Second order multi-object filtering with target interaction using determinantal point processes

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**NANYANG
TECHNOLOGICAL
UNIVERSITY**

SINGAPORE

**SECOND ORDER MULTI-OBJECT FILTERING WITH
TARGET INTERACTION USING DETERMINANTAL
POINT PROCESSES**

**TEOH ZHISHENG, TIMOTHY
SCHOOL OF PHYSICAL AND MATHEMATICAL
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TEOH ZHISHENG, TIMOTHY

SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

A thesis submitted to the Nanyang Technological University
in partial fulfilment of the requirement for the degree of
Doctor of Philosophy

2019

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Abstract

The Probability Hypothesis Density (PHD) filter is an algorithm that propagates the first-order moment of random finite sets for use in multi-target tracking based on sensor measurements. This algorithm usually assumes that targets behave independently, an hypothesis which may not hold in practice due to potential target interactions. We begin by discussing some general properties, functionals and branching dependency associated with spatial point processes that constitute the foundations in the construction of the filtering equations. Next, we show some extended computations incorporating Janossy densities in the derivation of closed-form solutions to the first and second order posterior factorial moment densities of the PHD filter. Implementation of the PHD filter using Sequential Monte Carlo (SMC) method on independent and interactive multi-target environments were explored, and the results obtained motivated the advancement to the Determinantal Point Processes (DPPs) setting.

In the main contribution of this thesis, we constructed a second order PHD filter based on determinantal point processes, which are able to capture some level of interactions among configuration points in domain space. We derived the factorial moments and Janossy densities of determinantal point processes and branched determinantal point processes from its associated probability generating functionals (PGFLs), which are described by the determinant of its kernel. Additionally, we derived combinatorial expressions of the marginal and joint Janossy densities of the general branched point processes. These results obtained are applied to derive closed-form first and second order factorial moment expressions of the determinantal PHD filtering equations. Our approach relies on posterior moment formulas for the estimation of a general hidden point process after a thinning operation and a superposition with a Poisson Point Process (PPP), and on suitable approximation formulas of the determinantal point process hypothesis. In our Determinantal PHD filter, the knowledge of first and second order moments of the prediction and posterior filtering equations suffices, in principle, to update the respective Janossy densities that characterize the underlying determinantal point process. The repulsive properties of determinantal point processes also apply to the modeling of negative correlation between distinct observation domains. An algorithm to implement the determinantal PHD filter using Sequential Monte Carlo (SMC) method is developed. Under specific scenarios, the numerical results obtained are shown to exhibit better performance when compared under the Poisson setting.

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List of Abbreviations

PHD	Probability Hypothesis Density.
CPHD	Cardinalized Probability Hypothesis Density.
PGF	Probability Generating Function.
r.v	Random Variable.
PGFl	Probability Generating Functional.
SMC	Sequential Monte Carlo.
PPP	Poisson Point Process.
DPP	Determinantal Point Process.

List of Symbols

$\mathbb{N}_0$	The set of natural numbers $\{0, 1, 2, 3, \dots\}$.
$\mathbb{N}_1$	The set of natural numbers $\{1, 2, 3, \dots\}$.
G_X	PGF of r.v X .
$G_{X,Y}$	Joint PGF of r.v X and Y .
$G_{X Y}$	Conditional PGF of r.v $X Y$.
$\mu_X^{(r)}$	The r^{th} order nonfactorial moment of r.v X .
$\rho_X^{(r)}$	The r^{th} order factorial moment of r.v X .
$\mathcal{G}_\Phi$	PGFl of point process Φ .
$\mathcal{L}_\Phi$	Laplace transform of a point process Φ .
$\mathcal{G}_{\Psi,\Phi}$	Joint PGFl of point processes Ψ and Φ .
$\mathcal{G}_{\Phi \Psi}$	Conditional PGFl of point process $\Phi \Psi$.
$\tilde{\mathcal{G}}_\Phi$	PGFl of a DPP Φ .
$\tilde{\mathcal{L}}_\Phi$	Laplace transform of a DPP Φ .
$x_{1:n}$	Shorthand notation to describe the point configuration $\{x_1, \dots, x_n\}$.
$j_\Phi^{(n)}(x_{1:n})$	Janossy density of a point process Φ with point configuration $x_{1:n}$.
$j_{\Psi,\Phi=y_{1:n}}^{(m,n)}(x_{1:m}),$ $j_{\Psi,\Phi}^{(m,n)}(x_{1:m}, y_{1:n})$	The joint Janossy density of point processes Ψ and Φ with point configurations $x_{1:m} \in \Psi$ and $y_{1:n} \in \Phi$.
$\mu_\Phi^{(n)}(x_{1:n})$	The n^{th} order nonfactorial moment density of a point process Φ with point configurations $x_{1:n}$.
$\rho_\Phi^{(n)}(x_{1:n})$	The n^{th} order factorial moment density of a point process Φ with point configurations $x_{1:n}$.
$\mu_{\Phi \Xi=z_{1:m}}^{(n)}(x_{1:n})$	The n^{th} order conditional nonfactorial moment density of the point process $\Phi \Xi = z_{1:m}$ at the point configurations $x_{1:n}$.
$\rho_{\Phi \Xi=z_{1:m}}^{(n)}(x_{1:n})$	The n^{th} order conditional factorial moment density of the point process $\Phi \Xi = z_{1:m}$ at the point configurations $x_{1:n}$.

Chapter 1

Introduction

Probability Hypothesis Density (PHD) filters have been introduced in [13] for multi-target tracking in cluttered environments, using Poisson point processes (PPP) for the modeling of predicted targets locations. The PHD filters are practical approaches to solve inherently nontrivial multitarget tracking problem to estimate the number and states of targets. Due to the Poisson nature of the PHD filtering equations which are linear equations, the PHD filters have low complexity, and they allow for closed-form update formulas, see e.g. [3] for a review.

Several extensions of the PHD filter to non-Poisson prior distributions have been proposed, using for example Gauss-Poisson point processes that generalizes the Poisson point process by allowing for two-point clusters [20]. Cardinalized Probability Hypothesis Density (CPHD) filters have been introduced in [14] as a generalization in which the target count is allowed to have an arbitrary distribution.

PHD filters usually approximate the distribution of the number of targets by a Poisson distribution estimated by a single mean (or variance) parameter, which can result in high variance estimates when the estimated mean is high. In other words, the mean and variance of the PPP are characterized by the same parameter. As such, when the PHD filter propagates a higher mean, it also implies a higher variance. Second-order PHD filters that can propagate separate information on mean and variance parameters have been recently proposed in [18], using e.g. the Panjer cardinality distribution.

A common feature of cardinalized filters is to assume that target locations are distributed as n independent random samples according to a reference intensity mea-

sure, given that the observation window contains n points. While this hypothesis is natural and facilitates an explicit derivation of prediction formulas, it does not reflect potential interaction between targets. Taking into account correlation via more general point process-based PHD filters poses several challenges linked to the derivation of closed form filtering formulas. In addition, the distribution of general point processes relies on Janossy densities which may not be characterized by the knowledge of a finite number of moments.

In this thesis we construct a second order PHD filter based on determinantal point processes [12], [19], [31], which are able to model repulsion among configuration points on a target domain Λ . Such processes are determined by their kernel functions $(K(x, y))_{x, y \in \Lambda}$ and their Janossy densities can be recovered from their first and second order moments. In this setting the knowledge of first and second moments suffices, in principle, to update the Janossy densities that characterize the underlying determinantal point process.

Determinantal point processes have also been used for other applications in multi-target tracking, see [8]. Permanental processes have been used in [15] to propagate a joint Poisson distribution, however this approach is distinct from the determinantal setting.

In Chapter 2, we give a review of some basic operations and properties that are associated with discrete random variables and spatial point processes. In Section 2.1, we focus on the notions about probability generating functions (PGFs), factorial moments and branching processes that constitute the fundamental basis to the construction of the integer-valued PHD filter that will be introduced in Chapter 3. In Section 2.2, we recall some useful properties about spatial point processes, the notion of probability generating functionals (PGFIs) and its associated relationship with factorial moment and Janossy densities. In addition, we review some operations that involve point processes as well as a special type of dependence between two point processes known as branching of point processes. These concepts associated with spatial point processes form the foundations to construct the spatial point process PHD filter that will be described in Section 3.2 of Chapter 3.

In Chapter 3, we give extended computations incorporating Janossy densities within associated PGFIs, and by properties linked with functional derivatives of the Poisson point process (PPP) characteristic PGFI to derive closed-form solutions to the first and second order posterior factorial moment densities of the PHD filter. Adapted

from [3], in Section 3.1 we review the integer-valued PHD filter and its construction. In Section 3.2, we describe the spatial point process PHD filter and present some extended insights to the construction of the PHD filtering equations in subsection 3.2.1 and subsection 3.2.3. In particular, in Proposition 3.2.4 and Proposition 3.2.5, we derive the Poisson closed-form expression for the first and second posterior factorial moment densities of the PHD filter. In subsection 3.2.4, we describe the *nearly constant turn-rate* ground truth motion and measurement dynamics. We also describe an extension of the motion dynamics to incorporate targets interaction among targets. In addition, we list the Poisson PHD filter algorithm and provide some simulation comparison plots based on sequential Monte Carlo (SMC) implementation. The simulation scenario and parameters are equivalent as of [9],[22]. In our simulation comparison plots of the SMC-PHD filtering algorithm, we show the outputs of independent and interactive ground truth dynamics among multiple targets in target space for illustration purposes. The idea of target interaction among multiple targets in target space motivates the interest of this thesis to construct a second order PHD filter based on Determinantal Point Processes (DPPs) which is presented in Chapter 5.

In Chapter 4, we focus on the computation and derivations of factorial moments and Janossy densities related to DPPs, branched DPPs and general branched point processes. In Section 4.1, we layout some properties of DPPs and provide explicit expressions for the associated PGFl that differ from the conventional PGFl expression of typical point processes stated in Chapter 2. In subsection 4.1.1, we recover the factorial moment densities using functional derivatives of the associated DPP PGFl formulas. We show that the general derivative of the DPP associated PGFl under specific conditions obtained in Corollary 4.1.9 independently recovers Proposition 4.1.2. In subsection 4.1.2, we present the general expression of the DPP Janossy densities in Proposition 4.1.12 for which the proof independently recovers the result in Equation (2.18) of Proposition 2.2.4. The Janossy expression derived is represented in terms of the associated factorial moment density. In subsection 4.1.3, we construct and show some useful properties and relationships associated with the branched DPP PGFl. We describe the results of the marginal first and second order factorial moment densities evaluated using the branched DPP PGFl expressions. The result of the marginal first order moment density is given in Proposition 4.1.14 and the result of the marginal second order factorial moment density is given in Proposition 4.1.15. In Section 4.2, we present a combinatorial expression of the PGFl for general branched point processes and derive the general marginal and

joint Janossy densities. In Proposition 4.2.8, we derive the result of the marginal Janossy densities. In Proposition 4.2.12, we derive the result of the joint Janossy densities. The main results derived in this chapter form part of the contribution in the construction of the Determinantal PHD filter in Chapter 5.

Chapter 5 constitutes the main original contribution to this thesis. In Section 5.1 we derive general formulas for the computation of the first and second moments of a posterior point process Φ given an observation process Ξ of targets constructed by branching. We relate the posterior distribution of branched point processes into the context of multi-object tracking problem. In subsection 5.1.1, we derive the general expression of the conditional Janossy density in Lemma 5.1.1 and recover the marginal Janossy density of the Poisson case under specific conditions. In subsection 5.1.2, we derive the general expression of the first and second order posterior factorial moment densities using extensions of the corrector terms introduced in [5], [21] for the cardinalized PHD filter as well as the Janossy densities of the general branched point processes in Proposition 5.1.2 and Proposition 5.1.3 respectively. In addition, under specific conditions, we show that the general expressions of the first and second order factorial moment densities recover to the Poisson case in Proposition 3.2.4 and Proposition 3.2.5. We present the general formula for the posterior covariance in Proposition 5.1.4. In Section 5.2 we present a PHD filtering algorithm based on determinantal point processes, with the computation of the prediction kernel $K_\Phi(x, y)$ and of the updated kernel $K_{\Phi|\Xi}(x, y)$. In subsection 5.2.1, we propose to use Janossy density approximation formula based on determinantal setting to obtain closed-form expressions. In Proposition 5.2.1 and Proposition 5.2.2, we derive the first and second order corrector term approximation formulas. In Proposition 5.2.3, we derive the approximate kernel update (posterior) formula under the determinantal setting. In Proposition 5.2.5 of subsection 5.2.2, we present the prediction phase first and second order PHD filtering equations for which will be applied to obtain the prediction determinantal kernel and Janossy kernel. In subsection 5.2.3, we present the determinantal kernel update approximation formula to the posterior phase PHD filtering equations. In subsection 5.3.1 of Section 5.3, we present an algorithm using the sequential Monte Carlo method for the implementation of the Determinantal PHD filter. The implementation of the algorithm relies on closed-form filter update expressions obtained from approximation formulas for corrector terms and Janossy densities as described in subsection 5.2.1. In subsection 5.3.2, we present the numerical simulation results of the Determinantal PHD filter in some specific simulated environments.

Chapter 2

Preliminaries

In this chapter, we present to the reader a review of the mathematical basis and first principles of the PHD filters for random variables and spatial point processes. Fundamental operations and properties associated with discrete random variables and spatial point processes are covered in this chapter. In Section 2.1, we focus on the notions about probability generating functions (PGFs), factorial moments and branching processes that constitute the fundamental basis to the construction of the integer-valued PHD filter that will be introduced in Section 3.1 of Chapter 3. In Section 2.2, we recall some useful properties about spatial point processes, the notion of probability generating functionals (PGFls) as a generalization to PGFs, and its associated relationship with factorial moment and Janossy densities. In addition, we review some operations that involve point processes as well as a special type of dependence between two point processes known as branching of point processes. These concepts associated with spatial point processes form the foundations to construct the spatial point process PHD filter that will be described in Section 3.2 of Chapter 3.

2.1 Operations of Integer-Valued Random Variables

In this section, we give a review of PGFs, factorial moments and branching processes that are associated with discrete random variables.

2.1.1 Probability Generating Functions (PGFs)

In the following, we review the definitions, operations and properties of Probability Generating Functions (PGFs).

Definition 2.1.1. Given a discrete random variable (r.v) X , we denote the cardinality distribution of X as $\{p_X(n)\}_{n \geq 0}$, where $p_X(n) = \mathbb{P}(X^{-1}(n))$ is the probability that $X = n$. Note that we have $\sum_{n=0}^{\infty} p_X(n) = 1$.

Definition 2.1.2. For a r.v X with cardinality distribution $\{p_X(n)\}_{n \geq 0}$, and, for any real variable u such that $0 \leq u \leq 1$, the probability generating function (PGF) is given by

$$G_X(u) = \mathbb{E}[u^X] = \sum_{n=0}^{\infty} p_X(n)u^n. \quad (2.1)$$

By setting $u = 0$,

$$G_X(0) = \sum_{n=0}^{\infty} p_X(n)0^n = p_X(0).$$

By setting $u = 1$,

$$G_X(1) = \sum_{n=0}^{\infty} p_X(n)1^n = \sum_{n=0}^{\infty} p_X(n) = 1.$$

In addition, since each $p_X(n) \geq 0$ and $0 \leq G_X(1) \leq 1$, the PGF of X is uniformly convergent and infinitely differentiable for $0 \leq u < 1$.

The r^{th} derivative of $G_X(u)$ is given by

$$\begin{aligned} G_X^{(r)}(u) &= \sum_{n=r}^{\infty} n(n-1) \cdots (n-r+1) p_X(n) u^{n-r} \\ &= \sum_{n=r}^{\infty} (n)_r p_X(n) u^{n-r}, \end{aligned}$$

where $(n)_r$ denotes the falling factorial. By convention, $(n)_0 = 1$ as it is an empty product, and when $n = r$, we have $(r)_r = r!$.

Remark 2.1.3. By setting $u = 0$ and $u = 1$ respectively in the r^{th} derivative of $G_X(u)$, we have the following standard results $p_X(r) = \frac{G_X^{(r)}(0)}{r!}$ and $G_X^{(r)}(1) = \mathbb{E}[(X)_r]$. In other words, the PGF of a r.v X uniquely extracts the cardinality probabilities and recovers the factorial moments through the applications of their derivatives.

Joint Probability Generating Functions

Definition 2.1.4. For r.v X, Y , with cardinality distributions given by $\{p_X(m)\}_{m \geq 0}$ and $\{p_Y(n)\}_{n \geq 0}$ respectively, and, for any real variables $0 \leq u, v \leq 1$ that are associated with X and Y respectively, their joint PGF is given by

$$G_{X,Y}(u, v) = \mathbb{E}[u^X v^Y] = \sum_{m,n=0}^{\infty} p_{X,Y}(m, n) u^m v^n. \quad (2.2)$$

When $u, v = 0$,

$$G_{X,Y}(0, 0) = p_{X,Y}(0, 0).$$

When $u, v = 1$,

$$G_{X,Y}(1, 1) = \sum_{m,n=0}^{\infty} p_{X,Y}(m, n) = 1.$$

Lemma 2.1.5. Consider the realization $X = m$, where $m \in \mathbb{N}_0$, the associated joint PGF is given by

$$G_{X=m,Y}(u, v)|_{u=0} = \frac{1}{m!} \frac{\partial^m}{\partial u^m} G_{X,Y}(u, v)|_{u=0}.$$

Remark 2.1.6. Consider the realizations $X = m$ and $Y = n$, where $m, n \in \mathbb{N}_0$. Then, the joint cardinality probability $p_{X,Y}(m, n)$ is given by

$$\begin{aligned} p_{X,Y}(m, n) &= G_{X=m,Y=n}(u, v)|_{u,v=0} \\ &= \frac{1}{m!} \frac{1}{n!} \frac{\partial^m}{\partial u^m} \frac{\partial^n}{\partial v^n} G_{X,Y}(u, v)|_{u,v=0}. \end{aligned}$$

Independent Random Variables

In the case where both X and Y are independent r.v, their joint PGF is given by

$$G_{X,Y}(u, v) = G_X(u)G_Y(v).$$

Superposition of Random Variables

For a r.v Z with cardinality probability distribution $\{p_Z(n)\}_{n \geq 0}$, such that $Z = X + Y$, where X and Y are independent r.v with known PGFs G_X and G_Y , the PGF $G_Z(s)$ is given by

$$G_Z(u) = G_X(u)G_Y(u).$$

Marginalization of Random Variables

For any $n \in \mathbb{N}_0$, the marginalization of Y over X in $\{p_{X,Y}(m, n)\}_{m, n \geq 0}$ is given by

$$p_Y(n) = \sum_{m=0}^{\infty} p_{X,Y}(m, n).$$

In addition, the marginalization of Y over X based on the joint PGF $G_{X,Y}(u, v)$ is given by

$$G_{X,Y}(u, v)|_{u=1} = G_Y(v). \quad (2.3)$$

2.1.2 Factorial Moments

In the following, we give a review of the relationship associated with nonfactorial and factorial moments of discrete random variables.

Definition 2.1.7. For any $r \in \mathbb{N}_0$, the r^{th} order nonfactorial moment of r.v X is given by

$$\mu_X^{(r)} = \mathbb{E}[X^r] = \sum_{n=0}^{\infty} n^r p_X(n).$$

The r^{th} order factorial moment of X is given by

$$\begin{aligned} \rho_X^{(r)} &= \mathbb{E}[(X)_r] \\ &= \mathbb{E}[X(X-1) \cdots (X-r+1)] \\ &= \sum_{n=r}^{\infty} n(n-1) \cdots (n-r+1) p_X(n) \\ &= \sum_{n=r}^{\infty} (n)_r p_X(n). \end{aligned}$$

The Stirling Numbers of The Second Kind

The Stirling number of the second kind is an important link to bridge the relationship between nonfactorial and factorial moments, see (2.5) and (2.50). In the following, we state the formal definition and give a review of its applications and properties.

Definition 2.1.8. *The Stirling numbers of the second kind (or Stirling partition number), classically denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$, counts the number of ways to partition a set of n elements into k nonempty subsets, and the formula is given by*

$$\begin{aligned} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} &= \frac{1}{k!} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i^n \\ &= \frac{1}{k!} \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n. \end{aligned} \quad (2.4)$$

The following are some initial conditions of the *Stirling numbers of the second kind*, where $n \in \mathbb{N}_0$,

$$\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\} = 0, \left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\} = 1, \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\} = 1 \text{ and } \left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\} = 1.$$

The Bell numbers, denoted by B_n , counts the total number of partitions of a set of n elements into any positive integer $k \leq n$ of non-empty subsets,

$$B_n = \sum_{k=1}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}.$$

In addition, by the following classical expression,

$$\begin{aligned} x^n &= \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} (x)_k \\ &= \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} x(x-1) \cdots (x-k+1), \end{aligned}$$

the r^{th} order nonfactorial moment of r.v X can be expressed in terms of its preceding factorial moments as

$$\mu_X^{(r)} = \sum_{k=0}^r \left\{ \begin{smallmatrix} r \\ k \end{smallmatrix} \right\} \rho_X^{(k)}. \quad (2.5)$$

Remark 2.1.9. *Note that the first order nonfactorial moment of X is the same as the first order factorial moment of X (see [3]), which is given by*

$$\mu_X^{(1)} = \sum_{n=0}^{\infty} n p_X(n) = \rho_X^{(1)}.$$

The second order nonfactorial moment of X is given by

$$\mu_X^{(2)} = \sum_{n=0}^{\infty} n^2 p_X(n) = \mu_X^{(1)} + \rho_X^{(2)}.$$

The variance of X is given by

$$v_X = \mu_X^{(2)} - (\mu_X^{(1)})^2.$$

The third order nonfactorial moment of X is given by

$$\mu_X^{(3)} = \sum_{n=0}^{\infty} n^3 p_X(n) = \mu_X^{(1)} + 3\rho_X^{(2)} + \rho_X^{(3)}.$$

For any r.v X, Y such that $X : (\Omega_X, \mathcal{F}_X, \mathbb{P}_X) \rightarrow \mathbb{N}_0$ and $Y : (\Omega_Y, \mathcal{F}_Y, \mathbb{P}_Y) \rightarrow \mathbb{N}_0$, with cardinality distributions given by $\{p_X(m)\}_{m \geq 0}$ and $\{p_Y(n)\}_{n \geq 0}$ respectively. By (2.5), the r^{th} order joint nonfactorial moment of X and Y is given by

$$\mu_{X,Y}^{(r)} = \mathbb{E}[(XY)^r] = \sum_{m,n=0}^{\infty} p_{X,Y}(m,n)(mn)^r.$$

In addition, their covariance between X and Y is given by

$$c_{X,Y} = \mu_{X,Y}^{(1)} - \mu_X^{(1)} \mu_Y^{(1)}.$$

2.1.3 Branching Processes

Definition 2.1.10. *Ancestor-Descendant Random Variables.*

Suppose the r.v X and Y have dependency of an ancestor-descendant relationship, where X yields Y . Then, X is known as the ascestor r.v and Y is the descendant r.v.

The branching process of X is described as, for any realization $X = m$, the conditional r.v $Y|X = m$ is a superposition of m independent and identical r.v Z that collectively yield Y , that is, $Y = \sum_{i=1}^m Z_i$. The conditional PGF is given by

$$G_{Y|X}(v|m) = (G_Z(v))^m \tag{2.6}$$

where G_Z is the PGF of r.v Z .

Proposition 2.1.11. *Consider the r.v X and Y such that X is the ancestor r.v and Y is the descendant r.v. Then, the joint PGF of X and Y is given by*

$$G_{X,Y}(u, v) = G_X(uG_Z(v)), \quad (2.7)$$

where $G_{X,Y}(u, v)$ translates into the composition of the PGF describing the branching process of X .

Remark 2.1.12. *As a consequence of (2.3), the marginalization of the joint PGF $G_{X,Y}(u, v)$ over X is given by*

$$\begin{aligned} G_Y(v) &= G_{X,Y}(u, v)|_{u=1} \\ &= G_X(uG_Z(v))|_{u=1} \\ &= G_X(G_Z(v)). \end{aligned} \quad (2.8)$$

Bernoulli Case

A Bernoulli random variable X is a basic integer-valued random variable. It is characterized with a probability parameter $0 \leq p \leq 1$ such that

$$\begin{aligned} p_X(0) &= 1 - p \\ p_X(1) &= p \\ p_X(n) &= 0, \quad \forall n \geq 2. \end{aligned}$$

From (2.1), the associated PGF $G_X(u)$ is given by

$$\begin{aligned} G_X(u) &= \sum_{n=0}^{\infty} p_X(n)u^n \\ &= p_X(0) + p_X(1)u + \sum_{n=2}^{\infty} p_X(n)u^n \\ &= 1 - p + pu. \end{aligned} \quad (2.9)$$

The first order moment of X is given by

$$\mu_X^{(1)} = G_X^{(1)}(u)|_{u=1} = p. \quad (2.10)$$

The r^{th} order factorial moment is given by

$$\rho_X^{(r)} = G_X^{(r)}(u)|_{u=1} = 0, \quad \forall r \geq 2.$$

The variance of X is given by

$$\begin{aligned} v_X &= \mu_X^{(2)} - (\mu_X^{(1)})^2 \\ &= (\mu_X^{(1)} + \rho_X^{(2)}) - (\mu_X^{(1)})^2 \\ &= p(1 - p). \end{aligned} \tag{2.11}$$

Poisson Random Variable

A Poisson random variable X is another basic integer-valued random variable. It is characterized with a rate parameter $\lambda_X \geq 0$ such that

$$p_X(n) = e^{-\lambda_X} \frac{\lambda_X^n}{n!}, \quad \forall n \geq 0. \tag{2.12}$$

The associated PGF $G_X(u)$ is given by

$$G_X(u) = \sum_{n=0}^{\infty} e^{-\lambda_X} \frac{\lambda_X^n}{n!} u^n = e^{\lambda_X(u-1)}. \tag{2.13}$$

A standard property that characterizes the Poisson random variable is such that the first order moment $\mu_X^{(1)}$ and the variance of X are the same as its rate parameter λ_X , hence we have

$$\mu_X^{(1)} = v_X = \lambda_X. \tag{2.14}$$

The r^{th} order factorial moment of X is given by

$$\rho_X^{(r)} = G_X^{(r)}(u)|_{u=1} = \lambda_X^r, \quad \forall r \geq 2. \tag{2.15}$$

2.2 Functionals and Operations of Spatial Point Processes

2.2.1 Construction and Properties

Consider a locally compact second countable Hausdorff space $\mathbb{X}$ endowed with its Borel σ -algebra $\mathbb{X}_\sigma$. For any subset $S \subseteq \mathbb{X}$, let $|S|$ denote the cardinality of S . In the case where S is not finite, $|S| = \infty$. On the other hand, for the case where S is finite, the set of locally finite point configurations on $\mathbb{X}$ is defined by

$$\mathfrak{N} := \{\varphi \subseteq \mathbb{X} : |\varphi \cap S| < \infty, \quad \forall \text{ relatively compact sets } S \subset \mathbb{X}\}.$$

The corresponding Borel σ -algebra of $\mathfrak{N}$ is denoted by $\mathfrak{N}_\sigma$. And, for any set $S \subseteq \mathbb{X}$, we write $\mathfrak{N}(S) := \{\varphi \in \mathfrak{N} : \varphi \subseteq S\}$. More generally, $\mathfrak{N}$ can be described as the set of all simple nonnegative integer-valued Radon measures on $\mathbb{X}$. The term simple is defined on an integer-valued Radon measure ϱ on $\mathbb{X}$ if for any $x \in \mathbb{X}$, $\varrho(\{x\}) \in \{0, 1\}$. For each configuration $\varphi \in \mathfrak{N}$, the integer-valued Radon measure on $\mathbb{X}_\sigma$ is given by $\sum_{x \in \varphi} \delta_x$. In the context of topology, $\mathfrak{N}$ is naturally topologized by the vague topology, that is, the weakest topology such that for any continuous and compactly supported function f on $\mathbb{X}$, the mapping $\varphi \mapsto \langle f, \varphi \rangle := \sum_{x \in \varphi} f(x)$ is continuous. For $\varphi \in \mathfrak{N}$, in order to represent an addition of a point at x_0 , we denote it as $\varphi \cup x_0 = \varphi \cup \{x_0\}$, and for a removal of a point at x_0 , we denote it as $\varphi \setminus x_0 = \varphi \setminus \{x_0\}$. The set of finite point configurations on $\mathbb{X}$ is defined by

$$\mathfrak{N}^f := \{\varphi \subseteq \mathbb{X} : |\varphi| < \infty\},$$

for which $\mathfrak{N}^f$ is endowed with the trace σ -algebra $\mathfrak{N}_\sigma^f = \mathfrak{N}_\sigma|_{\mathfrak{N}^f}$. For any Radon measure ν on $\mathbb{X}$, and for any measurable function $f : \mathfrak{N}^f \rightarrow \mathbb{R}^+$, the ν -sample measure $\mathfrak{L}^\nu$ on $(\mathfrak{N}^f, \mathfrak{N}_\sigma^f)$ is defined by the identity

$$\begin{aligned} \int_{\mathfrak{N}^f} f(\varphi) \mathfrak{L}^\nu(d\varphi) &:= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{X}^n} f(x_1, \dots, x_n) \nu(dx_1) \cdots \nu(dx_n) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{X}^n} f(x_{1:n}) \nu(dx_{1:n}), \end{aligned}$$

where $f_n(x_{1:n}) = f(x_1, \dots, x_n)$ and $\nu(dx_{1:n}) = \nu(dx_1) \cdots \nu(dx_n)$. By imposing a restriction to any relatively compact subset $\Lambda \subseteq \mathbb{X}$, let $\mathfrak{N}(\Lambda)$ be the set of all finite configurations on Λ , together with its corresponding trace σ -algebra denoted by $\mathfrak{N}_\sigma(\Lambda)$. In addition, let $\mathfrak{L}_\Lambda^\nu(d\varphi) = \mathbf{1}_{\{\varphi \subseteq \Lambda\}} \mathfrak{L}^\nu(d\varphi)$, where $\mathbf{1}_A$ is the indicator function of a set A . For any measurable function $f : \mathfrak{N}(\Lambda) \rightarrow \mathbb{R}^+$, the μ -sample measure $\mathfrak{L}_\Lambda^\nu$ on $(\mathfrak{N}(\Lambda), \mathfrak{N}_\sigma(\Lambda))$ is defined by

$$\int_{\mathfrak{N}(\Lambda)} f(\varphi) \mathfrak{L}_\Lambda^\nu(d\varphi) := \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} f(x_{1:n}) \nu(dx_{1:n}).$$

Point Process

Definition 2.2.1. A simple and locally finite point process Φ defined on $\mathbb{X}$ with probability space $(\Omega, \Omega_\sigma, \mathbb{P})$ is a measurable mapping given by

$$\Phi : (\Omega, \Omega_\sigma, \mathbb{P}) \rightarrow (\mathfrak{N}, \mathfrak{N}_\sigma).$$

The probability distribution of Φ is given by

$$P_\Phi(\varphi) = \mathbb{P}(\{\omega \in \Omega : \Phi(\omega) \in \varphi\}) = \mathbb{P}(\Phi^{-1}(\varphi)),$$

where $\varphi \in \mathfrak{N}_\sigma$. In addition, we have $\int_{\mathfrak{N}_\sigma} P_\Phi(d\varphi) = 1$.

Counting Measure

Definition 2.2.2. For any relatively compact subset $\Lambda \subseteq \mathbb{X}$, and, for any simple and locally finite subset $\varphi \in \mathfrak{N}$, the associated random measure, denoted by $\Phi(\Lambda)$, is a counting measure given by

$$\Phi(\varphi \cap \Lambda) = \sum_{x \in \varphi} \mathbf{1}_\Lambda(x).$$

A point process Φ is equipped with a factorial moment density function $\rho_\Phi : \mathfrak{N}^f \rightarrow \mathbb{R}^+$ with respect to a Radon measure ν on $(\mathbb{X}, \mathbb{X}_\sigma)$ if ρ_Φ is measurable and for any measurable nonnegative function $g : \mathfrak{N}^f \rightarrow \mathbb{R}^+$, we can write

$$\begin{aligned} \int_{\mathfrak{N}} \sum_{\alpha \subset \varphi, \alpha \in \mathfrak{N}^f} g(\alpha) P_\Phi(d\varphi) &= \int_{\mathfrak{N}^f} g(\alpha) \rho_\Phi(\alpha) \mathfrak{L}^\mu(d\alpha) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{X}^n} g_n(x_{1:n}) \rho_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}). \end{aligned}$$

Suppose $\alpha = \{x_1, \dots, x_n\}$, where $n \geq 1$, the n^{th} order factorial moment density function is given by $\rho_\Phi(\alpha) = \rho_\Phi^{(n)}(x_{1:n})$. In fact, under this setting, $\rho_\Phi^{(n)}$ is a symmetric function on $\mathbb{X}^n$. In other words, when the factorial moment density functions of Φ with respect to a Radon measure ν on $\mathbb{X}$ exist, the factorial moment density functions are measurable symmetric functions given by $\rho_\Phi^{(n)} : \mathbb{X}^n \rightarrow \mathbb{R}^+$ and possess the given expression

$$\mathbb{E} \left[\prod_{i=1}^n \Phi(B_i) \right] = \int_{B_1 \times \dots \times B_n} \rho_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}),$$

where $\{B_i\}_{i=1}^n$, $n \geq 1$ denotes any family of mutually disjoint bounded subsets of $\mathbb{X}$. The factorial moment density function ρ_Φ translates into a point density with respect to ν when $n = 1$, that is, $\rho_\Phi^{(1)}(x_1)$. In general, $\rho_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n})$ gives the probability of locating the respective point configurations in the vicinity of each x_i , $i \in \{1, \dots, n\}$. The following proposition is an extension of the above expression.

Proposition 2.2.3 (Proposition 2.1 of [4]). *Consider a family of mutually disjoint bounded subsets $\{B_i\}_{i=1}^n$ of $\mathbb{X}$, let $\nu_1, \dots, \nu_n \in \mathbb{Z}_0^+$ such that $\sum_{i=1}^n \nu_i = N$. Then,*

$$\mathbb{E} \left[\prod_{i=1}^n \frac{\Phi(B_i)!}{(\Phi(B_i) - \nu_i)!} \right] = \int_{B_1^{\nu_1} \times \dots \times B_n^{\nu_n}} \rho_{\Phi}^{(N)}(x_{1:N}) \nu(dx_{1:N}).$$

In addition, $\rho_{\Phi}^{(N)}(x_{1:N}) = 0$ whenever $x_i = x_j$ for some $i \neq j$.

For any relatively compact subset $\Lambda \subseteq \mathbb{X}$, the Janossy densities of a point process Φ with respect to a Radon measure ν on Λ are measurable functions which are given by $j_{\Phi}^{(n)} : \Lambda^n \rightarrow \mathbb{R}^+$, for $n \geq 1$. In addition, for any integrable random variable F given by

$$\begin{aligned} F(\Phi) &= f_0 \mathbf{1}_{\{\Phi(\Lambda)=0\}} + \sum_{n=1}^{\infty} \mathbf{1}_{\{\Phi(\Lambda)=n\}} f_n(X_1, \dots, X_n) \\ &= f_0 \mathbf{1}_{\{\Phi(\Lambda)=0\}} + \sum_{n=1}^{\infty} \mathbf{1}_{\{\Phi(\Lambda)=n\}} f_n(X_{1:n}), \end{aligned}$$

where f_n is a symmetric function on Λ^n , for each $n \geq 1$. Then, we have

$$\mathbb{E}[F(\Phi)] = f_0 j_{\Phi}^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} f_n(x_{1:n}) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}). \quad (2.16)$$

Given a functional $\mathcal{F}$, we will use the functional derivative $\frac{\partial_g}{\partial h} \mathcal{F}(h)$ defined as

$$\frac{\partial_g}{\partial h} \mathcal{F}(h) := \lim_{\epsilon \rightarrow 0} \frac{\mathcal{F}(h + \epsilon g) - \mathcal{F}(h)}{\epsilon}, \quad g \in L^\infty(\Lambda).$$

Given a point $x \in \Lambda$, we also let

$$\frac{\partial_{\delta_x}}{\partial h} \mathcal{F}(h) := \lim_{n \rightarrow \infty} \frac{\partial_{g_n}}{\partial h} \mathcal{F}(h),$$

where $(g_n)_{n \geq 1}$ is a sequence of bounded functions converging weakly to the Dirac measure δ_x at $x \in \Lambda$. In the subsequent part of the thesis, we work with the term Dirac delta function $\delta_x(\cdot)$ which follows the same convention and notation as in [18].

The Janossy densities of a point process possess the symmetric property, that is, for every permutation $\sigma(n)$ of $\{1, \dots, n\}$, we have $j_{\Phi}^{(n)}(x_{\sigma(1):\sigma(n)}) = j_{\Phi}^{(n)}(x_{1:n})$. Next, for every relatively compact subset $\Lambda \subseteq \mathbb{X}$, we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}) = 1. \quad (2.17)$$

Proposition 2.2.4 (see Theorem 5.4II of [1]). *The relationship between the expressions involving the correlation functions and the Janossy densities is given by the identities*

$$j_{\Phi}^{(n)}(x_{1:n}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{\Lambda^m} \rho_{\Phi}^{(n+m)}(x_{1:n}, y_{1:m}) \nu(dy_{1:m}), \quad (2.18)$$

$$\rho_{\Phi}^{(n)}(x_{1:n}) = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_{\Phi}^{(n+m)}(x_{1:n}, y_{1:m}) \nu(dy_{1:m}). \quad (2.19)$$

Janossy Measure

Definition 2.2.5. *Let $p_{\Phi}^{(n)}(x_{1:n})$ be the joint probability density of the point configuration α given that the point process Φ has exactly $|\alpha|$ points. Then, the Janossy measure $J_{\Phi}^{(n)}$ is given by*

$$\begin{aligned} J_{\Phi}^{(n)}(\Lambda^n) &= \int_{\Lambda^n} j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\ &= \sum_{\sigma(n)} \int_{\Lambda^n} p_{\Phi}^{(n)}(x_{\sigma_1:\sigma_n}) \nu(dx_{\sigma_1:\sigma_n}) \\ &= n! \int_{\Lambda^n} p_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}). \end{aligned}$$

For $n = 0$, $j_{\Phi}^{(0)}(\emptyset)$ is the probability of no points in Λ . For $n \geq 0$, the Janossy densities $j_{\Phi}^{(n)}(x_{1:n}) = n! p_{\Phi}^{(n)}(x_{1:n})$. It is important to note that the above expression provides a mean to describe the joint density of the point configuration α given that the point process Φ has exactly $|\alpha|$ points.

The equivalence for (2.16) is given by

$$\begin{aligned} \mathbb{E}[F(\Phi)] &= f_0 j_{\Phi}^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} f_n(x_{1:n}) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\ &= f_0 p_{\Phi}^{(0)} + \sum_{n=1}^{\infty} \int_{\Lambda^n} f_n(x_{1:n}) p_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}). \end{aligned}$$

Laplace Transform

Definition 2.2.6. *For any relatively compact subset $\Lambda \subseteq \mathbb{X}$, and for any measurable nonnegative function g on Λ , the point process Φ is characterized by its Laplace*

transform, which is explicitly given by

$$\begin{aligned}
\mathcal{L}_\Phi(g) &= \int_{\mathfrak{N}(\Lambda)} e^{-\langle g, \varphi \rangle} \mathfrak{L}_\Lambda^\mu(d\varphi) \\
&= \int_{\mathfrak{N}(\Lambda)} e^{-\sum_{x \in \varphi} g(x)} \mathfrak{L}_\Lambda^\mu(d\varphi) \\
&= e^{-g(\emptyset)} j_\Phi^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} e^{-\sum_{i=1}^n g(x_i)} j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&= e^{-g(\emptyset)} j_\Phi^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{i=1}^n e^{-g(x_i)} \right) j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&= j_\Phi^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{i=1}^n e^{-g(x_i)} \right) j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}). \tag{2.20}
\end{aligned}$$

Notably, for consistency purpose, it is understood that $g(\emptyset) = 0$, as such the term $e^{-g(\emptyset)} = 1$.

First Order Moment Density

Proposition 2.2.7. *For $\varphi = \{x_1, \dots, x_n\} \in \mathfrak{N}$, and consider the function $f_n(x_{1:n}) = \sum_{i=1}^n \delta_x(x_i)$ as in (2.16), where δ_x is the Dirac delta function at the point x , the first order moment density, denoted as $\mu_\Phi^{(1)}(x)$, is given by*

$$\mu_\Phi^{(1)}(x) = j_\Phi^{(1)}(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_\Phi^{(n+1)}(x, x_{1:n}) \nu(dx_{1:n}). \tag{2.21}$$

Proof. By substituting the function $f_n(x_{1:n})$ with $\sum_{i=1}^n \delta_x(x_i)$ in (2.16) and that $\delta_x(\emptyset) = 0$, we have

$$\begin{aligned}
\mu_\Phi^{(1)}(x) &= \delta_x(\emptyset) j_\Phi^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{i=1}^n \delta_x(x_i) \right) j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{i=1}^n \delta_x(x_i) \right) j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^{n-1}} n j_\Phi^{(n)}(x, x_{1:n-1}) \nu(dx_{1:n-1}) \\
&= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_{\Lambda^{n-1}} j_\Phi^{(n)}(x, x_{1:n-1}) \nu(dx_{1:n-1})
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_{\Phi}^{(n+1)}(x, x_{1:n}) \nu(dx_{1:n}) \\
&= j_{\Phi}^{(1)}(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_{\Phi}^{(n+1)}(x, x_{1:n}) \nu(dx_{1:n}).
\end{aligned}$$

□

Campbell's Theorem

The following is an essential result in point process theory that allows the expectation formula to be expressed as the integral with the first order moment density.

Proposition 2.2.8. *Consider a point process Φ . For any continuous, compactly support, nonnegative measurable and symmetric function f on $\mathbb{X}$ such that for any $\varphi \in \mathfrak{N}$, we have $\sum_{x \in \varphi} f(x)$. For any relatively compact subset $\Lambda \subseteq \mathbb{X}$, the Campbell's theorem is given by*

$$\mathbb{E} \left[\sum_{x \in \varphi} f(x) \right] = \int_{\Lambda} f(x) \mu_{\Phi}^{(1)}(x) \nu(dx). \quad (2.22)$$

Proof. From (2.16), we recall that

$$\mathbb{E}[F(\Phi)] = f_0 j_{\Phi}^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} f_n(x_{1:n}) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}).$$

By substituting $f_n(x_{1:n})$ with $\sum_{i=1}^n f(x_i)$, we have

$$\begin{aligned}
\mathbb{E} \left[\sum_{x \in \varphi} f(x) \right] &= f(\emptyset) j_{\Phi}^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{i=1}^n f(x_i) \right) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{i=1}^n \int_{\Lambda^n} f(x_i) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}) \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \left(n \int_{\Lambda^n} f(x_1) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}) \right) \\
&= \int_{\Lambda} f(x) \left(\sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_{\Lambda^{n-1}} j_{\Phi}^{(n)}(x, x_{1:n-1}) \nu(dx_{1:n-1}) \right) \nu(dx) \\
&= \int_{\Lambda} f(x) \left(\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_{\Phi}^{(n+1)}(x, x_{1:n}) \nu(dx_{1:n}) \right) \nu(dx)
\end{aligned}$$

$$\begin{aligned}
&= \int_{\Lambda} f(x) \left(j_{\Phi}^{(1)}(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_{\Phi}^{(n+1)}(x, x_{1:n}) \nu(dx_{1:n}) \right) \nu(dx) \\
&= \int_{\Lambda} f(x) \mu_{\Phi}^{(1)}(x) \nu(dx).
\end{aligned}$$

□

2.2.2 Probability Generating Functionals (PGFLs)

The notion of PGFL is an important mathematical expression that characterize a point process. In the following, we review the definition of PGFL and its associated properties.

Definition 2.2.9. *The probability generating functional (PGFL), denoted $\mathcal{G}_{\Phi}$, of a point process Φ with Janossy density j_{Φ} is given by*

$$\begin{aligned}
\mathcal{G}_{\Phi}(h) &= \mathbb{E} \left[\prod_{i=1}^{\Phi(\Lambda)} h(X_i) \right] \\
&= h(\emptyset) j_{\Phi}^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{i=1}^n h(x_i) \right) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&= j_{\Phi}^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{i=1}^n h(x_i) \right) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}),
\end{aligned}$$

where $h \in L^{\infty}(\Lambda)$ is a bounded measurable function on Λ .

