find a small enough neighbourhood of z_0 , say N , such that $\psi(z') \neq 0$ for all $z' \in N \setminus \{z_0\}$.

By (2.2.1), we have

$$W_{\psi,\varphi}^* K_{z_0} = \overline{\psi(z_0)} K_{\varphi(z_0)} = 0.$$

Since $W_{\psi,\varphi}$ is complex symmetric with respect to some conjugation, say $\mathcal{C}$, we have for all $z \in \mathcal{D}$,

$$\begin{aligned} \|W_{\psi,\varphi} \mathcal{C} K_{z_0}\| &= \|\mathcal{C} W_{\psi,\varphi}^* K_{z_0}\| \\ &= \|W_{\psi,\varphi}^* K_{z_0}\| = 0. \end{aligned}$$

This implies that $W_{\psi,\varphi} \mathcal{C} K_{z_0}$ is identically zero. Furthermore, for all $z' \in N \setminus \{z_0\}$,

$$W_{\psi,\varphi} \mathcal{C} K_{z_0}(z') = 0 \implies \psi(z') \mathcal{C} K_{\varphi(z_0)} = 0 \implies \mathcal{C} K_{\varphi(z_0)} = 0.$$

By the Identity Theorem for holomorphic functions, we have $\mathcal{C} K_{\varphi(z_0)}(z) = 0$ for all $z \in \mathcal{D}$. Thus

$$\|K_{\varphi(z_0)}\| = \|\mathcal{C} K_{\varphi(z_0)}\| = 0,$$

which is a contradiction. □

Obviously, if ψ is identically zero, $W_{\psi,\varphi}$ must also be identically zero. Such a case is trivial and we shall assume from now on, ψ is not identically zero. Thus Proposition 2.2.5 tells us $\psi(z)$ is never zero at any $z \in \mathcal{D}$.

A word of caution to the reader: thus far we have freely used $\mathcal{H}$ to denote a separable complex Hilbert space (in Chapter 1) and a complex RKHS (this chapter). We will continue to use $\mathcal{H}$ interchangeably without explicit mention, but it will be clear from the context which $\mathcal{H}$ are we referring to. In the case where $\mathcal{H}$ and K_w are mentioned together, it shall be known that $\mathcal{H}$ is a complex

RKHS. If there is no mention of K_w , it shall be known that $\mathcal{H}$ is merely a separable complex Hilbert space.

Care should also be taken to distinguish between K_w and $K_w^{(\gamma)}$. The former will be used to denote the reproducing kernel for $\mathcal{H}$ and the latter for $\mathcal{H}_\gamma(\mathbb{D})$.

Chapter 3

Weighted composition conjugations

In this chapter, we characterize anti-linear weighted composition operators on $\mathcal{H}_\gamma(\mathbb{D})$ that are conjugations, which we name as weighted composition conjugations. To do so, we require the $\mathcal{J}$ -symmetric structure of $W_{\psi,\varphi}$ on $\mathcal{H}_\gamma(\mathbb{D})$ and various other preliminary results.

3.1 The $\mathcal{J}$ -symmetry of $W_{\psi,\varphi}$ on $\mathcal{H}_\gamma(\mathbb{D})$

First, we find a criterion for the $\mathcal{J}$ -symmetry of $W_{\psi,\varphi}$ on $\mathcal{H}_\gamma(\mathbb{D})$, which allows us to study properties of weighted composition conjugations.

The following proposition is identical to Proposition 2.9 of [4] for the weighted Hardy spaces. However, we would like to emphasize the need for $\gamma \in \mathbb{N}$ in the proof of the sufficiency part. It should be noted that the requirement $\gamma \in \mathbb{N}$ is essential, as in general, for any $a, b \in \mathbb{C}$ and $\gamma > 0$, $(ab)^\gamma \neq a^\gamma b^\gamma$, while the equality holds when $\gamma \in \mathbb{N}$.

Proposition 3.1.1. *Let $\gamma \in \mathbb{N}$. Let $\psi : \mathbb{D} \rightarrow \mathbb{C}$ and $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions such that $W_{\psi,\varphi} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$. Then $W_{\psi,\varphi}$ is $\mathcal{J}$ -symmetric if and*

only if for all $z \in \mathbb{D}$,

$$\psi(z) = \frac{b}{(1 - a_0 z)^\gamma} \quad \text{and} \quad \varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z},$$

where $a_0, a_1 \in \mathbb{D}$ and $b \in \mathbb{C}$.

Proof. Necessity. Suppose that $W_{\psi, \varphi}$ is $\mathcal{J}$ -symmetric. By Proposition 2.2.4, we can write

$$W_{\psi, \varphi} \mathcal{J} K_w^{(\gamma)}(z) = \mathcal{J} W_{\psi, \varphi}^* K_w^{(\gamma)}(z) \quad (3.1.1)$$

for all $w, z \in \mathbb{D}$.

We simplify the left hand side of (3.1.1) to get

$$\begin{aligned} W_{\psi, \varphi} \mathcal{J} K_w^{(\gamma)}(z) &= \psi(z) \overline{K_w^{(\gamma)}(\overline{\varphi(z)})} \\ &= \frac{\psi(z)}{(1 - w\varphi(z))^\gamma} \end{aligned} \quad \text{by (1.3.1)}$$

and simplify the right hand side of (3.1.1) to get

$$\begin{aligned} \mathcal{J} W_{\psi, \varphi}^* K_w^{(\gamma)}(z) &= \mathcal{J} \left(\overline{\psi(w)} K_{\varphi(w)}(z) \right) \quad \text{by (2.2.1)} \\ &= \psi(w) \overline{K_{\varphi(w)}(\overline{z})} \\ &= \frac{\psi(w)}{(1 - z\varphi(w))^\gamma} \end{aligned} \quad \text{by (1.3.1)}$$

Hence (3.1.1) reduces to

$$\frac{\psi(z)}{(1 - w\varphi(z))^\gamma} = \frac{\psi(w)}{(1 - z\varphi(w))^\gamma} \quad (3.1.2)$$

for all $w, z \in \mathbb{D}$.

Since (3.1.2) holds for all $w \in \mathbb{D}$, we set $w = 0$ to get

$$\psi(z) = \frac{b}{(1 - a_0 z)^\gamma} \quad (3.1.3)$$

for all $z \in \mathbb{D}$, where $a_0 = \varphi(0)$ and $b = \psi(0)$.

Now we substitute (3.1.3) back into (3.1.2) to get

$$(1 - a_0 w)^\gamma (1 - z\varphi(w))^\gamma = (1 - a_0 z)^\gamma (1 - w\varphi(z))^\gamma$$

for all $w, z \in \mathbb{D}$. Since φ is holomorphic, we differentiate the above equation with respect to w to get

$$\begin{aligned} & (1 - a_0 w)^\gamma \gamma (1 - z\varphi(w))^{\gamma-1} (-z\varphi'(w)) + (1 - z\varphi(w))^\gamma \gamma (1 - a_0 w)^{\gamma-1} (-a_0) \\ &= (1 - a_0 z)^\gamma \gamma (1 - w\varphi(z))^{\gamma-1} (-\varphi(z)). \end{aligned}$$

Finally, we set $w = 0$ again to get

$$\begin{aligned} & \gamma(1 - a_0 z)^{\gamma-1} (-a_1 z) + (1 - a_0 z)^\gamma (-\gamma a_0) = (1 - a_0 z)^\gamma (-\gamma \varphi(z)) \\ & \iff \varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z} \end{aligned}$$

for all $z \in \mathbb{D}$, where $a_1 = \varphi'(0)$.

