Investigation and application of nonlinear ultrasonic guided wave in closed and open waveguides

Zuo, Peng

2019

Zuo, P. (2019). Investigation and application of nonlinear ultrasonic guided wave in closed and open waveguides. Doctoral thesis, Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/88854>

<https://doi.org/10.32657/10220/47639>

**NANYANG
TECHNOLOGICAL
UNIVERSITY**

SINGAPORE

**INVESTIGATION AND APPLICATION OF
NONLINEAR ULTRASONIC GUIDED WAVE
IN CLOSED AND OPEN WAVEGUIDES**

Peng Zuo

School of Mechanical and Aerospace Engineering

2019

INVESTIGATION AND APPLICATION OF NONLINEAR ULTRASONIC GUIDED WAVE IN CLOSED AND OPEN WAVEGUIDES

by

Peng Zuo

School of Mechanical and Aerospace Engineering
Nanyang Technological University

A thesis submitted to the Nanyang Technological University
in fulfilment of the requirement for the degree of
Doctor of Philosophy

2019

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research, is free of plagiarised materials, and has not been submitted for a higher degree to any other University or Institution.

11 Feb 2019
Date

Zuo W
Peng Zuo

Supervisor Declaration Statement

I have reviewed the content and presentation style of this thesis and declare it is free of plagiarism and of sufficient grammatical clarity to be examined. To the best of my knowledge, the research and writing are those of the candidate except as acknowledged in the Author Attribution Statement. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

11 Feb 2019
Date

Zheng Fan

Authorship Attribution Statement

This thesis contains materials from 5 papers published in the following peer-reviewed journals where I was the first author.

Chapter 3 is published as P. Zuo, X. Yu and Z. Fan. Numerical modeling of embedded solid waveguides using SAFE-PML approach using a commercially available finite element package. *NDT&E International*, **90**, 11-23 (2017). DOI: 10.1016/j.ndteint.2017.04.003. P. Zuo and Z. Fan. SAFE-PML approach for modal study of waveguides with arbitrary cross sections immersed in inviscid fluid. *Journal of Sound and Vibration*, **406**, 181-196 (2017). DOI: 10.1016/j.jsv.2017.06.001.

The contributions of the co-authors are as follows:

- Prof Fan provided the initial project direction and edited the manuscript drafts.
- Dr Yu assisted in the computation of dispersion curve by using Disperse.
- I carried out the work and prepared the manuscript drafts.

Chapter 4 is published as P. Zuo, Y. Zhou and Z. Fan. Numerical studies of nonlinear ultrasonic guided waves in uniform waveguides with arbitrary cross sections. *AIP Advances*, **6**, 075207 (2016). DOI: 10.1063/1.4959005

The contributions of the co-authors are as follows:

- Prof Fan and Dr Zhou provided the initial project direction and edited the manuscript drafts.
- I carried out the work and prepared the manuscript drafts.

Chapter 5 is published as P. Zuo, Y. Liu, Y. Zhou and Z. Fan. Investigation of nonlinear ultrasonic guided waves in open waveguides based on perfectly matched layer *The Journal of the Acoustical Society of America*, **144**(3), 1572-1588 (2018). DOI: 10.1121/1.5055210.

The contributions of the co-authors are as follows:

- Prof Fan and Dr Zhou provided the initial project direction and edited the manuscript drafts.
- Dr Liu provided the design of the magnetostrictive transducers.
- I carried out the work and prepared the manuscript drafts.

Chapter 6 is published as P. Zuo, Y. Zhou and Z. Fan. Numerical and experimental investigation of nonlinear ultrasonic Lamb waves at low frequency. *Applied Physics Letters*, **109**, 021902 (2016). DOI: 10.1063/1.4958705.

The contributions of the co-authors are as follows:

- Prof Fan and Dr Zhou provided the initial project direction and edited the manuscript drafts.
- I carried out the work and prepared the manuscript drafts.

11 Feb 2019
Date

Zuo H
Peng Zuo

Acknowledgements

I would like to express my hearties gratitude to my supervisor Professor Zheng Fan for his excellent guidance, useful discussion and support throughout the course of my Ph.D. study at Nanyang Technological University. He is a gracious and enthusiastic mentor, always showing excitement and encouragement in our conversations of regular meetings, even though most of time I make little progress. He is also a knowledgeable supervisor, I have learned a lot of knowledge about acoustics from him, which definitely widens my horizon. My supervisor sets a good example in my research career.

I also wish to bring my deep gratitude to my co-supervisor Dr. Yu Zhou from Advanced Remanufacturing and Technology Center for her inspiration and encouragement. I have vastly benefited from discussion with Dr. Zhou in our regular meetings. Most of my research has been advised by her with invaluable suggestions and feedback.

My thanks also go to Dr. Yang Liu from Schlumberger-Doll Research for providing the design of magnetostrictive transducers and helpful discussions. I am pleased to coauthor one of my paper with him. I also appreciate Zheda Jingyi Electromechanical Technology Engineering Co., Ltd. for gratuitously providing iron-cobalt foil for magnetostrictive transducer.

I would like to extend my thanks to all my former and current colleagues (Dr. Madis, Dr. Maria, Dr. Amir, Dr. Jian Chen, Dr. Jing Xiao, Dr. Xudong Yu, Swami, Danylo, Jing Rao, Xingfang, Zhaoyu, Madhir, Abhishek, Lei Hu) in the NDT lab, for their help and creating such friendly working environment.

Finally, I am most grateful to my parent, my dear wife and my sister for their unfailing support, endless patience and encouragement. Without their support and encouragement, my thesis would never have been accomplished.

Contents

Contents	1
Abstract	6
Nomenclature	8
List of Figures	14
List of Tables	21
1 Introduction	22
1.1 Motivation	22
1.2 Outline of Thesis	25
2 Literature Review	28
2.1 Background	28
2.2 Nonlinear Ultrasonic Bulk Waves	29
2.3 Nonlinear Ultrasonic Guided Waves	32
2.4 Semi Analytical Finite Element Method	35

2.5	Mode Orthogonality Relation in Closed and Open Waveguides	37
2.6	Summary	39
3	SAFE-PML Method	41
3.1	Background	41
3.2	Mathematical Framework of the SAFE-PML Method	42
3.2.1	Mathematical framework for embedded waveguides	44
3.2.2	Mathematical framework for immersed waveguides	47
3.2.2.1	Wave equations in the fluid	47
3.2.2.2	Wave equations in the solid waveguide core structure	48
3.2.2.3	Fluid loading interface conditions	49
3.2.3	FE formula and implementation of the SAFE-PML formulas .	50
3.2.4	PML function, PML length and mode filtering	52
3.3	Validation of the SAFE-PML Method	56
3.3.1	Aluminum plate attached to semi-infinite half-space of elastomer material	57
3.3.1.1	Model description	57
3.3.1.2	Results	58
3.3.2	A cylindrical steel bar immersed in water	60
3.3.2.1	Model description	60
3.3.2.2	Results	61
3.4	Application of the SAFE-PML Method	65

3.4.1	Steel pipe embedded in infinite concrete	65
3.4.1.1	Model description	65
3.4.1.2	Results	66
3.4.2	A rectangular steel bar immersed in water	71
3.4.2.1	Modal description	71
3.4.2.2	Results	72
3.5	Summary	74
4	Nonlinear Ultrasonic Guided Waves in Closed Waveguides with Arbitrary Cross Sections	76
4.1	Background	76
4.2	Mathematical Framework of Nonlinear Ultrasonic Guided Waves in Closed Waveguides	77
4.3	Numerical Methods	83
4.3.1	Nonlinear SAFE method	83
4.3.2	Time domain FE model	84
4.4	Applications	85
4.4.1	Nonlinear ultrasonic guided waves in a plate	85
4.4.2	Nonlinear ultrasonic guided waves in a rectangular bar	91
4.5	Summary	98
5	Nonlinear Ultrasonic Guided Waves in Open Waveguides	99
5.1	Background	99

5.2	Mathematical Framework of Nonlinear Ultrasonic Guided Waves in Open Waveguides	101
5.3	Numerical Methods	111
5.3.1	Nonlinear SAFE method	111
5.3.2	Time domain FE model	112
5.4	Case Studies	114
5.4.1	An aluminum plate attached to a semi-infinite half-space of an elastomer	114
5.4.1.1	Nonlinear SAFE method computations	114
5.4.1.2	Time domain FE simulations	116
5.4.2	An aluminum plate with water-loaded on one side	120
5.4.2.1	Nonlinear SAFE method computations	120
5.4.2.2	Time domain FE simulations	122
5.5	Experimental Validations	126
5.5.1	Experimental setup	126
5.5.2	Calibration and experimental results	128
5.6	Discussions	131
5.7	Summary	132
6	Applications of Nonlinear Lamb Waves at Low Frequency Regions	134
6.1	Background	134
6.2	Nonlinear SAFE Modeling	136

6.3	FE Modeling	137
6.4	Experiments	140
6.5	Discussions	142
6.6	Summary	143
7	Conclusions	144
7.1	Thesis Review	144
7.2	Summary of Findings	146
7.2.1	Developments of SAFE-PML method	146
7.2.2	Nonlinear ultrasonic guided waves in open waveguides	148
7.2.3	Nonlinear Lamb waves at low frequencies	149
7.3	Future Work	150
A	The Coefficients of the FE Formalism for Embedded Waveguides	152
B	Finite Deformation Method	156
C	Derivations of Eq. (5.27)	158
	References	160
	List of Publications	174

Abstract

Nonlinear ultrasonic guided waves, combining the advantages of nonlinear ultrasound and ultrasonic guided waves, are interesting for Non-Destructive Testing (NDT) since they are sensitive to incipient damage as well as offering the potential for rapid inspections of large structures. For waveguides with regular cross sections, e.g. plates, rods or pipes, analytical solutions can be obtained to study the nonlinear guided wave propagation. For waveguides with arbitrary cross sections, numerical methods are usually used due to the complexity to solve nonlinear Navier equation analytically with arbitrary boundary conditions. All above waveguides are commonly called closed waveguides, where energy of the wave only concentrates in the waveguides. Nonlinear ultrasonic guided waves have been investigated widely in closed waveguides, especially in regular waveguides. However, it will be more challenging when the waveguides are embedded/immersed in another medium (usually called open waveguides), where interaction between the waveguides and the surrounding material may lead to energy leaking away from the waveguides, which makes the description of nonlinear ultrasonic guided waves even more challenging for open waveguides.

Studies of nonlinear ultrasonic guided waves in open waveguides require understandings of modal properties of these structures. A number of analytical or numerical models have been developed to understand the behavior of guided waves in open waveguides, among which one of the attractive methods is to combine the Semi-Analytical Finite Element (SAFE) method with Perfectly Matched Layer (PML). But most of the studies on SAFE-PML method focus on embedded cases, which only contain solid medium, and most of them are based on specific Finite Element (FE)

codes that are not accessible to public and time consuming to develop. In this thesis, SAFE-PML method is developed for both embedded and immersed waveguides, and implement it into a commercial FE package. As no source code is required, the presented method will be attractive to a wide range of researchers in NDT. This method is first demonstrated and validated in two cases with analytical solutions. It is then applied to practically important problems, showing the potential of the method.

The properties of nonlinear ultrasonic guided waves in open waveguides are investigated. Mathematical framework is first established based on real reciprocity relation and modal expansion with PMLs. Numerical models are then implemented, including nonlinear SAFE method to predict the properties of nonlinear ultrasonic guided waves, and time domain FE models to simulate the nonlinear guided wave propagation and cross validate the predictions from nonlinear SAFE method. Two examples, an aluminum plate attached to an elastomer and an aluminum plate with water-loaded on one side are studied to demonstrate the proposed methods and reveal some interesting phenomena that only exist in open waveguides. It is interesting to find that the amplitude of the attenuated second harmonic wave (SHW) in immersed waveguides can become constant with propagation distance, only if the primary wave is non-leaky, which may bring potential NDT applications for underwater inspections. Such feature is also validated by well-designed experiments in immersed plates.

This thesis also explores the possibility to use low frequency symmetric Lamb mode (S_0) for nonlinear applications, as the widely used internal resonant Lamb mode pair (S_1 , s_2) suffers from two associated complications: inherent dispersive and multi-mode natures. At low frequency region, the S_0 mode shows little dispersivity and it is easier to generate pure S_0 mode. Also, the secondary mode grows linearly in a significant distance. Numerical modelings including the nonlinear SAFE method and time domain FE models, as well as experiments are used to validate the nonlinear features and therefore demonstrate an easy alternative for nonlinear Lamb wave applications.

Nomenclature

Symbols

$ \cdot $	absolute value of quantity
$\text{Re}(\cdot)$	real part of the complex
$\text{Im}(\cdot)$	imaginary part of the complex
$\tilde{\nabla}(\cdot)$	gradient operator with respect to stretched coordinates
$\nabla(\cdot)$	gradient operator with respect to real coordinates
$O(\cdot)$	higher order term
$\text{tr}(\cdot)$	trace of tensor
$\det(\cdot)$	determinant of tensor
$(\cdot)^T$	transpose operator
$(\cdot)^*$	conjugate complex
A, B, C	third order elastic constants (Landau-Lifshitz model)
A_1, A_2	amplitude of the primary and secondary mode
A_m	modal amplitude
$\hat{A}_1$	compensated amplitude
a_1, a_2, b_1, b_2	constant value in the PML function
α_L, α_S	attenuation of the longitudinal and shear wave
β	nonlinearity parameter
b	a constant

C_{imjl}	component of stiffness tensor
$\tilde{C}_{imjl}$	component of stiffness tensor in the stretched coordinates
$C_{\alpha\beta}$	component of stiffness tensor with index contraction
C_{11}, C_{111}	second and third order elastic constant
C_L	longitudinal wave velocity
C_S	shear wave velocity
c_p	phase velocity
$c.c.$	complex conjugate
d_{x_1}, d_{x_2}	start position of the PML
d	thickness of a plate
δ	deviation factor
δ_{ij}	Kronecker delta
E	Young's modulus
$\mathbf{E}$	Green-Lagrange strain tensor
E_{ij}	Green-Lagrange strain tensor component
$\mathbf{F}$	deformation gradient tensor
$\tilde{f}$	an arbitrary function
f	frequency
$f_{resonant}$	resonant frequency
$\bar{\mathbf{f}}, \bar{\mathbf{f}}^{(1)}$	nonlinear term
$\bar{f}_i$	component of the nonlinear term
$\mathbf{f}_1, \mathbf{f}_2$	volume force
f_m^{vol}, f_m^{surf}	energy power flux through volume and surface
$\bar{f}_{-m}^{vol}, \bar{f}_{-m}^{surf}$	energy power flux
$\gamma, \gamma_1, \gamma_2$	PML function
$\Gamma_{A/B}, \Gamma_B$	interface and surface of the cross section

h_{x_1}, h_{x_2}	length of the PML
$\mathbf{I}$	identity matrix
I_1, I_2, I_3	first three invariants of the Green-Lagrange tensor
i, j, m, l, n	integral indices
i	imaginary unit
K	compressibility coefficient
k	wavenumber
k_l, k_m, k_n	wavenumber of the l th, m th and n th mode
k_{leak}, k_L	wavenumber of leaky and longitudinal wave
k_{x_3}	wavenumber in the x_3 direction
L, M, N	third order elastic constants (Murnaghan model)
L_d	dispersion length
$\lambda, \lambda_A, \lambda_B$	Lamé constant
μ, μ_A, μ_B	Lamé constant
$\tilde{\mathbf{n}}^f, \mathbf{n}^s, \mathbf{n}_r$	unit vector
n_i^f, n_i^s	unit vector component
$n_{\tilde{m}}, n_m$	unit vector component
$\mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z$	unit vector
$\mathbf{n}_A, \mathbf{n}_B^{\text{inn}}, \mathbf{n}_B^{\text{out}}$	unit vector
η	user-defined parameter
$\Omega, \Omega_A, \Omega_B$	cross section domain
$\partial\Omega_{\text{inner}}, \partial\Omega_{\text{outer}}, \partial\Omega$	interface and outside boundary of the cross section
ω	angular frequency
p	acoustic pressure

P	acoustic pressure in the cross section
P_{x_3}, P_{mn}	energy power flux in the x_3 direction
$\bar{P}, \bar{P}_{guide}, \bar{P}_{tot}$	average energy power flux over the cross section
Φ	parabolic function
$Q_{n,m}$	energy power flux
Q	number of modes
ρ, ρ^s, ρ^f	material density (s :solid, f :fluid)
$\rho_0, \rho_{0A}, \rho_{0B}$	material density in the reference configuration
S	area of the cross section
$\mathbf{S}$	the second P-K stress
$\mathbf{s}$	linear part of the first P-K stress
$\bar{\mathbf{s}}, \bar{\mathbf{s}}^{(1)}$	nonlinear part of the first P-K stress
$\bar{s}_{ij}$	component of the nonlinear part of the first P-K stress
$\mathbf{s}_1, \mathbf{s}_2$	stress tensors
$\mathbf{s}_A, \mathbf{s}_B$	linear part of the first P-K stress
$\mathbf{s}_m, \mathbf{s}_n$	stress tensor of the m th and n th mode
$\sigma_{ij}^s, \sigma_{ij}^f$	component of the stress tensor in solid and fluid material
σ_{xx}	normal stress in the x direction
$\mathbf{T}$	stress tensor in SAFE-PML model
T_i	component of stress tensor in SAFE-PML model
t	time
θ	leaky angle
$\mathbf{u}, \mathbf{u}^s, \mathbf{u}^f$	particle displacement vector
u_i^s, u_i^f	particle displacement vector component
$\mathbf{U}$	particle displacement vector in the cross section
U_i^s	particle displacement vector component

U_r, U_θ, U_3	component of the displacement vector in the cross section
$\mathbf{u}^{(1)}, u^{(1)}$	primary displacement field
$\mathbf{U}^{(1)}$	displacement field in the cross section
$\mathbf{u}^{(2)}, u^{(2)}$	secondary displacement field
$\mathbf{u}_A, \mathbf{u}_B$	particle displacement vector
$\mathbf{u}_A^{(1)}, \mathbf{u}_B^{(1)}$	primary displacement field
$\mathbf{u}_A^{(2)}, \mathbf{u}_B^{(2)}$	secondary displacement field
u	particle displacement in one dimension
V_{leak}	leaky wave velocity
V_m	guided wave velocity
$\mathbf{v}_1, \mathbf{v}_2$	velocity vector
V_j^s	a variable
$\mathbf{v}_m, \mathbf{v}_n$	velocity vector of the m th and n th mode
w	strain energy density function
$\mathbf{X}$	coordinate vector in the reference configuration
$\mathbf{x}$	coordinate vector in the current configuration
$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$	stretched complex coordinates
x_1, x_2, x_3	real coordinates
x, y, z	real coordinates

Acronyms

SAFE	Semi-Analytical Finite Element
PML	Perfectly Matched Layer
FE	Finite Element
NDT	Non-Destructive Test
SHM	Structural Health Monitoring
NWMS	nonlinear wave modulation spectroscopy

NRUS	nonlinear resonant ultrasound spectroscopy
CAN	contact acoustic nonlinearity
SH	shear-horizontal
AR	absorbing region
BEM	Boundary Element Method
P-K	Piola-Kirchhoff
STFT	short time Fourier transform
FFT	fast Fourier transform
MST	magnetostrictive transducer
SHW	second harmonic wave

List of Figures

3.1	<i>Schematics of (a) an arbitrary waveguide embedded/immersed in an infinite medium and (b) a two dimensional model with PML for the calculation of modal properties of the waveguide, where Ω_A represents the waveguide core structure; Ω_B includes surrounding medium and the PML; $\partial\Omega_{inner}$ and $\partial\Omega_{outer}$ are the interface between the waveguide core structure and the surrounding medium and the outer surface of the open waveguide, respectively; d_{x_1} and d_{x_2} denote the start position of the PML in the x_1 and x_2 direction, respectively; h_{x_1} and h_{x_2} represent the length of the PML in the x_1 and x_2 direction, respectively.</i>	43
3.2	<i>Schematic of plane wave propagation model.</i>	55
3.3	<i>Schematic of the SAFE-PML model used for the aluminum plate attached to an elastomer.</i>	58
3.4	<i>Energy flux ratio of the guided waves in the aluminum plate attached to an elastomer material obtained by the SAFE-PML method at 200 kHz.</i>	59
3.5	<i>(a) Phase velocity and (b) attenuation dispersion curves for an aluminum plate attached to an elastomer material. The dots represent the solutions from the SAFE-PML method, and the solid lines are the solutions obtained from the matrix method program [1].</i>	60

3.6	<i>Schematic of the SAFE-PML model for a cylindrical steel bar immersed in water.</i>	61
3.7	<i>(a) Phase velocity and (b) attenuation dispersion curves for a cylindrical steel bar immersed in water. The dots represent the solutions from the SAFE-PML method, and the solid lines are the solutions obtained from the matrix method program [1]. The horizontal dash lines in (a) indicate the longitudinal velocity of the water.</i>	62
3.8	<i>Mode shapes (on the left) and displacements (on the right) on the x_2 axis from the ordinate origin to the endpoint of the PML for the cylindrical bar immersed in water at 1000 kHz: (a) mode shapes and (b) displacements of the $F(1,1)$ mode with $k = 2338.35 + 24.20i \text{ m}^{-1}$; (c) mode shapes and (d) displacements of the $T(0,1)$ mode with $k = 1927.36 \text{ m}^{-1}$; (e) mode shapes and (f) displacements of the $L(0,1)$ mode with $k = 1255.66 + 8.21i \text{ m}^{-1}$. U_r and U_θ represent the radial displacement and the circumferential displacement in the cross section, respectively; U_3 is the displacement in the axial direction. . .</i>	64
3.9	<i>Schematic of the SAFE-PML model for a steel pipe (a) fully embedded and (b) half embedded in infinite concrete.</i>	66
3.10	<i>(a) Phase velocity and (b) attenuation dispersion curves for the circumferential order zero and order one modes for the steel pipe fully embedded in concrete; the two horizontal dash lines in (a) represent the shear and longitudinal bulk velocity of the concrete, respectively. .</i>	67
3.11	<i>Mode shapes for the fully embedded pipe at 50 kHz for (a) $L(0,1)$ mode with $k = 154.428 + 0.00023i \text{ m}^{-1}$, (b) $F(1,1)$ mode with $k = 154.167 + 0.00014i \text{ m}^{-1}$, (c) $T(0,1)$ mode with $k = 95.928 + 8.246i \text{ m}^{-1}$, (d) $F(1,2)$ mode with $k = 95.451 + 8.283i \text{ m}^{-1}$, (e) $L(0,2)$ mode with $k = 57.7 + 4.748i \text{ m}^{-1}$ and (f) $F(1,3)$ mode with $k = 56.91 + 4.8374i \text{ m}^{-1}$. .</i>	68

3.12	(a) Phase velocity and (b) attenuation dispersion curves for the circumferential order zero and order one modes for the steel pipe half embedded in concrete; the two horizontal dash lines in (a) represent the shear and longitudinal bulk velocity of the concrete, respectively.	69
3.13	Mode shapes for the half embedded pipe at 50 kHz for (a) $F(1,1)$ mode with $k = 154.269 + 0.0018i \text{ m}^{-1}$, (b) $F'(1,1)$ mode with $k = 169.939 \text{ m}^{-1}$, (c) $F(1,2)$ mode with $k = 95.532 + 8.208i \text{ m}^{-1}$, (d) $F'(1,2)$ mode with $k = 95.979 + 0.0666i \text{ m}^{-1}$, (e) $F(1,3)$ mode with $k = 57.147 + 4.618i \text{ m}^{-1}$ and (f) $F'(1,3)$ mode with $k = 56.066 + 0.196i \text{ m}^{-1}$	70
3.14	Schematic of the SAFE-PML model for a rectangular steel bar immersed in water.	72
3.15	(a) Phase velocity and (b) attenuation dispersion curves for a rectangular steel bar immersed in water. The horizontal dash lines in (a) indicate the longitudinal velocity of the water.	73
3.16	Mode shapes for the rectangular bar immersed in water at 500 kHz for (a) B_x^1 mode with $k = 1646.23 + 85.56i \text{ m}^{-1}$, (b) B_y^1 mode with $k = 1328.72 + 16.03i \text{ m}^{-1}$, (c) $T1$ mode with $k = 1292.22 + 20.46i \text{ m}^{-1}$ and (d) $L1$ mode with $k = 606.40 + 1.05i \text{ m}^{-1}$	74
4.1	Schematic of a closed waveguide with arbitrary geometry, where Ω and $\partial\Omega$ represent the waveguide domain and the stress free boundaries, respectively.	77
4.2	Phase velocity dispersion curves for Rayleigh Lamb modes and SH modes in the aluminate plate.	86
4.3	Behaviors of the modal amplitude for the $s2$ mode with different deviation from the resonant frequency.	87

4.4	(a) Schematic of the plate for the FE simulation, numbers in the left side of the plate are scale factors of the displacement of the node. (b) Displacement in the z direction of the $S1$ mode at 3561.2 kHz. (c) Displacement in the z direction of the $s2$ mode at 7122.4 kHz.	89
4.5	(a) Time domain signal received at a propagation distance of 100 mm; (b) STFT's of the time domain signal; (c), (d) Slices at the primary and second harmonic frequencies as a function of time.	90
4.6	Relative nonlinearity parameter with respected to the propagation distance.	91
4.7	Schematic of the rectangular bar in the modeling.	92
4.8	Phase velocity dispersion curves for the rectangular bar.	93
4.9	In-plane displacements (marked as vector arrows) of (a) the primary mode B_x^2 at 110.6 kHz and (b) the secondary mode l^1 at 221.2 kHz. Wave structures of (c) the primary mode and (d) the secondary mode extracted along the cutline, where u , v , w represent the normalized displacement in the x , y , z direction, respectively.	94
4.10	Schematic of the rectangular bar for the FE simulation. Numbers in (b) represent the scale factor of the displacement of the node.	95
4.11	Time domain signals for (a) out of plane displacement of the primary mode and (b) in-plane displacement of the secondary mode received at a propagation distance of 410 mm.	96
4.12	Displacement component in the z direction for (a) the primary mode at 110.6 kHz and (b) the secondary modes at 221.2 kHz.	97
4.13	Relative nonlinearity parameter with respect to the propagation distance for the excitation of the B_x^2 mode.	97

5.1	<i>Schematics of (a) an infinite open waveguide and (b) the corresponding finite open waveguide with PML, where $\Gamma_{A/B}$ and Γ_B are the interface between the waveguide core structure and the surrounding material and the outer surface of the finite open waveguide, respectively; $\mathbf{n}_A$ and $\mathbf{n}_B^{\text{inn}}$ represent the outward-pointing unit normal vector of the core structure and the surrounding material on the interface, respectively, with holding a relationship as $\mathbf{n}_A = -\mathbf{n}_B^{\text{inn}}$; $\mathbf{n}_B^{\text{out}}$ is the outward-pointing unit normal vector of the finite open waveguide on Γ_B; d_x and d_y denote the start position of the PML in the x and y direction, respectively; h_x and h_y represent the length of the PML in the x and y direction, respectively..</i>	102
5.2	<i>Nonlinear SAFE method computations with (a) the schematic of the SAFE-PML model for the aluminum plate attached to an elastomer, (b) phase velocity and (c) attenuation dispersion curves for the embedded plate and (d) variation of the modal amplitude of the SHW with propagation distance.</i>	114
5.3	<i>Schematic of the embedded plate in the FE simulation.</i>	116
5.4	<i>Time domain signals received at $z = 50$ mm in (a) the x direction and (b) the z direction; Frequency spectra by using FFT to the signals received in (c) the x direction and (d) the z direction.</i>	118
5.5	<i>A comparison on the normalized modal amplitude of the SHW in the embedded plate between the FE simulation and the nonlinear SAFE method computation.</i>	119
5.6	<i>Nonlinear SAFE method computations with (a) the schematic of the SAFE-PML model for the aluminum plate with water-loaded on one side, (b) phase velocity and (c) attenuation dispersion curves for the one-sidedly immersed plate and (d) variation of the modal amplitude of the SHW with propagation distance.</i>	120

5.7	<i>Schematic of the immersed plate in the FE modeling.</i>	122
5.8	<i>Time domain signals received at $z = 70$ mm in (a) the x direction and (b) the y direction; Frequency spectra by using FFT to the signals received in (c) the x direction and (d) the y direction.</i>	124
5.9	<i>A comparison on the normalized modal amplitude of the SHW in the immersed plate between the FE simulation and the nonlinear SAFE method computation.</i>	125
5.10	<i>Schematic of experimental setup.</i>	126
5.11	<i>(a) In-plane displacement signals and (b) out-of-plane displacement signals received by lasers at $z = 210$ mm; Frequency spectra of (c) the in-plane displacement signals and (d) the out-of-plane displacement signals.</i>	128
5.12	<i>Normalized nonlinearity parameter as a function of amplifier output level.</i>	130
5.13	<i>The compensated amplitude of (a) the primary mode and (b) the secondary mode with propagation distance.</i>	131
5.14	<i>The amplitude of the SHW changes with proportional variation of the third order elastic constants.</i>	132
6.1	<i>Phase velocity dispersion curves for Rayleigh Lamb modes in aluminum plates. Primary waves are shown in solid curves with upper case labels while SHWs are shown in dashed curves with lower case labels at their half frequencies.</i>	135
6.2	<i>Modal amplitudes for propagative secondary modes.</i>	137
6.3	<i>(a) Schematic of the plate for the FE simulation. Displacement in the z direction of the $S0$ mode at (b) 300 kHz, (c) 600 kHz and (d) 1200 kHz.</i>	138

6.4	(a) Time domain signal received at a propagation distance of 90 mm. (b) FFT's of the time domain signal.	139
6.5	Relative nonlinearity parameter with respect to the propagation distance with excitation frequency of (a) 300 kHz and (b) 600 kHz. . . .	140
6.6	Experimental setup.	141
6.7	Relative nonlinearity parameter measured over increasing propagation distance for the aluminum plate with excitation frequency of (a) 300 kHz and (b) 600 kHz.	142
6.8	Dispersion length with respect to excitation frequencies for the low frequency S0 wave.	143

List of Tables

3.1	Material mechanical properties used in the models	57
4.1	Material properties used in the models [2, 3].	85
4.2	Modal amplitudes of SHW.	87
4.3	Modal amplitudes of SHW.	95
5.1	Mechanical properties of materials used in the models.	116
6.1	Elastic constants used in numerical simulation.	137

Chapter 1

Introduction

1.1 Motivation

The assessment of material damage prior to macro-crack formation, typically during the incipient stage of material deterioration, is necessary to develop accurate and efficient strategies for the life prediction and maintenance of critical industrial components with engineering applications. From the perspective of continuum mechanics, damage evolution in metallic materials is usually initiated by microstructural changes on the microscopic scale, exacerbated by crack initiation and crack growth to a macroscopic size, and terminated by material or structure failure [4]. As damage evolution on different scales is controlled by distinct different mechanism, it can be roughly categorized into microscale and macroscale evolution periods on the basis of spatial scales. Material damage in the micro-scale evolution period, including microstructural changes and crack initiation, remains invisible and drastically affects nonlinear parameters, such as third-order elastic constants, but exerts little or no influence on linear material parameters, such as Young's modulus and Poisson ratio [5]. Moreover, microstructural changes account for a considerable part of the total lifetime (even up to 90% of the life in some special cases [6]) of the material. The second stage includes crack growth and material failure on a macroscale level and is distinct from the first stage because the variation in the linear nature

of the material dominants during this stage. The second stage accounts for a relatively small percentage of the material lifespan. Therefore, once material damage evolves into macroscale damage and becomes detectable, the time for implementing an effective maintenance strategy becomes limited. This limitation is particularly problematic in the aerospace industry.

Ultrasonic techniques for defect detection include ultrasonic bulk wave and guided wave techniques and are most powerful nondestructive method for material degradation evaluation. Traditional bulk wave testing provides point-by-point measurements, and defect detection in large areas requires scanning with a transducer over the region of interest. The ultrasonic guided wave method is an attractive alternative and rapid approach for large area inspection because the wave is guided by structural boundaries, causing energy to concentrate in the waveguide. However, both of these methods normally rely on measurement of particular properties of the waves (transmission and reflection coefficients, velocity, attenuation) for damage detection [7]. Given that all of the measured parameters have a direct relation to linear elastic properties of the material, the conventional methods based on ultrasonic waves are only sensitive to macroscale damage, such as open crack and large scale corrosion.

Nonlinear ultrasound techniques have been shown to be able to detect and monitor microscopic damage. The underlying principle of this method can be understood simply as a nonlinear elastic response when an incident wave propagates in the material and its physical mechanism can be demonstrated in such a way: a propagating sinusoidal ultrasonic wave with a relatively high amplitude can be distorted by the nonlinear elastic properties of the material along the propagation path when it propagates in the material, leading to the generation of higher order harmonics of the initial waveform, wherein the second harmonic wave (SHW) component generally dominants [8]. The amplitude of the generated SHW relates to the third-order elastic constants of the material, thus degradation of the material can be quantified through measurement of the amplitude of the SHW [8]. This method has proven to be sensitive to early stages of microstructural evolution, particularly to

fatigue-generated dislocation substructures [9], persistent slip bands [10] and thermal aging [11]. However, the collection of bulk wave based nonlinear measurements is tedious and time-consuming because it requires point-by-point measurements.

Nonlinear ultrasonic guided wave combines the advantages of nonlinear ultrasound and ultrasonic guided waves, so that it is sensitive to microstructural change and can propagate over a long distance for large area inspections. Thus it has become a useful tool to characterize the incipient damages for large waveguide structures [8, 12]. Comparing with ultrasonic waves in bulk medium, wave propagation in waveguides is much more complex: many modes exist in waveguides and most of them are dispersive. Therefore, it is significant to identify conditions under which the nonlinear response of the waveguide can be maximized. The theory of nonlinear ultrasonic guided waves was first investigated in plate-like structures [2, 13–16], with two conditions (usually called internal resonant conditions) being summarized using perturbation method and modal expansion analysis. Under these two conditions, the SHW grows cumulatively along propagation distance and the nonlinearity parameter, defined by the amplitude of the SHW over the square of the amplitude of the primary wave, was used to characterize the state of the material. The theoretical analysis of nonlinear ultrasonic guided waves was then extended from plate-like structures to cylindrical rods and shells where although much more guided modes exist, they present the same internal resonant conditions as plates [17–19]. Solutions for nonlinear guided wave propagation in plate-like structures, rods and shell structures can be derived analytically by solving nonlinear Navier equations due to relatively simple boundary conditions. Numerical method is necessary for understanding modal properties of nonlinear ultrasonic guided waves in complex waveguides with arbitrary cross-sectional geometry. For such case, an interesting attempt to predict the properties of SHW is to use Semi-Analytical Finite Element (SAFE) method to analyze modal properties of waveguide together with nonlinear analysis [20, 21]. Following theoretical analysis, a number of applications using nonlinear ultrasonic guided waves were developed. These applications include characterization of material nonlinearity and evaluation of incipient damages in plate-like structures [22], the investigation of the material decay [23], and the assessment of thermal fatigue

damages in isotropic pipes [24].