In a similar sense as in the Laplace transform, for consistency purposes, it is understood that $h(\emptyset) = 1$.

The following is an identity between the PGFL and the Laplace transform of a point process Φ when we replace $h = e^{-g}$, which is given by

$$\begin{aligned}
\mathcal{G}_{\Phi}(h) &= h(\emptyset) j_{\Phi}^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{i=1}^n h(x_i) \right) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&= e^{-g(\emptyset)} j_{\Phi}^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{i=1}^n e^{-g(x_i)} \right) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&= \mathcal{L}_{\Phi}(g).
\end{aligned}$$

Remark 2.2.10. Suppose $h : \mathfrak{N}(\Lambda) \rightarrow [0, 1]$. By setting $h = 0$ and $h = 1$ to $\mathcal{G}_\Phi(h)$, we have $\mathcal{G}_\Phi(h)|_{h=0} = j_\Phi^{(0)}$ and

$$\mathcal{G}_\Phi(h)|_{h=1} = \mathcal{L}_\Phi(g)|_{g=0} = j_\Phi^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}) = 1. \quad (2.23)$$

Joint Probability Generating Functionals (PGFLs)

Point processes can involve joint behavior and are characterized by their joint PGFLs.

Definition 2.2.11. For two point processes Ψ and Φ , the joint PGFL of (Ψ, Φ) is given by

$$\begin{aligned} \mathcal{G}_{\Psi, \Phi}(g, h) &= \mathbb{E} \left[\prod_{m=1}^{\Psi(\Lambda)} g(X_m) \prod_{n=1}^{\Phi(\Lambda)} h(Y_n) \right] \\ &= \sum_{m \geq 0} \sum_{n \geq 0} \frac{1}{m!} \frac{1}{n!} \int_{\Lambda^m} \int_{\Lambda^n} \left(\prod_{i=0}^m g(x_i) \right) \left(\prod_{j=0}^n h(y_j) \right) j_{\Psi, \Phi}^{(m, n)}(x_{1:m}, y_{1:n}) \nu(dx_{1:m}) \nu(dy_{1:n}). \end{aligned} \quad (2.24)$$

where g is the nonnegative measurable function associated with Ψ and h is the nonnegative measurable function associated with Φ . The notation $j_{\Psi, \Phi}^{(m, n)}(x_{1:m}, y_{1:n})$ denotes the $(m, n)^{\text{th}}$ joint Janossy density of (Ψ, Φ) where $\Psi = x_{1:m} = (x_1, \dots, x_m)$ and $\Phi = y_{1:n} = (y_1, \dots, y_n)$.

Remark 2.2.12. As in (2.17), we have

$$\sum_{m \geq 0} \sum_{n \geq 0} \frac{1}{m!} \frac{1}{n!} \int_{\Lambda^m} \int_{\Lambda^n} j_{\Psi, \Phi}^{(m, n)}(x_{1:m}, y_{1:n}) \nu(dx_{1:m}) \nu(dy_{1:n}) = 1.$$

In addition, by the partial derivatives evaluated in the directions of corresponding nonnegative measurable functions g and h with Dirac delta functions $\delta_{x_1}, \dots, \delta_{x_m}, \delta_{y_1}, \dots, \delta_{y_n}$, we have

$$j_{\Psi, \Phi=y_{1:n}}^{(m, n)}(x_{1:m}) := j_{\Psi, \Phi}^{(m, n)}(x_{1:m}, y_{1:n}) = \frac{\partial_{\delta_{x_1}}}{\partial g} \dots \frac{\partial_{\delta_{x_m}}}{\partial g} \frac{\partial_{\delta_{y_1}}}{\partial h} \dots \frac{\partial_{\delta_{y_n}}}{\partial h} \mathcal{G}_{\Psi, \Phi}(g, h)|_{g, h=0}. \quad (2.25)$$

The functional derivatives of PGFL will be discussed in the later part of this chapter.

The joint PGFl $\mathcal{G}_{\Psi, \Phi}(g, h)$ can also be expressed as

$$\begin{aligned}
& \mathcal{G}_{\Psi, \Phi}(g, h) \\
&= g(\emptyset)h(\emptyset)j_{\Psi, \Phi}^{(0)} + \sum_{n \geq 1} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{i=1}^n h(y_i) \right) j_{\Psi, \Phi}^{(n)}(\emptyset, y_{1:n}) \nu(dy_{1:n}) \\
&+ \sum_{m \geq 1} \frac{1}{m!} \int_{\Lambda^m} \left(\prod_{i=1}^m g(x_i) \right) j_{\Psi, \Phi}^{(m)}(x_{1:m}, \emptyset) \nu(dx_{1:m}) \\
&+ \sum_{m \geq 1} \sum_{n \geq 1} \frac{1}{m!} \frac{1}{n!} \int_{\Lambda^m} \int_{\Lambda^n} \left(\prod_{i=1}^m g(x_i) \right) \left(\prod_{j=1}^n h(y_j) \right) j_{\Psi, \Phi}^{(m,n)}(x_{1:m}, y_{1:n}) \nu(dx_{1:m}) \nu(dy_{1:n}).
\end{aligned} \tag{2.26}$$

When the two point processes, Ψ and Φ , are independent of one another, we have

$$\begin{aligned}
\mathcal{G}_{\Psi, \Phi}(g, h) &= \mathbb{E} \left[\prod_{m=1}^{\Psi(\Lambda)} g(X_m) \prod_{n=1}^{\Phi(\Lambda)} h(Y_n) \right] \\
&= \mathbb{E} \left[\prod_{m=1}^{\Psi(\Lambda)} g(X_m) \right] \mathbb{E} \left[\prod_{n=1}^{\Phi(\Lambda)} h(Y_n) \right] \\
&= \mathcal{G}_{\Psi}(g) \mathcal{G}_{\Phi}(h).
\end{aligned}$$

Superposition of Point Processes

Lemma 2.2.13. *Consider a point process $\Xi = \Phi_1 \cup \dots \cup \Phi_N$ where $\Phi_1, \dots, \Phi_N$ are N independent point processes with known PGFls, denoted by $\mathcal{G}_{\Phi_1}, \dots, \mathcal{G}_{\Phi_N}$. The point process Ξ is a superposition of point processes $\Phi_1, \dots, \Phi_N$ with its PGFl given by*

$$\mathcal{G}_{\Xi}(h) = \mathbb{E} \left[\prod_{x \in \Xi} h(x) \right] = \mathcal{G}_{\Phi_1}(h) \cdots \mathcal{G}_{\Phi_N}(h). \tag{2.27}$$

Proof. For a nonnegative measurable function h that corresponds to the point process $\Xi = \Phi_1 \cup \dots \cup \Phi_N$, where $\Phi_1, \dots, \Phi_N$ are N independent point processes with known PGFls denoted $\mathcal{G}_{\Phi_1}, \dots, \mathcal{G}_{\Phi_N}$, the PGFl of Ξ is given by

$$\begin{aligned}
\mathcal{G}_{\Xi}(h) &= \mathbb{E} \left[\prod_{x \in \Xi} h(x) \right] \\
&= \mathbb{E} \left[\prod_{x \in \Phi_1 \cup \dots \cup \Phi_N} h(x) \right]
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E} \left[\prod_{x_1 \in \Phi_1} h(x_1) \cdots \prod_{x_N \in \Phi_N} h(x_N) \right] \\
&= \mathbb{E} \left[\prod_{x_1 \in \Phi_1} h(x_1) \right] \cdots \mathbb{E} \left[\prod_{x_N \in \Phi_N} h(x_N) \right] \\
&= \mathcal{G}_{\Phi_1}(h) \cdots \mathcal{G}_{\Phi_N}(h).
\end{aligned}$$

□

Marginalization of Point Processes

By marginalization of the joint Janossy densities, we have

$$j_{\Psi}^{(m)}(x_{1:m}) = \sum_{n \geq 0} \int_{\Lambda^n} \frac{1}{n!} j_{\Psi, \Phi}^{(m,n)}(x_{1:m}, y_{1:n}) \nu(dy_{1:n}). \quad (2.28)$$

$$j_{\Phi}^{(n)}(y_{1:n}) = \sum_{m \geq 0} \int_{\Lambda^m} \frac{1}{m!} j_{\Psi, \Phi}^{(m,n)}(x_{1:m}, y_{1:n}) \nu(dx_{1:m}). \quad (2.29)$$

Lemma 2.2.14. *The marginalization over Ψ of $\mathcal{G}_{\Psi, \Phi}(g, h)$ is given by*

$$\mathcal{G}_{\Psi, \Phi}(g, h)|_{g=1} = \mathcal{G}_{\Phi}(h). \quad (2.30)$$

The marginalization over Φ of $\mathcal{G}_{\Psi, \Phi}(g, h)$ is given by

$$\mathcal{G}_{\Psi, \Phi}(g, h)|_{h=1} = \mathcal{G}_{\Psi}(g), \quad (2.31)$$

where g corresponds to Ψ and h corresponds to Φ .

Proof. By applying (2.29) and setting $g = 1$ into (2.24), we have

$$\begin{aligned}
&\mathcal{G}_{\Psi, \Phi}(g, h)|_{g=1} \\
&= \sum_{m \geq 0} \sum_{n \geq 0} \frac{1}{m!} \frac{1}{n!} \int_{\Lambda^m} \int_{\Lambda^n} \left(\prod_{i=0}^m 1 \right) \left(\prod_{j=0}^n h(y_j) \right) j_{\Psi, \Phi}^{(m,n)}(x_{1:m}, y_{1:n}) \nu(dx_{1:m}) \nu(dy_{1:n}) \\
&= \sum_{m \geq 0} \sum_{n \geq 0} \frac{1}{m!} \frac{1}{n!} \int_{\Lambda^m} \int_{\Lambda^n} \left(\prod_{j=0}^n h(y_j) \right) j_{\Psi, \Phi}^{(m,n)}(x_{1:m}, y_{1:n}) \nu(dx_{1:m}) \nu(dy_{1:n}) \\
&= \sum_{n \geq 0} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{j=0}^n h(y_j) \right) \left(\sum_{m \geq 0} \frac{1}{m!} \int_{\Lambda^m} j_{\Psi, \Phi}^{(m,n)}(x_{1:m}, y_{1:n}) \nu(dx_{1:m}) \right) \nu(dy_{1:n})
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n \geq 0} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{j=0}^n h(y_j) \right) j_{\Phi}^{(n)}(y_{1:n}) \nu(dy_{1:n}) \\
&= \mathcal{G}_{\Phi}(h).
\end{aligned}$$

Similarly, by applying (2.28) and setting $h = 1$, we have

$$\begin{aligned}
&\mathcal{G}_{\Psi, \Phi}(g, h)|_{h=1} \\
&= \sum_{m \geq 0} \sum_{n \geq 0} \frac{1}{m!} \frac{1}{n!} \int_{\Lambda^m} \int_{\Lambda^n} \left(\prod_{i=0}^m g(x_i) \right) \left(\prod_{j=0}^n 1 \right) j_{\Psi, \Phi}^{(m, n)}(x_{1:m}, y_{1:n}) \nu(dx_{1:m}) \nu(dy_{1:n}) \\
&= \sum_{m \geq 0} \sum_{n \geq 0} \frac{1}{m!} \frac{1}{n!} \int_{\Lambda^m} \int_{\Lambda^n} \left(\prod_{i=0}^m g(x_i) \right) j_{\Psi, \Phi}^{(m, n)}(x_{1:m}, y_{1:n}) \nu(dx_{1:m}) \nu(dy_{1:n}) \\
&= \sum_{m \geq 0} \frac{1}{m!} \int_{\Lambda^m} \left(\prod_{i=0}^m g(x_i) \right) \left(\sum_{n \geq 0} \frac{1}{n!} \int_{\Lambda^n} j_{\Psi, \Phi}^{(m, n)}(x_{1:m}, y_{1:n}) \nu(dy_{1:n}) \right) \nu(dx_{1:m}) \\
&= \sum_{m \geq 0} \frac{1}{m!} \int_{\Lambda^m} \left(\prod_{i=0}^m g(x_i) \right) j_{\Psi}(x_{1:m}) \nu(dx_{1:m}) \\
&= \mathcal{G}_{\Psi}(g).
\end{aligned}$$

□

Remark 2.2.15. For any realization of finite point configuration $\Psi = x_{1:m} = (x_1, \dots, x_m)$, where $m \geq 0$, the associated joint PGFl can be expressed as

$$\begin{aligned}
\mathcal{G}_{\Psi=x_{1:m}, \Phi}(h) &= \mathcal{G}_{\Psi=x_{1:m}, \Phi}(g, h)|_{g=0} \\
&= \sum_{n \geq 0} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{i=0}^n h(y_i) \right) j_{\Psi, \Phi}^{(m, n)}(x_{1:m}, y_{1:n}) \nu(dy_{1:n}) \\
&= \frac{\partial_{\delta_{x_1}}}{\partial g} \dots \frac{\partial_{\delta_{x_m}}}{\partial g} \mathcal{G}_{\Psi, \Phi}(g, h)|_{g=0}.
\end{aligned}$$

2.2.3 Janossy and Moment Densities

In this subsection, we describe the important properties such as the Janossy densities, factorial and nonfactorial moment densities, that can be extracted from the derivatives of PGFls and the Laplace transform expressions of spatial point processes. In addition, we show the relationship between factorial and nonfactorial moment densities.

Functional Derivatives of Probability Generating Functionals (PGFLs)

In the following, we evaluate the functional derivatives of the PGFL expression to extract information about the Janossy and factorial moment densities related to the point process Φ .

First Order Functional Derivative of the PGFL

Proposition 2.2.16. *For any relatively compact subset $\Lambda \subseteq \mathbb{X}$, and for a nonnegative measurable function $f : \Lambda \rightarrow \mathbb{R}^+$, the first order functional derivative of $\mathcal{G}_\Phi(h)$, evaluated in the direction of h by a nonnegative measurable function f , is given by*

$$\frac{\partial_f}{\partial h} \mathcal{G}_\Phi(h) = \mathbb{E} \left[\sum_{i=1}^{\Phi(\Lambda)} f(X_i) \prod_{j \in \{1, \dots, \Phi(\Lambda)\} \setminus \{i\}} h(X_j) \right] \quad (2.32)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{i=1}^n f(x_i) \prod_{\substack{j=1 \\ j \neq i}}^n h(x_j) \right) j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}). \quad (2.33)$$

Proof. Recall from [3] about functional derivatives, for a functional $\mathcal{G}(h)$ of some nonnegative measurable function h , the functional derivative of $\mathcal{G}(h)$ evaluated in the direction of h with a nonnegative measurable function $f : \Lambda \rightarrow \mathbb{R}^+$, where Λ is the domain space, is given by

$$\frac{\partial_f}{\partial h} \mathcal{G}(h) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\mathcal{G}(h + \epsilon f) - \mathcal{G}(h) \right) \quad (2.34)$$

In addition, for a linear functional $\mathcal{G}(h) = \int_\Lambda h(x) u(x) \nu(dx)$, the functional derivative of $\mathcal{G}(h)$ evaluated in the direction of f for some function u on Λ is given by

$$\begin{aligned} \frac{\partial_f}{\partial h} \mathcal{G}(h) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\int_\Lambda (h(x) + \epsilon f(x)) u(x) \nu(dx) - \int_\Lambda h(x) u(x) \nu(dx) \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_\Lambda \epsilon f(x) u(x) \nu(dx) \\ &= \int_\Lambda f(x) u(x) \nu(dx). \end{aligned} \quad (2.35)$$

The first order functional derivative of $\mathcal{G}_\Phi(h)$ evaluated in the direction of h with function f is given by

$$\frac{\partial_f}{\partial h} \mathcal{G}_\Phi(h)$$

$$\begin{aligned}
&= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\mathcal{G}_\Phi(h + \epsilon f) - \mathcal{G}_\Phi(h) \right) \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\mathbb{E} \left[\prod_{i=1}^{\Phi(\Lambda)} (h(X_i) + \epsilon f(X_i)) \right] - \mathbb{E} \left[\prod_{i=1}^{\Phi(\Lambda)} h(X_i) \right] \right) \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[\prod_{i=1}^{\Phi(\Lambda)} h(X_i) + \sum_{k=1}^{\Phi(\Lambda)} \epsilon^k \sum_{\substack{\alpha \subset \{1, \dots, \Phi(\Lambda)\} \\ |\alpha|=k}} \prod_{i \in \alpha} f(X_i) \prod_{j \in \{1, \dots, \Phi(\Lambda)\} \setminus \alpha} h(X_j) - \prod_{i=1}^{\Phi(\Lambda)} h(X_i) \right] \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[\sum_{k=1}^{\Phi(\Lambda)} \epsilon^k \sum_{\substack{\alpha \subset \{1, \dots, \Phi(\Lambda)\} \\ |\alpha|=k}} \prod_{i \in \alpha} f(X_i) \prod_{j \in \{1, \dots, \Phi(\Lambda)\} \setminus \alpha} h(X_j) \right] \\
&= \mathbb{E} \left[\sum_{\substack{\alpha \subset \{1, \dots, \Phi(\Lambda)\} \\ |\alpha|=1}} \prod_{i \in \alpha} f(X_i) \prod_{j \in \{1, \dots, \Phi(\Lambda)\} \setminus \alpha} h(X_j) \right] \\
&= \mathbb{E} \left[\sum_{i=1}^{\Phi(\Lambda)} f(X_i) \prod_{j \in \{1, \dots, \Phi(\Lambda)\} \setminus \{i\}} h(X_j) \right] \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{i=1}^n f(x_i) \prod_{\substack{j=1 \\ j \neq i}}^n h(x_j) \right) j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}).
\end{aligned}$$

□

Remark 2.2.17. By setting $h = 0$ in (2.33) of Proposition 2.2.16, the first order functional derivative of $\mathcal{G}_\Phi(h)$ is given by

$$\frac{\partial_f}{\partial h} \mathcal{G}_\Phi(h)|_{h=0} = \int_{\Lambda} f(x_1) j_\Phi^{(1)}(x_1) \nu(dx_1).$$

In addition, $\frac{\partial_f}{\partial h} \mathcal{G}_\Phi(h)$ recovers (2.22) of Proposition 2.2.8 when we set $h = 1$, for which we have

$$\begin{aligned}
\frac{\partial_f}{\partial h} \mathcal{G}_\Phi(h)|_{h=1} &= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{i=1}^n f(x_i) \prod_{\substack{j=1 \\ j \neq i}}^n 1 \right) j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{i=1}^n f(x_i) \right) j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&= \int_{\Lambda} f(x) \mu_\Phi^{(1)}(x) \nu(dx)
\end{aligned} \tag{2.36}$$

$$= \mathbb{E} \left[\sum_{x \in \varphi} f(x) \right].$$

First Order Janossy and Moment Densities

Corollary 2.2.18. *By setting $h = 0$ and substituting the function f as a Dirac delta function at a point x , denoted δ_x , in (2.33) of Proposition 2.2.16, we have the first order Janossy density at a point $x \in \Lambda$,*

$$\begin{aligned} \frac{\partial_{\delta_x}}{\partial h} \mathcal{G}_{\Phi}(h)|_{h=0} &= \int_{\Lambda} \delta_x(x_1) j_{\Phi}^{(1)}(x_1) \nu(dx_1) \\ &= j_{\Phi}^{(1)}(x). \end{aligned}$$

Also, by applying (2.22) of Proposition 2.2.8 after substituting the function f with Dirac delta function δ_x and setting $h = 1$, we recover the first order moment density $\mu_{\Phi}^{(1)}(x)$,

$$\begin{aligned} \frac{\partial_{\delta_x}}{\partial h} \mathcal{G}_{\Phi}(h)|_{h=1} &= \int_{\Lambda} \delta_x(x_1) \mu_{\Phi}^{(1)}(x_1) \nu(dx_1) \\ &= \mu_{\Phi}^{(1)}(x). \end{aligned} \tag{2.37}$$

Proof. By setting $h = 0$, the computation of the first order Janossy density at a point $x \in \Lambda$ is given by

$$\begin{aligned} \frac{\partial_{\delta_x}}{\partial h} \mathcal{G}_{\Phi}(h)|_{h=0} &= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{i=1}^n \delta_x(x_i) \prod_{\substack{j=1 \\ j \neq i}}^n h(x_j) \right) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n})|_{h=0} \\ &= \int_{\Lambda} \delta_x(x_1) j_{\Phi}^{(1)}(x_1) \nu(dx_1) \\ &= j_{\Phi}^{(1)}(x). \end{aligned}$$

By using similar technique in the derivation of (2.22) of Proposition 2.2.8 and by setting $h = 1$, the computation of the first order moment density at a point $x \in \Lambda$ is given by

$$\begin{aligned} \frac{\partial_{\delta_x}}{\partial h} \mathcal{G}_{\Phi}(h)|_{h=1} &= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{i=1}^n \delta_x(x_i) \prod_{\substack{j=1 \\ j \neq i}}^n h(x_j) \right) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n})|_{h=1} \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{i=1}^n \delta_x(x_i) \prod_{\substack{j=1 \\ j \neq i}}^n 1 \right) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{i=1}^n \delta_x(x_i) \right) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i=1}^n \int_{\Lambda^n} \delta_x(x_i) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_{\Lambda^n} \delta_x(x_1) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&= \int_{\Lambda} \delta_x(y) \left(\sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_{\Lambda^{n-1}} j_{\Phi}^{(n)}(y, x_{1:n-1}) \nu(dx_{1:n-1}) \right) \nu(dy) \\
&= \int_{\Lambda} \delta_x(y) \left(\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_{\Phi}^{(n+1)}(y, x_{1:n}) \nu(dx_{1:n}) \right) \nu(dy) \\
&= \int_{\Lambda} \delta_x(y) \left(j_{\Phi}^{(1)}(y) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_{\Phi}^{(n+1)}(y, x_{1:n}) \nu(dx_{1:n}) \right) \nu(dy) \\
&= \int_{\Lambda} \delta_x(y) \mu_{\Phi}^{(1)}(y) \nu(dy) \\
&= \mu_{\Phi}^{(1)}(x).
\end{aligned}$$

□

Second Order Functional Derivative of the PGFl

Proposition 2.2.19. *The second order derivative of $\mathcal{G}_{\Phi}(h)$ in the directions of measurable nonnegative functions $f, g : \Lambda \rightarrow \mathbb{R}^+$ is given by*

$$\frac{\partial_g}{\partial h} \frac{\partial_f}{\partial h} \mathcal{G}_{\Phi}(h) = \sum_{n=2}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{\substack{i,j=1 \\ i \neq j}}^n f(x_i) g(x_j) \prod_{\substack{k=1 \\ k \neq i,j}}^n h(x_k) \right) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}). \quad (2.38)$$

Proof. To derive (2.38), we evaluate the second order derivative of $\mathcal{G}_{\Phi}(h)$ in the direction of h with measurable nonnegative functions $f, g : \Lambda \rightarrow \mathbb{R}^+$, which is given by

$$\begin{aligned}
&\frac{\partial_g}{\partial h} \frac{\partial_f}{\partial h} \mathcal{G}_{\Phi}(h) \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\frac{\partial_f}{\partial h} \mathcal{G}_{\Phi}(h + \epsilon g) - \frac{\partial_f}{\partial h} \mathcal{G}_{\Phi}(h) \right)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[\sum_{i=1}^{\Phi(\Lambda)} f(X_i) \prod_{j \in \{1, \dots, \Phi(\Lambda)\} \setminus \{i\}} (h(X_j) + \epsilon g(X_j)) \right. \\
&\quad \left. - \sum_{i=1}^{\Phi(\Lambda)} f(X_i) \prod_{j \in \{1, \dots, \Phi(\Lambda)\} \setminus \{i\}} h(X_j) \right] \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[\sum_{i=1}^{\Phi(\Lambda)} f(X_i) \sum_{k=1}^{\Phi(\Lambda)-1} \epsilon^k \sum_{\substack{\alpha \subset \{1, \dots, \Phi(\Lambda)\} \setminus \{i\} \\ |\alpha|=k}} \prod_{j \in \alpha} g(X_j) \prod_{l \in \{1, \dots, \Phi(\Lambda)\} \setminus \{i, j\}} h(X_l) \right] \\
&= \mathbb{E} \left[\sum_{i=1}^{\Phi(\Lambda)} f(X_i) \sum_{\substack{\alpha \subset \{1, \dots, \Phi(\Lambda)\} \setminus \{i\} \\ |\alpha|=1}} \prod_{j \in \alpha} g(X_j) \prod_{l \in \{1, \dots, \Phi(\Lambda)\} \setminus \{i, j\}} h(X_l) \right] \\
&= \mathbb{E} \left[\sum_{\substack{i, j=1 \\ i \neq j}}^{\Phi(\Lambda)} f(X_i) g(X_j) \prod_{k \in \{1, \dots, \Phi(\Lambda)\} \setminus \{i, j\}} h(X_k) \right] \\
&= \sum_{n=2}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{\substack{i, j=1 \\ i \neq j}}^n f(x_i) g(x_j) \prod_{\substack{k=1 \\ k \neq i, j}}^n h(x_k) \right) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}).
\end{aligned}$$

□

Second Order Janossy and Factorial Moment Densities

Corollary 2.2.20. *By setting $h = 0$ and substituting the functions f and g with Dirac delta functions evaluated at two distinct points $x, y \in \Lambda$ respectively in (2.38) of Proposition 2.2.19, we have the second order Janossy density of Φ at two distinct points $x, y \in \Lambda$, $x \neq y$,*

$$\frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_{\Phi}(h)|_{h=0} = j_{\Phi}^{(2)}(x, y). \quad (2.39)$$

Similarly, by setting $h = 1$ and substituting the functions f and g with Dirac delta functions evaluated at two distinct points $x, y \in \Lambda$ respectively in (2.38), we have the second order factorial moment density of Φ at $x, y \in \Lambda$, $x \neq y$,

$$\begin{aligned}
\rho_{\Phi}^{(2)}(x, y) &= \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_{\Phi}(h)|_{h=1} \\
&= j_{\Phi}^{(2)}(x, y) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_{\Phi}^{(n+2)}(x, y, x_{1:n}) \nu(dx_{1:n}).
\end{aligned} \quad (2.40)$$

Proof. Since the second order derivative involves two distinct points, that is $x, y \in \Lambda$, $x \neq y$, by setting $h = 0$ and by letting the respective functions f to be the Dirac delta function at a point x , denoted δ_x , and the function g to be the Dirac delta function at a point y , denoted δ_y , respectively, we derive (2.39) which is given by

$$\begin{aligned} \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_\Phi(h)|_{h=0} &= \sum_{n=2}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{\substack{i,j=1 \\ i \neq j}}^n \delta_x(x_i) \delta_y(x_j) \prod_{\substack{k=1 \\ k \neq i,j}}^n 0 \right) j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\ &= \frac{1}{2!} \int_{\Lambda^2} (\delta_x(x_1) \delta_y(x_2) + \delta_x(x_2) \delta_y(x_1)) j_\Phi^{(2)}(x_1, x_2) \nu(dx_1) \nu(dx_2) \\ &= j_\Phi^{(2)}(x, y). \end{aligned}$$

By setting $h = 1$, together with the respective Dirac delta functions δ_x and δ_y , and by similar technique used in the derivation of (2.21), we derive (2.40),

$$\begin{aligned} \rho_\Phi^{(2)}(x, y) &= \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_\Phi(h)|_{h=1} \\ &= \sum_{n=2}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{\substack{i,j=1 \\ i \neq j}}^n \delta_x(x_i) \delta_y(x_j) \prod_{\substack{k=1 \\ k \neq i,j}}^n 1 \right) j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\ &= \sum_{n=2}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{\substack{i,j=1 \\ i \neq j}}^n \delta_x(x_i) \delta_y(x_j) \right) j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\ &= \sum_{n=2}^{\infty} \frac{1}{n!} \int_{\Lambda^{n-2}} n(n-1) j_\Phi^{(n)}(x, y, x_{1:n-2}) \nu(dx_{1:n-2}) \\ &= \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \int_{\Lambda^{n-2}} j_\Phi^{(n)}(x, y, x_{1:n-2}) \nu(dx_{1:n-2}) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_\Phi^{(n+2)}(x, y, x_{1:n}) \nu(dx_{1:n}) \\ &= j_\Phi^{(2)}(x, y) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_\Phi^{(n+2)}(x, y, x_{1:n}) \nu(dx_{1:n}). \end{aligned}$$

□

Higher Order Functional Derivative of the PGFl

For the general case and by similar argument used in the derivation of the first and second order functional derivative of the PGFl, the n^{th} order functional derivative

of $\mathcal{G}_\Phi(h)$ is given by the following proposition.

Proposition 2.2.21. *For any n measurable nonnegative functions $f_1, \dots, f_n : \Lambda \rightarrow \mathbb{R}^+$, the n^{th} order derivative of $\mathcal{G}_\Phi(h)$ is given by*

$$\begin{aligned} \frac{\partial f_1}{\partial h} \cdots \frac{\partial f_n}{\partial h} \mathcal{G}_\Phi(h) &= \mathbb{E} \left[\sum_{\substack{i_1, \dots, i_n=1 \\ i_1 \neq \dots \neq i_n}}^{\Phi(\Lambda)} f_1(X_{i_1}) \cdots f_n(X_{i_n}) \prod_{j \in \{1, \dots, \Phi(\Lambda)\} \setminus \{i_1, \dots, i_n\}} h(X_j) \right] \\ &= \sum_{k=n}^{\infty} \frac{1}{k!} \int_{\Lambda^k} \left(\sum_{\substack{i_1, \dots, i_n=1 \\ i_1 \neq \dots \neq i_n}}^k f_1(x_{i_1}) \cdots f_n(x_{i_n}) \prod_{j \in \{1, \dots, k\} \setminus \{i_1, \dots, i_n\}} h(x_j) \right) \\ &\quad \times j_\Phi^{(k)}(x_{1:k}) \nu(dx_{1:k}). \end{aligned} \quad (2.41)$$

Higher Order Janossy Densities

Proposition 2.2.22. *For a sequence of distinct point configurations $(y_1, \dots, y_n) = y_{1:n} \in \mathfrak{N}$, the n^{th} order Janossy density, denoted $j_\Phi^{(n)}(y_{1:n})$, is given by*

$$j_\Phi^{(n)}(y_{1:n}) = \frac{\partial \delta_{y_1}}{\partial h} \cdots \frac{\partial \delta_{y_n}}{\partial h} \mathcal{G}_\Phi(h)|_{h=0}. \quad (2.42)$$

Proof. By substituting the corresponding functions f_i for $1 \leq i \leq n$ with Dirac delta functions evaluated at n distinct points $y_1, \dots, y_n \in \Lambda$ and by setting $h = 0$ to (2.41), we have the respective Janossy densities, which is given by

$$\begin{aligned} \frac{\partial \delta_{y_1}}{\partial h} \cdots \frac{\partial \delta_{y_n}}{\partial h} \mathcal{G}_\Phi(h)|_{h=0} &= \sum_{k=n}^{\infty} \frac{1}{k!} \int_{\Lambda^k} \left(\sum_{\substack{i_1, \dots, i_n=1 \\ i_1 \neq \dots \neq i_n}}^k \delta_{y_1}(x_{i_1}) \cdots \delta_{y_n}(x_{i_n}) \prod_{j \in \{1, \dots, k\} \setminus \{i_1, \dots, i_n\}} 0 \right) \\ &\quad \times j_\Phi^{(k)}(x_{1:k}) \nu(dx_{1:k}) \\ &= \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{\substack{i_1, \dots, i_n=1 \\ i_1 \neq \dots \neq i_n}} \delta_{y_1}(x_{i_1}) \cdots \delta_{y_n}(x_{i_n}) \right) j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\ &= j_\Phi^{(n)}(y_{1:n}). \end{aligned}$$

□

Higher Order Factorial Moment Densities

The following Proposition recovers the expression in (2.19) of Proposition 2.2.4.

Proposition 2.2.23. *For a sequence of distinct point configurations $(y_1, \dots, y_n) = y_{1:n} \in \mathfrak{N}$, the n^{th} order factorial moment density, denoted $\rho_{\Phi}^{(n)}(y_{1:n})$, is given by*

$$\begin{aligned} \rho_{\Phi}^{(n)}(y_{1:n}) &= \frac{\partial_{\delta_{y_1}}}{\partial h} \cdots \frac{\partial_{\delta_{y_n}}}{\partial h} \mathcal{G}_{\Phi}(h)|_{h=1} \\ &= j_{\Phi}^{(n)}(y_{1:n}) + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\Lambda^{k+n}} j_{\Phi}^{(k+n)}(y_{1:n}, x_{1:k}) \nu(dx_{1:k}). \end{aligned} \quad (2.43)$$

Proof. By substituting the corresponding functions f_i for $1 \leq i \leq n$ with Dirac delta functions evaluated at n distinct points $y_1, \dots, y_n \in \Lambda$ and by setting $h = 1$ to (2.41), we have

$$\begin{aligned} \rho_{\Phi}^{(n)}(y_{1:n}) &= \frac{\partial_{\delta_{y_1}}}{\partial h} \cdots \frac{\partial_{\delta_{y_n}}}{\partial h} \mathcal{G}_{\Phi}(h)|_{h=1} \\ &= \sum_{k=n}^{\infty} \frac{1}{k!} \int_{\Lambda^k} \left(\sum_{\substack{i_1, \dots, i_n=1 \\ i_1 \neq \dots \neq i_n}}^k \delta_{y_1}(x_{i_1}) \cdots \delta_{y_n}(x_{i_n}) \prod_{j \in \{1, \dots, k\} \setminus \{i_1, \dots, i_n\}} 1 \right) j_{\Phi}^{(k)}(x_{1:k}) \nu(dx_{1:k}) \\ &= \sum_{k=n}^{\infty} \frac{1}{k!} \int_{\Lambda^k} \left(\sum_{\substack{i_1, \dots, i_n=1 \\ i_1 \neq \dots \neq i_n}}^k \delta_{y_1}(x_{i_1}) \cdots \delta_{y_n}(x_{i_n}) \right) j_{\Phi}^{(k)}(x_{1:k}) \nu(dx_{1:k}) \\ &= \sum_{k=n}^{\infty} \frac{1}{k!} \int_{\Lambda^{k-n}} k(k-1) \cdots (k-n+1) j_{\Phi}^{(k)}(y_{1:n}, x_{1:k-n}) \nu(dx_{1:k-n}) \\ &= \sum_{k=n}^{\infty} \frac{1}{(k-n)!} \int_{\Lambda^{k-n}} j_{\Phi}^{(k)}(y_{1:n}, x_{1:k-n}) \nu(dx_{1:k-n}) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\Lambda^{k+n}} j_{\Phi}^{(k+n)}(y_{1:n}, x_{1:k}) \nu(dx_{1:k}) \\ &= j_{\Phi}^{(n)}(y_{1:n}) + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\Lambda^{k+n}} j_{\Phi}^{(k+n)}(y_{1:n}, x_{1:k}) \nu(dx_{1:k}). \end{aligned}$$

□

Functional Derivatives of Laplace Transform

In the following, we evaluate the functional derivatives of the Laplace transform expression to extract information about the nonfactorial moment densities related to the point process Φ .

First Order Functional Derivative of the Laplace Transform

Proposition 2.2.24. *The first order functional derivative of the Laplace transform $\mathcal{L}_\Phi(g)$ evaluated in the direction of g with a measurable nonnegative function $f : \Lambda \rightarrow \mathbb{R}^+$ is given by*

$$\frac{\partial_f}{\partial g} \mathcal{L}_\Phi(g) = (-1) \cdot \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{i=1}^n f(x_i) \prod_{i=1}^n e^{-g(x_i)} \right) j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}). \quad (2.44)$$

Proof. In a similar setting as in the derivatives of PGFl, and from (2.20), for a measurable nonnegative function $f : \Lambda \rightarrow \mathbb{R}^+$, the first order functional derivative of the Laplace transform $\mathcal{L}_\Phi(g)$, evaluated in the direction of f , is given by

$$\begin{aligned} \frac{\partial_f}{\partial g} \mathcal{L}_\Phi(g) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\mathcal{L}_\Phi(g + \epsilon f) - \mathcal{L}_\Phi(g) \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\mathbb{E} \left[\prod_{i=1}^{\Phi(\Lambda)} e^{-(g(X_i) + \epsilon f(X_i))} \right] - \mathbb{E} \left[\prod_{i=1}^{\Phi(\Lambda)} e^{-g(X_i)} \right] \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[\prod_{i=1}^{\Phi(\Lambda)} e^{-(g(X_i) + \epsilon f(X_i))} - \prod_{i=1}^{\Phi(\Lambda)} e^{-g(X_i)} \right] \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[e^{-\sum_{i=1}^{\Phi(\Lambda)} (g(X_i) + \epsilon f(X_i))} - e^{-\sum_{i=1}^{\Phi(\Lambda)} g(X_i)} \right] \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[e^{-\sum_{i=1}^{\Phi(\Lambda)} g(X_i)} \left(e^{-\epsilon \sum_{i=1}^{\Phi(\Lambda)} f(X_i)} - 1 \right) \right] \\ &= \mathbb{E} \left[- \sum_{i=1}^{\Phi(\Lambda)} f(X_i) e^{-\sum_{i=1}^{\Phi(\Lambda)} g(X_i)} \right] \\ &= (-1) \cdot \mathbb{E} \left[\sum_{i=1}^{\Phi(\Lambda)} f(X_i) \prod_{i=1}^{\Phi(\Lambda)} e^{-g(X_i)} \right] \\ &= (-1) \cdot \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{i=1}^n f(x_i) \prod_{i=1}^n e^{-g(x_i)} \right) j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}). \end{aligned}$$

□

First Order (Nonfactorial) Moment Density

Corollary 2.2.25. *By multiplying (2.44) with a factor of (-1) and setting $g = 0$, we recover (2.22) of Proposition 2.2.8, for which the following computes as*

$$\begin{aligned} (-1) \cdot \frac{\partial_f}{\partial g} \mathcal{L}_\Phi(g)|_{g=0} &= (-1) \cdot \mathbb{E} \left[- \sum_{i=1}^{\Phi(\Lambda)} f(X_i) \prod_{i=1}^{\Phi(\Lambda)} 1 \right] \\ &= \mathbb{E} \left[\sum_{i=1}^{\Phi(\Lambda)} f(X_i) \right] \\ &= \int_{\Lambda} f(x) \mu_{\Phi}^{(1)}(x) \nu(dx). \end{aligned}$$

By substituting the function f with Dirac delta function at a point $x \in \Lambda$, the first order (nonfactorial) moment density of the Laplace transform is equivalent to the first order moment density of Φ as in (2.37) of Corollary 2.2.18,

$$\begin{aligned} (-1) \cdot \frac{\partial_{\delta_x}}{\partial g} \mathcal{L}_\Phi(g)|_{g=0} &= \int_{\Lambda} \delta_x(x_1) \mu_{\Phi}^{(1)}(x_1) \nu(dx_1) \\ &= \mu_{\Phi}^{(1)}(x). \end{aligned}$$

Second Order Functional Derivative of the Laplace Transform

Proposition 2.2.26. *The second order derivative of the Laplace transform evaluated in the direction of g with functions $f_1, f_2 : \Lambda \rightarrow \mathbb{R}^+$ is given by*

$$\frac{\partial_{f_1}}{\partial g} \frac{\partial_{f_2}}{\partial g} \mathcal{L}_\Phi(g) = \mathbb{E} \left[\left(\sum_{i=1}^{\Phi(\Lambda)} f_1(X_i) \right) \left(\sum_{i=1}^{\Phi(\Lambda)} f_2(X_i) \right) \prod_{i=1}^{\Phi(\Lambda)} e^{-g(X_i)} \right]. \quad (2.45)$$

In addition, by setting $g = 0$, we have

$$\frac{\partial_{f_1}}{\partial g} \frac{\partial_{f_2}}{\partial g} \mathcal{L}_\Phi(g)|_{g=0} = \mathbb{E} \left[\left(\sum_{i=1}^{\Phi(\Lambda)} f_1(X_i) \right) \left(\sum_{i=1}^{\Phi(\Lambda)} f_2(X_i) \right) \right]. \quad (2.46)$$

Proof. The second order derivative of the Laplace transform in the direction of g with functions $f_1, f_2 : \Lambda \rightarrow \mathbb{R}^+$ is constructed as

$$\begin{aligned} &\frac{\partial_{f_1}}{\partial g} \frac{\partial_{f_2}}{\partial g} \mathcal{L}_\Phi(g) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\frac{\partial_{f_1}}{\partial g} \mathcal{L}_\Phi(g + \epsilon f_2) - \frac{\partial_{f_1}}{\partial g} \mathcal{L}_\Phi(g) \right) \end{aligned}$$

$$\begin{aligned}
&= (-1) \cdot \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\mathbb{E} \left[\sum_{i=1}^{\Phi(\Lambda)} f_1(X_i) \prod_{i=1}^{\Phi(\Lambda)} e^{-(g(X_i) + \epsilon f_2(X_i))} \right] - \mathbb{E} \left[\sum_{i=1}^{\Phi(\Lambda)} f_1(X_i) \prod_{i=1}^{\Phi(\Lambda)} e^{-g(X_i)} \right] \right) \\
&= (-1) \cdot \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\mathbb{E} \left[\sum_{i=1}^{\Phi(\Lambda)} f_1(X_i) \prod_{i=1}^{\Phi(\Lambda)} e^{-(g(X_i) + \epsilon f_2(X_i))} - \sum_{i=1}^{\Phi(\Lambda)} f_1(X_i) \prod_{i=1}^{\Phi(\Lambda)} e^{-g(X_i)} \right] \right) \\
&= (-1) \cdot \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\mathbb{E} \left[\sum_{i=1}^{\Phi(\Lambda)} f_1(X_i) e^{-\sum_{i=1}^{\Phi(\Lambda)} (g(X_i) + \epsilon f_2(X_i))} - \sum_{i=1}^{\Phi(\Lambda)} f_1(X_i) e^{-\sum_{i=1}^{\Phi(\Lambda)} g(X_i)} \right] \right) \\
&= (-1) \cdot \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\mathbb{E} \left[\sum_{i=1}^{\Phi(\Lambda)} f_1(X_i) e^{-\sum_{i=1}^{\Phi(\Lambda)} g(X_i)} \left(e^{-\epsilon \sum_{i=1}^{\Phi(\Lambda)} f_2(X_i)} - 1 \right) \right] \right) \\
&= (-1) \cdot \mathbb{E} \left[\sum_{i=1}^{\Phi(\Lambda)} f_1(X_i) \left(- \sum_{i=1}^{\Phi(\Lambda)} f_2(X_i) \right) e^{-\sum_{i=1}^{\Phi(\Lambda)} g(X_i)} \right] \\
&= (-1)^2 \cdot \mathbb{E} \left[\left(\sum_{i=1}^{\Phi(\Lambda)} f_1(X_i) \right) \left(\sum_{i=1}^{\Phi(\Lambda)} f_2(X_i) \right) e^{-\sum_{i=1}^{\Phi(\Lambda)} g(X_i)} \right] \\
&= \mathbb{E} \left[\left(\sum_{i=1}^{\Phi(\Lambda)} f_1(X_i) \right) \left(\sum_{i=1}^{\Phi(\Lambda)} f_2(X_i) \right) \prod_{i=1}^{\Phi(\Lambda)} e^{-g(X_i)} \right].
\end{aligned}$$

□

Second Order Nonfactorial Moment Density

Corollary 2.2.27. *The second order nonfactorial moment density at points $x, y \in \Lambda$ (not necessarily distinct) is given by*

$$\begin{aligned}
\mu_{\Phi}^{(2)}(x, y) &= \frac{\partial_{\delta_x}}{\partial g} \frac{\partial_{\delta_y}}{\partial g} \mathcal{L}_{\Phi}(g)|_{g=0} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_{\Phi}^{(n+1)}(x, x_{1:n}) \nu(dx_{1:n}) + \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_{\Phi}^{(n+2)}(x, y, x_{1:n}) \nu(dx_{1:n}) \\
&= \mu_{\Phi}^{(1)}(x) + \rho_{\Phi}^{(2)}(x, y).
\end{aligned} \tag{2.47}$$

Proof. By substituting the functions f_1 and f_2 of (2.46) with Dirac delta functions at points $x, y \in \Lambda$ (not necessarily distinct), we have

$$\mathbb{E} \left[\left(\sum_{i=1}^{\Phi(\Lambda)} \delta_x(X_i) \right) \left(\sum_{i=1}^{\Phi(\Lambda)} \delta_y(X_i) \right) \right]$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{i=1}^n \delta_x(x_i) \right) \left(\sum_{i=1}^n \delta_y(x_i) \right) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{i=1}^n (\delta_x(x_i))^2 \right) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&\quad + \sum_{n=2}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\sum_{\substack{i,j=1 \\ i \neq j}}^n \delta_x(x_i) \delta_y(x_j) \right) j_{\Phi}^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^{n-1}} n j_{\Phi}^{(n)}(x, x_{1:n-1}) \nu(dx_{1:n-1}) \\
&\quad + \sum_{n=2}^{\infty} \frac{1}{n!} \int_{\Lambda^{n-2}} n(n-1) j_{\Phi}^{(n)}(x, y, x_{1:n-2}) \nu(dx_{1:n-2}) \\
&= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_{\Lambda^{n-1}} j_{\Phi}^{(n)}(x, x_{1:n-1}) \nu(dx_{1:n-1}) \\
&\quad + \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \int_{\Lambda^{n-2}} j_{\Phi}^{(n)}(x, y, x_{1:n-2}) \nu(dx_{1:n-2}) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_{\Phi}^{(n+1)}(x, x_{1:n}) \nu(dx_{1:n}) + \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_{\Phi}^{(n+2)}(x, y, x_{1:n}) \nu(dx_{1:n}).
\end{aligned}$$

In addition, by (2.21) and (2.40), we have

$$\mathbb{E} \left[\left(\sum_{i=1}^{\Phi(\Lambda)} \delta_x(X_i) \right) \left(\sum_{i=1}^{\Phi(\Lambda)} \delta_y(X_i) \right) \right] = \mu_{\Phi}^{(1)}(x) + \rho_{\Phi}^{(2)}(x, y).$$

□

Higher Order Functional Derivative of the Laplace Transform

For the general case and by a similar argument used in the derivation of the first and second order functional derivatives of the Laplace transform, the n^{th} order functional derivative of $\mathcal{L}_{\Phi}(g)$ is given by following proposition.