Sufficiency. Suppose that for all $z \in \mathbb{D}$

$$\psi(z) = \frac{b}{(1 - a_0 z)^\gamma} \quad \text{and} \quad \varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z},$$

where $a_0, a_1 \in \mathbb{D}$ and $b \in \mathbb{C}$.

Since $W_{\psi,\varphi}$ is bounded on $\mathcal{H}_\gamma(\mathbb{D})$, it suffices to prove that (3.1.1) holds for all $w, z \in \mathbb{D}$.

On one hand, we have

$$\begin{aligned} W_{\psi,\varphi} \mathcal{J}K_w^{(\gamma)}(z) &= \frac{\psi(z)}{(1 - w\varphi(z))^\gamma} \\ &= \frac{b}{(1 - a_0 z)^\gamma} \cdot \left(1 - w \left(a_0 + \frac{a_1 z}{1 - a_0 z} \right) \right)^{-\gamma} \\ &= b((1 - a_0 z) - a_0 w(1 - a_0 z) - a_1 w z)^{-\gamma} \\ &= b(1 - a_0(w + z) + wz(a_0^2 - a_1))^{-\gamma}. \end{aligned}$$

Here it is essential that the third equality holds, due to $\gamma \in \mathbb{N}$.

On the other hand, we have

$$\begin{aligned} \mathcal{J}W_{\psi,\varphi}^* K_w^{(\gamma)}(z) &= \frac{\psi(w)}{(1 - z\varphi(w))^\gamma} \\ &= b(1 - a_0(w + z) + wz(a_0^2 - a_1))^{-\gamma}. \end{aligned}$$

Thus (3.1.1) holds for all $w, z \in \mathbb{D}$. □

We shall no longer mention the fact $\gamma \in \mathbb{N}$ in the other proofs to keep the exposition short. We also remark that Proposition 3.1.1 contains the results of [11] as a particular case. Simply set $\gamma = 1$.

3.2 Preliminary results

Let $\gamma \in \mathbb{N}$. Let $u : \mathbb{D} \rightarrow \mathbb{C}$ and $v : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions. For all $f \in \mathcal{H}_\gamma(\mathbb{D})$ and $z \in \mathbb{D}$, the functions u and v induce an anti-linear operator on $\mathcal{H}_\gamma(\mathbb{D})$ defined by

$$\mathcal{A}_{u,v}f(z) = u(z) \cdot \overline{f(\overline{v(z)})}.$$

The operator $\mathcal{A}_{u,v}$ is known as an anti-linear weighted composition operator on $\mathcal{H}_\gamma(\mathbb{D})$. The standard conjugation $\mathcal{J}f(z) = \overline{f(\bar{z})}$ is a specific case of $\mathcal{A}_{u,v}$. Simply set $u(z) = 1$ and $v(z) = z$.

From this point on, we always assume that $\mathcal{A}_{u,v} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$. It is easy to see that $W_{u,v} = \mathcal{A}_{u,v}\mathcal{J}$, thus $W_{u,v}$ is invariant (hence bounded) on $\mathcal{H}_\gamma(\mathbb{D})$. Equivalently, we have

$$\mathcal{A}_{u,v} = W_{u,v}\mathcal{J}, \tag{3.2.1}$$

which is a composition of two bounded operators, thus $\mathcal{A}_{u,v}$ is bounded too.

We note the following property which relates the adjoint of composite operators to the composition of adjoints.

Proposition 3.2.1. *Let $S, T : \mathcal{H} \rightarrow \mathcal{H}$ be bounded operators that are either linear or anti-linear. The property $(ST)^* = T^*S^*$ holds.*

Proof. Let $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{C}$.

Case 1. Suppose that both S and T are linear. Then

$$ST(\alpha x + y) = S(\alpha T(x) + T(y)) = \alpha ST(x) + ST(y),$$

i.e. ST is linear. Hence

$$\langle x, (ST)^*y \rangle = \langle STx, y \rangle = \langle Tx, S^*y \rangle = \langle x, T^*S^*y \rangle,$$

which implies $(ST)^* = T^*S^*$.

Case 2. Suppose that both S and T are anti-linear. Then

$$ST(\alpha x + y) = S(\bar{\alpha}T(x) + T(y)) = \alpha ST(x) + ST(y),$$

i.e. ST is linear. Hence

$$\langle x, (ST)^*y \rangle = \langle STx, y \rangle = \langle S^*y, Tx \rangle = \langle x, T^*S^*y \rangle,$$

which implies $(ST)^* = T^*S^*$.

Case 3. Without loss of generality assume that S is linear and T is anti-linear. Then

$$ST(\alpha x + y) = S(\bar{\alpha}T(x) + T(y)) = \bar{\alpha}ST(x) + ST(y),$$

i.e. ST is anti-linear. Hence

$$\langle (ST)^*y, x \rangle = \langle STx, y \rangle = \langle Tx, S^*y \rangle = \langle T^*S^*y, x \rangle,$$

which implies $(ST)^* = T^*S^*$. □

We now establish the relationship between $\mathcal{A}_{u,v}$ and $W_{u,v}$ when $\mathcal{A}_{u,v}$ is either self-adjoint or unitary.

Proposition 3.2.2. *Let $u : \mathbb{D} \rightarrow \mathbb{C}$ and $v : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions such that $W_{u,v} : \mathcal{H} \rightarrow \mathcal{H}$. Then $\mathcal{A}_{u,v}$ is self-adjoint if and only if $W_{u,v}$ is $\mathcal{J}$ -symmetric.*

Proof. By (3.2.1), we have

$$\mathcal{A}_{u,v} = \mathcal{A}_{u,v}^* \iff W_{u,v}\mathcal{J} = (W_{u,v}\mathcal{J})^*.$$

By Proposition 3.2.1 and self-adjointness of $\mathcal{J}$ respectively, we get

$$(W_{u,v}\mathcal{J})^* = \mathcal{J}^*W_{u,v}^* = \mathcal{J}W_{u,v}^*.$$

Now, the involutivity of $\mathcal{J}$ gives us

$$W_{u,v}\mathcal{J} = \mathcal{J}W_{u,v}^* \iff W_{u,v} = \mathcal{J}W_{u,v}^*\mathcal{J}.$$

□

Proposition 3.2.3. *Let $u : \mathbb{D} \rightarrow \mathbb{C}$ and $v : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions such that $W_{u,v} : \mathcal{H} \rightarrow \mathcal{H}$. Then $\mathcal{A}_{u,v}$ is unitary if and only if $W_{u,v}$ is unitary.*