However, all currently available studies on nonlinear ultrasonic guided waves have been limited to closed waveguides (waveguides in a vacuum), where all the energy is confined in the waveguides. Civil engineering and geophysics present highly challenging cases in which waveguides are embedded/immersed in another infinite medium, such as embedded fibers or underwater pipes. Such embedded/immersed waveguides are also known as open waveguides. In these cases, interactions between the waveguides and the surrounding material may cause the energy in the waveguide to leak into the surrounding medium, which further distorts the behavior of the primary wave and the SHW. Thus the amplitude of the SHW may not be linearly cumulative as in closed waveguides. In order to seek potential applications of nonlinear ultrasonic guided waves in open waveguides, the behavior of the SHW in such structure must be understood.

1.2 Outline of Thesis

This thesis investigates properties of nonlinear ultrasonic guided waves in closed and open waveguides with arbitrary cross sections and suggests their potential nonlinear applications in defect detection in industrial structure. The remainder of this thesis is organized into six chapters.

A review of the related literature on nonlinear ultrasonic bulk wave, nonlinear ultrasonic guided wave, and SAFE method is provided in Chapter 2. Mode orthogonality relations in closed and open waveguides are also reviewed to provide the foundation for the study of nonlinear ultrasonic guided wave propagation in closed and open waveguides.

The development of the SAFE-PML method for the modal study of embedded and immersed waveguides with arbitrary cross-sectional geometry is discussed in Chapter 3. The mathematical equations of the SAFE-PML method are derived by combining the SAFE method with Perfectly Matched Layer (PML). Then the equations are

implemented into a standard Finite Element (FE) eigenvalue formalism that can be solved by using a commercial software. Subsequently, the developed method is validated on waveguides with regular cross sections by comparing the results of SAFE-PML with those of analytical models. The validated method is then applied to some important practical examples to illustrate its potential uses. The SAFE-PML method developed in this chapter is used as the basic tool for studying the nonlinear ultrasonic guided wave propagation in open waveguides in Chapter 5.

The investigate of the propagation of the nonlinear ultrasonic guided wave in closed waveguides is discussed in Chapter 4. In the investigation, the propagation of nonlinear ultrasonic guided wave in closed waveguides is first mathematically described following by the introduction of the nonlinear SAFE method to predict the internal resonant modes in waveguides with arbitrary cross sections, and time domain FE simulations to confirm the existence of internal resonant modes and validate their properties. The numerical models are then demonstrated in an aluminum plate and compared with analytical solutions for validation. Lastly, the models are applied to study properties of nonlinear ultrasonic guided waves in a rectangular steel bar.

The extension of the study on nonlinear ultrasonic guided waves from closed waveguides to open waveguides based on PMLs is investigated in Chapter 5. This chapter starts with mathematical framework by using real reciprocity relation and modal expansion with PMLs. Numerical models are then implemented, including nonlinear SAFE method to predict the properties of nonlinear ultrasonic guided waves, and FE models to simulate the propagation of nonlinear ultrasonic guided waves and cross validate the predictions from nonlinear SAFE method. Following the numerical models, two examples are given to illustrate the proposed methods and reveal some interesting properties that only occur in the open waveguides, including an aluminum plate attached to an elastomer and an aluminum plate loaded with water on one side. Finally, well designed experiments are carried out to validate the properties of nonlinear ultrasonic guided wave in one-sidedly immersed plates, showing potential Non-Destructive Test (NDT) applications of nonlinear ultrasonic guided waves for underwater inspections.

The exploration of the possibility to use low frequency symmetric Lamb mode (S0) for nonlinear applications by using the numerical methods developed in Chapter 4 is discussed in Chapter 6, in order to overcome two associated complications in widely used nonlinear Lamb mode: inherent dispersive and multimode natures. It is demonstrated that although the exact phase velocity matching condition is no longer valid, the amplitude of the SHW still increases cumulatively for a significant distance. Numerical simulations and experiments are used to validate the nonlinear features and therefore demonstrate an easy alternative for nonlinear Lamb wave applications.

The findings of the thesis and discussions of future works are presented in Chapter 7.

Chapter 2

Literature Review

2.1 Background

Linear ultrasonic wave propagation in solid can be described by linear constitutive equations that represent the relationship between stress and strain and wherein stresses are only linear functions of the strains. The fundamental property of the linear waves is that any wave can be obtained through the simple superposition of separate monochromatic waves that individually and independently propagate. Such linear wave properties have been widely exploited in NDT and Structural Health Monitoring (SHM). However, waves will no longer be linear if the contribution of microscale damage evolution to the constitutive equations is considered, which will lead to a nonlinear stress-strain relationship and nonlinear motion equations. In this scenario, nonlinear effects, though very small, may become significant in potential applications in incipient damage detection. There are numerous nonlinear ultrasonic related techniques, such as acousto-elasticity measurements [25], wherein changes in stress stage of the nonlinear elastic medium result in changes in wave velocity; nonlinear wave modulation spectroscopy (NWMS) techniques [26], which combine components with low and high frequencies and search for a modulation frequency; nonlinear resonant ultrasound spectroscopy (NRUS) techniques [27], where resonant frequency will shift with increasing amplitude excitation; contact acoustic nonlin-

earity (CAN) [28] in which higher harmonic waves will be generated through the repeated collisions of two surfaces under an incident ultrasonic wave; and nonlinear ultrasound methods [29], where second or higher order harmonic wave will be generated as a result of the distortion of nonlinear elastic properties of the solid. However, the discussions in this thesis will only concentrate on the nonlinear ultrasound method.

2.2 Nonlinear Ultrasonic Bulk Waves

Nonlinear elastic wave propagation was first discussed by Landau and Lifshitz in their famous book "Theory of Elasticity" [30], where they introduced a nonlinear stress-strain relationship with a quadratic nonlinearity and proposed the concept of the anharmonic effect in wave propagation. Subsequently, numerous works have investigated the propagation of finite amplitude elastic waves and the ability to produce a SHW. Theoretical studies have mostly focused on the pure longitudinal wave propagating in an isotropic, nonlinear elastic medium because the governing equations of wave propagation can be simplified into one dimension. In general, when a longitudinal wave is fed into an isotropic, nonlinear elastic medium, the governing equation can be written as

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_{xx}}{\partial x}, \quad (2.1)$$

where ρ is the material density; u means the particle displacement; t is time; x is the wave propagation direction and σ_{xx} presents the normal stress in this direction. The constitutive equation with a quadratic nonlinearity for a nonlinear elastic solid can be written as

$$\sigma_{xx} = E \left(\frac{\partial u}{\partial x} \right) + \beta E \left(\frac{\partial u}{\partial x} \right)^2, \quad (2.2)$$

where E is the Young's modulus with β being the nonlinearity parameter. By substituting Eq. (2.2) into Eq. (2.1), the nonlinear wave equation can be derived as

$$\rho \frac{\partial^2 u}{\partial t^2} = E \frac{\partial^2 u}{\partial x^2} + 2E\beta \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2}. \quad (2.3)$$

To solve the nonlinear equation, the displacement u is assumed as $u = u^{(1)} + u^{(2)}$ by using perturbation method, where $u^{(1)}$ and $u^{(2)}$ represent the initially excited wave and the SHW generated from the first-order perturbation, respectively. The initial wave excited in the solid can be assumed to be a sinusoidal wave

$$u^{(1)} = A_1 \cos(kx - \omega t). \quad (2.4)$$

Then the solution can be obtained as

$$u = u^{(1)} + u^{(2)} = A_1 \cos(kx - \omega t) - A_2 \sin(2kx - 2\omega t), \quad (2.5)$$

where $A_2 = \frac{\beta}{8} A_1^2 k^2 x$ is the amplitude of the SHW. From the expression, it is easy to see that the amplitude of the SHW is proportional to the nonlinearity parameter, the square of the amplitude of the primary wave and the propagation distance. Thus, the amplitude of the SHW grows cumulatively along propagation distance if the small decrement in the primary wave amplitude is ignored. When the longitudinal wave propagates in the (100) direction in an isotropic or crystalline solid, β can be expressed in terms of the elastic constants as [8]

$$\beta = -\frac{3C_{11} + C_{111}}{3C_{11}}, \quad (2.6)$$

where C_{11} and C_{111} represent second and third order elastic constants, respectively. The nonlinearity parameter consists of a combination of second order and third order elastic constants. Therefore, the measurement of SHW offers a unique method for quantifying the extent of the distortion of acoustic waveforms, and consequently, the magnitude of nonlinearity of solids [8]. Furthermore, the theoretical studies of nonlinear ultrasonic bulk wave have been extended by Gerlich and Breazeale [31]

and Cantrell [32]. These researchers studied the propagation of longitudinal wave along different directions in crystalline systems and derived the relationship between the amplitude of the SHW and the second and third order elastic moduli for solids with arbitrary crystalline symmetry. The nonlinearity parameters of such solids are strongly dependent on crystal structure and symmetries. The above works discuss only one type of nonlinear interaction in the propagation of one longitudinal wave. Other types of interaction between two waves have been widely investigated [33–38]. In this case, two elastic waves with different wave types and different frequencies propagate and interact in a nonlinear elastic material to produce a scattering wave with sum or difference frequencies of the primary waves. Under certain conditions, the generated scatter wave will resonate at a certain direction such that its amplitude is large enough to be detected, which is also found to be proportional to the wave interaction zone volume and strongly dependent on the third order elastic constants. Therefore, the measurement of the amplitude of the scattered wave provides an alternative approach for third order elastic constants determination.

From the microscopic perspective, nonlinear effects originate from microstructure of the material, such as dislocations and lattice anharmonicity. In order to define quantitative relation between the magnitude of the nonlinearity parameter and the microstructural feature, the contributions of basic structures to the nonlinearity parameter must be understood. Thus, analytical models have been developed to quantify contributions to the nonlinearity parameter. Dislocation models describe the contributions from dislocation glide motion and dislocation dipoles and have been widely used for interpretation in numerous experiments [9, 39–41]. Although precipitates by themselves exerts little effects on the nonlinearity parameter, their interactions with dislocations can change the magnitude of the nonlinearity parameter. Precipitate models demonstrate this mechanism as well as the dependence of the nonlinearity parameter on the presence of precipitates [42, 43]. In addition, microcrack models have been used to study the contributions of nonlinear stress-strain relationships that originate from asperity contact [44, 45]. Even though a comprehensive understanding of the relationship between the nonlinearity parameter and microstructure damage remains to be unachieved, the obtained analytical models

effectively stimulate applications of the harmonic generation to NDT.

A straightforward application of nonlinear ultrasonic bulk wave is the characterization of the material properties of a crystalline system based on the explicit relation between the amplitude of the SHW and the second and third order elastic constants [46–49]. In addition, given the close link between the SHW and dislocations, the nonlinear ultrasonic bulk wave has been extensively applied to detect fatigue damage in various materials. These applications were first carried out in aluminum alloys [50,51] and then extended to nickel based superalloys [10], carbon and stainless steels [52,53], titanium alloys [54], and crystal coppers [55]. The literature indicates an approximately monotonic growth trend for the nonlinearity parameter during fatigue life. Second harmonic generations have also been used to monitor the effect of microstructural changes during the thermal aging of various materials, which range from alloys [42,56,57] to steels [43,58,59]. In these cases, the main contribution to the nonlinearity parameter originates from the precipitate during thermal aging and this microstructural change during thermal aging has shown to be detectable with harmonic generations. Furthermore, creep damage [60] and radiation damage [61] have also been studied on the basis of harmonic generations.

2.3 Nonlinear Ultrasonic Guided Waves

Bulk wave based nonlinear ultrasound has considerable potential applications in the detection of incipient damage in a wide range of materials. However, these bulk wave based techniques are time consuming and tedious because they require scanning the whole region of interest with the transducer via point-by-point through thickness inspection. Nonlinear ultrasonic guided waves, which combine the advantages of nonlinear ultrasound and ultrasonic guided waves, have shown to be a powerful tool for the characterization of incipient damages in large structures, as guided waves can propagate over a long distance. However, hundreds of modes exist in the waveguides because waves have to continually interact with boundaries to ensure each mode satisfies the displacement and stress boundary conditions. Furthermore,

most of the modes are dispersive with the shape of the wave packet changing during wave propagation.

Deng [13, 62–64] first studied the theory of nonlinear ultrasonic guided waves in plate-like structures by investigating the SHW generation of Rayleigh Lamb and shear-horizontal (SH) wave propagation. It is found that SHW can be cumulatively increased with propagation distance under certain conditions and the cumulatively increased second harmonic fields need to be symmetrical regardless of the symmetry of the primary wave. Subsequently, de Lima et al. [14] developed an analysis framework via perturbation method and modal analysis. Their framework summarizes the two conditions for the generation of the cumulative SHW: (1) phase velocity matching condition, which means the primary wave and the SHW have the same phase velocity; (2) non-zero energy transfer from the primary wave to the SHW. To comprehensively understand the properties of cumulative SHW in plates, Müller et al. [15] obtained analytical asymptotic solutions to Lamb waves and proved that the secondary mode must be symmetric. In addition, Müller claimed that group velocity matching condition should be an additional criterion for the practical generation of a cumulatively increased secondary mode. They also identified five mode types which satisfy all three conditions for the development of an experimental procedure. However, through theoretical analysis and experimental examination, Deng et al. [65] concluded that group velocity matching condition is not necessary. In order to properly select resonant mode pairs in plates, Liu et al. [2] demonstrated a method for the selection of appropriate primary modes that can generate cumulative second harmonics in accordance with the internal resonance conditions of homogeneous isotropic plates. Moreover, higher order harmonic generations were also explored. Srivastava et al. [16] studied the higher order harmonic Lamb wave generation and found that antisymmetric Lamb mode is impossible at all the even order harmonics, whereas all the higher odd order harmonics allow symmetric and antisymmetric Lamb modes. Liu et al. [66] carried out an analysis on the properties of third order harmonic wave generation in nonlinear plates and concluded that when a primary SH wave is excited, third order harmonic SH waves can be holo-internal resonant with it. More recently, these theoretical studies on plate-like

structures have provided guidance for the application of nonlinear ultrasonic guided waves in SHM, which aims to accomplish online damage detection by using built-in transducers that require permanent bonding layers. Shan et al. [67, 68] proposed a nonlinear theoretical framework to assess the contribution of the adhesive bonding layer to second harmonics generation and proposed a refined nonlinear parameter for characterizing nonlinear sources of different natures for SHM systems. Hong et al. [69] incorporated the linearly increased feature into a probabilistic model to evaluate barely visible impact damage in SHM systems.

The mathematics of nonlinear ultrasonic guided waves in cylindrical rod and pipe structures are more complex than those in plate-like structures. de Lima et al. [17] extended the approach for the analysis of cumulative second harmonic generation from plate-like structures to cylindrical rod structures and proposed the internal resonant criteria. Chillara et al. [19] studied the axisymmetric longitudinal guided wave propagation and SHW generation in pipe structures with assuming that the pipe structures satisfy a large radius asymptotic approximation. In the study, they found that only asymptotic symmetric modes can be efficiently induced by the longitudinal modes. Liu et al. [70] analyzed the SHW generation from the interactions of the axisymmetric wave including torsional and longitudinal waves in a circular cylinder by forming the nonlinear forcing terms for the wave equation in a cylindrical coordinate system. From the analysis, they proved that either torsional or longitudinal SHW can be generated and cumulative by a specific type of guided wave mode interactions. In addition to second harmonic generation, Srivastava et al. [18] theoretically studied the higher order harmonic generation in nonlinear, isotropic rods. They indicated that the angular order of the primary mode restricts mode generation at either even or odd higher harmonics. Moreover, Liu et al. [3, 71] formulated a mathematical framework for analyzing the interaction of two collimated waves in hollow circular cylinders where it is found that the linearly cumulative amplitude of the higher order harmonic wave results from the scattering of the two collimated waves. It is known that theoretical analysis of the nonlinear ultrasonic guided wave is possible in plates and rods/pipes given the simple boundary conditions when solving nonlinear Navier equations [72]. Numerical methods are

necessary for the analysis of complex waveguides with arbitrary cross sections, such as rectangular bars and railroad tracks. For such cases, a numerical algorithm were proposed which combines the SAFE method to numerically calculate the propagating modes with nonlinear analysis for solving the amplitude of the SHW based on the internal resonant condition [20, 21, 73].

Following theoretical analysis, a number of applications using nonlinear ultrasonic guided waves have been developed. These applications include the material nonlinearity characterization in elastic plates by using nonlinear Lamb waves [22, 74, 75]; the evaluation of incipient damage, such as creep [11], fatigue [76–78] and plasticity [79, 80], in plate like structures; the investigation of the material decay [23, 81] and the assessment of thermal fatigue damage in isotropic pipes [24].

2.4 Semi Analytical Finite Element Method

Understanding modal properties of ultrasonic guided waves in waveguides is of significance not only in applications of guided wave propagation but also in developments of useful inspection techniques. For regular waveguides (such as plates, rods and pipes), analytical solutions can be extracted through analytical methods, i.e. Transfer Matrix Method [82], Global Matrix Method [83] and dispersion equations [84]. For waveguides with arbitrary cross sections, numerical method is necessarily required. Semi Analytical Finite Element (SAFE) method is particularly efficient for modal studies in such waveguides from a computation perspective because it uses a harmonic wave propagation assumption together with a FE discretization of the cross section. As a consequence, the three dimension problem becomes a two dimensional one, considerably decreasing computational cost.

The SAFE method was first developed in 1973 for computing the eigenmodes of waveguides of arbitrary cross sections [85, 86]. Then, the method was extended to anisotropic waveguides by Huang and Dong [87], where an anisotropic composite cylinder was considered for both propagating waves and edge vibrations studies. Using essentially the same principle, the SAFE method has been applied to a wide

variety of waveguides, such as railways [88, 89], square rods [90], grooved plates [91] and welded plates [92]. Moreover, Bartoli et al. [93] further extended the SAFE model from lossless material to lossy medium allowing for viscoelastic damping, which is represented by the imaginary part of the wavenumber.

However, most of the existing research work is based on specific FE codes, which are not available to the public and are time consuming to develop. Wilcox et al. [94] have attempted to implement an approximation of the SAFE method in a standard FE package (FE77) by using a cyclic axial symmetry condition. But the method cannot find non-propagating waves because constraints from the cyclic axial symmetry approximation implicitly force the wavenumber to be real. Predoi et al. [91] have reorganized the SAFE equations in a standard FE eigenvalue formalism and implemented it into COMSOL, a commercial software, where the eigenvalue formalism can be solved by the eigenvalue solver. In addition, the periodic boundary conditions have been introduced to the SAFE method to enable the modeling of infinitely wide waveguides.

Recently, the SAFE method has been extended to embedded/immersed waveguides [95–97], where the energy in the waveguides may leak into the surrounding materials. Castaings and Lowe [98] introduced a finite absorbing region (AR) with damping properties and the length of the AR is proportional to the largest wavelength of the leaky waves. By coupling the SAFE method and the AR, leaky waves can be absorbed with no reflection from exterior boundaries and thus this can simulate an infinite extent of the surrounding material. Fan et al. [95] extended the SAFE-AR to immersed case where an infinite extent of the inviscid fluid medium is represented by a finite absorbing region. An alternative approach has been proposed by Mazzotti et al. [99, 100] through combining the SAFE method with a regularized 2.5D Boundary Element Method (BEM) for model study of embedded/immersed waveguides. In this method, the infinite material is described by BEM and converted into a single SAFE-like element which is dependent on wavenumber and frequency. In addition, Treysède et al. [97, 101] have introduced PML in the SAFE method, whose impedance perfectly matches with the neighbor medium and absorbs

almost all propagating waves of non-zero frequencies and non-tangential angles-of-incidence. However, most of the studies on SAFE-PML method focus on embedded waveguides, which only contains solid medium.

2.5 Mode Orthogonality Relation in Closed and Open Waveguides

It is known that modal analysis is powerful and widely used to handle wave problems in waveguide, for which a procedure for expanding arbitrary wave field by superposition of waveguide modes (modal basis) has to be developed. This requires proof that the waveguide modes can form a set of orthogonal functions which is in the same sense that the sine and cosine functions are orthogonal to each other serving as orthogonal basis functions in a Fourier series expansion. That means the waveguide modes must satisfy orthogonality relation. In mathematics, orthogonality relation can be derived from reciprocity relation and different reciprocity relations lead to different orthogonality relations, which are applicable to different kind of problem. In closed waveguides, there are three kinds of modes existing. They are propagating modes, non-propagating modes and evanescent modes. It has been proved that they are orthogonal to each other and present an orthogonality relation of [102]

$$P_{mn} = 0 \text{ if } k_m \neq k_n^*, \quad (2.7)$$

where k_m and k_n represent wavenumber of the m th and n th modes, respectively; $*$ indicate the complex conjugate; P_{mn} is interpreted as energy power flux on the cross section, which is defined as

$$P_{mn} = \frac{1}{4} \int_{\Omega} \{ -\mathbf{v}_n^* \cdot \mathbf{s}_m - \mathbf{v}_m \cdot \mathbf{s}_n^* \} \cdot \mathbf{n}_z d\Omega, \quad (2.8)$$

where $\mathbf{v}_n$ and $\mathbf{v}_m$ are velocity of the n th and m th modes, respectively; $\mathbf{s}_n$ and $\mathbf{s}_m$ represent stress tensor of the n th and m th modes, respectively; $\mathbf{n}_z$ is the unit vector in the wave propagation direction and Ω represents the waveguide cross section.

However, the application of modal analysis to open waveguides is more intricate since the orthogonality relation in these structures is obscure because of the unboundedness of the surrounding material. Theoretically, the unboundedness of the surrounding material in open waveguides leads to three types of modes forming [101,103,104]. They are trapped modes where energy concentrates in the waveguide core structure, leaky modes which propagate with energy leaking into the surrounding medium and radiation modes that resonate in the surrounding material. In mathematics, trapped modes and radiation modes have been proven to be the modal basis of an open waveguide, but the continuum radiation modes are difficult to handle in mathematical manipulation as a large number of sampled radiation modes have to be included in the expansion [105,106]. In electromagnetic field, the discrete leaky modes have been proposed to approximate the contribution of the radiation modes to the modal expansion [107], even though they do not belong to the modal basis due to their amplitude growing to infinity in transverse direction [104]. In order to circumvent this problem, two methods have been suggested [107]. One approach is to truncate the leaky modes within a restricted space very close to the waveguide core structure so that they are approximately orthogonal. The other one is to retain the full wave field of leaky modes but to replace the orthogonality relation with a more general, mathematically exact orthogonality by deforming the real coordinates into a complex plane. Even though the two methods have shown to be satisfied sensibly on not only open waveguides for optical applications [108,109], but also on an embedded elastic waveguide for acoustic scattering studies [110], the application of orthogonality for leaky modes in open waveguides still remains challenging.

More recently, an approach has been proposed to address the applicability of leaky modes in open waveguides by introducing a finite PML to truncate the infinite surrounding medium [106,111]. With PMLs, the leaky wave can be totally damped in a limited space, efficiently simulating the infinite surrounding medium, so that the infinite open waveguide is converted to a finite one. In addition, the finite PML transform the continuum radiation modes existing in the infinite open waveguides into discrete sets of PML modes which only resonate in the PML [106]. Following

this, an orthogonality relation has been derived [111]

$$Q_{n,m} = 0 \text{ if } n \neq -m, \quad (2.9)$$

in which $Q_{n,m} = -\frac{1}{4} \int_{\Omega} \{\mathbf{v}_n \cdot \mathbf{s}_m - \mathbf{v}_m \cdot \mathbf{s}_n\} \cdot \mathbf{n}_z d\Omega$ and Ω is the integral area which includes the PML region. This orthogonality relation can be applied to any types of modes, including trapped modes, leaky modes and PML modes. Hence a new modal basis has been defined by these three types of modes and leaky modes are applicable in modal expansion to describe any wave field in open waveguides through superposition of the modal basis. The new orthogonality relation will be utilized in nonlinear wave propagation in open waveguides.

2.6 Summary

In this chapter, theoretical analyses and applications of nonlinear ultrasound (including bulk waves and guided waves) have been reviewed. In bulk medium, only two type of non-dispersive waves exist: longitudinal wave and shear wave. When a longitudinal wave propagates in the medium, a SHW is induced due to distortions of the nonlinear properties of the material. The amplitude of the SHW contains information of the nonlinearity parameter and grows cumulatively with the propagation distance. Thus material characterization and incipient damage detections become possible through measurements of the secondary mode.

In waveguides, because of boundedness of boundaries, the wave can propagate over a long distance with little energy loss. But many modes exist in waveguides and most of them are dispersive. Therefore, selections of the primary mode that can generate a cumulatively increased secondary mode are of significance. Two conditions have been summarized to select the mode pair: phase velocity matching and non-zero energy transfer from primary wave to the SHW. The application of the nonlinear ultrasonic guided wave shows promising in NDT and SHM.

For waveguides with arbitrary cross sections, analytical solutions cannot be found,

and numerical method is therefore necessarily required. SAFE method being a powerful tool on modal property study of waveguides is revisited in this chapter. SAFE method uses a FE to represent the cross section, together with harmonic wave propagation in the axial direction of the waveguide. SAFE method has been extended to open waveguides by coupling other techniques which can absorb leaky waves, i.e. ARs, BEM and PMLs.

Mode orthogonality relations are also recalled in both closed waveguide and open waveguides, as modal analysis is useful in waveguide studies. In closed waveguides, three type of modes (propagating modes, non-propagating modes and evanescent modes) can satisfy orthogonality relation. In open waveguides where three kind of modes exist: trapped modes, leaky modes and radiation modes, orthogonality relation is intricate because of unboundedness of the waveguide. By introducing a PML to truncate the infinite surrounding material into finite one, the continuum radiation modes are transformed to be discrete sets of PML modes. In this configuration, an orthogonality relation can be derived, which can be applied to any types of modes.

Chapter 3

SAFE-PML Method

3.1 Background

Understanding properties of ultrasonic guided waves in open waveguides (including embedded waveguides and immersed waveguides) is important for various industries. On one hand, ultrasonic guided waves can be used as an efficient tool for NDT over a large area, as guided waves can propagate for a long distance from a single generation position. On the other hand, interactions between guided waves and the surrounding material make it possible to sense the presence and nature of the surrounding medium [95, 112, 113]. In both cases, understanding modal properties of the open waveguides, especially the attenuation of the guided wave is essential.

SAFE-PML method is an efficient way to study the modal properties of open waveguides by coupling the SAFE method with PML method. Bérenger [114] first proposed the concept of the PML in the context of electromagnetic waves and then it was shown that the PML equations can be understood as a complex valued coordinate stretching in the electromagnetic wave equations [115, 116]. The same ideas are immediately applicable to other wave equations, such as Helmholtz equations [117], Euler equations [118] and even elastodynamic wave equations [119–122]. SAFE-PML method is a further extension of the PML method in elastodynamic wave equations. By combining of the SAFE and the PML method, leaky waves can be

damped, while the attenuation in the axial direction can be extracted and represented by the imaginary part of the wavenumber. However, most of the studies on SAFE-PML method focus on embedded waveguides and based on specific FE codes, which are not available to the public and time consuming to develop.

In this chapter, the SAFE-PML method is developed for both embedded and immersed waveguides. It starts with mathematical derivations of the coupled SAFE-PML equations, with the equations reorganized in a standard FE eigenvalue formalism which can be solved by the commercial software. The method is validated by two examples which can be solved by analytical solutions and then used to study important NDT applications. As it based on a commercial FE package, the developed method will be attractive to a wide range of researchers in NDT. In addition, the developed method can provide a fundamental tool for studies of nonlinear ultrasonic guided wave in open waveguides in Chapter 6.

3.2 Mathematical Framework of the SAFE-PML Method

In this section, the SAFE-PML equations have been derived and implemented into a commercial package. The PML method can be viewed as a result of complex coordinate stretching, where the ordinary solutions of wave equations in real coordinate space are mapped to a complex coordinate space in which the wave field decays exponentially. Thus, the original wave propagating eigenfunctions in an infinite medium can be mapped to exponentially decaying eigenfunctions in the model with a finite length of PML [123].

This can be understood in a simple one dimensional wave propagation case. Considering a harmonic wave traveling along the positive direction of the complex coordinate $\tilde{x}_1$, the solution of the propagating wave can be written as $u(\tilde{x}_1, t) = \bar{A}e^{ik\tilde{x}_1}e^{-i\omega t}$, where i and k denote the imaginary unit and wavenumber, respectively; ω represents the angular frequency. If $\tilde{x}_1$ is defined in terms of a complex function,

$\tilde{x}_1 = \text{Re}(\tilde{x}_1) + i\text{Im}(\tilde{x}_1)$, the wave can be expressed as $u(\tilde{x}_1) = \bar{A}e^{ik\text{Re}(\tilde{x}_1)}e^{-k\text{Im}(\tilde{x}_1)}$ (excluding the time harmonic part). Thus, if $\text{Im}(\tilde{x}_1) > 0$, $u(\tilde{x}_1)$ is a positive direction propagating wave that is attenuated in that direction.

The SAFE method starts with the three-dimensional elasticity approach. With an assumption of a harmonic wave propagating in the waveguide axial direction, the method can be reduced to a two dimensional one where only cross section needs to be solved (as shown in Fig. 3.1). Hence, the coupled SAFE-PML method just needs to do the complex coordinate stretching in the cross section.

Figure 3.1: Schematics of (a) an arbitrary waveguide embedded/immersed in an infinite medium and (b) a two dimensional model with PML for the calculation of modal properties of the waveguide, where Ω_A represents the waveguide core structure; Ω_B includes surrounding medium and the PML; $\partial\Omega_{inner}$ and $\partial\Omega_{outer}$ are the interface between the waveguide core structure and the surrounding medium and the outer surface of the open waveguide, respectively; d_{x_1} and d_{x_2} denote the start position of the PML in the x_1 and x_2 direction, respectively; h_{x_1} and h_{x_2} represent the length of the PML in the x_1 and x_2 direction, respectively.

The stretched coordinates in the waveguide are defined as

$$\begin{cases} \tilde{x}_1(x_1) = \int_0^{x_1} \gamma_1(\xi) d\xi \\ \tilde{x}_2(x_2) = \int_0^{x_2} \gamma_2(\xi) d\xi \\ \tilde{x}_3(x_3) = x_3 \end{cases}, \quad (3.1)$$

where $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_3$ denote the stretched coordinates. $\gamma_1(x_1)$ and $\gamma_2(x_2)$ represent two complex-valued coordinate stretching functions, also called PML function, which are non-zero and continuous, satisfying

$$\gamma_1(x_1) = 1 \text{ for } |x_1| \leq d_{x_1} \text{ and } \text{Im}\{\gamma_1\} > 0 \text{ for } |x_1| > d_{x_1}, \quad (3.2)$$

$$\gamma_2(x_2) = 1 \text{ for } |x_2| \leq d_{x_2} \text{ and } \text{Im}\{\gamma_2\} > 0 \text{ for } |x_2| > d_{x_2}. \quad (3.3)$$

Therefore, this change of variable yields for any function $\tilde{f}$

$$\begin{cases} \frac{\partial \tilde{f}}{\partial \tilde{x}_1} = \frac{1}{\gamma_1} \frac{\partial f}{\partial x_1} \\ \frac{\partial \tilde{f}}{\partial \tilde{x}_2} = \frac{1}{\gamma_2} \frac{\partial f}{\partial x_2} \\ \frac{\partial \tilde{f}}{\partial \tilde{x}_3} = \frac{\partial f}{\partial x_3} \end{cases}, \quad (3.4)$$

where $\tilde{f}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, t) = f(x_1, x_2, x_3, t)$.

Embedded waveguides only deal with solid, while immersed waveguides contain both solid and fluid. Therefore, the mathematical derivations for embedded waveguides and immersed waveguides are discussed in separate subsections.