Proposition 2.2.28. *For n measurable nonnegative functions $f_1, \dots, f_n : \Lambda \rightarrow \mathbb{R}^+$, the n^{th} order derivative of the Laplace transform is given by*

$$\frac{\partial f_1}{\partial g} \dots \frac{\partial f_n}{\partial g} \mathcal{L}_{\Phi}(g) = \mathbb{E} \left[\prod_{k=1}^n \left(\sum_{i=1}^{\Phi(\Lambda)} f_k(X_i) \right) \prod_{i=1}^{\Phi(\Lambda)} e^{-g(X_i)} \right].$$

And, by setting $g = 0$, we have

$$\frac{\partial f_1}{\partial g} \cdots \frac{\partial f_n}{\partial g} \mathcal{L}_\Phi(g)|_{g=0} = \mathbb{E} \left[\prod_{k=1}^n \left(\sum_{i=1}^{\Phi(\Lambda)} f_k(X_i) \right) \right]. \quad (2.48)$$

Higher Order Nonfactorial Moment Densities

By setting $g = 0$ and substituting the corresponding functions $f_1, \dots, f_n$ in (2.48) with Dirac delta functions at the corresponding sequence of points $(x_1, \dots, x_n) = x_{1:n} \in \Lambda$, the n^{th} order derivative of the Laplace transform $\mathcal{L}_\Phi(g)$ yields the n^{th} order nonfactorial moment density. In addition, by the Stirling number of the second kind as in Definition 2.1.8 and (2.5), the higher order nonfactorial moment densities are composed from the sum of their preceding factorial moment densities.

Proposition 2.2.29. *The general n^{th} order nonfactorial moment density evaluated at a sequence of points $(x_1, \dots, x_n) = x_{1:n} \in \Lambda$ (not necessarily distinct) is given by*

$$\mu_\Phi^{(n)}(x_{1:n}) = \frac{\partial_{\delta_{x_1}}}{\partial g} \cdots \frac{\partial_{\delta_{x_n}}}{\partial g} \mathcal{L}_\Phi(g)|_{g=0} = \mathbb{E} \left[\prod_{k=1}^n \left(\sum_{i=1}^{\Phi(\Lambda)} \delta_{x_k}(X_i) \right) \right]. \quad (2.49)$$

In addition, by the properties of the Stirling number of the second kind as in (2.4) and (2.5), the nonfactorial moment density of a point process $\mu_\Phi^{(n)}(x_{1:n})$ can be expressed as

$$\mu_\Phi^{(n)}(x_{1:n}) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \rho_\Phi^{(k)}(x_{1:k}), \quad (2.50)$$

where $\rho_\Phi^{(k)}(x_{1:k})$ is the k^{th} order factorial moment density of Φ at the point configuration $x_{1:k} \in \Lambda$.

Remark 2.2.30. *The first order nonfactorial moment density derived from (2.20) at a point $x \in \Lambda$ is the same as the first order factorial moment density, which is given by*

$$\mu_\Phi^{(1)}(x) = \rho_\Phi^{(1)}(x).$$

The second order nonfactorial moment density at two points $x, y \in \Lambda$ (not necessarily distinct) is given by

$$\begin{aligned} \mu_\Phi^{(2)}(x, y) &= \rho_\Phi^{(1)}(x) + \rho_\Phi^{(2)}(x, y) \\ &= \mu_\Phi^{(1)}(x) + \rho_\Phi^{(2)}(x, y). \end{aligned}$$

By (2.5), the third order nonfactorial moment density at points $x, y, z \in \Lambda$ (not necessarily distinct) is given by

$$\begin{aligned}\mu_{\Phi}^{(3)}(x, y, z) &= \rho_{\Phi}^{(1)}(x) + 3\rho_{\Phi}^{(2)}(x, y) + \rho_{\Phi}^{(3)}(x, y, z) \\ &= \mu_{\Phi}^{(1)}(x) + 3\rho_{\Phi}^{(2)}(x, y) + \rho_{\Phi}^{(3)}(x, y, z).\end{aligned}$$

2.2.4 Branching of Point Processes

The branching process of random variable has been introduced in Definition 2.1.10. In a similar setting for point processes and as in [3], the branching of point processes is a special kind of dependence that is used to describe a relationship between two point processes Ψ and Φ . Suppose the point processes Ψ and Φ have dependency of an ancestor-descendant relationship, where Ψ is the ancestor point process and Φ is the descendant point process. The branching process of Ψ is described as, for any realization of finite point configuration $\Psi = x_{1:m} = (x_1, \dots, x_m)$, the conditional point process $\Phi|\Psi$ is a superposition of m independent point processes, denoted by $\Upsilon|x$ where the point $x \in (x_1, \dots, x_m)$. By applying (2.27), the associated conditional PGFl of $\Phi|\Psi = x_{1:m}$ is given by

$$\mathcal{G}_{\Phi|\Psi}(h|x_{1:m}) := \prod_{i=1}^m \mathcal{G}_{\Upsilon}(h|x_i). \quad (2.51)$$

where $\mathcal{G}_{\Upsilon}(h|x)$ is the PGFl of $\Upsilon|x$.

From Bayes theorem, the conditional Janossy density of Φ given that $\Psi = x_{1:m} = (x_1, \dots, x_m)$ is given by

$$j_{\Phi|\Psi=x_{1:m}}^{(n)}(y_{1:n}) = \frac{j_{\Psi,\Phi}^{(m,n)}(x_{1:m}, y_{1:n})}{j_{\Psi}^{(m)}(x_{1:m})}. \quad (2.52)$$

Proposition 2.2.31. *For two point processes Ψ and Φ such that Ψ is the ancestor point process and Φ is the descendant point process of Ψ , their joint PGFl is given by*

$$\mathcal{G}_{\Psi,\Phi}(g, h) = \mathcal{G}_{\Psi}(g(\cdot))\mathcal{G}_{\Upsilon}(h|\cdot), \quad (2.53)$$

where g corresponds to Ψ and h corresponds to Φ .

Proof. By a similar fashion used to derive (2.7), and by applying (2.52) into (2.24), we have

$$\begin{aligned}
& \mathcal{G}_{\Psi, \Phi}(g, h) \\
&= \sum_{m \geq 0} \sum_{n \geq 0} \frac{1}{m!} \frac{1}{n!} \int_{\Lambda^m} \int_{\Lambda^n} \left(\prod_{i=0}^m g(x_i) \right) \left(\prod_{j=0}^n h(y_j) \right) j_{\Psi, \Phi}^{(m, n)}(x_{1:m}, y_{1:n}) \nu(dx_{1:m}) \nu(dy_{1:n}) \\
&= \sum_{m \geq 0} \sum_{n \geq 0} \frac{1}{m!} \frac{1}{n!} \int_{\Lambda^m} \int_{\Lambda^n} \left(\prod_{i=0}^m g(x_i) \right) \left(\prod_{j=0}^n h(y_j) \right) \\
&\quad \times j_{\Psi}^{(m)}(x_{1:m}) j_{\Phi|\Psi=x_{1:m}}^{(n)}(y_{1:n}) \nu(dx_{1:m}) \nu(dy_{1:n}) \\
&= \sum_{m \geq 0} \frac{1}{m!} \int_{\Lambda^m} \left(\prod_{i=0}^m g(x_i) \right) \left(\sum_{n \geq 0} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{j=0}^n h(y_j) \right) j_{\Phi|\Psi=x_{1:m}}^{(n)}(y_{1:n}) \nu(dy_{1:n}) \right) \\
&\quad \times j_{\Psi}^{(m)}(x_{1:m}) \nu(dx_{1:m}) \\
&= \sum_{m \geq 0} \frac{1}{m!} \int_{\Lambda^m} \left(\prod_{i=0}^m g(x_i) \right) \mathcal{G}_{\Phi|\Psi}(h|x_{1:m}) j_{\Psi}^{(m)}(x_{1:m}) \nu(dx_{1:m}) \\
&= \sum_{m \geq 0} \frac{1}{m!} \int_{\Lambda^m} \left(\prod_{i=0}^m g(x_i) \right) \left(\prod_{i=0}^m \mathcal{G}_{\Upsilon}(h|x_i) \right) j_{\Psi}^{(m)}(x_{1:m}) \nu(dx_{1:m}) \\
&= \sum_{m \geq 0} \frac{1}{m!} \int_{\Lambda^m} \left(\prod_{i=0}^m g(x_i) \mathcal{G}_{\Upsilon}(h|x_i) \right) j_{\Psi}^{(m)}(x_{1:m}) \nu(dx_{1:m}) \\
&= \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)).
\end{aligned}$$

Hence, the joint PGFl $\mathcal{G}_{\Psi, \Phi}(g, h)$ translates into the composition of the PGFl describing the branching process of Ψ . $\square$

Remark 2.2.32. From (2.30) and (2.53), the marginalization over Ψ in $\mathcal{G}_{\Psi, \Phi}(g, h)$ is given by

$$\mathcal{G}_{\Psi, \Phi}(g, h)|_{g=1} = \mathcal{G}_{\Phi}(h) = \mathcal{G}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot)), \quad (2.54)$$

where the explicit form is given by

$$\begin{aligned}
\mathcal{G}_{\Phi}(h) &= \mathcal{G}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot)) \\
&= \sum_{m \geq 0} \frac{1}{m!} \int_{\Lambda^m} \left(\prod_{i=0}^m \mathcal{G}_{\Upsilon}(h|x_i) \right) j_{\Psi}^{(m)}(x_{1:m}) \nu(dx_{1:m}).
\end{aligned} \quad (2.55)$$

Bernoulli Point Process

The Bernoulli point process Υ , also known as thinned point process in [2], is a simple point process with parameter $p \in [0, 1]$ and spatial distribution $l(x)$ that is used to describe binary events of a single point $x \in \Lambda$. As in [18], it is a specific case of independent and identically distributed (i.i.d.) point process, which has size 1 with probability p thereafter subject to a spatial distribution $l(x)$ with probability $pl(x)$, and size 0 with probability $q := 1 - p$. The associated PGFl is given by

$$\mathcal{G}_\Upsilon(h) = q + p \int_\Lambda h(x)l(x)\nu(dx). \quad (2.56)$$

The construction in (2.56) is given by

$$\begin{aligned} \mathcal{G}_\Upsilon(h) &= j_\Upsilon^{(0)} + \int_\Lambda h(x)j_\Upsilon^{(1)}(x)\nu(dx) + \sum_{n=2}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{i=1}^n h(x_i) \right) j_\Upsilon^{(n)}(x_{1:n})\nu(dx_{1:n}) \\ &= q + p \int_\Lambda h(x)l(x)\nu(dx), \end{aligned}$$

where $j_\Upsilon^{(0)} = q$, $j_\Upsilon^{(1)}(x) = pl(x)$ and $j_\Upsilon^{(n)}(x_{1:n}) = 0$ for $n \geq 2$.

I.I.D. Point Process

As in [18], [24], the i.i.d point process Φ is a general point process that is endowed with a cardinality distribution ϑ on $\mathbb{N}_0$ and i.i.d spatial distribution $l(x)$ for each point $x \in \Lambda$. The associated PGFl is given by

$$\mathcal{G}_\Phi(h) = \sum_{n=0}^{\infty} \vartheta(n) \left(\int_\Lambda h(x)l(x)\nu(dx) \right)^n.$$

For cluster point processes, readers can refer to [27], [28] for more details.

Poisson Point Process (PPP)

The Poisson point process (PPP) Φ is a i.i.d point process with cardinality distribution ϑ on $\mathbb{N}_0$ described by a Poisson distribution with rate $\lambda \geq 0$, and i.i.d spatial distribution $l(x)$ for each point $x \in \Lambda$. The Poisson point process is characterized by the first order moment density $\mu_\Phi^{(1)}(x) := \lambda l(x)$, also known as the intensity function in [25]. The associated PGFl is given by

$$\mathcal{G}_\Phi(h) = e^{\int_\Lambda (h(x)-1)\mu_\Phi^{(1)}(x)\nu(dx)}. \quad (2.57)$$

The construction of (2.57) is given by

$$\begin{aligned}
\mathcal{G}_\Phi(h) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{i=1}^n h(x_i) \right) j_\Phi^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{i=1}^n h(x_i) \right) n! e^{-\lambda} \frac{\lambda^n}{n!} \left(\prod_{i=1}^n l(x_i) \right) \nu(dx_{1:n}) \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_{\Lambda^n} \left(\prod_{i=1}^n h(x_i) l(x_i) \right) \nu(dx_{1:n}) \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left(\int_{\Lambda} h(x) l(x) \nu(dx) \right)^n \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\lambda \int_{\Lambda} h(x) l(x) \nu(dx) \right)^n \\
&= e^{-\lambda} e^{\lambda \int_{\Lambda} h(x) l(x) \nu(dx)} \\
&= e^{\lambda (\int_{\Lambda} h(x) l(x) \nu(dx) - 1)}.
\end{aligned}$$

The first order moment density $\mu_\Phi^{(1)}$ evaluated at the point $y \in \Lambda$ is given by

$$\begin{aligned}
\mu_\Phi^{(1)}(y) &= \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_\Phi(h) |_{h=1} \\
&= \frac{\partial_{\delta_y}}{\partial h} \left(e^{\lambda (\int_{\Lambda} h(x) l(x) \nu(dx) - 1)} \right) |_{h=1} \\
&= \frac{\partial_{\delta_y}}{\partial h} \left(\lambda \left(\int_{\Lambda} h(x) l(x) \nu(dx) - 1 \right) \right) \mathcal{G}_\Phi(h) |_{h=1} \\
&= \lambda \int_{\Lambda} \delta_y(x) l(x) \nu(dx) \\
&= \lambda l(y).
\end{aligned}$$

Since $l(y) = \frac{\mu_\Phi^{(1)}(y)}{\lambda}$ and, as in [25], the first moment measure in Λ is given by $\mu_\Phi^{(1)}(\Lambda) = \int_{\Lambda} \mu_\Phi^{(1)}(x) \nu(dx) = \lambda$, hence we have

$$\begin{aligned}
\mathcal{G}_\Phi(h) &= e^{\lambda (\int_{\Lambda} h(x) l(x) \nu(dx) - 1)} \\
&= e^{\int_{\Lambda} h(x) \lambda l(x) \nu(dx) - \lambda} \\
&= e^{\int_{\Lambda} h(x) \lambda l(x) \nu(dx) - \int_{\Lambda} \mu_\Phi^{(1)}(x) \nu(dx)} \\
&= e^{\int_{\Lambda} h(x) \mu_\Phi^{(1)}(x) \nu(dx) - \int_{\Lambda} \mu_\Phi^{(1)}(x) \nu(dx)}
\end{aligned}$$

$$\begin{aligned}
&= e^{\int_{\Lambda} h(x) \mu_{\Phi}^{(1)}(x) - \mu_{\Phi}^{(1)}(x) \nu(dx)} \\
&= e^{\int (h(x)-1) \mu_{\Phi}^{(1)}(x) \nu(dx)}.
\end{aligned}$$

The second order factorial moment density $\rho_{\Phi}^{(2)}$ evaluated at the points $x_1, x_2 \in \Lambda$ is given by

$$\begin{aligned}
\rho_{\Phi}^{(2)}(x_1, x_2) &= \frac{\partial_{\delta_{x_1}}}{\partial h} \frac{\partial_{\delta_{x_2}}}{\partial h} \mathcal{G}_{\Phi}(h)|_{h=1} \\
&= \frac{\partial_{\delta_{x_1}}}{\partial h} \frac{\partial_{\delta_{x_2}}}{\partial h} \left(e^{\int (h(x)-1) \mu_{\Phi}^{(1)}(x) \nu(dx)} \right) |_{h=1} \\
&= \left(\int \delta_{x_1}(x) \mu_{\Phi}^{(1)}(x) \nu(dx) \right) \left(\int \delta_{x_2}(x) \mu_{\Phi}^{(1)}(x) \nu(dx) \right) \mathcal{G}_{\Phi}(h)|_{h=1} \\
&= \mu_{\Phi}^{(1)}(x_1) \mu_{\Phi}^{(1)}(x_2) \\
&= \lambda^2 l(x_1) l(x_2).
\end{aligned}$$

Hence, the second order factorial moment measure in Λ is given by

$$\begin{aligned}
\rho_{\Phi}^{(2)}(\Lambda, \Lambda) &= \int_{\Lambda^2} \mu_{\Phi}^{(1)}(x_1) \mu_{\Phi}^{(1)}(x_2) \nu(dx_1) \nu(dx_2) \\
&= \lambda^2.
\end{aligned}$$

By (2.5), the second order nonfactorial moment measure is given by

$$\begin{aligned}
\mu_{\Phi}^{(2)}(\Lambda, \Lambda) &= \rho_{\Phi}^{(2)}(\Lambda, \Lambda) + \mu_{\Phi}^{(1)}(\Lambda) \\
&= \lambda^2 + \lambda.
\end{aligned}$$

The first order moment and the second order factorial moment are required to evaluate the variance, as in [3], where in general variance does not admit a density. Hence, the variance of the Poisson point process is given by

$$\begin{aligned}
v_{\Phi}(\Lambda) &= \rho_{\Phi}^{(2)}(\Lambda, \Lambda) + \mu_{\Phi}^{(1)}(\Lambda) - (\mu_{\Phi}^{(1)}(\Lambda))^2 \\
&= \lambda^2 + \lambda - \lambda^2 \\
&= \lambda.
\end{aligned}$$

Remark 2.2.33. *At this point, we have reviewed the mathematical basis, operations and first principles associated with discrete random variables in Section 2.1 and with spatial point processes in Section 2.2. The foundations established are the main tools that will be extensively applied in the construction of the integer-valued PHD filter*

in Section 3.1 and of spatial point process PHD filter in Section 3.2 respectively. In the next chapter, we will also introduce the multitarget tracking problem statement, bayesian filtering framework and the modeling assumptions. The PHD filter is essentially a point process Bayesian filtering approach that jointly estimates the number of targets and their dynamic states in cluttered environments by propagating the first order moment, or intensity function, of a point process. It is capable of solving the multitarget tracking problem based on sensor measurements that are also modeled as point processes.

Chapter 3

Probability Hypothesis Density (PHD) Filter

In this chapter, we give extended computations incorporating Janossy densities within associated PGFLs, and by properties linked with functional derivatives of the Poisson point process (PPP) characteristic PGFL to derive closed-form solutions to the first and second order posterior factorial moment densities of the PHD filter. In the early chapter, the multitarget problem statement and Bayesian filtering framework are described. Adapted from [3], in Section 3.1 we review the integer-valued PHD filter and its construction. In Section 3.2, we describe the spatial point process PHD filter and present some extended insights to the construction of the PHD filtering equations in subsection 3.2.1 and subsection 3.2.3. In particular, in Proposition 3.2.4 and Proposition 3.2.5, we derive the Poisson closed-form expression for the first and second posterior factorial moment densities of the PHD filter. In subsection 3.2.4, we state the *nearly constant turn-rate* ground truth motion and measurement dynamics of [9], [22]. We also give an extension of the motion dynamics to incorporate targets interaction among targets. In addition, we give some simulation comparison plots after implementing the sequential Monte Carlo (SMC) Poisson PHD filtering algorithm (see [5], [22] and [25]). The simulation scenario and parameters are equivalent as of [9],[22]. In our simulation comparison plots of the SMC-PHD filtering algorithm, we show the outputs of independent and interactive ground truth dynamics among multiple targets in target space for illustration purposes. The idea of target interaction among multiple targets in target space motivates the interest of this thesis to construct a second order PHD filter based on Determinantal Point Processes (DPPs) which is presented in Chapter 5.

Multitarget Tracking

The objective of multitarget tracking addresses the problem of estimating the unknown and time-varying number of targets as well as their dynamic state trajectories in some observation space. For a given target space, each target may either survive and stay on its trajectory or vanish. Concurrently, new birth targets may enter into the scene. A sequence of observation sets are usually made available by sensors at discrete time interval. Constrained by sensor limitations, the sensors often generate noisy measurements of the environment that reflect partial observations of the targets and some are spurious measurements. In this sequel, we adopt the spatial point process approach to describe the inherently stochastic nature of multitarget tracking problem.

Suppose there are $N \in \mathbb{N}_0$ targets in target state space $\mathcal{X} \subseteq \mathbb{R}^{d_x}$ at discrete time step $t \geq 0$, the multitarget state set representation X_t can be described as

$$X_t = \{x_{t,1}, \dots, x_{t,N}\} \in \mathcal{X}_\sigma,$$

where $\mathcal{X}_\sigma$ denotes the collection of all finite subsets of the state space $\mathcal{X}$.

The measurement process can be described in a similar fashion. Suppose there are $M \in \mathbb{N}_0$ observation points in measurement space $\mathcal{Z} \subseteq \mathbb{R}^{d_z}$ at time step t , the measurement set representation Z_t is described as

$$Z_t = \{z_{t,1}, \dots, z_{t,m}\} \in \mathcal{Z}_\sigma,$$

where $\mathcal{Z}_\sigma$ denotes the collection of all finite subsets of the measurement space $\mathcal{Z}$.

In order to solve the multitarget tracking problem and to provide meaningful information of the targets given the observations, the Bayesian filtering principle is applied to the multitarget framework. The filtering principle aims to provide probabilistic description of the tracking scenario and to update the distribution whenever new observation sets are available. In the following, we review the Bayesian filtering framework used to address the multitarget estimation problem.

Bayesian Filtering Framework

The Bayesian filter (also known as a two-step recursion process) is an optimal filter that constitutes a prediction step and an update step. It is a well-known probabilistic approach for estimating unknown probability distributions recursively in time

through some known characteristic measurement models and stochastic processes. It is an essential framework that is used to construct the PHD filter. The multitarget Bayesian filter is given by the following proposition.

Proposition 3.0.1. *Multitarget Bayesian Filtering Equations.*

The prediction formula at discrete time step $t \geq 0$, which is adapted from the Chapman-Kolmogorov equation, is given by

$$P(X_t|Z_{1:t-1}) = \int f_{t|t-1}(X_t|X)P(X|Z_{1:t-1})\nu(dX), \quad (3.1)$$

where $f_{t|t-1}(X_t|X)$ is the multitarget Markov transition kernel that describes the state-set X_t at time step t given that the state set X at time step $t - 1$, and $P(X_{t-1}|Z_{1:t-1})$ is the multitarget posterior distribution at time step $t - 1$ conditioned on the sequence of measurement sets $Z_{1:t-1}$.

The update formula at time step t , which is adapted from Bayes theorem, is given by

$$P(X_t|Z_{1:t}) = \frac{f_t(Z_t|X_t)P(X_t|Z_{1:t-1})}{\int f_t(Z_t|X)P(X|Z_{1:t-1})\nu(dX)}, \quad (3.2)$$

where $f_t(Z_t|X_t)$ is the multitarget measurement likelihood kernel that describes the measurement set Z_t conditioned on the state set X_t at time step t .

The Bayesian filtering equations stated above are general result that is not tractable in practice, see [34]. In order to ensure tractability of filtering equations in the context of multitarget tracking, the main tools in spatial point processes introduced in previous chapter are applicable and are required in the construction of the PHD filter. Equivalent expressions of the multitarget Bayesian filtering equations can be adapted into the context of PGFLs as in [29], for which the PGFL-based filtering equations are described in subsections 3.2.1 and 3.2.2.

Multitarget Modeling Assumptions

The multitarget modeling assumptions for the prediction step accounts for new birth targets, surviving target motion trajectories and death of targets, for which the multitarget state set representation Y is described as a point process according to the following equation,

$$Y = \sum_{x \in X} T_s(x) + Y_b,$$

where $(T_s, x \in X)$ is a family of Bernoulli point processes that describes the track evolution of the targets from the point process X of the previous time step, and Y_b is the birth point process. Details about the PHD prediction phase are described in subsection 3.2.1.

The update step relies on the current measurement information to provide meaningful information about the targets after accounting for noise, partial observations and false alarms in the modeling assumptions. The measurement point process Z is dependent on the target point process Y , for which we can model Z according to the following equation,

$$Z = \sum_{x \in Y} T_d(x) + Z_c,$$

where $(T_d, x \in Y)$ is a family of Bernoulli point processes that describes the sensor observation of the targets, and Z_c is the clutter point process that describes false alarm observations. Details about the posterior phase are described in subsection 3.2.2, and in subsection 3.2.3 for the Poisson assumption. In the next section, we review the construction of a restricted “Cardinality only” PHD filter which focuses on projections and estimations of the mean number of target in a multitarget tracking context.

3.1 Integer-Valued PHD Filter

The integer-valued PHD filter is a specialized “cardinality only” Bayesian filter, as introduced in [3], for which its intention is to provide projections and estimations to the mean number of target in a multi-object tracking scenario subject to some known cardinal measurement characteristics. It is a restricted PHD filter without spatial information about the targets, and can be seen as a discrete random variable PHD filter. This section provides a review of the basic concepts and construction involved to obtain the Integer-valued PHD filtering equations.

3.1.1 Prediction Phase

By the prediction step modeling assumptions introduced in [3], where the targets in the tracking domain are independent, each target survives with probability p_s , or gets deleted with probability $1 - p_s$ at each transition step, and birth of new targets follows a random variable with some known characteristics. Consider two r.v X and

Y such that X is the ancestor r.v and Y is the descendant r.v. In addition, consider a birth r.v Y_b such that X and Y_b yield Y . The r.v Y_b models the random birth event that is independent of X . Hence, Y is subject to a branching process of X as in (2.7) and a birth r.v Y_b . As such, the joint PGF of X and Y is constructed as

$$G_{X,Y}(u, v) = G_X(uG_{T_s}(v))G_{Y_b}(v),$$

where T_s is the Bernoulli r.v with parameter p_s and its respective PGF is given by

$$G_{T_s}(v) = 1 - p_s + p_s v,$$

the real variables $0 \leq u \leq 1$ and $0 \leq v \leq 1$ are associated with G_X and G_Y respectively, and, $G_{Y_b}(v)$ is the PGF of Y_b .

The prediction PGF of the integer-valued PHD filter is constructed by the superposition of the marginal over X of the joint PGF $G_{X,Y}(u, v)$ and a birth PGF $G_{Y_b}(v)$, hence, by setting $u = 1$, the prediction PGF is given by

$$\begin{aligned} G_Y(v) &= G_{X,Y}(u, v)|_{u=1} \\ &= G_X(uG_{T_s}(v))G_{Y_b}(v)|_{u=1} \\ &= G_X(G_{T_s}(v))G_{Y_b}(v) \\ &= G_X(1 - p_s + p_s v)G_{Y_b}(v). \end{aligned} \tag{3.3}$$

Prediction Phase First Order Moment of Random Variables

Proposition 3.1.1. *The first order prediction moment is given by*

$$\mu_Y^{(1)} = p_s \mu_X^{(1)} + \mu_{Y_b}^{(1)}, \tag{3.4}$$

where $\mu_X^{(1)}$ and $\mu_{Y_b}^{(1)}$ are the first order moment of X and Y_b respectively.

Proof. By differentiating (3.3) with respect to v , we have

$$\begin{aligned} G_Y^{(1)}(v) &= \frac{\partial}{\partial v} G_Y(v) \\ &= \frac{\partial}{\partial v} \left(G_X(1 - p_s + p_s v) G_{Y_b}(v) \right) \\ &= G_{Y_b}(v) \frac{\partial}{\partial v} G_X(1 - p_s + p_s v) + G_X(1 - p_s + p_s v) \frac{\partial}{\partial v} G_{Y_b}(v) \\ &= p_s G_X^{(1)}(1 - p_s + p_s v) G_{Y_b}(v) + G_X(1 - p_s + p_s v) G_{Y_b}^{(1)}(v). \end{aligned}$$

By setting $v = 1$, we yield the prediction first order moment

$$\begin{aligned}
 \mu_Y^{(1)} &= G_Y^{(1)}(1) \\
 &= p_s G_X^{(1)}(1) G_{Y_b}(1) + G_X(1) G_{Y_b}^{(1)}(1) \\
 &= p_s G_X^{(1)}(1) + G_{Y_b}^{(1)}(1) \\
 &= p_s \mu_X^{(1)} + \mu_{Y_b}^{(1)}.
 \end{aligned}$$

□

Remark 3.1.2. *The above expression is derived in a tractable form that requires no filtering approximation, as a result, the prediction model is not limited to a single probability model for X and Y_b .*

3.1.2 Posterior Phase

Following the posterior update step assumptions introduced in [3], where a measurement associated with a single detected target is generated independently, each target is detected with probability p_d , or is undetected with probability $1 - p_d$, and clutter measurements are described by a random variable with some known characteristics. Consider the measurement r.v Z such that Z has a descendant dependency on Y . Then, Z is subject to a branching process of Y as in (2.7). In addition, consider the clutter r.v Z_c where Z_c models the random clutter event that is independent of Z . Hence, the joint PGF of Y and Z is constructed as

$$G_{Y,Z}(v, w) = G_Y(v G_{T_d}(w)) G_{Z_c}(w),$$

where T_d is the Bernoulli r.v with parameter p_d and its respective PGF is given by

$$G_{T_d}(w) = 1 - p_d + p_d w,$$

the real variable $0 \leq w \leq 1$ is associated with G_Z and G_{Z_c} .

Lemma 3.1.3. *Given the realization of r.v $Z = m$, the posterior PGF of $Y|Z = m$ is given by*

$$G_{Y|Z}(v|m) = \frac{\frac{\partial^m}{\partial w^m} G_Y(v G_{T_d}(w)) G_{Z_c}(w)|_{w=0}}{\frac{\partial^m}{\partial w^m} G_Y(G_{T_d}(w)) G_{Z_c}(w)|_{w=0}}. \quad (3.5)$$

Proof. By using Bayes theorem, the conditional PGF is constructed as

$$\begin{aligned}
 \sum_{n=0}^{\infty} p_{Y|Z}(n|m)v^n &= \frac{\sum_{n=0}^{\infty} p_{Y,Z}(n,m)v^n}{p_Z(m)} \\
 &= \frac{G_{Y,Z=m}(v,w)|_{w=0}}{G_Z^{(m)}(w)|_{w=0}} \\
 &= \frac{\frac{\partial^m}{\partial w^m} G_{Y,Z}(v,w)|_{w=0}}{\frac{\partial^m}{\partial w^m} G_{Y,Z}(\mathbf{1},w)|_{w=0}} \\
 &= \frac{\frac{\partial^m}{\partial w^m} G_Y(vG_{T_d}(w))G_{Z_c}(w)|_{w=0}}{\frac{\partial^m}{\partial w^m} G_Y(G_{T_d}(w))G_{Z_c}(w)|_{w=0}}.
 \end{aligned}$$

Hence, we have

$$G_{Y|Z}(v|m) = \sum_{n=0}^{\infty} p_{Y|Z}(n|m)v^n = \frac{\frac{\partial^m}{\partial w^m} G_Y(vG_{T_d}(w))G_{Z_c}(w)|_{w=0}}{\frac{\partial^m}{\partial w^m} G_Y(G_{T_d}(w))G_{Z_c}(w)|_{w=0}}.$$

□

Posterior Phase First Order Moment of Random Variables

The first order posterior moment is derived from the first order derivative of (3.5) with respect to v . From (3.5), only the numerator part of $G_{Y|Z}(v|m)$ has dependence on v , it suffices to evaluate the derivative of $\frac{\partial^m}{\partial w^m} G_Y(vG_{T_d}(w))G_{Z_c}(w)|_{w=0}$, which is given by

$$\frac{\partial}{\partial v} \frac{\partial^m}{\partial w^m} G_Y(vG_{T_d}(w))G_{Z_c}(w)|_{w=0} = \frac{\partial^{m+1}}{\partial v \partial w^m} G_Y(vG_{T_d}(w))G_{Z_c}(w)|_{w=0}.$$

The first order partial derivative of (3.5) with respect to v is given by

$$\frac{\partial}{\partial v} G_{Y|Z}(v|m) = \frac{\frac{\partial^{m+1}}{\partial v \partial w^m} G_Y(vG_{T_d}(w))G_{Z_c}(w)|_{w=0}}{\frac{\partial^m}{\partial w^m} G_Y(G_{T_d}(w))G_{Z_c}(w)|_{w=0}}.$$

Hence, the posterior first order moment of $Y|Z = m$, denoted as $\mu_{Y|Z=m}^{(1)}$, is given by

$$\mu_{Y|Z=m}^{(1)} = \frac{\partial}{\partial v} G_{Y|Z}(v|m)|_{v=1}$$

$$= \frac{\frac{\partial^{m+1}}{\partial v \partial w^m} G_Y(v G_{T_d}(w)) G_{Z_c}(w)|_{w=0, v=1}}{\frac{\partial^m}{\partial w^m} G_Y(G_{T_d}(w)) G_{Z_c}(w)|_{w=0}}, \quad (3.6)$$

where the resulting expression exists as a general but intractable form. The PHD filter requires a filtering approximation to provide a tractable result to the above posterior update expression. Hence, by assuming that X and Y_b follows the Poisson distribution with properties as in (2.12), (2.13) and (2.14), the approximation enables a simplified and closed form expression.

3.1.3 Closed-Form Poisson Expressions

Proposition 3.1.4. *Under the Poisson modeling assumption, the first order posterior moment of $Y|Z = m$ is given by*

$$\mu_{Y|Z=m}^{(1)} = m \frac{p_d \mu_Y^{(1)}}{p_d \mu_Y^{(1)} + \mu_{Z_c}^{(1)}} + (1 - p_d) \mu_Y^{(1)}, \quad (3.7)$$

where $\mu_Y^{(1)}$ and $\mu_{Z_c}^{(1)}$ are the Poisson first order moment of Y and Z_c respectively.

Proof. Following (2.13), we have

$$\begin{aligned} G_Y(v G_{T_d}(w)) G_{Z_c}(w) &= G_Y(v(1 - p_d + p_d w)) G_{Z_c}(w) \\ &= e^{(v(1-p_d+p_d w)-1)\mu_Y^{(1)}} \times e^{(w-1)\mu_{Z_c}^{(1)}} \\ &= e^{(v(1-p_d+p_d w)-1)\mu_Y^{(1)} + (w-1)\mu_{Z_c}^{(1)}}. \end{aligned} \quad (3.8)$$

By taking the first order derivative of (3.8) with respect to w , we have

$$\begin{aligned} \frac{\partial}{\partial w} G_Y(v G_{T_d}(w)) G_{Z_c}(w) &= \frac{\partial}{\partial w} e^{(v(1-p_d+p_d w)-1)\mu_Y^{(1)} + (w-1)\mu_{Z_c}^{(1)}} \\ &= (p_d v \mu_Y^{(1)} + \mu_{Z_c}^{(1)}) e^{(v(1-p_d+p_d w)-1)\mu_Y^{(1)} + (w-1)\mu_{Z_c}^{(1)}} \\ &= (p_d v \mu_Y^{(1)} + \mu_{Z_c}^{(1)}) G_Y(v G_{T_d}(w)) G_{Z_c}(w). \end{aligned}$$

From the standard differentiation result on exponential functions, the general m^{th} order derivative of (3.8) with respect to w is given by

$$\frac{\partial^m}{\partial w^m} G_Y(v G_{T_d}(w)) G_{Z_c}(w) = (p_d v \mu_Y^{(1)} + \mu_{Z_c}^{(1)})^m G_Y(v G_{T_d}(w)) G_{Z_c}(w). \quad (3.9)$$

By setting $v = 1$, we recover the marginal of (3.9),

$$\begin{aligned} \frac{\partial^m}{\partial w^m} G_Y(G_{T_d}(w)) G_{Z_c}(w) &= (p_d \mu_Y^{(1)} + \mu_{Z_c}^{(1)})^m e^{(w-1)(p_d \mu_Y^{(1)} + \mu_{Z_c}^{(1)})} \\ &= (p_d \mu_Y^{(1)} + \mu_{Z_c}^{(1)})^m G_Y(G_{T_d}(w)) G_{Z_c}(w). \end{aligned}$$

Then, by setting $w = 0$ to the above expression, we have the denominator of (3.6),

$$\begin{aligned} \frac{\partial^m}{\partial w^m} G_Y(G_{T_d}(w)) G_{Z_c}(w)|_{w=0} &= (p_d \mu_Y^{(1)} + \mu_{Z_c}^{(1)})^m G_Y(G_{T_d}(w)) G_{Z_c}(w)|_{w=0} \\ &= (p_d \mu_Y^{(1)} + \mu_{Z_c}^{(1)})^m e^{-(p_d \mu_Y^{(1)} + \mu_{Z_c}^{(1)})}. \end{aligned} \quad (3.10)$$

By differentiating (3.9) with respect to v , we have

$$\begin{aligned} &\frac{\partial^{m+1}}{\partial v \partial w^m} G_Y(v G_{T_d}(w)) G_{Z_c}(w) \\ &= \frac{\partial}{\partial v} \left((p_d v \mu_Y^{(1)} + \mu_{Z_c}^{(1)})^m G_Y(v G_{T_d}(w)) G_{Z_c}(w) \right) \\ &= m p_d \mu_Y^{(1)} (p_d v \mu_Y^{(1)} + \mu_{Z_c}^{(1)})^{m-1} G_Y(v G_{T_d}(w)) G_{Z_c}(w) \\ &\quad + (p_d v \mu_Y^{(1)} + \mu_{Z_c}^{(1)})^m \frac{\partial}{\partial v} G_Y(v G_{T_d}(w)) G_{Z_c}(w) \\ &= m p_d \mu_Y^{(1)} (p_d v \mu_Y^{(1)} + \mu_{Z_c}^{(1)})^{m-1} G_Y(v G_{T_d}(w)) G_{Z_c}(w) \\ &\quad + (p_d v \mu_Y^{(1)} + \mu_{Z_c}^{(1)})^m (1 - p_d + p_d w) \mu_Y^{(1)} G_Y(v G_{T_d}(w)) G_{Z_c}(w). \end{aligned}$$

By setting $w = 0$ and $v = 1$, we recover the numerator of (3.6),

$$\frac{\partial^{m+1}}{\partial v \partial w^m} G_Y(v G_{T_d}(w)) G_{Z_c}(w)|_{w=0, v=1} \quad (3.11)$$

$$\begin{aligned} &= m p_d \mu_Y^{(1)} (p_d \mu_Y^{(1)} + \mu_{Z_c}^{(1)})^{m-1} e^{-(p_d \mu_Y^{(1)} + \mu_{Z_c}^{(1)})} \\ &\quad + (p_d \mu_Y^{(1)} + \mu_{Z_c}^{(1)})^m (1 - p_d) \mu_Y^{(1)} e^{-(p_d \mu_Y^{(1)} + \mu_{Z_c}^{(1)})}. \end{aligned} \quad (3.12)$$

Finally, by substituting the results of (3.10) and (3.12) into (3.6), we have

$$\begin{aligned} \mu_{Y|Z=m}^{(1)} &= \frac{\frac{\partial^{m+1}}{\partial v \partial w^m} G_Y(v G_{T_d}(w)) G_{Z_c}(w)|_{w=0, v=1}}{\frac{\partial^m}{\partial w^m} G_Y(G_{T_d}(w)) G_{Z_c}(w)|_{w=0}} \\ &= m \frac{p_d \mu_Y^{(1)}}{p_d \mu_Y^{(1)} + \mu_{Z_c}^{(1)}} + (1 - p_d) \mu_Y^{(1)}. \end{aligned}$$

□

Integer Valued PHD Filtering Equations

Remark 3.1.5. *The prediction integer-valued PHD filtering equation is given by*

$$\mu_Y^{(1)} = p_s \mu_X^{(1)} + \mu_{Y_b}^{(1)}.$$

The posterior integer-valued PHD filtering equation is given by

$$\mu_{Y|Z=m}^{(1)} = m \frac{p_d \mu_Y^{(1)}}{p_d \mu_Y^{(1)} + \mu_{Z_c}^{(1)}} + (1 - p_d) \mu_Y^{(1)}.$$

3.2 Spatial Point Process PHD Filter

The spatial point process PHD filter is a specialized Bayesian filter and as the name implies, incorporates spatial information about the targets in a multi-object tracking scenario (see [26] for more review). The objective of this filter is to project and estimate the mean number of target (also known as the intensity or first order moment) in any space domain, subject to some known measurement characteristics. It is an application of branched point processes as described in 2.2.4. This section provides a review and some extended insights to the construction of the spatial point process PHD filtering equations. In Proposition 3.2.1 and Proposition 3.2.2, we evaluate the first and second order prediction factorial moment densities by the means of functional derivatives and Janossy densities within the Poisson prediction PGFl. In Proposition 3.2.4 and Proposition 3.2.5, we derive the Poisson closed-form expression for the first and second posterior factorial moment densities that forms the posterior phase filtering equations of the PHD filter by means of functional derivatives and Janossy densities within the Poisson posterior PGFl.

3.2.1 Prediction Phase

The prediction point process Φ is constructed by the superposition of a branching process on prior point process Ψ and a birth point process Φ_b , as in [3]. The branching process on prior point process Ψ is invoked with Bernoulli point processes Φ_s with probability of survival $p_s(x)$ at the point $x \in \Psi$ and spatial transition density $l_s(y|x)$ from point x to point y . Hence, by (2.54), the probability generating functional of Φ is given by

$$\mathcal{G}_\Phi(h) := \mathcal{G}_{\Phi_b}(h) \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot)), \quad (3.13)$$

for which we have

$$\mathcal{G}_{\Phi_s}(h|\cdot) := q_s(\cdot) + \int_{\Lambda} h(y) \tilde{l}_s(y|\cdot) \nu(dy),$$

where for the purpose of compactness, we represent $q_s(\cdot) := 1 - p_s(\cdot)$ and $\tilde{l}_s(y|\cdot) := p_s(\cdot) l_s(y|\cdot)$.

Next in line, we present the first and second prediction moment densities. We consider Janossy densities and adopt the functional derivative approach to derive Proposition 3.2.1 and Proposition 3.2.2. The result in Proposition 3.2.1 is consistent with [3], in addition, we extend [3] to incorporate the second order prediction factorial moment density.