Proof. By (3.2.1), we have

$$\begin{aligned} \mathcal{A}_{u,v} \mathcal{A}_{u,v}^* &= \mathcal{A}_{u,v}^* \mathcal{A}_{u,v} = \mathcal{I} \\ \iff (W_{u,v} \mathcal{J})(W_{u,v} \mathcal{J})^* &= (W_{u,v} \mathcal{J})^* (W_{u,v} \mathcal{J}) = \mathcal{I}. \end{aligned}$$

By Proposition 3.2.1, we get

$$W_{u,v} \mathcal{J} \mathcal{J}^* W_{u,v}^* = \mathcal{I} \text{ and } \mathcal{J}^* W_{u,v}^* W_{u,v} \mathcal{J} = \mathcal{I}$$

respectively. Since $\mathcal{J}$ is self-adjoint and $\mathcal{J}^2 = \mathcal{I}$, we have

$$W_{u,v} W_{u,v}^* = \mathcal{I} \text{ and } W_{u,v}^* W_{u,v} = \mathcal{I}.$$

□

As a result of Propositions 3.2.2 and 3.2.3, we are now able to study self-adjoint $\mathcal{A}_{u,v}$ and unitary $\mathcal{A}_{u,v}$ in terms of $W_{u,v}$, which was already investigated in [12] and Proposition 3.1.1. An automorphism on $\mathbb{D}$ is a holomorphic function $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ that is also bijective. Recall that the set of all automorphisms of $\mathbb{D}$ is

$$\text{Aut}(\mathbb{D}) = \left\{ \beta \cdot \frac{\alpha - z}{1 - \bar{\alpha}z} : \alpha \in \mathbb{D}, \beta \in \partial\mathbb{D} \right\}.$$

Corollary 3.6 of [12] gives equivalent assertions for bounded (or equivalently, invariant) $W_{\psi,\varphi}$. We restate the corollary in the context of our work.

Proposition 3.2.4 ([12]). *Let $\gamma \in \mathbb{N}$. Let $\psi : \mathbb{D} \rightarrow \mathbb{C}$ and $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions such that $W_{\psi,\varphi} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$. Then $W_{\psi,\varphi}$ is unitary if and only if for all $z \in \mathbb{D}$,*

$$\psi = \lambda k_{\varphi^{-1}(0)}^{(\gamma)} \quad \text{and} \quad \varphi \in \text{Aut}(\mathbb{D}),$$

where $\lambda \in \partial\mathbb{D}$.

The next result gives an explicit condition for φ when it is both of the form in Proposition 3.1.1 and $\varphi \in \text{Aut}(\mathbb{D})$.

Proposition 3.2.5. *Let $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic for all $z \in \mathbb{D}$ and*

$$\varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z},$$

where $a_0, a_1 \in \mathbb{D}$. If $\varphi \in \text{Aut}(\mathbb{D})$, then we have either of the following:

1. *there exist $\beta \in \partial\mathbb{D}$ such that*

$$\varphi(z) = \beta z. \tag{3.2.2}$$

2. *there exists $\alpha \in \mathbb{D} \setminus \{0\}$ such that*

$$\varphi(z) = \frac{\bar{\alpha}}{\alpha} \cdot \frac{\alpha - z}{1 - \bar{\alpha}z}. \tag{3.2.3}$$

Proof. By our initial assumption,

$$\varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z} = \frac{a_0 + (a_1 - a_0^2)z}{1 - a_0 z}$$

for all $z \in \mathbb{D}$. Now suppose that $\varphi \in \text{Aut}(\mathbb{D})$. So there exist $\alpha \in \mathbb{D}$ and $\beta \in \partial\mathbb{D}$ such that for all $z \in \mathbb{D}$,

$$\varphi(z) = \frac{\beta\alpha - \beta z}{1 - \bar{\alpha}z}.$$

Equating both expressions for φ give us

$$\begin{aligned} \frac{a_0 + (a_1 - a_0^2)z}{1 - a_0 z} &= \frac{\beta\alpha - \beta z}{1 - \bar{\alpha}z} \\ \iff (1 - \bar{\alpha}z)(a_0 + (a_1 - a_0^2)z) &= (1 - a_0 z)(\beta\alpha - \beta z) \\ \iff a_0 + (a_1 - a_0^2 - \bar{\alpha}a_0)z + \bar{\alpha}(a_0^2 - a_1)z^2 \\ &= \beta\alpha - \beta(a_0\alpha + 1)z + a_0\beta z^2. \end{aligned}$$

Comparing the constants and the coefficients of z and z^2 respectively,

$$a_0 = \beta\alpha, \tag{3.2.4}$$

$$a_1 - a_0^2 - \bar{\alpha}a_0 = -\beta(a_0\alpha + 1), \tag{3.2.5}$$

$$\bar{\alpha}(a_0^2 - a_1) = a_0\beta. \tag{3.2.6}$$

Case 1. If $\alpha = 0$, then $a_0 = 0$ by (3.2.4) and $a_1 = -\beta$ by (3.2.5). Hence $\varphi(z) = -\beta z$. This proves the case of (3.2.2) since $-\beta \in \partial\mathbb{D}$ too.

Case 2. Now suppose that $\alpha \in \mathbb{D} \setminus \{0\}$. We substitute $a_0 = \beta\alpha$ into (3.2.6) to get

$$a_1 = \frac{\beta^2\alpha(|\alpha|^2 - 1)}{\bar{\alpha}}$$

and further substitute a_0, a_1 into (3.2.5) to get

$$\begin{aligned} \frac{\beta^2\alpha(|\alpha|^2 - 1)}{\bar{\alpha}} - (\beta\alpha)^2 - \bar{\alpha}\beta\alpha &= -\beta(\beta\alpha^2 + 1) \\ \iff \beta^2\alpha(|\alpha|^2 - 1) &= \beta\bar{\alpha}(|\alpha|^2 - 1). \end{aligned}$$

Since $\beta \neq 0$ and $|\alpha|^2 - 1 \neq 0$, we divide both sides by $\beta(|\alpha|^2 - 1)$ to get $\beta = \frac{\bar{\alpha}}{\alpha}$. Hence we have

$$\begin{aligned} \varphi(z) &= \frac{a_0 + (a_1 - a_0^2)z}{1 - a_0z} = \beta \cdot \frac{\alpha + \left(\frac{\beta\alpha(|\alpha|^2 - 1)}{\bar{\alpha}} - \beta\alpha^2 \right)z}{1 - \beta\alpha z} \\ &= \frac{\bar{\alpha}}{\alpha} \cdot \frac{\alpha + (|\alpha|^2 - 1)z - \alpha\bar{\alpha}z}{1 - \bar{\alpha}z} \\ &= \frac{\bar{\alpha}}{\alpha} \cdot \frac{\alpha - z}{1 - \bar{\alpha}z}. \end{aligned}$$

This proves the case of (3.2.3). □

3.3 Criterion for $\mathcal{A}_{u,v}$ to be a conjugation

We first find an explicit condition for $\mathcal{A}_{u,v}$ to be self-adjoint.