3.2.1 Mathematical framework for embedded waveguides

Considering a guided wave traveling in the $\tilde{x}_3$ direction, the displacement vector in the waveguide can be written as

$$u_i^s(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, t) = U_i^s(\tilde{x}_1, \tilde{x}_2) e^{i(k\tilde{x}_3 - \omega t)}, \quad (3.5)$$

in which $\omega = 2\pi f$ denotes the angular frequency and f being the frequency; the subscript $i = 1, 2, 3$; the superscript s means solid medium and U_i^s represents the displacement in the cross section. Substituting Eq. (3.5) into Eq. (3.4), the displacement gradients deduced

$$\begin{cases} \frac{\partial u_i^s}{\partial \tilde{x}_1} = \frac{1}{\gamma_1} \frac{\partial U_i^s}{\partial x_1} e^{i(kx_3 - \omega t)} \\ \frac{\partial u_i^s}{\partial \tilde{x}_2} = \frac{1}{\gamma_2} \frac{\partial U_i^s}{\partial x_2} e^{i(kx_3 - \omega t)} \\ \frac{\partial u_i^s}{\partial \tilde{x}_3} = ik U_i^s e^{i(kx_3 - \omega t)} \end{cases} \quad (3.6)$$

The differential equations of motion in the open waveguide domain Ω ($\Omega = \Omega_A + \Omega_B$) are

$$\sum_{j,m,l=1}^3 \left[C_{imjl} \frac{\partial^2 u_j^s}{\partial \tilde{x}_m \partial \tilde{x}_l} \right] + \rho^s \omega^2 u_i^s = 0 \quad \text{for } i = 1, 2, 3, \quad (3.7)$$

where ρ^s is the mass density in the domain, and C_{imjl} are the components of the elastic constant tensor, which can be related to the component of the stiffness matrix $C_{\alpha\beta}$ ($1 \leq \alpha, \beta \leq 6$) by the index contraction: $\alpha = i = m$ if $i = m$, and $\alpha = 9 - (i + m)$ otherwise; $\beta = j = l$ if $j = l$, and $\beta = 9 - (j + l)$ otherwise. Using some intermediary transformations, Eq. (3.7) can be written as

$$\begin{aligned} & \tilde{C}_{imjl} \frac{\partial^2 U_j^s}{\partial x_m \partial x_l} + ik \left(\tilde{C}_{i3jm} + \tilde{C}_{imj3} \right) \frac{\partial U_j^s}{\partial x_m} + C_{i1j1} \left(\frac{1}{\gamma_1} \right) \left(\frac{1}{\gamma_1} \right)' \frac{\partial U_j^s}{\partial x_1} \\ & + C_{i2j2} \left(\frac{1}{\gamma_2} \right) \left(\frac{1}{\gamma_2} \right)' \frac{\partial U_j^s}{\partial x_2} - k^2 C_{i3j3} U_j^s + \rho^s \omega^2 \delta_{ij} U_i^s = 0 \end{aligned} \quad (3.8)$$

with summation over the indices $j = 1, 2, 3$ and $m, l = 1, 2$, where the apostrophe means the derivative with respect to the independent variable. The coefficients are defined in terms of the component of the elastic constant tensor and the coordinate stretching function

$$\begin{aligned} \tilde{C}_{imjl} &= \frac{1}{\gamma_m \gamma_l} C_{imjl}; \quad \tilde{C}_{i3jm} = \frac{1}{\gamma_m} C_{i3jm}; \\ \tilde{C}_{imj3} &= \frac{1}{\gamma_m} C_{imj3} \quad \text{no summation on } m, l \end{aligned} \quad (3.9)$$

On the boundary $\partial\Omega_{outer}$, the stress vector is

$$\begin{aligned} T_i &= \sum_{j,m,l=1}^3 C_{imjl} \frac{\partial u_j^s}{\partial \tilde{x}_l} n_{\tilde{m}} \\ &= \sum_{j=1}^3 \left[\sum_{m,l=1}^2 \frac{1}{\gamma_l \gamma_m} C_{imjl} \frac{\partial U_j^s}{\partial x_l} n_m + \sum_{m=1}^2 \frac{1}{\gamma_m} i k C_{imj3} U_j^s n_m \right], \end{aligned} \quad (3.10)$$

where $n_{\tilde{m}}$ and n_m are the component of the outward unit vector normal to the boundary $\partial\Omega_{outer}$ in the complex stretched coordinates and real coordinates, respectively, and they are related by $n_{\tilde{m}} = \frac{1}{\gamma_m} n_m$ (no summation on m). Eq. (3.10) can be expressed as

$$T_i = \tilde{C}_{imjl} \frac{\partial U_j^s}{\partial x_l} n_m + i k \tilde{C}_{imj3} U_j^s n_m, \quad (3.11)$$

with summation over the indices $j = 1, 2, 3$ and $m, l = 1, 2$.

According to Eq. (3.8), this is a quadratic eigenvalue problem, where the eigensolutions are the wavenumber k and mode shapes of the waveguide at the chosen angular frequency ω . In order to cast it into a general linear form, a new variable should be introduced [91, 98, 101, 124]

$$V_j^s = k U_j^s. \quad (3.12)$$

Thus, the governing equations of the system can be rewritten as

$$\begin{aligned} &\tilde{C}_{imjl} \frac{\partial^2 U_j^s}{\partial x_m \partial x_l} + i \left(\tilde{C}_{i3jm} + \tilde{C}_{imj3} \right) \frac{\partial V_j^s}{\partial x_m} + C_{i1j1} \left(\frac{1}{\gamma_1} \right) \left(\frac{1}{\gamma_1} \right)' \frac{\partial U_j^s}{\partial x_1} \\ &+ C_{i2j2} \left(\frac{1}{\gamma_2} \right) \left(\frac{1}{\gamma_2} \right)' \frac{\partial U_j^s}{\partial x_2} - k C_{i3j3} V_j^s + \rho^s \omega^2 \delta_{ij} U_i^s = 0 \text{ in } \Omega, \end{aligned} \quad (3.13)$$

$$\rho^s \omega^2 \delta_{ij} V_j^s - k \rho^s \omega^2 \delta_{ij} U_j^s = 0 \text{ in } \Omega, \quad (3.14)$$

$$T_i = \tilde{C}_{imjl} \frac{\partial U_j^s}{\partial x_l} n_m + i k \tilde{C}_{imj3} V_j^s n_m \text{ on } \partial\Omega_{outer}, \quad (3.15)$$

in which δ_{ij} is the Kronecker delta.

3.2.2 Mathematical framework for immersed waveguides

3.2.2.1 Wave equations in the fluid

In the fluid, a harmonic wave propagation along the $\tilde{x}_3$ direction is assumed, thus the acoustic pressure in the fluid can be written as

$$p(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, t) = P(\tilde{x}_1, \tilde{x}_2) e^{i(k\tilde{x}_3 - \omega t)}. \quad (3.16)$$

Considering a compressive fluid with compressibility coefficient K , the acoustic pressure p is related to the displacement $\mathbf{u}^f$ (the superscript f means fluid medium) through the constitutive equations written as

$$p = -K \tilde{\nabla} \cdot \mathbf{u}^f, \quad (3.17)$$

where $\tilde{\nabla}$ denotes the gradient operator in the stretched coordinates. The motion equations in the fluid domain are

$$\rho^f \omega^2 \mathbf{u}^f = \tilde{\nabla} p, \quad (3.18)$$

where ρ^f is the density of the fluid. Combining Eq. (3.17) and Eq. (3.18), the motion equations of the fluid can be written as

$$\tilde{\nabla} \cdot (K \tilde{\nabla} p) + \rho^f \omega^2 p = 0. \quad (3.19)$$

Using some intermediary transformations, Eq. (3.19) can be rewritten as

$$\sum_{i=1}^2 \left[K \left(\frac{1}{\gamma_i} \right)^2 \frac{\partial^2 P}{\partial x_i^2} + K \left(\frac{1}{\gamma_i} \right) \left(\frac{1}{\gamma_i} \right)' \frac{\partial P}{\partial x_i} \right] - K k^2 P + \rho^f \omega^2 P = 0, \quad (3.20)$$

in which the apostrophe means the derivative with respect to the independent variable.

3.2.2.2 Wave equations in the solid waveguide core structure

In the solid waveguide core structure, the real coordinate space is coincident with the complex coordinate space ($\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_2$ and $\tilde{x}_3 = x_3$), as the coordinate stretching functions are equal to one ($\gamma_1 = 1$ and $\gamma_2 = 1$). Thus equation derivation for the solid core structure is placed in the real coordinate space for simplicity. The harmonic wave propagation in the solid core structure in the x_3 direction is also assumed, thus the displacement in the solid core structure can be written as

$$u_i^s(x_1, x_2, x_3, t) = U_i^s(x_1, x_2) e^{i(kx_3 - \omega t)}, \quad (3.21)$$

where U_i^s is the displacement of the waveguide core structure in the cross section. The differential equations of motion in the core structure can be presented as

$$\sum_{j,m,l=1}^3 \left[C_{imjl} \frac{\partial^2 u_j^s}{\partial x_m \partial x_l} \right] + \rho^s \omega^2 u_i^s = 0 \text{ for } i = 1, 2, 3, \quad (3.22)$$

Using some intermediary transformations and Einstein's convention of summation for economy of presentation, Eq. (3.22) can be written as

$$C_{imjl} \frac{\partial^2 U_j^s}{\partial x_m \partial x_l} + i(C_{i3jm} + C_{imj3}) k \frac{\partial U_j^s}{\partial x_m} - C_{i3j3} k^2 U_j^s + \rho^s \omega^2 \delta_{ij} U_j^s = 0, \quad (3.23)$$

with summation over the indices $j = 1, 2, 3$ and $m, l = 1, 2$. Eq. (3.23) is a quadratic eigenvalue problem, where the eigensolutions are the wavenumber k and mode shape of the waveguide at the chosen angular frequency ω . In order to cast it into a general linear form, a new variable should be introduced [91, 98, 101, 124]

$$V_j^s = k U_j^s. \quad (3.24)$$

Thus, the governing equations of the solid core structure can be cast into

$$C_{imjl} \frac{\partial^2 U_j^s}{\partial x_m \partial x_l} + i(C_{i3jm} + C_{imj3}) \frac{\partial V_j^s}{\partial x_m} - k C_{i3j3} V_j^s + \rho^s \omega^2 \delta_{ij} U_j^s = 0, \quad (3.25)$$

$$\rho^s \omega^2 \delta_{ij} V_j^s = k \rho^s \omega^2 \delta_{ij} U_j^s, \quad (3.26)$$

with summation over $j = 1, 2, 3$ and $m, l = 1, 2$.

3.2.2.3 Fluid loading interface conditions

For inviscid fluid, continuity of the normal displacement as well as stress is applied as the interface boundary condition.

In the fluid domain, the following relations can be obtained

$$\tilde{\mathbf{n}}^f \cdot (K \tilde{\nabla} p) = \rho^f \omega^2 K \tilde{\mathbf{n}}^f \cdot \mathbf{u}^s, \quad (3.27)$$

where $\tilde{\mathbf{n}}^f$ is the outward unit vector of the fluid domain on the interface in the complex stretched coordinate; $\mathbf{u}^s$ is the displacement of the interface calculated in the solid domain. On the solid-fluid interface, the PML function should be equal to 1, which means that the real coordinate is coincident with the complex coordinate. Therefore $\tilde{\mathbf{n}}^f = \mathbf{n}^f$ is applied, where $\mathbf{n}^f$ is the outward unit vector in the real coordinate. With the consideration of $n_3^f = 0$, the interface condition can be cast into

$$K \left(n_1^f \frac{\partial P}{\partial x_1} + n_2^f \frac{\partial P}{\partial x_2} \right) = \rho^f \omega^2 K \left(n_1^f U_1^s + n_2^f U_2^s \right). \quad (3.28)$$

In the solid waveguide domain, the boundary conditions at the interface can be written as

$$\mathbf{T} = -p \mathbf{n}^s, \quad (3.29)$$

where $\mathbf{n}^s$ is the outward unit vector of the solid domain, related to the outward unit vector of the fluid domain by $\mathbf{n}^s = -\mathbf{n}^f$; $\mathbf{T}$ is the stress vector at the interface which can be presented by

$$T_i = C_{imjl} \frac{\partial U_j^s}{\partial x_l} n_m^s + i C_{imj3} V_j^s n_m^s = -P n_i^s, \quad (3.30)$$

with summation over the indices $j = 1, 2, 3$ and $m, l = 1, 2$.

3.2.3 FE formula and implementation of the SAFE-PML formulas

In a commercial FE package [125], the input formula for eigenvalues problems has the general expression

$$\nabla \cdot (c \nabla \mathbf{u} + \alpha \mathbf{u} - r) - a \mathbf{u} - \beta \cdot \nabla \mathbf{u} + d_a \lambda \mathbf{u} - e_a \lambda^2 \mathbf{u} = f \text{ in } \Omega, \quad (3.31)$$

in which all matrix coefficients can take real or complex values for elastic or viscoelastic materials; $\mathbf{u}$ represents the set of variables to be determined and Ω represents the computational domain. The generalized Neumann boundary condition and Dirichlet boundary condition applied on the boundary of the computational domain $\partial\Omega$ are expressed in the same FE code as

$$\mathbf{n} \cdot (c \nabla \mathbf{u} + \alpha \mathbf{u} - r) + q \mathbf{u} = g \text{ on } \partial\Omega, \quad (3.32)$$

$$\mathbf{u} = h \text{ on } \partial\Omega. \quad (3.33)$$

For the embedded waveguides, in order to cast the governing equations (Eqs. (3.13) and (3.14)) into the general FE formalism (Eq. (3.31)), the variables in the FE software has to be

$$\mathbf{u} = [U_1^s, U_2^s, U_3^s, V_1^s, V_2^s, V_3^s]^T, \quad (3.34)$$

and the FE coefficients must be

$$\begin{aligned}
 c &= \begin{bmatrix} \mathbf{c}^{UU} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}; \alpha = \begin{bmatrix} \mathbf{0} & i\alpha^{UV} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}; \gamma = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}; \\
 a &= \begin{bmatrix} \mathbf{a}^{UU} & i\mathbf{a}^{UV} \\ \mathbf{0} & \mathbf{a}^{VV} \end{bmatrix}; \beta = \begin{bmatrix} \beta^{UU} & -i\beta^{UV} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}; \\
 d_a &= \begin{bmatrix} \mathbf{0} & \mathbf{d}^{UV} \\ \mathbf{d}^{VU} & \mathbf{0} \end{bmatrix}; e_a = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}; f = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}.
 \end{aligned} \tag{3.35}$$

It is assumed that $\mathbf{0}$ represents a zero matrix of appropriate dimension. The submatrices are given in Appendix A.

It can be seen that the Neumann boundary conditions (Eq. (3.32)) with $q = 0$ and $g = 0$ correspond to stress free boundaries in the waveguides, and rigidly fixed boundaries can also be implemented by use of the Dirichlet boundary conditions (Eq. (3.33)) $\mathbf{u} = \mathbf{0}$ on the corresponding boundaries.

For immersed waveguides, in the fluid domain, Eq. (3.20) is compared with Eq. (3.31). The acoustic pressure needs to be chosen as the finite element variable, $\mathbf{u} = P$, and the matrix coefficients become

$$\begin{aligned}
 c &= \begin{bmatrix} K\left(\frac{1}{\gamma_1}\right)^2 & 0 \\ 0 & K\left(\frac{1}{\gamma_2}\right)^2 \end{bmatrix}; a = -\rho^f \omega^2; e_a = K; \\
 \beta &= \begin{bmatrix} K\left(\frac{1}{\gamma_1}\right)\left(\frac{1}{\gamma_1}\right)' \\ K\left(\frac{1}{\gamma_2}\right)\left(\frac{1}{\gamma_2}\right)' \end{bmatrix}; \alpha = r = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; d_a = f = 0.
 \end{aligned} \tag{3.36}$$

In the solid domain, the new variables $\mathbf{u} = \begin{bmatrix} U_1^s & U_2^s & U_3^s & V_1^s & V_2^s & V_3^s \end{bmatrix}^T$ should be chosen as the variable and the matrix coefficients can be obtained from Eqs. (3.35).

Regarding to the fluid loading conditions at the interface between the solid core structure and the fluid, Neumann conditions are used at the interface with

$$q = 0 \text{ and } g = \rho^f \omega^2 K (n_1 U_1^s + n_2 U_2^s), \tag{3.37}$$

in the fluid domain and

$$q = 0 \text{ and } g = \begin{bmatrix} Pn_1 & Pn_2 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \quad (3.38)$$

in the solid domain, where n_1 and n_2 are the component of the unit vector normal to the interface which are predefined in the package and can be called directly (here, $\mathbf{n} = \mathbf{n}^f$ is assumed. If $\mathbf{n} = -\mathbf{n}^f$, the sign of the interface conditions will change). On the outer boundaries, the Neumann boundary conditions with $q = 0$ and $g = 0$ correspond to sound hard boundaries, which means the normal derivative of the acoustic pressure is zero; and sound soft boundaries, where the acoustic pressure vanishes, can also be implemented by use of the Dirichlet boundary condition $\mathbf{u} = 0$ on the corresponding boundaries.

3.2.4 PML function, PML length and mode filtering

In SAFE-PML method, properly defined PML function (γ) and the length for the PML are crucial to absorb leaky waves within the PML. For the PML function, it must satisfy $\gamma = 1$ outside the PML and $\text{Im}\{\gamma\} > 0$ inside the layer. It is usually to use a constant function as a PML function in the literature [126, 127]. However, it may suffer from the artificial discontinuity between the physical domain and the PML region. A continuous function, e.g. $\gamma = 1 + i\Phi/\omega$, where Φ is a parabolic function inside the PML, is usually chosen as the PML function in problems in the time domain [128–130]. For frequency domain, a PML function with real and imaginary parts being presented by a continuous parabolic function is often used and shows good performance [101]. In our cases, the following parabolic PML functions are adopted [101]

$$\gamma_1(x_1) = \begin{cases} 1 & \text{if } x_1 \leq d_{x_1} \\ 1 + \hat{\gamma}_1 \left(\frac{|x_1| - d_{x_1}}{h_{x_1}} \right)^2 & \text{if } x_1 > d_{x_1} \end{cases}; \quad (3.39)$$

$$\gamma_2(x_2) = \begin{cases} 1 & \text{if } x_2 \leq d_{x_2} \\ 1 + \hat{\gamma}_2 \left(\frac{|x_2| - d_{x_2}}{h_{x_2}} \right)^2 & \text{if } x_2 > d_{x_2} \end{cases}, \quad (3.40)$$

where $\hat{\gamma}_1 = a_1 + ib_1$ ($b_1 > 0$) and $\hat{\gamma}_2 = a_2 + ib_2$ ($b_2 > 0$) quantify the PML absorption and will be given explicitly in each case study.

In order to estimate the length of the PML, a simplified 2D plane wave propagation model with an assumption of infinite length in the x_1 direction is studied (shown in Fig. 3.2). The guided wave propagates in the x_3 direction with wavenumber k_{x_3} . When the guided wave velocity is higher than that of bulk waves (longitudinal or shear wave) in the surrounding material, some energy of the guided wave will leak into the surrounding medium with an angle determined by the Snell's Law,

$$\sin \theta = \frac{V_{leak}}{V_m}, \quad (3.41)$$

where V_{leak} and V_m are the phase velocities of the leaky wave and the guided wave, respectively. In order to estimate the effective length of the PML, an extreme case is considered here, when guided wave velocity is much larger than the leaky wave ($\theta \approx 0$), thus the leaky wave propagates along the x_2 direction. The amplitude of the leaky propagating wave can be represented as

$$u = \bar{A} e^{ik_{leak} x_2}, \quad (3.42)$$

where $\bar{A}$ is a constant. When the wave travels into the PML, Eq. (3.42) becomes

$$u = \bar{A} e^{ik_{leak} \tilde{x}_2}. \quad (3.43)$$

In the PML, the explicit expression of the stretched coordinate is obtained by substituting Eq. (3.40) into Eq. (3.1): $\tilde{x}_2 = x_2 + \frac{1}{3} h_{x_2} (a_2 + ib_2) \left(\frac{x_2 - d_{x_2}}{h_{x_2}} \right)^3$, so that Eq. (3.43) can be cast into

$$u = \bar{A} e^{-\frac{1}{3} k_{leak} b_2 h_{x_2} \left(\frac{x_2 - d_{x_2}}{h_{x_2}} \right)^3} e^{ik_{leak} \left(x_2 + \frac{1}{3} a_2 h_{x_2} \left(\frac{x_2 - d_{x_2}}{h_{x_2}} \right)^3 \right)}. \quad (3.44)$$

The amplitude of the wave traveling in the PML is

$$\bar{\bar{A}} = \bar{A} e^{-\frac{1}{3} k_{leak} b_2 h_{x_2} \left(\frac{x_2 - d_{x_2}}{h_{x_2}} \right)^3}. \quad (3.45)$$

Therefore, the amplitude of the reflected wave from the outer boundary of the PML can be presented as

$$\bar{\bar{A}} = \bar{A} e^{-\frac{2}{3} k_{leak} b_2 h_{x_2}}. \quad (3.46)$$

Assuming $\bar{\bar{A}} \leq 0.01 \bar{A}$ gives

$$\frac{2}{3} k_{leak} b_2 h_{x_2} \geq 4.6. \quad (3.47)$$

Thus the length of the PML can be calculated by the following relation

$$h_{x_2} \geq \frac{6.9}{k_{leak} b_2}. \quad (3.48)$$

If the PML function is given, b_2 can be determined. Then the PML length can be obtained by calculating the minimum wavenumber of the longitudinal wave or the shear wave in the frequency range of interest.

It is worth noting that, the calculation above was based on a simplified 2D model, where the leaky wave is considered as a plane wave. However, in the actual 3D guided wave problem, the solution is more complex and the leaky waves are conical waves. Therefore, by considering the extreme case in the mathematical derivation, only a rough estimation of the length of the PML is provided.

Figure 3.2: *Schematic of plane wave propagation model.*

As mentioned in Ref. [101,106], the wave modes in the open waveguides with PMLs can be divided into three categories: trapped modes, leaky modes and PML modes. Trapped modes experience no attenuation when they propagate in the waveguide. The energy of the mode only concentrates in the waveguide and does not radiate into the infinite medium. Leaky modes propagate with attenuation due to the energy leakage from the waveguide to the surrounding medium. By contrast, PML modes resonate mainly in the PML region. Although PML modes are valid mathematical solutions, they are of less interest to practical applications. Thus, it is desirable to filtrate the PML modes in post-processing to obtain dispersion curves in the SAFE-PML method.

A filter condition is applied in post-processing to identify and remove the PML modes. As these modes resonate in the PML region, the filtering condition is defined as a ratio of the energy flux between the waveguide core structure region and the whole cross section. The criterion of the model is identified via a user-defined parameter η

$$\frac{\bar{P}_{guide}}{\bar{P}_{tot}} > \eta, \quad (3.49)$$

where $\bar{P}$ represents the average energy flux along the wave propagation direction x_3

and is defined as

$$\bar{P} = \frac{\int_S |P_{x_3}| \, ds}{\int_S ds}, \quad (3.50)$$

in which S denotes the area of the cross section. $|P_{x_3}|$ represents the absolute value of the energy flux, defined by [102]

$$P_{x_3} = -\text{Re} \left[\left(\frac{i\omega}{2} \right) (u_1^{s*} \sigma_{31}^s + u_2^{s*} \sigma_{32}^s + u_3^{s*} \sigma_{33}^s) \right], \quad (3.51)$$

where the asterisk stands for the complex conjugate, and the component of the stress tensor can be calculated by

$$\begin{aligned} \sigma_{31}^s &= C_{55} \left(iku_1^s + \frac{\partial u_3^s}{\partial x_1} \right); \\ \sigma_{32}^s &= C_{44} \left(\frac{\partial u_3^s}{\partial x_2} + iku_2^s \right); \\ \sigma_{33}^s &= C_{13} \frac{\partial u_1^s}{\partial x_1} + C_{23} \frac{\partial u_2^s}{\partial x_2} + C_{33} iku_3^s. \end{aligned} \quad (3.52)$$

In the fluid, the energy flux can be calculate by the following formula

$$P_{x_3} = -\text{Re} \left[\left(\frac{i\omega}{2} \right) u_3^{f*} \sigma_{33}^f \right], \quad (3.53)$$

where $\sigma_{33}^f = -p$ and the displacement u_3^f can be calculated by

$$u_3^f = \frac{ikp}{\rho^f \omega^2}. \quad (3.54)$$

All of these quantities used in the post-processing can be easily extracted from the eigensolutions of the FE solver.

3.3 Validation of the SAFE-PML Method

In this section, two example cases with relatively simple geometries are chosen to validate the SAFE-PML method. Solutions from analytical or other FE methods are used for comparison.

3.3.1 Aluminum plate attached to semi-infinite half-space of elastomer material

The first example consists of an aluminum plate attached to a semi-infinite half-space of an elastomer material. This is a representative geometry that has been used by other researchers [98] to validate the FE computation of leaky guided waves from the plate to the elastomer, as it has a simple geometry that can also be solved by a commercial matrix method program [1].

3.3.1.1 Model description

To model the propagation of waves along an infinitely wide plate, a narrow strip of the structure with periodic boundary are used to represent continuity of displacements and stresses between the two edges [91], as shown in Fig. 3.3. The narrow strip of the structure is 1 mm in width, and the thickness of the plate is 4 mm. As it was mentioned in [101], the PML works better when it is close to the core structures. Thus, a 1 mm thick layer of the elastomer is attached to the aluminum plate. According to Eq. (3.48), the thickness of the PML is dependent on the PML function and the minimum wavenumber of the leaky wave in the surrounding medium at the frequencies of interest. In the simulation, the PML function is $\hat{\gamma}_2 = 3 + 12i$ and 15 frequencies will be solved from 50 to 400 kHz with increments of 25 kHz. The minimum wavenumber of the leaky wave can be obtained by calculating for the longitudinal wave at $f = 50$ kHz: $k_{leak} = \min(k_L) = 329.48 \text{ m}^{-1}$, where k_L is the

Table 3.1: Material mechanical properties used in the models

Material	ρ (kg/m ³)	C_L (m/s)	C_S (m/s)	α_L and α_S (dB/ λ)
Aluminum	2780	6347.3	3116.4	...
Elastomer	1100	953.5	522.2	2.73
Steel	7932	5960.5	3260	...
Concrete	2300	4222.1	2637.5	...
Water	1000	1500	...	...

wavenumber of the longitudinal wave in the elastomer. From Eq. (3.48) it can be obtained that the length of the PML should be larger than 1.75 mm. However, the length of the PML is chosen to be 1 mm in this model, as the viscoelastic elastomer can also absorb the leaky waves before they reach the PML. The material properties are given in Table 5.1.

Interface boundary conditions between the plate and the elastomer are applied by imposing continuity of displacements and stresses. Stress-free conditions are imposed at the outer surface of the structure. In the whole geometry, 1186 triangular elements are generated with side length of 0.11 mm and the number of degrees of freedom is 14970. In the post-processing, $\eta = 1.3$ is used to identify and filter the PML modes.

Figure 3.3: Schematic of the SAFE-PML model used for the aluminum plate attached to an elastomer.

3.3.1.2 Results

In order to illustrate the selection of the parameter η used in the post-processing, the energy flux ratio ($\bar{P}_{guide}/\bar{P}_{tot}$) of the guided waves calculated by the SAFE-PML method at 200 kHz are shown in Fig. 3.4. As the PML modes only resonant within the PML region, their energy flux ratio is close to zero, while the trapped and the

leaky modes have significantly higher ratio. The threshold η should be chosen to be a value between the highest ratio of the PML modes and the lowest one of the trapped or the leaky modes at all frequencies. In this case, $\eta = 1.3$ turns out to be satisfactory.

Figure 3.4: *Energy flux ratio of the guided waves in the aluminum plate attached to an elastomer material obtained by the SAFE-PML method at 200 kHz.*

Figure 3.5 presents dispersion curves for the phase velocity and the attenuation, which are computed from the real and imaginary part of the wavenumber. Only three fundamental guided modes, i.e. A0, S0 and SH0, are calculated and plotted in the figure. A great agreement can be observed between the results from the SAFE-PML method and the analytical solutions.

Comparing with the SAFE-AR model used in Ref. [98], the damping area is reduced significantly from 30 mm to 1 mm, which obviously leads to huge reduction of the computational cost. In addition, due to the short length of the PML region, the number of the PML modes (unwanted modes) are significantly reduced, which greatly simplifies the post-processing of the mode filtering.

Figure 3.5: (a) Phase velocity and (b) attenuation dispersion curves for an aluminum plate attached to an elastomer material. The dots represent the solutions from the SAFE-PML method, and the solid lines are the solutions obtained from the matrix method program [1].

3.3.2 A cylindrical steel bar immersed in water

The second validation case includes a cylindrical steel bar immersed in water. This problem can also be calculated by the matrix method program and has been used as a validation case for the SAFE model in the literature [95].

3.3.2.1 Model description

The schematic of the second validation case is shown in Fig. 3.6. The steel bar with radial thickness of 1 mm is immersed in water which is modeled by a square with length of 4 mm. The PML functions in this model are $\hat{\gamma}_1 = 3 + 12i$ and $\hat{\gamma}_2 = 3 + 12i$, and the model is solved for 39 frequencies from 100 kHz to 2000 kHz with an increment of 50 kHz. Based on Eq. (3.48), the minimum length of the PML is $h_{x_1} = h_{x_2} \geq 1.37$ mm. Thus a layer of 1.5 mm is used as the PML attached to the water. The material properties are listed in Table 5.1.

Water loading interface conditions to describe the interaction between the cylindrical bar and the surrounding water are implemented based on Eqs. (3.37) and (3.38).

Neumann boundary conditions with $q = 0$ and $g = 0$ are imposed on the outside boundaries of the PML. 2206 triangular elements are used to mesh the whole geometry and the number of degrees of freedom is 6326. In the post-processing, $\eta = 2$ is used to identify and filter the PML modes.

Figure 3.6: Schematic of the SAFE-PML model for a cylindrical steel bar immersed in water.

3.3.2.2 Results

Figure 3.7 presents the dispersion curves for leaky guided waves in the cylindrical bar. Excellent agreements can be found between the SAFE-PML method and the analytical solutions in the dispersion curves for both phase velocity and attenuation. It can be seen that the F(1,1) mode goes across the longitudinal velocity of the water around 250 kHz. When the phase velocity of the F(1,1) mode is slower than that of the water, no energy leaks into the water; while when it goes across the velocity of the water, some energy leaks into the water, and the F(1,1) mode changes from

non-leaky to leaky [83].

Figure 3.7: (a) Phase velocity and (b) attenuation dispersion curves for a cylindrical steel bar immersed in water. The dots represent the solutions from the SAFE-PML method, and the solid lines are the solutions obtained from the matrix method program [1]. The horizontal dash lines in (a) indicate the longitudinal velocity of the water.

Figure 3.8 presents the mode shape of the guided modes at 1000 kHz and the corresponding displacements on the x_2 axis in the cross section, which can give an intuitive interpretation of the absorption efficiency of the PML and attenuation feature of the modes shown in Fig. 3.7(b). The arrows in the mode shapes represent displacements in the cross section and the color indicates the energy flux in the x_3 direction. According to the displacements, it is obvious that leaky waves can be totally damped in a very short distance when they go into the PML, showing the PML works properly, being an efficient tool to attenuate leaky waves. It also can be seen that the $F(1,1)$ mode have antisymmetric mode shape with energy concentrates in the two opposite corners where the radial displacements dominate and strongly couple into the water, leading to a relatively high attenuation. As no displacement coupling occurs between the cylindrical bar and the water in the $T(0,1)$ mode (shown in Fig. 3.8(c) and (d)), there is no displacement in the surrounding water in this mode, thus the wave propagates without attenuation. The symmetrical longitudinal mode $L(0,1)$ is dominant in the axial direction, with most of the energy concentrating in the center of the bar, thus this mode is weakly coupled with the water, showing low attenuation at this frequency.

By comparison to the SAFE-AR model in Ref. [95], it is obviously that, only 1.5 mm PML can effectively damp the leaky waves over the frequency region, which is about one fourth of the length of the absorbing region in the SAFE-AR model, therefore leading to a significant reduction in the computational cost as well as the number of the PML modes.

From the two validation cases, it can be concluded that the SAFE-PML method is an efficient method to investigate modal properties of leaky guided waves in embedded/immersed waveguides.

Figure 3.8: Mode shapes (on the left) and displacements (on the right) on the x_2 axis from the ordinate origin to the endpoint of the PML for the cylindrical bar immersed in water at 1000 kHz: (a) mode shapes and (b) displacements of the F(1,1) mode with $k = 2338.35 + 24.20i \text{ m}^{-1}$; (c) mode shapes and (d) displacements of the T(0,1) mode with $k = 1927.36 \text{ m}^{-1}$; (e) mode shapes and (f) displacements of the L(0,1) mode with $k = 1255.66 + 8.21i \text{ m}^{-1}$. U_r and U_θ represent the radial displacement and the circumferential displacement in the cross section, respectively; U_3 is the displacement in the axial direction.

3.4 Application of the SAFE-PML Method

In this section, practical examples are solved to illustrate the potential of SAFE-PML method in NDT applications. The examples focus on embedded pipes and immersed rectangular bars. They are interesting to researchers developing advanced inspection techniques but difficult to be analyzed by analytical methods.

3.4.1 Steel pipe embedded in infinite concrete

Pipe structures are widely used for the transportation, distribution of gas, water, chemicals and petroleum products. Long range inspection is achieved for free pipes from the understanding of modal properties. However the inspection of embedded pipelines are still challenging, as the energy will leak into the surrounding medium leading to high attenuation rate thus significantly decreasing the wave propagation distance [131, 132]. This will get even more complicated, when only part of the circumference region of the pipe contacts surrounding medium, resulting in unpredictable test-ranges [133]. In this section, the SAFE-PML method is used to understand the modal properties for two cases - pipes completely or half embedded in concrete, which demonstrates the great potential of the developed method.

3.4.1.1 Model description

The geometries of the model are shown in Fig. 3.9. An 8 inch pipe is studied which has a inner radius of 101.6 mm and the thickness of 9 mm. In the fully embedded case, concrete material is fully attached to the pipe in a square shape with the width of 230 mm, while for the half embedded case the bottom half of the pipe does not contact the concrete. The PML functions in the simulation are chosen as $\hat{\gamma}_1 = 3 + 12i$ and $\hat{\gamma}_2 = 3 + 12i$. The SAFE-PML model is solved for 19 frequencies from 10 to 100 kHz with increments of 5 kHz. The minimum length of the PML can be calculated based on Eq. (3.48): $h_{x_1} = h_{x_2} \geq 38.64$ mm. In this study, 45 mm is used as the thickness of the PML. All of the material parameters are taken from Table 5.1.

Between the steel pipe and the concrete, an interface boundary condition to ensure continuity of displacements and stresses is imposed. Stress-free conditions are imposed at the inner surface of the pipe and the outside boundaries of the PML. For the fully embedded case, 4576 triangular elements are used to mesh the whole geometry and the number of degrees of freedom is 56712. 4721 triangular elements are generated for the half embedded geometry with the number of degrees of freedom being 59184. The mode filter parameter $\eta = 3.5$ is used in the post-processing to pick up the guided modes propagating in the pipe.

Figure 3.9: Schematic of the SAFE-PML model for a steel pipe (a) fully embedded and (b) half embedded in infinite concrete.

3.4.1.2 Results

Figure 3.10 shows dispersion curves of wave modes in the fully embedded case propagating along the steel pipe and radiating energy into the surrounding concrete. Although the dispersion curves of both axisymmetric and nonaxisymmetric wave modes were calculated, only solutions for circumferential order zero (axisymmetric modes) and circumferential order one (nonaxisymmetric modes) are plotted in the figure, as these modes are of particular interest to practical applications.

It is well known that, in the free pipes, the L(0,2) mode [134] and the T(0,1)

mode [135] at specific frequency range are attractive for long range inspections, since they are non-dispersive and easy to be generated and separated by time-domain gating. However, when the pipe is fully embedded in concrete, it can be seen from Fig. 3.10(b) that both modes suffer from high attenuation due to the energy leaking into the concrete. Therefore the inspection distance is significantly reduced.

It is worth noting that the $L(0,1)$ mode may be attractive for the inspection of the fully embedded pipe in concrete as there exists a non-leaky region. It can be seen from Fig. 3.10, when the phase velocity of the $L(0,1)$ mode is below the shear bulk velocity of the concrete (between 20-60 kHz), no energy leaks into the concrete. As this mode is also little dispersive in that frequency range, it could be explored as a potential candidate for the long distance inspection.

Figure 3.10: (a) Phase velocity and (b) attenuation dispersion curves for the circumferential order zero and order one modes for the steel pipe fully embedded in concrete; the two horizontal dash lines in (a) represent the shear and longitudinal bulk velocity of the concrete, respectively.