First Order Prediction Moment Density

Proposition 3.2.1. *The first order prediction moment density of Φ is given by*

$$\begin{aligned} \mu_{\Phi}^{(1)}(x) &= \frac{\partial_{\delta_x} \mathcal{G}_{\Phi}(h)|_{h=1}}{\partial h} = \frac{\partial_{\delta_x} \mathcal{G}_{\Phi_b}(h) \mathcal{G}_{\Psi}(\mathcal{G}_{\Phi_s}(h|\cdot))|_{h=1}}{\partial h} \\ &= \int_{\Lambda} \tilde{l}_s(x|u) \mu_{\Psi}^{(1)}(u) \nu(du) + \mu_{\Phi_b}^{(1)}(x), \end{aligned} \quad (3.14)$$

where $\mu_{\Phi_b}^{(1)}(x)$ is the first order moment density of Φ_b .

Proof. We evaluate the first order derivative of $\mathcal{G}_{\Phi}(h) := \mathcal{G}_{\Phi_b}(h) \mathcal{G}_{\Psi}(\mathcal{G}_{\Phi_s}(h|\cdot))$ in the direction of h , by a measurable nonnegative function $f : \Lambda \rightarrow \mathbb{R}^+$, which is given by

$$\begin{aligned} \frac{\partial_f}{\partial h} \mathcal{G}_{\Phi}(h) &= \frac{\partial_f}{\partial h} \left(\mathcal{G}_{\Phi_b}(h) \mathcal{G}_{\Psi}(\mathcal{G}_{\Phi_s}(h|\cdot)) \right) \\ &= \mathcal{G}_{\Psi}(\mathcal{G}_{\Phi_s}(h|\cdot)) \frac{\partial_f}{\partial h} \mathcal{G}_{\Phi_b}(h) + \mathcal{G}_{\Phi_b}(h) \frac{\partial_f}{\partial h} \mathcal{G}_{\Psi}(\mathcal{G}_{\Phi_s}(h|\cdot)). \end{aligned} \quad (3.15)$$

By setting $h = 1$ in the above expression, we have

$$\begin{aligned} \frac{\partial_f}{\partial h} \mathcal{G}_{\Phi}(h)|_{h=1} &= \left(\mathcal{G}_{\Psi}(\mathcal{G}_{\Phi_s}(h|\cdot)) \frac{\partial_f}{\partial h} \mathcal{G}_{\Phi_b}(h) + \mathcal{G}_{\Phi_b}(h) \frac{\partial_f}{\partial h} \mathcal{G}_{\Psi}(\mathcal{G}_{\Phi_s}(h|\cdot)) \right) \Big|_{h=1} \\ &= \frac{\partial_f}{\partial h} \mathcal{G}_{\Phi_b}(h)|_{h=1} + \frac{\partial_f}{\partial h} \mathcal{G}_{\Psi}(\mathcal{G}_{\Phi_s}(h|\cdot))|_{h=1} \\ &= \int_{\Lambda} f(\bar{x}) \mu_{\Phi_b}^{(1)}(\bar{x}) \nu(d\bar{x}) + \frac{\partial_f}{\partial h} \mathcal{G}_{\Psi}(\mathcal{G}_{\Phi_s}(h|\cdot))|_{h=1}. \end{aligned}$$

At this point, it remains to evaluate the term $\frac{\partial_f}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot))$. From (2.55), we have the following expression

$$\mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot)) = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \left(\prod_{i=0}^m \mathcal{G}_{\Phi_s}(h|x_i) \right) j_\Psi^{(m)}(x_{1:m}) \nu(dx_{1:m}).$$

By applying (2.33), we have

$$\begin{aligned} & \frac{\partial_f}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot)) \\ &= \frac{\partial_f}{\partial h} \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \left(\prod_{i=0}^m \mathcal{G}_{\Phi_s}(h|x_i) \right) j_\Psi^{(m)}(x_{1:m}) \nu(dx_{1:m}) \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \frac{\partial_f}{\partial h} \left(\prod_{i=0}^m \mathcal{G}_{\Phi_s}(h|x_i) \right) j_\Psi^{(m)}(x_{1:m}) \nu(dx_{1:m}) \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \left(\sum_{i=1}^m \frac{\partial_f}{\partial h} \mathcal{G}_{\Phi_s}(h|x_i) \prod_{\substack{j=1 \\ j \neq i}}^m \mathcal{G}_{\Phi_s}(h|x_j) \right) j_\Psi^{(m)}(x_{1:m}) \nu(dx_{1:m}) \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \left(\sum_{i=1}^m \left(\int_{\Lambda} f(y) \tilde{l}_s(y|x_i) \nu(dy) \right) \prod_{\substack{j=1 \\ j \neq i}}^m \mathcal{G}_{\Phi_s}(h|x_j) \right) j_\Psi^{(m)}(x_{1:m}) \nu(dx_{1:m}). \end{aligned}$$

Next, by setting $h = 1$, we have

$$\begin{aligned} & \frac{\partial_f}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot))|_{h=1} \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \left(\sum_{i=1}^m \left(\int_{\Lambda} f(y) \tilde{l}_s(y|x_i) \nu(dy) \right) \right) j_\Psi^{(m)}(x_{1:m}) \nu(dx_{1:m}) \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} \left(\sum_{i=1}^m \int_{\Lambda^m} \left(\int_{\Lambda} f(y) \tilde{l}_s(y|x_i) \nu(dy) \right) j_\Psi^{(m)}(x_{1:m}) \nu(dx_{1:m}) \right) \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} \left(m \int_{\Lambda^m} \left(\int_{\Lambda} f(y) \tilde{l}_s(y|x_1) \nu(dy) \right) j_\Psi^{(m)}(x_{1:m}) \nu(dx_{1:m}) \right) \\ &= \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \left(\int_{\Lambda^m} \left(\int_{\Lambda} f(y) \tilde{l}_s(y|x_1) \nu(dy) \right) j_\Psi^{(m)}(x_{1:m}) \nu(dx_{1:m}) \right) \\ &= \int_{\Lambda} \left(\int_{\Lambda} f(y) \tilde{l}_s(y|\bar{x}) \nu(dy) \right) \left(\sum_{m=1}^{\infty} \frac{1}{(m-1)!} \int_{\Lambda^{m-1}} j_\Psi^{(m)}(\bar{x}, x_{1:m-1}) \nu(dx_{1:m-1}) \right) \nu(d\bar{x}) \\ &= \int_{\Lambda} \left(\int_{\Lambda} f(y) \tilde{l}_s(y|\bar{x}) \nu(dy) \right) \left(\sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_\Psi^{(m+1)}(\bar{x}, x_{1:m}) \nu(dx_{1:m}) \right) \nu(d\bar{x}) \end{aligned}$$

$$\begin{aligned}
&= \int_{\Lambda} \left(\int_{\Lambda} f(y) \tilde{l}_s(y|\bar{x}) \nu(dy) \right) \left(j_{\Psi}^{(1)}(\bar{x}) + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_{\Psi}^{(m+1)}(\bar{x}, x_{1:m}) \nu(dx_{1:m}) \right) \nu(d\bar{x}) \\
&= \int_{\Lambda} \left(\int_{\Lambda} f(y) \tilde{l}_s(y|\bar{x}) \nu(dy) \right) \mu_{\Psi}^{(1)}(\bar{x}) \nu(d\bar{x}).
\end{aligned}$$

Hence, we have

$$\frac{\partial f}{\partial h} \mathcal{G}_{\Phi}(h)|_{h=1} = \int_{\Lambda} f(\bar{x}) \mu_{\Phi_b}^{(1)}(\bar{x}) \nu(d\bar{x}) + \int_{\Lambda} \left(\int_{\Lambda} f(y) \tilde{l}_s(y|\bar{x}) \nu(dy) \right) \mu_{\Psi}^{(1)}(\bar{x}) \nu(d\bar{x}).$$

By substituting the function f with Dirac delta function δ_x , we have

$$\begin{aligned}
\frac{\partial \delta_x}{\partial h} \mathcal{G}_{\Phi}(h)|_{h=1} &= \int_{\Lambda} \delta_x(\bar{x}) \mu_{\Phi_b}^{(1)}(\bar{x}) \nu(d\bar{x}) + \frac{\partial \delta_x}{\partial h} \mathcal{G}_{\Psi}(\mathcal{G}_{\Phi_s}(h|\cdot))|_{h=1} \\
&= \mu_{\Phi_b}^{(1)}(x) + \int_{\Lambda} \left(\int_{\Lambda} \delta_x(y) \tilde{l}_s(y|\bar{x}) \nu(dy) \right) \mu_{\Psi}^{(1)}(\bar{x}) \nu(d\bar{x}) \\
&= \mu_{\Phi_b}^{(1)}(x) + \int_{\Lambda} \tilde{l}_s(x|\bar{x}) \mu_{\Psi}^{(1)}(\bar{x}) \nu(d\bar{x}).
\end{aligned}$$

□

Second Order Prediction Factorial Moment Density

Proposition 3.2.2. *The second order prediction factorial moment density of Φ is given by*

$$\begin{aligned}
\rho_{\Phi}^{(2)}(x, y) &= \frac{\partial \delta_x}{\partial h} \frac{\partial \delta_y}{\partial h} \mathcal{G}_{\Phi}(h)|_{h=1} = \frac{\partial \delta_x}{\partial h} \frac{\partial \delta_y}{\partial h} \left(\mathcal{G}_{\Phi_b}(h) \mathcal{G}_{\Psi}(\mathcal{G}_{\Phi_s}(h|\cdot)) \right)|_{h=1} \\
&= \rho_{\Phi_b}^{(2)}(x, y) + \mu_{\Phi_b}^{(1)}(x) \int_{\Lambda} \tilde{l}_s(y|u) \mu_{\Psi}^{(1)}(u) \nu(du) + \mu_{\Phi_b}^{(1)}(y) \int_{\Lambda} \tilde{l}_s(x|u) \mu_{\Psi}^{(1)}(u) \nu(du) \\
&\quad + \int_{\Lambda^2} \tilde{l}_s(x|u) \tilde{l}_s(y|v) \rho_{\Psi}^{(2)}(u, v) \nu(du) \nu(dv), \tag{3.16}
\end{aligned}$$

where $x, y \in \Lambda$ and $x \neq y$.

Proof. By evaluating the partial derivative of (3.15) in the direction of h by a measurable nonnegative function $g : \Lambda \rightarrow \mathbb{R}^+$, the second order derivative of $\mathcal{G}_{\Phi}(h)$ is given by

$$\begin{aligned}
&\frac{\partial g}{\partial h} \frac{\partial f}{\partial h} \mathcal{G}_{\Phi}(h) \\
&= \frac{\partial g}{\partial h} \frac{\partial f}{\partial h} \left(\mathcal{G}_{\Phi_b}(h) \mathcal{G}_{\Psi}(\mathcal{G}_{\Phi_s}(h|\cdot)) \right)
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
&= \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot)) \frac{\partial_g}{\partial h} \frac{\partial_f}{\partial h} \mathcal{G}_{\Phi_b}(h) + \frac{\partial_g}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot)) \frac{\partial_f}{\partial h} \mathcal{G}_{\Phi_b}(h) \\
&\quad + \frac{\partial_g}{\partial h} \mathcal{G}_{\Phi_b}(h) \frac{\partial_f}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot)) + \mathcal{G}_{\Phi_b}(h) \frac{\partial_g}{\partial h} \frac{\partial_f}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot)). \tag{3.18}
\end{aligned}$$

By setting $h = 1$ in the above expression, we have

$$\begin{aligned}
&\frac{\partial_g}{\partial h} \frac{\partial_f}{\partial h} \mathcal{G}_\Phi(h)|_{h=1} \\
&= \frac{\partial_g}{\partial h} \frac{\partial_f}{\partial h} \mathcal{G}_{\Phi_b}(h)|_{h=1} + \frac{\partial_g}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot))|_{h=1} \frac{\partial_f}{\partial h} \mathcal{G}_{\Phi_b}(h)|_{h=1} \\
&\quad + \frac{\partial_g}{\partial h} \mathcal{G}_{\Phi_b}(h)|_{h=1} \frac{\partial_f}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot))|_{h=1} + \frac{\partial_g}{\partial h} \frac{\partial_f}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot))|_{h=1}.
\end{aligned}$$

By substituting the function f with Dirac delta function δ_x and the function g with Dirac delta function δ_y at each component of the sum, we have

$$\begin{aligned}
&\rho_\Phi^{(2)}(x, y) \\
&= \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_\Phi(h)|_{h=1} \\
&= \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_{\Phi_b}(h)|_{h=1} + \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot))|_{h=1} \frac{\partial_{\delta_x}}{\partial h} \mathcal{G}_{\Phi_b}(h)|_{h=1} \\
&\quad + \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_{\Phi_b}(h)|_{h=1} \frac{\partial_{\delta_x}}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot))|_{h=1} + \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot))|_{h=1} \\
&= \rho_{\Phi_b}^{(2)}(x, y) + \mu_{\Phi_b}^{(1)}(x) \int_\Lambda \tilde{l}_s(y|u) \mu_\Psi^{(1)}(u) \nu(du) \\
&\quad + \mu_{\Phi_b}^{(1)}(y) \int_\Lambda \tilde{l}_s(x|u) \mu_\Psi^{(1)}(u) \nu(du) + \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot))|_{h=1}.
\end{aligned}$$

At this point, it remains to evaluate the component $\frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot))|_{h=1}$. We consider the term $\frac{\partial_g}{\partial h} \frac{\partial_f}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot))$, and by applying (2.38), we have

$$\begin{aligned}
&\frac{\partial_g}{\partial h} \frac{\partial_f}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot)) \\
&= \sum_{m=2}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \left(\sum_{\substack{i,j=1 \\ i \neq j}}^m \frac{\partial_f}{\partial h} \mathcal{G}_{\Phi_s}(h|x_i) \frac{\partial_g}{\partial h} \mathcal{G}_{\Phi_s}(h|x_j) \prod_{\substack{k=1 \\ k \neq i,j}}^m \mathcal{G}_{\Phi_s}(h|x_k) \right) j_\Psi^{(m)}(x_{1:m}) \nu(dx_{1:m}),
\end{aligned}$$

where $\frac{\partial_f}{\partial h} \mathcal{G}_{\Phi_s}(h|x) = \int_\Lambda f(y) \tilde{l}_s(y|x) \nu(dy)$ and $\frac{\partial_g}{\partial h} \mathcal{G}_{\Phi_s}(h|x) = \int_\Lambda g(y) \tilde{l}_s(y|x) \nu(dy)$.

Next, by setting $h = 1$, and by applying the result found in (2.40), we have

$$\frac{\partial_g}{\partial h} \frac{\partial_f}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_{\Phi_s}(h|\cdot))|_{h=1}$$

$$\begin{aligned}
&= \sum_{m=2}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \left(\sum_{\substack{i,j=1 \\ i \neq j}}^m \frac{\partial f}{\partial h} \mathcal{G}_{\Phi_s}(h|x_i) \frac{\partial g}{\partial h} \mathcal{G}_{\Phi_s}(h|x_j) \right) j_{\Psi}^{(m)}(x_{1:m}) \nu(dx_{1:m}) \\
&= \sum_{m=2}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \left(\sum_{\substack{i,j=1 \\ i \neq j}}^m \left(\int_{\Lambda} f(y_1) \tilde{l}_s(y_1|x_i) \nu(dy_1) \right) \left(\int_{\Lambda} g(y_2) \tilde{l}_s(y_2|x_j) \nu(dy_2) \right) \right) \\
&\quad \times j_{\Psi}^{(m)}(x_{1:m}) \nu(dx_{1:m}) \\
&= \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{\substack{i,j=1 \\ i \neq j}}^m \int_{\Lambda^m} \left(\int_{\Lambda} f(y_1) \tilde{l}_s(y_1|x_i) \nu(dy_1) \right) \left(\int_{\Lambda} g(y_2) \tilde{l}_s(y_2|x_j) \nu(dy_2) \right) \\
&\quad \times j_{\Psi}^{(m)}(x_{1:m}) \nu(dx_{1:m}) \\
&= \sum_{m=2}^{\infty} \frac{m(m-1)}{m!} \int_{\Lambda^m} \left(\int_{\Lambda} f(y_1) \tilde{l}_s(y_1|x_1) \nu(dy_1) \right) \left(\int_{\Lambda} g(y_2) \tilde{l}_s(y_2|x_2) \nu(dy_2) \right) \\
&\quad \times j_{\Psi}^{(m)}(x_{1:m}) \nu(dx_{1:m}) \\
&= \sum_{m=2}^{\infty} \frac{1}{(m-2)!} \int_{\Lambda^m} \left(\int_{\Lambda} f(y_1) \tilde{l}_s(y_1|x_1) \nu(dy_1) \right) \left(\int_{\Lambda} g(y_2) \tilde{l}_s(y_2|x_2) \nu(dy_2) \right) \\
&\quad \times j_{\Psi}^{(m)}(x_{1:m}) \nu(dx_{1:m}) \\
&= \int_{\Lambda^2} \left(\int_{\Lambda} f(y_1) \tilde{l}_s(y_1|u) \nu(dy_1) \right) \left(\int_{\Lambda} g(y_2) \tilde{l}_s(y_2|v) \nu(dy_2) \right) \\
&\quad \times \left(\sum_{m=2}^{\infty} \frac{1}{(m-2)!} \int_{\Lambda^{m-2}} j_{\Psi}^{(m)}(u, v, x_{1:m-2}) \nu(dx_{1:m-2}) \right) \nu(du) \nu(dv) \\
&= \int_{\Lambda^2} \left(\int_{\Lambda} f(y_1) \tilde{l}_s(y_1|u) \nu(dy_1) \right) \left(\int_{\Lambda} g(y_2) \tilde{l}_s(y_2|v) \nu(dy_2) \right) \\
&\quad \times \left(\sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_{\Psi}^{(m+2)}(u, v, x_{1:m}) \nu(dx_{1:m}) \right) \nu(du) \nu(dv) \\
&= \int_{\Lambda^2} \left(\int_{\Lambda} f(y_1) \tilde{l}_s(y_1|u) \nu(dy_1) \right) \left(\int_{\Lambda} g(y_2) \tilde{l}_s(y_2|v) \nu(dy_2) \right) \rho_{\Psi}^{(2)}(u, v) \nu(du) \nu(dv).
\end{aligned}$$

By substituting the function f with Dirac delta function δ_x and the function g with Dirac delta function δ_y , we have

$$\begin{aligned}
&\frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_{\Psi}(\mathcal{G}_{\Phi_s}(h|\cdot))|_{h=1} \\
&= \int_{\Lambda^2} \left(\int_{\Lambda} \delta_x(y_1) \tilde{l}_s(y_1|u) \nu(dy_1) \right) \left(\int_{\Lambda} \delta_y(y_2) \tilde{l}_s(y_2|v) \nu(dy_2) \right) \rho_{\Psi}^{(2)}(u, v) \nu(du) \nu(dv)
\end{aligned}$$

$$= \int_{\Lambda^2} \tilde{l}_s(x|u) \tilde{l}_s(y|v) \rho_{\Psi}^{(2)}(u, v) \nu(du) \nu(dv).$$

Hence, the second order prediction factorial moment density of Φ is given by

$$\begin{aligned} \rho_{\Phi}^{(2)}(x, y) &= \rho_{\Phi_b}^{(2)}(x, y) + \mu_{\Phi_b}^{(1)}(x) \int_{\Lambda} \tilde{l}_s(y|u) \mu_{\Psi}^{(1)}(u) \nu(du) + \mu_{\Phi_b}^{(1)}(y) \int_{\Lambda} \tilde{l}_s(x|u) \mu_{\Psi}^{(1)}(u) \nu(du) \\ &\quad + \int_{\Lambda^2} \tilde{l}_s(x|u) \tilde{l}_s(y|v) \rho_{\Psi}^{(2)}(u, v) \nu(du) \nu(dv). \end{aligned}$$

□

3.2.2 Posterior Phase

The objective of establishing the posterior point process $\Phi|\Xi$ is to incorporate the information generated by a measurement point process Ξ with the prediction point process Φ in order to provide meaningful and improved description of the spatial point configurations. This phase forms the basis of the update step in the Bayesian approach in point process filtering.

Firstly, we consider the joint point processes between Φ and Ξ , which is characterized by their joint probability generating functional $\mathcal{G}_{\Phi, \Xi}(h, g)$. Their joint relationship is described as a dependency of an ancestor-descendant genealogy, where Φ is the ancestor point process that results to Ξ , the descendant point process. Hence, the special dependence between Φ and Ξ is constructed as a superposition of a branching process on Φ and a clutter point process Ξ_c with PGFl $\mathcal{G}_{\Xi_c}(g)$. The branching process on Φ is invoked with Bernoulli point processes Ξ_d with probability of detection $p_d(x)$ at the point $x \in \Phi$ and measurement likelihood density $l_d(z|x)$ where $z \in \Xi_d$ originates from the point $x \in \Phi$.

As such, by (2.53), the joint probability generating functional of Φ and Ξ is given by

$$\mathcal{G}_{\Phi, \Xi}(h, g) := \mathcal{G}_{\Phi}(h(\cdot)) \mathcal{G}_{\Xi_d}(g|\cdot) \mathcal{G}_{\Xi_c}(g), \quad (3.19)$$

for which we have

$$\mathcal{G}_{\Xi_d}(g|\cdot) := q_d(\cdot) + \int_{\Lambda} g(z) \tilde{l}_d(z|\cdot) \nu(dz),$$

where for the purpose of compactness, we represent $q_d(\cdot) := 1 - p_d(\cdot)$ and $\tilde{l}_d(z|\cdot) := p_d(\cdot)l_s(z|\cdot)$.

Recall from (2.52), conditioned on the realization of $\Xi = z_{1:m} = (z_1, \dots, z_m)$, the n^{th} conditional Janossy density of Φ is given by

$$j_{\Phi|\Xi=z_{1:m}}^{(n)}(x_1, \dots, x_n) = \frac{j_{\Phi, \Xi=z_{1:m}}^{(n,m)}(x_1, \dots, x_n)}{j_{\Xi}^{(m)}(z_{1:m})},$$

where $x_1, \dots, x_n \in \Lambda$.

Lemma 3.2.3. *The posterior PGFl of $\Phi|\Xi = z_{1:m}$ is given by*

$$\mathcal{G}_{\Phi|\Xi}(h|z_{1:m}) = \frac{\frac{\partial_{\delta_{z_1}} \dots \partial_{\delta_{z_m}} \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0}}{\partial g}}{\frac{\partial_{\delta_{z_1}} \dots \partial_{\delta_{z_m}} \mathcal{G}_{\Phi}(\mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0}}{\partial g}}. \quad (3.20)$$

Proof. From (2.52), we have

$$\begin{aligned} \mathcal{G}_{\Phi|\Xi}(h|z_{1:m}) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{i=1}^n h(x_i) \right) j_{\Phi|\Xi=z_{1:m}}^{(n)}(x_{1:n}) \nu(dx_{1:n}) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{i=1}^n h(x_i) \right) \frac{j_{\Phi, \Xi=z_{1:m}}^{(n,m)}(x_{1:n})}{j_{\Xi}^{(m)}(z_{1:m})} \nu(dx_{1:n}) \\ &= \frac{\mathcal{G}_{\Phi, \Xi=z_{1:m}}(h, g)|_{g=0}}{j_{\Xi}^{(m)}(z_{1:m})} \\ &= \frac{\frac{\partial_{\delta_{z_1}} \dots \partial_{\delta_{z_m}} \mathcal{G}_{\Phi, \Xi}(h, g)|_{g=0}}{\partial g}}{\frac{\partial_{\delta_{z_1}} \dots \partial_{\delta_{z_m}} \mathcal{G}_{\Phi, \Xi}(\mathbf{1}, g)|_{g=0}}{\partial g}} \\ &= \frac{\frac{\partial_{\delta_{z_1}} \dots \partial_{\delta_{z_m}} \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0}}{\partial g}}{\frac{\partial_{\delta_{z_1}} \dots \partial_{\delta_{z_m}} \mathcal{G}_{\Phi}(\mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0}}{\partial g}}. \end{aligned}$$

□

First Order Posterior Moment Density

The first order posterior moment density of the conditional point process $\Phi|\Xi = z_{1:m}$ is computed by taking a partial derivative of (3.20) evaluated in the direction of h

by a Dirac delta function at a point $x \in \Lambda$,

$$\frac{\partial_{\delta_x}}{\partial h} \mathcal{G}_{\Phi|\Xi}(h|z_{1:m}) = \frac{\frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0}}{\frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Phi}(\mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0}}.$$

Similar to the derivation of first order posterior moment under the integer-valued PHD filter found in (3.6), only the numerator component of $\mathcal{G}_{\Phi|\Xi}(h|z_{1:m})$ has dependence on function h . Thus, it suffices to evaluate the partial derivative of $\frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0}$ in the direction of h by a Dirac delta function δ_x . By setting $h = 1$, the first order posterior moment density of the conditional point process $\Phi|\Xi = z_{1:m}$ at a point $x \in \Lambda$ is given by

$$\begin{aligned} \mu_{\Phi|\Xi=z_{1:m}}^{(1)}(x) &= \frac{\partial_{\delta_x}}{\partial h} \mathcal{G}_{\Phi|\Xi}(h|z_{1:m})|_{h=1} \\ &= \frac{\frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0, h=1}}{\frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Phi}(\mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0}}. \end{aligned} \quad (3.21)$$

Second Order Posterior Factorial Moment Density

The second order posterior moment density of the conditional point process $\Phi|\Xi = z_{1:m}$ is given by

$$\begin{aligned} \rho_{\Phi|\Xi=z_{1:m}}^{(2)}(x, y) &= \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_{\Phi|\Xi}(h|z_{1:m})|_{h=1} \\ &= \frac{\frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0, h=1}}{\frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Phi}(\mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0}}, \end{aligned} \quad (3.22)$$

where the points $x, y \in \Lambda$ and $x \neq y$.

At this point, the first and second order posterior factorial moment densities derived are of a general form. As such, by assuming that Φ and Ξ_c are Poisson point processes (PPP) with PGFl taking the form of (2.57), the exponential term inherent under the property of PPP enables a simplification that lead to a tractable closed form expression to both (3.21) and (3.22).

3.2.3 Closed-Form Poisson Expressions

By the Poisson point process assumption, we derive the tractable closed-form expression of the first and second order posterior factorial moment densities using functional derivative approach and the properties of PPP. For extended details about the Poisson PHD filter, readers can refer to [26]. The derivation of the first order posterior moment density is consistent to [3], and in addition, we extend the derivation to the second order posterior factorial moment density.

Poisson First Order Posterior Moment Density

Proposition 3.2.4. *Under the Poisson point process (PPP) assumption, the first order posterior moment density of the conditional point process $\Phi|\Xi = z_{1:m}$ with respect to $\nu(dx)$ at a point $x \in \Lambda$ is given by*

$$\mu_{\Phi|\Xi=z_{1:m}}^{(1)}(x) = q_d(x)\mu_{\Phi}^{(1)}(x) + \sum_{z \in z_{1:m}} \frac{\tilde{l}_d(z|x)\mu_{\Phi}^{(1)}(x)}{l_c(z) + \int_{\Lambda} \tilde{l}_d(z|u)\mu_{\Phi}^{(1)}(u)\nu(du)} \quad (3.23)$$

where $\mu_{\Phi}^{(1)}(\cdot)$ and $l_c(\cdot)$ are the Poisson intensity functions of Φ and Ξ_c respectively.

Proof. Recall from (2.57), the expression of $\mathcal{G}_{\Phi}(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g)$ under the Poisson setting is given by

$$\begin{aligned} & \mathcal{G}_{\Phi}(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \\ &= e^{\int_{\Lambda} (h(u)\mathcal{G}_{\Xi_d}(g|u)-1)\mu_{\Phi}^{(1)}(u)\nu(du)} \times e^{\int_{\Lambda} (g(z)-1)l_c(z)\nu(dz)} \\ &= e^{\int_{\Lambda} (h(u)\mathcal{G}_{\Xi_d}(g|u)-1)\mu_{\Phi}^{(1)}(u)\nu(du) + \int_{\Lambda} (g(z)-1)l_c(z)\nu(dz)} \end{aligned} \quad (3.24)$$

where $\mathcal{G}_{\Xi_d}(g|\cdot) = q_d(\cdot) + \int_{\Lambda} g(z)\tilde{l}_d(z|\cdot)\nu(dz)$.

By taking the first order functional derivative of (3.24) in the direction of g with a Dirac delta function on z_1 , we have

$$\begin{aligned} & \frac{\partial_{\delta_{z_1}}}{\partial g} \mathcal{G}_{\Phi}(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \\ &= \left(l_c(z_1) + \int_{\Lambda} h(u)\tilde{l}_d(z_1|u)\mu_{\Phi}^{(1)}(u)\nu(du) \right) \mathcal{G}_{\Phi}(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g). \end{aligned}$$

As a consequence of the exponential terms in $\mathcal{G}_{\Phi}(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g)$, the general m^{th} order functional derivative of $\mathcal{G}_{\Phi}(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g)$ is given by

$$\frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Phi}(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) = \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} h(u)\tilde{l}_d(z_k|u)\mu_{\Phi}^{(1)}(u)\nu(du) \right)$$

$$\times \mathcal{G}_\Phi(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g). \quad (3.25)$$

By setting $g = 0$ and $h = 1$ on (3.25), we get the denominator term of (3.21), which is given by

$$\begin{aligned} \frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_\Phi(\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g)|_{g=0} &= \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} \tilde{l}_d(z_k|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \\ &\times \mathcal{G}_\Phi(\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g)|_{g=0}, \end{aligned} \quad (3.26)$$

where, for the purpose of completeness, the term $\mathcal{G}_\Phi(\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g)|_{g=0}$ is given by

$$\begin{aligned} \mathcal{G}_\Phi(\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g)|_{g=0} &= e^{\int_{\Lambda} (\mathcal{G}_{\Xi_d}(g|u)-1) \mu_{\Phi}^{(1)}(u) \nu(du) + \int_{\Lambda} (g(z)-1) l_c(z) \nu(dz)} \Big|_{g=0} \\ &= e^{-(\int_{\Lambda} p_d(u) \mu_{\Phi}^{(1)}(u) \nu(du) + \int_{\Lambda} l_c(z) \nu(dz))}. \end{aligned}$$

Next, by taking another functional derivative of (3.25) in the direction of h with a Dirac delta function on a point $x \in \Lambda$, we have

$$\begin{aligned} &\frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_\Phi(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \\ &= \frac{\partial_{\delta_x}}{\partial h} \left(\prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} h(u) \tilde{l}_d(z_k|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \mathcal{G}_\Phi(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \right) \\ &= \sum_{k=1}^m \tilde{l}_d(z_k|x) \mu_{\Phi}^{(1)}(x) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} h(u) \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \\ &\quad \times \mathcal{G}_\Phi(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \\ &\quad + \frac{\partial_{\delta_x}}{\partial h} \mathcal{G}_\Phi(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} h(u) \tilde{l}_d(z_k|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \\ &= \sum_{k=1}^m \tilde{l}_d(z_k|x) \mu_{\Phi}^{(1)}(x) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} h(u) \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \\ &\quad \times \mathcal{G}_\Phi(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \\ &\quad + \left(q_d(x) + \int_{\Lambda} g(z) \tilde{l}_d(z|x) \nu(dz) \right) \mu_{\Phi}^{(1)}(x) \mathcal{G}_\Phi(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \\ &\quad \times \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} h(u) \tilde{l}_d(z_k|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \\ &= \sum_{k=1}^m \tilde{l}_d(z_k|x) \mu_{\Phi}^{(1)}(x) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} h(u) \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \end{aligned}$$

$$\begin{aligned}
& \times \mathcal{G}_\Phi(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \\
& + \mathcal{G}_{\Xi_d}(g|x)\mu_\Phi^{(1)}(x)\mathcal{G}_\Phi(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \\
& \times \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} h(u)\tilde{l}_d(z_k|u)\mu_\Phi^{(1)}(u)\nu(du) \right). \tag{3.27}
\end{aligned}$$

By setting $g = 0$ and $h = 1$, we have the numerator term of (3.21), which is given by

$$\begin{aligned}
& \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_\Phi(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g)|_{g=0, h=1} \\
& = \sum_{k=1}^m \tilde{l}_d(z_k|x)\mu_\Phi^{(1)}(x) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} \tilde{l}_d(z_l|u)\mu_\Phi^{(1)}(u)\nu(du) \right) \mathcal{G}_\Phi(\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g)|_{g=0} \\
& + q_d(x)\mu_\Phi^{(1)}(x) \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} \tilde{l}_d(z_k|u)\mu_\Phi^{(1)}(u)\nu(du) \right) \mathcal{G}_\Phi(\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g)|_{g=0}. \tag{3.28}
\end{aligned}$$

Hence, by substituting the respective terms of (3.26) and (3.28) into (3.21), we have (3.23). $\square$

Poisson Second Order Posterior Factorial Moment Density

Proposition 3.2.5. *Under the Poisson assumption, the second order posterior moment density of the conditional point process $\Phi|\Xi = z_{1:m}$ with respect to $\nu(dx)$ at two points $x, y \in \Lambda$, $x \neq y$, is given by*

$$\begin{aligned}
& \rho_{\Phi|\Xi=z_{1:m}}^{(2)}(x, y) \\
& = q_d(x)q_d(y)\mu_\Phi^{(1)}(x)\mu_\Phi^{(1)}(y) + \sum_{k=1}^m \frac{q_d(x)\mu_\Phi^{(1)}(x)\mu_\Phi^{(1)}(y)\tilde{l}_d(z_k|y)}{l_c(z_k) + \int_{\Lambda} \tilde{l}_d(z_k|u)\mu_\Phi^{(1)}(u)\nu(du)} \\
& + \sum_{k=1}^m \frac{q_d(y)\mu_\Phi^{(1)}(x)\mu_\Phi^{(1)}(y)\tilde{l}_d(z_k|x)}{l_c(z_k) + \int_{\Lambda} \tilde{l}_d(z_k|u)\mu_\Phi^{(1)}(u)\nu(du)} \\
& + \sum_{\substack{k,l=1 \\ k \neq l}}^m \frac{\mu_\Phi^{(1)}(x)\mu_\Phi^{(1)}(y)\tilde{l}_d(z_k|x)\tilde{l}_d(z_l|y)}{\left(l_c(z_k) + \int_{\Lambda} \tilde{l}_d(z_k|u)\mu_\Phi^{(1)}(u)\nu(du) \right) \left(l_c(z_l) + \int_{\Lambda} \tilde{l}_d(z_l|u)\mu_\Phi^{(1)}(u)\nu(du) \right)}. \tag{3.29}
\end{aligned}$$

Proof. By taking another functional derivative of (3.27) in the direction of h with a Dirac delta function on another point $y \in \Lambda$ such that $x \neq y$, we have

$$\begin{aligned}
& \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \frac{\partial_{\delta_{z_1}}}{\partial g} \cdots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g) \\
&= \frac{\partial_{\delta_y}}{\partial h} \sum_{k=1}^m \tilde{l}_d(z_k|x) \mu_{\Phi}^{(1)}(x) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} h(u) \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \\
&\quad \times \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g) \\
&\quad + \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_{\Xi_d}(g|x) \mu_{\Phi}^{(1)}(x) \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g) \\
&\quad \times \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} h(u) \tilde{l}_d(z_k|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right). \tag{3.30}
\end{aligned}$$

Considering the first term of sum in (3.30), we have

$$\begin{aligned}
& \frac{\partial_{\delta_y}}{\partial h} \sum_{k=1}^m \tilde{l}_d(z_k|x) \mu_{\Phi}^{(1)}(x) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} h(u) \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \\
&\quad \times \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g) \\
&= \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g) \sum_{k=1}^m \tilde{l}_d(z_k|x) \mu_{\Phi}^{(1)}(x) \sum_{\substack{l=1 \\ l \neq k}}^m \tilde{l}_d(z_l|y) \mu_{\Phi}^{(1)}(y) \\
&\quad \times \prod_{\substack{s=1 \\ s \neq k, l}}^m \left(l_c(z_s) + \int_{\Lambda} h(u) \tilde{l}_d(z_s|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \\
&\quad + \left(\sum_{k=1}^m \tilde{l}_d(z_k|x) \mu_{\Phi}^{(1)}(x) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} h(u) \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \right) \\
&\quad \times \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g) \\
&= \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g) \sum_{k=1}^m \tilde{l}_d(z_k|x) \mu_{\Phi}^{(1)}(x) \sum_{\substack{l=1 \\ l \neq k}}^m \tilde{l}_d(z_l|y) \mu_{\Phi}^{(1)}(y) \\
&\quad \times \prod_{\substack{s=1 \\ s \neq k, l}}^m \left(l_c(z_s) + \int_{\Lambda} h(u) \tilde{l}_d(z_s|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right)
\end{aligned}$$

$$\begin{aligned}
& + \left(\sum_{k=1}^m \tilde{l}_d(z_k|x) \mu_{\Phi}^{(1)}(x) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} h(u) \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \right) \\
& \quad \times \left(q_d(y) + \int_{\Lambda} g(z) \tilde{l}_d(z|y) \nu(dz) \right) \mu_{\Phi}^{(1)}(y) \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g) \\
& = \mu_{\Phi}^{(1)}(x) \mu_{\Phi}^{(1)}(y) \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g) \\
& \quad \times \left(\sum_{k=1}^m \tilde{l}_d(z_k|x) \sum_{\substack{l=1 \\ l \neq k}}^m \tilde{l}_d(z_l|y) \prod_{\substack{s=1 \\ s \neq k, l}}^m \left(l_c(z_s) + \int_{\Lambda} h(u) \tilde{l}_d(z_s|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \right. \\
& \quad \left. + \mathcal{G}_{\Xi_d}(g|y) \left(\sum_{k=1}^m \tilde{l}_d(z_k|x) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} h(u) \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \right) \right). \tag{3.31}
\end{aligned}$$

By setting $g = 0$ and $h = 1$ in (3.31), we have

$$\begin{aligned}
& \frac{\partial_{\delta_y}}{\partial h} \left(\sum_{k=1}^m \tilde{l}_d(z_k|x) \mu_{\Phi}^{(1)}(x) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} h(u) \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \right. \\
& \quad \left. \times \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g) \right) \Big|_{g=0, h=1} \\
& = \mu_{\Phi}^{(1)}(x) \mu_{\Phi}^{(1)}(y) \mathcal{G}_{\Phi}(\mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g) \Big|_{g=0} \\
& \quad \times \left(\sum_{k=1}^m \tilde{l}_d(z_k|x) \sum_{\substack{l=1 \\ l \neq k}}^m \tilde{l}_d(z_l|y) \prod_{\substack{s=1 \\ s \neq k, l}}^m \left(l_c(z_s) + \int_{\Lambda} \tilde{l}_d(z_s|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \right. \\
& \quad \left. + q_d(y) \left(\sum_{k=1}^m \tilde{l}_d(z_k|x) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \right) \right). \tag{3.32}
\end{aligned}$$

Next, considering the second term of the sum in (3.30), we have

$$\begin{aligned}
& \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_{\Xi_d}(g|x) \mu_{\Phi}^{(1)}(x) \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g) \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} h(u) \tilde{l}_d(z_k|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \\
& = \mathcal{G}_{\Xi_d}(g|x) \mu_{\Phi}^{(1)}(x) \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} h(u) \tilde{l}_d(z_k|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \\
& \quad \times \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)
\end{aligned}$$

$$\begin{aligned}
& + \mathcal{G}_{\Xi_d}(g|x)\mu_{\Phi}^{(1)}(x)\mathcal{G}_{\Phi}(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \\
& \quad \times \frac{\partial_{\delta_y}}{\partial h} \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} h(u)\tilde{l}_d(z_k|u)\mu_{\Phi}^{(1)}(u)\nu(du) \right) \\
& = \mathcal{G}_{\Xi_d}(g|x)\mathcal{G}_{\Xi_d}(g|y)\mu_{\Phi}^{(1)}(x)\mu_{\Phi}^{(1)}(y)\mathcal{G}_{\Phi}(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \\
& \quad \times \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} h(u)\tilde{l}_d(z_k|u)\mu_{\Phi}^{(1)}(u)\nu(du) \right) \\
& \quad + \mathcal{G}_{\Xi_d}(g|x)\mu_{\Phi}^{(1)}(x)\mathcal{G}_{\Phi}(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \\
& \quad \times \frac{\partial_{\delta_y}}{\partial h} \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} h(u)\tilde{l}_d(z_k|u)\mu_{\Phi}^{(1)}(u)\nu(du) \right) \\
& = \mathcal{G}_{\Xi_d}(g|x)\mathcal{G}_{\Xi_d}(g|y)\mu_{\Phi}^{(1)}(x)\mu_{\Phi}^{(1)}(y)\mathcal{G}_{\Phi}(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \\
& \quad \times \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} h(u)\tilde{l}_d(z_k|u)\mu_{\Phi}^{(1)}(u)\nu(du) \right) \\
& \quad + \mathcal{G}_{\Xi_d}(g|x)\mu_{\Phi}^{(1)}(x)\mathcal{G}_{\Phi}(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \\
& \quad \times \sum_{k=1}^m \tilde{l}_d(z_k|y)\mu_{\Phi}^{(1)}(y) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} h(u)\tilde{l}_d(z_l|u)\mu_{\Phi}^{(1)}(u)\nu(du) \right) \\
& = \mathcal{G}_{\Xi_d}(g|x)\mathcal{G}_{\Xi_d}(g|y)\mu_{\Phi}^{(1)}(x)\mu_{\Phi}^{(1)}(y)\mathcal{G}_{\Phi}(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \\
& \quad \times \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} h(u)\tilde{l}_d(z_k|u)\mu_{\Phi}^{(1)}(u)\nu(du) \right) \\
& \quad + \mathcal{G}_{\Xi_d}(g|x)\mu_{\Phi}^{(1)}(x)\mu_{\Phi}^{(1)}(y)\mathcal{G}_{\Phi}(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \\
& \quad \times \sum_{k=1}^m \tilde{l}_d(z_k|y) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} h(u)\tilde{l}_d(z_l|u)\mu_{\Phi}^{(1)}(u)\nu(du) \right). \tag{3.33}
\end{aligned}$$

By setting $g = 0$ and $h = 1$ in (5.1), we have

$$\begin{aligned}
& \frac{\partial_{\delta_y}}{\partial h} \left(\mathcal{G}_{\Xi_d}(g|x)\mu_{\Phi}^{(1)}(x)\mathcal{G}_{\Phi}(h(\cdot)\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \right. \\
& \quad \left. \times \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} h(u)\tilde{l}_d(z_k|u)\mu_{\Phi}^{(1)}(u)\nu(du) \right) \right) \Big|_{g=0, h=1} \\
& = q_d(x)q_d(y)\mu_{\Phi}^{(1)}(x)\mu_{\Phi}^{(1)}(y)\mathcal{G}_{\Phi}(\mathcal{G}_{\Xi_d}(g|\cdot))\mathcal{G}_{\Xi_c}(g) \Big|_{g=0} \\
& \quad \times \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} \tilde{l}_d(z_k|u)\mu_{\Phi}^{(1)}(u)\nu(du) \right)
\end{aligned}$$

$$\begin{aligned}
& + q_d(x) \mu_{\Phi}^{(1)}(x) \mu_{\Phi}^{(1)}(y) \mathcal{G}_{\Phi}(\mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0} \\
& \quad \times \sum_{k=1}^m \tilde{l}_d(z_k|y) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \\
& = q_d(x) \mu_{\Phi}^{(1)}(x) \mu_{\Phi}^{(1)}(y) \mathcal{G}_{\Phi}(\mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0} \\
& \quad \times \left(\sum_{k=1}^m \tilde{l}_d(z_k|y) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \right. \\
& \quad \left. + q_d(y) \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} \tilde{l}_d(z_k|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \right). \tag{3.34}
\end{aligned}$$