Theorem 3.3.1. *Let $\gamma \in \mathbb{N}$. Let $u : \mathbb{D} \rightarrow \mathbb{C}$ and $v : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions. Then $\mathcal{A}_{u,v} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$ is self-adjoint if and only if for all $z \in \mathbb{D}$,*

$$u(z) = \frac{b}{(1 - a_0z)^\gamma} \quad \text{and} \quad v(z) = a_0 + \frac{a_1z}{1 - a_0z},$$

where $a_0, a_1 \in \mathbb{D}$ and $b \in \mathbb{C}$.

Proof. We simply combine Propositions 3.1.1 and 3.2.2. □

Similarly, we find an explicit condition for $\mathcal{A}_{u,v}$ to be unitary.

Theorem 3.3.2. *Let $\gamma \in \mathbb{N}$. Let $u : \mathbb{D} \rightarrow \mathbb{C}$ and $v : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions. Then $\mathcal{A}_{u,v} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$ is unitary if and only if for all $z \in \mathbb{D}$,*

$$u = \lambda k_{v^{-1}(0)}^{(\gamma)} \quad \text{and} \quad v \in \text{Aut}(\mathbb{D}),$$

where $\lambda \in \partial\mathbb{D}$.

Proof. We simply combine Propositions 3.2.3 and 3.2.4. □

We shall now use Proposition 1.2.3, Theorems 3.3.1 and 3.3.2 to characterize all weighted composition conjugations.

Theorem 3.3.3 (Criterion for conjugation). *Let $\gamma \in \mathbb{N}$. Let $u : \mathbb{D} \rightarrow \mathbb{C}$ and $v : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions. Then $\mathcal{A}_{u,v} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$ is a conjugation if and only if either of the following holds:*

1. *there exist $\alpha, \lambda \in \partial\mathbb{D}$ such that for all $z \in \mathbb{D}$,*

$$u(z) = \lambda \quad \text{and} \quad v(z) = \alpha z. \tag{3.3.1}$$

2. *there exist $\alpha \in \mathbb{D} \setminus \{0\}$ and $\lambda \in \partial\mathbb{D}$ such that for all $z \in \mathbb{D}$,*

$$u(z) = \lambda k_\alpha^{(\gamma)}(z) \quad \text{and} \quad v(z) = \frac{\bar{\alpha}}{\alpha} \cdot \frac{\alpha - z}{1 - \bar{\alpha}z}. \tag{3.3.2}$$

Proof. Necessity. Suppose that $\mathcal{A}_{u,v}$ is a conjugation. Then $\mathcal{A}_{u,v}$ is both self-adjoint and unitary. By Theorems 3.3.1 and 3.3.2 respectively, we have

$$v(z) = a_0 + \frac{a_1 z}{1 - \bar{a}_0 z} \quad \text{and} \quad v \in \text{Aut}(\mathbb{D}).$$

By Proposition 3.2.5, v is either of the form (3.2.2) or (3.2.3).

Case 1. There exist $\alpha \in \partial\mathbb{D}$ such that for all $z \in \mathbb{D}$, v is of the form (3.2.2).

Then

$$v(0) = 0 \iff 0 = v^{-1}(0)$$

and hence Theorem 3.3.2 tells us $u(z) = \lambda k_0^{(\gamma)}(z) = \lambda$, which is also consistent with Theorem 3.3.1. This proves (3.3.1).

Case 2. There exist $\alpha \in \mathbb{D} \setminus \{0\}$ such that for all $z \in \mathbb{D}$, v is of the form (3.2.3). Then

$$v(\alpha) = 0 \iff \alpha = v^{-1}(0)$$

and hence Theorem 3.3.2 tells us $u(z) = \lambda k_\alpha^{(\gamma)}(z)$, which is also consistent with Theorem 3.3.1. This proves (3.3.2).

Sufficiency. Now suppose that we have either (3.3.1) or (3.3.2).

Case 1. There exist $\alpha, \lambda \in \partial\mathbb{D}$ such that for all $z \in \mathbb{D}$, we have (3.3.1). Clearly, $v \in \text{Aut}(\mathbb{D})$ and $v^{-1}(0) = 0$. We can write

$$u(z) = \lambda k_0^{(\gamma)}(z) = \lambda k_{v^{-1}(0)}^{(\gamma)}(z)$$

and according to Theorem 3.3.2, $\mathcal{A}_{u,v}$ is unitary. Moreover, we can write u and v into the forms given in Theorem 3.3.1 by setting $b = \lambda, a_0 = 0$ and $a_1 = \alpha$. Hence $\mathcal{A}_{u,v}$ is self-adjoint.

Case 2. There exist $\alpha \in \mathbb{D} \setminus \{0\}$ and $\lambda \in \partial\mathbb{D}$ such that for all $z \in \mathbb{D}$, we have (3.3.2). Clearly, $v \in \text{Aut}(\mathbb{D})$ and $v^{-1}(0) = \alpha$. We can write

$$u(z) = \lambda k_\alpha^{(\gamma)}(z) = \lambda k_{v^{-1}(0)}^{(\gamma)}(z)$$

and according to Theorem 3.3.2, $\mathcal{A}_{u,v}$ is unitary. Moreover, we can write u and v into the forms given in Theorem 3.3.1 by setting $b = \lambda(1 - |\alpha|^2)^{\frac{\gamma}{2}}, a_0 = \bar{\alpha}$ and $a_1 = \frac{\bar{\alpha}}{\alpha}(|\alpha|^2 - 1)$. Hence $\mathcal{A}_{u,v}$ is self-adjoint.

Therefore, $\mathcal{A}_{u,v}$ is a conjugation in both cases. $\square$

We denote $\mathcal{A}_{u,v}$ with u and v of the form (3.3.1) by $\mathcal{C}_1$ and $\mathcal{A}_{u,v}$ with u and v of the form (3.3.2) by $\mathcal{C}_2$.

Chapter 4

Complex symmetric weighted composition operators

As a direct consequence of Theorem 3.3.3, there are only two classes of conjugations $\mathcal{A}_{u,v}$ on $\mathcal{H}_\gamma(\mathbb{D})$, either $\mathcal{C}_1$ or $\mathcal{C}_2$. To obtain explicit conditions for $W_{\psi,\varphi}$ to be complex symmetric, it suffices to find holomorphic functions $\psi : \mathbb{D} \rightarrow \mathbb{C}$ and $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ such that $W_{\psi,\varphi}$ is either **(a)** $\mathcal{C}_1$ -symmetric or **(b)** $\mathcal{C}_2$ -symmetric.

4.1 $\mathcal{C}_1$ -symmetry of $W_{\psi,\varphi}$

The next theorem is our first significant result which characterizes complex symmetric $W_{\psi,\varphi}$ with respect to the weighted composition conjugation $\mathcal{C}_1$.