The mode shape of the guided wave modes at 50 kHz are presented in Fig. 3.11, where the arrows represent displacements in the cross section, and the color indicates the energy flux in the x_3 direction. It can be observed that, comparing with the $T(0,1)$, $F(1,2)$, $L(0,2)$ and $F(1,3)$ modes, there is almost no energy leaking into the surrounding concrete in the $L(0,1)$ and $F(1,1)$ modes.

Figure 3.11: Mode shapes for the fully embedded pipe at 50 kHz for (a) $L(0,1)$ mode with $k = 154.428 + 0.00023i \text{ m}^{-1}$, (b) $F(1,1)$ mode with $k = 154.167 + 0.00014i \text{ m}^{-1}$, (c) $T(0,1)$ mode with $k = 95.928 + 8.246i \text{ m}^{-1}$, (d) $F(1,2)$ mode with $k = 95.451 + 8.283i \text{ m}^{-1}$, (e) $L(0,2)$ mode with $k = 57.7 + 4.748i \text{ m}^{-1}$ and (f) $F(1,3)$ mode with $k = 56.91 + 4.8374i \text{ m}^{-1}$.

The dispersion curves for the phase velocity and attenuation in the half embedded case are presented in Fig. 3.12. Comparing to the dispersion curves in fully em-

bedded case, two significant features can be identified. Firstly, as the system is non-symmetrical, the axisymmetric modes, such as $L(0,1)$, $T(0,1)$ and $L(0,2)$ mode, do not exist in this case. Secondly, each of the non-axisymmetric modes, e.g. $F(1,1)$, $F(1,2)$ and $F(1,3)$ mode, splits into two independent modes. These two modes have similar mode shapes as their corresponding mode in the fully embedded case, but with energy focusing in the embedded section (named F' modes) and the free section (named F'' modes), respectively. From Fig. 3.12(b), it can be seen that the attenuation behavior of F' modes is very similar to their corresponding modes in the fully embedded case, while F'' modes have little attenuation. These phenomena can be understood from the mode shapes of these modes.

Figure 3.12: (a) Phase velocity and (b) attenuation dispersion curves for the circumferential order zero and order one modes for the steel pipe half embedded in concrete; the two horizontal dash lines in (a) represent the shear and longitudinal bulk velocity of the concrete, respectively.

Fig. 3.13 presents the mode shapes of a 8 inch pipe half embedded in concrete at 50 kHz, with the arrows representing displacements in the cross-section, and the color indicating the energy flux in the x_3 direction. For F' modes, strong coupling exists between the pipe and the surrounding concrete, thus leading to the attenuation similar to that in the fully embedded case. For the F'' modes, most of the energy concentrates in the free part, and very little leaks into the concrete, which results in little attenuation as indicated in Fig. 3.12(b).

Figure 3.13: Mode shapes for the half embedded pipe at 50 kHz for (a) $F'(1,1)$ mode with $k = 154.269 + 0.0018i \text{ m}^{-1}$, (b) $F''(1,1)$ mode with $k = 169.939 \text{ m}^{-1}$, (c) $F'(1,2)$ mode with $k = 95.532 + 8.208i \text{ m}^{-1}$, (d) $F''(1,2)$ mode with $k = 95.979 + 0.0666i \text{ m}^{-1}$, (e) $F'(1,3)$ mode with $k = 57.147 + 4.618i \text{ m}^{-1}$ and (f) $F''(1,3)$ mode with $k = 56.066 + 0.196i \text{ m}^{-1}$.

It is worth noting that, these desired features discovered in pipes embedded in concrete may change when the surrounding material properties or the connection

conditions between pipes and the surrounding materials are changed, e.g. pipes buried in sand or soil, embedded pipes coated with bitumen and partially embedded pipes with different extent of concrete cover. This is because the modal properties of the propagating modes are sensitive to the physical conditions, and may vary significantly from case to case [131]. However, the calculation for the fully and half embedded pipes in concrete has demonstrated the efficiency of the SAFE-PML method.

3.4.2 A rectangular steel bar immersed in water

Rectangular bars are typical structures with a wide application in fluid characterization [95, 136–138], therefore the study of modal properties of the rectangular bar is practically important. However, only particular guided modes in such waveguide were studied in previous work, while no effort has been attempted to understand modal properties of all guided modes due to lack of efficient models.

3.4.2.1 Modal description

Fig. 3.14 shows the geometry of the model. A rectangular steel bar of $1.1 \text{ mm} \times 2.2 \text{ mm}$ immersed in water is studied. The surrounding water is modeled as a square shape with length of 4 mm . The PML functions in the simulation are chosen as $\hat{\gamma}_1 = 3 + 12i$ and $\hat{\gamma}_2 = 3 + 12i$. The SAFE-PML model is solved for 39 frequencies from 50 to 1000 kHz with increments of 25 kHz. The minimum length of the PML can be calculated based on Eq. (3.48): $h_{x_1} = h_{x_2} \geq 2.75 \text{ mm}$. In this study, 3 mm is used as the length of the PML. All of the material properties are given in Table 5.1.

Fluid loading conditions are imposed at the interface between the rectangular bar and the water. Neumann boundary conditions with $q = 0$ and $g = 0$ are used on the outside boundaries of the PML. 2064 triangular elements are generated to mesh the whole geometry and the number of degrees of freedom is 4872. The mode filter parameter $\eta = 2.5$ is used in the post-processing to pick up the guided mode

propagating in the rectangular bar.

Figure 3.14: *Schematic of the SAFE-PML model for a rectangular steel bar immersed in water.*

3.4.2.2 Results

Figure 3.15 shows the dispersion curves of leaky guided waves for the rectangular bar. The modes can be named as in Ref. [139] according to the convention for modes in a rectangular bar in vacuum.

It can be seen that the first order longitudinal mode L1 is little dispersive and almost non-leaky at a relatively low frequency range (0 – 700 kHz). These properties make this mode a potential candidate for NDT application in immersed rectangular bars.

The two features have also been observed in the first order torsional mode T1, the difference being that the T1 mode experiences almost no attenuation in a relatively narrow frequency range (0–200 kHz). In this range, the fluid can be trapped around

the corner of the cross section because of the non-circular cross section geometry, which thus will affect the propagation of the torsional wave. Therefore, this mode can sense the presence and nature of the surrounding fluid and can be used as a sensitive tool for fluid characterizations [95].

The two bending modes (B_x^1 and B_y^1) are similar to the F(1,1) mode in the cylindrical bar. They have non-leaky feature when the phase velocity of the mode is slower than that of the water. However, as their phase velocities go above the velocity of the water, the attenuation increases rapidly due to energy leaking into the water.

Figure 3.15: (a) Phase velocity and (b) attenuation dispersion curves for a rectangular steel bar immersed in water. The horizontal dash lines in (a) indicate the longitudinal velocity of the water.

The corresponding mode shape of the guided modes at 500 kHz are presented in Fig. 3.16. The arrows represent displacements in the cross section, and the color indicates the energy flux in the x_3 direction. The bending modes (B_x^1 and B_y^1) dominate in the x_1 and x_2 direction and are strongly coupled with the water in the dominant vibration direction, thus leading to a lot of energy radiating into the water. It also can be seen that the torsional mode (T1) cause displacements in the water and has a relatively high attenuation at the frequency, which is in contrast to the case in the cylindrical bar where the torsional mode is non-attenuative. In the longitudinal mode (L1), as the dominant displacement is in the x_3 direction, the vibration in the x_1 and x_2 direction becomes weak, so that it is slightly attenuated

at the frequency.

Figure 3.16: Mode shapes for the rectangular bar immersed in water at 500 kHz for (a) B_x^1 mode with $k = 1646.23 + 85.56i \text{ m}^{-1}$, (b) B_y^1 mode with $k = 1328.72 + 16.03i \text{ m}^{-1}$, (c) $T1$ mode with $k = 1292.22 + 20.46i \text{ m}^{-1}$ and (d) $L1$ mode with $k = 606.40 + 1.05i \text{ m}^{-1}$.

3.5 Summary

The SAFE-PML method has been developed for open waveguides including embedded waveguides and immersed waveguides. The governing equations are derived by combining the SAFE method with the PML method. As it is implemented in a commercial software, the SAFE-PML method provides a fast and efficient way to investigate the modal properties of the open waveguides where energy can leak into the surrounding medium. The developed method has been first validated on two examples with regular cross sections, including an aluminum plate attached

to elastomer material and a cylindrical steel bar immersed in water, showing good agreements with analytical solutions. The method is then applied on two practically important problems, including a half embedded pipe and immersed rectangular bar demonstrating that such method can effectively predict modal properties of guided waves for practical applications. This developed method is also utilized in Chapter 6 as a fundamental tool for studies of nonlinear ultrasonic guided wave in open waveguides.

It should be noted that, the main drawback of the SAFE-PML method is to select optimal PML parameters, including the PML function and the length of the PML. It is clear that an optimal PML function can reduce the length of the PML leading to a relatively low computational cost as well as the number of the PML modes. Even though the PML function used in this paper shows good performance, and the length of the PML can be predicted by a 2D plane wave propagation model, the selection of the optimal PML parameters is still an open problem, which requires extra effort in future work.

Chapter 4

Nonlinear Ultrasonic Guided Waves in Closed Waveguides with Arbitrary Cross Sections

4.1 Background

Linear ultrasonic methods, widely used in NDT, are only sensitive to macro-damages in the order of the wavelength of the ultrasound but they are not capable of detecting micro-damages such as micro-cracks or material degradation [7], which usually occur at very early stage of the operation. Such challenge may be overcome by using nonlinear ultrasonic methods, as they are much more sensitive to incipient damages [5, 8, 12, 140]. In recent years, nonlinear ultrasonic guided waves, combining advantages of nonlinear ultrasound and guided waves, have shown to be a useful tool to characterize incipient damages in large structures, as guided waves have the ability to propagate for a long distance and therefore provide capability for the remote inspection of areas with difficult access [12].

However, most studies on nonlinear ultrasonic guided waves to date have been limited to regular waveguides, such as plates, rods or pipes. It is also known that

experimental investigation of nonlinear ultrasonic guided waves is very challenging due to weak harmonic waves, noise, and the system nonlinearity. Therefore in order to further explore nonlinear guided waves in various waveguides, it is important to develop a numerical framework that is capable of computing resonant modes in complex waveguides efficiently, and simulate the propagation of nonlinear guided waves using optimized generation and signal processing methods.

This chapter starts with a review of the theoretical study on nonlinear ultrasonic guided waves, following by the introduction of the numerical framework to predict the internal resonant modes in a complex waveguide with arbitrary cross section using an improved nonlinear SAFE method, and time domain FE simulations to confirm the existence of internal resonant modes and validate their properties. The numerical models were first demonstrated in an aluminum plate which can be compared with analytical solutions, and then they were applied on a rectangular steel bar which can only be analyzed by numerical methods.

4.2 Mathematical Framework of Nonlinear Ultrasonic Guided Waves in Closed Waveguides

Figure 4.1: *Schematic of a closed waveguide with arbitrary geometry, where Ω and $\partial\Omega$ represent the waveguide domain and the stress free boundaries, respectively.*

In this section, a brief review of generalized nonlinear elasticity wave propagation theory in closed waveguides (shown in Fig. 4.1) using finite deformations (see Appendix B) and nonlinear material properties is presented. Mathematical derivations start from nonlinear Navier equation; perturbation method is then utilized to transform the nonlinear equation into two linear equations; following that, the nonlinear wave propagation solution can be obtained through complex reciprocity relation and modal analysis based on the fact that the wave mode satisfy orthogonality relations in closed waveguides.

The governing equations with stress free boundary conditions can be formulated as a concise form by using the displacement $\mathbf{u}$,

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u}) + \bar{\mathbf{f}} \text{ in } \Omega, \quad (4.1)$$

$$(\mathbf{s}(\mathbf{u}) + \bar{\mathbf{s}}(\mathbf{u})) \cdot \mathbf{n}_r = \mathbf{0} \text{ on } \partial\Omega, \quad (4.2)$$

in which $\mathbf{n}_r$ is a unit vector which is normal to the waveguide; $\mathbf{s}(\mathbf{u})$ represents the corresponding linear part of the first Piola-Kirchhoff (P-K) stress, being written as

$$\mathbf{s}(\mathbf{u}) = \frac{\lambda}{2} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \mathbf{I} + \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right). \quad (4.3)$$

$\bar{\mathbf{f}}$ and $\bar{\mathbf{s}}(\mathbf{u})$ represent nonlinear terms and they can be explicitly expressed using tensor components as

$$\begin{aligned} \bar{f}_i &= \left(\mu_A + \frac{A}{4} \right) (u_{l,kk} u_{l,i} + u_{l,kk} u_{i,l} + 2u_{i,lk} u_{l,k}) + \left(\lambda_A + \mu_A + \frac{A}{4} + B \right) \\ &\quad \times (u_{l,ik} u_{l,k} + u_{k,lk} u_{i,l}) + (\lambda_A + B) (u_{i,kk} u_{l,l}) + \left(\frac{A}{4} + B \right) \\ &\quad \times (u_{k,lk} u_{l,i} + u_{k,ik} u_{k,l}) + (B + 2C) (u_{k,ik} u_{l,l}) + O(u_i^3) \end{aligned} \quad (4.4)$$

$$\begin{aligned} \bar{s}_{ij} &= \left(\frac{\lambda}{2} u_{k,l} u_{k,l} + C u_{k,k} u_{l,l} \right) \delta_{ij} + B u_{k,k} u_{j,i} + \frac{A}{4} u_{j,k} u_{k,i} \\ &\quad + \frac{B}{2} (u_{k,l} u_{k,l} + u_{k,l} u_{l,k}) \delta_{ij} + (\lambda + B) u_{k,k} u_{i,j} \\ &\quad + \left(\mu + \frac{A}{4} \right) (u_{i,k} u_{j,k} + u_{k,i} u_{k,j} + u_{i,k} u_{k,j}) + O(u_i^3), \end{aligned} \quad (4.5)$$

where δ_{ij} is the Kronecker delta, $O(u_i^3)$ is the higher order term.

This nonlinear wave propagation equation is solved by using the perturbation theory where the displacement field $\mathbf{u}$ can be divided into two parts: the main field ($\mathbf{u}^{(1)}$) and the small perturbation field ($\mathbf{u}^{(2)}$) [17]

$$\mathbf{u} = \mathbf{u}^{(1)} + \mathbf{u}^{(2)}. \quad (4.6)$$

The main displacement field represents the primary wave field and the perturbation displacement field demonstrates the secondary wave field, which is much smaller than the primary wave field. Combining Eqs. (4.1), (4.2) and (4.6), the nonlinear problem can be cast into two linear problems where the governing equation and boundary condition are given in the first-order approximation by

$$\rho_0 \frac{\partial^2 \mathbf{u}^{(1)}}{\partial t^2} = (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}^{(1)}) - \mu \nabla \times (\nabla \times \mathbf{u}^{(1)}) \quad \text{in } \Omega, \quad (4.7)$$

$$\mathbf{s}(\mathbf{u}^{(1)}) \cdot \mathbf{n}_r = \mathbf{0} \quad \text{on } \partial\Omega, \quad (4.8)$$

where $\mathbf{s}(\mathbf{u}^{(1)})$ is a function of $\mathbf{u}^{(1)}$. In the second order approximation, the governing equation and the boundary condition can be given by

$$\rho_0 \frac{\partial^2 \mathbf{u}^{(2)}}{\partial t^2} = (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}^{(2)}) - \mu \nabla \times (\nabla \times \mathbf{u}^{(2)}) + \bar{\mathbf{f}}^{(1)} \quad \text{in } \Omega, \quad (4.9)$$

$$\mathbf{s}(\mathbf{u}^{(2)}) \cdot \mathbf{n}_r = -\bar{\mathbf{s}}^{(1)} \cdot \mathbf{n}_r \quad \text{on } \partial\Omega, \quad (4.10)$$

where $\bar{\mathbf{f}}^{(1)}$ and $\bar{\mathbf{s}}^{(1)}$ are functions of the primary wave field ($\mathbf{u}^{(1)}$), which can be calculated by substituting the primary wave field into Eqs. (4.4) and (4.5). The physical meaning of the secondary wave field ($\mathbf{u}^{(2)}$) can be understood as a specific solution with a volume force, $\bar{\mathbf{f}}^{(1)}$, applied in the waveguide, and a surface force, $\bar{\mathbf{s}}^{(1)}$, applied on the boundary of the waveguide. Thus, the original nonlinear problem reduces to seeking for two separate linear solutions.

4. Nonlinear Ultrasonic Guided Waves in Closed Waveguides with Arbitrary Cross Sections

In the waveguide, the primary wave field can be numerically calculated through the SAFE method, which assumes a harmonic wave propagation in the z direction. So that the appropriate solution form for the primary wave field can be written as

$$\mathbf{u}^{(1)}(x, y, z, t) = \mathbf{U}^{(1)}(x, y) e^{i(kz - \omega t)}, \quad (4.11)$$

where z is the wave propagation direction; $\mathbf{U}^{(1)}$ means the displacement field in the cross section of wave propagation; k and $\omega = 2\pi f$ represent the wavenumber and the circular frequency, respectively; i demonstrates the imaginary unit. The complex reciprocity relation and modal expansion method are used to solve the secondary wave field with the former being written as [102]

$$\nabla \cdot (-\mathbf{v}_1^* \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_1^*) = \mathbf{v}_1^* \cdot \mathbf{f}_2 + \mathbf{v}_2 \cdot \mathbf{f}_1^*, \quad (4.12)$$

where "1" and "2" represent two possible solutions of the waveguide; $\mathbf{v}_1$ and $\mathbf{v}_2$ represent the velocity tensors; $\mathbf{s}_1$ and $\mathbf{s}_2$ are the stress tensors; $\mathbf{f}_1$ and $\mathbf{f}_2$ demonstrate the volume forces and $*$ indicate the complex conjugate.

Solution "1" can be considered as the n th mode of the waveguide without external force fields and solution "2" is a solution excited by the external forces applied by the primary wave. Thus the two possible solutions can be written as

$$\mathbf{v}_1^* = \mathbf{v}_n^* e^{-ik_n^* z}, \quad \mathbf{s}_1^* = \mathbf{s}_n^* e^{-ik_n^* z}, \quad \mathbf{f}_1^* = 0, \quad (4.13)$$

$$\mathbf{v}_2 = \mathbf{v}_2(x, y, z), \quad \mathbf{s}_2 = \mathbf{s}_2(x, y, z), \quad \mathbf{f}_2 = \bar{\mathbf{f}}^{(1)}. \quad (4.14)$$

Substituting the possible solutions into Eq. (4.12), Eq. (4.12) can be expressed as

$$\begin{aligned} & \nabla_{\perp} \cdot (-\mathbf{v}_n^* e^{-ik_n^* z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n^* e^{-ik_n^* z}) + \\ & \frac{\partial}{\partial z} (-\mathbf{v}_n^* e^{-ik_n^* z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n^* e^{-ik_n^* z}) \cdot \mathbf{n}_z = \mathbf{v}_n^* e^{-ik_n^* z} \cdot \bar{\mathbf{f}}^{(1)}, \end{aligned} \quad (4.15)$$

4. Nonlinear Ultrasonic Guided Waves in Closed Waveguides with Arbitrary Cross Sections

where $\nabla_{\perp} = \frac{\partial}{\partial x} () \cdot \mathbf{n}_x + \frac{\partial}{\partial y} () \cdot \mathbf{n}_y$ and $\mathbf{n}_z$ means the unit vector in the z direction. Integrating Eq. (4.15) over the cross section leads to

$$\begin{aligned} & \int_{\Omega} \nabla_{\perp} \cdot \left(-\mathbf{v}_n^* e^{-ik_n^* z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n^* e^{-ik_n^* z} \right) d\Omega \\ & + \int_{\Omega} \frac{\partial}{\partial z} \left(-\mathbf{v}_n^* e^{-ik_n^* z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n^* e^{-ik_n^* z} \right) \cdot \mathbf{n}_z d\Omega = \int_{\Omega} \mathbf{v}_n^* e^{-ik_n^* z} \cdot \bar{\mathbf{f}}^{(1)} d\Omega \end{aligned} \quad (4.16)$$

It is known that wave modes in the closed waveguides satisfy orthogonality relations. Thus the SHW field can be written as a superposition of modal basis

$$\mathbf{v}_2(x, y, z) = \frac{1}{2} \sum_{m=1}^{\infty} A_m(z) \mathbf{v}_m(x, y) + c.c., \quad (4.17)$$

$$\mathbf{s}_2(x, y, z) = \frac{1}{2} \sum_{m=1}^{\infty} A_m(z) \mathbf{s}_m(x, y) + c.c., \quad (4.18)$$

where $c.c.$ represents complex conjugate; $A_m(z)$ is the modal amplitude for the m th mode which is to quantify the contribution of the m th mode in the mode expansion.

The first term on the left hand side of Eq. (4.16) can be calculated by using Gauss divergence theorem

$$\begin{aligned} & \int_{\Omega} \nabla_{\perp} \cdot \left(-\mathbf{v}_n^* e^{-ik_n^* z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n^* e^{-ik_n^* z} \right) d\Omega \\ & = \int_{\partial\Omega} \left(-\mathbf{v}_n^* e^{-ik_n^* z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n^* e^{-ik_n^* z} \right) \cdot \mathbf{n}_r d\Gamma \end{aligned} \quad (4.19)$$

With considering the boundary conditions $\mathbf{s}_2 \cdot \mathbf{n}_r|_{\partial\Omega} = -\bar{\mathbf{s}}^{(1)} \cdot \mathbf{n}_r$ and $\mathbf{s}_n^* \cdot \mathbf{n}_r|_{\partial\Omega} = 0$, Eq. (4.19) can be simplified as

$$\begin{aligned} & \int_{\Omega} \nabla_{\perp} \cdot \left(-\mathbf{v}_n^* e^{-ik_n^* z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n^* e^{-ik_n^* z} \right) d\Omega \\ & = - \int_{\partial\Omega} \mathbf{v}_n^* e^{-ik_n^* z} \cdot \left(-\bar{\mathbf{s}}^{(1)} \right) \cdot \mathbf{n}_r d\Gamma \end{aligned} \quad (4.20)$$

Substituting Eqs. (4.17) and (4.18) into Eq. (4.16), the second term on the left hand

side of Eq. (4.16) can be expressed as

$$\begin{aligned} & \int_{\Omega} \frac{\partial}{\partial z} \left(-\mathbf{v}_n^* e^{-ik_n^* z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n^* e^{-ik_n^* z} \right) \cdot \mathbf{n}_z d\Omega \\ &= P_{mn} \frac{\partial}{\partial z} \left(e^{-ik_n^* z} \sum_{m=1}^{\infty} 2A_m(z) \right) \end{aligned} \quad (4.21)$$

where $P_{mn} = \int_{\Omega} \left(-\frac{1}{4} \mathbf{v}_n^* \cdot \mathbf{s}_m - \frac{1}{4} \mathbf{v}_m \cdot \mathbf{s}_n^* \right) \cdot \mathbf{n}_z d\Omega$, which means flux energy in the z direction.

So the governing equation (Eq. (4.16)) can be written as

$$\begin{aligned} & P_{mn} \frac{\partial}{\partial z} \left(e^{-ik_n^* z} \sum_{m=1}^{\infty} 2A_m(z) \right) \\ &= e^{-ik_n^* z} \left(\int_{\Omega} \mathbf{v}_n^* \cdot \bar{\mathbf{f}}^{(1)} d\Omega + \int_{\partial\Omega} \mathbf{v}_n^* \cdot (-\bar{\mathbf{s}}^{(1)}) \cdot \mathbf{n}_r d\Gamma \right) \end{aligned} \quad (4.22)$$

For propagating mode, the orthogonality relation is given by [102]

$$P_{mn} = 0, \text{ if } m \neq n. \quad (4.23)$$

Therefore, Eq. (4.22) can be simplified as an ordinary differential equation

$$2P_{mm} \left(\frac{d}{dz} - ik_m^* \right) A_m(z) = \int_{\Omega} \mathbf{v}_m^* \cdot \bar{\mathbf{f}}^{(1)} d\Omega + \int_{\partial\Omega} \mathbf{v}_m^* \cdot (-\bar{\mathbf{s}}^{(1)}) \cdot \mathbf{n}_r d\Gamma. \quad (4.24)$$

Solving the equation, the amplitude of the SHW can be obtained

$$A_m(z) = \frac{i(f_m^{vol} + f_m^{surf})}{4P_{mm}(2k_l - k_m^*)} e^{ik_m^* z} (1 - e^{i(2k_l - k_m^*)z}), \quad (4.25)$$

where $f_m^{vol} = \frac{1}{2} \int_{\Omega} \mathbf{v}_m^* \cdot \bar{\mathbf{f}}^{(1)} d\Omega$ and $f_m^{surf} = \frac{1}{2} \int_{\partial\Omega} \mathbf{v}_m^* \cdot (-\bar{\mathbf{s}}^{(1)}) \cdot \mathbf{n}_r d\Gamma$ represent the power flux through the volume and through the surface of the waveguide, respectively; k_l is the wavenumber of the primary wave.

From Eq. (4.25), it is evident that if the solution satisfy two conditions: (1) phase velocity matching: $k_m^* = 2k_l$; (2) non-zero energy transfer from the primary wave to the SHW: $f_m^{surf} + f_m^{vol} \neq 0$, the amplitude of the SHW can be simplified as

$$A_m(z) = \frac{f_m^{vol} + f_m^{surf}}{4P_{mm}} z e^{i2k_l z}, \quad (4.26)$$

showing that the modal amplitude of the m th secondary mode increases linearly with the propagation distance z . These two conditions are known as the internal resonance conditions. As a result, nonlinear guided waves only exist in particular frequencies.

4.3 Numerical Methods

4.3.1 Nonlinear SAFE method

As discussed in Section 4.2, it is necessary to identify internal resonant mode pair, in which the propagation of the primary wave can lead to the generation of the cumulatively increasing secondary wave. Eqs. (4.6-4.10) has shown that the nonlinear guided wave problem can be divided into two linear guided wave problems via perturbation theory, which describe the solutions of the primary wave and SHW, respectively. So that the nonlinear SAFE method (combining the SAFE method and the nonlinear analysis) can be divided into two stages corresponding to the two conditions in Eq. (4.25), based on the linear solutions obtained.

In the first stage, the dispersion curves and modal properties of a given waveguide are developed using the SAFE method and all potential internal resonant mode pairs are selected based on the phase velocity matching criterion between the primary wave and the SHW. A simplified method is implemented by plotting the dispersion curve of the SHWs at its half frequency. Therefore, by overlapping the phase velocity dispersion curves of the primary wave, all the intersection points between the two dispersion curves are selected as the potential resonant points (mode pairs) which satisfy the phase velocity matching criterion.

After identifying the potential internal resonant mode pairs, the second stage is to ensure there is non-zero energy transferred from the primary wave to the SHW. In this stage, an integral process is implemented to calculate the power flux, P_{mm} , f_m^{vol} and f_m^{surf} , where mode shapes obtained from the SAFE method are used to

calculate the velocity vector and the first P-K stress tensor. If the energy transfer is not zero, the modal amplitude for the secondary mode increases cumulatively with propagation distance and these mode pairs are identified as internal resonance.

4.3.2 Time domain FE model

The nonlinear SAFE method has provided modal solutions of nonlinear guided waves in complex structures, including their frequencies and mode shapes. Based on the prediction from the nonlinear SAFE method, time domain FE model is developed to simulate the propagation of the nonlinear ultrasonic guided wave, and the SHW is collected to validate the internal resonant modes. Comparing with the experimental investigations of nonlinear guided waves, which usually suffer from the noise and the system nonlinearity, FE models can provide accurate control of the material properties in a noise-free environment and is therefore an ideal tool to validate findings from the nonlinear SAFE method.

In the time domain FE model, the Murnaghan model is used which is equivalent to the Landau-Lifshitz model used in the nonlinear SAFE method. The Murnaghan model can be expressed by the strain energy function of the material [12]

$$w = \frac{1}{2}\lambda(\text{tr}(\mathbf{E}))^2 + \mu\text{tr}(\mathbf{E}^2) + \frac{1}{3}(L + 2M)(\text{tr}(\mathbf{E}))^3 - M\text{tr}(\mathbf{E})((\text{tr}(\mathbf{E}))^2 - \text{tr}(\mathbf{E}^2)) + M\det(\mathbf{E}), \quad (4.27)$$

where $\text{tr}()$ and $\det()$ denote trace and determinant of the tensor, respectively; L , M and N are third order elastic constants in the Murnaghan model related to the the Landau-Lifshitz constants by

$$L = B + C, \quad M = \frac{1}{2}A + B, \quad N = A. \quad (4.28)$$

In the present work, both the nonlinear SAFE method and the time domain FE model were carried out in a commercial FE software package [125]. In the following sections, two examples were studied using the simulation tool discussed above: an

Table 4.1: Material properties used in the models [2, 3].

Material	ρ_0 (kg/m ³)	λ (GPa)	μ (GPa)	A (GPa)	B (GPa)	C (GPa)
Aluminium	2700	55.27	25.95	-351.2	-149.4	-102.8
Steel	7932	107.8	84.7	-325	-310	-800

aluminum plate which can be compared with analytical solutions, and a steel bar with rectangular cross section which can only be investigated by FE models.

4.4 Applications

4.4.1 Nonlinear ultrasonic guided waves in a plate

In this section, a 1 mm thick aluminum plate was chosen to demonstrate the non-linear SAFE method and its solutions were compared with analytical solutions. Material properties for the plate are listed in Table 4.1. The behavior of the modal amplitude for the secondary mode was discussed at both resonant and non-resonant frequencies.

Figure 4.2 presents phase velocity dispersion curves of the aluminum plate, in which primary modes are shown in solid curves with upper case labels while second harmonic modes are plotted in dashed curves with lower case labels at their half frequencies. All intersection points of the two group of curves were identified as potential resonant points, following by the calculation of the energy flux transferred from the primary wave to the corresponding SHW. In Fig. 4.2, four resonant points were identified from 0 to 500 kHz marked with triangles: namely (SH0, s0) at 1710.2 kHz, (SH1, s1) at 1780.6 kHz, (S1, s2) and (SH2, s2) at 3561.2 kHz. The results agree very well with the ones derived from the theoretical analysis [2]. It can be noted that there are another three intersection points marked with dots, which are (S0, sh1) at 954.3 kHz, (S0, a1) at 2219.6 kHz and (A1, sh3) at 2649.5 kHz, and they correspond to non-resonant points due to zero energy transfer.

Figure 4.2: Phase velocity dispersion curves for Rayleigh Lamb modes and SH modes in the aluminate plate.

Table 4.2 demonstrates the energy transfer at all intersection points in Fig. 4.2. Modal amplitudes of the secondary modes are listed in the table. For internal resonant points ((SH0, s0), (SH1, s1), (S1, s2), (SH2, s2)), the modal amplitude of the SHW increases cumulatively with the propagation distance and the (S1, s2) pair has the highest energy transfer efficiency. For non-resonant points ((S0, sh1), (S0, a1), (A1, sh3)), the amplitude of the SHW is almost zero, indicating zero energy transfer.

It has been shown that nonlinear guided waves only exist in limited frequency points as it needs to follow two strict criteria. However, in practice, the uncertainty of the material properties make it not feasible to collect the exact excitation frequency. Fig. 4.3 demonstrates the behavior of the SHW, when the excitation frequency deviates slightly from the resonant frequency predicted from the nonlinear SAFE method. A deviation factor, $\delta = (f - f_{\text{resonant}})/f_{\text{resonant}}$ is used to quantify the difference from the excitation frequency of the resonant mode pair (S1, s2) at 3561.2 kHz. It can be seen from Fig. 4.3 that when there is a significant deviation the modal amplitude of the SHW is bounded and oscillates with a specific period, and this period increases when the deviation decreases. Such oscillation phenomenon for the

4. Nonlinear Ultrasonic Guided Waves in Closed Waveguides with Arbitrary Cross Sections

Table 4.2: Modal amplitudes of SHW.

mode pair \ distance (m)	0.1	0.3	0.5	0.7	0.9
(SH0, s0)	2490	7470	12450	17430	22410
(SH1, s1)	244.13	732.39	1220.65	1708.91	2197.17
(S1, s2)	7049	21147	35245	49343	70490
(SH2, s2)	492.11	1476.33	2460.55	3444.77	4428.99
(S0, sh1)	0.0019	0.0056	0.0093	0.0130	0.0168
(S0, a1)	0.0317	0.0928	0.1474	0.1916	0.2223
(A1, sh3)	0.1616	0.4847	0.8079	1.131	1.4541

secondary mode has also been mentioned in references [14, 15, 141, 142]. When the deviation is going to zero, the modal amplitude of the SHW approaches the resonant one. This suggests that in the real experiment, when the exact resonant frequency is not certain, it is still possible to collect the cumulatively increased secondary mode within a limited propagation distance. Such distance decreases with larger mismatch between the excitation and the resonant frequency.

Figure 4.3: Behaviors of the modal amplitude for the $s2$ mode with different deviation from the resonant frequency.

Time domain FE simulation was carried out in this section, using the internal resonant mode pair (S1, s2) at 3561.2 kHz. This pair has been widely investigated among researchers [11, 22, 23, 74–81] for nonlinear guided waves, as it satisfies both phase velocity and group velocity matching conditions and therefore easier for the signal post processing. Fig. 4.4(a) shows the plate model. In the model, plane strain condition was applied with the length of the plate being set to be 400 mm. The primary wave was generated at the left side of the plate through prescribed displacement boundary conditions, using a 20 cycle Hann windowed tone burst with central frequency of 3561.2 kHz. To maximize the generation of the primary mode (S1), the input time signal at each specified node was scaled according to the mode shape of the primary mode (shown in Fig. 4.4(b)) [83]. Other boundaries were applied with stress free boundary conditions. In the simulation model, the whole geometry was meshed by rectangular elements with a maximum element size of 0.09 mm, and a maximum time step of 0.0025 μ s was used in calculation. Since both S1 and s2 mode are dominant by the in-plane displacement [72] with the maximum displacement occurring on the two surface of the plate shown in Fig. 4.4(b) and (c), the monitor points were placed on the top surface of the plate to pick up the in-plane displacement, at propagation distances from 10 mm to 300 mm in an increment of 10 mm.

4. Nonlinear Ultrasonic Guided Waves in Closed Waveguides with Arbitrary Cross Sections

Figure 4.4: (a) Schematic of the plate for the FE simulation, numbers in the left side of the plate are scale factors of the displacement of the node. (b) Displacement in the z direction of the S1 mode at 3561.2 kHz. (c) Displacement in the z direction of the s2 mode at 7122.4 kHz.