Thereafter, by taking the sum of (3.32) and (3.34), we have the numerator term of (3.22), which is given by

$$\begin{aligned}
& \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0, h=1} \\
& = \mu_{\Phi}^{(1)}(x) \mu_{\Phi}^{(1)}(y) \mathcal{G}_{\Phi}(\mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0} \\
& \quad \times \left(\sum_{k=1}^m \tilde{l}_d(z_k|x) \sum_{\substack{l=1 \\ l \neq k}}^m \tilde{l}_d(z_l|y) \prod_{\substack{s=1 \\ s \neq k, l}}^m \left(l_c(z_s) + \int_{\Lambda} \tilde{l}_d(z_s|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \right. \\
& \quad \left. + q_d(y) \left(\sum_{k=1}^m \tilde{l}_d(z_k|x) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \right) \right) \\
& \quad + q_d(x) \mu_{\Phi}^{(1)}(x) \mu_{\Phi}^{(1)}(y) \mathcal{G}_{\Phi}(\mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0} \\
& \quad \times \left(\sum_{k=1}^m \tilde{l}_d(z_k|y) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \right. \\
& \quad \left. + q_d(y) \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} \tilde{l}_d(z_k|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \right) \\
& = \mu_{\Phi}^{(1)}(x) \mu_{\Phi}^{(1)}(y) \mathcal{G}_{\Phi}(\mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0} \\
& \quad \times \left(\sum_{k=1}^m \tilde{l}_d(z_k|x) \sum_{\substack{l=1 \\ l \neq k}}^m \tilde{l}_d(z_l|y) \prod_{\substack{s=1 \\ s \neq k, l}}^m \left(l_c(z_s) + \int_{\Lambda} \tilde{l}_d(z_s|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \right. \\
& \quad \left. + q_d(y) \left(\sum_{k=1}^m \tilde{l}_d(z_k|x) \prod_{\substack{l=1 \\ l \neq k}}^m \left(l_c(z_l) + \int_{\Lambda} \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \right) \right)
\end{aligned}$$

$$\begin{aligned}
& + q_d(x) \left(\sum_{k=1}^m \tilde{l}_d(z_k|y) \prod_{\substack{l=1 \\ l \neq k}}^m (l_c(z_l) + \int_{\Lambda} \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du)) \right) \\
& + q_d(x) q_d(y) \prod_{k=1}^m \left(l_c(z_k) + \int_{\Lambda} \tilde{l}_d(z_k|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right). \tag{3.35}
\end{aligned}$$

Finally, by substituting (3.26) and (3.35) into (3.22), we have the second order posterior factorial moment density of $x, y \in \Lambda$, $x \neq y$, which is given by

$$\begin{aligned}
& \rho_{\Phi|\Xi=z_{1:m}}^{(2)}(x, y) \\
& = \frac{\frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0, h=1}}{\frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Phi}(\mathcal{G}_{\Xi_d}(g|\cdot)) \mathcal{G}_{\Xi_c}(g)|_{g=0}} \\
& = q_d(x) q_d(y) \mu_{\Phi}^{(1)}(x) \mu_{\Phi}^{(1)}(y) + \sum_{k=1}^m \frac{q_d(x) \mu_{\Phi}^{(1)}(x) \mu_{\Phi}^{(1)}(y) \tilde{l}_d(z_k|y)}{l_c(z_k) + \int_{\Lambda} \tilde{l}_d(z_k|u) \mu_{\Phi}^{(1)}(u) \nu(du)} \\
& + \sum_{k=1}^m \frac{q_d(y) \mu_{\Phi}^{(1)}(x) \mu_{\Phi}^{(1)}(y) \tilde{l}_d(z_k|x)}{l_c(z_k) + \int_{\Lambda} \tilde{l}_d(z_k|u) \mu_{\Phi}^{(1)}(u) \nu(du)} \\
& + \sum_{\substack{k, l=1 \\ k \neq l}}^m \frac{\mu_{\Phi}^{(1)}(x) \mu_{\Phi}^{(1)}(y) \tilde{l}_d(z_k|x) \tilde{l}_d(z_l|y)}{\left(l_c(z_k) + \int_{\Lambda} \tilde{l}_d(z_k|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \left(l_c(z_l) + \int_{\Lambda} \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right)}.
\end{aligned}$$

□

Spatial Point Process Poisson PHD Filtering Equations

Following next, we state the spatial point process Poisson PHD filtering equations. The Poisson prediction first order moment density of the PHD filtering equation (3.14) is given by

$$\mu_{\Phi}^{(1)}(x) = \mu_{\Phi_b}^{(1)}(x) + \int_{\Lambda} \tilde{l}_s(x|u) \mu_{\Psi}^{(1)}(u) \nu(du).$$

The Poisson prediction second order factorial moment density of the PHD filtering equation (3.16) is given by

$$\rho_{\Phi}^{(2)}(x, y) = \rho_{\Phi_b}^{(2)}(x, y) + \mu_{\Phi_b}^{(1)}(x) \int_{\Lambda} \tilde{l}_s(y|u) \mu_{\Psi}^{(1)}(u) \nu(du) + \mu_{\Phi_b}^{(1)}(y) \int_{\Lambda} \tilde{l}_s(x|u) \mu_{\Psi}^{(1)}(u) \nu(du)$$

$$+ \int_{\Lambda^2} \tilde{l}_s(x|u) \tilde{l}_s(y|v) \rho_{\Psi}^{(2)}(u, v) \nu(du) \nu(dv).$$

The Poisson posterior first order moment density of the PHD filtering equation (3.23) is given by

$$\mu_{\Phi|\Xi=z_{1:m}}^{(1)}(x) = q_d(x) \mu_{\Phi}^{(1)}(x) + \sum_{z \in z_{1:m}} \frac{p_d(x) \mu_{\Phi}^{(1)}(x)}{\mu_{\Xi}^{(1)}(z) + \int_{\Lambda} \tilde{l}_d(z|u) \mu_{\Phi}^{(1)}(u) \nu(du)}.$$

The Poisson posterior second order factorial moment density of the PHD filtering equation (3.29) is given by

$$\begin{aligned} \rho_{\Phi|\Xi=z_{1:m}}^{(2)}(x, y) &= q_d(x) q_d(y) \mu_{\Phi}^{(1)}(x) \mu_{\Phi}^{(1)}(y) + \sum_{k=1}^m \frac{q_d(x) \mu_{\Phi}^{(1)}(x) \mu_{\Phi}^{(1)}(y) \tilde{l}_d(z_k|y)}{l_c(z_k) + \int_{\Lambda} \tilde{l}_d(z_k|u) \mu_{\Phi}^{(1)}(u) \nu(du)} \\ &+ \sum_{k=1}^m \frac{q_d(y) \mu_{\Phi}^{(1)}(x) \mu_{\Phi}^{(1)}(y) \tilde{l}_d(z_k|x)}{l_c(z_k) + \int_{\Lambda} \tilde{l}_d(z_k|u) \mu_{\Phi}^{(1)}(u) \nu(du)} \\ &+ \sum_{\substack{k,l=1 \\ k \neq l}}^m \frac{\mu_{\Phi}^{(1)}(x) \mu_{\Phi}^{(1)}(y) \tilde{l}_d(z_k|x) \tilde{l}_d(z_l|y)}{\left(l_c(z_k) + \int_{\Lambda} \tilde{l}_d(z_k|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right) \left(l_c(z_l) + \int_{\Lambda} \tilde{l}_d(z_l|u) \mu_{\Phi}^{(1)}(u) \nu(du) \right)}, \end{aligned}$$

where $x, y \in \Lambda$, $x \neq y$.

3.2.4 Sequential Monte Carlo Implementation

In this subsection, we describe and state the *nearly constant turn-rate* ground truth motion and measurement dynamics. We also give an extension of the motion dynamics to incorporate targets interaction among targets in Remark 3.2.6. In addition, we give some simulation comparison plots after implementing the sequential Monte Carlo (SMC) Poisson PHD filtering algorithm. The simulation scenario and parameters are equivalent as of [9], [22]. In our simulation comparison plots of the SMC-PHD filtering algorithm, we describe the outputs of independent and interactive ground truth dynamics among multiple targets in target space.

Ground Truth and Measurement Dynamics

The state of each target at time t is given by $\mathbf{x}_t = (x_t, \dot{x}_t, y_t, \dot{y}_t, \theta_t)^\top$, where x_t, y_t are the cartesian coordinates, $\dot{x}_t, \dot{y}_t$ are the respective velocities, and θ_t is the turn

PHD Filter Initialization (time $t = 0$)

Set $\gamma_{\Phi_b,0} \in \mathbb{R}^+$, $\eta \in (0, 1)$ and $P_p \in \mathbb{N}$.
 Sample $N_{\Phi_b,0}$ particle state dynamics $\{\mathbf{x}_i\}_{i=1}^{N_{\Phi_b,0}}$ for initial birth process Φ_b uniformly distributed within state space Λ .
 Initialize the (prediction) first order moment density kernel $\mu_{\Phi,0}^{(1)}$ stated below.
for $1 \leq i, j \leq N_{\Phi_b,0}$ **do**
 $\mu_{\Phi,0}^{(1)}(\mathbf{x}_i) := \gamma_{\Phi_b,0}/N_{\Phi_b,0}$
end for
 Compute the Poisson (posterior) first order

moment density kernel $\mu_{\Phi|\Xi,0}^{(1)}$ using (3.23).
 Estimate the number of targets as $\gamma_0 := \sum_{i=1}^{N_{\Phi_b,0}} \mu_{\Phi|\Xi,0}^{(1)}(\mathbf{x}_i)$.
 Perform resampling as in [10] to obtain the particle state dynamics $\{\mathbf{x}_i\}_{i=1}^{N_0}$ where $N_0 := P_p \times \lfloor \gamma_0 \rfloor$.
 Reassign uniform weight to the resampled particles $\{x_i\}_{i=1}^{N_0}$ stated below.
for $1 \leq i, j \leq N_0$ **do**
 $\mu_{\Phi|\Xi,0}^{(1)}(\mathbf{x}_i) := \gamma_0/N_0$.
end for

PHD Filter (time $t + 1 \geq 1$)

Compute the (prediction) state transition dynamics $\{\mathbf{x}_{t+1|t}^{(i)}\}_{i=1}^{N_t}$ from $\{\mathbf{x}_t^{(i)}\}_{i=1}^{N_t}$ based on (3.36).
 Compute the Poisson (prediction) first order moment density kernel $\mu_{\Phi,t+1|t}^{(1)}$ from $\mu_{\Psi,t+1}^{(1)} := \mu_{\Phi|\Xi,t}^{(1)}$ using (3.14).
 Sample $N_{\Phi_b,t+1}$ new particle state dynamics for birth process Φ_b at time $t + 1$ uniformly distributed within state space Λ to generate $\{\mathbf{x}_i^{(i)}\}_{i=1}^{N_{\Phi_b,t+1}}$ where $N_{\Phi_b,t+1} := P_b \times \lfloor \gamma_t \rfloor$.
 Extend the Poisson (prediction) first order moment density kernel $\mu_{\Phi,t+1|t}^{(1)}$ to dimension $N_{\Phi,t+1} := N_t + N_{\Phi_b,t+1}$ with state dynamics $\{\mathbf{x}_i\}_{i=1}^{N_{\Phi,t+1}} := \{\mathbf{x}_{t+1|t}^{(i)}\}_{i=1}^{N_t} \cup \{\mathbf{x}_i^{(i)}\}_{i=1}^{N_{\Phi_b,t+1}}$, where the following indexes and assigned weights are allocated to the new particles.

for $N_t + 1 \leq i, j \leq N_{\Phi,t+1}$ **do**
 $\mu_{\Phi,t+1|t}^{(1)}(\mathbf{x}_i) := \eta \gamma_t / N_{\Phi_b,t+1}$
end for
 Compute the Poisson (posterior) first order moment density kernel $\mu_{\Phi|\Xi,t+1}^{(1)}$ using (3.23).
 Estimate the number of targets as $\gamma_{t+1} := \sum_{i=1}^{N_{\Phi,t+1}} \mu_{\Phi|\Xi,t+1}^{(1)}(\mathbf{x}_i)$.
 Perform resampling as in [10] to obtain the particle state dynamics $\{\mathbf{x}_i\}_{i=1}^{N_{t+1}}$ where $N_{t+1} := P_p \times \lfloor \gamma_{t+1} \rfloor$ (capped at 800 particles).
 Reassign uniform weight to the resampled particles $\{x_i\}_{i=1}^{N_{t+1}}$ stated below.
for $1 \leq i, j \leq N_{t+1}$ **do**
 $\mu_{\Phi|\Xi,t+1}^{(1)}(\mathbf{x}_i) := \gamma_{t+1}/N_{t+1}$.
end for

rate. At time $t + 1$, the location of every target is given by the following equation

$$\mathbf{x}_{t+1} = \mathbf{F}(\theta_t)\mathbf{x}_t + \mathbf{G}\mathbf{v}_t, \quad (3.36)$$

for which $\mathbf{v}_t = (v_x, v_y, v_\theta)^\top$ is the acceleration noise vector distributed according to the 3-dimensional multivariate Gaussian distribution with mean vector $\boldsymbol{\mu}_0$ and

covariance matrix Σ denoted by

$$\mathbf{v}_t \sim \mathcal{N}(\boldsymbol{\mu}_0, \Sigma), \text{ where } \boldsymbol{\mu}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_{v_x}^2 & 0 & 0 \\ 0 & \sigma_{v_y}^2 & 0 \\ 0 & 0 & \sigma_{v_\theta}^2 \end{pmatrix},$$

and

$$\mathbf{F} = \begin{pmatrix} 1 & \sin(\tau\theta_t)/\theta_t & 0 & (\cos(\tau\theta_t) - 1)/\theta_t & 0 \\ 0 & \cos(\tau\theta_t) & 0 & -\sin(\tau\theta_t) & 0 \\ 0 & (\cos(\tau\theta_t) - 1)/\theta_t & 1 & \sin(\tau\theta_t)/\theta_t & 0 \\ 0 & \sin(\tau\theta_t) & 0 & \cos(\tau\theta_t) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{G} = \begin{pmatrix} \tau^2/2 & 0 & 0 \\ \tau & 0 & 0 \\ 0 & \tau^2/2 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau \end{pmatrix},$$

with τ the time sampling period.

Remark 3.2.6. *More specific ground truth dynamics such as repulsive interaction motion model can be traced back to the stationary distribution of a DPP, a simple case of a Ginibre DPP, see for example Equation (2.19) in § 2.2 of [17]. Repulsive dynamics within motion model can be introduced to enhance interaction among multiple targets within a target space. Suppose there are N targets in the target space, the motion model with repulsive dynamics for each target i for $i \in \{1, \dots, N\}$, is given by*

$$\mathbf{x}_{t+1}^i = \mathbf{F}(\theta_t)\mathbf{x}_t^i + \mathbf{G}\mathbf{v}_t^i + \mathbf{s}_t^i \quad (3.37)$$

where

$$\mathbf{s}_t^i = \begin{pmatrix} \alpha_x \sum_{j=1}^N \frac{x_t^i - x_t^j}{|\mathbf{x}_t^i - \mathbf{x}_t^j|} \\ 0 \\ \alpha_y \sum_{j=1}^N \frac{y_t^i - y_t^j}{|\mathbf{x}_t^i - \mathbf{x}_t^j|} \\ 0 \\ 0 \end{pmatrix}$$

is a repulsion term which avoids collisions amongst targets. In the case $\alpha_x = \alpha_y = 0$, the modified approach recovers the standard motion dynamics in (3.36).

The measurement vector of each target at time t is written as $\mathbf{m}_t = (m_{r_t}, m_{\omega_t})^\top$ using the range and bearing components m_{r_t} and m_{ω_t} . The measurement generated by every target at time $t + 1$ is then given by the following equation

$$\mathbf{m}_{t+1} = \mathbf{p}_{t+1} + \mathbf{w}_{t+1}, \quad (3.38)$$

for which $\mathbf{w}_{t+1} = (w_{r_{t+1}}, w_{\omega_{t+1}})^\top$ is the measurement noise vector distributed according to the 2-dimensional multivariate Gaussian distribution with mean vector $\boldsymbol{\mu}_{0,m}$ and covariance matrix $\boldsymbol{\Sigma}_m$ denoted by

$$\mathbf{w}_{t+1} \sim \mathcal{N}(\boldsymbol{\mu}_{0,m}, \boldsymbol{\Sigma}_m), \text{ where } \boldsymbol{\mu}_{0,m} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{\Sigma}_m = \begin{pmatrix} \sigma_{w_r}^2 & 0 \\ 0 & \sigma_{w_\omega}^2 \end{pmatrix},$$

$$\text{and } \mathbf{p}_{t+1} = \begin{pmatrix} \sqrt{x_{t+1}^2 + y_{t+1}^2} \\ \arctan(y_{t+1}/x_{t+1}) \end{pmatrix}.$$

SMC-Poisson PHD Filter Outputs

We provide simulation comparison plots after implementing the sequential Monte Carlo (SMC) Poisson PHD filtering algorithm (see [5], [22] and [25]) in one trial of 50 time steps for the purpose of illustration and motivation of this thesis. In Figure 3.1, we give the SMC-PHD filter output of independent targets with motion trajectories following (3.36) over 50 time steps. In Figure 3.2, we give the SMC-PHD filter output of interactive targets with repulsive motion trajectories following (3.37) over 50 time steps. The measurement model for both figures follow (3.38). The simulation scenario and parameters are equivalent as of [9],[22] and 4 ground truth targets are initiated at 4 randomly appointed locations within a uniform $[-2000, 2000]$ by $[-2000, 2000]$ square boundary target space. Corresponding starting locations of the 4 targets for both figures are the same and are indicated by enlarged black “+”-signs. Target motion trajectories are indicated in black “.” dots. The probability of survival for each target is set at 1. Clutter measurements are generated according to Poisson with rate $l_c = 1$. The clutter points in the figures are indicated by black “ $\times$ ”-sign. Measurement generated are in red “+”-signs, and the probability of detection associated with each target is set at 0.9 for both figures.

Our SMC implementation uses 1000 discretized particles at initialization, 100 resampling particles per expected target, and 100 new particles per expected target birth, which follows a time-dependent Poisson birth process. The total number of particles are capped at 800 and the minimum number of particles are benchmarked

Figure 3.1: SMC-PHD Filter Performance on Independent Targets.

at 200 every time step. There is no birth of new targets in the ground truth trajectories for both simulations. The resampling step required in SMC-PHD filter to prevent sample impoverishment follows the modified universal resampling scheme introduced in [10]. The posterior state estimates of target locations follow the prescribed threshold weighted average.

Our code performs data association between target-originated measurements and posterior state estimates. For this, given a measurement we select its associated estimate by minimizing the distance between the measurement and all candidate estimates. The association of state estimates are indicated by an edge joining the ground truth in the figures. Blue edges indicate that the state estimates performs better than the associated measurements in terms of Euclidean distance to the ground truth, red edges indicate poorer performance for the state estimates when compared with the measurement to the ground truth. The ratio of overall performance of good state estimates against poor state estimates are reflected at the bottom of each figure. More details about the SMC-Poisson PHD filter implementation and applications, readers can refer to [26].

In Figure 3.1, the performance ratio of state estimates is 56%. In Figure 3.2, the

Figure 3.2: SMC-PHD Filter Performance on Repulsive Target Interactions.

performance ratio of state estimates is 47%, for which the SMC-Poisson PHD filter fairs poorer when targets exhibit some form of interaction. The overall state estimate performance for both figures are expected to show poorer performance due to the fact that PHD filter only produces first order moment density (or intensity function about the target space) and is not equipped with reliable point estimates extraction property as in [25]. However, at this stage, the idea of target interaction among multiple targets in target space motivated the interest of this thesis to construct a second order PHD filter based on Determinantal Point Processes (DPPs) which is presented in Chapter 5.

Chapter 4

Combinatorial Approach to Point Process Moments and Densities

In this chapter, we focus on the computation and derivations of factorial moments and Janossy densities related to determinantal point processes, branch determinantal point processes and general branched point processes by combinatorial and functional differential approach. The main results derived in this chapter form part of the contribution in the construction of the determinantal PHD filter in Chapter 5.

In Section 4.1, we layout some properties of DPPs and provide explicit expressions for the associated PGFl of which is described by the determinant of its kernel, a form that is disparate from the general PGFl of point processes stated in Chapter 2. In subsection 4.1.1, we recover the factorial moment densities using functional derivatives of the associated DPP PGFl formulas. We show that the general derivative of the DPP associated PGFl under specific conditions obtained in Corollary 4.1.9 independently recovers Proposition 4.1.2. In subsection 4.1.2, we present the general expression of the DPP Janossy densities in Proposition 4.1.12 for which the proof independently recovers the result in Equation (2.18) of Proposition 2.2.4. The Janossy expression derived is represented in terms of the associated factorial moment density. In subsection 4.1.3, we construct and show some useful properties and relationships associated with the branched DPP PGFl. We describe the results of the marginal first and second order factorial moment densities evaluated using the branched DPP PGFl expressions. The result of the marginal first order moment density is given in Proposition 4.1.14 and the result of the marginal second order factorial moment density is given in Proposition 4.1.15.

In Section 4.2, we present a combinatorial expression of the PGFl for general branched point processes and derive the general marginal and joint Janossy densities. In Proposition 4.2.8, we derive the result of the marginal Janossy densities. In Proposition 4.2.12, we derive the result of the joint Janossy densities. In Proposition 4.2.12, we derive the result of the joint Janossy densities.

4.1 Determinantal Point Processes (DPPs)

Determinantal point processes (DPP), also known as fermion point processes, model point configurations that exhibit repulsive interaction among points in a domain space (see [12],[19],[23],[31] and reference therein). The DPPs in recent years have attracted much research interest with applications such as machine learning in [32], wireless network communication (see [16],[30],[33]) and multi-target tracking in [8]. Such point processes possess several eminent mathematical properties such as Laplace transform, PGFls, factorial moment densities and Janossy densities in closed-form expressions that is characterized by its kernel. We begin this section with the definition of determinantal point processes and establish its associated properties including its PGFl expression.

For a locally compact second countable Hausdorff space Λ , let I_d be the identity operator on $L^2(\Lambda, \nu)$, where $L^2(\Lambda, \nu)$ is the Hilbert space of square integrable functions *w.r.t.* the Radon measure ν . And let $\mathfrak{K}$ be a trace class operator on $L^2(\Lambda, \nu)$,

$$\begin{aligned} \mathfrak{K} : L^2(\Lambda, \nu) &\rightarrow L^2(\Lambda, \nu) \\ f(x) &\mapsto \mathfrak{K}[f](x) = \int_{\Lambda} K(x, y) f(y) \nu(dy), \end{aligned}$$

with $\text{tr} K = \int_{\Lambda} K(x, x) \nu(dx) < +\infty$. The expression below is known as the Fredholm determinant of the operator $I_d - \mathfrak{K}$,

$$\text{Det}(I_d - \mathfrak{K}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det(K(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}),$$

where K is the kernel of $\mathfrak{K}$ and $\det(K(x_i, x_j)_{1 \leq i, j \leq n})$ is the determinant of the $n \times n$ square matrix $K(x_i, x_j)_{1 \leq i, j \leq n}$.

Definition 4.1.1. Let $\mathfrak{K}_{\Psi}$ be a Hermitian trace class operator on $L^2(\Lambda, \nu)$, with spectrum contained in $[0, 1)$, a kernel $K_{\Psi} : \Lambda \times \Lambda \rightarrow \mathbb{R}^+$, and $\alpha \in \{-\frac{1}{m} : m \in \mathbb{N}_1\}$.

A point process Ψ on Λ is an α -determinantal point process (α -DPP) with kernel $K_\Psi(x, y)$, $x, y \in \Psi$ if Ψ has characteristic Laplace functional given by

$$\tilde{\mathcal{L}}_\Psi(g) = \text{Det} \left(I_d + \alpha \left(\sqrt{1 - e^{-g(x)}} K_\Psi(x, y) \sqrt{1 - e^{-g(y)}} \right)_{x, y \in \Lambda} \right)^{-\frac{1}{\alpha}},$$

for any nonnegative measurable function $g \in \mathcal{B}_+(\Lambda)$, where $\mathcal{B}_+(\Lambda)$ denotes the set of bounded measurable functions from Λ to $\mathbb{R}^+$.

The associated PGFl of α -DPP is given by

$$\tilde{\mathcal{G}}_\Psi(h) = \tilde{\mathcal{L}}_\Psi(-\ln h) = \text{Det} \left(I_d + \alpha \left(\sqrt{1 - h(x)} K_\Psi(x, y) \sqrt{1 - h(y)} \right)_{x, y \in \Lambda} \right)^{-\frac{1}{\alpha}}, \quad (4.1)$$

where $h = e^{-g}$ is a nonnegative measurable function from $[0, 1)$ on Λ , see Definition 2.2.6 and Definition 2.2.9.

In this sequel, we only consider the case of $\alpha = -1$; that is, the (-1) -DPP, also classically known as the determinantal point process (DPP).

In the following proposition, we recover the classical result of the factorial moment densities of DPP independently in Corollary 4.1.9 by the derivatives of the DPP associated PGFl.

Proposition 4.1.2 (See Theorem 3.6 of [19] and Theorem 2.3 of [4]). *For the case $\alpha = -1$, the point process Ψ is a determinantal point process (DPP) with kernel K_Ψ and is equipped with factorial moment densities of the form*

$$\rho_\Psi^{(n)}(x_{1:n}) = \det(K_\Psi(x_i, x_j))_{1 \leq i, j \leq n}, \quad (4.2)$$

where $\rho_\Psi^{(n)}(x_{1:n})$ satisfy the relation

$$\mathbb{E}[\Psi(B_1) \cdots \Psi(B_n)] = \int_{B_1} \cdots \int_{B_n} \rho_\Psi^{(n)}(x_{1:n}) \nu(dx_{1:n}),$$

for $B_1, \dots, B_n$ are disjoint subsets of Λ .

The associated PGFl of the DPP Ψ is given by

$$\tilde{\mathcal{G}}_\Psi(h) = \text{Det} \left(I_d - \left(\sqrt{1 - h(x)} K_\Psi(x, y) \sqrt{1 - h(y)} \right)_{x, y \in \Lambda} \right)$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \det \left(- \left(\sqrt{1-h(x_i)} K_{\Psi}(x_i, x_j) \sqrt{1-h(x_j)} \right)_{1 \leq i, j \leq n} \right) \nu(dx_{1:n}) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det \left(\left(\sqrt{1-h(x_i)} K_{\Psi}(x_i, x_j) \sqrt{1-h(x_j)} \right)_{1 \leq i, j \leq n} \right) \nu(dx_{1:n})
\end{aligned} \tag{4.3}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \prod_{i=1}^n (1-h(x_i)) \det(K_{\Psi}(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}). \tag{4.4}$$

Proposition 4.1.3 (Proposition 2.4 of [4] and Lemma 3.3 of [19]). *The determinantal point process Ψ on Λ with kernel K_{Ψ} has its Janossy densities given by*

$$j_{\Psi}^{(0)} = \text{Det}(I_d - K_{\Psi}), \quad j_{\Psi}^{(1)}(x) = \text{Det}(I_d - K_{\Psi}) J_{\Psi}(x, x), \quad x \in \Lambda, \tag{4.5}$$

and

$$j_{\Psi}^{(n)}(x_{1:n}) := \text{Det}(I_d - K_{\Psi}) \det(J_{\Psi}(x_i, x_j)_{1 \leq i, j \leq n}), \quad x_{1:n} := (x_1, \dots, x_n) \in \Lambda,$$

where J_{Ψ} is the kernel of the operator $\mathfrak{J}_{\Psi}$ defined by

$$\mathfrak{J}_{\Psi} := (I_d - \mathfrak{K}_{\Psi})^{-1} \mathfrak{K}_{\Psi},$$

By setting $h = 1$ in $\tilde{\mathcal{G}}_{\Psi}(h)$, we have $\tilde{\mathcal{G}}_{\Psi}(1) = 1$. And by setting $h = 0$, we have

$$\tilde{\mathcal{G}}_{\Psi}(0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det(K_{\Psi}(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}) = \text{Det}(I_d - K_{\Psi}) = j_{\Psi}^{(0)}.$$

4.1.1 Derivatives and Factorial Moment Densities

In this subsection, we derive functional derivatives from the associated PGFl of DPP. By the derivative results obtained, we recover factorial moment densities of the DPP. We show that the general derivative of the DPP associated PGFl under specific conditions obtained in Corollary 4.1.9 recovers Proposition 4.1.2.

First Order Moment Densities of DPPs

In the following proposition, we present the first order functional derivative of $\tilde{\mathcal{G}}_{\Psi}(h)$.

Proposition 4.1.4. *For a DPP Ψ with PGFl $\tilde{\mathcal{G}}_{\Psi}(h)$, the first order functional derivative of $\tilde{\mathcal{G}}_{\Psi}(h)$ evaluated in the direction of h with a nonnegative measurable function f is given by*

$$\frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_{\Psi}(h)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \int_{\Lambda^n} \left(\sum_{p=1}^n f(x_p) \prod_{\substack{q=1 \\ p \neq q}}^n (1 - h(x_q)) \right) \det(K_{\Psi}(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}). \quad (4.6)$$

Proof. By evaluating the functional derivative of (4.4) in the direction of h with nonnegative measurable function f , we have

$$\begin{aligned} & \frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_{\Psi}(h) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \prod_{i=1}^n (1 - h(x_i) - \epsilon f(x_i)) \det(K_{\Psi}(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}) \right. \\ & \quad \left. - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \prod_{i=1}^n (1 - h(x_i)) \det(K_{\Psi}(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}) \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \sum_{k=1}^n (-\epsilon)^k \sum_{\substack{\alpha \subset \{1, \dots, n\} \\ |\alpha|=k}} \prod_{p \in \alpha} f(x_p) \\ & \quad \times \prod_{q \in \{1, \dots, n\} \setminus \alpha} (1 - h(x_q)) \det(K_{\Psi}(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \int_{\Lambda^n} \left(\sum_{p=1}^n f(x_p) \prod_{\substack{q=1 \\ p \neq q}}^n (1 - h(x_q)) \right) \det(K_{\Psi}(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}) \\ &= \int_{\Lambda} f(x_1) \det(K_{\Psi}(x_1, x_1)) \nu(dx_1) \\ & \quad + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n!} \int_{\Lambda^n} \left(\sum_{p=1}^n f(x_p) \prod_{\substack{q=1 \\ p \neq q}}^n (1 - h(x_q)) \right) \det(K_{\Psi}(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}). \end{aligned}$$

□

From Proposition 4.1.4, we can recover the first order moment density of the DPP Ψ by substituting the function f with Dirac delta function at a point $x \in \Lambda$ and setting the corresponding function $h = 1$.

Corollary 4.1.5. *The first order moment density of the DPP Ψ at a point $x \in \Lambda$ is given by*

$$\mu_{\Psi}^{(1)}(x) = \frac{\partial_{\delta_x}}{\partial h} \tilde{\mathcal{G}}_{\Psi}(h)|_{h=1} = K_{\Psi}(x, x). \quad (4.7)$$

Proof. By setting $h = 1$ into (4.6) of Proposition 4.1.4, we have,

$$\frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_\Psi(h)|_{h=1} = \int_\Lambda f(x_1) \det(K_\Psi(x_1, x_1)) \nu(dx_1), \quad (4.8)$$

where the sum for $n \geq 2$ vanish to zero. By substituting the function f with Dirac delta function at the point $x \in \Lambda$, we recover (4.7), which is given by

$$\mu_\Psi^{(1)}(x) = \frac{\partial_{\delta_x}}{\partial h} \tilde{\mathcal{G}}_\Psi(h)|_{h=1} = \int_\Lambda \delta_x(x_1) \det(K_\Psi(x_1, x_1)) \nu(dx_1) = K_\Psi(x, x).$$

□

Second Order Factorial Moment Densities of DPPs

In the following, we present the second order functional derivative of $\tilde{\mathcal{G}}_\Psi(h)$.

Proposition 4.1.6. *For a DPP Ψ with PGFl $\tilde{\mathcal{G}}_\Psi(h)$, the second order functional derivative of $\tilde{\mathcal{G}}_\Psi(h)$ evaluated in the direction of h with nonnegative measurable functions f and g is given by*

$$\begin{aligned} & \frac{\partial_f}{\partial h} \frac{\partial_g}{\partial h} \tilde{\mathcal{G}}_\Psi(h) \\ &= \sum_{n=2}^{\infty} \frac{(-1)^{n+2}}{n!} \int_{\Lambda^n} \sum_{\substack{p,q=1 \\ p \neq q}}^n f(x_p) g(x_q) \prod_{\substack{r=1 \\ r \neq p,q}}^n (1 - h(x_r)) \det(K_\Psi(x_i, x_j)_{1 \leq i,j \leq n}) \nu(dx_{1:n}). \end{aligned} \quad (4.9)$$

Proof. By evaluating the functional derivative of (4.6) in Proposition 4.1.4 evaluated in the direction of h with a nonnegative measurable function g , we have

$$\begin{aligned} & \frac{\partial_f}{\partial h} \frac{\partial_g}{\partial h} \tilde{\mathcal{G}}_\Psi(h) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_\Psi(h + \epsilon g) - \frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_\Psi(h) \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \int_{\Lambda^n} \sum_{p=1}^n f(x_p) \prod_{\substack{q=1 \\ p \neq q}}^n (1 - h(x_q) - \epsilon g(x_q)) \right. \\ & \quad \times \det(K_\Psi(x_i, x_j)_{1 \leq i,j \leq n}) \nu(dx_{1:n}) \\ & \quad \left. - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \int_{\Lambda^n} \left(\sum_{p=1}^n f(x_p) \prod_{\substack{q=1 \\ p \neq q}}^n (1 - h(x_q)) \right) \det(K_\Psi(x_i, x_j)_{1 \leq i,j \leq n}) \nu(dx_{1:n}) \right) \end{aligned}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^{n+2}}{n!} \int_{\Lambda^n} \sum_{\substack{p,q=1 \\ p \neq q}}^n f(x_p)g(x_q) \prod_{\substack{r=1 \\ r \neq p,q}}^n (1 - h(x_r)) \det(K_{\Psi}(x_i, x_j)_{1 \leq i,j \leq n}) \nu(dx_{1:n}).$$

□

From Proposition 4.1.6, we can recover the second order factorial moment density of the DPP Ψ by substituting the functions f and g with Dirac delta functions at two distinct points $x, y \in \Lambda$, and setting the corresponding function $h = 1$.

Corollary 4.1.7. *The second order factorial moment density of the DPP Ψ at two distinct points $x, y \in \Lambda$ is given by*

$$\rho_{\Psi}^{(2)}(x, y) = \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \tilde{\mathcal{G}}_{\Psi}(h)|_{h=1} = K_{\Psi}(x, x)K_{\Psi}(y, y) - (K_{\Psi}(x, y))^2. \quad (4.10)$$

Proof. By setting $h = 1$ in (4.9) of Proposition 4.1.6, we have

$$\begin{aligned} & \frac{\partial_f}{\partial h} \frac{\partial_g}{\partial h} \tilde{\mathcal{G}}_{\Psi}(h)|_{h=1} \\ &= \frac{1}{2!} \int_{\Lambda^2} \left(f(x_1)g(x_2) + f(x_2)g(x_1) \right) \det(K_{\Psi}(x_i, x_j)_{1 \leq i,j \leq 2}) \nu(dx_1) \nu(dx_2) \\ &= \frac{1}{2!} \int_{\Lambda^2} \left(f(x_1)g(x_2) + f(x_2)g(x_1) \right) \\ & \quad \times \left(K_{\Psi}(x_1, x_1)K_{\Psi}(x_2, x_2) - K_{\Psi}(x_1, x_2)K_{\Psi}(x_2, x_1) \right) \nu(dx_1) \nu(dx_2). \end{aligned}$$

Hence, by substituting $f = \delta_x$, $g = \delta_y$, and applying the symmetric property of K_{Ψ} , we recovered (4.10), which is given by

$$\begin{aligned} & \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \tilde{\mathcal{G}}_{\Psi}(h)|_{h=1} \\ &= \frac{1}{2!} \int_{\Lambda^2} \left(\delta_x(x_1)\delta_y(x_2) + \delta_x(x_2)\delta_y(x_1) \right) \\ & \quad \times \left(K_{\Psi}(x_1, x_1)K_{\Psi}(x_2, x_2) - (K_{\Psi}(x_1, x_2))^2 \right) \nu(dx_1) \nu(dx_2) \\ &= K_{\Psi}(x, x)K_{\Psi}(y, y) - (K_{\Psi}(x, y))^2. \end{aligned}$$

□

Higher Order Factorial Moment Densities of DPPs

In the following, we extend Proposition 4.1.6 to the general functional derivatives of the DPP associated PGFl $\tilde{\mathcal{G}}_{\Psi}(h)$.

Proposition 4.1.8. *For a DPP Ψ with PGFl $\tilde{\mathcal{G}}_\Psi(h)$, the general r^{th} order functional derivative of $\tilde{\mathcal{G}}_\Psi(h)$ evaluated in the direction of h with nonnegative measurable functions $f_1, \dots, f_r$ is given by*

$$\begin{aligned} & \frac{\partial_{f_1}}{\partial h} \dots \frac{\partial_{f_r}}{\partial h} \tilde{\mathcal{G}}_\Psi(h) \\ &= \sum_{n=r}^{\infty} \frac{(-1)^{n+r}}{n!} \int_{\Lambda^n} \left(\sum_{\substack{p_1, \dots, p_r=1 \\ p_1 \neq \dots \neq p_r}}^n f_1(x_{p_1}) \dots f_r(x_{p_r}) \prod_{q \in \{1, \dots, n\} \setminus \{p_1, \dots, p_r\}} (1 - h(x_q)) \right) \\ & \quad \times \det(K_\Psi(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}). \end{aligned} \quad (4.11)$$

Proof. Recall from the expression in (4.6), the first order derivative of (4.4) evaluated in the direction of h with function f_1 is given by

$$\begin{aligned} & \frac{\partial_{f_1}}{\partial h} \tilde{\mathcal{G}}_\Psi(h) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \int_{\Lambda^n} \left(\sum_{p=1}^n f_1(x_p) \prod_{\substack{q=1 \\ p \neq q}}^n (1 - h(x_q)) \right) \det(K_\Psi(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}). \end{aligned}$$

From the expression in (4.9), the second order derivative of (4.4) evaluated in the direction of h with functions f_1 and f_2 is given by

$$\begin{aligned} & \frac{\partial_{f_1}}{\partial h} \frac{\partial_{f_2}}{\partial h} \tilde{\mathcal{G}}_\Psi(h) \\ &= \sum_{n=2}^{\infty} \frac{(-1)^{n+2}}{n!} \int_{\Lambda^n} \left(\sum_{\substack{p, q=1 \\ p \neq q}}^n f_1(x_p) f_2(x_q) \prod_{\substack{r=1 \\ r \neq p, q}}^n (1 - h(x_r)) \right) \\ & \quad \times \det(K_\Psi(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}). \end{aligned}$$

By similar argument, the expression of the general r^{th} order derivative of (4.4) evaluated in the direction of h with functions $f_1, \dots, f_r$ is given by

$$\begin{aligned} & \frac{\partial_{f_1}}{\partial h} \dots \frac{\partial_{f_r}}{\partial h} \tilde{\mathcal{G}}_\Psi(h) \\ &= \frac{\partial_{f_r}}{\partial h} \left(\sum_{n=r-1}^{\infty} \frac{(-1)^{n+r-1}}{n!} \right. \\ & \quad \times \int_{\Lambda^n} \left(\sum_{\substack{p_1, \dots, p_{r-1}=1 \\ p_1 \neq \dots \neq p_{r-1}}}^n f_1(x_{p_1}) \dots f_{r-1}(x_{p_{r-1}}) \prod_{q \in \{1, \dots, n\} \setminus \{p_1, \dots, p_{r-1}\}} (1 - h(x_q)) \right) \end{aligned}$$

$$\begin{aligned}
& \times \det(K_\Psi(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}) \Big) \\
&= \sum_{n=r}^{\infty} \frac{(-1)^{n+r}}{n!} \int_{\Lambda^n} \left(\sum_{\substack{p_1, \dots, p_r=1 \\ p_1 \neq \dots \neq p_r}}^n f_1(x_{p_1}) \cdots f_r(x_{p_r}) \prod_{q \in \{1, \dots, n\} \setminus \{p_1, \dots, p_r\}} (1 - h(x_q)) \right) \\
& \times \det(K_\Psi(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}).
\end{aligned}$$

□

From Proposition 4.1.8, we can recover the general factorial moment density of the DPP Ψ as in (4.2) of Proposition 4.1.2. By substituting the functions $f_1, \dots, f_r$ with Dirac delta functions $\delta_{y_1}, \dots, \delta_{y_r}$ at the sequel of distinct points $y_1, \dots, y_r \in \Lambda$ and by setting $h = 1$, we have the following corollary.

Corollary 4.1.9. *The r^{th} order factorial moment density of a DPP Ψ with kernel K_Ψ evaluated at the sequel of distinct points $y_1, \dots, y_r \in \Lambda$ is given by*

$$\rho_\Psi^{(r)}(y_1, \dots, y_r) = \det(K_\Psi(y_i, y_j)_{1 \leq i, j \leq r}). \quad (4.12)$$

Proof. By setting $h = 1$ and substituting the functions $f_1, \dots, f_r$ with Dirac delta functions $\delta_{y_1}, \dots, \delta_{y_r}$ respectively, we have the r^{th} order factorial moment of Ψ , which is given by

$$\begin{aligned}
& \rho_\Psi^{(r)}(y_1, \dots, y_r) \\
&= \frac{\partial}{\partial h} \cdots \frac{\partial}{\partial h} \tilde{\mathcal{G}}_\Psi(h) \Big|_{h=1} \\
&= \frac{(-1)^{2r}}{r!} \int_{\Lambda^r} \left(\sum_{\substack{p_1, \dots, p_r=1 \\ p_1 \neq \dots \neq p_r}}^r \delta_{y_1}(x_{p_1}) \cdots \delta_{y_r}(x_{p_r}) \right) \det(K_\Psi(x_i, x_j)_{1 \leq i, j \leq r}) \nu(dx_{1:r}) \\
&= \int_{\Lambda^r} \delta_{y_1}(x_1) \cdots \delta_{y_r}(x_r) \det(K_\Psi(x_i, x_j)_{1 \leq i, j \leq r}) \nu(dx_{1:r}) \\
&= \det(K_\Psi(y_i, y_j)_{1 \leq i, j \leq r}).
\end{aligned}$$

□

4.1.2 Janossy Densities

In this subsection, we recover the Janossy densities of DPP from the derivatives of its associated PGFl. In addition, we establish the link between Janossy densities and factorial moment densities of DPP.

First Order Janossy Density

The following proposition presents the first order Janossy density at a point $x \in \Lambda$ of the DPP Ψ .

Proposition 4.1.10. *The Janossy density of the DPP Ψ evaluated at a point $x \in \Lambda$ is given by*

$$\begin{aligned} j_{\Psi}^{(1)}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det(K_{\Psi}(x_i, x_j)_{\substack{1 \leq i, j \leq n+1 \\ |x_{n+1}=x}}) \nu(dx_{1:n}) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \rho_{\Psi}^{(n+1)}(x_{1:n}, x) \nu(dx_{1:n}) \end{aligned} \quad (4.13)$$

where the notation $K_{\Psi}(x_i, x_j)_{\substack{1 \leq i, j \leq n \\ |x_i=x}}$ refers to having the i^{th} component entry of K_{Ψ} equals to the point $x \in \Lambda$, that is, $K_{\Psi}(x_i, x_i) = K_{\Psi}(x, x)$.