Theorem 4.1.1 ($\mathcal{C}_1$ -symmetry of $W_{\psi,\varphi}$). *Let $\gamma \in \mathbb{N}$. Let $\psi : \mathbb{D} \rightarrow \mathbb{C}$ and $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions such that $W_{\psi,\varphi} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$. Suppose that $\mathcal{C}_1 := \mathcal{A}_{u,v}$ is a conjugation with u and v of the form (3.3.1), i.e. there exist $\alpha, \lambda \in \partial\mathbb{D}$ such that*

$$u(z) = \lambda \quad \text{and} \quad v(z) = \alpha z.$$

Then $W_{\psi,\varphi}$ is $\mathcal{C}_1$ -symmetric if and only if for all $z \in \mathbb{D}$

$$\psi(z) = \frac{d}{(1 - \alpha c_0 z)^\gamma} \quad \text{and} \quad \varphi(z) = c_0 + \frac{c_1 z}{1 - \alpha c_0 z},$$

where $c_0, c_1 \in \mathbb{D}$ and $d \in \mathbb{C}$.

Proof. Necessity. Suppose that $W_{\psi,\varphi}$ is $\mathcal{C}_1$ -symmetric. By Proposition 2.2.4, we can write

$$W_{\psi,\varphi} \mathcal{C}_1 K_w^{(\gamma)}(z) = \mathcal{C}_1 W_{\psi,\varphi}^* K_w^{(\gamma)}(z) \quad (4.1.1)$$

for all $w, z \in \mathbb{D}$.

We simplify the left hand side of (4.1.1) to get

$$\begin{aligned} W_{\psi,\varphi} \mathcal{C}_1 K_w^{(\gamma)}(z) &= \lambda \psi(z) \overline{K_w^{(\gamma)}(\overline{\alpha\varphi(z)})} \\ &= \frac{\lambda \psi(z)}{(1 - \alpha w \varphi(z))^\gamma} \end{aligned} \quad \text{by (1.3.1)}$$

and simplify the right hand side of (4.1.1) to get

$$\begin{aligned} \mathcal{C}_1 W_{\psi,\varphi}^* K_w^{(\gamma)}(z) &= \mathcal{C}_1 \left(\overline{\psi(w)} K_{\varphi(w)}(z) \right) \quad \text{by (2.2.1)} \\ &= \lambda \psi(w) \overline{K_{\varphi(w)}(\overline{\alpha z})} \\ &= \frac{\lambda \psi(w)}{(1 - \alpha z \varphi(w))^\gamma} \end{aligned} \quad \text{by (1.3.1)}$$

Hence (4.1.1) becomes

$$\frac{\lambda \psi(z)}{(1 - \alpha w \varphi(z))^\gamma} = \frac{\lambda \psi(w)}{(1 - \alpha z \varphi(w))^\gamma}$$

and thus

$$\psi(z) = \psi(w) \left(\frac{1 - \alpha w \varphi(z)}{1 - \alpha z \varphi(w)} \right)^\gamma \quad (4.1.2)$$

for all $w, z \in \mathbb{D}$. Since (4.1.2) holds for all $w \in \mathbb{D}$, we set $w = 0$ to get

$$\psi(z) = \frac{d}{(1 - \alpha c_0 z)^\gamma} \quad (4.1.3)$$

for all $z \in \mathbb{D}$, where $c_0 = \varphi(0)$ and $d = \psi(0)$.

Now we substitute (4.1.3) back into (4.1.2) to get

$$\begin{aligned} \frac{d}{(1 - \alpha c_0 z)^\gamma} &= \frac{d}{(1 - \alpha c_0 w)^\gamma} \left(\frac{1 - \alpha w \varphi(z)}{1 - \alpha z \varphi(w)} \right)^\gamma \\ \iff (1 - \alpha c_0 w)^\gamma (1 - \alpha z \varphi(w))^\gamma &= (1 - \alpha c_0 z)^\gamma (1 - \alpha w \varphi(z))^\gamma \end{aligned}$$

for all $w, z \in \mathbb{D}$. Since φ is holomorphic, we differentiate the above equation with respect to w to get

$$\begin{aligned} & (1 - \alpha c_0 w)^\gamma \gamma (1 - \alpha z \varphi(w))^{\gamma-1} (-\alpha z \varphi'(w)) \\ & + (1 - \alpha z \varphi(w))^\gamma \gamma (1 - \alpha c_0 w)^{\gamma-1} (-\alpha c_0) \\ & = (1 - \alpha c_0 z)^\gamma \gamma (1 - \alpha w \varphi(z))^{\gamma-1} (-\alpha \varphi(z)). \end{aligned}$$

Now we set $w = 0$ again to get

$$\begin{aligned} & z \varphi'(0) (1 - \alpha c_0 z)^{\gamma-1} + c_0 (1 - \alpha c_0 z)^\gamma = \varphi(z) (1 - \alpha c_0 z)^\gamma \\ \iff & \varphi(z) = c_0 + \frac{c_1 z}{1 - \alpha c_0 z} \end{aligned}$$

for all $z \in \mathbb{D}$, where $c_1 = \varphi'(0)$.

Sufficiency. Suppose that for all $z \in \mathbb{D}$,

$$\psi(z) = \frac{d}{(1 - \alpha c_0 z)^\gamma} \quad \text{and} \quad \varphi(z) = c_0 + \frac{c_1 z}{1 - \alpha c_0 z},$$

where $c_0, c_1 \in \mathbb{D}$ and $d \in \mathbb{C}$.

By our assumption, $W_{\psi,\varphi}$ is bounded on $\mathcal{H}_\gamma(\mathbb{D})$. Hence it suffices to prove that (4.1.1) holds for all $w, z \in \mathbb{D}$. Indeed we have

$$\begin{aligned} W_{\psi,\varphi} \mathcal{C}_1 K_w^{(\gamma)}(z) &= \psi(z) \cdot \frac{\lambda}{(1 - w \alpha \varphi(z))^\gamma} \\ &= \frac{d \lambda}{(1 - \alpha c_0 z)^\gamma} \left(1 - w \alpha \left(c_0 + \frac{c_1 z}{1 - \alpha c_0 z} \right) \right)^{-\gamma} \\ &= \frac{d \lambda}{(1 - \alpha c_0 z - w \alpha (c_0 (1 - \alpha c_0 z) + c_1 z))^\gamma} \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_1 W_{\psi,\varphi}^* K_w^{(\gamma)}(z) &= \psi(w) \cdot \frac{\lambda}{(1 - \varphi(w) \alpha z)^\gamma} \\ &= \frac{d \lambda}{(1 - \alpha c_0 w)^\gamma} \left(1 - \left(c_0 + \frac{c_1 w}{1 - \alpha c_0 w} \right) \alpha z \right)^{-\gamma} \\ &= \frac{d \lambda}{(1 - \alpha c_0 w - z \alpha (c_0 (1 - \alpha c_0 w) + c_1 w))^\gamma} \\ &= \frac{d \lambda}{(1 - \alpha c_0 z - w \alpha (c_0 (1 - \alpha c_0 z) + c_1 z))^\gamma}. \end{aligned}$$

Thus (4.1.1) holds for all $w, z \in \mathbb{D}$. □

Note that Theorem 4.1.1 contains Proposition 3.1.1 as a particular case, which in turn contains the results of [11]. Simply set $\alpha = \lambda = 1$ and we have $A_{u,v} = \mathcal{J}$.