Figure 4.5(a) shows received signals monitored at 100 mm away from the generator. S1 and S0 modes at the primary frequency (3561.2 kHz) and s2 mode at the second harmonic frequency (7122.4 kHz) were identified by using time-frequency analysis, in which short time Fourier transform (STFT) was used to identify mode properties and resolve the amplitude of specific modes in the frequency domain, due to its capability to provide accurate results within a short propagation distance. Fig. 4.5(b) shows the STFT results, in which the primary and second harmonic frequencies are marked with dotted horizontal lines. Slices at these two frequencies are shown in Fig. 4.5(c) and (d), respectively. Fig. 4.5(c) shows received wave packets in the primary frequency. Both S1 and S0 mode were identified, while the S0 mode has slower group velocity and much less amplitude due to the specific excitation method. In Fig. 4.5(d), only the s2 mode were identified in the frequency of the SHW. The amplitude of the primary and the secondary mode (A_1 and A_2) can be obtained by identifying the maximum value in the corresponding wave packets.

4. Nonlinear Ultrasonic Guided Waves in Closed Waveguides with Arbitrary Cross Sections

Figure 4.5: (a) Time domain signal received at a propagation distance of 100 mm; (b) STFT's of the time domain signal; (c), (d) Slices at the primary and second harmonic frequencies as a function of time.

The modal amplitude ratio, A_2/A_1^2 , also named as the relative nonlinearity parameter, is plotted with respect to the propagation distance in Fig. 4.6. A clear linearly increased feature of the relative nonlinearity parameter can be seen, which confirms that the (S1, s2) pair at 3561.2 kHz is an internal resonant mode pair.

Figure 4.6: *Relative nonlinearity parameter with respected to the propagation distance.*

4.4.2 Nonlinear ultrasonic guided waves in a rectangular bar

A steel bar with a rectangular cross section ($10 \text{ mm} \times 2 \text{ mm}$) was considered in this section. Material properties of steel are listed in Table 4.1. The schematic of the rectangular bar is shown in Fig. 4.7, with the coordinate shown at the center of the rectangular cross section.

Because of the complex cross section geometry, no analytical primary field is available. However, propagation modes in a rectangular bar have been studied and classified via approximate theories [139]. Because of the twofold symmetry nature of the rectangular cross section, the modes in a rectangular bar can be classified into three categories: longitudinal modes, torsional modes and bending modes. Longitudinal modes have symmetrical mode shapes and stress distributions in relation to the x and y axis. Torsional modes demonstrate a shearing deformation of the bar with displacements and stresses being asymmetrical in relation to the x and y axis. In this study, a superscript n is used to represent the order of the L or the T mode. In bending modes, the displacement and the stress are symmetric about either x axis or y axis, and asymmetric about the other axis. The bending modes can be classified as B_y^n and B_x^n , in which the subscript describes their axis of asymmetry

and the superscript denotes their order.

Figure 4.7: *Schematic of the rectangular bar in the modeling.*

Figure 4.8 presents the phase velocity dispersion curves for the rectangular bar in 0-500 kHz, including the primary modes in solid line and second harmonic modes (at half frequency) in dashed line. Two internal resonant points were identified and marked with triangles in Fig. 4.8. They are (B_x^2, l^1) at the excitation frequency of 110.6 kHz and (T^2, l^2) at 226.5 kHz, where the upper case letter and lower case letter represent the primary and the secondary mode, respectively. Other intersection points between the solid line and dashed line were identified as non-resonant points as there is no energy transfer, and one of them, (L^1, b_y^2) at 229 kHz, was marked in Fig. 4.8 with black dot as an example.

Figure 4.8: *Phase velocity dispersion curves for the rectangular bar.*

Figure 4.9(a) and (b) show the mode shapes for the first internal resonant mode pair, the primary mode (B_x^2) at 110.6 kHz and the secondary mode (I^1) at 221.2 kHz, where the vectors present the in-plane displacement. Fig. 4.9(c) and (d) show the relative displacement in the three axes along a through thickness cutline which bisects the bar, showing that the primary mode is dominated by the vertical displacement at the center of the cross section, while the secondary mode is dominated by the axial displacement in the mid-plane of the bar.

4. Nonlinear Ultrasonic Guided Waves in Closed Waveguides with Arbitrary Cross Sections

Figure 4.9: *In-plane displacements (marked as vector arrows) of (a) the primary mode B_x^2 at 110.6 kHz and (b) the secondary mode l^1 at 221.2 kHz. Wave structures of (c) the primary mode and (d) the secondary mode extracted along the cutline, where u , v , w represent the normalized displacement in the x , y , z direction, respectively.*

Table 4.3 presents the distance variation of the modal amplitude of the SHW for the three intersection points in Fig. 4.8. For the internal resonant mode pair, (B_x^2, l^1) at 110.6 kHz and (T^2, l^2) at 226.5 kHz, the modal amplitude of the secondary modes increases cumulatively along the propagation distance. For the example of the non-resonant mode pair, i.e. (L^1, b_y^2) at 229 kHz, the modal amplitude of the secondary mode, b_y^2 , is almost zero, which confirms no energy transferring from the primary wave to the SHW.

In the FE simulation, the internal resonant mode pair (B_x^2, l^1) at 110.6 kHz was selected. The schematic of the rectangular bar is shown in Fig. 4.10(a). The length of the rectangular bar was 3000 mm. In order to maximize the generation of the primary mode, prescribed amplitude which properly matched with the mode shape of the primary wave (Fig. 4.9(a)), was applied at specified nodes on the cross section of the bar, as shown in Fig. 4.10(b). A 20 cycle Hann windowed tone burst with

4. Nonlinear Ultrasonic Guided Waves in Closed Waveguides with Arbitrary Cross Sections

Table 4.3: Modal amplitudes of SHW.

distance (m)	0.1	0.3	0.5	0.7	0.9
mode pair					
(B_x^2, l^1)	592.85	1778.55	2964.25	4149.95	5335.65
(T^2, l^2)	27.84	83.52	139.20	194.88	250.56
(L^1, b_y^2)	0.00025	0.00078	0.0013	0.0018	0.0023

central frequency of 110.6 kHz was used in the simulation. Other boundaries were applied with stress free boundary conditions. In the simulation, the geometry was meshed by tetrahedron elements with a maximum element size of 2 mm and a maximum time step of $0.1 \mu s$ was used in calculation. The propagation of the wave is monitored by points in the center of the cross section ($x = 0, y = 0$), from 10 mm to 1000 mm along the z axis with an increment of 10 mm. The primary mode B_x^2 and the secondary mode l^1 were analysed by recording the displacements in the y and the z direction respectively, according to their mode shapes.

Figure 4.10: Schematic of the rectangular bar for the FE simulation. Numbers in (b) represent the scale factor of the displacement of the node.

Figure 4.11 shows time domain signals obtained for the primary wave (v) and the SHW (w) at 410 mm away from the generation. As illustrated in Fig. 4.11(a), two modes, B_x^1 and B_x^2 , were generated due to the similarity in their mode shapes, although the amplitude of the B_x^2 mode was significantly larger due to the mode-shape-matching excitation method.

Figure 4.11(b) shows the displacement obtained in the z direction. A number of overlapping wave packets were found in the time domain. In order to identify these

modes, the mode shape along the z direction of both primary and secondary modes are analyzed in Fig. 4.12. It can be seen from Fig. 4.12(a) that the excited two primary modes, B_x^1 and B_x^2 , have no displacement at the center of the bar. Fig. 4.12(b) demonstrates that among all possible secondary modes, only l^1 mode has displacement at the center of the bar. Therefore it can be confirmed that the packets received at the monitor points is the l^1 mode. It should be noted that although the phase velocities of the B_x^2 and the l^1 mode match with each other as indicated from the nonlinear SAFE method, a mismatch on the group velocities of the two modes exists. Therefore the received SHW appears to have a number of wave packets overlapping with each other. Such phenomenon has also been reported by Deng *et al.* [65]. In order to analyze the nonlinearity of the bar, the integrated amplitude of the SHW need to be used, and therefore fast Fourier transform (FFT) was applied for the signal processing instead of STFT method.

Figure 4.11: Time domain signals for (a) out of plane displacement of the primary mode and (b) in-plane displacement of the secondary mode received at a propagation distance of 410 mm.

4. Nonlinear Ultrasonic Guided Waves in Closed Waveguides with Arbitrary Cross Sections

Figure 4.12: Displacement component in the z direction for (a) the primary mode at 110.6 kHz and (b) the secondary modes at 221.2 kHz.

Since the wave packets of the two bending modes start to separate completely at the propagation distance of 410 mm, signals were obtained between 410 mm and 810 mm for post processing. Fig. 4.13 shows the relative nonlinearity parameter A_2/A_1^2 , in which A_1 and A_2 are amplitudes of the dominant displacement of the primary wave and the SHW. Almost linear increase of the relative nonlinearity parameter can be observed, confirming that (B_x^2, l^1) at 110.6 kHz is an internal resonant mode pair.

Figure 4.13: Relative nonlinearity parameter with respect to the propagation distance for the excitation of the B_x^2 mode.

4.5 Summary

In this chapter, numerical models have been developed to investigate nonlinear guided waves in closed waveguides with arbitrary cross-sectional geometry. It has been shown that nonlinear guided waves only exist at particular internal resonant frequencies where the phase velocities of the primary waves and SHWs match with each other, and with non-zero energy transferring from the former to the latter. The nonlinear SAFE method was used for the modal study of the complex waveguide, and resonant frequencies were identified from the calculation of dispersion curves and the power flux in the propagation direction. With the results from the nonlinear SAFE method, FE simulations were performed at the resonant frequencies using the particular mode shapes of the primary mode.

The numerical models were first demonstrated in an aluminum plate, and the predicted internal resonant frequencies were compared with results from analytical solutions, showing perfect agreement. The models were then applied to a steel rectangular bar, which is chosen to represent complex waveguides in general. FE simulations in both cases have clearly shown the accumulative feature of the SHW, indicating the strong nonlinearity at internal resonant frequencies.

Chapter 5

Nonlinear Ultrasonic Guided Waves in Open Waveguides

5.1 Background

This chapter extends the nonlinear ultrasonic guided waves propagation from closed waveguides to open waveguides. It is known that all of the studies are only limited to closed waveguides, where all the energy is confined in the waveguides. However, in civil engineering and geophysics, there are more challenging cases where the waveguides are embedded/immersed in another infinite medium. In these cases, the waveguide interacting with the surrounding material may lead to energy leaking away from the waveguide, which further distort the behavior of the primary wave and the SHW, resulting in the amplitude of the SHW having no linearly cumulative nature as in closed waveguides. Therefore, understanding the behavior of the SHW in open waveguides becomes a necessary prerequisite to seeking for potential applications of nonlinear ultrasonic guided waves in such structures.

In Chapter 4, it can be known that, modal expansion method is useful on closed waveguides studies to derive the SHW field [14, 64], which stems from the fact that the wave modes, including propagating modes, non-propagating modes and evanescent modes, satisfy the orthogonality relation [143]. However, the application

of this method to open waveguides is more intricate since the orthogonality relation in these structures is obscure because of the unboundedness of the surrounding material. Theoretically, the unboundedness of the surrounding material in open waveguides leads to three types of modes forming [101, 103, 104]: trapped modes, leaky modes and radiation modes. In mathematics, trapped modes and radiation modes form the modal basis but leaky modes do not belong to the modal basis. Even some methods have been suggested to make the leaky modes approximately orthogonal [107], the application of orthogonality for leaky modes in open waveguides still remain challenging.

Recently, an approach has been proposed to address the applicability of leaky modes by introducing a finite PML to truncate the infinite surrounding medium in open waveguides [106, 111]. With PMLs, the leaky wave can be totally damped in a limited space, thus eliminating echoes from outsider boundaries and efficiently simulating the infinite surrounding medium, so that the infinite open waveguide is converted to a finite one. In addition, the finite PML transforms the continuum radiation modes existing in the infinite open waveguides into discrete sets of PML modes which only resonate in the PML [106]. Following this, an orthogonality relationship has been derived, which can be applied to any types of modes, including trapped modes, leaky modes and PML modes [111]. Hence a new modal basis has been defined by these three types of modes and leaky modes are applicable in modal expansion to describe any wave field in open waveguides through superposition of the modal basis.

In this chapter, mathematical framework is developed by using real reciprocity relation and modal expansion with PMLs. Numerical models are then implemented, including the nonlinear SAFE method and FE models. The nonlinear SAFE method is to predict the properties of nonlinear guided waves in frequency domain, while FE models are to simulate the nonlinear guided wave propagation and cross validate the predictions from nonlinear SAFE method. Following the numerical models, two examples are given to illustrate the proposed methods and reveal some interesting properties only occur in the open waveguides, including an aluminum plate attached to an elastomer and an aluminum plate with water loaded on one side.

Finally, well designed experiments are carried out to validate the properties of nonlinear ultrasonic guided wave in one-sidedly immersed plates, showing potential NDT applications of nonlinear ultrasonic guided waves for underwater inspections.

5.2 Mathematical Framework of Nonlinear Ultrasonic Guided Waves in Open Waveguides

In this section, the theoretical foundation of nonlinear ultrasonic guided wave propagation in open waveguides is developed through finite deformation method and nonlinear elastic material model (see Appendix B).

Mathematical derivations of the nonlinear ultrasonic guided waves start with infinite open waveguides. As shown in Fig. 5.1(a), the open waveguide consists of two main parts, the solid core structure, occupying Ω_A and the infinite surrounding material in Ω_B , which could be a solid or an inviscid fluid. In this thesis, only the nonlinearity properties of the core structure are considered, thus a homogeneous, isotropic and hyper-elastic material model with holding Landau-Lifshitz constitutive equation is applied to the core structures and a homogeneous, isotropic and elastic material model is applied to the surrounding material.

Figure 5.1: Schematics of (a) an infinite open waveguide and (b) the corresponding finite open waveguide with PML, where $\Gamma_{A/B}$ and Γ_B are the interface between the waveguide core structure and the surrounding material and the outer surface of the finite open waveguide, respectively; $\mathbf{n}_A$ and $\mathbf{n}_B^{\text{inn}}$ represent the outward-pointing unit normal vector of the core structure and the surrounding material on the interface, respectively, with holding a relationship as $\mathbf{n}_A = -\mathbf{n}_B^{\text{inn}}$; $\mathbf{n}_B^{\text{out}}$ is the outward-pointing unit normal vector of the finite open waveguide on Γ_B ; d_x and d_y denote the start position of the PML in the x and y direction, respectively; h_x and h_y represent the length of the PML in the x and y direction, respectively..

In the open waveguide, the governing equations and boundary conditions can be expressed as

$$\rho_{0A} \frac{\partial^2 \mathbf{u}_A}{\partial t^2} = (\lambda_A + 2\mu_A) \nabla (\nabla \cdot \mathbf{u}_A) - \mu_A \nabla \times (\nabla \times \mathbf{u}_A) + \bar{\mathbf{f}} \text{ in } \Omega_A, \quad (5.1)$$

$$\rho_{0B} \frac{\partial^2 \mathbf{u}_B}{\partial t^2} = (\lambda_B + 2\mu_B) \nabla (\nabla \cdot \mathbf{u}_B) - \mu_B \nabla \times (\nabla \times \mathbf{u}_B) \text{ in } \Omega_B, \quad (5.2)$$

$$(\mathbf{s}_A + \bar{\mathbf{s}}) \cdot \mathbf{n}_A = -\mathbf{s}_B \cdot \mathbf{n}_B^{\text{inn}}, \quad \mathbf{u}_A = \mathbf{u}_B \text{ on } \Gamma_{A/B}, \quad (5.3)$$

$$\mathbf{u}_B \rightarrow 0, \quad \mathbf{s}_B \rightarrow 0 \quad \text{at infinity}, \quad (5.4)$$

where $\mathbf{u}_A$ and $\mathbf{u}_B$ represent the displacement in Ω_A and Ω_B , respectively; λ_A and μ_A are the Lamé constants of core structure in Ω_A with λ_B and μ_B being the Lamé constants of the surrounding material in Ω_B ; $\mathbf{s}_B$ is the first P-K stress in Ω_B and $\mathbf{s}_A$ represents the corresponding linear part of the first P-K stress in Ω_A , given by

$$\mathbf{s}_A = \frac{\lambda_A}{2} \text{tr} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \mathbf{I} + \mu_A \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right). \quad (5.5)$$

$\bar{\mathbf{f}}$ and $\bar{\mathbf{s}}$ collect all nonlinear terms and they can be explicitly expressed with tensor components as

$$\begin{aligned} \bar{f}_i &= \left(\mu_A + \frac{A}{4} \right) (u_{l,kk} u_{l,i} + u_{l,kk} u_{i,l} + 2u_{i,lk} u_{l,k}) + \left(\lambda_A + \mu_A + \frac{A}{4} + B \right) \\ &\quad \times (u_{l,ik} u_{l,k} + u_{k,lk} u_{i,l}) + (\lambda_A + B) (u_{i,kk} u_{l,l}) + \left(\frac{A}{4} + B \right) \\ &\quad \times (u_{k,lk} u_{l,i} + u_{k,ik} u_{k,l}) + (B + 2C) (u_{k,ik} u_{l,l}) + O(u_i^3) \end{aligned} \quad (5.6)$$

$$\begin{aligned} \bar{s}_{ij} &= \left(\frac{\lambda_A}{2} u_{k,l} u_{k,l} + C u_{k,k} u_{l,l} \right) \delta_{ij} + B u_{k,k} u_{j,i} + \frac{A}{4} u_{j,k} u_{k,i} \\ &\quad + \frac{B}{2} (u_{k,l} u_{k,l} + u_{k,l} u_{l,k}) \delta_{ij} + (\lambda_A + B) u_{k,k} u_{i,j} \\ &\quad + \left(\mu_A + \frac{A}{4} \right) (u_{i,k} u_{j,k} + u_{k,i} u_{k,j} + u_{i,k} u_{k,j}) + O(u_i^3) \end{aligned} \quad (5.7)$$

In order to solve the nonlinear wave propagation equations, the perturbation theory is used to divide the displacement field $\mathbf{u}$ into two parts [19]

$$\mathbf{u}_A = \mathbf{u}_A^{(1)} + \mathbf{u}_A^{(2)}, \quad (5.8)$$

$$\mathbf{u}_B = \mathbf{u}_B^{(1)} + \mathbf{u}_B^{(2)}, \quad (5.9)$$

where $\mathbf{u}_A^{(1)}$ and $\mathbf{u}_B^{(1)}$ are the primary displacement fields; $\mathbf{u}_A^{(2)}$ and $\mathbf{u}_B^{(2)}$ represent the secondary displacement fields, which are much smaller than the primary fields. Substituting Eqs. (5.8) and (5.9) into Eqs. (5.1)-(5.4), the nonlinear problem can

be cast into two linear problems. In the first order perturbation equations, the governing equations and boundary conditions become

$$\rho_{0A} \frac{\partial^2 \mathbf{u}_A^{(1)}}{\partial t^2} = (\lambda_A + 2\mu_A) \nabla (\nabla \cdot \mathbf{u}_A^{(1)}) - \mu_A \nabla \times (\nabla \times \mathbf{u}_A^{(1)}) \quad \text{in } \Omega_A, \quad (5.10)$$

$$\rho_{0B} \frac{\partial^2 \mathbf{u}_B^{(1)}}{\partial t^2} = (\lambda_B + 2\mu_B) \nabla (\nabla \cdot \mathbf{u}_B^{(1)}) - \mu_B \nabla \times (\nabla \times \mathbf{u}_B^{(1)}) \quad \text{in } \Omega_B, \quad (5.11)$$

$$\mathbf{s}_A^{(1)} \cdot \mathbf{n}_A + \mathbf{s}_B^{(1)} \cdot \mathbf{n}_B^{\text{inn}} = 0, \quad \mathbf{u}_A^{(1)} = \mathbf{u}_B^{(1)} \quad \text{on } \Gamma_{A/B}, \quad (5.12)$$

$$\mathbf{u}_B^{(1)} \rightarrow 0, \quad \mathbf{s}_B^{(1)} \rightarrow 0 \quad \text{at infinity.} \quad (5.13)$$

In the second order perturbation equations,

$$\rho_{0A} \frac{\partial^2 \mathbf{u}_A^{(2)}}{\partial t^2} = (\lambda_A + 2\mu_A) \nabla (\nabla \cdot \mathbf{u}_A^{(2)}) - \mu_A \nabla \times (\nabla \times \mathbf{u}_A^{(2)}) + \bar{\mathbf{f}}^{(1)} \quad \text{in } \Omega_A, \quad (5.14)$$

$$\rho_{0B} \frac{\partial^2 \mathbf{u}_B^{(2)}}{\partial t^2} = (\lambda_B + 2\mu_B) \nabla (\nabla \cdot \mathbf{u}_B^{(2)}) - \mu_B \nabla \times (\nabla \times \mathbf{u}_B^{(2)}) \quad \text{in } \Omega_B, \quad (5.15)$$

$$\mathbf{s}_A^{(2)} \cdot \mathbf{n}_A + \mathbf{s}_B^{(2)} \cdot \mathbf{n}_B^{\text{inn}} = -\bar{\mathbf{s}}^{(1)} \cdot \mathbf{n}_A, \quad \mathbf{u}_A^{(2)} = \mathbf{u}_B^{(2)} \quad \text{on } \Gamma_{A/B}, \quad (5.16)$$

$$\mathbf{u}_B^{(2)} \rightarrow 0, \quad \mathbf{s}_B^{(2)} \rightarrow 0 \quad \text{at infinity.} \quad (5.17)$$

It should be noted that $\bar{\mathbf{f}}^{(1)}$ and $\bar{\mathbf{s}}^{(1)}$ are functions of the primary displacement field. Eqs. (5.10)-(5.13), the homogeneous first order perturbation equations, describe the primary wave propagation in the open waveguide. Once the solution is obtained, the terms $\bar{\mathbf{f}}^{(1)}$ and $\bar{\mathbf{s}}^{(1)}$ can be calculated by substituting the primary solution into

Eqs. (5.6) and (5.7). Eqs. (5.14)-(5.17), the inhomogeneous second order perturbation equations, control the behavior of the SHW in the open waveguide, which can be interpreted physically as an excited solution by an external body force $\bar{\mathbf{f}}^{(1)}$ applied in Ω_A and a traction $\bar{\mathbf{s}}^{(1)}$ on $\Gamma_{A/B}$. Therefore, the original nonlinear problem reduces to seeking for two separate linear solutions.

For the primary wave field, analytical solutions can only be extracted for regular waveguides (e.g. plates, rods and pipes) through Transfer Matrix Method [82], Global Matrix method [83] and dispersion equations [84]. In order to solve the wave propagation in open waveguides with arbitrary cross-sectional geometries, numerical methods are usually required. Three main methods have been developed to extract wave modes in open waveguides, SAFE method coupled with AR (SAFE-AR) [95, 98], SAFE method incorporating PML (SAFE-PML) [97, 144, 145] and combining the SAFE method and BEM (SAFE-BEM) [99].

For the second wave field, it should be noted that complex reciprocity relation and modal expansion analysis are useful to derive the secondary field. However, their application to open waveguides remains intricate due to the fact that leaky modes cannot form the modal basis in open waveguides [106]. As indicated in Section 5.1, this difficulty can be addressed by introducing PMLs truncating the infinite surrounding medium within a restricted space, under which the trapped modes, leaky modes and PML modes form a set of modal basis and any arbitrary wave field can be expressed by superposition of the modal basis [111]. Accordingly, an equivalent open waveguide with PMLs is indeed necessary to be built in replacement of the infinite open waveguide not only for extracting the primary wave field via SAFE-PML method, but for the appropriate application of modal expansion in open waveguides.

Figure 5.1(b) presents an equivalent open waveguide with a PML around the surrounding medium. Ω_B now consists of the surrounding medium and the PML. The main function of the PML is to damp the leaky wave in order to simulate the infinite surrounding medium, as well as transform the radiation modes existing in the infinite surrounding medium into PML modes that resonate in the PML. The damping property of the PML comes from a mathematical coordinate mapping from

real space to complex plane by introducing a continuous, complex-valued stretching function [114]. Therefore the PML has the same material properties with surrounding medium making its impedance perfectly matched with surrounding material. As the complex coordinate mapping only occurs in the PML, the governing equations (Eqs. (5.10), (5.11), (5.14) and (5.15)) and the interface boundary conditions (Eqs. (5.12), (5.16)) in the infinite open waveguide are still available in the finite open waveguide. The governing equations in the PML appear a bit complex due to complex coordinate mapping. However, such coordinate mapping in the PML will not complicate the problem, as only interface and boundary conditions will be utilized during the following mathematical derivation. The governing equations of the PML is therefore not necessary to be explicitly written here. The boundary conditions at infinity have to be converted to boundary conditions on Γ_B ,

$$\mathbf{u}_B^{(1)} = 0, \mathbf{s}_B^{(1)} = 0 \text{ on } \Gamma_B, \quad (5.18)$$

$$\mathbf{u}_B^{(2)} = 0, \mathbf{s}_B^{(2)} = 0 \text{ on } \Gamma_B. \quad (5.19)$$

In the PML coupled open waveguide as shown in Fig. 5.1(b), the primary wave field can be numerically solved through SAFE-PML method. In order to solve the secondary field, the real reciprocity relation and modal expansion method are used. The real reciprocity relation can be written as [102]

$$\nabla \cdot (\mathbf{v}_1 \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_1) = \mathbf{v}_2 \cdot \mathbf{f}_1 - \mathbf{v}_1 \cdot \mathbf{f}_2, \quad (5.20)$$

where subscripts 1 and 2 represent two possible solutions in the open waveguide; $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocity tensors; $\mathbf{s}_1$ and $\mathbf{s}_2$ demonstrate the stress tensors and $\mathbf{f}_1$ and $\mathbf{f}_2$ represent the volume forces.

Solution "1" is considered to be the n th mode of the open waveguide without external force fields. Solution "2" is assumed to be a solution having the external forces applied by the primary wave shown in Eqs. (5.14) and (5.16), which means that

solution "2" is actually the secondary field. Solution "1" and solution "2" can be written as

$$\mathbf{v}_1 = \mathbf{v}_n e^{ik_n z}, \quad \mathbf{s}_1 = \mathbf{s}_n e^{ik_n z}, \quad \mathbf{f}_1 = 0, \quad (5.21)$$

$$\mathbf{v}_2 = \mathbf{v}_2(x, y, z), \quad \mathbf{s}_2 = \mathbf{s}_2(x, y, z), \quad \mathbf{f}_2 = \bar{\mathbf{f}}^{(1)}. \quad (5.22)$$

Substituting the solutions into Eq. (5.20), the equation can be expressed as

$$\begin{aligned} & \nabla_{\perp} \cdot (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) + \frac{\partial}{\partial z} (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \cdot \mathbf{n}_z \\ &= -\mathbf{v}_n e^{ik_n z} \cdot \bar{\mathbf{f}}^{(1)} \end{aligned}, \quad (5.23)$$

where $\nabla_{\perp} = \frac{\partial}{\partial x} (\cdot) \cdot \mathbf{n}_x + \frac{\partial}{\partial y} (\cdot) \cdot \mathbf{n}_y$, $\mathbf{n}_z$ is the unit vector in the z direction. Integrating Eq. (5.23) over the cross section as shown in Fig. 5.1(b), the formula becomes

$$\begin{aligned} & \int_{\Omega} \nabla_{\perp} \cdot (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) d\Omega \\ &+ \int_{\Omega} \frac{\partial}{\partial z} (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \cdot \mathbf{n}_z d\Omega = - \int_{\Omega} \mathbf{v}_n e^{ik_n z} \cdot \bar{\mathbf{f}}^{(1)} d\Omega \end{aligned}, \quad (5.24)$$

where $\Omega = \Omega_A + \Omega_B$.

The secondary field can be expressed as a linear combination of the waveguide modes

$$\mathbf{v}_2(x, y, z) = \sum_{m=1}^Q A_m(z) \mathbf{v}_m(x, y), \quad (5.25)$$

$$\mathbf{s}_2(x, y, z) = \sum_{m=1}^Q A_m(z) \mathbf{s}_m(x, y), \quad (5.26)$$

where Q is the number of modes. It should be noted that the mode subscript m and n can be positive and negative values, representing the positive-going and negative-going waves in the z direction, respectively. Here, only positive-going modes are considered for secondary field. In addition, it has been demonstrated that the PML

modes mainly oscillate in the PML region and only contribute to long-term diffraction, geometrical decay of the field [111] which is of little interest in the context of NDT, so that the sums can only run over all trapped and leaky modes.

The first term on the left-hand side of Eq. (5.24) can be calculated by substituting Eqs. (5.25) and (5.26) into Eq. (5.24) and using Gauss divergence theorem

$$\int_{\Omega} \nabla_{\perp} \cdot (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) d\Omega = \int_{\Gamma_{A/B}} \mathbf{v}_n e^{ik_n z} \cdot (-\bar{\mathbf{s}}^{(1)}) \cdot \mathbf{n}_A d\Gamma. \quad (5.27)$$

Additional steps for derivation of Eq. (5.27) are given in Appendix C.

For the second term on the left-hand side of Eq. (5.24), it can be casted into

$$\begin{aligned} & \int_{\Omega} \frac{\partial}{\partial z} (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \cdot \mathbf{n}_z d\Omega \\ &= -4Q_{n,m} \frac{\partial}{\partial z} \left(e^{ik_n z} \sum_{m=1}^Q A_m(z) \right), \end{aligned} \quad (5.28)$$

where $Q_{n,m} = -\frac{1}{4} \int_{\Omega} (\mathbf{v}_n \cdot \mathbf{s}_m - \mathbf{v}_m \cdot \mathbf{s}_n) \cdot \mathbf{n}_z d\Omega$, is the general orthogonality relation in open waveguides [111], which is similar with Auld's real orthogonality relation [102].

As the external volume force $\bar{\mathbf{f}}^{(1)}$ only exists in Ω_A , the term on the right-hand side of Eq. (5.24) can be expressed as

$$-\int_{\Omega} \mathbf{v}_n e^{ik_n z} \cdot \bar{\mathbf{f}}^{(1)} d\Omega = -\int_{\Omega_A} \mathbf{v}_n e^{ik_n z} \cdot \bar{\mathbf{f}}^{(1)} d\Omega. \quad (5.29)$$

Hence, Eq. (5.24) can be rewritten as

$$4Q_{n,m} \left(\frac{d}{dz} + ik_n \right) \sum_{m=1}^Q A_m(z) = \int_{\Omega_A} \mathbf{v}_n \cdot \bar{\mathbf{f}}^{(1)} d\Omega + \int_{\Gamma_{A/B}} \mathbf{v}_n \cdot (-\bar{\mathbf{s}}^{(1)}) \cdot \mathbf{n}_A d\Gamma \quad (5.30)$$

The general orthogonality relation is given by [111]

$$Q_{n,m} = 0, \text{ if } n \neq -m. \quad (5.31)$$

That means the positive-going m th mode is orthogonal to all modes except the negative-going m th mode in the general orthogonality relation. Therefore, Eq. (5.30) can be transformed into an ordinary differential equation whose solution (A_m) is the amplitude of the secondary mode

$$4Q_{-m,m} \left(\frac{d}{dz} + ik_{-m} \right) A_m(z) = e^{i2k_l z} \left(\bar{f}_{-m}^{vol} + \bar{f}_{-m}^{surf} \right), \quad (5.32)$$

where $\bar{f}_{-m}^{vol} = \int_{\Omega_A} \mathbf{v}_{-m} \cdot \hat{\mathbf{f}}^{(1)} d\Omega$ and $\bar{f}_{-m}^{surf} = \int_{\Gamma_{A/B}} \mathbf{v}_{-m} \cdot (-\hat{\mathbf{s}}^{(1)}) \cdot \mathbf{n}_A d\Omega$ having $\bar{\mathbf{f}}^{(1)} = e^{i2k_l z} \hat{\mathbf{f}}^{(1)}$ and $\bar{\mathbf{s}}^{(1)} = e^{i2k_l z} \hat{\mathbf{s}}^{(1)}$ represents the wavenumber of the primary wave which is assumed to be the l th mode of the open waveguide. Solving the equation by placing a relationship between positive-going and negative-going modes, $k_{-m} = -k_m$, into Eq. (5.32), and following an initial condition, $A_m(0) = 0$, the amplitude of the secondary mode can be obtained,

$$A_m(z) = \frac{i \left(\bar{f}_{-m}^{vol} + \bar{f}_{-m}^{surf} \right)}{4Q_{-m,m} (2k_l - k_m)} e^{ik_m z} \left(1 - e^{i(2k_l - k_m)z} \right). \quad (5.33)$$

It can be seen that, calculations of $\bar{f}_{-m}^{vol}$, $\bar{f}_{-m}^{surf}$ and $Q_{-m,m}$ require for the modal properties of the negative-going modes, which is not convenient in numerical computations. For propagating modes, the fields of the modes must satisfy the following relations [102],

$$\mathbf{v}_{-m}(x, y) = -\mathbf{v}_m^*(x, y), \quad \mathbf{s}_{-m}(x, y) = \mathbf{s}_m^*(x, y), \quad (5.34)$$

where $\mathbf{v}_m^*(x, y)$ and $\mathbf{s}_m^*(x, y)$ represent the complex conjugation. Substituting these relationship into definitions of $\bar{f}_{-m}^{vol}$, $\bar{f}_{-m}^{surf}$ and $Q_{-m,m}$ leads to the following equations,

$$\bar{f}_{-m}^{vol} = - \int_{\Omega_A} \mathbf{v}_m^* \cdot \hat{\mathbf{f}}^{(1)} d\Omega = -f_m^{vol}, \quad (5.35)$$

$$\bar{f}_{-m}^{surf} = - \int_{\Gamma_{A/B}} \mathbf{v}_m^* \cdot (-\hat{\mathbf{s}}^{(1)}) \cdot \mathbf{n}_A d\Gamma = -f_m^{surf}, \quad (5.36)$$

$$Q_{-m,m} = -\frac{1}{4} \int_{\Omega} \{-\mathbf{v}_m^* \cdot \mathbf{s}_m - \mathbf{v}_m \cdot \mathbf{s}_m^*\} \cdot \mathbf{n}_z d\Omega = -P_{mm}, \quad (5.37)$$

where f_m^{vol} , f_m^{surf} and P_{mm} represent the external power flux due to the volume source and surface source and the power flux of the m th mode in the wave propagation direction, respectively, which have the same expression as those of closed waveguides in Chapter 4. It is worth noting that $Q_{-m,m}$ is calculated on the whole cross section including the PML region. Therefore, the amplitude of the SHW can be rewritten as

$$A_m(z) = \frac{i(f_m^{vol} + f_m^{surf})}{4P_{mm}(2k_l - k_m)} e^{ik_m z} (1 - e^{i(2k_l - k_m)z}). \quad (5.38)$$

Although this formula appears similar in form to that from closed waveguides in previous studies [14, 17], the wavenumbers of the primary and secondary modes can be complex taking account of energy leakage.