Proof. By setting $h = 0$ and substituting f with δ_x in (4.6) of Proposition 4.1.4, we have

$$\begin{aligned} j_{\Psi}^{(1)}(x) &= \frac{\partial_{\delta_x}}{\partial h} \tilde{\mathcal{G}}_{\Psi}(h)|_{h=0} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \int_{\Lambda^n} \left(\sum_{p=1}^n \delta_x(x_p) \right) \det(K_{\Psi}(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}) \\ &= \int_{\Lambda} \delta_x(x_1) K_{\Psi}(x_1, x_1) \nu(dx_1) \\ &\quad - \frac{1}{2} \int_{\Lambda^2} \delta_x(x_1) \left(K_{\Psi}(x_1, x_1) K_{\Psi}(x_2, x_2) - (K_{\Psi}(x_1, x_2))^2 \right) \nu(dx_{1:2}) \\ &\quad - \frac{1}{2} \int_{\Lambda^2} \delta_x(x_2) \left(K_{\Psi}(x_1, x_1) K_{\Psi}(x_2, x_2) - (K_{\Psi}(x_1, x_2))^2 \right) \nu(dx_{1:2}) \\ &\quad + \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n!} \int_{\Lambda^n} \left(\sum_{p=1}^n \delta_x(x_p) \right) \det(K_{\Psi}(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}) \\ &= K_{\Psi}(x, x) - \frac{1}{2} \int_{\Lambda} \left(K_{\Psi}(x, x) K_{\Psi}(x_2, x_2) - (K_{\Psi}(x, x_2))^2 \right) \nu(dx_2) \\ &\quad - \frac{1}{2} \int_{\Lambda} \left(K_{\Psi}(x_1, x_1) K_{\Psi}(x, x) - (K_{\Psi}(x_1, x))^2 \right) \nu(dx_1) \\ &\quad + \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n!} \sum_{i=1}^n \int_{\Lambda^{n-1}} \det(K_{\Psi}(x_i, x_j)_{\substack{1 \leq i, j \leq n \\ |x_i=x}}) \nu(dx_{1:i-1}) \nu(dx_{i+1:n}) \end{aligned}$$

$$\begin{aligned}
&= K_\Psi(x, x) - \int_\Lambda \left(K_\Psi(x_1, x_1) K_\Psi(x, x) - (K_\Psi(x_1, x))^2 \right) \nu(dx_1) \\
&\quad + \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{(n-1)!} \int_{\Lambda^{n-1}} \det(K_\Psi(x_i, x_j)_{\substack{1 \leq i, j \leq n \\ |x_n = x}}) \nu(dx_{1:n-1}) \\
&= K_\Psi(x, x) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det(K_\Psi(x_i, x_j)_{\substack{1 \leq i, j \leq n+1 \\ |x_{n+1} = x}}) \nu(dx_{1:n}) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det(K_\Psi(x_i, x_j)_{\substack{1 \leq i, j \leq n+1 \\ |x_{n+1} = x}}) \nu(dx_{1:n}),
\end{aligned}$$

By (4.2) of Proposition 4.1.2, we have

$$\begin{aligned}
\frac{\partial_{\delta_x}}{\partial h} \tilde{\mathcal{G}}_\Psi(h)|_{h=0} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det(K_\Psi(x_i, x_j)_{\substack{1 \leq i, j \leq n+1 \\ |x_{n+1} = x}}) \nu(dx_{1:n}) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \rho_\Psi^{(n+1)}(x_{1:n}, x) \nu(dx_{1:n})
\end{aligned}$$

Note that at this point, the derivation of this result does not have an immediate representation that leads to (4.5). $\square$

Second Order Janossy Density

In the next proposition, we present the second order Janossy density at two distinct points $x, y \in \Lambda$ of the DPP Ψ .

Proposition 4.1.11. *The Janossy density of the DPP Ψ evaluated at two distinct points $x, y \in \Lambda$ is given by*

$$\begin{aligned}
j_\Psi^{(2)}(x, y) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det(K_\Psi(x_i, x_j)_{\substack{1 \leq i, j \leq n+2 \\ |x_{n+1} = x \\ |x_{n+2} = y}}) \nu(dx_{1:n}) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \rho_\Psi^{(n+2)}(x_{1:n}, x, y) \nu(dx_{1:n}), \tag{4.14}
\end{aligned}$$

where the notation $K_\Psi(x_i, x_j)_{\substack{1 \leq i, j \leq n \\ |x_i = x \\ |x_j = y}}$ refers to having the i^{th} and j^{th} component entries of K_Ψ equals to the point $x, y \in \Lambda$ respectively, that is, $K_\Psi(x_i, x_i) = K_\Psi(x, x)$, $K_\Psi(x_j, x_j) = K_\Psi(y, y)$ and $K_\Psi(x_i, x_j) = K_\Psi(x, y)$.

$$\begin{aligned}
j_{\Psi}^{(2)}(x, y) &= \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \tilde{\mathcal{G}}_{\Psi}(h)|_{h=0} \\
&= \sum_{n=2}^{\infty} \frac{(-1)^{n+2}}{n!} \int_{\Lambda^n} \left(\sum_{\substack{p,q=1 \\ p \neq q}}^n \delta_x(x_p) \delta_y(x_q) \right) \det(K_{\Psi}(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}) \\
&= \frac{1}{2} \int_{\Lambda^2} \left(\delta_x(x_1) \delta_y(x_2) + \delta_x(x_2) \delta_y(x_1) \right) \\
&\quad \times \left(K_{\Psi}(x_1, x_1) K_{\Psi}(x_2, x_2) - (K_{\Psi}(x_1, x_2))^2 \right) \nu(dx_1) \nu(dx_2) \\
&\quad + \sum_{n=3}^{\infty} \frac{(-1)^{n+2}}{n!} \int_{\Lambda^n} \left(\sum_{\substack{p,q=1 \\ p \neq q}}^n \delta_x(x_p) \delta_y(x_q) \right) \det(K_{\Psi}(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}) \\
&= \frac{1}{2} \int_{\Lambda^2} \delta_x(x_1) \delta_y(x_2) \left(K_{\Psi}(x_1, x_1) K_{\Psi}(x_2, x_2) - (K_{\Psi}(x_1, x_2))^2 \right) \nu(dx_1) \nu(dx_2) \\
&\quad + \frac{1}{2} \int_{\Lambda^2} \delta_x(x_2) \delta_y(x_1) \left(K_{\Psi}(x_1, x_1) K_{\Psi}(x_2, x_2) - (K_{\Psi}(x_1, x_2))^2 \right) \nu(dx_1) \nu(dx_2) \\
&\quad + \sum_{n=3}^{\infty} \frac{(-1)^{n+2}}{n!} \int_{\Lambda^n} \left(\sum_{\substack{p,q=1 \\ p \neq q}}^n \delta_x(x_p) \delta_y(x_q) \right) \det(K_{\Psi}(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}) \\
&= \frac{1}{2} \left(K_{\Psi}(x, x) K_{\Psi}(y, y) - (K_{\Psi}(x, y))^2 \right) + \frac{1}{2} \left(K_{\Psi}(y, y) K_{\Psi}(x, x) - (K_{\Psi}(y, x))^2 \right) \\
&\quad + \sum_{n=3}^{\infty} \frac{(-1)^{n+2}}{n!} \int_{\Lambda^n} \left(\sum_{\substack{p,q=1 \\ p \neq q}}^n \delta_x(x_p) \delta_y(x_q) \right) \det(K_{\Psi}(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}) \\
&= K_{\Psi}(x, x) K_{\Psi}(y, y) - (K_{\Psi}(x, y))^2 \\
&\quad + \sum_{n=3}^{\infty} \frac{(-1)^{n+2}}{n!} \sum_{\substack{i,j=1 \\ i \neq j}}^n \int_{\Lambda^{n-2}} \det(K_{\Psi}(x_i, x_j)_{\substack{1 \leq i, j \leq n \\ |x_i=x \\ |x_j=y}}}) \nu(dx_{1:n} \setminus dx_i dx_j) \\
&= K_{\Psi}(x, x) K_{\Psi}(y, y) - (K_{\Psi}(x, y))^2 \\
&\quad + \sum_{n=3}^{\infty} \frac{(-1)^{n+2}}{(n-2)!} \int_{\Lambda^{n-2}} \det(K_{\Psi}(x_i, x_j)_{\substack{1 \leq i, j \leq n \\ |x_{n-1}=x \\ |x_n=y}}}) \nu(dx_{1:n-2}) \\
&= K_{\Psi}(x, x) K_{\Psi}(y, y) - (K_{\Psi}(x, y))^2 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det(K_{\Psi}(x_i, x_j)_{\substack{1 \leq i, j \leq n+2 \\ |x_{n+1}=x \\ |x_{n+2}=y}}}) \nu(dx_{1:n})
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det(K_{\Psi}(x_i, x_j)_{\substack{1 \leq i, j \leq n+2 \\ |x_{n+1}=x \\ |x_{n+2}=y}}) \nu(dx_{1:n}).$$

By (4.2) of Proposition 4.1.2, we have

$$\begin{aligned} \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \tilde{G}_{\Psi}(h)|_{h=0} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det(K_{\Psi}(x_i, x_j)_{\substack{1 \leq i, j \leq n+2 \\ |x_{n+1}=x \\ |x_{n+2}=y}}) \nu(dx_{1:n}) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \rho_{\Psi}^{(n+2)}(x_{1:n}, x, y) \nu(dx_{1:n}). \end{aligned}$$

□

Higher Order Janossy Densities

In the following proposition, we present the general Janossy density expression of the DPP at a sequence of distinct point configurations, for which the derived expression independently recovers the result in Equation (2.18) of Proposition 2.2.4.

Proposition 4.1.12. *For a sequence of distinct point configurations $\Psi = y_{1:r} = (y_1, \dots, y_r)$, the r^{th} order Janossy density of the DPP Ψ is given by*

$$\begin{aligned} j_{\Psi}^{(r)}(y_{1:r}) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det(K_{\Psi}(x_i, x_j)_{\substack{1 \leq i, j \leq n+r \\ |x_{n+k}=y_k, 1 \leq k \leq r}}) \nu(dx_{1:n}) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \rho_{\Psi}^{(n+r)}(x_{1:n}, y_{1:r}) \nu(dx_{1:n}). \end{aligned} \quad (4.15)$$

Proof. By setting $h=0$ and substituting the functions $f_1, \dots, f_r$ with Dirac delta functions $\delta_{y_1}, \dots, \delta_{y_r}$ respectively in (4.11) of Proposition 4.1.8, we have

$$\begin{aligned} j_{\Psi}^{(r)}(y_{1:r}) &= \frac{\partial_{\delta_{y_1}}}{\partial h} \dots \frac{\partial_{\delta_{y_r}}}{\partial h} \tilde{G}_{\Psi}(h)|_{h=0} \\ &= \sum_{n=r}^{\infty} \frac{(-1)^{n+r}}{n!} \int_{\Lambda^n} \left(\sum_{\substack{p_1, \dots, p_r=1 \\ p_1 \neq \dots \neq p_r}}^n \delta_{y_1}(x_{p_1}) \dots \delta_{y_r}(x_{p_r}) \right) \det(K_{\Psi}(x_i, x_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}) \\ &= \det(K_{\Psi}(y_i, y_j)_{1 \leq i, j \leq r}) \\ &\quad + \sum_{n=r+1}^{\infty} \frac{(-1)^{n+r}}{n!} \sum_{\substack{p_1, \dots, p_r=1 \\ p_1 \neq \dots \neq p_r}}^n \int_{\Lambda^{n-r}} \det(K_{\Psi}(x_i, x_j)_{\substack{1 \leq i, j \leq n \\ |x_{p_k}=y_k, 1 \leq k \leq r}}) \nu(dx_{1:n} \setminus dx_{p_1:p_r}) \end{aligned}$$

$$\begin{aligned}
&= \det(K_\Psi(y_i, y_j)_{1 \leq i, j \leq r}) \\
&\quad + \sum_{n=r+1}^{\infty} \frac{(-1)^{n+r}}{(n-r)!} \int_{\Lambda^{n-r}} \det(K_\Psi(x_i, x_j)_{\substack{1 \leq i, j \leq n \\ |x_{n-r+k}=y_k, 1 \leq k \leq r}}) \nu(dx_{1:n-r}) \\
&= \det(K_\Psi(y_i, y_j)_{1 \leq i, j \leq r}) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det(K_\Psi(x_i, x_j)_{\substack{1 \leq i, j \leq n+r \\ |x_{n+k}=y_k, 1 \leq k \leq r}}) \nu(dx_{1:n}) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det(K_\Psi(x_i, x_j)_{\substack{1 \leq i, j \leq n+r \\ |x_{n+k}=y_k, 1 \leq k \leq r}}) \nu(dx_{1:n})
\end{aligned}$$

By applying (4.2) of Proposition 4.1.2, we have

$$\begin{aligned}
\frac{\partial_{\delta_{y_1}}}{\partial h} \cdots \frac{\partial_{\delta_{y_r}}}{\partial h} \tilde{\mathcal{G}}_\Psi(h)|_{h=0} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det(K_\Psi(x_i, x_j)_{\substack{1 \leq i, j \leq n+r \\ |x_{n+k}=y_k, 1 \leq k \leq r}}) \nu(dx_{1:n}) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \rho_\Psi^{(n+r)}(x_{1:n}, y_{1:r}) \nu(dx_{1:n}).
\end{aligned}$$

□

4.1.3 Branched DPPs and Factorial Moment Densities

In this subsection, we consider the setting of branching in determinantal point processes, and our objective is to derive factorial moments from the joint PGFl. From Proposition 2.2.31 and the expression in (4.4), suppose Ψ is the ancestor determinantal point process and Φ is the descendant determinantal point process. In addition, let Υ be the independent point process that is applied under superposition to describe the conditional point process $\Phi|\Psi$, as in (2.51). The explicit form of the joint determinantal PGFl of Ψ and Φ on Λ is given by

$$\begin{aligned}
\tilde{\mathcal{G}}_{\Psi, \Phi}(g, h) &= \tilde{\mathcal{G}}_\Psi(g(\cdot) \mathcal{G}_\Upsilon(h|\cdot)) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda} \prod_{i=1}^n \left(1 - g(u_i) \mathcal{G}_\Upsilon(h|u_i)\right) \det(K_\Psi(u_i, u_j)_{1 \leq i, j \leq n}) \nu(u_{1:n}) \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda} \prod_{i=1}^n \left(1 - g(u_i) \mathcal{G}_\Upsilon(h|u_i)\right) \det(K_\Psi(u_i, u_j)_{1 \leq i, j \leq n}) \nu(u_{1:n}).
\end{aligned}$$

By applying (2.54) and (2.55) into $\tilde{\mathcal{G}}_{\Psi, \Phi}(g, h)$, the marginal of $\tilde{\mathcal{G}}_{\Psi, \Phi}(g, h)$ over Ψ is given by

$$\tilde{\mathcal{G}}_\Phi(h) = \tilde{\mathcal{G}}_{\Psi, \Phi}(\mathbf{1}, h) = \tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(h|\cdot))$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \prod_{i=1}^n \left(1 - \mathcal{G}_{\Upsilon}(h|u_i)\right) \det(K_{\Psi}(u_i, u_j)_{1 \leq i, j \leq n}) \nu(du_{1:n}) \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \prod_{i=1}^n \left(1 - \mathcal{G}_{\Upsilon}(h|u_i)\right) \det(K_{\Psi}(u_i, u_j)_{1 \leq i, j \leq n}) \nu(du_{1:n}).
\end{aligned}$$

By imposing the specific branching operations of thinning and superposition with a Poisson point process as introduced in (3.13), we construct the joint PGFl of the special dependence between Ψ and Φ on Λ with the following expression

$$\begin{aligned}
\tilde{\mathcal{G}}_{\Psi, \Phi}(g, h) &= \mathcal{G}_{\Omega}(h) \tilde{\mathcal{G}}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \\
&= \mathcal{G}_{\Omega}(h) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \prod_{i=1}^n \left(1 - g(u_i) \mathcal{G}_{\Upsilon}(h|u_i)\right) \det(K_{\Psi}(u_i, u_j)_{1 \leq i, j \leq n}) \nu(du_{1:n}),
\end{aligned} \tag{4.16}$$

where $\tilde{\mathcal{G}}_{\Psi, \Phi}(g, h)$ is described as the superposition of a Poisson point process Ω and the branching on Ψ with Bernoulli point processes $\Upsilon|x$, $x \in \Psi$. We use similar notations as in before, and for simplicity, we take $p(\cdot) := p$, $q = 1 - p$ as constants, $\tilde{l}(z|\cdot) = pl(z|\cdot)$ and $\mathcal{G}_{\Upsilon}(h|\cdot) = q + \int_{\Lambda} h(u) \tilde{l}(u|\cdot) \nu(du)$.

The marginal of $\tilde{\mathcal{G}}_{\Psi, \Phi}(g, h)$ over Ψ is given by

$$\begin{aligned}
\tilde{\mathcal{G}}_{\Phi}(h) &= \mathcal{G}_{\Omega}(h) \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot)) \\
&= \mathcal{G}_{\Omega}(h) \sum_{n \geq 0} \frac{(-1)^n}{n!} \int_{\Lambda^n} \prod_{i=1}^n \left(1 - \mathcal{G}_{\Upsilon}(h|u_i)\right) \det(K_{\Psi}(u_i, u_j)_{1 \leq i, j \leq n}) \nu(du_{1:n}).
\end{aligned} \tag{4.17}$$

The following proposition gives a useful property that allows simplification in the evaluation of the PGFl and its associated derivatives when $h = 1$, and is consistent with the result in (2.23) of Remark 2.2.10. In addition, the proposition gives the Janossy density of the marginal $\mathcal{G}_{\Phi}(h)$ with empty point.

Proposition 4.1.13. *By setting $h = 1$ to the marginal PGFl $\tilde{\mathcal{G}}_{\Phi}(h)$ as in (4.17), we have*

$$\tilde{\mathcal{G}}_{\Phi}(h)|_{h=1} = \mathcal{G}_{\Omega}(\mathbf{1}) \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(\mathbf{1}|\cdot)) = 1, \tag{4.18}$$

a useful property for simplification in the analysis of DPP PGFl and its associated derivatives when $h = 1$.

In addition, by setting $h = 0$ we recover the Janossy density of the marginal PGFl $\mathcal{G}_\Phi(h)$ with empty point and is given by

$$j_\Phi^{(0)} = \tilde{\mathcal{G}}_\Phi(h)|_{h=0} = \mathcal{G}_\Omega(\mathbf{0})\tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(\mathbf{0}|\cdot)) = j_\Omega^{(0)} \sum_{n=0}^{\infty} \frac{q^n}{n!} \int_{\Lambda^n} j_\Psi^{(n)}(u_{1:n}) \nu(du_{1:n}), \quad (4.19)$$

where $j_\Omega^{(0)}$ is the Janossy density of the PPP Ω with empty point.

Proof. For (4.18), by setting $h = 1$, we have $\tilde{\mathcal{G}}_\Phi(h)|_{h=1} = \mathcal{G}_\Omega(\mathbf{1})\tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(\mathbf{1}|\cdot))$. Since the point process Ω is a PPP and by (2.23), we have $\mathcal{G}_\Omega(\mathbf{1}) = 1$.

For the second component $\tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(\mathbf{1}|\cdot))$, we have

$$\begin{aligned} & \tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(\mathbf{1}|\cdot)) \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \prod_{i=1}^n \left(1 - \mathcal{G}_\Upsilon(\mathbf{1}|u_i)\right) \det(K_\Psi(u_i, u_j)_{1 \leq i, j \leq n}) \nu(du_{1:n}) = 1, \end{aligned}$$

where the term $\prod_{i=1}^n \left(1 - \mathcal{G}_\Upsilon(\mathbf{1}|u_i)\right)$ within the sum above is evaluated as

$$\begin{aligned} \prod_{i=1}^n \left(1 - \mathcal{G}_\Upsilon(\mathbf{1}|u_i)\right) &= \prod_{i=1}^n \left(1 - (1 - p + \int_{\Lambda} \tilde{l}(u|u_i) \nu(du))\right) \\ &= \prod_{i=1}^n \left(1 - (1 - p + p)\right) = 0. \end{aligned}$$

Hence $\tilde{\mathcal{G}}_\Phi(h)|_{h=1} = \mathcal{G}_\Omega(\mathbf{1})\tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(\mathbf{1}|\cdot)) = 1$.

For (4.19), by setting $h = 0$, we have $\tilde{\mathcal{G}}_\Phi(h)|_{h=0} = \mathcal{G}_\Omega(\mathbf{0})\tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(\mathbf{0}|\cdot))$. Since Ω is a PPP and by Remark 2.2.10, we have $\mathcal{G}_\Omega(\mathbf{0}) = j_\Omega^{(0)}$.

For the second component $\tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(\mathbf{0}|\cdot))$, we have

$$\begin{aligned} \tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(\mathbf{0}|\cdot)) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \prod_{i=1}^n \left(1 - \mathcal{G}_\Upsilon(\mathbf{0}|u_i)\right) \det(K_\Psi(u_i, u_j)_{1 \leq i, j \leq n}) \nu(dx_{1:n}) \\ &= 1 - p \int_{\Lambda} K_\Psi(u_1, u_1) \nu(du_1) \\ &\quad + \frac{p^2}{2} \int_{\Lambda^2} \left(K_\Psi(u_1, u_1)K_\Psi(u_2, u_2) - (K_\Psi(u_1, u_2))^2\right) \nu(du_{1:2}) \end{aligned}$$

$$\begin{aligned}
& + \sum_{n=3}^{\infty} \frac{(-1)^n p^n}{n!} \int_{\Lambda^n} \det(K_{\Psi}(u_i, u_j)_{1 \leq i, j \leq n}) \nu(du_{1:n}) \\
& = \sum_{n=0}^{\infty} \frac{(-1)^n p^n}{n!} \int_{\Lambda^n} \det(K_{\Psi}(u_i, u_j)_{1 \leq i, j \leq n}) \nu(du_{1:n}) \\
& = \sum_{n=0}^{\infty} \frac{(-1)^n (1-q)^n}{n!} \int_{\Lambda^n} \det(K_{\Psi}(u_i, u_j)_{1 \leq i, j \leq n}) \nu(du_{1:n}) \\
& = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{k=0}^n \frac{(-q)^k n!}{(n-k)! k!} \int_{\Lambda^n} \det(K_{\Psi}(u_i, u_j)_{1 \leq i, j \leq n}) \nu(du_{1:n}) \\
& = \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{\infty} \frac{q^k (-1)^k}{(n-k)! k!} \int_{\Lambda^n} \det(K_{\Psi}(u_i, u_j)_{1 \leq i, j \leq n}) \nu(du_{1:n}) \\
& = \sum_{k=0}^{\infty} \frac{q^k}{k!} \sum_{n=k}^{\infty} \frac{(-1)^{n-k}}{(n-k)!} \int_{\Lambda^n} \det(K_{\Psi}(u_i, u_j)_{1 \leq i, j \leq n}) \nu(du_{1:n}) \\
& = \sum_{k=0}^{\infty} \frac{q^k}{k!} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^{n+k}} \det(K_{\Psi}(u_i, u_j)_{1 \leq i, j \leq n+k}) \nu(du_{1:n+k}).
\end{aligned}$$

By (4.15) of Proposition 4.1.12 or (2.18) of Proposition 2.2.4, we can express the above result as

$$\begin{aligned}
\tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(\mathbf{0}|\cdot)) & = \sum_{k=0}^{\infty} \frac{q^k}{k!} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^{n+k}} \det(K_{\Psi}(u_i, u_j)_{\substack{1 \leq i, j \leq n+k \\ |u_{n+r}=w_r, 1 \leq r \leq k}}) \nu(du_{1:n}) \nu(dw_{1:k}) \\
& = \sum_{k=0}^{\infty} \frac{q^k}{k!} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^{n+k}} \rho_{\Psi}^{(n+k)}(u_{1:n}, w_{1:k}) \nu(du_{1:n}) \nu(dw_{1:k}) \\
& = \sum_{k=0}^{\infty} \frac{q^k}{k!} \int_{\Lambda^k} j_{\Psi}^{(k)}(w_{1:k}) \nu(dw_{1:k}).
\end{aligned}$$

And finally, with a change of algebra notations for the purpose of consistency, we have

$$j_{\Phi}^{(0)} = \tilde{\mathcal{G}}_{\Phi}(h)|_{h=0} = \mathcal{G}_{\Omega}(\mathbf{0}) \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(\mathbf{0}|\cdot)) = j_{\Omega}^{(0)} \sum_{n=0}^{\infty} \frac{q^n}{n!} \int_{\Lambda^n} j_{\Psi}^{(n)}(u_{1:n}) \nu(du_{1:n}).$$

□

Marginal Factorial Moment Densities of Branched DPPs

In the remaining subsection, we compute the marginal first and second order factorial moment densities using the PGFl of the branched DPP.

Marginal First Order Moment Density

Proposition 4.1.14. *The marginal first order moment density of the branched DPP Φ is given by*

$$\mu_{\Phi}^{(1)}(x) = \mu_{\Omega}^{(1)}(x) + \int_{\Lambda} \tilde{l}(x|u) K_{\Psi}(u, u) \nu(du), \quad (4.20)$$

where $\mu_{\Omega}^{(1)}(x)$ is the first order moment density of the Poisson point process Ω evaluated at the point $x \in \Lambda$.

Proof. Recall from (2.37) of Corollary 2.2.18 and (2.54), the marginal first order moment density is given by

$$\mu_{\Phi}^{(1)}(x) = \frac{\partial_{\delta_x}}{\partial h} \tilde{\mathcal{G}}_{\Phi}(h)|_{h=1} = \frac{\partial_{\delta_x}}{\partial h} \tilde{\mathcal{G}}_{\Psi, \Phi}(\mathbf{1}, h)|_{h=1} = \frac{\partial_{\delta_x}}{\partial h} \mathcal{G}_{\Omega}(h) \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot)).$$

Firstly, we consider the PGFl of the marginal of the branched DPP as in (4.17),

$$\begin{aligned} \tilde{\mathcal{G}}_{\Phi}(h) &= \mathcal{G}_{\Omega}(h) \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot)) \\ &= \mathcal{G}_{\Omega}(h) \sum_{n \geq 0} \frac{(-1)^n}{n!} \int_{\Lambda^n} \prod_{i=1}^n \left(1 - \mathcal{G}_{\Upsilon}(h|u_i)\right) \det(K_{\Psi}(u_i, u_j)_{1 \leq i, j \leq n}) \nu(du_{1:n}). \end{aligned}$$

From (3.15), the first order derivative of $\tilde{\mathcal{G}}_{\Phi}(h)$ evaluated in the direction of h by a nonnegative measurable function f is given by

$$\frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_{\Phi}(h) = \mathcal{G}_{\Omega}(h) \frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot)) + \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_f}{\partial h} \mathcal{G}_{\Omega}(h).$$

By setting $h = 1$, we have

$$\begin{aligned} \frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_{\Phi}(h)|_{h=1} &= \mathcal{G}_{\Omega}(\mathbf{1}) \frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot))|_{h=1} + \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(\mathbf{1}|\cdot)) \frac{\partial_f}{\partial h} \mathcal{G}_{\Omega}(h)|_{h=1} \\ &= \frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot))|_{h=1} + \frac{\partial_f}{\partial h} \mathcal{G}_{\Omega}(h)|_{h=1} \\ &= \frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot))|_{h=1} + \int_{\Lambda} f(u) \mu_{\Omega}^{(1)}(u) \nu(du), \end{aligned} \quad (4.21)$$

where $\tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(\mathbf{1}|\cdot)) = 1$ follows from (4.18) and $\frac{\partial_f}{\partial h} \mathcal{G}_{\Omega}(h)|_{h=1} = \int_{\Lambda} f(u) \mu_{\Omega}^{(1)}(u) \nu(du)$ follows from (2.36).

Hence, it remains to evaluate the component $\frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(h|\cdot))|_{h=1}$. By adapting from (4.6) of Proposition 4.1.4, we have

$$\begin{aligned}
& \frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(h|\cdot)) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \frac{\partial_f}{\partial h} \prod_{i=1}^n \left(1 - \mathcal{G}_\Upsilon(h|u_i)\right) \det(K_\Psi(u_i, u_j)_{1 \leq i, j \leq n}) \nu(du_{1:n}) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \int_{\Lambda^n} \sum_{p=1}^n \left(\int_{\Lambda} f(v) \tilde{l}(v|u_p) \nu(dv) \right) \prod_{\substack{q=1 \\ q \neq p}}^n \left(1 - \mathcal{G}_\Upsilon(h|u_q)\right) \\
&\quad \times \det(K_\Psi(u_i, u_j)_{1 \leq i, j \leq n}) \nu(du_{1:n}) \\
&= \int_{\Lambda} \left(\int_{\Lambda} f(v) \tilde{l}(v|u_1) \nu(dv) \right) \det(K_\Psi(u_1, u_1)) \nu(du_1) \\
&\quad + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n!} \int_{\Lambda^n} \sum_{p=1}^n \left(\int_{\Lambda} f(v) \tilde{l}(v|u_p) \nu(dv) \right) \prod_{\substack{q=1 \\ q \neq p}}^n \left(1 - \mathcal{G}_\Upsilon(h|u_q)\right) \\
&\quad \times \det(K_\Psi(u_i, u_j)_{1 \leq i, j \leq n}) \nu(du_{1:n}) \\
&= \int_{\Lambda} \left(\int_{\Lambda} f(v) \tilde{l}(v|u) \nu(dv) \right) K_\Psi(u, u) \nu(du) \\
&\quad + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n!} \int_{\Lambda^n} \sum_{p=1}^n \left(\int_{\Lambda} f(v) \tilde{l}(v|u_p) \nu(dv) \right) \prod_{\substack{q=1 \\ q \neq p}}^n \left(1 - \mathcal{G}_\Upsilon(h|u_q)\right) \\
&\quad \times \det(K_\Psi(u_i, u_j)_{1 \leq i, j \leq n}) \nu(du_{1:n}).
\end{aligned}$$

By setting $h = 1$, we have

$$\frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(h|\cdot))|_{h=1} = \int_{\Lambda} \left(\int_{\Lambda} f(v) \tilde{l}(v|u) \nu(dv) \right) K_\Psi(u, u) \nu(du).$$

where, it is easy to verify that by setting $h = 1$ for the case of $n \geq 2$, we have the term $\prod_u (1 - \mathcal{G}_\Upsilon(h|u))|_{h=1} = 0$.

Finally, we substitute the relevant terms into (4.21) and replace the function f with Dirac delta function at a point $x \in \Lambda$, we conclude by the following

$$\begin{aligned}
\mu_\Phi^{(1)}(x) &= \frac{\partial_{\delta_x}}{\partial h} \tilde{\mathcal{G}}_\Phi(h)|_{h=1} \\
&= \int_{\Lambda} \delta_x(u) \mu_\Omega^{(1)}(u) \nu(du) + \frac{\partial_{\delta_x}}{\partial h} \tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(h|\cdot))|_{h=1}
\end{aligned}$$

$$\begin{aligned}
&= \mu_{\Omega}^{(1)}(x) + \int_{\Lambda} \left(\int_{\Lambda} \delta_x(v) \tilde{l}(v|u) \nu(dv) \right) K_{\Psi}(u, u) \nu(du) \\
&= \mu_{\Omega}^{(1)}(x) + \int_{\Lambda} \tilde{l}(x|u) K_{\Psi}(u, u) \nu(du).
\end{aligned}$$

□

Marginal Second Order Factorial Moment Density

Proposition 4.1.15. *The marginal second order factorial moment density of the branched DPP Φ is given by*

$$\begin{aligned}
&\rho_{\Phi}^{(2)}(x, y) \\
&= \int_{\Lambda^2} \tilde{l}(x|u) \tilde{l}(y|v) \left(K_{\Psi}(u, u) K_{\Psi}(v, v) - (K_{\Psi}(u, v))^2 \right) \nu(du) \nu(dv) \\
&\quad + \mu_{\Omega}^{(1)}(y) \int_{\Lambda} \tilde{l}(x|u) K_{\Psi}(u, u) \nu(du) + \mu_{\Omega}^{(1)}(x) \int_{\Lambda} \tilde{l}(y|v) K_{\Psi}(v, v) \nu(dv) + \rho_{\Omega}^{(2)}(x, y),
\end{aligned} \tag{4.22}$$

where $\rho_{\Omega}^{(2)}(x, y)$ is the second order factorial moment density of the Poisson point process Ω evaluated at the points $x, y \in \Lambda$ and $x \neq y$.

Proof. The marginal second order factorial moment density of a point process Φ over Ψ evaluated at any two unique points $x, y \in \Lambda$ is given by

$$\rho_{\Phi}^{(2)}(x, y) = \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \tilde{\mathcal{G}}_{\Phi}(h) = \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \tilde{\mathcal{G}}_{\Psi, \Phi}(\mathbf{1}, h)|_{h=1} = \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_{\Omega}(h) \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot))|_{h=1}.$$

Recall from (3.18), the second order derivative of $\tilde{\mathcal{G}}_{\Phi}(h)$ evaluated in the direction of h by functions f and g is given by

$$\begin{aligned}
\frac{\partial_g}{\partial h} \frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_{\Phi}(h) &= \frac{\partial_g}{\partial h} \frac{\partial_f}{\partial h} \left(\mathcal{G}_{\Omega}(h) \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot)) \right) \\
&= \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_g}{\partial h} \frac{\partial_f}{\partial h} \mathcal{G}_{\Omega}(h) + \frac{\partial_g}{\partial h} \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_f}{\partial h} \mathcal{G}_{\Omega}(h) \\
&\quad + \frac{\partial_g}{\partial h} \mathcal{G}_{\Omega}(h) \frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot)) + \mathcal{G}_{\Omega}(h) \frac{\partial_g}{\partial h} \frac{\partial_f}{\partial h} \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot)).
\end{aligned}$$

By setting $h = 1$, and substituting the functions f with Dirac delta function evaluated at the point $x \in \Lambda$ and g with Dirac delta function evaluated at the point $y \in \Lambda$, where $x \neq y$, we have

$$\frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \tilde{\mathcal{G}}_{\Phi}(h)|_{h=1}$$

$$\begin{aligned}
&= \tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(\mathbf{1}|\cdot)) \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_\Omega(h)|_{h=1} + \frac{\partial_{\delta_y}}{\partial h} \tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(h|\cdot))|_{h=1} \frac{\partial_{\delta_x}}{\partial h} \mathcal{G}_\Omega(h)|_{h=1} \\
&\quad + \frac{\partial_{\delta_y}}{\partial h} \mathcal{G}_\Omega(h)|_{h=1} \frac{\partial_{\delta_x}}{\partial h} \tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(h|\cdot))|_{h=1} + \mathcal{G}_\Omega(\mathbf{1}) \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(h|\cdot))|_{h=1} \\
&= \rho_\Omega^{(2)}(x, y) + \left(\int_\Lambda \left(\int_\Lambda \delta_y(w) \tilde{l}(w|v) \nu(dw) \right) K_\Psi(v, v) \nu(dv) \right) \left(\int_\Lambda \delta_x(u) \mu_\Omega^{(1)}(u) \nu(du) \right) \\
&\quad + \left(\int_\Lambda \delta_y(u) \mu_\Omega^{(1)}(u) \nu(du) \right) \left(\int_\Lambda \left(\int_\Lambda \delta_x(w) \tilde{l}(w|u) \nu(dw) \right) K_\Psi(u, u) \nu(du) \right) \\
&\quad + \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(h|\cdot))|_{h=1} \\
&= \rho_\Omega^{(2)}(x, y) + \mu_\Omega^{(1)}(x) \int_\Lambda \tilde{l}(y|v) K_\Psi(v, v) \nu(dv) + \mu_\Omega^{(1)}(y) \int_\Lambda \tilde{l}(x|u) K_\Psi(u, u) \nu(du) \\
&\quad + \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(h|\cdot))|_{h=1}.
\end{aligned}$$

From the above, it remains to evaluate the term $\frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(h|\cdot))|_{h=1}$. By applying (4.9) of Proposition 4.1.6 to $\frac{\partial_f}{\partial h} \frac{\partial_g}{\partial h} \tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(h|\cdot))$, we have

$$\begin{aligned}
&\frac{\partial_f}{\partial h} \frac{\partial_g}{\partial h} \tilde{\mathcal{G}}_\Psi(\mathcal{G}_\Upsilon(h|\cdot)) \\
&= \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \sum_{\substack{p,q=1 \\ p \neq q}}^n \left(\int_\Lambda f(v) \tilde{l}(v|u_p) \nu(dv) \right) \left(\int_\Lambda g(w) \tilde{l}(w|u_q) \nu(dw) \right) \\
&\quad \times \prod_{\substack{r=1 \\ r \neq p,q}}^n \left(1 - \mathcal{G}_\Upsilon(h|u_r) \right) \det(K_\Psi(u_i, u_j)_{1 \leq i,j \leq n}) \nu(du_{1:n}) \\
&= \frac{1}{2!} \int_{\Lambda^2} \sum_{\substack{p,q=1 \\ p \neq q}}^2 \left(\int_\Lambda f(v) \tilde{l}(v|u_p) \nu(dv) \right) \left(\int_\Lambda g(w) \tilde{l}(w|u_q) \nu(dw) \right) \\
&\quad \times \det(K_\Psi(u_i, u_j)_{1 \leq i,j \leq 2}) \nu(du_{1:2}) \\
&\quad + \sum_{n=3}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \sum_{\substack{p,q=1 \\ p \neq q}}^n \left(\int_\Lambda f(v) \tilde{l}(v|u_p) \nu(dv) \right) \left(\int_\Lambda g(w) \tilde{l}(w|u_q) \nu(dw) \right) \\
&\quad \times \prod_{\substack{r=1 \\ r \neq p,q}}^n \left(1 - \mathcal{G}_\Upsilon(h|u_r) \right) \det(K_\Psi(u_i, u_j)_{1 \leq i,j \leq n}) \nu(du_{1:n}).
\end{aligned} \tag{4.23}$$

By setting $h = 1$ and substituting the functions f and g with Dirac delta functions

at the points $x, y \in \Lambda$, $x \neq y$, respectively, we have

$$\begin{aligned}
& \frac{\partial_{\delta_x}}{\partial h} \frac{\partial_{\delta_y}}{\partial h} \tilde{\mathcal{G}}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot))|_{h=1} \\
&= \frac{1}{2!} \int_{\Lambda^2} \sum_{\substack{p,q=1 \\ p \neq q}}^2 \left(\int_{\Lambda} \delta_x(v) \tilde{l}(v|u_p) \nu(dv) \right) \left(\int_{\Lambda} \delta_y(w) \tilde{l}(w|u_q) \nu(dw) \right) \\
&\quad \times \det(K_{\Psi}(u_i, u_j)_{1 \leq i,j \leq 2}) \nu(du_{1:2}) \\
&= \frac{1}{2!} \int_{\Lambda^2} \left(\tilde{l}(x|u_1) \tilde{l}(y|u_2) + \tilde{l}(x|u_2) \tilde{l}(y|u_1) \right) \det(K_{\Psi}(u_i, u_j)_{1 \leq i,j \leq 2}) \nu(du_{1:2}) \\
&= \int_{\Lambda^2} \tilde{l}(x|u_1) \tilde{l}(y|u_2) \left(K_{\Psi}(u_1, u_1) K_{\Psi}(u_1, u_1) - (K_{\Psi}(u_1, u_2))^2 \right) \nu(du_{1:2}),
\end{aligned}$$

where the remaining terms at $n \geq 3$ vanish to zero. By substituting the relevant terms, we recover (4.22). $\square$

4.2 Branched Point Processes

The main objective of this section is to derive Proposition 4.2.8 on the marginal Janossy densities and Proposition 4.2.12 on the joint Janossy densities of the general branched point processes. In this setting, we depend on the explicit PGFl expression of the general branched point processes to derive general formulas of the respective Janossy densities. In subsection 4.2.1, we present a natural combinatorial interpretation of the PGFl associated with branched point processes that are described under the superposition of a Poisson point process and branching under independent Bernoulli point processes (also known as thinning operation). In subsection 4.2.2, we give a combinatorial interpretation of the marginal and joint Janossy densities derived from the associated PGFl.

4.2.1 Combinatorial Interpretation of PGFls

We recall from (3.19) and present the following Proposition that gives the natural combinatorial interpretation of the joint PGFl of the ancestor point process Ψ and the descendant point process Φ that is described under the superposition of a Poisson point process Ω and the branching of Ψ with independent Bernoulli point processes $\Upsilon|u$, $u \in \Psi$. Note that the nonnegative measurable functions g and h correspond to Ψ and Φ respectively. The following proposition is the starting point to derive the Janossy densities of general branched point processes which will be presented in the

remaining section. Proposition 4.2.8 and Proposition 4.2.12 will form the basis in the construction of the second order PHD filter using determinantal point processes in Chapter 5.

Proposition 4.2.1. *The joint probability generating functional (PGFl) of the general branched point process that is described under the superposition of a Poisson point process Ω and the branching of Ψ with independent Bernoulli point processes $\Upsilon|u$, $u \in \Psi$ is given by*

$$\begin{aligned} \mathcal{G}_{\Psi, \Phi}(g, h) &= \mathcal{G}_{\Omega}(h) \mathcal{G}_{\Psi}(g(\cdot)) \mathcal{G}_{\Upsilon}(h|\cdot) \\ &= \mathcal{G}_{\Omega}(h) \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \prod_{i=1}^m \left(g(u_i) \left(q + \int_{\Lambda} h(z) \tilde{l}(z|u_i) \nu(dz) \right) \right) \nu(du_{1:m}) \end{aligned} \quad (4.24)$$

$$\begin{aligned} &= \mathcal{G}_{\Omega}(h) \sum_{k=0}^{\infty} \int_{\Lambda^k} \prod_{j=1}^k \int_{\Lambda} h(z) \tilde{l}(z|u_j) \nu(dz) \\ &\quad \times \sum_{m=0}^{\infty} \frac{q^m}{m!k!} \int_{\Lambda^m} j_{\Psi}^{k+m}(u_{1:k+m}) \prod_{i=1}^{k+m} g(u_i) \nu(du_{1:k+m}), \end{aligned} \quad (4.25)$$

where we take $p(u) := p$, $1 - p := q$ as constants, $pl(z|u) := \tilde{l}(z|u)$ for $z \in \Phi$ and $u \in \Psi$. In addition, the PGFl of the Bernoulli point process $\Upsilon|u$, $u \in \Psi$ is given by

$$\mathcal{G}_{\Upsilon}(h|u) = q + \int_{\Lambda} h(z) \tilde{l}(z|u) \nu(dz).$$

Proof. The expression in (4.25) is given by

$$\begin{aligned} \mathcal{G}_{\Psi}(g(\cdot)) \mathcal{G}_{\Upsilon}(h|\cdot) &= \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \left(\prod_{i=1}^m g(u_i) \mathcal{G}_{\Upsilon}(h|u_i) \right) j_{\Psi}^{(m)}(u_{1:m}) \nu(du_{1:m}) \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \prod_{i=1}^m \left(g(u_i) \left(q + \int_{\Lambda} h(z) \tilde{l}(z|u_i) \nu(dz) \right) \right) \nu(du_{1:m}) \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \prod_{i=1}^m g(u_i) \sum_{k=0}^m \binom{m}{k} q^{m-k} \prod_{j=1}^k \int_{\Lambda} h(z) \tilde{l}(z|u_j) \nu(dz) \nu(du_{1:m}) \\ &= \sum_{k=0}^{\infty} \int_{\Lambda^k} \prod_{j=1}^k \int_{\Lambda} h(z) \tilde{l}(z|u_j) \nu(dz) \sum_{m=0}^{\infty} \frac{q^m}{m!k!} \int_{\Lambda^m} j_{\Psi}^{k+m}(x_{1:k+m}) \prod_{i=1}^{k+m} g(u_i) \nu(dx_{1:k+m}). \end{aligned}$$

□

In order to prepare for the PGFl derivatives of the general branched point processes, we recall from (3.20) that for a sequel of distinct point configurations $z_{1:m} := (z_1, \dots, z_m) \in \Phi$, where Φ is the descendant point process and Ψ is the ancestor point process, the PGFl of the conditional point process $\Psi|\Phi = z_{1:m}$ is given by

$$\mathcal{G}_{\Psi|\Phi}(g|z_{1:m}) = \frac{\frac{\partial_{\delta_{z_1}}}{\partial h} \dots \frac{\partial_{\delta_{z_m}}}{\partial h} \left(\mathcal{G}_{\Omega}(h) \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \right) \Big|_{h=0}}{\frac{\partial_{\delta_{z_1}}}{\partial h} \dots \frac{\partial_{\delta_{z_m}}}{\partial h} \left(\mathcal{G}_{\Omega}(h) \mathcal{G}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot)) \right) \Big|_{h=0}}.$$

We define the following convention that explains the combinatorial mapping involved in the next lemma.

Definition 4.2.2. *For a given set $S \subseteq \{1, \dots, m\}$, the elements of S are denoted by $S(1), \dots, S(\pi)$ in increasing order, where S is identified to be a mapping $S : \{1, \dots, \pi\} \rightarrow \{1, \dots, m\}$ and $\pi = |S|$.*

In the following lemma, we apply the Faà di Bruno's formula (see [29]) to the specific of $\mathcal{G}_{\Psi,\Phi}(g, h) = \mathcal{G}_{\Omega}(h) \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot))$ to obtain the general derivatives form of the joint PGFl $\mathcal{G}_{\Psi,\Phi}(g, h)$ evaluated in the direction of h with nonnegative measurable functions $f_1, \dots, f_m$. The lemma is required to address the combinatorial problem involved in differentiating the product of two PGFls in the aforementioned specific.