4.2 $\mathcal{C}_2$ -symmetry of $W_{\psi,\varphi}$

Now we present our second and final significant result which characterizes complex symmetric $W_{\psi,\varphi}$ with respect to the weighted composition conjugation $\mathcal{C}_2$. The proof of the next theorem is similar to that of Theorem 4.1.1, but contains lengthier algebraic manipulations.

Theorem 4.2.1 ($\mathcal{C}_2$ -symmetry of $W_{\psi,\varphi}$). *Let $\gamma \in \mathbb{N}$. Let $\psi : \mathbb{D} \rightarrow \mathbb{C}$ and $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions such that $W_{\psi,\varphi} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$. Suppose that $\mathcal{C}_2 := \mathcal{A}_{u,v}$ is a conjugation with u and v of the form (3.3.2), i.e. there exist $\alpha \in \mathbb{D} \setminus \{0\}$ and $\lambda \in \partial\mathbb{D}$ such that*

$$u(z) = \lambda k_\alpha^{(\gamma)}(z) \quad \text{and} \quad v(z) = \frac{\bar{\alpha}}{\alpha} \cdot \frac{\alpha - z}{1 - \bar{\alpha}z}.$$

Then $W_{\psi,\varphi}$ is $\mathcal{C}_2$ -symmetric if and only if for all $z \in \mathbb{D}$

$$\psi(z) = \frac{d(c_0^2 - \alpha c_1)^\gamma}{(c_0^2 - \alpha c_1 - (c_1 - c_2)z)^\gamma} \quad \text{and}$$

$$\varphi(z) = \frac{\alpha(\bar{\alpha}c_0^2 - c_1) - (|\alpha|^2 c_1 - c_2)z}{\bar{\alpha}(c_0^2 - \alpha c_1) - \bar{\alpha}(c_1 - c_2)z},$$

where $c_0, c_1, c_2, d \in \mathbb{C}$.

Proof. Necessity. Suppose that $W_{\psi,\varphi}$ is $\mathcal{C}_2$ -symmetric. By Proposition 2.2.4, we can write

$$W_{\psi,\varphi} \mathcal{C}_2 K_w^{(\gamma)}(z) = \mathcal{C}_2 W_{\psi,\varphi}^* K_w^{(\gamma)}(z) \tag{4.2.1}$$

for all $w, z \in \mathbb{D}$.

We simplify the left hand side of (4.2.1) to get

$$\begin{aligned}
W_{\psi,\varphi}\mathcal{C}_2K_w^{(\gamma)}(z) &= \frac{\psi(z)u(\varphi(z))}{(1-w \cdot v(\varphi(z)))^\gamma} && \text{by (1.3.1)} \\
&= \frac{\lambda(1-|\alpha|^2)^{\frac{\gamma}{2}}}{(1-\bar{\alpha}\varphi(z))^\gamma} \cdot \frac{\psi(z)}{\left(1-w\left(\frac{\bar{\alpha}}{\alpha} \cdot \frac{\alpha-\varphi(z)}{1-\bar{\alpha}\varphi(z)}\right)\right)^\gamma} && \text{by (1.3.2)} \\
&= \frac{\alpha^\gamma \lambda \psi(z) (1-|\alpha|^2)^{\frac{\gamma}{2}}}{(\alpha(1-\bar{\alpha}\varphi(z)) - \bar{\alpha}w(\alpha - \varphi(z)))^\gamma}
\end{aligned}$$

and simplify the right hand side of (4.2.1) to get

$$\begin{aligned}
\mathcal{C}_2W_{\psi,\varphi}^*K_w^{(\gamma)}(z) &= \mathcal{C}_2\left(\overline{\psi(w)}K_{\varphi(w)}(z)\right) && \text{by (2.2.1)} \\
&= \frac{u(z)\psi(w)}{(1-\varphi(w)v(z))^\gamma} && \text{by (1.3.1)} \\
&= \frac{\lambda(1-|\alpha|^2)^{\frac{\gamma}{2}}}{(1-\bar{\alpha}z)^\gamma} \cdot \frac{\psi(w)}{\left(1-\varphi(w)\left(\frac{\bar{\alpha}}{\alpha} \cdot \frac{\alpha-z}{1-\bar{\alpha}z}\right)\right)^\gamma} && \text{by (1.3.2)} \\
&= \frac{\alpha^\gamma \lambda \psi(w) (1-|\alpha|^2)^{\frac{\gamma}{2}}}{(\alpha(1-\bar{\alpha}\varphi(w)) - \bar{\alpha}z(\alpha - \varphi(w)))^\gamma}.
\end{aligned}$$

In order to simplify the exposition, we introduce for any $w, z \in \mathbb{D}$, the following functions

$$p_z(w) := \alpha(1 - \bar{\alpha}\varphi(z)) - \bar{\alpha}w(\alpha - \varphi(z))$$

and

$$q_z(w) := \alpha(1 - \bar{\alpha}\varphi(w)) - \bar{\alpha}z(\alpha - \varphi(w)).$$

We note that by the above notation, we have $p_z(w) = q_w(z)$ for any $w, z \in \mathbb{D}$.

Now, we can write (4.2.1) as

$$\frac{\alpha^\gamma \lambda \psi(z) (1-|\alpha|^2)^{\frac{\gamma}{2}}}{p_z(w)^\gamma} = \frac{\alpha^\gamma \lambda \psi(w) (1-|\alpha|^2)^{\frac{\gamma}{2}}}{q_z(w)^\gamma}$$

and thus

$$\frac{\psi(z)}{p_z(w)^\gamma} = \frac{\psi(w)}{q_z(w)^\gamma} \tag{4.2.2}$$

for all $w, z \in \mathbb{D}$. Since (4.2.2) holds for all $w \in \mathbb{D}$, we set $w = 0$ and thus

$$\psi(z) = \frac{\psi(0)p_z(0)^\gamma}{q_z(0)^\gamma} \tag{4.2.3}$$

for all $z \in \mathbb{D}$.