As a side remark, the same results can be obtained for immersed waveguides which is immersed in a fluid only by replacing the governing equations of the surrounding medium with acoustic wave equations and following the same mathematical procedures presented above.

It is known that, a linearly cumulative SHW in closed waveguides can be selected as the dominant contribution to the secondary wave field only if it satisfies the internal resonance conditions: (1) phase velocity matching; (2) non-zero power flux. Under these conditions, the other secondary modes are negligible due to minimal contribution to the expansion. Therefore, it is significant to select the mode pair, including the primary wave and the dominant SHW, to have a maximum nonlinear response of the waveguide. This situation in open waveguides is complicated due to energy leakage, which usually makes the amplitude of both the primary wave and the SHW intricate and unpredictable. However, it is also possible to select the dominant term in the modal expansion by considering two general cases similar to closed waveguides. In the first case, if there is no energy transfer from primary wave to SHW, the amplitude of the SHW must be zero, leading to no contribution to the

expansion. In the second case, if the phase velocity of the primary wave mismatch with the SHW, the amplitude of the SHW is bounded in value and oscillates with propagation distance, whose contribution is usually minimal with being of little interest for application purposes. Consequently, two conditions are proposed in open waveguides which seem exactly the same with that in closed waveguides: (1) phase velocity matching, $\text{Re}(2k_l) = \text{Re}(k_m)$; (2) non-zero power flux, $f_m^{vol} + f_m^{surf} \neq 0$. As long as these two criteria are met, the SHW is dominant although its amplitude with propagation distance may not be linearly cumulative and has to be calculated numerically.

5.3 Numerical Methods

5.3.1 Nonlinear SAFE method

According to the mathematical derivation, two conditions have to be satisfied in order to calculate the SHW. Accordingly, the computational procedure can be divided into two stages corresponding to the two conditions.

In the first stage, the phase velocity dispersion curves and attenuation dispersion curves of a given open waveguide can be calculated through the SAFE-PML method, which is developed in Chapter 3. Then the phase velocity matching condition is utilized to identify the mode pair by plotting the phase velocity dispersion curves of the primary wave which then overlap the dispersion curves of the SHW at its half frequency, where all the intersection points between the two dispersion curves are selected as the potential mode pair which satisfy the phase velocity matching condition. Moreover, modal properties of the primary and secondary mode, like wavenumber, attenuations and mode shapes, can also be extracted from the SAFE-PML method, that will be used in next stage.

The second stage is to calculate the amplitude of the SHW. In this stage, according to Eqs. (5.35)-(5.37), the power flux can be obtained via numerical integration, during

which mode shapes obtained in the first stage are used to compute the velocity vector and the first P-K stress tensor. If the power flux through the surface and volume is non-zero, the amplitude of the SHW is not zero and can be calculated numerically by substituting the power flux and wavenumbers of the primary waves and SHWs into Eq. (5.38).

5.3.2 Time domain FE model

The nonlinear SAFE method can provide modal properties of nonlinear guided waves in open waveguides, including the excitation frequency and the amplitude of the SHW. Based on the prediction from the nonlinear SAFE method, time domain FE model is developed to simulate the propagation of the nonlinear guided wave, and the SHW is collected to cross validate the prediction of the nonlinear SAFE method.

Embedded waveguide models consist of a nonlinear elastic core structure and an elastic surrounding material which is usually set to be large enough to avoid boundary reflections. In the core structure, the Murnaghan model is used to describe the material behavior, which is equivalent to the Landau-Lifshitz model in the nonlinear SAFE method, written as

$$w = \frac{1}{2}\lambda(\text{tr}(\mathbf{E}))^2 + \mu\text{tr}(\mathbf{E}^2) + \frac{1}{3}(L + 2M)(\text{tr}(\mathbf{E}))^3 - M\text{tr}(\mathbf{E})((\text{tr}(\mathbf{E}))^2 - \text{tr}(\mathbf{E}^2)) + N\det(\mathbf{E}), \quad (5.39)$$

where $\text{tr}()$ and $\det()$ denote trace and determinant of the tensor, respectively; L , M and N are third order elastic constants in the Murnaghan model, which relate to the Landau-Lifshitz constants by

$$L = B + C, \quad M = \frac{1}{2}A + B, \quad N = A. \quad (5.40)$$

In the surrounding material, a linear elastic material model is used together with a continuum boundary condition being applied between the core structure and the surrounding material.

Immersed waveguide models are termed fluid-solid interaction systems due to the coupling between the core structure and the surrounding fluid. The Murnaghan model is also used to simulate the solid core structure, while an acoustic model is applied to the surrounding fluid. In order to simulate the coupling, acoustic-structure boundary conditions have to be implemented between the core structure and the surrounding fluid.

Both the nonlinear SAFE method and the time domain FE model were implemented in a commercial FE software package [125]. In the following section, two case studies were carried out using the simulation tools discussed above: an aluminum plate attached to an elastomer, and an aluminum plate with water-loaded on one side.

5.4 Case Studies

5.4.1 An aluminum plate attached to a semi-infinite half-space of an elastomer

5.4.1.1 Nonlinear SAFE method computations

Figure 5.2: Nonlinear SAFE method computations with (a) the schematic of the SAFE-PML model for the aluminum plate attached to an elastomer, (b) phase velocity and (c) attenuation dispersion curves for the embedded plate and (d) variation of the modal amplitude of the SHW with propagation distance.

The first example is to study a 4 mm thick aluminum plate attached to a semi-infinite half-space of an elastomer. The schematic of the SAFE-PML model is shown in Fig. 5.2(a). A narrow strip of the structure with width of 1 mm was imposed

with periodic boundary conditions, where the displacements and stresses between the two edges are continuous. The elastomer was set to be 1 mm in thickness with considering the fact that PMLs work better when it is close to the core structures. A 1 mm thick layer of the PML was attached to the elastomer, which was thick enough to absorb all of the leaky waves. In the simulation, the PML function was chosen as $\hat{\gamma}_2 = 3 + 12i$ and it was solved from 50 kHz to 1000 kHz with $\eta = 1.3$ being used to identify and filter the PML modes during post-processing. The interface between the plate and the elastomer was imposed with a continuity condition on displacements and stresses and the outer surface of the structure were applied with stress-free conditions. In the whole geometry, 1186 triangular elements were automatically generated by the software with side length of 0.11 mm and the number of degree of freedom was 14970. All of the material properties were listed in Table 5.1.

Figure 5.2(b) presents the phase velocity dispersion curves for the aluminum plate attached to an elastomer in 50-500 kHz, including the primary modes in solid lines and the second harmonic modes (at half frequency) in dashed lines. A few mode pairs can be found at the intersections of the solid lines and the dashed lines, that satisfy the phase velocity matching condition, among which, only one mode pair marked with a triangle can satisfy the additional condition, non-zero power flux. It is mode pair (SH0, s0) at the excitation frequency of 427.55 kHz, where the upper case letter and the lower case letter represent the primary and secondary mode, respectively.

Attenuation dispersion curves for the embedded plate in 50-1000 kHz are shown in Fig. 5.2(c) with attenuations of the primary wave and the SHW can be obtained and indicated with two arrows showing that the attenuation of the SH0 mode is predicted to be 71 dB/m and that of the s0 mode is calculated as 279 dB/m. It is clear that both the primary wave and the SHW are strongly attenuative.

The distance variation of the modal amplitude of the SHW for the mode pair (SH0, s0) is shown in Fig. 5.2(d). It can be seen that, the amplitude of the s0 mode increases and then decreases rapidly to zero in a very short distance. This phenomenon can be understood from the view point of energy transfer. In the beginning, even

Table 5.1: Mechanical properties of materials used in the models.

Material	ρ_0 (kg/m ³)	C_L (m/s)	C_S (m/s)	A (GPa)	B (GPa)	C (GPa)	α_L and α_S (dB/ λ)
Aluminum	2780	6347.3	3116.4	-351.2	-149.4	-102.8	...
Elastomer	1100	953.5	522.2	...	...	...	2.73
Water	1000	1500	...	...	...	...	...

though both the primary wave and the SHW are leaky, the energy transferred from the SH0 mode to the s0 mode is larger than the energy that leaks away from the s0 mode, thus the amplitude of the s0 mode increases progressively. With the wave propagating, the leakage of the SH0 mode leads to the decay of its amplitude, which further greatly reduces the energy transfer to the SHW, as it is proportional to the square of the amplitude. Once the energy transfer is less than the energy leakage of the s0 mode up to a certain distance, the amplitude of the s0 mode starts to decrease and finally goes to zero as shown in Fig. 5.2(d).

5.4.1.2 Time domain FE simulations

Figure 5.3: Schematic of the embedded plate in the FE simulation.

Time domain FE simulations are carried out in this section for the mode pair (SH0, s0) at excitation frequency of 427.55 kHz in a 4 mm thick plate. The schematic of the embedded plate model is shown in Fig. 5.3. Three dimensional modeling should be used in FE simulations to generate SH wave propagation [146]. The plate was assumed to be 300 mm in length attached to the elastomer with thickness of 46 mm

in order to avoid boundary reflections. Both the plate and the elastomer were set to be 0.1 mm in width with periodic conditions being applied to simulate infinity in the x direction and reduce computational cost. Continuity boundary condition was imposed between the plate and the elastomer to force the displacement and the stress to be continuous and other boundaries were applied with stress free boundary conditions. The primary wave (SH0) was generated from the left side of the plate by using a 10 cycle Hann windowed tone burst with central frequency of 427.55 kHz in a manner of prescribed displacement boundary condition in the x direction and monitors were placed at the center of the plate to receive displacements from 10 mm to 250 mm along the z direction with an increment of 10 mm. The displacements in the x and z directions were recorded for the analysis of the primary mode SH0 and the secondary mode s0 respectively, as they are dominant in the directions. In the simulation, the whole geometry were meshed with tetrahedron elements with a maximum element size of 0.45 mm and a maximum time step of 0.05 μ s was used. All of the material parameters used in the FE simulation were taken from Table 5.1.

Figure 5.4: Time domain signals received at $z = 50$ mm in (a) the x direction and (b) the z direction; Frequency spectra by using FFT to the signals received in (c) the x direction and (d) the z direction.

Figure 5.4(a) plots time domain signals obtained in the x direction for the primary mode at $z = 50$ mm. It is apparent that a pure SH0 mode has been excited in the plate because of the mode-shape-matching excitation. In Fig. 5.4(b), displacements obtained in the z direction in which the secondary mode is dominant are displayed. It should be noted that the complex wave pattern consists of the s0 mode and the edge resonance [147] which is induced by the s0 mode propagation around the secondary frequency. This can be understood that, with the primary wave (SH0) propagation, the nonlinear response of the plate continuously transfers a small part of the energy from the primary wave to the SHW, forming the SHW s0, whose propagation further leads to the edge resonance. For such complex wave pattern, it is not easy to identify the s0 mode. However, considering that all the energy of the s0 is transferred from the SH0 mode, fast Fourier transform (FFT) is applied on the

windowed signal, whose interval has the same width of the SH0 mode, even though a small part of the s0 mode may be outside of the window due to the group velocity mismatch and the small dispersion properties of the s0 mode.

Figure 5.4(c) and (d) show frequency spectra calculated by converting the time domain signals in Fig. 5.4(a) and (b) into frequency domain via FFT, respectively. As indicated by Fig. 5.4(c), the distinct peak existing at the primary frequency represents the primary mode. In Fig. 5.4(d), the peak at the second harmonic frequency reveals that most of the energy concentrates on the secondary wave motion, meaning the vibration in the z direction actually comes from the nonlinear response of the material nonlinearity and finite elasticity. The amplitude of the s0 mode can be collected from the frequency spectrum at second harmonic frequency.

Figure 5.5: A comparison on the normalized modal amplitude of the SHW in the embedded plate between the FE simulation and the nonlinear SAFE method computation.

The normalized amplitude variation of the s0 mode (normalized by the maximum value over the propagation distance) with propagation distance from FE simulation is plotted in comparison with that from nonlinear SAFE method shown in Fig. 5.5. Very good agreements can be observed between the two results, which not only validate the predictions from the nonlinear SAFE method but also show a complicated

behavior of the nonlinear guided wave in embedded plates.

5.4.2 An aluminum plate with water-loaded on one side

5.4.2.1 Nonlinear SAFE method computations

Figure 5.6: Nonlinear SAFE method computations with (a) the schematic of the SAFE-PML model for the aluminum plate with water-loaded on one side, (b) phase velocity and (c) attenuation dispersion curves for the one-sidedly immersed plate and (d) variation of the modal amplitude of the SHW with propagation distance.

The second case includes a 4 mm thick aluminum plate with water-loaded on one side. Fig. 5.6(a) presents the schematic of the SAFE-PML model. Similarly with the previous case, a narrow strip of the structure with periodic boundary conditions was used to ensure displacements and stresses are continuous between the two edges.

The water layer was set to be 1 mm in thickness. In order to damp the leaky waves in the water layer, a 1.5 mm thick layer of the PML was attached to the water. $\hat{\gamma}_2 = 3 + 12i$ was chosen as the PML function together with $\eta = 1.5$ being used for post-processing in the simulation calculated from 20 kHz to 1000 kHz. Water loading interface conditions [145] were imposed between the plate and the water and the other boundaries were applied with stress free conditions. 814 triangular elements were automatically generated with the number of degree of freedom being 7479. All of the material properties used were taken from Table 5.1.

Phase velocity dispersion curves for the aluminum plate with water loaded on one side in solid curves, overlapping the second harmonic modes at its half frequency in dashed curves are shown in Fig. 5.6(b) in the frequency range of 20-500 kHz. The mode pair (SH0, s0) at excitation frequency of 425 kHz was also selected for studies in this case, as the primary mode and the secondary mode in this mode pair can satisfy the phase velocity matching condition and the non-zero energy transfer.

Figure 5.6(c) shows the attenuation dispersion curves for the immersed plate in 20-1000 kHz. As water cannot support shear stress, the SH0 mode and other shear-horizontal modes are non-leaky. The s0 mode is dominant in the y direction which is strongly coupled with the water layer leading to energy leakage. Therefore, the s0 mode is a strongly leaky wave with attenuation being 413 dB/m at the second harmonic frequency. In this case, the primary mode is non-leaky, but the secondary mode is strongly leaky.

The variation of the modal amplitude of the s0 mode with propagation distance is calculated with nonlinear SAFE method for the mode pair (SH0, s0) and shown in Fig. 5.6(d). An interesting phenomenon can be observed, in which the amplitude of the s0 mode increases and then keeps constant with propagation distance. Such phenomenon can also be understood from the view point of energy transfer. As the primary wave is non-leaky, the energy transfer from the SH0 mode to the s0 mode can be considered as constant. For the s0 mode, the energy transfer (input energy) is prevail and larger than the leakage (energy consuming) at the beginning propagation distance, so that the amplitude of the s0 mode increases cumulatively.

At a certain distance, the energy transfer becomes nearly equal to the leakage, leading to a balanced activity of the input energy and the energy consumed by the s0 mode. The amplitude of the s0 mode is therefore keeping constant with propagation distance.

Additionally, the constant amplitude phenomenon can also be mathematically derived. As the primary mode is non-leaky, the wavenumber of the primary wave is assumed to be a positive number without imaginary part as $k_l = a_1$. To present a leaky secondary mode, having the same phase velocity with the primary mode, the wavenumber of the SHW has to be assumed as a complex number: $k_m = 2a_1 + bi$, where the imaginary part is positive, demonstrating the attenuation of the SHW. Substituting the wavenumbers into Eq. (5.38), the amplitude of the secondary mode can be obtained as

$$A_m(z) = \frac{(f_m^{vol} + f_m^{surf})}{4P_{mm}b} (1 - e^{-bz}) e^{2a_1zi}. \quad (5.41)$$

When $z \rightarrow \infty$, $e^{-bz} \rightarrow 0$. The absolute value of the modal amplitude goes to

$$|A_m| = \frac{|f_m^{vol} + f_m^{surf}|}{4P_{mm}b}, \quad (5.42)$$

which is a constant, whose value is proportional to the power flux transferred from the primary mode through surface and volume of the waveguide and inversely proportional to the power flux and the attenuation of the secondary mode.

5.4.2.2 Time domain FE simulations

Figure 5.7: Schematic of the immersed plate in the FE modeling.

The mode pair (SH0, s0) at 425 kHz in a 4 mm thick plate with water-loaded on one side is selected for FE simulation. The schematic of the three dimensional model is shown in Fig. 5.7. The length of the plate was assumed to be 600 mm with the water being set to be 96 mm in thickness which is thick enough to avoid boundary reflections. Similarly with the previous case study, both the plate and the water were set to be 0.2 mm in width with periodic conditions being applied to represent infinity in the x direction. In order to model the coupling between the plate and the water, acoustic structure boundary condition was imposed between the plate and the water, with displacements and stresses in the normal direction of the solid-fluid interface being continuous. Apart from the interface, other boundaries were applied with stress free boundary conditions. The primary mode (SH0) was excited by applying a 10 cycle Hann windowed tone burst signal with central frequency of 425 kHz at the left end of the plate through prescribed displacement boundary condition. Monitors were placed at the free surface of the plate to record the dominant displacement for the primary mode in the x direction, and the secondary mode in the y direction from 10 mm to 450 mm along the z direction with an increment of 10 mm. In the simulation, tetrahedron elements were used to mesh the geometry with a maximum element size of 0.3 mm and a maximum time step of $0.05 \mu\text{s}$ was applied. The material parameters in the FE simulation were taken from Table 5.1.

Figure 5.8: Time domain signals received at $z = 70$ mm in (a) the x direction and (b) the y direction; Frequency spectra by using FFT to the signals received in (c) the x direction and (d) the y direction.

Figure 5.8(a) and (b) present time domain signals obtained in the x and y directions at $z = 70$ mm, respectively. The SH0 mode and the s0 mode can be identified in the time domain and it is interesting to find that the s0 mode remains in the same location as the primary mode in the time trace even if it has a slightly slower group velocity (predicted to be 2380 m/s). This is mainly due to the strong attenuation. It is known that the s0 mode is generated by the propagating SH0 mode and has a number of wave packets overlapping with each other [65, 148] because of group velocity mismatch. As the s0 is strongly attenuated, the first-generated wave packet will first leak into the water leaving the newly generated wave packet within the SH0 packet. Therefore, the s0 mode appears to move as the same speed as the SH0 mode. In addition, it is also interesting to find the edge resonance behind the s0 mode in time domain. However, the amplitude of the edge resonance is too small to

be visible.

FFT was applied to the time domain signal in Fig. 5.8(a) and (b) to obtain the frequency spectra shown in Fig. 5.8(c) and (d), respectively. The amplitude of the SH0 mode and the s0 mode can be determined and collected at the primary and the second harmonic frequency, respectively.

Figure 5.9: A comparison on the normalized modal amplitude of the SHW in the immersed plate between the FE simulation and the nonlinear SAFE method computation.

The normalized amplitude variation of the s0 mode (normalized with maximum value) with propagation distance was extracted from FE simulation and compared with the nonlinear SAFE computation, shown in Fig. 5.9. Obviously, the FE simulation captures the constant amplitude feature of the s0 mode, agreeing very well with that from the nonlinear SAFE computation.

It can be seen from the two case studies that the nonlinear ultrasonic guided wave properties in open waveguides are quite different from that in closed waveguides and it is interesting to find that the secondary modal amplitude increases and then keeps constant during wave propagation in the 4 mm thick one-sidedly immersed aluminum plate for the mode pair (SH0, s0) at 425 kHz. However, the constant amplitude feature of the attenuated secondary mode is not limited to one-sidedly immersed

plates. It can be extended to any immersed waveguides only if the primary wave is non-leaky as demonstrated mathematically in Section 5.4.2.1, which may bring potential applications for underwater inspections. Thus, experiments are carried out in next section to verify the constant amplitude feature of the s_0 mode in immersed plates.

5.5 Experimental Validations

5.5.1 Experimental setup

Figure 5.10: Schematic of experimental setup.

Experiments were designed to verify the constant amplitude of the SHW for the mode pair (SH_0 , s_0) at $fd = 1700 \text{ kHz} \cdot \text{mm}$ in a plate with water-loaded on one

side. The schematic of the experimental setup is shown in Fig. 5.10. A 3 mm thick aluminum plate (Al 1100) was placed in a water tank with water loaded on the bottom surface. It is known that Magnetostrictive transducers (MSTs) can provide a high efficient way to excite the SH0 mode in plates, thus MSTs were used for pure SH0 mode generation, which consist of a thin iron-cobalt foil, rigidly bounded to the aluminum plate with cyanoacrylate, a meandering electric coil, placed upon the foil and a permanent magnet, whose field was aligned with the current direction in the coil. The spacing of the coil elements was 2.74 mm corresponding to the half wavelength of the primary wave.

A high voltage 10 cycle Hann windowed tone burst signal at central frequency of 566.7 kHz was generated by a high power gated amplifier (RITEC RAM-5000 SNAP) with output level of 90% and fed into a 600 kHz low pass filter to filtrate electronic higher harmonics. The purified signal was then sent to the MST via a matching network which matches the inductive impedance of the MST allowing it to be driven efficiently. Ultrasonic waves were measured using two laser vibrometers (Polytec OFV 5000) to pick up both in-plane displacements (u , in the x direction) and out-of-plane displacements (v , in the y direction) along the wave propagation direction (z direction) from 60 mm to 450 mm with an increment of 30 mm. An oscilloscope was used to store the time trace of the signal with 3000 averages to achieve a signal with a relatively high signal-to-noise ratio and the data were transferred to computers for post-processing where FFT was used to extract the amplitude of the primary wave and the SHW. In each experiment, each set of measurements was repeated ten times.

5.5.2 Calibration and experimental results

Figure 5.11: (a) In-plane displacement signals and (b) out-of-plane displacement signals received by lasers at $z = 210$ mm; Frequency spectra of (c) the in-plane displacement signals and (d) the out-of-plane displacement signals.

It is known that the system nonlinearity always exists in transducer setups. A successful harmonic generation experiments must ensure the material nonlinearity is dominant in the measurement. Thus a calibration procedure is necessary before measurement are carried out. During the calibration, the lasers were fixed at the propagation distance of 210 mm for signal receiving. The output level of the gated amplifier increased from 20% to 90% with a step of 10%. For each output level, the lasers recorded the in-plane displacements and out-of-plane displacements.

Figure 5.11(a) and (b) plot the in-plane displacements, measuring the SH0 mode, and out-of-plane displacements, demonstrating the s0 mode at output level of 90%. The SH0 mode was identified according to the group velocity (3100 m/s) in the

time domain. Despite the SH0 mode, the strongly dispersive SH1 mode was also generated in the plate due to the similarity in their mode shapes. As demonstrated in Section 5.4.2.2, the secondary mode (s0 mode) should arrive at the same time as the primary mode, even it has a slightly slower group velocity. Fig. 5.11(c) and (d) demonstrates the frequency spectra for the Hann windowed SH0 and s0 signals in semi-log coordinates, respectively. The amplitude of the primary wave (A_1) was collected at the primary frequency, although there was a small component at the second harmonic frequency due to the instrumentation nonlinearity. The amplitude of the SHW (A_2) was picked up from Fig. 5.11(d) at the second harmonic frequency.

Figure 5.12 plots the normalized modal amplitude ratio (nonlinearity parameter), A_2/A_1^2 , as a function of the excitation output, where the measurement system nonlinearity can be indicated. It can be seen that the modal amplitude ratio exists two stages with varying the output level of the gated amplifier: stage 1, a rapid decrease of A_2/A_1^2 with increasing the output level; stage 2, A_2/A_1^2 remains almost constant with varying input output. It should be noted that the amplitude of the SHW is expected to increase in proportion to the square of the primary mode. Therefore, the existence of stage 2 means that the instrumentation nonlinearity, although present to an extent, is so small that its amplification by higher input voltage does not exert significant influence on the material nonlinearity measurement [75]. Based on the calibration, a 90% output level in the measurements can ensure the contribution of the material nonlinearity is dominant in measurements.

Figure 5.12: *Normalized nonlinearity parameter as a function of amplifier output level..*

In order to assess the variation of the s0 mode with propagation distance, the amplitudes of the SH0 mode and the s0 mode were collected and plotted with respect to propagation distance. However, it is known that geometric divergence of the wave front in plates leads to decay of the wave amplitude, introducing difficulties in estimating the variation of the s0 mode. A simplified compensation manipulation for the geometric divergence is therefore used to correct the wave amplitudes. As waves in the plate follow two dimensional divergence case, the amplitude of the primary wave decays approximately with a factor of $1/\sqrt{z}$, where z is the propagation distance [149]. Thus the amplitude of the primary wave is compensated by $\sqrt{z}$, meaning that $\hat{A}_1 = A_1 \cdot \sqrt{z}$, where $\hat{A}_1$ is the compensated amplitude and A_1 is the measurement from experiments. According to the mathematical derivation, the amplitude of the SHW is proportional to the energy transfer from the primary wave. Therefore, the amplitude of the SHW is compensated by z , as the energy transfer is proportional to the square of the amplitude of the primary wave. It also should be noted that the decay of the secondary mode itself was neglected in this case. The compensated amplitudes of the SH0 mode and the s0 mode as well as the standard deviation for each propagation distance are shown in Fig. 5.13. It can be seen that,

as the SH0 is non-leaky, there is almost no decay in the amplitude of this mode. For the secondary mode, even though the amplitude of this mode shows a relatively big variation and they cannot capture the sharp increase in the beginning propagation distances existing in theory and simulations, the current results can give an intuitive indication of the constant amplitude with propagation distance in far field.

Figure 5.13: The compensated amplitude of (a) the primary mode and (b) the secondary mode with propagation distance.

5.6 Discussions

It is known that, in closed waveguides, the relative nonlinearity parameter, defined as the amplitude of the SHW over the square of the amplitude of the primary wave, can linearly increase with respect to the propagation distance with its slope being strongly related to the material higher order elastic constants, so that the slope is usually used to characterize the material of the closed waveguides. Using essentially the same principle, the constant amplitude of the secondary mode in immersed waveguides can also provide useful applications. According to Eq. (5.42), the constant value of the secondary modal amplitude is proportional to the power flux transferred from the primary mode and inversely proportional to the power flux and the attenuation of the secondary mode. The power flux and the attenuation of the secondary mode, according to Eq. (5.38), remain the same even though there are changes of the higher order elastic constants in the core structure which however

leads to a considerable variation of the power flux transferred from the primary wave. In this scenario, a relationship between the higher order elastic constants and the constant amplitude value can be formulated which makes it possible to identify material states by the constant modal amplitudes, as shown in Fig. 5.14, where the constant modal amplitude of the SHW changes with proportional variation of the third order elastic constants. It should be noted that, the amplitude of the secondary mode is usually very weak especially in leaky cases, proper methods have to be proposed to enhance the secondary mode to obtain a reasonable signal to noise ratio.

Figure 5.14: *The amplitude of the SHW changes with proportional variation of the third order elastic constants.*

5.7 Summary

In this chapter, the properties of nonlinear ultrasonic guided waves in open waveguides have been studied based on PMLs. Mathematical framework has been established based on real reciprocity relation and modal expansion by introducing PMLs truncating the infinite open waveguide into a finite one, in which the amplitude of the secondary mode has been derived and two conditions have been proposed

to select the dominant mode in the secondary field. Numerical models have been developed, including nonlinear SAFE method to predict the properties of nonlinear ultrasonic guided waves, and FE models to simulate the wave propagation and cross validate the predictions from the nonlinear SAFE method. Two examples, an aluminum plate attached to an elastomer and an aluminum plate with water-loaded on one side have been studied to demonstrate the proposed methods, as well as show modal properties of the secondary mode in open waveguides. Even though only embedded/immersed plates were studied in this paper, the methods developed are applicable to any open waveguides with arbitrary cross sections. It is interesting to find that the amplitude of the SHW keeps constant during the wave propagation when the primary wave is non-leaky, e.g. SH wave in immersed plates. Experiments were carried out to validate the constant amplitude feature of the SHW in immersed plates, which may provide potential applications of the nonlinear ultrasonic guided waves for underwater inspections.

Chapter 6

Applications of Nonlinear Lamb Waves at Low Frequency Regions

6.1 Background

In this chapter, nonlinear ultrasonic guided wave propagation developed in Chapter 4 is used to explore the nonlinear application of the S0 mode at low frequencies because this mode shows little dispersion and easy to be generated at low frequency region.

It has been known that in order to achieve a linearly increased SHW, the nonlinear guided wave must satisfy two strict conditions, including the phase velocity matching condition and non-zero energy transferred from the primary wave to the SHW [14, 15, 64]. As shown in Fig. 6.1, which plots phase velocity dispersion curves for an aluminum plate, the mode pair (S1, s2) can satisfy the two conditions at frequency thickness product of $fd = 3.57 \text{ kHz} \cdot \text{mm}$. Therefore nonlinear guided waves only exist in particular frequencies, and they appear in mode pairs. It is also known from theoretical study that an infinite number of mode pairs can satisfy the above two conditions, although in the literature the mode pair (S1, s2) are most widely used, as they also synchronous group velocities [74, 77, 79, 150]. However, it is usually difficult to generate the primary mode (S1 mode) in practice as

the uncertainty of the material properties exists in experiments. In addition, this mode pair suffers from two associated complications: inherent dispersive and multi-mode natures. That means there are challenges in the signal processing as multiple modes exist in that frequency and both S1 and s2 modes are dispersive. Although time frequency analysis, e.g. STFT, can be used to separate Lamb modes that are closely spaced, Heisenberg uncertainty principle make it impossible to perfectly resolve time and frequency at the same time, and its resolutions depend strongly on the user's experience in determining the type and size of the window [151]. Because of these uncertainties, the post processing results for the time domain signal may be distorted.

Figure 6.1: Phase velocity dispersion curves for Rayleigh Lamb modes in aluminum plates. Primary waves are shown in solid curves with upper case labels while SHWs are shown in dashed curves with lower case labels at their half frequencies.

It has been discussed in Chapter 3 that mismatch of the phase velocity, $c_p(2\omega) \neq c_p(\omega)$, where $c_p = \omega/k$ and k is the wavenumber, plus the nonzero energy transmission results in the sinusoidal behavior for the SHW with a spatial periodicity [14,141]. Here $k_d = k(2\omega) - 2k(\omega)$ is interpreted as the deviation from exact phase velocity matching and the period of the oscillation, called dispersion length L_d , can be described by a simple formula, $L_d = 2\pi/|k_d|$ [14]. In general, the sinusoidal behavior

is not desired, because of the boundedness and oscillation nature of the amplitude of the SHW. However, a special case would be practically useful: if k_d is very small, the sinusoidal solution approaches closely to the resonant solution. In this case, the amplitude of the SHW grows linearly for a certain distance, which creates the possibility for practical applications. As shown in Fig. 6.1, the S0 mode can be found to be very little dispersive at low frequency region, leading to small k_d and thus cumulative increase of the SHW for a certain distance. This is particularly interesting as the S0 mode is much easier to generate, and there is a wide range of frequencies can be used.

This chapter is to explore the nonlinear features of the low frequency region S0 mode in plates via numerical method developed in Chapter 4 and experimental validations, and therefore demonstrate an easy alternative for nonlinear Lamb wave applications.

6.2 Nonlinear SAFE Modeling

In this study, the nonlinear SAFE method is applied, which is developed in Chapter 4 to analyze the modal properties of nonlinear ultrasonic guided wave in closed waveguides [20, 21].

A 1 mm thick aluminum plate was calculated with all of the material properties being taken from Table 6.1. The modal amplitude of the secondary modes with respect to the propagation distance was calculated through the nonlinear SAFE method with the S0 mode being the primary excitation. Fig. 6.2(a) plots the modal amplitudes of all possible secondary modes with the primary wave being excited at 300 kHz and it is clear that the modal amplitudes of both sh0 and a0 modes almost equal to zero along the propagation distance, which is because the primary wave transfers no energy to these two secondary modes. For the s0 mode, although it is not an internally resonant point, the modal amplitude increases almost linearly up to 500 mm (around 28 wavelengths, marked by an arrow in Fig. 6.2(a)) in the beginning part of its dispersion length of 2000 mm. From a practical application perspective, such distance is significant for applications of nonlinear Lamb waves. Fig. 6.2(b)

Table 6.1: Elastic constants used in numerical simulation.

ρ_0 (kg/m ³)	λ (GPa)	μ (GPa)	A (GPa)	B (GPa)	C (GPa)
2700	55.27	25.95	-351.2	-149.4	-102.8

plots the modal amplitudes of the propagative secondary modes with the S0 mode being excited at 600 kHz. In this situation, as the deviation of the phase velocity between the primary wave and the SHW becomes larger, the dispersion length of the s0 mode decreases to 200 mm which results in a very short distance (50 mm, around 5.6 wavelengths, marked by an arrow in Fig. 6.2(b)) where the amplitude of the s0 mode grows linearly.

Figure 6.2: Modal amplitudes for propagative secondary modes.

6.3 FE Modeling

FE numerical simulations were carried out in a commercial software [125] to validate the predictions from the nonlinear SAFE method. In the simulation, the Murnaghan model was used, which is equivalent to the Landau-Lifshitz model in the nonlinear SAFE method [70] and plane strain condition was applied. The plate model is shown in Fig. 6.3 with the thickness being chosen to be 1 mm and the length being assumed to be 600 mm. The primary mode was excited from the left side of the plate through prescribed displacement boundary conditions, using a 10 cycle Hann

6. Applications of Nonlinear Lamb Waves at Low Frequency Regions

windowed tone burst with central frequencies of 300 kHz and 600 kHz respectively. Stress free boundary conditions were applied to other boundaries. Rectangular elements ($0.2 \text{ mm} \times 0.1 \text{ mm}$) were used to mesh the geometry and a maximum time step of $0.01 \mu\text{s}$ was applied in calculations. In addition, as the maximum displacement of both the primary wave and the SHW occurs at the center of the plate (as shown in Fig. 6.3) [72], the in-plane displacement was picked up through monitors placed at the center of the plate along the propagation distances in a step of 10 mm. FFT was used in post-processing to resolve the amplitudes of the primary wave (A_1) and the SHW (A_2). Then the nonlinearity of the material can be measured by using the amplitude of the primary and secondary mode to calculate the relative nonlinearity parameter, A_2/A_1^2 , with respect to the propagation distance.