Lemma 4.2.3. *The m^{th} derivative of the joint PGFl $\mathcal{G}_{\Psi,\Phi}(g, h)$ evaluated in the direction of h with nonnegative measurable functions $f_1, \dots, f_m$ is given by*

$$\begin{aligned} & \frac{\partial_{f_1}}{\partial h} \dots \frac{\partial_{f_m}}{\partial h} \mathcal{G}_{\Psi,\Phi}(g, h) \\ &= \frac{\partial_{f_1}}{\partial h} \dots \frac{\partial_{f_m}}{\partial h} \left(\mathcal{G}_{\Omega}(h) \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \right) \\ &= \sum_{\substack{S \subseteq \{1, \dots, m\} \\ \pi = |S| \\ Q = \{1, \dots, m\} \setminus S}} \frac{\partial_{f_{S(1)}}}{\partial h} \dots \frac{\partial_{f_{S(\pi)}}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_{f_{Q(1)}}}{\partial h} \dots \frac{\partial_{f_{Q(m-\pi)}}}{\partial h} \mathcal{G}_{\Omega}(h) \end{aligned} \quad (4.26)$$

$$\begin{aligned} &= \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_{f_1}}{\partial h} \dots \frac{\partial_{f_m}}{\partial h} \mathcal{G}_{\Omega}(h) + \mathcal{G}_{\Omega}(h) \frac{\partial_{f_1}}{\partial h} \dots \frac{\partial_{f_m}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \\ &+ \sum_{\substack{S \subseteq \{1, \dots, m\} \\ \pi = |S|, \pi \geq 1 \\ Q = \{1, \dots, m\} \setminus S}} \frac{\partial_{f_{S(1)}}}{\partial h} \dots \frac{\partial_{f_{S(\pi)}}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_{f_{Q(1)}}}{\partial h} \dots \frac{\partial_{f_{Q(m-\pi)}}}{\partial h} \mathcal{G}_{\Omega}(h), \end{aligned} \quad (4.27)$$

where the index set $S \subseteq \{1, \dots, m\}$ covers the entire list of 2^m subsets of $\{1, \dots, m\}$.

Remark 4.2.4. We iterate the first 3 examples of (4.27), which are given by

$$\begin{aligned}
& \frac{\partial_{f_1}}{\partial h} \mathcal{G}_{\Psi, \Phi}(g, h) \\
&= \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_{f_1}}{\partial h} \mathcal{G}_{\Omega}(h) + \mathcal{G}_{\Omega}(h) \frac{\partial_{f_1}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)). \\
\\
& \frac{\partial_{f_1}}{\partial h} \frac{\partial_{f_2}}{\partial h} \mathcal{G}_{\Psi, \Phi}(g, h) \\
&= \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_{f_1}}{\partial h} \frac{\partial_{f_2}}{\partial h} \mathcal{G}_{\Omega}(h) + \frac{\partial_{f_2}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_{f_1}}{\partial h} \mathcal{G}_{\Omega}(h) \\
&+ \frac{\partial_{f_1}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_{f_2}}{\partial h} \mathcal{G}_{\Omega}(h) + \mathcal{G}_{\Omega}(h) \frac{\partial_{f_1}}{\partial h} \frac{\partial_{f_2}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)). \\
\\
& \frac{\partial_{f_1}}{\partial h} \frac{\partial_{f_2}}{\partial h} \frac{\partial_{f_3}}{\partial h} \mathcal{G}_{\Psi, \Phi}(g, h) \\
&= \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_{f_1}}{\partial h} \frac{\partial_{f_2}}{\partial h} \frac{\partial_{f_3}}{\partial h} \mathcal{G}_{\Omega}(h) + \frac{\partial_{f_3}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_{f_1}}{\partial h} \frac{\partial_{f_2}}{\partial h} \mathcal{G}_{\Omega}(h) \\
&+ \frac{\partial_{f_2}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_{f_1}}{\partial h} \frac{\partial_{f_3}}{\partial h} \mathcal{G}_{\Omega}(h) + \frac{\partial_{f_2}}{\partial h} \frac{\partial_{f_3}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_{f_1}}{\partial h} \mathcal{G}_{\Omega}(h) \\
&+ \frac{\partial_{f_1}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_{f_2}}{\partial h} \frac{\partial_{f_3}}{\partial h} \mathcal{G}_{\Omega}(h) + \frac{\partial_{f_1}}{\partial h} \frac{\partial_{f_3}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_{f_2}}{\partial h} \mathcal{G}_{\Omega}(h) \\
&+ \mathcal{G}_{\Omega}(h) \frac{\partial_{f_1}}{\partial h} \frac{\partial_{f_2}}{\partial h} \frac{\partial_{f_3}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) + \frac{\partial_{f_1}}{\partial h} \frac{\partial_{f_2}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_{f_3}}{\partial h} \mathcal{G}_{\Omega}(h).
\end{aligned}$$

4.2.2 Derivatives and Janossy Densities

In this subsection, the objective is to derive the marginal and the joint Janossy densities associated with the PGFl expression in Proposition 4.2.1 using Lemma 4.2.3. The expression of the marginal Janossy densities is presented in Proposition 4.2.8. The expression of the joint Janossy densities is presented in Proposition 4.2.12. In the following, we begin with the derivatives of the general branching PGFls.

Functional Derivatives of General Branching PGFl

The next Lemma gives the first derivative of $\mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot))$ evaluated in the direction of h with a nonnegative measurable function f_1 , for which will be used in proving Lemma 4.2.6 to get the second derivative.

Lemma 4.2.5. *The first derivative of $\mathcal{G}_\Psi(g(\cdot)\mathcal{G}_\Upsilon(h|\cdot))$ evaluated in the direction of h with function f_1 is given by*

$$\begin{aligned} & \frac{\partial_{f_1}}{\partial h} \mathcal{G}_\Psi(g(\cdot)\mathcal{G}_\Upsilon(h|\cdot)) \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \sum_{s_1=1}^m g(u_{s_1}) \int_{\Lambda} f_1(z) \tilde{l}(z|u_{s_1}) \nu(dz) \\ & \quad \times \prod_{\substack{s=1 \\ s \neq s_1}}^m \left(g(u_s) \mathcal{G}_\Upsilon(h|u_s) \right) \nu(du_{1:m}). \end{aligned} \quad (4.28)$$

By setting $g = 1$, $h = 0$ and substituting f_1 with Dirac delta function δ_{z_1} at the point $z_1 \in \Phi$, we have

$$\frac{\partial_{\delta_{z_1}}}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_\Upsilon(h|\cdot))|_{h=0} = \sum_{m=1}^{\infty} \frac{q^{m-1}}{(m-1)!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \tilde{l}(z_1|u_1) \nu(du_{1:m}). \quad (4.29)$$

Proof. The first derivative of $\mathcal{G}_\Psi(g(\cdot)\mathcal{G}_\Upsilon(h|\cdot))$ evaluated in the direction of h with function f_1 as in (4.28) is computed as

$$\begin{aligned} & \frac{\partial_{f_1}}{\partial h} \mathcal{G}_\Psi(g(\cdot)\mathcal{G}_\Upsilon(h|\cdot)) \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \frac{\partial_{f_1}}{\partial h} \prod_{i=1}^m \left(g(u_i) \left(q + \int_{\Lambda} h(z) \tilde{l}(z|u_i) \nu(dz) \right) \right) \nu(du_{1:m}) \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \sum_{s_1=1}^m g(u_{s_1}) \int_{\Lambda} f_1(z) \tilde{l}(z|u_{s_1}) \nu(dz) \\ & \quad \times \prod_{\substack{s=1 \\ s \neq s_1}}^m \left(g(u_s) \mathcal{G}_\Upsilon(h|u_s) \right) \nu(du_{1:m}). \end{aligned}$$

By setting $g = 1$, $h = 0$ and substituting f_1 with Dirac delta function δ_{z_1} at the point $z_1 \in \Phi$, we have (4.29),

$$\begin{aligned} & \frac{\partial_{\delta_{z_1}}}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_\Upsilon(h|\cdot))|_{h=0} \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) q^{m-1} \sum_{s_1=1}^m \tilde{l}(z_1|u_{s_1}) \nu(du_{1:m}) \\ &= \sum_{m=1}^{\infty} \frac{q^{m-1}}{m!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \left(\tilde{l}(z_1|u_1) + \cdots + \tilde{l}(z_1|u_m) \right) \nu(du_{1:m}) \end{aligned}$$

$$= \sum_{m=1}^{\infty} \frac{q^{m-1}}{(m-1)!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \tilde{l}(z_1|u_1) \nu(du_{1:m}).$$

□

By taking the partial derivative of (4.28) in Lemma 4.2.5 with a nonnegative measurable function f_2 , the following Lemma shows the second derivative of $\mathcal{G}_{\Psi}(g(\cdot)\mathcal{G}_{\Upsilon}(h|\cdot))$ evaluated in the direction of h with nonnegative measurable functions f_1 and f_2 .

Lemma 4.2.6. *The second derivative of $\mathcal{G}_{\Psi}(g(\cdot)\mathcal{G}_{\Upsilon}(h|\cdot))$ evaluated in the direction of h with nonnegative measurable functions f_1 and f_2 is given by*

$$\begin{aligned} & \frac{\partial_{f_1}}{\partial h} \frac{\partial_{f_2}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot)\mathcal{G}_{\Upsilon}(h|\cdot)) \\ &= \sum_{m=2}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \sum_{\substack{s_1, s_2=1 \\ s_1 \neq s_2}}^m \left(g(u_{s_1}) \int_{\Lambda} f_1(z) \tilde{l}(z|u_{s_1}) \nu(dz) \right) \\ & \quad \times \left(g(u_{s_2}) \int_{\Lambda} f_2(z) \tilde{l}(z|u_{s_2}) \nu(dz) \right) \prod_{\substack{s=1 \\ s \neq s_1, s_2}}^m \left(g(u_s) \mathcal{G}_{\Upsilon}(h|u_s) \right) \nu(du_{1:m}). \end{aligned} \quad (4.30)$$

By setting $g = 1$, $h = 0$ and substituting f_1 and f_2 with Dirac delta functions δ_{z_1} and δ_{z_2} at two distinct points $z_1, z_2 \in \Phi$ respectively, we have

$$\frac{\partial_{\delta_{z_1}}}{\partial h} \frac{\partial_{\delta_{z_2}}}{\partial h} \mathcal{G}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot))|_{h=0} = \sum_{m=2}^{\infty} \frac{q^{m-2}}{(m-2)!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \tilde{l}(z_1|u_1) \tilde{l}(z_2|u_2) \nu(du_{1:m}). \quad (4.31)$$

Proof. The second derivative of $\mathcal{G}_{\Psi}(g(\cdot)\mathcal{G}_{\Upsilon}(h|\cdot))$ as in (4.30) is computed as

$$\begin{aligned} & \frac{\partial_{f_1}}{\partial h} \frac{\partial_{f_2}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot)\mathcal{G}_{\Upsilon}(h|\cdot)) \\ &= \frac{\partial_{f_2}}{\partial h} \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \sum_{s_1=1}^m g(u_{s_1}) \int_{\Lambda} f_1(z) \tilde{l}(z|u_{s_1}) \nu(dz) \\ & \quad \times \prod_{\substack{s=1 \\ s \neq s_1}}^m \left(g(u_s) \mathcal{G}_{\Upsilon}(h|u_s) \right) \nu(du_{1:m}) \\ &= \sum_{m=2}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \sum_{\substack{s_1, s_2=1 \\ s_1 \neq s_2}}^m \left(g(u_{s_1}) \int_{\Lambda} f_1(z) \tilde{l}(z|u_{s_1}) \nu(dz) \right) \end{aligned}$$

$$\times \left(g(u_{s_2}) \int_{\Lambda} f_2(z) \tilde{l}(z|u_{s_2}) \nu(dz) \right) \prod_{\substack{s=1 \\ s \neq s_1, s_2}}^m \left(g(u_s) \mathcal{G}_{\Upsilon}(h|u_s) \right) \nu(du_{1:m}).$$

By setting $g = 1$, $h = 0$ and substituting f_1 and f_2 with Dirac delta functions δ_{z_1} and δ_{z_2} at two distinct points $z_1, z_2 \in \Phi$ respectively, we have (4.31),

$$\begin{aligned} & \frac{\partial_{\delta_{z_1}}}{\partial h} \frac{\partial_{\delta_{z_2}}}{\partial h} \mathcal{G}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot))|_{h=0} \\ &= \sum_{m=2}^{\infty} \frac{q^{m-2}}{m!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \sum_{\substack{s_1, s_2=1 \\ s_1 \neq s_2}}^m \tilde{l}(z_1|u_{s_1}) \tilde{l}(z_2|u_{s_2}) \nu(du_{1:m}) \\ &= \sum_{m=2}^{\infty} \frac{m(m-1)q^{m-2}}{m!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \tilde{l}(z_1|u_1) \tilde{l}(z_2|u_2) \nu(du_{1:m}) \\ &= \sum_{m=2}^{\infty} \frac{q^{m-2}}{(m-2)!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \tilde{l}(z_1|u_1) \tilde{l}(z_2|u_2) \nu(du_{1:m}). \end{aligned}$$

□

In the next proposition, we present the general r^{th} order derivative of $\mathcal{G}_{\Psi}(g(\cdot)\mathcal{G}_{\Upsilon}(h|\cdot))$ evaluated in the direction of h with nonnegative measurable functions $f_1, \dots, f_r$.

Proposition 4.2.7. *The general r^{th} order derivative of $\mathcal{G}_{\Psi}(g(\cdot)\mathcal{G}_{\Upsilon}(h|\cdot))$ evaluated in the direction of h with functions $f_1, \dots, f_r$ is given by*

$$\begin{aligned} & \frac{\partial_{f_1}}{\partial h} \dots \frac{\partial_{f_r}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot)\mathcal{G}_{\Upsilon}(h|\cdot)) \\ &= \sum_{m=r}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \sum_{\substack{s_1, \dots, s_r=1 \\ s_1 \neq \dots \neq s_r}}^m \prod_{l=1}^r \left(g(u_{s_l}) \int_{\Lambda} f_l(z) \tilde{l}(z|u_{s_l}) \nu(dz) \right) \\ & \quad \times \prod_{s \in \{1, \dots, m\} \setminus \{s_1, \dots, s_r\}}^m \left(g(u_s) \mathcal{G}_{\Upsilon}(h|u_s) \right) \nu(du_{1:m}). \end{aligned} \tag{4.32}$$

By setting $g = 1$, $h = 0$ and substituting $f_1, \dots, f_r$ with Dirac delta functions $\delta_{z_1}, \dots, \delta_{z_r}$ at distinct points $z_1, \dots, z_r \in \Phi$, we have

$$\frac{\partial_{\delta_{z_1}}}{\partial h} \dots \frac{\partial_{\delta_{z_r}}}{\partial h} \mathcal{G}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot))|_{h=0} = \sum_{m=r}^{\infty} \frac{q^{m-r}}{(m-r)!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \prod_{l=1}^r \left(\tilde{l}(z_l|u_l) \right) \nu(du_{1:m}). \tag{4.33}$$

Proof. In a similar argument as in Lemma 4.2.5 and Lemma 4.2.6 and by applying (2.41) of Proposition 2.2.21, the general r^{th} order derivative of $\mathcal{G}_\Psi(g(\cdot)\mathcal{G}_\Upsilon(h|\cdot))$ as in (4.32) is computed as

$$\begin{aligned}
& \frac{\partial_{f_1}}{\partial h} \cdots \frac{\partial_{f_r}}{\partial h} \mathcal{G}_\Psi(g(\cdot)\mathcal{G}_\Upsilon(h|\cdot)) \\
&= \frac{\partial_{f_r}}{\partial h} \sum_{m=r-1}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \sum_{\substack{s_1, \dots, s_{r-1}=1 \\ s_1 \neq \dots \neq s_{r-1}}}^m \prod_{l=1}^{r-1} \left(g(u_{s_l}) \int_{\Lambda} f_l(z) \tilde{l}(z|u_{s_l}) \nu(dz) \right) \\
&\quad \times \prod_{s \in \{1, \dots, m\} \setminus \{s_1, \dots, s_{r-1}\}}^m \left(g(u_s) \mathcal{G}_\Upsilon(h|u_s) \right) \nu(du_{1:m}) \\
&= \sum_{m=r-1}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \sum_{\substack{s_1, \dots, s_{r-1}=1 \\ s_1 \neq \dots \neq s_{r-1}}}^m \prod_{l=1}^{r-1} \left(g(u_{s_l}) \int_{\Lambda} f_l(z) \tilde{l}(z|u_{s_l}) \nu(dz) \right) \\
&\quad \times \frac{\partial_{f_r}}{\partial h} \prod_{s \in \{1, \dots, m\} \setminus \{s_1, \dots, s_{r-1}\}}^m \left(g(u_s) \mathcal{G}_\Upsilon(h|u_s) \right) \nu(du_{1:m}) \\
&= \sum_{m=r}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \sum_{\substack{s_1, \dots, s_r=1 \\ s_1 \neq \dots \neq s_r}}^m \prod_{l=1}^r \left(g(u_{s_l}) \int_{\Lambda} f_l(z) \tilde{l}(z|u_{s_l}) \nu(dz) \right) \\
&\quad \times \prod_{s \in \{1, \dots, m\} \setminus \{s_1, \dots, s_r\}}^m \left(g(u_s) \mathcal{G}_\Upsilon(h|u_s) \right) \nu(du_{1:m}).
\end{aligned}$$

By setting $g = 1$, $h = 0$ and substituting $f_1, \dots, f_r$ with Dirac delta functions $\delta_{z_1}, \dots, \delta_{z_r}$ at distinct points $z_1, \dots, z_r \in \Phi$, we have (4.33),

$$\begin{aligned}
& \frac{\partial_{\delta_{z_1}}}{\partial h} \cdots \frac{\partial_{\delta_{z_r}}}{\partial h} \mathcal{G}_\Psi(\mathcal{G}_\Upsilon(h|\cdot))|_{h=0} \\
&= \sum_{m=r}^{\infty} \frac{q^{m-r}}{m!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \sum_{\substack{s_1, \dots, s_r=1 \\ s_1 \neq \dots \neq s_r}}^m \prod_{l=1}^r \left(\tilde{l}(z_l|u_{s_l}) \right) \nu(du_{1:m}) \\
&= \sum_{m=r}^{\infty} \frac{q^{m-r}}{m!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \frac{m!}{(m-r)!} \prod_{l=1}^r \left(\tilde{l}(z_l|u_l) \right) \nu(du_{1:m}) \\
&= \sum_{m=r}^{\infty} \frac{q^{m-r}}{(m-r)!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \prod_{l=1}^r \left(\tilde{l}(z_l|u_l) \right) \nu(du_{1:m}).
\end{aligned}$$

□

Combinatorial Interpretation of Marginal Janossy Densities

In the following proposition, we present the expression of the marginal Janossy densities of branched point processes.

Proposition 4.2.8. *The marginal Janossy densities of the branched point process that is described under the superposition of a Poisson point process and branching under independent Bernoulli point processes are given by*

$$j_{\Phi}^{(r)}(z_{1:r}) = \sum_{m=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, r\} \\ |S| \leq m}} \prod_{j \notin S} j_{\Omega}^{(1)}(z_j) \frac{q^{m-|S|}}{(m-|S|)!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \prod_{i \in S} \left(\tilde{l}(z_i | u_i) \right) \nu(du_{1:m}), \quad (4.34)$$

where $j_{\Omega}^{(1)}(z_j)$ is the first order moment density of the Poisson point process Ω at the point $z_j \in \Lambda$, and $j \notin S$.

Proof. We recall from (4.26) of Lemma 4.2.3, the general r^{th} derivative of $\mathcal{G}_{\Psi, \Phi}(g, h)$ evaluated in the direction of h with nonnegative measurable functions $f_1, \dots, f_r$ is given by

$$\begin{aligned} & \frac{\partial_{f_1}}{\partial h} \dots \frac{\partial_{f_r}}{\partial h} \mathcal{G}_{\Psi, \Phi}(g, h) \\ &= \sum_{\substack{S \subset \{1, \dots, r\} \\ \pi = |S| \\ Q = \{1, \dots, r\} \setminus S}} \frac{\partial_{f_{S(1)}}}{\partial h} \dots \frac{\partial_{f_{S(\pi)}}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \frac{\partial_{f_{Q(1)}}}{\partial h} \dots \frac{\partial_{f_{Q(m-\pi)}}}{\partial h} \mathcal{G}_{\Omega}(h). \end{aligned}$$

In addition, we recall from (4.33) of Proposition 4.2.7, we have

$$\frac{\partial_{\delta_{z_1}}}{\partial h} \dots \frac{\partial_{\delta_{z_r}}}{\partial h} \mathcal{G}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot))|_{h=0} = \sum_{m=r}^{\infty} \frac{q^{m-r}}{(m-r)!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \prod_{l=1}^r \left(\tilde{l}(z_l | u_l) \right) \nu(du_{1:m}).$$

By substituting (4.33) into (4.26) after taking care of the combinatorial allocation defined in Definition 4.2.2, and by setting $g = 1$, $h = 0$ together with substituting $f_1, \dots, f_r$ with Dirac delta functions $\delta_{z_1}, \dots, \delta_{z_r}$ at distinct points $z_1, \dots, z_r \in \Phi$, we recover the Janossy densities of the descendant point process Φ , which are given by

$$\begin{aligned} j_{\Phi}^{(r)}(z_{1:r}) &= \frac{\partial_{\delta_{z_1}}}{\partial h} \dots \frac{\partial_{\delta_{z_r}}}{\partial h} \mathcal{G}_{\Psi, \Phi}(\mathbf{1}, h)|_{h=0} \\ &= \frac{\partial_{\delta_{z_1}}}{\partial h} \dots \frac{\partial_{\delta_{z_r}}}{\partial h} \left(\mathcal{G}_{\Omega}(h) \mathcal{G}_{\Psi}(\mathcal{G}_{\Upsilon}(h|\cdot)) \right) \Big|_{h=0} \end{aligned}$$

$$= \sum_{m=0}^{\infty} \sum_{\substack{S \subseteq \{1, \dots, r\} \\ |S| \leq m}} \prod_{j \notin S} j_{\Omega}^{(1)}(z_j) \frac{q^{m-|S|}}{(m-|S|)!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \prod_{i \in S} \left(\tilde{l}(z_i | u_i) \right) \nu(du_{1:m}),$$

where $j_{\Omega}^{(1)}(z_j)$ is the first order moment density of the Poisson point process Ω at the point $z_j \in \Lambda$, and $j \notin S$. $\square$

Combinatorial Interpretation of Joint Janossy Densities

By (2.25) of Remark 2.2.12 and (4.26) of Proposition 4.2.3, the general expression of the joint Janossy density of Ψ and Φ under the branching process of Ψ with independent Bernoulli point processes $\Upsilon|x$, $x \in \Psi$ and superposition with a Poisson point process Ω is given by

$$\begin{aligned} j_{\Psi, \Phi=z_{1:r}}^{(n,r)}(x_{1:n}) &= \frac{\partial_{\delta_{x_1}}}{\partial g} \dots \frac{\partial_{\delta_{x_n}}}{\partial g} \frac{\partial_{\delta_{z_1}}}{\partial h} \dots \frac{\partial_{\delta_{z_r}}}{\partial h} \mathcal{G}_{\Psi, \Phi}(g, h)|_{g=0, h=0} \\ &= \frac{\partial_{\delta_{x_1}}}{\partial g} \dots \frac{\partial_{\delta_{x_n}}}{\partial g} \frac{\partial_{\delta_{z_1}}}{\partial h} \dots \frac{\partial_{\delta_{z_r}}}{\partial h} \left(\mathcal{G}_{\Omega}(h) \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \right)_{|g=0, h=0} \\ &= \frac{\partial_{\delta_{x_1}}}{\partial g} \dots \frac{\partial_{\delta_{x_n}}}{\partial g} \sum_{\substack{S \subseteq \{1, \dots, r\} \\ \pi=|S| \\ Q=\{1, \dots, r\} \setminus S}} \frac{\partial_{\delta_{z_{S(1)}}}}{\partial h} \dots \frac{\partial_{\delta_{z_{S(\pi)}}}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot)) \\ &\quad \times \frac{\partial_{\delta_{z_{Q(1)}}}}{\partial h} \dots \frac{\partial_{\delta_{z_{Q(r-\pi)}}}}{\partial h} \mathcal{G}_{\Omega}(h)|_{g=0, h=0} \\ &= \frac{\partial_{\delta_{x_1}}}{\partial g} \dots \frac{\partial_{\delta_{x_n}}}{\partial g} \sum_{\substack{S \subseteq \{1, \dots, r\} \\ \pi=|S| \\ Q=\{1, \dots, r\} \setminus S}} j_{\Omega}^{(r-\pi)}(z_{Q(1)}, \dots, z_{Q(r-\pi)}) \\ &\quad \times \frac{\partial_{\delta_{z_{S(1)}}}}{\partial h} \dots \frac{\partial_{\delta_{z_{S(\pi)}}}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot))_{|g=0, h=0} \\ &= \frac{\partial_{\delta_{x_1}}}{\partial g} \dots \frac{\partial_{\delta_{x_n}}}{\partial g} \sum_{\substack{S \subseteq \{1, \dots, r\} \\ \pi=|S| \\ Q=\{1, \dots, r\} \setminus S}} \prod_{i=1}^{r-\pi} j_{\Omega}^{(1)}(z_{Q(i)}) \frac{\partial_{\delta_{z_{S(1)}}}}{\partial h} \dots \frac{\partial_{\delta_{z_{S(\pi)}}}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot))_{|g=0, h=0}. \end{aligned} \tag{4.35}$$

In the following corollary, we give the expression of the r^{th} order derivative of $\mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot))|_{h=0}$ evaluated in the direction of h with Dirac delta functions $\delta_{z_1}, \dots, \delta_{z_r}$

at a sequence of distinct point configurations $(z_1, \dots, z_r) = z_{1:r} \in \Phi$. This result extends from (4.33) of Proposition 4.2.7 to accommodate the nonnegative measurable function g within the composition PGFl $\mathcal{G}_\Psi(g(\cdot)\mathcal{G}_\Upsilon(h|\cdot))$.

Corollary 4.2.9. *The r^{th} order derivative of $\mathcal{G}_\Psi(g(\cdot)\mathcal{G}_\Upsilon(h|\cdot))|_{h=0}$ evaluated in the direction of h with Dirac delta functions $\delta_{z_1}, \dots, \delta_{z_r}$ at a sequence of distinct point configurations $(z_1, \dots, z_r) = z_{1:r} \in \Phi$ is given by*

$$\begin{aligned} & \frac{\partial_{\delta_{z_1}}}{\partial h} \dots \frac{\partial_{\delta_{z_r}}}{\partial h} \mathcal{G}_\Psi(g(\cdot)\mathcal{G}_\Upsilon(h|\cdot))|_{h=0} \\ &= \sum_{m=r}^{\infty} \frac{q^{m-r}}{(m-r)!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \left(\prod_{i=1}^m g(u_i) \right) \left(\prod_{j=1}^r \tilde{l}(z_j|u_j) \right) \nu(du_{1:m}). \end{aligned} \quad (4.36)$$

Proof. We restate (4.32), which is given by

$$\begin{aligned} & \frac{\partial_{f_1}}{\partial h} \dots \frac{\partial_{f_r}}{\partial h} \mathcal{G}_\Psi(g(\cdot)\mathcal{G}_\Upsilon(h|\cdot)) \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \sum_{\substack{s_1, \dots, s_r=1 \\ s_1 \neq \dots \neq s_r}}^m \prod_{l=1}^r \left(g(u_{s_l}) \int_{\Lambda} f_l(z) \tilde{l}(z|u_{s_l}) \nu(dz) \right) \\ & \quad \times \prod_{s \in \{1, \dots, m\} \setminus \{s_1, \dots, s_r\}}^m \left(g(u_s) \mathcal{G}_\Upsilon(h|u_s) \right) \nu(du_{1:m}). \end{aligned}$$

By setting $h = 0$ and substituting $f_1, \dots, f_r$ to Dirac delta functions at $z_1, \dots, z_r$ respectively, we have

$$\begin{aligned} & \frac{\partial_{\delta_{z_1}}}{\partial h} \dots \frac{\partial_{\delta_{z_r}}}{\partial h} \mathcal{G}_\Psi(g(\cdot)\mathcal{G}_\Upsilon(h|\cdot))|_{h=0} \\ &= \sum_{m=r}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \\ & \quad \times \sum_{\substack{s_1, \dots, s_r=1 \\ s_1 \neq \dots \neq s_r}}^m \left(g(u_{s_1}) \int_{\Lambda} \delta_{z_1}(z) \tilde{l}(z|u_{s_1}) \nu(dz) \right) \dots \left(g(u_{s_r}) \int_{\Lambda} \delta_{z_r}(z) \tilde{l}(z|u_{s_r}) \nu(dz) \right) \\ & \quad \times \prod_{s \in \{1, \dots, m\} \setminus \{s_1, \dots, s_r\}}^m \left(g(u_s) \mathcal{G}_\Upsilon(\mathbf{0}|u_s) \right) \nu(du_{1:m}) \\ &= \sum_{m=r}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \end{aligned}$$

$$\begin{aligned}
& \times \sum_{\substack{s_1, \dots, s_r=1 \\ s_1 \neq \dots \neq s_r}}^m g(u_{s_1}) \tilde{l}(z_1|u_{s_1}) \cdots g(u_{s_r}) \tilde{l}(z_r|u_{s_r}) \prod_{s \in \{1, \dots, m\} \setminus \{s_1, \dots, s_r\}}^m (qg(u_s)) \nu(du_{1:m}) \\
& = \frac{1}{r!} \int_{\Lambda^r} j_{\Psi}^{(r)}(u_{1:r}) \sum_{\substack{s_1, \dots, s_r=1 \\ s_1 \neq \dots \neq s_r}}^r g(u_{s_1}) \tilde{l}(z_1|u_{s_1}) \cdots g(u_{s_r}) \tilde{l}(z_r|u_{s_r}) \nu(du_{1:r}) \\
& \quad + \frac{1}{(r+1)!} \int_{\Lambda^{r+1}} j_{\Psi}^{(r+1)}(u_{1:r+1}) \sum_{\substack{s_1, \dots, s_r=1 \\ s_1 \neq \dots \neq s_r}} g(u_{s_1}) \tilde{l}(z_1|u_{s_1}) \cdots g(u_{s_r}) \tilde{l}(z_r|u_{s_r}) \\
& \quad \times \prod_{s \in \{1, \dots, r+1\} \setminus \{s_1, \dots, s_r\}} (qg(u_s)) \nu(u_{1:r+1}) \\
& \quad + \frac{1}{(r+2)!} \int_{\Lambda^{r+2}} j_{\Psi}^{(r+2)}(u_{1:r+2}) \sum_{\substack{s_1, \dots, s_r=1 \\ s_1 \neq \dots \neq s_r}} g(u_{s_1}) \tilde{l}(z_1|u_{s_1}) \cdots g(u_{s_r}) \tilde{l}(z_r|u_{s_r}) \\
& \quad \times \prod_{s \in \{1, \dots, r+2\} \setminus \{s_1, \dots, s_r\}} (qg(u_s)) \nu(u_{1:r+2}) \\
& \quad + \sum_{m=r+3}^{\infty} \frac{1}{m!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \sum_{\substack{s_1, \dots, s_r=1 \\ s_1 \neq \dots \neq s_r}}^m g(u_{s_1}) \tilde{l}(z_1|u_{s_1}) \cdots g(u_{s_r}) \tilde{l}(z_r|u_{s_r}) \\
& \quad \times \prod_{s \in \{1, \dots, m\} \setminus \{s_1, \dots, s_r\}}^m (qg(u_s)) \nu(du_{1:m}) \\
& = \int_{\Lambda^r} j_{\Psi}^{(r)}(u_{1:r}) \left(\prod_{i=1}^r g(u_i) \right) \left(\prod_{j=1}^r \tilde{l}(z_j|u_j) \right) \nu(du_{1:r}) \\
& \quad + q \int_{\Lambda^{r+1}} j_{\Psi}^{(r+1)}(u_{1:r+1}) \left(\prod_{i=1}^{r+1} g(u_i) \right) \left(\prod_{j=1}^r \tilde{l}(z_j|u_j) \right) \nu(du_{1:r+1}) \\
& \quad + \frac{q^2}{2!} \int_{\Lambda^{r+2}} j_{\Psi}^{(r+2)}(u_{1:r+2}) \left(\prod_{i=1}^{r+2} g(u_i) \right) \left(\prod_{j=1}^r \tilde{l}(z_j|u_j) \right) \nu(du_{1:r+2}) \\
& \quad + \sum_{m=r+3}^{\infty} \frac{q^{m-r}}{(m-r)!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \left(\prod_{i=1}^m g(u_i) \right) \left(\prod_{j=1}^r \tilde{l}(z_j|u_j) \right) \nu(du_{1:m}) \\
& = \sum_{m=r}^{\infty} \frac{q^{m-r}}{(m-r)!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \left(\prod_{i=1}^m g(u_i) \right) \left(\prod_{j=1}^r \tilde{l}(z_j|u_j) \right) \nu(du_{1:m}).
\end{aligned}$$

□

We extend (4.36) of Corollary 4.2.9 by applying a partial derivative evaluated in

the direction of g with a nonnegative measurable function η_1 . The partial derivative computation is given in the following

$$\begin{aligned}
& \frac{\partial \eta_1}{\partial g} \frac{\partial \delta_{z_1}}{\partial h} \cdots \frac{\partial \delta_{z_r}}{\partial h} \mathcal{G}_\Psi(g(\cdot) \mathcal{G}_\Upsilon(h|\cdot))|_{h=0} \\
&= \sum_{m=r}^{\infty} \frac{q^{m-r}}{(m-r)!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \frac{\partial \eta_1}{\partial g} \left(\prod_{i=1}^m g(u_i) \right) \left(\prod_{j=1}^r \tilde{l}(z_j|u_j) \right) \nu(du_{1:m}) \\
&= \sum_{m=r}^{\infty} \frac{q^{m-r}}{(m-r)!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \left(\sum_{k=1}^m \eta_1(u_k) \prod_{\substack{l=1 \\ l \neq k}}^m g(u_l) \right) \left(\prod_{j=1}^r \tilde{l}(z_j|u_j) \right) \nu(du_{1:m}) \\
&= \int_{\Lambda^r} j_\Psi^{(r)}(du_{1:r}) \left(\sum_{k=1}^r \eta_1(u_k) \prod_{\substack{l=1 \\ l \neq k}}^r g(u_l) \right) \left(\prod_{j=1}^r \tilde{l}(z_j|u_j) \right) \nu(du_{1:r}) \\
&\quad + \sum_{m=r+1}^{\infty} \frac{q^{m-r}}{(m-r)!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \left(\sum_{k=1}^m \eta_1(u_k) \prod_{\substack{l=1 \\ l \neq k}}^m g(u_l) \right) \\
&\quad \times \left(\prod_{j=1}^r \tilde{l}(z_j|u_j) \right) \nu(du_{1:m}). \tag{4.37}
\end{aligned}$$

In the following proposition, we present the n^{th} partial derivative of the expression in (4.36) evaluated in the direction of g with nonnegative measurable functions $\eta_1, \dots, \eta_n$. Since we are taking the n^{th} partial derivative in the direction of g , we can adapt the standard differentiation argument as in Proposition 2.2.21.

Proposition 4.2.10. *The n^{th} partial derivative of (4.36) evaluated in the direction of g with nonnegative measurable functions $\eta_1, \dots, \eta_n$ is given by*

$$\begin{aligned}
& \frac{\partial \eta_1}{\partial g} \cdots \frac{\partial \eta_n}{\partial g} \frac{\partial \delta_{z_1}}{\partial h} \cdots \frac{\partial \delta_{z_r}}{\partial h} \mathcal{G}_\Psi(g(\cdot) \mathcal{G}_\Upsilon(h|\cdot))|_{h=0} \\
&= \int_{\Lambda^r} j_\Psi^{(r)}(u_{1:r}) \left(\sum_{\substack{k_1, \dots, k_n=1 \\ k_1 \neq \dots \neq k_n}}^r \eta_1(u_{k_1}) \cdots \eta_n(u_{k_n}) \prod_{l \in \{1, \dots, r\} \setminus \{k_1, \dots, k_n\}}^r g(u_l) \right) \\
&\quad \times \left(\prod_{j=1}^r \tilde{l}(z_j|u_j) \right) \nu(du_{1:r}) \\
&\quad + \sum_{m=r+1}^{\infty} \frac{q^{m-r}}{(m-r)!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m})
\end{aligned}$$

$$\times \left(\sum_{\substack{k_1, \dots, k_n=1 \\ k_1 \neq \dots \neq k_n}}^m \eta_1(u_{k_1}) \cdots \eta_n(u_{k_n}) \prod_{l \in \{1, \dots, m\} \setminus \{k_1, \dots, k_n\}}^m g(u_l) \right) \left(\prod_{j=1}^r \tilde{l}(z_j | u_j) \right) \nu(du_{1:m}). \quad (4.38)$$

Proof. By applying (2.41) of Proposition 2.2.21 and by the extension of (4.37) to the n^{th} partial derivative of (4.36), the n^{th} partial derivative of the expression in (4.36) of Corollary 4.2.9 evaluated in the direction of g with nonnegative measurable functions $\eta_1, \dots, \eta_n$ is computed as

$$\begin{aligned} & \frac{\partial_{\eta_1}}{\partial g} \cdots \frac{\partial_{\eta_n}}{\partial g} \frac{\partial_{\delta_{z_1}}}{\partial h} \cdots \frac{\partial_{\delta_{z_r}}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot))|_{h=0} \\ &= \sum_{m=r}^{\infty} \frac{q^{m-r}}{(m-r)!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \\ & \quad \times \left(\sum_{\substack{k_1, \dots, k_n=1 \\ k_1 \neq \dots \neq k_n}}^m \eta_1(du_{k_1}) \cdots \eta_n(u_{k_n}) \prod_{l \in \{1, \dots, m\} \setminus \{k_1, \dots, k_n\}}^m g(u_l) \right) \left(\prod_{j=1}^r \tilde{l}(z_j | u_j) \right) \nu(du_{1:m}) \\ &= \int_{\Lambda^r} j_{\Psi}^{(r)}(u_{1:r}) \left(\sum_{\substack{k_1, \dots, k_n=1 \\ k_1 \neq \dots \neq k_n}}^r \eta_1(u_{k_1}) \cdots \eta_n(u_{k_n}) \prod_{l \in \{1, \dots, r\} \setminus \{k_1, \dots, k_n\}}^r g(u_l) \right) \\ & \quad \times \left(\prod_{j=1}^r \tilde{l}(z_j | u_j) \right) \nu(du_{1:r}) \\ & \quad + \sum_{m=r+1}^{\infty} \frac{q^{m-r}}{(m-r)!} \int_{\Lambda^m} j_{\Psi}^{(m)}(u_{1:m}) \\ & \quad \times \left(\sum_{\substack{k_1, \dots, k_n=1 \\ k_1 \neq \dots \neq k_n}}^m \eta_1(u_{k_1}) \cdots \eta_n(u_{k_n}) \prod_{l \in \{1, \dots, m\} \setminus \{k_1, \dots, k_n\}}^m g(u_l) \right) \left(\prod_{j=1}^r \tilde{l}(z_j | u_j) \right) \nu(du_{1:m}). \end{aligned}$$

□

For the case of $n \geq r$, by setting $g = 0$ and substituting the functions $\eta_1, \dots, \eta_n$ with Dirac delta functions $\delta_{x_1}, \dots, \delta_{x_n}$ at a sequence of distinct point configurations $(x_1, \dots, x_n) = x_{1:n} \in \Psi$ respectively in (4.38) of Proposition 4.2.10, we present the following corollary.

Corollary 4.2.11. *By setting $g = 0$ and substituting the functions $\eta_1, \dots, \eta_n$ with Dirac delta functions $\delta_{x_1}, \dots, \delta_{x_n}$ at a sequence of distinct point configurations $x_{1:n} \in$*

Ψ respectively in (4.38), and for the case of $n \geq r$, we have

$$\frac{\partial_{\delta_{x_1}}}{\partial g} \dots \frac{\partial_{\delta_{x_n}}}{\partial g} \frac{\partial_{\delta_{z_1}}}{\partial h} \dots \frac{\partial_{\delta_{z_r}}}{\partial h} \mathcal{G}_\Psi(g(\cdot) \mathcal{G}_\Upsilon(h|\cdot))|_{g=0, h=0} = \frac{n! q^{n-r}}{(n-r)!} \left(\prod_{j=1}^r \tilde{l}(z_j | x_j) \right) j_\Psi^{(n)}(x_{1:n}). \quad (4.39)$$

Proof.

$$\begin{aligned} & \frac{\partial_{\delta_{x_1}}}{\partial g} \dots \frac{\partial_{\delta_{x_n}}}{\partial g} \frac{\partial_{\delta_{z_1}}}{\partial h} \dots \frac{\partial_{\delta_{z_r}}}{\partial h} \mathcal{G}_\Psi(g(\cdot) \mathcal{G}_\Upsilon(h|\cdot))|_{g=0, h=0} \\ &= \sum_{m=r}^{\infty} \frac{q^{m-r}}{(m-r)!} \int_{\Lambda^m} j_\Psi^{(m)}(u_{1:m}) \left(\sum_{\substack{k_1, \dots, k_n=1 \\ k_1 \neq \dots \neq k_n}}^m \delta_{x_1}(u_{k_1}) \dots \delta_{x_n}(u_{k_n}) \prod_{l \in \{1, \dots, m\} \setminus \{k_1, \dots, k_n\}}^m \mathbf{0} \right) \\ & \quad \times \left(\prod_{j=1}^r \tilde{l}(z_j | u_j) \right) \nu(du_{1:m}) \\ &= \frac{q^{n-r}}{(n-r)!} \int_{\Lambda^n} j_\Psi^{(n)}(u_{1:n}) \left(\sum_{\substack{k_1, \dots, k_n=1 \\ k_1 \neq \dots \neq k_n}}^n \delta_{x_1}(u_{k_1}) \dots \delta_{x_n}(u_{k_n}) \right) \left(\prod_{j=1}^r \tilde{l}(z_j | u_j) \right) \nu(du_{1:n}) \\ &= \frac{n!}{(n-r)!} q^{n-r} \int_{\Lambda^n} \delta_{x_1}(u_1) \dots \delta_{x_n}(u_n) j_\Psi^{(n)}(u_{1:n}) \left(\prod_{j=1}^r \tilde{l}(z_j | u_j) \right) \nu(du_{1:n}) \\ &= \frac{n! q^{n-r}}{(n-r)!} \left(\prod_{j=1}^r \tilde{l}(z_j | x_j) \right) j_\Psi^{(n)}(x_{1:n}). \end{aligned}$$

The case of $n \geq r$ is required in order to ensure the result is valid. Since there are n points in the ancestor point process Ψ and r points in the descendant point process Φ , and that the branching process is on Ψ , the n points in Ψ should at least be equal to or greater than r points in Φ subject to the composition of n independent Bernoulli point processes with parameter p . In the case of $n < r$, the excess $r - n$ points in Φ is assigned to the Poisson point process Ω defined in subsection 4.2.1. $\square$

In the final proposition of this chapter, we present the combinatorial expression of the joint Janossy density of $\Psi = x_{1:n}$ and $\Phi = z_{1:m}$ in the following proposition, which depends on the result of Corollary 4.2.11.

Proposition 4.2.12. *The combinatorial expression of the joint Janossy density of $\Psi = x_{1:n}$ and $\Phi = z_{1:m}$ described under the branching of independent Bernoulli point*

processes $\Upsilon|u$, $u \in \Psi$ and superposition with a Poisson point process Ω is given by

$$j_{\Psi, \Phi=z_{1:m}}^{(n,m)}(x_{1:n}) = j_{\Psi}^{(n)}(x_{1:n}) \sum_{\substack{S \subseteq \{1, \dots, m\} \\ |S| \leq n}} \frac{n!q^{n-|S|}}{(n-|S|)!} \prod_{j \notin S} j_{\Omega}(z_j) \sum_{\pi: S \rightarrow \{1, \dots, n\}} \prod_{i \in S} \tilde{l}(z_i | x_{\pi(i)}), \quad (4.40)$$

where the sum being over injective mappings $\pi: S \rightarrow \{1, \dots, n\}$. The nonnegative measurable functions g and h correspond with Ψ and Φ respectively.