Now we substitute (4.2.3) back into (4.2.2) to get

$$\frac{\psi(0)p_z(0)^\gamma}{p_z(w)^\gamma q_z(0)^\gamma} = \frac{\psi(0)p_w(0)^\gamma}{q_z(w)^\gamma q_w(0)^\gamma},$$

which can also be written as

$$\frac{\psi(0)p_z(0)^\gamma}{p_z(w)^\gamma q_z(0)^\gamma} = \frac{\psi(0)q_0(w)^\gamma}{q_z(w)^\gamma p_0(w)^\gamma}$$

and further reduced to

$$\left(\frac{p_z(0)}{q_z(0)}\right)^\gamma (q_z(w)p_0(w))^\gamma = (q_0(w)p_z(w))^\gamma$$

for all $w, z \in \mathbb{D}$. Since φ is holomorphic, we are able to differentiate $p_z(w)$ and $q_z(w)$, thus we can differentiate the above equation with respect to w to get

$$\begin{aligned} & \left(\frac{p_z(0)}{q_z(0)}\right)^\gamma \gamma (q_z(w)p_0(w))^{\gamma-1} (q_z(w)p'_0(w) + q'_z(w)p_0(w)) \\ &= \gamma (q_0(w)p_z(w))^{\gamma-1} (q_0(w)p'_z(w) + q'_0(w)p_z(w)) \end{aligned}$$

for all $w, z \in \mathbb{D}$. We further let $w = 0$ to get

$$\begin{aligned} & \left(\frac{p_z(0)}{q_z(0)}\right)^\gamma (q_z(0)p_0(0))^{\gamma-1} (q_z(0)p'_0(0) + q'_z(0)p_0(0)) \\ &= (q_0(0)p_z(0))^{\gamma-1} (q_0(0)p'_z(0) + q'_0(0)p_z(0)) \end{aligned}$$

which reduces to

$$p_z(0) (q_z(0)p'_0(0) + q'_z(0)p_0(0)) = q_z(0) (q_0(0)p'_z(0) + q'_0(0)p_z(0)) \quad (4.2.4)$$

for all $z \in \mathbb{D}$.

Denote $c_0 = \alpha(1 - \bar{\alpha}\varphi(0))$ and $\beta = \bar{\alpha}(\alpha - \varphi(0))$. Then we have

$$\begin{aligned} p_z(0) &= \alpha(1 - \bar{\alpha}\varphi(z)), & q'_z(0) &= -\bar{\alpha}\varphi'(0)(\alpha - z), \\ q_z(0) &= c_0 - \beta z, & p'_z(0) &= -\bar{\alpha}(\alpha - \varphi(z)), \\ p_0(0) &= q_0(0) = c_0, & q'_0(0) &= -|\alpha|^2 \varphi'(0), \\ p'_0(0) &= -\beta. \end{aligned}$$

We substitute the above expressions into (4.2.4) to get

$$\begin{aligned}
& \alpha(1 - \bar{\alpha}\varphi(z))(-\beta(c_0 - \beta z) - \bar{\alpha}c_0\varphi'(0)(\alpha - z)) \\
&= (c_0 - \beta z)(-\bar{\alpha}c_0(\alpha - \varphi(z)) - \alpha|\alpha|^2\varphi'(0)(1 - \bar{\alpha}\varphi(z))) \\
\iff & \alpha(1 - \bar{\alpha}\varphi(z))(-\beta(c_0 - \beta z) - \bar{\alpha}c_0\varphi'(0)(\alpha - z)) \\
&= -\bar{\alpha}c_0(c_0 - \beta z)(\alpha - \varphi(z)) \\
&\quad - \alpha(1 - \bar{\alpha}\varphi(z))(|\alpha|^2c_0\varphi'(0) - |\alpha|^2\beta\varphi'(0)z) \\
\iff & \alpha(1 - \bar{\alpha}\varphi(z))(-\beta c_0 + \beta^2 z + \bar{\alpha}c_0\varphi'(0)z - |\alpha|^2\beta\varphi'(0)z) \\
&= -|\alpha|^2c_0^2 + |\alpha|^2\beta c_0z + \bar{\alpha}\varphi(z)(c_0^2 - \beta c_0z) \\
\iff & \alpha(-\beta c_0 + \beta^2 z + \bar{\alpha}c_0\varphi'(0)z - |\alpha|^2\beta\varphi'(0)z) \\
&\quad - \bar{\alpha}\varphi(z) \cdot \alpha(-\beta c_0 + \beta^2 z + \bar{\alpha}c_0\varphi'(0)z - |\alpha|^2\beta\varphi'(0)z) \\
&= -|\alpha|^2c_0^2 + |\alpha|^2\beta c_0z + \bar{\alpha}\varphi(z)(c_0^2 - \beta c_0z) \\
\iff & |\alpha|^2c_0^2 - |\alpha|^2\beta c_0z - \alpha\beta c_0 + \alpha\beta^2 z + |\alpha|^2c_0\varphi'(0)z - \alpha|\alpha|^2\beta\varphi'(0)z \\
&= \bar{\alpha}\varphi(z)(c_0^2 - \beta c_0z - \alpha\beta c_0 + \alpha\beta^2 z + |\alpha|^2c_0\varphi'(0)z - \alpha|\alpha|^2\beta\varphi'(0)z)
\end{aligned}$$

for all $z \in \mathbb{D}$.

Denote $c_1 = \beta c_0$ and $c_2 = |\alpha|^2c_0\varphi'(0) - \alpha|\alpha|^2\beta\varphi'(0) + \alpha\beta^2$. Then we get

$$|\alpha|^2c_0^2 - \alpha c_1 - (|\alpha|^2c_1 - c_2)z = \bar{\alpha}\varphi(z)(c_0^2 - \alpha c_1 - (c_1 - c_2)z)$$

for all $z \in \mathbb{D}$. Hence we have

$$\varphi(z) = \frac{\alpha(\bar{\alpha}c_0^2 - c_1) - (|\alpha|^2c_1 - c_2)z}{\bar{\alpha}(c_0^2 - \alpha c_1) - \bar{\alpha}(c_1 - c_2)z} \quad (4.2.5)$$

for all $z \in \mathbb{D}$. Let $d = \psi(0)$. Finally, we substitute (4.2.5) into (4.2.3) to get

$$\begin{aligned}
\psi(z) &= \frac{d\alpha^\gamma}{(c_0 - \beta z)^\gamma} \left(1 - \frac{\alpha(\bar{\alpha}c_0^2 - c_1) - (|\alpha|^2c_1 - c_2)z}{c_0^2 - \alpha c_1 - (c_1 - c_2)z} \right)^\gamma \\
&= \frac{d\alpha^\gamma}{(c_0 - \beta z)^\gamma} \left(\frac{c_0^2(1 - |\alpha|^2) - c_1(1 - |\alpha|^2)z}{c_0^2 - \alpha c_1 - (c_1 - c_2)z} \right)^\gamma \\
&= \frac{d}{(c_0 - \beta z)^\gamma} \left(\alpha c_0(1 - |\alpha|^2) \cdot \frac{c_0 - \beta z}{c_0^2 - \alpha c_1 - (c_1 - c_2)z} \right)^\gamma \\
&= \frac{d(c_0^2 - \alpha c_1)^\gamma}{(c_0^2 - \alpha c_1 - (c_1 - c_2)z)^\gamma}
\end{aligned}$$

for all $z \in \mathbb{D}$.