Figure 6.3: (a) Schematic of the plate for the FE simulation. Displacement in the z direction of the S0 mode at (b) 300 kHz, (c) 600 kHz and (d) 1200 kHz.

Figure 6.4(a) shows the received signals monitored at 90 mm away from the generator when the primary wave is excited at 300 kHz. It can be seen that the shape of the wave packet does not change because of the little dispersive nature of the S0 mode at low frequency and only the S0 mode is excited in the plate. FFT is applied to the S0 mode to calculate the spectrum shown in Fig. 6.4(b) on a log-scale. It is evident that considerable second harmonic components appear as expected at the

twice primary frequency (600 kHz) and the amplitude of the SHW is much smaller than the primary mode.

Figure 6.4: (a) Time domain signal received at a propagation distance of 90 mm. (b) FFT's of the time domain signal.

The relative nonlinearity parameter with respect to the propagation distance is shown in Fig. 6.5. A great agreement can be observed between the numerical results and the nonlinear SAFE predictions. When the excitation frequency is 300 kHz, Fig. 6.5(a) demonstrates the linearly increased feature of the relative nonlinearity parameter at least from 10 to 500 mm. In Fig. 6.5(b), when the excitation frequency is 600 kHz, the relative nonlinearity parameter captures the sinusoidal behavior and the linear cumulative increase of the secondary mode is prominent at the beginning distance of each dispersion length, e.g. 0-50 mm and 200-250 mm.

Figure 6.5: *Relative nonlinearity parameter with respect to the propagation distance with excitation frequency of (a) 300 kHz and (b) 600 kHz.*

6.4 Experiments

Experiments were also performed on a 1 mm thick aluminum plate (Al 1100) to validate the nonlinear feature of the mode pair (S0, s0) at low frequency. The experimental setup is shown in Fig. 6.6. Two wide-band piezoelectric shear wave transducers (manufactured by Doppler Electronic Technologies Co.) with central frequency of 300 kHz and 600 kHz respectively and diameter of 30 mm were mounted on the plate for excitation of the primary S0 mode by using a 10 cycle Hann windowed tone burst signal. A high voltage tone burst signal was generated by the high power gated amplifier (RITEC RAM-5000 SNAP) and was fed into a low pass filter to filtrate the secondary component generated by the gated amplifier. Then the purified signal was then fed into the shear wave transducer. The ultrasonic waves were measured by using two laser vibrometers (Polytec OFV 5000) to pick up the in-plane displacement along the wave propagation direction. The laser vibrometer has a wide frequency band of approximately 0-2.5 MHz. Therefore, both the primary wave and the SHW can be detected in the same measurement. An oscilloscope was used to store the time trace of the signal with 1000 averages to achieve a relatively high signal-to-noise ratio signal and FFT was used in the post-processing to calculate the

6. Applications of Nonlinear Lamb Waves at Low Frequency Regions

spectra in the frequency domain. In each experiment, measurements were carried out in the far field from 90 to 390 mm with an increment of 30 mm, and five times were performed for each set of measurements.

Figure 6.6: *Experimental setup.*

Figure 6.7 plots the relative nonlinearity parameter as a function of the propagation distance with excitation frequency of 300 kHz and 600 kHz, respectively. In the first case, when the excitation frequency is 300 kHz, the linearly cumulative feature of the

relative nonlinearity parameter is clearly shown from 90 to 390 mm in Fig. 6.7(a), which demonstrates a cumulatively increased feature of the SHW in this distance. When the excitation frequency is 600 kHz, a type sinusoidal curve is shown in Fig. 6.7(b) between 200 and 400 mm, which means, the amplitude of the secondary mode increases first and then decreases with a sinusoidal behavior in the distance. Such results have a very close agreement with the predictions from the nonlinear SAFE calculation and the numerical simulations, indicating that the amplitude of the SHW indeed grows linearly in a certain distance at low frequency region.

Figure 6.7: *Relative nonlinearity parameter measured over increasing propagation distance for the aluminum plate with excitation frequency of (a) 300 kHz and (b) 600 kHz.*

6.5 Discussions

It has been known that a large dispersion length can be obtained for lower frequencies, thus lead to a long distance along which the SHW grows linearly (shown in Fig. 6.8). However, it should be noted that nonlinearity parameter [8], $\beta = 8A_2/k^2 x A_1^2$, is a material property dependent only on the microstructure. Therefore the amplitude of the SHW (A_2) is proportional to the square the amplitude of the primary wave (A_1). At very low frequency, where the wavenumber k is small, A_1 needs to be very high to generate noticeable signal of the SHW, which is practically difficult in experiments. Therefore, the selection of the excitation frequency

of the S0 mode needs to be a compromise between the dispersion length and the experimental complexity. On the other side, a quick way for selections of excitation frequencies in experiments for the low frequency S0 mode can be established with an assumption that the SHW increases almost linearly in the first quarter of the sinusoidal dispersion length. Therefore for a given inspection length, the frequency range can be selected according to Fig. 6.8.

Figure 6.8: *Dispersion length with respect to excitation frequencies for the low frequency S0 wave.*

6.6 Summary

In this chapter, low frequency S0 mode is shown to be an alternative for nonlinear Lamb wave applications. This mode has a good property of little dispersion and easy generation and the SHW grows linearly along a significant distance, which is usually enough for practical application. All of the features have been validated by numerical simulations and experiments. The excitation frequency can be selected by the computation of the dispersion length as well as practical concerns in the experiments.

Chapter 7

Conclusions

7.1 Thesis Review

Nonlinear ultrasonic guided waves, as a useful tool for evaluations of incipient damage over a large area, have attracted plenty of interest in NDT. However, most investigations on nonlinear ultrasonic guided waves are only limited to closed waveguides with simple geometries, such as plates, rods and pipes. In this thesis, the study of nonlinear ultrasonic guided waves has been extended from closed waveguides to open waveguides with arbitrary cross-sectional geometries and potential nonlinear ultrasonic guided wave application has also been provided.

Chapter 1 introduced the motivation of this research and outlined the structure of this thesis.

In Chapter 2, both the nonlinear ultrasonic bulk waves and nonlinear ultrasonic guided waves were first reviewed. Then the SAFE method was briefly introduced in order to study the modal properties of waveguides with arbitrary cross sections. The orthogonal relation in closed and open waveguides were also revisited which provide a foundation to study the nonlinear ultrasonic guided wave propagation in both closed and open waveguides.

To describe wave propagation with energy leaking in open waveguides, the SAFE-

PML method, as a powerful tool has been developed in Chapter 3 through combining the SAFE method with PML method. Estimation of the length of the PML was also demonstrated via a simplified 2D plane wave propagation model. The method has been first validated on an aluminum plate attached to an elastomer material, and then a cylindrical steel bar immersed in water, showing good agreements with analytical solutions. The developed SAFE-PML method was then applied on two practically important problems, including embedded pipes and an immersed rectangular bar, demonstrating that such method can effectively predict modal properties of open waveguides. As the equation of this method has been implemented in a commercial FE package, no source code being required, the SAFE-PML method can provide an easy and efficient way to investigate the modal properties of open waveguides, which will be attractive to a wide range of researchers in NDT.

Chapter 4 studied properties of nonlinear ultrasonic guided waves in closed waveguides with mathematical framework being reviewed first. Then the nonlinear SAFE method, coupling the SAFE method and nonlinear analysis, was developed based on the mathematical framework to predict the properties of the nonlinear ultrasonic guided waves. Time domain FE simulation has also been implemented to simulate nonlinear guided wave propagation to confirm the predictions from the nonlinear SAFE method. The numerical models has been demonstrated in an aluminum plate and compared with analytical solutions for validation, showing perfect agreement. Then the models has been applied on a rectangular bar, which is chosen to represent complex waveguides in general. A few internal resonant mode pairs can be found with strong nonlinearity at internal resonant frequencies.

Chapter 5 has extended the nonlinear ultrasonic guided wave from closed waveguides to open waveguides based on PMLs. Mathematical framework has been established based on real reciprocity relation and modal expansion by introducing PMLs truncating the infinite open waveguide into a finite one, in which the amplitude of the secondary mode has been derived and two conditions have been proposed to select the dominant mode in the secondary field. The nonlinear SAFE method has been developed by using both the SAFE-PML method developed in Chapter 3 and non-

linear analysis to predict the properties of nonlinear ultrasonic guided waves. Time domain FE simulations were implemented to simulate wave propagation and cross validate the predictions from the nonlinear SAFE method. Two examples, an aluminum plate attached to an elastomer and an aluminum plate with water-loaded on one side have been studied to demonstrate the proposed methods and it is interesting to find that the amplitude of the SHW keeps constant during the wave propagation when the primary wave is non-leaky, e.g. SH wave in immersed plates. Experiments were carried out to validate the constant amplitude feature of the SHW in immersed plates, which may provide potential applications of the nonlinear ultrasonic guided waves for underwater inspections.

In Chapter 6, the low frequency S0 mode has been demonstrated as an alternative for nonlinear Lamb wave applications. The widely used nonlinear Lamb mode has two associated complications: inherent dispersive and multimode natures. To overcome this, low frequency S0 mode has been proposed as it is little dispersive and easy to generate. The nonlinear SAFE method and the time domain FE simulation were carried out to study the properties of the SHW with the low frequency S0 mode as excitation. It is found that although the exact phase velocity matching condition is no longer valid, the amplitude of the SHW still increases cumulatively for a significant distance. Experiments were also used to validate this nonlinear feature and therefore provide a potential for nonlinear applications in plate like structures.

7.2 Summary of Findings

7.2.1 Developments of SAFE-PML method

It is known that modal properties in regular waveguides (such as plates, rods and pipes) can be extracted through analytical method, i.e. Transfer Matrix Method [82], Global Matrix Method [83] and dispersion equations [84]. For waveguides with irregular cross sections, SAFE method has been widely used because only the cross

section needs to be meshed, and the wave is assumed to propagate harmonically along the propagation direction. However, it is much more challenging when the waveguides embedded/immersed in another medium since the waveguide interacting with the surrounding material may lead to energy leaking away from the waveguide. For these cases, three methods have been proposed by extending the SAFE method with taking account of the energy leakage. They are SAFE-AR method, which couples the SAFE method and the absorbing region, SAFE-BEM method, that combines the SAFE method and boundary element method, and SAFE-PML method, which integrates the PML method into the SAFE method to absorb the leaky waves. Among them, the SAFE-PML method is efficient and relatively easy to operate in mathematics. However, most of the studies on SAFE-PML method are limited to embedded waveguides, which only contains solid surrounding medium and most of them are based on specific FE codes, which are not available to the public and time consuming to develop.

In this thesis, the SAFE-PML method has been developed for both embedded and immersed waveguides through coupling the SAFE method and the PML method. Then the governing equations of the SAFE-PML method have been reorganized in a standard FE eigenvalue formalism and implemented into a commercial FE package, which provides an easily accessible approach for practical use to investigate the guided waves in embedded/immersed waveguides. A 2D plane wave propagation model has also been developed for roughly estimating the length of the PML.

The developed method was first validated with analytical solutions, showing good agreement. Then it was applied to two cases-pipes fully and half embedded in concrete. It is known that, in free pipes, the $L(0,2)$ mode and the $T(0,1)$ mode at specific frequency range are attractive for long range inspections, since they are non-dispersive and easy to be generated and separated by time-domain gating. However, when the pipe is fully embedded in concrete, both modes suffer from high attenuation due to the energy leaking into the concrete. However, it is worth noting that the $L(0,1)$ mode may be attractive for the inspection of the fully embedded pipe in concrete as there is a non-leaky region where this mode is also little dispersive.

In half embedded pipes, two significant features can be identified: as the system is non-symmetrical, the axisymmetric modes, such as $L(0,1)$, $T(0,1)$ and $L(0,2)$ mode, do not exist in this case; each of the non-axisymmetric modes, e.g. $F(1,1)$, $F(1,2)$ and $F(1,3)$ mode, splits into two independent modes. Such method was also applied to a rectangular steel bar immersed in water, showing that the first order longitudinal mode $L1$ is little dispersive and almost non-leaky at a relatively low frequency range, thus this mode can be a potential candidate for NDT application; the first order torsional mode $T1$ experiences almost no attenuation in a relatively narrow frequency range, and the fluid can be trapped around the corner of the cross section due to the non-circular cross section, thus this mode can sense the presence and nature of the surrounding fluid.

7.2.2 Nonlinear ultrasonic guided waves in open waveguides

Nonlinear ultrasonic guided wave, combining the advantages of nonlinear ultrasound and ultrasonic guided waves, has been attractive for NDT for a few decades, as it is sensitive to the incipient damage and can propagate for a long distance for large area inspections. However, all of the studies on nonlinear ultrasonic guided wave are only limited to closed waveguides, where all the energy is confined in the waveguides. There are more challenging cases, typically existing in civil engineer and geophysics, in which the waveguides are embedded/immersed in another infinite medium, such as embedded fibers, underwater pipes, etc. In these cases, the waveguide interacting with the surrounding material may lead to energy in the waveguide leaking into the surrounding medium, which further distorts the behavior of the primary waves and the SHWs. Thus the amplitude of the SHW may not be linearly cumulative as in closed waveguides.

In this thesis, nonlinear ultrasonic guided wave in open waveguides has been studied. Mathematical framework was first formulated by using real reciprocity relation and modal expansion through introducing PMLs truncating the infinite open waveguide into a finite one where the amplitude of the secondary mode has been derived and two conditions have been proposed to select the dominant mode in the secondary

field. Then nonlinear SAFE method was developed through combining the SAFE-PML method and nonlinear analysis to predict the properties of nonlinear ultrasonic guided waves. Following that, FE model has been implemented to simulate the propagation of nonlinear ultrasonic guided waves and cross validate the predictions from the nonlinear SAFE method.

Two examples were given to illustrate the proposed methods and reveal some interesting properties only occur in the open waveguides, including an aluminum plate attached to an elastomer and an aluminum plate with water loaded on one side. It is interesting to find that in immersed case the amplitude of the secondary mode keeps constant during the wave propagation when the primary wave is non-leaky, e.g. SH mode in immersed plates. Experiments were carried out to validate the constant amplitude feature of the SHW in immersed plates. Therefore, this constant amplitude property of the SHW may provide potential applications of the nonlinear ultrasonic guided waves for underwater inspections.

7.2.3 Nonlinear Lamb waves at low frequencies

It is known that existence of nonlinear ultrasonic guided waves must satisfy two strict conditions, including phase velocity matching condition and non-zero energy transfer from the primary wave to the SHW. Therefore, nonlinear ultrasonic guided wave exist only in particular frequencies and they appear in mode pairs. Mode pair (S1, s2) is widely used in plate like structures as they satisfied the internal resonant conditions as well as the group velocity matching condition. However, generating the S1 mode at the exact resonant frequency is usually difficult, because of the uncertainty in the material properties used in the experiment. This mode pair also suffers from two associated complications: inherent dispersive and multimode natures, which bring challenges in the signal processing as multiple modes exist and both S1 and s2 modes are dispersive. Although STFT can be used to separate Lamb modes that are closely spaced, Heisenberg uncertainty principle makes it impossible to perfectly resolve time and frequency at the same time, and its resolutions depend strongly on the user's experience in determining the type and size of the window.

Due to these uncertainties, the post processing results for the time domain signal may be distorted.

In this thesis, the possibility to use low frequency symmetric Lamb mode (S0) for nonlinear applications was explored to overcome these uncertainties presented above, as the S0 mode is little dispersive, relatively easier to generate, and there are a wide range of frequencies that can be used. Numerical simulations including nonlinear SAFE method and time domain FE simulation have been carried out to demonstrate that even though the exact phase velocity matching criterion is not valid, the SHW grows linearly along a significant distance. Experiments has also been conducted to validate this features and the selection of excitation frequency has been discussed based on both the computation of the dispersion length and practical concerns in the experiments.

7.3 Future Work

In our studies, nonlinearity ultrasonic guided wave in both closed and open waveguides has been developed and validated in simple cases. In future, it will be useful to apply the studies to a few realistic industry structures for incipient damage detection. Specific applications, such as bearings and welded plates, can be investigated. Bearing is a critical component in industries as it is a major part for all moving structures. When a bearing operates under high loads, the contact pressure between the rolling element and the ring raceway will induced a high stress field just below the surface of the ring which can result in micro-plastic damage occurring. Conventional ultrasonic testing cannot detect such defects due to the small scale. However nonlinear Rayleigh surface waves are possible to be adopted for testing in the bearing structure, as most of the energy of Rayleigh surface waves is concentrated around the surface, which makes the nonlinear Rayleigh surface wave an ideal tool for surface and subsurface damage detection in bearing structures. Welded plate is also widely used in industries and attracts growing interest in NDT. This structure can be considered as a typical open waveguide, as energy of the guided wave can

be trapped in the weld cap and some energy may leak into the plate becoming the plate modes. In reality, defects are commonly located in the weld region, which is usually named the heat affected zone. So that, it is expected to find an internal resonant mode pair which is sensitive to the incipient damage in the heat affected zone.

In Chapter 6, the linearly increased feature of the relative nonlinearity parameter (defined by A_2/A_1^2) for the S0 mode at low frequency has been validated by experiments. According to nonlinear wave propagation studies, the slope of the relative nonlinearity parameter with respect to the propagation distance is proportional to the acoustic nonlinearity parameter of the material. However, our studies assumed the wave to be a plane wave and did not take into account the effects of geometry diffraction and material attenuation for both the primary wave and the SHW. The accurate measurement of the relative nonlinearity parameter generally requires corrections for geometry diffraction and material attenuation. In this case, future work can be carried out to investigate the correction of diffraction and attenuation for nonlinear Lamb waves in plate like structures.

It is known from Chapter 6 that the mode pair (S0, s0) at low frequency is more advantageous, compared with the widely used internal resonant mode pair in plates, since it is little dispersive and easy to generate and the amplitude of the secondary mode increase linearly in significant distance. In order to make this mode pair practically useful, it should be possible to apply the (S0, s0) mode pair to plate like structures for the purpose of detection of incipient damage such as fatigue. As fatigue damage can lead to the change of the third order elastic constants and these constants are also strongly related to the amplitude of the SHW, it is expected to see the variation of the amplitude of the SHW with the increasing of fatigue damage.

Appendix A

The Coefficients of the FE Formalism for Embedded Waveguides

According to Eq. (3.35), the submatrices of the coefficients in the FE formalism are given as follows.

$$\mathbf{c}^{UU} = \begin{bmatrix} \mathbf{c}_{11}^{UU} & \mathbf{c}_{12}^{UU} & \mathbf{c}_{13}^{UU} \\ \mathbf{c}_{21}^{UU} & \mathbf{c}_{22}^{UU} & \mathbf{c}_{23}^{UU} \\ \mathbf{c}_{31}^{UU} & \mathbf{c}_{32}^{UU} & \mathbf{c}_{33}^{UU} \end{bmatrix}. \quad (\text{A.1})$$

$$\begin{aligned}
 \mathbf{c}_{11}^{UU} &= \begin{bmatrix} \left(\frac{1}{\gamma_1}\right)^2 C_{11} & \left(\frac{1}{\gamma_1 \gamma_2}\right) C_{16} \\ \left(\frac{1}{\gamma_2 \gamma_1}\right) C_{61} & \left(\frac{1}{\gamma_2}\right)^2 C_{66} \end{bmatrix}; \mathbf{c}_{12}^{UU} = \begin{bmatrix} \left(\frac{1}{\gamma_1}\right)^2 C_{16} & \left(\frac{1}{\gamma_1 \gamma_2}\right) C_{12} \\ \left(\frac{1}{\gamma_2 \gamma_1}\right) C_{66} & \left(\frac{1}{\gamma_2}\right)^2 C_{62} \end{bmatrix}; \\
 \mathbf{c}_{13}^{UU} &= \begin{bmatrix} \left(\frac{1}{\gamma_1}\right)^2 C_{15} & \left(\frac{1}{\gamma_1 \gamma_2}\right) C_{14} \\ \left(\frac{1}{\gamma_2 \gamma_1}\right) C_{65} & \left(\frac{1}{\gamma_2}\right)^2 C_{64} \end{bmatrix}; \mathbf{c}_{21}^{UU} = \begin{bmatrix} \left(\frac{1}{\gamma_1}\right)^2 C_{61} & \left(\frac{1}{\gamma_1 \gamma_2}\right) C_{66} \\ \left(\frac{1}{\gamma_2 \gamma_1}\right) C_{21} & \left(\frac{1}{\gamma_2}\right)^2 C_{26} \end{bmatrix}; \\
 \mathbf{c}_{22}^{UU} &= \begin{bmatrix} \left(\frac{1}{\gamma_1}\right)^2 C_{66} & \left(\frac{1}{\gamma_1 \gamma_2}\right) C_{62} \\ \left(\frac{1}{\gamma_2 \gamma_1}\right) C_{26} & \left(\frac{1}{\gamma_2}\right)^2 C_{22} \end{bmatrix}; \mathbf{c}_{23}^{UU} = \begin{bmatrix} \left(\frac{1}{\gamma_1}\right)^2 C_{65} & \left(\frac{1}{\gamma_1 \gamma_2}\right) C_{64} \\ \left(\frac{1}{\gamma_2 \gamma_1}\right) C_{25} & \left(\frac{1}{\gamma_2}\right)^2 C_{24} \end{bmatrix}; \\
 \mathbf{c}_{31}^{UU} &= \begin{bmatrix} \left(\frac{1}{\gamma_1}\right)^2 C_{51} & \left(\frac{1}{\gamma_1 \gamma_2}\right) C_{56} \\ \left(\frac{1}{\gamma_2 \gamma_1}\right) C_{41} & \left(\frac{1}{\gamma_2}\right)^2 C_{46} \end{bmatrix}; \mathbf{c}_{32}^{UU} = \begin{bmatrix} \left(\frac{1}{\gamma_1}\right)^2 C_{56} & \left(\frac{1}{\gamma_1 \gamma_2}\right) C_{52} \\ \left(\frac{1}{\gamma_2 \gamma_1}\right) C_{46} & \left(\frac{1}{\gamma_2}\right)^2 C_{42} \end{bmatrix}; \\
 \mathbf{c}_{33}^{UU} &= \begin{bmatrix} \left(\frac{1}{\gamma_1}\right)^2 C_{55} & \left(\frac{1}{\gamma_1 \gamma_2}\right) C_{54} \\ \left(\frac{1}{\gamma_2 \gamma_1}\right) C_{45} & \left(\frac{1}{\gamma_2}\right)^2 C_{44} \end{bmatrix}.
 \end{aligned} \tag{A.2}$$

$$\alpha^{UV} = \begin{bmatrix} \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{15} \\ \left(\frac{1}{\gamma_2}\right) C_{65} \end{bmatrix} & \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{14} \\ \left(\frac{1}{\gamma_2}\right) C_{64} \end{bmatrix} & \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{13} \\ \left(\frac{1}{\gamma_2}\right) C_{63} \end{bmatrix} \\ \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{65} \\ \left(\frac{1}{\gamma_2}\right) C_{25} \end{bmatrix} & \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{64} \\ \left(\frac{1}{\gamma_2}\right) C_{24} \end{bmatrix} & \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{63} \\ \left(\frac{1}{\gamma_2}\right) C_{23} \end{bmatrix} \\ \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{55} \\ \left(\frac{1}{\gamma_2}\right) C_{45} \end{bmatrix} & \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{54} \\ \left(\frac{1}{\gamma_2}\right) C_{44} \end{bmatrix} & \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{53} \\ \left(\frac{1}{\gamma_2}\right) C_{43} \end{bmatrix} \end{bmatrix}. \tag{A.3}$$

$$\mathbf{a}^{UU} = \begin{bmatrix} -\rho\omega^2 & 0 & 0 \\ 0 & -\rho\omega^2 & 0 \\ 0 & 0 & -\rho\omega^2 \end{bmatrix}. \tag{A.4}$$

$$\mathbf{a}^{VV} = \begin{bmatrix} -\rho\omega^2 & 0 & 0 \\ 0 & -\rho\omega^2 & 0 \\ 0 & 0 & -\rho\omega^2 \end{bmatrix}. \tag{A.5}$$

$$\mathbf{a}^{UV} = \begin{bmatrix} \left(\frac{1}{\gamma_1}\right)' C_{15} + \left(\frac{1}{\gamma_2}\right)' C_{65} & \left(\frac{1}{\gamma_1}\right)' C_{14} + \left(\frac{1}{\gamma_2}\right)' C_{64} & \left(\frac{1}{\gamma_1}\right)' C_{13} + \left(\frac{1}{\gamma_2}\right)' C_{63} \\ \left(\frac{1}{\gamma_1}\right)' C_{65} + \left(\frac{1}{\gamma_2}\right)' C_{25} & \left(\frac{1}{\gamma_1}\right)' C_{64} + \left(\frac{1}{\gamma_2}\right)' C_{24} & \left(\frac{1}{\gamma_1}\right)' C_{63} + \left(\frac{1}{\gamma_2}\right)' C_{23} \\ \left(\frac{1}{\gamma_1}\right)' C_{55} + \left(\frac{1}{\gamma_2}\right)' C_{45} & \left(\frac{1}{\gamma_1}\right)' C_{54} + \left(\frac{1}{\gamma_2}\right)' C_{44} & \left(\frac{1}{\gamma_1}\right)' C_{53} + \left(\frac{1}{\gamma_2}\right)' C_{43} \end{bmatrix} \quad (\text{A.6})$$

$$\beta^{UU} = \begin{bmatrix} \beta_{11}^{UU} & \beta_{12}^{UU} & \beta_{13}^{UU} \\ \beta_{21}^{UU} & \beta_{22}^{UU} & \beta_{23}^{UU} \\ \beta_{31}^{UU} & \beta_{32}^{UU} & \beta_{33}^{UU} \end{bmatrix}. \quad (\text{A.7})$$

$$\begin{aligned} \beta_{11}^{UU} &= \begin{bmatrix} \frac{1}{\gamma_1} \left(\frac{1}{\gamma_1}\right)' C_{11} + \frac{1}{\gamma_1} \left(\frac{1}{\gamma_2}\right)' C_{61} \\ \frac{1}{\gamma_2} \left(\frac{1}{\gamma_1}\right)' C_{16} + \frac{1}{\gamma_2} \left(\frac{1}{\gamma_2}\right)' C_{66} \end{bmatrix}; \beta_{12}^{UU} = \begin{bmatrix} \frac{1}{\gamma_1} \left(\frac{1}{\gamma_1}\right)' C_{16} + \frac{1}{\gamma_1} \left(\frac{1}{\gamma_2}\right)' C_{66} \\ \frac{1}{\gamma_2} \left(\frac{1}{\gamma_1}\right)' C_{12} + \frac{1}{\gamma_2} \left(\frac{1}{\gamma_2}\right)' C_{62} \end{bmatrix}; \\ \beta_{13}^{UU} &= \begin{bmatrix} \frac{1}{\gamma_1} \left(\frac{1}{\gamma_1}\right)' C_{15} + \frac{1}{\gamma_1} \left(\frac{1}{\gamma_2}\right)' C_{65} \\ \frac{1}{\gamma_2} \left(\frac{1}{\gamma_1}\right)' C_{14} + \frac{1}{\gamma_2} \left(\frac{1}{\gamma_2}\right)' C_{64} \end{bmatrix}; \beta_{21}^{UU} = \begin{bmatrix} \frac{1}{\gamma_1} \left(\frac{1}{\gamma_1}\right)' C_{61} + \frac{1}{\gamma_1} \left(\frac{1}{\gamma_2}\right)' C_{21} \\ \frac{1}{\gamma_2} \left(\frac{1}{\gamma_1}\right)' C_{66} + \frac{1}{\gamma_2} \left(\frac{1}{\gamma_2}\right)' C_{26} \end{bmatrix}; \\ \beta_{22}^{UU} &= \begin{bmatrix} \frac{1}{\gamma_1} \left(\frac{1}{\gamma_1}\right)' C_{66} + \frac{1}{\gamma_1} \left(\frac{1}{\gamma_2}\right)' C_{26} \\ \frac{1}{\gamma_2} \left(\frac{1}{\gamma_1}\right)' C_{62} + \frac{1}{\gamma_2} \left(\frac{1}{\gamma_2}\right)' C_{22} \end{bmatrix}; \beta_{23}^{UU} = \begin{bmatrix} \frac{1}{\gamma_1} \left(\frac{1}{\gamma_1}\right)' C_{65} + \frac{1}{\gamma_1} \left(\frac{1}{\gamma_2}\right)' C_{25} \\ \frac{1}{\gamma_2} \left(\frac{1}{\gamma_1}\right)' C_{64} + \frac{1}{\gamma_2} \left(\frac{1}{\gamma_2}\right)' C_{24} \end{bmatrix}; \\ \beta_{31}^{UU} &= \begin{bmatrix} \frac{1}{\gamma_1} \left(\frac{1}{\gamma_1}\right)' C_{51} + \frac{1}{\gamma_1} \left(\frac{1}{\gamma_2}\right)' C_{41} \\ \frac{1}{\gamma_2} \left(\frac{1}{\gamma_1}\right)' C_{56} + \frac{1}{\gamma_2} \left(\frac{1}{\gamma_2}\right)' C_{46} \end{bmatrix}; \beta_{32}^{UU} = \begin{bmatrix} \frac{1}{\gamma_1} \left(\frac{1}{\gamma_1}\right)' C_{56} + \frac{1}{\gamma_1} \left(\frac{1}{\gamma_2}\right)' C_{46} \\ \frac{1}{\gamma_2} \left(\frac{1}{\gamma_1}\right)' C_{52} + \frac{1}{\gamma_2} \left(\frac{1}{\gamma_2}\right)' C_{42} \end{bmatrix}; \\ \beta_{33}^{UU} &= \begin{bmatrix} \frac{1}{\gamma_1} \left(\frac{1}{\gamma_1}\right)' C_{55} + \frac{1}{\gamma_1} \left(\frac{1}{\gamma_2}\right)' C_{45} \\ \frac{1}{\gamma_2} \left(\frac{1}{\gamma_1}\right)' C_{54} + \frac{1}{\gamma_2} \left(\frac{1}{\gamma_2}\right)' C_{44} \end{bmatrix}. \end{aligned} \quad (\text{A.8})$$

$$\beta^{UV} = \begin{bmatrix} \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{51} \\ \left(\frac{1}{\gamma_2}\right) C_{56} \end{bmatrix} & \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{56} \\ \left(\frac{1}{\gamma_2}\right) C_{52} \end{bmatrix} & \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{55} \\ \left(\frac{1}{\gamma_2}\right) C_{54} \end{bmatrix} \\ \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{41} \\ \left(\frac{1}{\gamma_2}\right) C_{46} \end{bmatrix} & \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{46} \\ \left(\frac{1}{\gamma_2}\right) C_{42} \end{bmatrix} & \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{45} \\ \left(\frac{1}{\gamma_2}\right) C_{44} \end{bmatrix} \\ \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{31} \\ \left(\frac{1}{\gamma_2}\right) C_{36} \end{bmatrix} & \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{36} \\ \left(\frac{1}{\gamma_2}\right) C_{32} \end{bmatrix} & \begin{bmatrix} \left(\frac{1}{\gamma_1}\right) C_{35} \\ \left(\frac{1}{\gamma_2}\right) C_{34} \end{bmatrix} \end{bmatrix}. \quad (\text{A.9})$$

$$\mathbf{d}^{UV} = \begin{bmatrix} -C_{55} & -C_{54} & -C_{53} \\ -C_{45} & -C_{44} & -C_{43} \\ -C_{35} & -C_{34} & -C_{33} \end{bmatrix}. \quad (\text{A.10})$$

$$\mathbf{d}^{VU} = \begin{bmatrix} -\rho\omega^2 & 0 & 0 \\ 0 & -\rho\omega^2 & 0 \\ 0 & 0 & -\rho\omega^2 \end{bmatrix}. \quad (\text{A.11})$$

Appendix B

Finite Deformation Method

In the finite deformation method, the description of the motion of a material body has been given by considering the relation between two configurations, reference configuration and current configuration, in which each material particle of the body is named with a marker labeled by position $\mathbf{X}$ in the reference configuration and $\mathbf{x} = \varphi(\mathbf{X}, t)$ in the current configuration at a given time t . The change in shape of the infinitesimal volume elements can be measured by the deformation gradient tensor, defined by

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}. \quad (\text{B.1})$$

The Green-Lagrange strain tensor is defined

$$\mathbf{E} = \frac{1}{2} (\mathbf{F}^T \mathbf{F} - \mathbf{I}), \quad (\text{B.2})$$

where $\mathbf{F}^T$ represents transposition of $\mathbf{F}$, and $\mathbf{I}$ represents the identity tensor. To describe the nonlinear material properties, Landau-Lifshitz hyperelastic constitutive model is given through a strain energy density function including third order elastic constants

$$w = \frac{1}{2} \lambda I_1^2 + \mu I_2 + \frac{1}{3} C I_1^3 + B I_1 I_2 + \frac{1}{3} A I_3, \quad (\text{B.3})$$

where λ and μ are Lamé constants; A , B and C represent third order elastic constants; I_1 , I_2 and I_3 are defined by $I_1 = E_{ii}$, $I_2 = E_{ij} E_{ji}$ and $I_3 = E_{ij} E_{jk} E_{ki}$, representing the first three invariants of the Green-Lagrange tensor. Basing on the first P-K stress tensor, the

momentum equation describing the dynamic behavior of the solid is given in the reference configuration by

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \mathbf{s}, \quad (\text{B.4})$$

where ρ_0 is the density in the reference configuration, $\mathbf{u}$ is the displacement and ∇ is the gradient operator with respect to the reference configuration. The second P-K stress tensor is given by

$$\mathbf{S} = \frac{\partial w}{\partial \mathbf{E}}. \quad (\text{B.5})$$

The first P-K stress is related to the second P-K stress by

$$\mathbf{s} = \mathbf{F}\mathbf{S}. \quad (\text{B.6})$$

Appendix C

Derivations of Eq. (5.27)

Gauss divergence theorem is used to derive the first term on the left-hand side of Eq. (5.24),

$$\begin{aligned}
& \int_{\Omega} \nabla_{\perp} \cdot (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) d\Omega \\
&= \int_{\Gamma_{A/B}} (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \cdot \mathbf{n}_A d\Gamma \\
&+ \int_{\Gamma_{A/B}} (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \cdot \mathbf{n}_B^{\text{inn}} d\Gamma \\
&+ \int_{\Gamma_B} (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \cdot \mathbf{n}_B^{\text{out}} d\Gamma
\end{aligned} \tag{C.1}$$

As the wave field vanish on Γ_B (Eqs. (5.18) and (5.19)), the third term on the right-hand side of Eq. (C.1) goes to zero, so that Eq. (C.1) can be rewritten as

$$\begin{aligned}
& \int_{\Omega} \nabla_{\perp} \cdot (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) d\Omega \\
&= \int_{\Gamma_{A/B}} (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \cdot \mathbf{n}_A d\Gamma \\
&+ \int_{\Gamma_{A/B}} (\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z}) \cdot \mathbf{n}_B^{\text{inn}} d\Gamma \\
&= \int_{\Gamma_{A/B}} \mathbf{v}_n e^{ik_n z} \cdot (\mathbf{s}_2|_A \cdot \mathbf{n}_A + \mathbf{s}_2|_B \cdot \mathbf{n}_B^{\text{inn}}) d\Gamma \\
&+ \int_{\Gamma_{A/B}} -\mathbf{v}_2 e^{ik_n z} \cdot (\mathbf{s}_n|_A \cdot \mathbf{n}_A + \mathbf{s}_n|_B \cdot \mathbf{n}_B^{\text{inn}}) d\Gamma
\end{aligned} \tag{C.2}$$

According to the interface boundary conditions on $\Gamma_{A/B}$, the following equations are satisfied,

$$\mathbf{s}_2|_A \cdot \mathbf{n}_A + \mathbf{s}_2|_B \cdot \mathbf{n}_B^{\text{inn}} = -\bar{\mathbf{s}}^{(1)} \cdot \mathbf{n}_A, \tag{C.3}$$

$$\mathbf{s}_n|_A \cdot \mathbf{n}_A + \mathbf{s}_n|_B \cdot \mathbf{n}_B^{\text{inn}} = 0. \quad (\text{C.4})$$

Substituting Eqs. (C.3) and (C.4) into Eq. (C.2), this equation can be simplified as

$$\int_{\Omega} \nabla_{\perp} \cdot \left(\mathbf{v}_n e^{ik_n z} \cdot \mathbf{s}_2 - \mathbf{v}_2 \cdot \mathbf{s}_n e^{ik_n z} \right) d\Omega = \int_{\Gamma_{A/B}} \mathbf{v}_n e^{ik_n z} \cdot \left(-\bar{\mathbf{s}}^{(1)} \right) \cdot \mathbf{n}_A d\Gamma. \quad (\text{C.5})$$

References

- [1] B. N. Pavlakovic and M. J. S. Lowe. *DISPERSE: A System for Generating Dispersion Curves*. User's Manual, 2003.
- [2] Y. Liu, V. K. Chillara, and C. J. Lissenden. On selection of primary modes for generation of strong internally resonant second harmonics in plate. *J. Sound Vib.*, 332(19):4517–4528, 2013.
- [3] Y. Liu, C. J. Lissenden, and J. L. Rose. Higher order interaction of elastic waves in weakly nonlinear hollow circular cylinders. II. Physical interpretation and numerical results. *J. Appl. Phys.*, 115:214902, 2014.
- [4] J. Schijve. *Fatigue of structures and materials*. Springer Science & Business Media, 2001.
- [5] K. Y. Jhang. Nonlinear ultrasonic techniques for nondestructive assessment of micro damage in material: A review. *Int. J. Precis. Eng. Man.*, 10(1):123–135, 2009.
- [6] D. G. Harlow, J. Nardiello, and J. Payne. The effect of constituent particles in aluminum alloys on fatigue damage evolution: Statistical observations. *Int. J. Fatigue*, 32(3):505–511, 2010.
- [7] A. S. Birks and R. E. Green. *Nondestructive testing handbook*, volume 7. American Society for Nondestructive Testing, Columbus OH, 1991.
- [8] K. H. Matlack, J. Y. Kim, L. J. Jacobs, and J. Qu. Review of second harmonic generation measurement techniques for material state determination in metals. *J. Nondestruct. Eval.*, 34(1):1–23, 2014.
- [9] A. Hikata, B. B. Chick, and C. Elbaum. Effect of dislocations on finite amplitude ultrasonic waves in aluminum. *Appl. Phys. Lett.*, 3(11):195–197, 1963.