Sufficiency. Suppose that for all $z \in \mathbb{D}$,

$$\begin{aligned}\psi(z) &= \frac{d(c_0^2 - \alpha c_1)^\gamma}{(c_0^2 - \alpha c_1 - (c_1 - c_2)z)^\gamma} \quad \text{and} \\ \varphi(z) &= \frac{\alpha(\bar{\alpha}c_0^2 - c_1) - (|\alpha|^2 c_1 - c_2)z}{\bar{\alpha}(c_0^2 - \alpha c_1) - \bar{\alpha}(c_1 - c_2)z},\end{aligned}$$

where $c_0, c_1, c_2, d \in \mathbb{C}$.

By our assumption, $W_{\psi, \varphi}$ is bounded on $\mathcal{H}_\gamma(\mathbb{D})$. Hence it suffices to prove that (4.2.1) holds for all $w, z \in \mathbb{D}$. Clearly we have

$$W_{\psi, \varphi} \mathcal{C}_2 K_w^{(\gamma)}(z) = \frac{d(c_0^2 - \alpha c_1)^\gamma}{(c_0^2 - \alpha c_1 - (c_1 - c_2)z)^\gamma} \cdot \frac{\alpha^\gamma \lambda (1 - |\alpha|^2)^{\frac{\gamma}{2}}}{(\alpha(1 - \bar{\alpha}\varphi(z)) - \bar{\alpha}w(\alpha - \varphi(z)))^\gamma}$$

and

$$\mathcal{C}_2 W_{\psi, \varphi}^* K_w^{(\gamma)}(z) = \frac{d(c_0^2 - \alpha c_1)^\gamma}{(c_0^2 - \alpha c_1 - (c_1 - c_2)w)^\gamma} \cdot \frac{\alpha^\gamma \lambda (1 - |\alpha|^2)^{\frac{\gamma}{2}}}{(\alpha(1 - \bar{\alpha}\varphi(w)) - \bar{\alpha}z(\alpha - \varphi(w)))^\gamma}.$$

Note that the numerators of $W_{\psi, \varphi} \mathcal{C}_2 K_w^{(\gamma)}(z)$ and $\mathcal{C}_2 W_{\psi, \varphi}^* K_w^{(\gamma)}(z)$ are the same thus it suffices to check if their denominators are the same. We denote

$$D := (c_0^2 - \alpha c_1 - (c_1 - c_2)z)(\alpha(1 - \bar{\alpha}\varphi(z)) - \bar{\alpha}w(\alpha - \varphi(z)))$$

and

$$D_* := (c_0^2 - \alpha c_1 - (c_1 - c_2)w)(\alpha(1 - \bar{\alpha}\varphi(w)) - \bar{\alpha}z(\alpha - \varphi(w))).$$

Substituting φ gives us

$$\begin{aligned}D &= (c_0^2 - \alpha c_1 - (c_1 - c_2)z) \left(\alpha(1 - |\alpha|^2) \cdot \frac{c_0^2 - c_1 z}{c_0^2 - \alpha c_1 - (c_1 - c_2)z} \right. \\ &\quad \left. - \alpha \bar{\alpha} w + \bar{\alpha} w \frac{\alpha(\bar{\alpha}c_0^2 - c_1) - (|\alpha|^2 c_1 - c_2)z}{\bar{\alpha}(c_0^2 - \alpha c_1) - \bar{\alpha}(c_1 - c_2)z} \right) \\ &= \alpha(1 - |\alpha|^2)(c_0^2 - c_1 z) - |\alpha|^2 w(c_0^2 - \alpha c_1 - (c_1 - c_2)z) \\ &\quad + w(\alpha(\bar{\alpha}c_0^2 - c_1) - (|\alpha|^2 c_1 - c_2)z) \\ &= \alpha c_0^2(1 - |\alpha|^2) - \alpha c_1(1 - |\alpha|^2)z - |\alpha|^2(c_0^2 - \alpha c_1)w \\ &\quad + |\alpha|^2(c_1 - c_2)wz + \alpha(\bar{\alpha}c_0^2 - c_1)w - (|\alpha|^2 c_1 - c_2)wz \\ &= \alpha c_0^2(1 - |\alpha|^2) - \alpha c_1(1 - |\alpha|^2)(w + z) \\ &\quad - \bar{\alpha}(|\alpha|^2 c_1 - c_2 - \alpha c_1 + \alpha c_2)wz\end{aligned}$$

Likewise, we have

$$\begin{aligned} D_* &= \alpha c_0^2 (1 - |\alpha|^2) - \alpha c_1 (1 - |\alpha|^2) (z + w) \\ &\quad - \bar{\alpha} (|\alpha|^2 c_1 - c_2 - \alpha c_1 + \alpha c_2) zw \end{aligned}$$

by interchanging the variables w and z . So we indeed have $D = D_*$ for all $w, z \in \mathbb{D}$. □

Chapter 5

Conclusion and further study

In this thesis, we have managed to characterize weighted composition conjugations into two classes, namely $\mathcal{C}_1$ and $\mathcal{C}_2$ (see Theorem 3.3.3). We then proceeded to characterize $W_{\psi,\varphi}$ on $\mathcal{H}_\gamma(\mathbb{D})$ that are $\mathcal{C}_1$ -symmetric and $\mathcal{C}_2$ -symmetric (see Theorems 4.1.1 and 4.2.1), thereby completely characterizing $W_{\psi,\varphi}$ on $\mathcal{H}_\gamma(\mathbb{D})$ which are complex symmetric with respect to all weighted composition conjugations.

5.1 Open questions

Let $\mathcal{D} \subset \mathbb{C}$ be a domain and $\beta = (\beta_j)$ be a sequence of positive real numbers. Consider the following space

$$\mathcal{H}(\mathcal{D}, \beta) := \left\{ f(z) = \sum_{j=0}^{\infty} a_j z^j \in \mathcal{O}(\mathcal{D}) : \|f\|^2 = \sum_{j=0}^{\infty} |a_j|^2 \beta_j^2 < \infty \right\}.$$

If $\mathcal{D} = \mathbb{C}$, we have the Hilbert space of entire functions. Further, if $\beta_j = \sqrt{j!}$, we have the Fock space, which was studied in [8]. If $\mathcal{D} = \mathbb{D}$, then we have the weighted Hardy spaces. Further, if $\beta_j = 1$, we have the Hardy space; if $\beta_j = \frac{1}{\sqrt{j+1}}$, we have the Bergman space; if $\beta_j = \sqrt{j+1}$, we have the Dirichlet space.

Question 5.1.1. *What is the complex symmetric structure of $W_{\psi,\varphi}$ with respect to weighted composition conjugations on $\mathcal{H}(\mathcal{D}, \beta)$?*

In the statement of Theorems 4.1.1 and 4.2.1, we note that φ is independent of $\gamma \in \mathbb{N}$. This leads us to suspect that even if we have $\gamma > 0$, we will arrive at the same formula for φ . However as mentioned earlier, our proof requires $\gamma \in \mathbb{N}$ due to the multiplication of complex numbers raised to power of γ . Is there a way to arrive at the formulae of φ and ψ without considering such multiplication?

Question 5.1.2. *Do Theorems 4.1.1 and 4.2.1 hold true for real positive γ ?*

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