- [10] J. H. Cantrell. Ultrasonic harmonic generation from fatigue-induced dislocation substructures in planar slip metals and assessment of remaining fatigue life. *J. Appl. Phys.*, 106(9):093516, 2009.
- [11] Y. Xiang, M. Deng, F. Z. Xuan, and C. J. Liu. Effect of precipitate-dislocation interactions on generation of nonlinear Lamb waves in creep-damaged metallic alloys. *J. Appl. Phys.*, 111(10):104905, 2012.
- [12] V. K. Chillara and C. J. Lissenden. Review of nonlinear ultrasonic guided wave non-destructive evaluation: theory, numerics, and experiments. *Opt. Eng.*, 55(1):011002–011002, 2016.
- [13] M. Deng. Cumulative second-harmonic generation accompanying nonlinear shear horizontal mode propagation in a solid plate. *J. Appl. Phys.*, 84(7):3500–3505, 1998.
- [14] W. J. N. de Lima and M. F. Hamilton. Finite-amplitude waves in isotropic elastic plates. *J. Sound Vib.*, 265(4):819–839, 2003.
- [15] M. F. Müller, J. Y. Kim, J. Qu, and L. J. Jacobs. Characteristics of second harmonic generation of Lamb waves in nonlinear elastic plates. *J. Acoust. Soc. Am.*, 127(4):2141–2152, 2010.
- [16] A. Srivastava and F. L. di Scalea. On the existence of antisymmetric or symmetric Lamb waves at nonlinear higher harmonics. *J. Sound Vib.*, 323(3):932–943, 2009.
- [17] W. J. N. de Lima and M. F. Hamilton. Finite amplitude waves in isotropic elastic waveguides with arbitrary constant cross-sectional area. *Wave Motion*, 41(1):1–11, 2005.
- [18] A. Srivastava and F. L. di Scalea. On the existence of longitudinal or flexural waves in rods at nonlinear higher harmonics. *J. Sound Vib.*, 329(9):1499–1506, 2010.
- [19] V. K. Chillara and C. J. Lissenden. Analysis of second harmonic guided waves in pipes using a large-radius asymptotic approximation for axis-symmetric longitudinal modes. *Ultrasonics*, 53(4):862–869, 2013.
- [20] C. Nucera and F. L. di Scalea. Nonlinear semianalytical finite-element algorithm for the analysis of internal resonance conditions in complex waveguides. *J. Eng. Mech.*, 140(3):502–522, 2014.

-
- [21] C. Nucera and F. L. di Scalea. Nondestructive measurement of neutral temperature in continuous welded rails by nonlinear ultrasonic guided waves. *J. Acoust. Soc. Am.*, 136(5):2561–2574, 2014.
- [22] M. Deng, P. Wang, and X. Lv. Experimental observation of cumulative second-harmonic generation of Lamb-wave propagation in an elastic plate. *J. Phys. D: Appl. Phys.*, 38(2):344, 2005.
- [23] Y. Xiang, M. Deng, F. Z. Xuan, and C. J. Liu. Cumulative second-harmonic analysis of ultrasonic Lamb waves for ageing behavior study of modified-HP austenite steel. *Ultrasonics*, 51(8):974–981, 2011.
- [24] W. Li and Y. Cho. Thermal fatigue damage assessment in an isotropic pipe using nonlinear ultrasonic guided waves. *Exp. Mech.*, 54(8):1309–1318, 2014.
- [25] M. Hirao, A. Tomizawa, and H. Fukuoka. Nonlinear resonance interaction of ultrasonic waves under applied stress. *J. Appl. Phys.*, 56(1):235–237, 1984.
- [26] E. M. Ballad, S. Y. Vezirov, K. Pfeleiderer, I. Y. Solodov, and G. Busse. Nonlinear modulation technique for NDE with air-coupled ultrasound. *Ultrasonics*, 42(1):1031–1036, 2004.
- [27] C. Payan, V. Garnier, J. Moysan, and P. A. Johnson. Applying nonlinear resonant ultrasound spectroscopy to improving thermal damage assessment in concrete. *J. Acoust. Soc. Am.*, 121(4):EL125–EL130, 2007.
- [28] J. D. Achenbach and A. N. Norris. Loss of specular reflection due to nonlinear crack-face interaction. *J. Nondestruct. Eval.*, 3(4):229–239, 1982.
- [29] M. A. Breazeale and D. O. Thompson. Finite-amplitude ultrasonic waves in aluminum. *Appl. Phys. Lett.*, 3(5):77–78, 1963.
- [30] L. D. Landau and E. M. Lifshitz. *Theory of Elasticity*, 3rd. Pergamon Press, 1959.
- [31] D. Gerlich and M. A. Breazeale. Ultrasonic second-harmonic generation in various crystalline systems: Coupling parameters in terms of elastic moduli and propagation directions. *J. Appl. Phys.*, 63(12):5712–5717, 1988.
- [32] J. H. Cantrell. Crystalline structure and symmetry dependence of acoustic nonlinearity parameters. *J. Appl. Phys.*, 76(6):3372–3380, 1994.
-

-
- [33] G. L. Jones and D. R. Kobett. Interaction of elastic waves in an isotropic solid. *J. Acoust. Soc. Am.*, 35(1):5–10, 1963.
- [34] J. D. Childress and C. G. Hambrick. Interactions between elastic waves in an isotropic solid. *Phys. Rev.*, 136(2A):A411, 1964.
- [35] I. A. Beresnev. Interaction of two spherical elastic waves in a nonlinear five-constant medium. *J. Acoust. Soc. Am.*, 94(6):3400–3404, 1993.
- [36] B. N. Kuvshinov, T. J. H. Smit, and X. H. Campman. Non-linear interaction of elastic waves in rocks. *Geophys. J. Int.*, 194(3):1920–1940, 2013.
- [37] G. Tang, M. Liu, L. J. Jacobs, and J. Qu. Detecting localized plastic strain by a scanning collinear wave mixing method. *J. Nondestruct. Eval.*, 33(2):196–204, 2014.
- [38] V. A. Korneev and A. Demčenko. Possible second-order nonlinear interactions of plane waves in an elastic solid. *J. Acoust. Soc. Am.*, 135(2):591–598, 2014.
- [39] W. D. Cash and W. Cai. Dislocation contribution to acoustic nonlinearity: The effect of orientation-dependent line energy. *J. Appl. Phys.*, 109(1):014915, 2011.
- [40] J. Zhang, F. Xuan, and Y. Xiang. Dislocation characterization in cold rolled stainless steel using nonlinear ultrasonic techniques: A comprehensive model. *Europhys. Lett.*, 103(6):68003, 2013.
- [41] Z. Chen and J. Qu. Dislocation-induced acoustic nonlinearity parameter in crystalline solids. *J. Appl. Phys.*, 114(16):164906, 2013.
- [42] J. H. Cantrell and W. T. Yost. Determination of precipitate nucleation and growth rates from ultrasonic harmonic generation. *Appl. Phys. Lett.*, 77(13):1952–1954, 2000.
- [43] D. C. Hurley, D. Balzar, and P. T. Purtscher. Nonlinear ultrasonic assessment of precipitation hardening in ASTM A710 steel. *J. Mater. Res.*, 15(09):2036–2042, 2000.
- [44] S. Hirose and J. D. Achenbach. Higher harmonics in the far field due to dynamic crack-face contacting. *J. Acoust. Soc. Am.*, 93(1):142–147, 1993.
- [45] V. E. Nazarov and A. M. Sutin. Nonlinear elastic constants of solids with cracks. *J. Acoust. Soc. Am.*, 102(6):3349–3354, 1997.
-

-
- [46] H. J. McSkimin and P. Andreatch Jr. Measurement of third-order moduli of silicon and germanium. *J. Appl. Phys.*, 35(11):3312–3319, 1964.
- [47] M. A. Breazeale and J. Ford. Ultrasonic studies of the nonlinear behavior of solids. *J. Appl. Phys.*, 36(11):3486–3490, 1965.
- [48] W. B. Gauster and M. A. Breazeale. Ultrasonic measurement of the nonlinearity parameters of copper single crystals. *Phys. Rev.*, 168(3):655, 1968.
- [49] W. T. Yost, J. H. Cantrell Jr, and M. A. Breazeale. Ultrasonic nonlinearity parameters and third-order elastic constants of copper between 300 and 3 K. *J. Appl. Phys.*, 52(1):126–128, 1981.
- [50] W. L. Morris, O. Buck, and R. V. Inman. Acoustic harmonic generation due to fatigue damage in high-strength aluminum. *J. Appl. Phys.*, 50(11):6737–6741, 1979.
- [51] W. T. Yost and J. H. Cantrell. The effects of fatigue on acoustic nonlinearity in aluminum alloys. In *Ultrasonics Symposium, 1992. Proceedings., IEEE 1992*, pages 947–955. IEEE, 1992.
- [52] K. Y. Jhang and K. C. Kim. Evaluation of material degradation using nonlinear acoustic effect. *Ultrasonics*, 37(1):39–44, 1999.
- [53] J. K. Na, J. H. Cantrell, and W. T. Yost. Linear and nonlinear ultrasonic properties of fatigued 410CB stainless steel. In *Review of progress in quantitative nondestructive evaluation*, pages 1347–1352. Springer, 1996.
- [54] J. Frouin, S. Sathish, T. E. Matikas, and J. K. Na. Ultrasonic linear and nonlinear behavior of fatigued Ti-6Al-4V. *J. Mater. Res.*, 14(04):1295–1298, 1999.
- [55] T. M. Apple, J. H. Cantrell, C. M. Amaro, C. R. Mayer, W. T. Yost, S. R. Agnew, and J. M. Howe. Acoustic harmonic generation from fatigue-generated dislocation substructures in copper single crystals. *Philos. Mag.*, 93(21):2802–2825, 2013.
- [56] W. Li, Y. Cho, J. Lee, and J. D. Achenbach. Assessment of heat treated inconel X-750 alloy by nonlinear ultrasonics. *Exp. Mech.*, 53(5):775–781, 2013.
- [57] A. Mukhopadhyay, R. Sarkar, S. Punnoose, J. Valluri, T. K. Nandy, and K. Balasubramaniam. Scatter in nonlinear ultrasonic measurements due to crystallographic
-

- orientation change induced anisotropy in harmonics generation. *J. Appl. Phys.*, 111(5):054905, 2012.
- [58] A. Ruiz, N. Ortiz, A. Medina, J. Y. Kim, and L. J. Jacobs. Application of ultrasonic methods for early detection of thermal damage in 2205 duplex stainless steel. *NDT & E Int.*, 54:19–26, 2013.
- [59] A. Viswanath, B. P. C. Rao, S. Mahadevan, T. Jayakumar, and B. Raj. Microstructural characterization of M250 grade maraging steel using nonlinear ultrasonic technique. *J. Mater. Sci.*, 45(24):6719–6726, 2010.
- [60] S. Baby, B. N. Kowmudi, C. M. Omprakash, D. V. V. Satyanarayana, K. Balasubramaniam, and V. Kumar. Creep damage assessment in titanium alloy using a nonlinear ultrasonic technique. *Scr. Mater.*, 59(8):818–821, 2008.
- [61] K. H. Matlack, J. J. Wall, J. Y. Kim, J. Qu, L. J. Jacobs, and H. W. Viehrig. Evaluation of radiation damage using nonlinear ultrasound. *J. Appl. Phys.*, 111(5):054911, 2012.
- [62] M. Deng. Cumulative second-harmonic generation of Lamb-mode propagation in a solid plate. *J. Appl. Phys.*, 85(6):3051–3058, 1999.
- [63] M. Deng. Cumulative second-harmonic generation of generalized Lamb-wave propagation in a solid waveguide. *J. Phys. D: Appl. Phys.*, 33(3):207, 2000.
- [64] M. Deng. Analysis of second-harmonic generation of Lamb modes using a modal analysis approach. *J. Appl. Phys.*, 94(6):4152–4159, 2003.
- [65] M. Deng, Y. Xiang, and L. Liu. Time-domain analysis and experimental examination of cumulative second-harmonic generation by primary Lamb wave propagation. *J. Appl. Phys.*, 109(11):113525, 2011.
- [66] Y. Liu, V. K. Chillara, C. J. Lissenden, and J. L. Rose. Third harmonic shear horizontal and Rayleigh Lamb waves in weakly nonlinear plates. *J. Appl. Phys.*, 114(11):114908, 2013.
- [67] S. Shan, L. Cheng, and P. Li. Adhesive nonlinearity in Lamb-wave-based structural health monitoring systems. *Smart Mater. Struct.*, 26(2):025019, 2016.

- [68] S. Shan, L. Cheng, and F. Wen. Characterization of nonplanar second harmonic lamb waves with a refined nonlinear parameter. *Journal of Nondestructive Evaluation, Diagnostics and Prognostics of Engineering Systems*, 1(1):011004, 2018.
- [69] M. Hong, Z. Mao, M. D. Todd, and Z. Su. Uncertainty quantification for acoustic nonlinearity parameter in Lamb wave-based prediction of barely visible impact damage in composites. *Mech. Syst. Signal Pr.*, 82:448–460, 2017.
- [70] Y. Liu, E. Khajeh, C. J. Lissenden, and J. L. Rose. Interaction of torsional and longitudinal guided waves in weakly nonlinear circular cylinders. *J. Acoust. Soc. Am.*, 133(5):2541–2553, 2013.
- [71] Y. Liu, C. J. Lissenden, and J. L. Rose. Higher order interaction of elastic waves in weakly nonlinear hollow circular cylinders. I. Analytical foundation. *J. Appl. Phys.*, 115:214901, 2014.
- [72] K. F. Graff. *Wave Motion in Elastic Solids*. Courier Corporation, 2012.
- [73] C. Nucera and F. L. di Scalea. Higher-harmonic generation analysis in complex waveguides via a nonlinear semianalytical finite element algorithm. *Math. Probl. Eng.*, 2012(Article ID 365630):16 pages, 2012.
- [74] C. Bermes, J. Y. Kim, J. Qu, and L. J. Jacobs. Experimental characterization of material nonlinearity using Lamb waves. *Appl. Phys. Lett.*, 90(2):021901, 2007.
- [75] C. Bermes, J. Y. Kim, J. Qu, and L. J. Jacobs. Nonlinear Lamb waves for the detection of material nonlinearity. *Mech. Syst. Signal Pr.*, 22(3):638–646, 2008.
- [76] M. Deng and J. Pei. Assessment of accumulated fatigue damage in solid plates using nonlinear Lamb wave approach. *Appl. Phys. Lett.*, 90(12):121902, 2007.
- [77] C. Pruell, J. Y. Kim, J. Qu, and L. J. Jacobs. Evaluation of fatigue damage using nonlinear guided waves. *Smart Mater. Struct.*, 18(3):035003, 2009.
- [78] W. Li, Y. Cho, and J. D. Achenbach. Detection of thermal fatigue in composites by second harmonic Lamb waves. *Smart Mater. Struct.*, 21(8):085019, 2012.
- [79] C. Pruell, Jin Y. K., J. Qu, and L. J. Jacobs. Evaluation of plasticity driven material damage using Lamb waves. *Appl. Phys. Lett.*, 91(23):231911, 2007.

- [80] C. Pruell, J. Y. Kim, J. Qu, and L. J. Jacobs. A nonlinear-guided wave technique for evaluating plasticity-driven material damage in a metal plate. *NDT & E Int.*, 42(3):199–203, 2009.
- [81] Y. Xiang, M. Deng, F. Z. Xuan, and C. J. Liu. Experimental study of thermal degradation in ferritic Cr–Ni alloy steel plates using nonlinear Lamb waves. *NDT & E Int.*, 44(8):768–774, 2011.
- [82] A. H. Nayfeh and P. B. Nagy. General study of axisymmetric waves in layered anisotropic fibers and their composites. *J. Acoust. Soc. Am.*, 99(2):931–941, 1996.
- [83] B. N. Pavlakovic. *Leaky guided ultrasonic waves in NDT*. Imperial College, London, PhD Thesis, 1998.
- [84] R. N. Thurston. Elastic waves in rods and clad rods. *J. Acoust. Soc. Am.*, 64(1):1–37, 1978.
- [85] P. E. Lagasse. Higher-order finite-element analysis of topographic guides supporting elastic surface waves. *J. Acoust. Soc. Am.*, 53(4):1116–1122, 1973.
- [86] B. Aalami. Waves in prismatic guides of arbitrary cross section. *J. Appl. Mech.-T. ASME*, 40(4):1067–1072, 1973.
- [87] K. H. Huang and S. B. Dong. Propagating waves and edge vibrations in anisotropic composite cylinders. *J. Sound Vib.*, 96(3):363–379, 1984.
- [88] L. Gavrić. Computation of propagative waves in free rail using a finite element technique. *J. Sound Vib.*, 185(3):531–543, 1995.
- [89] T. Hayashi, W. J. Song, and J. L. Rose. Guided wave dispersion curves for a bar with an arbitrary cross-section, a rod and rail example. *Ultrasonics*, 41(3):175–183, 2003.
- [90] T. Hayashi, C. Tamayama, and M. Murase. Wave structure analysis of guided waves in a bar with an arbitrary cross-section. *Ultrasonics*, 44(1):17–24, 2006.
- [91] M. V. Predoi, M. Castaings, B. Hosten, and C. Bacon. Wave propagation along transversely periodic structures. *J. Acoust. Soc. Am.*, 121(4):1935–1944, 2007.
- [92] Z. Fan and M. J. S. Lowe. Elastic waves guided by a welded joint in a plate. *P. Roy. Soc. A-Math Phys.*, 465(2107):2053–2068, 2009.

-
- [93] I. Bartoli, A. Marzani, F. L. di Scalea, and E. Viola. Modeling wave propagation in damped waveguides of arbitrary cross-section. *J. Sound Vib.*, 295(3):685–707, 2006.
- [94] P. Wilcox, M. Evans, O. Diligent, M. Lowe, and P. Cawley. Dispersion and excitability of guided acoustic waves in isotropic beams with arbitrary cross section. In *AIP Conference Proceedings*, volume 615, pages 203–210. AIP, 2002.
- [95] Z. Fan, M. J. S. Lowe, M. Castaings, and C. Bacon. Torsional waves propagation along a waveguide of arbitrary cross section immersed in a perfect fluid. *J. Acoust. Soc. Am.*, 124(4):2002–2010, 2008.
- [96] A. C. Hladky-Hennion, P. Langlet, and M. de Billy. Finite element analysis of the propagation of acoustic waves along waveguides immersed in water. *J. Sound Vib.*, 200(4):519–530, 1997.
- [97] F. Treyssède, K. L. Nguyen, A. S. Bonnet-BenDhia, and C. Hazard. Finite element computation of trapped and leaky elastic waves in open stratified waveguides. *Wave Motion*, 51(7):1093–1107, 2014.
- [98] M. Castaings and M. Lowe. Finite element model for waves guided along solid systems of arbitrary section coupled to infinite solid media. *J. Acoust. Soc. Am.*, 123(2):696–708, 2008.
- [99] M. Mazzotti, I. Bartoli, A. Marzani, and E. Viola. A coupled SAFE-2.5D BEM approach for the dispersion analysis of damped leaky guided waves in embedded waveguides of arbitrary cross-section. *Ultrasonics*, 53(7):1227–1241, 2013.
- [100] M. Mazzotti, A. Marzani, and I. Bartoli. Dispersion analysis of leaky guided waves in fluid-loaded waveguides of generic shape. *Ultrasonics*, 54(1):408–418, 2014.
- [101] K. L. Nguyen, F. Treyssède, and C. Hazard. Numerical modeling of three-dimensional open elastic waveguides combining semi-analytical finite element and perfectly matched layer methods. *J. Sound Vib.*, 344:158–178, 2015.
- [102] B. A. Auld. *Acoustic Fields and Waves in Solids*, volume 2. R. E. Krieger, 1990.
- [103] D. Marcuse. *Theory of dielectric optical waveguides*. Academic Press, 1974.
- [104] R. E. Collin. *Field theory of guided waves*. IEEE Press, Wiley, New York, 1991.
-

-
- [105] V. V. Shevchenko. The expansion of the fields of open waveguides in proper and improper modes. *Radiophys. Quant. Electron.*, 14(8):972–977, 1971.
- [106] M. Gallezot, F. Treyssède, and L. Laguerre. Contribution of leaky modes in the modal analysis of unbounded problems with perfectly matched layers. *J. Acoust. Soc. Am.*, 141(1):EL16–EL21, 2017.
- [107] R. Sammut and A. W. Snyder. Leaky modes on a dielectric waveguide: orthogonality and excitation. *Appl. Optics*, 15(4):1040–1044, 1976.
- [108] A. W. Snyder. Leaky-ray theory of optical waveguides of circular cross section. *Appl. Phys.*, 4(4):273–298, 1974.
- [109] S. L. Lee, Y. Chung, L. A. Coldren, and N. Dagli. On leaky mode approximations for modal expansion in multilayer open waveguides. *IEEE J. Quantum Elect.*, 31(10):1790–1802, 1995.
- [110] T. Vogt, M. Lowe, and P. Cawley. The scattering of guided waves in partly embedded cylindrical structures. *J. Acoust. Soc. Am.*, 113(3):1258–1272, 2003.
- [111] M. Gallezot, F. Treyssède, and L. Laguerre. A modal approach based on perfectly matched layers for the forced response of elastic open waveguides. *J. Comput. Phys.*, 356:391–409, 2018.
- [112] T. K. Vogt, J. S. Lowe, and P. Cawley. Measurement of the material properties of viscous liquids using ultrasonic guided waves. *IEEE Trans. Ultrason. Ferroelectr. Freq. Control*, 51(6):737–747, 2004.
- [113] F. B. Cegla, P. Cawley, and M. J. S. Lowe. Material property measurement using the quasi-Scholte mode - A waveguide sensor. *J. Acoust. Soc. Am.*, 117(3):1098–1107, 2005.
- [114] J. P. Bérenger. A perfectly matched layer for the absorption of electromagnetic waves. *J. Comput. Phys.*, 114(2):185–200, 1994.
- [115] C. M. Rappaport. Perfectly matched absorbing boundary conditions based on anisotropic lossy mapping of space. *IEEE Microw. Guided Wave Lett.*, 5(3):90–92, 1995.
-

-
- [116] F. L. Teixeira and W. C. Chew. General closed-form PML constitutive tensors to match arbitrary bianisotropic and dispersive linear media. *IEEE Microw. Guided Wave Lett.*, 8(6):223–225, 1998.
- [117] I. Harari, M. Slavutin, and E. Turkel. Analytical and numerical studies of a finite element PML for the Helmholtz equation. *J. Comput. Acoust.*, 8(01):121–137, 2000.
- [118] F. Q. Hu. On absorbing boundary conditions for linearized Euler equations by a perfectly matched layer. *J. Comput. Phys.*, 129(1):201–219, 1996.
- [119] W. C. Chew and Q. H. Liu. Perfectly matched layers for elastodynamics: a new absorbing boundary condition. *J. Comput. Acoust.*, 4(04):341–359, 1996.
- [120] F. D. Hastings, J. B. Schneider, and S. L. Broschat. Application of the perfectly matched layer (PML) absorbing boundary condition to elastic wave propagation. *J. Acoust. Soc. Am.*, 100(5):3061–3069, 1996.
- [121] U. Basu and A. K. Chopra. Perfectly matched layers for time-harmonic elastodynamics of unbounded domains: theory and finite-element implementation. *Comput. Meth. Appl. Mech. Eng.*, 192(11):1337–1375, 2003.
- [122] S. François, M. Schevenels, G. Lombaert, and G. Degrande. A two-and-a-half-dimensional displacement-based PML for elastodynamic wave propagation. *Int. J. Numer. Methods Eng.*, 90(7):819–837, 2012.
- [123] F. L. Teixeira and W. C. Chew. Complex space approach to perfectly matched layers: a review and some new developments. *Int. J. Numer. Model.-Electron. Netw. Device Fields*, 13:441–455, 2000.
- [124] T. Hayashi and D. Inoue. Calculation of leaky Lamb waves with a semi-analytical finite element method. *Ultrasonics*, 54(6):1460–1469, 2014.
- [125] COMSOL. *User’s Guide and Introduction*. COMSOL MULTIPHYSICS, <http://www.comsol.com/>, accessed 20th October 2017.
- [126] Y. O. Agha, F. Zolla, A. Nicollet, and S. Guenneau. On the use of PML for the computation of leaky modes. *Compel*, 27:95–109, 2008.
- [127] A. Pelat, S. Felix, and V. Pagneux. A coupled modal-finite element method for the wave propagation modeling in irregular open waveguides. *J. Acoust. Soc. Am.*, 129(3):1240–1249, 2011.
-

- [128] I. Harari and U. Albocher. Studies of FE/PML for exterior problems of time-harmonic elastic waves. *Comput. Meth. Appl. Mech. Eng.*, 195(29):3854–3879, 2006.
- [129] F. Collino and C. Tsogka. Application of the perfectly matched absorbing layer model to the linear elastodynamic problem in anisotropic heterogeneous media. *Geophysics*, 66(1):294–307, 2001.
- [130] A. Bermúdez, L. Hervella-Nieto, A. Prieto, and R. Rodríguez. An optimal perfectly matched layer with unbounded absorbing function for time-harmonic acoustic scattering problems. *J. Comput. Phys.*, 223(2):469–488, 2007.
- [131] E. Leinov, M. J. S. Lowe, and P. Cawley. Investigation of guided wave propagation and attenuation in pipe buried in sand. *J. Sound Vib.*, 347:96–114, 2015.
- [132] E. Leinov, M. J. S. Lowe, and P. Cawley. Ultrasonic isolation of buried pipes. *J. Sound Vib.*, 363:225–239, 2016.
- [133] E. Leinov, M. J. S. Lowe, and P. Cawley. Investigation of guided wave propagation in pipes fully and partially embedded in concrete. *J. Acoust. Soc. Am.*, 140(6):4528–4539, 2016.
- [134] D. N. Alleyne, M. J. S. Lowe, and P. Cawley. The reflection of guided waves from circumferential notches in pipes. *J. Appl. Mech.-T. ASME*, 65(3):635–641, 1998.
- [135] Z. Liu, C. He, B. Wu, X. Wang, and S. Yang. Circumferential and longitudinal defect detection using T(0,1) mode excited by thickness shear mode piezoelectric elements. *Ultrasonics*, 44:e1135–e1138, 2006.
- [136] H. H. Bau. Torsional wave sensor - A theory. *J. Appl. Mech.-Trans. ASME*, 53(4):846–848, 1986.
- [137] C. C. J. Smit and E. D. Smith. The analysis and results of a continuous wave ultrasonic densitometer. *J. Acoust. Soc. Am.*, 104(3):1413–1417, 1998.
- [138] C. L. Shepard, B. J. Burghard, M. A. Friesel, B. P. Hildebrand, X. Moua, A. A. Diaz, and C. W. Enderlin. Measurements of density and viscosity of one-and two-phase fluids with torsional waveguides. *IEEE Trans. Ultrason. Ferroelectr. Freq. Control*, 46(3):536–548, 1999.

- [139] W. B. Fraser. Stress wave propagation in rectangular bars. *Int. J. Solids Struct.*, 5(4):379–397, 1969.
- [140] Y. Zheng, R. G. Maev, and I. Y. Solodov. Review/synthèse nonlinear acoustic applications for material characterization: A review. *Can. J. Phys.*, 77(12):927–967, 2000.
- [141] Y. Xiang, M. Deng, and F. Z. Xuan. Analysis of second-harmonic generation of Lamb waves using a combined method in a two-layered solid waveguide. *J. Appl. Phys.*, 106(2):024902, 2009.
- [142] N. Matsuda and S. Biwa. Frequency dependence of second-harmonic generation in Lamb waves. *J. Nondestruct. Eval.*, 33(2):169–177, 2014.
- [143] A. Pau. Derivation of wave mode orthogonality from reciprocity in direct notation. *ASME J. Nondestr. Eval.*, 1(2):024501, 2018.
- [144] P. Zuo, X. Yu, and Z. Fan. Numerical modeling of embedded solid waveguides using SAFE-PML approach using a commercially available finite element package. *NDT&E Int.*, 90:11–23, 2017.
- [145] P. Zuo and Z. Fan. SAFE-PML approach for modal study of waveguides with arbitrary cross sections immersed in inviscid fluid. *J. Sound Vib.*, 406:181–196, 2017.
- [146] P. Belanger. High order shear horizontal modes for minimum remnant thickness. *Ultrasonics*, 54(4):1078–1087, 2014.
- [147] B. A. Auld and E. M. Tsao. A variational analysis of edge resonance in a semi-infinite plate. *IEEE Trans. Sonics Ultrason.*, 24(5):317–326, 1977.
- [148] P. Zuo, Y. Zhou, and Z. Fan. Numerical studies of nonlinear ultrasonic guided waves in uniform waveguides with arbitrary cross sections. *AIP Adv.*, 6(7):075207, 2016.
- [149] J. B. Keller. Geometrical theory of diffraction. *J. Opt. Soc. Am.*, 52(2):116–130, 1962.
- [150] N. Matsuda and S. Biwa. Phase and group velocity matching for cumulative harmonic generation in lamb waves. *J. Appl. Phys.*, 109(9):094903, 2011.

- [151] M. Niethammer, L. J. Jacobs, J. Qu, and J. Jarzynski. Time-frequency representations of Lamb waves. *J. Acoust. Soc. Am.*, 109(5):1841–1847, 2001.

List of Publications

- [1] **P. Zuo**, Y. Zhou, Z. Fan. Numerical studies of nonlinear ultrasonic guided waves in uniform waveguides with arbitrary cross sections. *AIP Advances*, **6**, 075207 (2016).

- [2] **P. Zuo**, Y. Zhou, Z. Fan. Numerical and experimental investigation of nonlinear ultrasonic Lamb waves at low frequency. *Applied Physics Letters*, **109**, 021902 (2016).

- [3] **P. Zuo**, X. Yu, Z. Fan. Numerical modeling of embedded solid waveguides using SAFE-PML approach using a commercially available finite element package. *NDT&E International*, **90**, 11-23 (2017).

- [4] **P. Zuo**, Z. Fan. SAFE-PML approach for modal study of waveguides with arbitrary cross sections immersed in inviscid fluid. *Journal of Sound and Vibration*, **406**, 181-196 (2017).

- [5] **P. Zuo**, Y. Liu, Y. Zhou, Z. Fan. Investigation of nonlinear ultrasonic guided waves in open waveguides based on perfectly matched layers. *The Journal of the Acoustical Society of America*, **144**(3), 1572-1588 (2018).