Proof. We recall the expression in (4.35),

$$\begin{aligned} j_{\Psi, \Phi=z_{1:m}}^{(n,m)}(x_{1:n}) &= \frac{\partial_{\delta_{x_1}} \dots \partial_{\delta_{x_n}}}{\partial g} \sum_{\substack{S \subseteq \{1, \dots, m\} \\ \pi: S \rightarrow \{1, \dots, n\} \\ Q = \{1, \dots, m\} \setminus S}} \prod_{i=1}^{m-\pi} j_{\Omega}^{(1)}(z_{Q(i)}) \frac{\partial_{\delta_{z_{S(1)}}}}{\partial h} \dots \frac{\partial_{\delta_{z_{S(\pi)}}}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot))|_{g=0, h=0} \\ &= \frac{\partial_{\delta_{x_1}} \dots \partial_{\delta_{x_n}}}{\partial g} \sum_{\substack{S \subseteq \{1, \dots, m\} \\ \pi: S \rightarrow \{1, \dots, n\} \\ Q = \{1, \dots, m\} \setminus S}} \prod_{i=1}^{m-|S|} j_{\Omega}^{(1)}(z_{Q(i)}) \frac{\partial_{\delta_{z_{\pi(1)}}}}{\partial h} \dots \frac{\partial_{\delta_{z_{\pi(|S|)}}}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot))|_{g=0, h=0}. \end{aligned}$$

In the following, we recall from (4.39) of Corollary 4.2.11,

$$\frac{\partial_{\delta_{x_1}} \dots \partial_{\delta_{x_n}}}{\partial g} \frac{\partial_{\delta_{z_1}}}{\partial h} \dots \frac{\partial_{\delta_{z_r}}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot))|_{g=0, h=0} = \frac{n!q^{n-r}}{(n-r)!} \left(\prod_{j=1}^r \tilde{l}(z_j | x_j) \right) j_{\Psi}^{(n)}(x_{1:n}).$$

By substituting (4.39) into (4.35) after taking care of the combinatorial allocation defined in Definition 4.2.2, the joint Janossy density of $\Psi = x_{1:n}$ and $\Phi = z_{1:m}$ described under the branching of independent Bernoulli point processes $\Upsilon|u$, $u \in \Psi$ and superposition with a Poisson point process Ω is evaluated as

$$\begin{aligned} j_{\Psi, \Phi=z_{1:m}}^{(n,m)}(x_{1:n}) &= \frac{\partial_{\delta_{x_1}} \dots \partial_{\delta_{x_n}}}{\partial g} \sum_{\substack{S \subseteq \{1, \dots, m\} \\ \pi: S \rightarrow \{1, \dots, n\} \\ Q = \{1, \dots, m\} \setminus S}} \prod_{i=1}^{m-|S|} j_{\Omega}^{(1)}(z_{Q(i)}) \frac{\partial_{\delta_{z_{\pi(1)}}}}{\partial h} \dots \frac{\partial_{\delta_{z_{\pi(|S|)}}}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot))|_{g=0, h=0} \\ &= \frac{\partial_{\delta_{x_1}} \dots \partial_{\delta_{x_n}}}{\partial g} \sum_{\substack{S \subseteq \{1, \dots, m\} \\ \pi: S \rightarrow \{1, \dots, n\}}} \prod_{j \notin S} j_{\Omega}^{(1)}(z_j) \frac{\partial_{\delta_{z_{\pi(1)}}}}{\partial h} \dots \frac{\partial_{\delta_{z_{\pi(|S|)}}}}{\partial h} \mathcal{G}_{\Psi}(g(\cdot) \mathcal{G}_{\Upsilon}(h|\cdot))|_{g=0, h=0} \end{aligned}$$

$$= j_{\Psi}^{(n)}(x_{1:n}) \sum_{\substack{S \subset \{1, \dots, n\} \\ |S| \leq n}} \frac{n! q^{n-|S|}}{(n-|S|)!} \prod_{j \notin S} j_{\Omega}(z_j) \sum_{\pi: S \rightarrow \{1, \dots, n\}} \prod_{i \in S} \tilde{l}(z_i | x_{\pi(i)}).$$

where $j_{\Omega}^{(1)}(z_j)$ is the first order moment density of the Poisson point process Ω at the point $z_j \in \Lambda$, and $j \notin S$. The second sum being over injective mappings $\pi: S \rightarrow \{1, \dots, n\}$. $\square$

Chapter 5

Second Order Multi-Object Filtering with Target Interaction using Determinantal Point Processes

This chapter constitutes the main original contribution to this thesis. In this chapter we construct a second-order PHD filter based on Determinantal Point Processes (DPPs) which are able to model repulsion between targets. Such processes are characterized by their first and second-order moments, which allows the algorithm to propagate variance and covariance information in addition to first-order target count estimates. Our approach relies on posterior moment formulas for the estimation of a general hidden point process after a thinning operation and a superposition with a Poisson Point Process (PPP), and on suitable approximation formulas in the determinantal point process setting. The repulsive properties of determinantal point processes also apply to the modeling of negative correlation between distinct observation domains. Monte Carlo simulations with correlation estimates are provided.

In Section 5.1 we derive general formulas for the computation of the first and second moments of a posterior point process Φ given an observation process Ξ of targets constructed by branching. We relate the posterior distribution of branched point processes into the context of multi-object tracking problem. In subsection 5.1.1, we derive the general expression of the conditional Janossy density in Lemma 5.1.1 and recover the marginal Janossy density of the Poisson case under specific conditions. In subsection 5.1.2, we derive the general expression of the first and second order posterior factorial moment densities using extensions of the corrector terms intro-

duced in [5], [21] for the cardinalized PHD filter as well as the Janossy densities of the general branched point processes in Proposition 5.1.2 and Proposition 5.1.3 respectively. In addition, under specific conditions, we show that the general expressions of the first and second order factorial moment densities recover to the Poisson case in Proposition 3.2.4 and Proposition 3.2.5. We present the general formula for the posterior covariance in Proposition 5.1.4.

In Section 5.2 we present a PHD filtering algorithm based on determinantal point processes, with the computation of the prediction kernel $K_{\Phi}(x, y)$ and of the updated kernel $K_{\Phi|\Xi}(x, y)$. In subsection 5.2.1, we propose to use the Janossy density approximation formula based on determinantal setting to obtain closed-form expressions. In Proposition 5.2.1 and Proposition 5.2.2, we derive the first and second order corrector term approximation formulas. In Proposition 5.2.3, we derive the approximate kernel update (posterior) formula under the determinantal setting. In Proposition 5.2.5 of subsection 5.2.2, we present the prediction phase first and second order PHD filtering equations for which will be applied to obtain the prediction determinantal kernel and Janossy kernel. In subsection 5.2.3, we present the determinantal kernel update approximation formula to the posterior phase PHD filtering equations.

In subsection 5.3.1 of Section 5.3, we present an algorithm using the sequential Monte Carlo method for the implementation of the Determinantal PHD filter. The implementation of the algorithm relies on closed-form filter update expressions obtained from approximation formulas for corrector terms and Janossy densities as described in subsection 5.2.1. In subsection 5.3.2, we present the numerical simulation results of the Determinantal PHD filter in some specific simulated environments.

5.1 Posterior Distribution of Branched Point Processes

In this section we put into context the problem of multi-object tracking, see [13], and compute the conditional first and second moments of a posterior point process of targets Φ given sensor observations Ξ , using extensions of the corrector terms introduced in [5], [21] for the cardinalized PHD filter, see Propositions 5.1.2 and 5.1.3. We start with a review of point process properties, see e.g. [3] and references therein for details.

Branching of Point Processes

Consider a point process Φ on a domain $\Lambda \subset \mathbb{R}^d$, that has Janossy densities $j_\Phi^{(n)}(x_1, \dots, x_n)$, $n \geq 0$. Namely, for any integrable functional F of Φ of the form

$$F(\Phi) = f_0 \mathbf{1}_{\{\Phi(\Lambda)=0\}} + \sum_{n=1}^{\infty} \mathbf{1}_{\{\Phi(\Lambda)=n\}} f_n(X_1, \dots, X_n),$$

where f_n is a symmetric function on Λ^n for each $n \geq 1$, we have

$$E[F(\Phi)] = f_0 j_\Phi^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} f_n(x_1, \dots, x_n) j_\Phi^{(n)}(x_1, \dots, x_n) \nu(dx_1) \cdots \nu(dx_n),$$

where $\nu(dx)$ is a reference measure on Λ .

The factorial moment density $\rho_\Phi^{(n)}(x_1, \dots, x_n)$ of Φ , $x_1, \dots, x_n \in \Lambda$, $x_i \neq x_j$, $1 \leq i < j \leq n$, is defined from the relation

$$\int_{A_1 \times \cdots \times A_n} \rho_\Phi^{(n)}(x_1, \dots, x_n) \nu(dx_1) \cdots \nu(dx_n) = E \left[\sum_{x_1, \dots, x_n \in \Phi} \mathbf{1}_{A_1}(x_1) \cdots \mathbf{1}_{A_n}(x_n) \right],$$

for mutually disjoint measurable subsets $A_1, \dots, A_n \subset \Lambda$.

The PGFl of the point process Φ is defined by

$$\begin{aligned} h \mapsto \mathcal{G}_\Phi(h) &:= E \left[\prod_{i=1}^{\Phi(\Lambda)} h(X_i) \right] \\ &= j_\Phi^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_\Phi^{(n)}(x_1, \dots, x_n) \prod_{i=1}^n h(x_i) \nu(dx_1) \cdots \nu(dx_n), \end{aligned}$$

where h is a nonnegative measurable function on Λ .

Recall that the Janossy densities $j_\Phi^{(n)}(x_1, \dots, x_n)$ and factorial moment densities $\rho_\Phi^{(n)}(x_1, \dots, x_n)$ of Φ can be recovered from the PGFl $\mathcal{G}_\Phi(h)$ as

$$j_\Phi^{(n)}(x_1, \dots, x_n) = \frac{\partial_{\delta_{x_1}}}{\partial h} \cdots \frac{\partial_{\delta_{x_n}}}{\partial h} \mathcal{G}_\Phi(h)|_{h=0}, \quad x_1, \dots, x_n \in \Lambda,$$

see e.g. § 2.4 of [3], and as

$$\rho_\Phi^{(n)}(x_1, \dots, x_n) = \frac{\partial_{\delta_{x_1}}}{\partial h} \cdots \frac{\partial_{\delta_{x_n}}}{\partial h} \mathcal{G}_\Phi(h)|_{h=1}, \quad x_1, \dots, x_n \in \Lambda, \quad (5.1)$$

with $x_i \neq x_j$, $1 \leq i < j \leq n$, see e.g. § 3.4 of [3].

The point process Ξ of sensor observations is constructed via the following steps.

- (i) Thinning and shifting. Every target point $x \in \Phi$ is kept with probability $p_d(x) \in (0, 1]$ and shifted according to the probability density function $l_d(\cdot|x)$, by branching the hidden point process Φ with a Bernoulli point process Ξ_s with PGFl

$$\begin{aligned} g \mapsto \mathcal{G}_{\Xi_s}(g | x) &:= q_d(x) + p_d(x) \int_{\Lambda} g(z) l_d(z|x) \nu(dz) \\ &= q_d(x) + \int_{\Lambda} g(z) \tilde{l}_d(z|x) \nu(dz), \end{aligned}$$

where for compactness of notation we take $q_d(x) := 1 - p_d(x)$, and let $\tilde{l}_d(z|x) := p_d(x) l_d(z|x)$, $x \in \Lambda$.

- (ii) Superposition with a Poisson point process Ξ_c with intensity function $l_c(\cdot)$ representing clutter.

In the sequel we use the shorthand notation

$$x_{1:n} = (x_1, \dots, x_n) \in \Lambda^n, \quad \text{and} \quad \nu(dx_{1:n}) := \nu(dx_1) \cdots \nu(dx_n),$$

see [5], [18]. The joint PGFl of the point process (Ξ, Φ) is given by

$$(h, g) \mapsto \mathcal{G}_{\Xi, \Phi}(g, h) := \mathcal{G}_{\Xi_c}(g) \mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_s}(g | \cdot)), \quad (5.2)$$

where, taking $p_d(x) := p_d$ for simplicity, $x \in \Lambda$, we have

$$\begin{aligned} &\mathcal{G}_{\Phi}(h(\cdot) \mathcal{G}_{\Xi_s}(g | \cdot)) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_{\Phi}^{(n)}(x_{1:n}) \prod_{i=1}^n \left(h(x_i) \left(q_d + \int_{\Lambda} g(z) \tilde{l}_d(z|x_i) \nu(dz) \right) \right) \nu(dx_{1:n}) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_{\Phi}^{(n)}(x_{1:n}) \prod_{j=1}^n h(x_j) \sum_{k=0}^n \binom{n}{k} q_d^{n-k} \prod_{i=1}^k \int_{\Lambda} g(z) \tilde{l}_d(z|x_i) \nu(dz) \nu(dx_{1:n}) \\ &= \sum_{k=0}^{\infty} \int_{\Lambda^k} \prod_{i=1}^k \int_{\Lambda} g(z) \tilde{l}_d(z|x_i) \nu(dz) \sum_{n=0}^{\infty} \frac{q_d^n}{n! k!} \int_{\Lambda^n} j_{\Phi}^{(k+n)}(x_{1:k+n}) \prod_{j=1}^{k+n} h(x_j) \nu(dx_{1:k+n}). \end{aligned}$$

More details about the above combinatorial interpretation of the branched PGFl can be found in the proof of Proposition 4.2.1.

The marginal PGFl of the branched process Ξ is given by

$$\begin{aligned} g \mapsto \mathcal{G}_\Xi(g) &= \mathcal{G}_{\Xi, \Phi}(g, \mathbf{1}) \\ &= \mathcal{G}_{\Xi_c}(g) \mathcal{G}_\Phi(\mathcal{G}_{\Xi_s}(g | \cdot)) \\ &= \mathcal{G}_{\Xi_c}(g) \mathcal{G}_\Phi \left(q_d(\cdot) + p_d(\cdot) \int_{\Lambda} g(y) l_d(y | \cdot) \nu(dy) \right). \end{aligned}$$

Marginal Moments of Branched Point Processes

The first derivative of $\mathcal{G}_\Xi(g)$ is given by

$$\begin{aligned} \frac{\partial_{\delta_x}}{\partial g} \mathcal{G}_\Xi(g) &= \frac{\partial_{\delta_x}}{\partial g} (\mathcal{G}_\Phi(\mathcal{G}_{\Xi_s}(g | \cdot)) \mathcal{G}_{\Xi_b}(g)) \\ &= \mathcal{G}_{\Xi_b}(g) \frac{\partial_{\delta_x}}{\partial g} \mathcal{G}_\Phi(\mathcal{G}_{\Xi_s}(g | \cdot)) + \mathcal{G}_\Phi(\mathcal{G}_{\Xi_s}(g | \cdot)) \frac{\partial_{\delta_x}}{\partial g} \mathcal{G}_{\Xi_b}(g) \\ &= \mathcal{G}_{\Xi_b}(g) \frac{\partial_{\delta_x}}{\partial g} \mathcal{G}_\Phi \left(1 - p_d(\cdot) + p_d(\cdot) \int_{\Lambda} g(y) l_d(y | \cdot) \nu(dy) \right) \\ &\quad + \mathcal{G}_\Phi \left(1 - p_d(\cdot) + p_d(\cdot) \int_{\Lambda} g(y) l_d(y | \cdot) \nu(dy) \right) \frac{\partial_{\delta_x}}{\partial g} \mathcal{G}_{\Xi_b}(g), \end{aligned}$$

from which the first order moment density of Ξ can be computed after setting $g = 1$ as

$$\mu_\Xi^{(1)}(x) = \frac{\partial_{\delta_x}}{\partial g} \mathcal{G}_\Xi(g) |_{g=1} = \int_{\Lambda} p_d(y) l_d(x | y) \mu_\Phi^{(1)}(y) \nu(dy) + \mu_{\Xi_b}^{(1)}(x). \quad (5.3)$$

Similarly, the second derivative of $\mathcal{G}_\Xi(g)$ is given by

$$\begin{aligned} &\frac{\partial_{\delta_x}}{\partial g} \frac{\partial_{\delta_y}}{\partial g} \left(\mathcal{G}_\Phi(\mathcal{G}_{\Xi_s}(g | \cdot)) \mathcal{G}_{\Xi_b}(g) \right) \\ &= \mathcal{G}_{\Xi_b}(g) \frac{\partial_{\delta_x}}{\partial g} \frac{\partial_{\delta_y}}{\partial g} \mathcal{G}_\Phi(\mathcal{G}_{\Xi_s}(g | \cdot)) + \frac{\partial_{\delta_x}}{\partial g} \mathcal{G}_\Phi(\mathcal{G}_{\Xi_s}(g | \cdot)) \frac{\partial_{\delta_y}}{\partial g} \mathcal{G}_{\Xi_b}(g) \\ &\quad + \frac{\partial_{\delta_y}}{\partial g} \mathcal{G}_\Phi(\mathcal{G}_{\Xi_s}(g | \cdot)) \frac{\partial_{\delta_x}}{\partial g} \mathcal{G}_{\Xi_b}(g) + \mathcal{G}_\Phi(\mathcal{G}_{\Xi_s}(g | \cdot)) \frac{\partial_{\delta_x}}{\partial g} \frac{\partial_{\delta_y}}{\partial g} (\mathcal{G}_{\Xi_b}(g)), \end{aligned}$$

from which the second order factorial moment density of Ξ can be computed after setting $g = 1$ as

$$\begin{aligned} \rho_\Xi^{(2)}(x, y) &= \int_{\Lambda^2} p_d(u) l_d(x | u) p_d(v) l_d(y | v) \rho_\Phi^{(2)}(u, v) \nu(du) \nu(dv) + \rho_{\Xi_b}^{(2)}(x, y) \\ &\quad + \mu_{\Xi_b}^{(1)}(y) \int_{\Lambda} p_d(u) l_d(x | u) \mu_\Phi^{(1)}(u) \nu(du) + \mu_{\Xi_b}^{(1)}(x) \int_{\Lambda} p_d(v) l_d(y | v) \mu_\Phi^{(1)}(v) \nu(dv), \end{aligned} \quad (5.4)$$

where $x, y \in \Lambda$, $x \neq y$.

5.1.1 Posterior Janossy Densities

Posterior Distribution

In the next lemma, which is consistent with Theorem 1 of [11], we derive the general expression of the PGFl of the posterior point process Φ given the sensor observation Ξ . In the sequel we let $|S|$ denote the cardinality of subsets $S \subset \{1, \dots, m\}$, and we use the notation $z_{1:m} = (z_1, \dots, z_m)$, while $z_{1:m} \setminus z$ denotes the sequence $(z_1, \dots, z_m)$ with the omission of z if $z \in \{z_1, \dots, z_m\}$.

Lemma 5.1.1. *The n -th conditional Janossy density of Φ given that $\Xi = z_{1:m} = (z_1, \dots, z_m)$ satisfies*

$$j_{\Phi|\Xi=z_{1:m}}^{(n)}(x_1, \dots, x_n) = \frac{j_{\Phi, \Xi=z_{1:m}}^{(n,m)}(x_1, \dots, x_n)}{j_{\Xi}^{(m)}(z_{1:m})}, \quad x_1, \dots, x_n \in \Lambda, \quad (5.5)$$

$m, n \geq 0$, where

(i) the (n, m) -th joint Janossy density of (Φ, Ξ) is given by

$$\begin{aligned} & j_{\Phi, \Xi=z_{1:m}}^{(n,m)}(x_1, \dots, x_n) \\ &= j_{\Phi}^{(n)}(x_1, \dots, x_n) \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq n}} \frac{n! q_d^{n-|S|}}{(n-|S|)!} \prod_{j \notin S} l_c(z_j) \sum_{\pi: S \rightarrow \{1, \dots, n\}} \prod_{i \in S} \tilde{l}_d(z_i | x_{\pi(i)}), \end{aligned} \quad (5.6)$$

the sum being over injective mappings $\pi: S \rightarrow \{1, \dots, n\}$,

(ii) the Janossy densities of the measurement process Ξ are given by

$$j_{\Xi}^{(m)}(z_1, \dots, z_m) = \sum_{n=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq n}} \prod_{j \notin S} l_c(z_j) \frac{q_d^{n-|S|}}{(n-|S|)!} \int_{\Lambda^n} j_{\Phi}^{(n)}(x_{1:n}) \prod_{i \in S} \tilde{l}_d(z_i | x_i) \nu(dx_{1:n}), \quad (5.7)$$

for $m \geq 0$.

Proof. By (5.2), the partial derivative of the joint PGFl of Φ when $\Xi = z_{1:m} = (z_1, \dots, z_m)$ is given by

$$\frac{\partial_{\delta_{z_1}} \dots \partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Phi, \Xi}(h, g)|_{g=0}$$

$$= \sum_{n=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq n}} \prod_{j \notin S} l_c(z_j) \frac{q_d^{n-|S|}}{(n-|S|)!} \int_{\Lambda^n} j_{\Phi}^{(n)}(y_{1:n}) \prod_{k=1}^n h(y_k) \prod_{i \in S} \tilde{l}_d(z_i | y_i) \nu(dy_{1:n}),$$

which yields (5.7) from the relation

$$\begin{aligned} j_{\Xi}^{(m)}(z_1, \dots, z_m) &= \frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Xi}(g)|_{g=0} \\ &= \frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Phi, \Xi}(\mathbf{1}, g)|_{g=0}. \end{aligned}$$

More details about derivation of (5.7) can be found in Proposition 4.2.8.

Similarly, (5.6) is obtained from

$$j_{\Phi, \Xi=z_{1:m}}^{(n,m)}(x_1, \dots, x_n) = \frac{\partial_{\delta_{x_1}}}{\partial h} \dots \frac{\partial_{\delta_{x_n}}}{\partial h} \frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Phi, \Xi}(h, g)|_{h,g=0},$$

and a standard differentiation argument. More details about derivation of (5.6) can be found in Proposition 4.2.12.

Finally, (5.5) follows from the Bayes formula. $\square$

Note that (5.6) and (5.7) admit natural combinatorial interpretations by identifying $S^c = \{1, \dots, m\} \setminus S$ to the set of points created according to a Poisson point process with intensity function $l_c(z)$, and by letting $n - |S|$ denote the number of points in Φ deleted with probability q_d by the Bernoulli process.

Poisson Case

In case Φ is a Poisson point process with intensity measure $\nu(dx)$ we have $j_{\Phi}^{(n)} = e^{-\nu(\Lambda)}$, $n \geq 0$, hence (5.7) recovers the classical expression

$$\begin{aligned} j_{\Xi}^{(m)}(z_1, \dots, z_m) &= e^{-\nu(\Lambda)} \sum_{n=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq n}} \prod_{j \notin S} l_c(z_j) \frac{(q_d \nu(\Lambda))^{n-|S|}}{(n-|S|)!} \prod_{i \in S} \int_{\Lambda} \tilde{l}_d(z_i | x_i) \nu(dx_i) \\ &= e^{-\nu(\Lambda)} \sum_{S \subset \{1, \dots, m\}} \prod_{j \notin S} l_c(z_j) \prod_{i \in S} \int_{\Lambda} \tilde{l}_d(z_i | x_i) \nu(dx_i) \sum_{n=|S|}^{\infty} \frac{(q_d \nu(\Lambda))^{n-|S|}}{(n-|S|)!} \\ &= e^{-p_d \nu(\Lambda)} \sum_{S \subset \{1, \dots, m\}} \prod_{j \notin S} l_c(z_j) \prod_{i \in S} \int_{\Lambda} \tilde{l}_d(z_i | x_i) \nu(dx_i) \\ &= e^{-p_d \nu(\Lambda)} \prod_{j=1}^m \left(l_c(z_j) + \int_{\Lambda} \tilde{l}_d(z_j | x) \nu(dx) \right), \quad m \geq 0. \end{aligned}$$

5.1.2 Posterior Factorial Moment Densities

First Order Posterior Moment Density

In the next proposition we estimate the first order conditional moment density of Φ given the sensor observation $\Xi = z_{1:m} = (z_1, \dots, z_m)$, using extensions of the corrector terms $l_{z_{1:m}}^{(1)}$ introduced in [5], [21] for the cardinalized PHD filter.

Proposition 5.1.2. *For a conditional point process $\Phi|\Xi = (z_1, \dots, z_m)$, the first order conditional moment density is given by*

$$\mu_{\Phi|\Xi=z_{1:m}}^{(1)}(x) = q_d l_{z_{1:m}}^{(1)}(x) + \sum_{z \in z_{1:m}} \tilde{l}_d(x|z) l_{z_{1:m}}^{(1)}(x; z), \quad (5.8)$$

with respect to $\nu(dx)$, $m \geq 0$, where

$$l_{z_{1:m}}^{(1)}(x) := \frac{\Upsilon_{z_{1:m}}^{(1)}(x)}{j_{\Xi}^{(m)}(z_{1:m})}, \quad l_{z_{1:m}}^{(1)}(x; z) := \frac{\Upsilon_{z_{1:m} \setminus z}^{(1)}(x)}{j_{\Xi}^{(m)}(z_{1:m})}, \quad (5.9)$$

are corrector terms, $j_{\Xi}^{(m)}(z_{1:m}) = j_{\Xi}^{(m)}(z_1, \dots, z_m)$ is given by (5.7), and

$$\Upsilon_{z_{1:m}}^{(1)}(x) := \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \int_{\Lambda^p} j_{\Phi}^{(p+1)}(x_{1:p}, x) \prod_{i \in S} \tilde{l}_d(z_i|x_i) \nu(dx_{1:p}), \quad (5.10)$$

$m \geq 0$.

Proof. The first order joint moment density of Φ with $\Xi = (z_1, \dots, z_m)$ can be obtained from the PGFl (5.2) as

$$\mu_{\Phi, \Xi=z_{1:m}}^{(1)}(x) = \frac{\partial_{\delta_x}}{\partial h} \mathcal{G}_{\Phi, \Xi=z_{1:m}}(h)_{h=1},$$

or, using the joint Janossy densities (5.6) and denoting by $d\hat{x}_r$ the absence of dx_r , as

$$\begin{aligned} & \mu_{\Phi, \Xi=z_{1:m}}^{(1)}(x) \\ &= \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{r=1}^p \int_{\Lambda^{p-1}} j_{\Phi, \Xi=z_{1:m}}^{(p,m)}(x_{1:p})|_{x_r=x} \nu(dx_1) \cdots \nu(d\hat{x}_r) \cdots \nu(dx_p) \\ &= \sum_{p=1}^{\infty} \sum_{r=1}^p \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \int_{\Lambda^{p-1}} j_{\Phi}^{(p)}(x_{1:p})|_{x_r=x} \end{aligned}$$

$$\begin{aligned}
& \times \sum_{\pi: S \rightarrow \{1, \dots, p\}} \prod_{i \in S} \tilde{l}_d(z_i | x_{\pi(i)})|_{x_r=x} \nu(dx_{1:p} \setminus dx_r) \\
&= \sum_{p=1}^{\infty} \sum_{r=1}^p \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p-1}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \\
& \quad \times \sum_{\pi: S \rightarrow \{1, \dots, p\} \setminus \{r\}} \int_{\Lambda^{p-1}} j_{\Phi}^{(p)}(x_{1:p})|_{x_r=x} \prod_{i \in S} \tilde{l}_d(z_i | x_{\pi(i)}) \nu(dx_{1:p} \setminus dx_r) \\
&+ \sum_{p=1}^{\infty} \sum_{r=1}^m \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p, r \in S}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \\
& \quad \times \sum_{\pi: S \rightarrow \{1, \dots, p\}} \int_{\Lambda^{p-1}} \prod_{i \in S} \tilde{l}_d(z_i | x_i)|_{x_{\pi(r)}=x} j_{\Phi}^{(p)}(x_{1:p})|_{x_{\pi(r)}=x} \nu(dx_{1:p} \setminus dx_{\pi(r)}) \\
&= \sum_{p=1}^{\infty} \sum_{r=1}^p \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p-1}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \\
& \quad \times \sum_{\pi: S \rightarrow \{1, \dots, p\} \setminus \{r\}} \int_{\Lambda^{p-1}} \prod_{i \in S} \tilde{l}_d(z_i | x_{\pi(i)}) j_{\Phi}^{(p)}(x_{1:p})|_{x_r=x} \nu(dx_{1:p} \setminus dx_r) \\
&+ \sum_{p=1}^{\infty} \sum_{r=1}^m \tilde{l}_d(z_r | x) \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p, r \in S}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \\
& \quad \times \sum_{\pi: S \rightarrow \{1, \dots, p\}} \int_{\Lambda^{p-1}} \prod_{i \in S \setminus \{r\}} \tilde{l}_d(z_i | x_i) j_{\Phi}^{(p)}(x_{1:p})|_{x_{\pi(r)}=x} \nu(dx_{1:p} \setminus dx_{\pi(r)}) \\
&= q_d \sum_{p=1}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p-1}} \frac{q_d^{p-1-|S|}}{(p-|S|-1)!} \prod_{j \notin S} l_c(z_j) \\
& \quad \times \sum_{\pi: S \rightarrow \{1, \dots, p-1\}} \int_{\Lambda^{p-1}} \prod_{i \in S} \tilde{l}_d(z_i | x_{\pi(i)}) j_{\Phi}^{(p)}(x_{1:p-1}, x) \nu(dx_{1:p-1}) \\
&+ \sum_{r=1}^m \sum_{p=1}^{\infty} \tilde{l}_d(z_r | x) \sum_{\substack{S \subset \{1, \dots, m\} \\ |S \setminus \{r\}| \leq p-1, r \in S}} \frac{q_d^{p+1-|S \setminus \{r\}|}}{(p-|S|+1)!} \prod_{j \notin S} l_c(z_j) \\
& \quad \times \sum_{\pi: S \setminus \{r\} \rightarrow \{1, \dots, p-1\}} \int_{\Lambda^{p-1}} \prod_{i \in S \setminus \{r\}} \tilde{l}_d(z_i | x_{\pi(i)}) j_{\Phi}^{(p)}(x_{1:p-1}, x) \nu(dx_{1:p-1}) \\
&= q_d \Upsilon_{z_{1:m}}^{(1)}(x) + \sum_{z \in z_{1:m}} \tilde{l}_d(x | z) \Upsilon_{z_{1:m} \setminus z}^{(1)}(x),
\end{aligned}$$

and it remains to divide by $j_{\Xi}^{(m)}(z_{1:m})$. $\square$

Second Order Posterior Moment Density

Similarly, the second partial moment of the first order integral of Φ when $\Xi = z_{1:m} = (z_1, \dots, z_m)$ is obtained in the next proposition, which uses an extension of the corrector terms $l_{z_{1:m}}^{(2)}$ introduced in [5], [21] for the cardinalized PHD filter.

Proposition 5.1.3. *The second order conditional factorial moment of Φ given that $\Xi = z_{1:m} = (z_1, \dots, z_m)$ is given by its density*

$$\begin{aligned} \rho_{\Phi|\Xi=z_{1:m}}^{(2)}(x, y) &= q_d^2 l_{z_{1:m}}^{(2)}(x, y) + q_d \sum_{z \in z_{1:m}} (\tilde{l}_d(z|x) + \tilde{l}_d(z|y)) l_{z_{1:m}}^{(2)}(x, y; z) \\ &\quad + \sum_{\substack{z, z' \in z_{1:m} \\ z \neq z'}} \tilde{l}_d(z|x) \tilde{l}_d(z'|y) l_{z_{1:m}}^{(2)}(x, y; z, z'), \quad x, y \in \Lambda, \quad x \neq y, \end{aligned} \quad (5.11)$$

with respect to $\nu(dx)\nu(dy)$, with the corrector terms

$$l_{z_{1:m}}^{(2)}(x, y) := \frac{\Upsilon_{z_{1:m}}^{(2)}(x, y)}{j_{\Xi}^{(m)}(z_{1:m})}, \quad l_{z_{1:m}}^{(2)}(x, y; z) := \frac{\Upsilon_{z_{1:m} \setminus z}^{(2)}(x, y)}{j_{\Xi}^{(m)}(z_{1:m})}, \quad (5.12)$$

and

$$l_{z_{1:m}}^{(2)}(x, y; z, z') := \frac{\Upsilon_{z_{1:m} \setminus \{z, z'\}}^{(2)}(x, y)}{j_{\Xi}^{(m)}(z_{1:m})}, \quad (5.13)$$

where $j_{\Xi}^{(m)}(z_1, \dots, z_m)$ is as in (5.7), and

$$\Upsilon_{z_{1:m}}^{(2)}(x, y) := \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \int_{\Lambda^p} j_{\Phi}^{(p+2)}(x_{1:p}, x, y) \prod_{i \in S} \tilde{l}_d(z_i | x_i) \nu(dx_{1:p}), \quad (5.14)$$

where $x, y \in \Lambda$, $m \geq 0$.

Proof. Factorial moments can be computed using the second derivative of the conditional PGFl (5.2), see (5.1), or equivalently using the joint Janossy densities (5.6) as in the proof of Proposition 5.1.2. We have

$$\rho_{\Phi, \Xi=z_{1:m}}^{(2)}(x, y)$$

$$\begin{aligned}
&= \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{\substack{r,u=1 \\ r \neq u}}^p \int_{\Lambda^{p-2}} j_{\Phi, \Xi=z_{1:m}}^{(p,m)}(x_{1:p}) \Big|_{\substack{x_r=x \\ x_u=y}} \nu(dx_1) \cdots \nu(d\hat{x}_r) \cdots \nu(d\hat{x}_u) \cdots \nu(dx_p) \\
&= \sum_{p=2}^{\infty} \sum_{\substack{r,u=1 \\ r \neq u}}^p \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) j_{\Phi}^{(p)}(x_{1:p}) \Big|_{\substack{x_r=x \\ x_u=y}} \\
&\quad \times \sum_{\pi: S \rightarrow \{1, \dots, p\}} \int_{\Lambda^{p-1}} \prod_{i \in S} \tilde{l}_d(z_i | x_{\pi(i)}) \Big|_{\substack{x_r=x \\ x_u=y}} \nu(dx_1) \cdots \nu(d\hat{x}_r) \cdots \nu(d\hat{x}_u) \cdots \nu(dx_p) \\
&= q_d^2 \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \\
&\quad \times \sum_{\pi: S \rightarrow \{1, \dots, p\}} \int_{\Lambda^p} j_{\Phi}^{(p+2)}(x_{1:p}, x, y) \prod_{i \in S} \tilde{l}_d(z_i | x_{\pi(i)}) \nu(dx_{1:p}) \\
&+ q_d \sum_{r=1}^m \tilde{l}_d(z_r | x) \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p+1, r \in S}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \\
&\quad \times \sum_{\pi: S \setminus \{r\} \rightarrow \{1, \dots, p\}} \int_{\Lambda^p} j_{\Phi}^{(p+2)}(x_{1:p}, x, y) \prod_{i \in S \setminus \{r\}} \tilde{l}_d(z_i | x_{\pi(i)}) \nu(dx_{1:p}) \\
&+ q_d \sum_{r=1}^m \tilde{l}_d(z_r | y) \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p+1, r \in S}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \\
&\quad \times \sum_{\pi: S \setminus \{r\} \rightarrow \{1, \dots, p\}} \int_{\Lambda^p} j_{\Phi}^{(p+2)}(x_{1:p}, x, y) \prod_{i \in S \setminus \{r\}} \tilde{l}_d(z_i | x_{\pi(i)}) \nu(dx_{1:p}) \\
&+ \sum_{\substack{r,u=1 \\ r \neq u}}^m \tilde{l}_d(z_r | x) \tilde{l}_d(z_u | y) \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p+2, r, u \in S}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \\
&\quad \times \sum_{\pi: S \setminus \{r, u\} \rightarrow \{1, \dots, p\}} \int_{\Lambda^p} j_{\Phi}^{(p+2)}(x_{1:p}, x, y) \prod_{i \in S \setminus \{r, u\}} \tilde{l}_d(z_i | x_{\pi(i)}) \nu(dx_{1:p}). \\
&= q_d^2 \Upsilon_{z_{1:m}}^{(2)}(x, y) + q_d \sum_{z \in z_{1:m}} (\tilde{l}_d(z | x) + \tilde{l}_d(z | y)) \Upsilon_{z_{1:m} \setminus z}^{(2)}(x, y) \\
&\quad + \sum_{\substack{z, z' \in z_{1:m} \\ z \neq z'}} \tilde{l}_d(z | x) \tilde{l}_d(z' | y) \Upsilon_{z_{1:m} \setminus \{z, z'\}}^{(2)}(x, y),
\end{aligned}$$

where $x, y \in \Lambda$, $x \neq y$, and it remains to divide by $j_{\Xi}^{(m)}(z_{1:m})$. $\square$

Poisson Case

In the case of a Poisson point process with $j_{\Phi}^{(n)} = e^{-\nu(\Lambda)}$, $n \geq 0$, (5.10) reads

$$\begin{aligned}
\Upsilon_{z_{1:m}}^{(1)}(x) &= e^{-\nu(\Lambda)} \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \nu(\Lambda)^{p-|S|} \prod_{i \in S} \int_{\Lambda} \tilde{l}_d(z_i|u) \nu(du) \\
&= e^{-\nu(\Lambda)} \sum_{S \subset \{1, \dots, m\}} \prod_{j \notin S} l_c(z_j) \prod_{i \in S} \int_{\Lambda} \tilde{l}_d(z_i|u) \nu(du) \sum_{p=|S|}^{\infty} \frac{q_d^{p-|S|}}{(p-|S|)!} \nu(\Lambda)^{p-|S|} \\
&= e^{-p_d \nu(\Lambda)} \sum_{S \subset \{1, \dots, m\}} \prod_{j \notin S} l_c(z_j) \prod_{i \in S} \int_{\Lambda} \tilde{l}_d(z_i|u) \nu(du) \\
&= e^{-p_d \nu(\Lambda)} \prod_{j=1}^m \left(l_c(z_j) + \int_{\Lambda} \tilde{l}_d(z_j|u) \nu(du) \right) \\
&= j_{\Xi}^{(m)}(z_{1:m}), \quad m \geq 0,
\end{aligned}$$

and similarly from (5.14) we find

$$\Upsilon_{z_{1:m}}^{(2)}(x, y) = \Upsilon_{z_{1:m}}^{(1)}(x) = j_{\Xi}^{(m)}(z_{1:m}).$$

Hence in the Poisson case the corrector terms are given by $l_{z_{1:m}}^{(1)}(x) = l_{z_{1:m}}^{(2)}(x, y) = 1$ and

$$l_{z_{1:m}}^{(1)}(x; z) = l_{z_{1:m}}^{(2)}(x, y; z) = \frac{1}{l_c(z) + \int_{\Lambda} \tilde{l}_d(z|u) \nu(du)},$$

with

$$\begin{aligned}
l_{z_{1:m}}^{(2)}(x, y; z, z') &= l_{z_{1:m}}^{(1)}(x; z) l_{z_{1:m}}^{(1)}(y, z') \\
&= \frac{1}{\left(l_c(z) + \int_{\Lambda} \tilde{l}_d(z|u) \nu(du) \right) \left(l_c(z') + \int_{\Lambda} \tilde{l}_d(z'|u) \nu(du) \right)},
\end{aligned}$$

hence the first and second (factorial) moment densities (5.8), (5.11) of Φ with respect to $\nu(dx)$ given the branched process Ξ reduce to the classical expressions

$$\mu_{\Phi|\Xi=z_{1:m}}^{(1)}(x) = q_d + \sum_{z \in z_{1:m}} \frac{\tilde{l}_d(z|x)}{l_c(z) + \int_{\Lambda} \tilde{l}_d(z|u) \nu(du)},$$

and

$$\rho_{\Phi|\Xi=z_{1:m}}^{(2)}(x, y) = q_d^2 + q_d \sum_{z \in z_{1:m}} \frac{\tilde{l}_d(z|x) + \tilde{l}_d(z|y)}{l_c(z) + \int_{\Lambda} \tilde{l}_d(z|u) \nu(du)}$$

$$+ \sum_{\substack{r,p=1 \\ r \neq p}}^m \frac{\tilde{l}_d(z_r|x)\tilde{l}_d(z_p|y)}{\left(l_c(z_r) + \int_{\Lambda} \tilde{l}_d(z_r|u)\nu(du)\right) \left(l_c(z_p) + \int_{\Lambda} \tilde{l}_d(z_p|v)\nu(dv)\right)},$$

$x, y \in \Lambda$, $x \neq y$, $m \geq 0$.

Posterior Covariance

Let

$$c_{\Phi|\Xi=(z_{1:m})}^{(2)}(A, B) := \mu_{\Phi|\Xi=z_{1:m}}^{(2)}(A, B) - \mu_{\Phi|\Xi=z_{1:m}}^{(1)}(A)\mu_{\Phi|\Xi=z_{1:m}}^{(1)}(B),$$

denote the posterior covariance, where $\mu_{\Phi|\Xi=z_{1:m}}^{(1)}(A)$ is the posterior first moment

$$\mu_{\Phi|\Xi=z_{1:m}}^{(1)}(A) := \int_A \mu_{\Phi|\Xi=z_{1:m}}^{(1)}(x)\nu(dx),$$

and $\mu_{\Phi|\Xi=z_{1:m}}^{(2)}(A, B)$ is the posterior second moment

$$\mu_{\Phi|\Xi=z_{1:m}}^{(2)}(A, B) = \int_{A \cap B} \mu_{\Phi|\Xi=z_{1:m}}^{(1)}(x)\nu(dx) + \int_{A \times B} \rho_{\Phi|\Xi=z_{1:m}}^{(2)}(x, y)\nu(dx)\nu(dy).$$

Using (5.8), (5.11) and Propositions 5.1.2-5.1.3, we obtain the following representation of the posterior covariance in a form similar to the variance update formula obtained for the Panjer-based filter of Theorem IV.7-(42) in [18].

Proposition 5.1.4. *The posterior covariance $c_{\Phi|\Xi=z_{1:m}}^{(2)}(A, B)$ of Φ given that $\Xi = (z_1, \dots, z_m)$ is given by*

$$\begin{aligned} & c_{\Phi|\Xi=z_{1:m}}^{(2)}(A, B) \\ &= q_d \int_{A \cap B} l_{z_{1:m}}^{(1)}(x)\nu(dx) + q_d^2 \int_{A \times B} (l_{z_{1:m}}^{(2)}(x, y) - l_{z_{1:m}}^{(1)}(x)l_{z_{1:m}}^{(1)}(y))\nu(dx)\nu(dy) \\ &+ q_d \sum_{z \in z_{1:m}} \int_{A \times B} \tilde{l}_d(z|x) (l_{z_{1:m}}^{(2)}(x, y; z) - l_{z_{1:m}}^{(2)}(x, y)l_{z_{1:m}}^{(1)}(x; z))\nu(dx)\nu(dy) \\ &+ q_d \sum_{z \in z_{1:m}} \int_{A \times B} \tilde{l}_d(z|y) (l_{z_{1:m}}^{(2)}(x, y; z) - l_{z_{1:m}}^{(1)}(y; z)l_{z_{1:m}}^{(2)}(x, y))\nu(dx)\nu(dy) \\ &+ \sum_{z \in z_{1:m}} \left(\int_{A \cap B} \tilde{l}_d(z|x)l_{z_{1:m}}^{(1)}(x; z)\nu(dx) \right. \\ &\quad \left. - \int_A \tilde{l}_d(z|x)l_{z_{1:m}}^{(1)}(x; z)\nu(dx) \int_B \tilde{l}_d(z|y)l_{z_{1:m}}^{(1)}(y; z)\nu(dy) \right) \end{aligned}$$

$$+ \sum_{\substack{z, z' \in z_{1:m} \\ z \neq z'}} \tilde{l}_d(z|x) \tilde{l}_d(z'|y) \int_{A \times B} (l_{z_{1:m}}^{(2)}(x, y; z, z') - l_{z_{1:m}}^{(1)}(x; z) l_{z_{1:m}}^{(1)}(y; z')) \nu(dx) \nu(dy),$$

where $m \geq 0$.

Poisson Case

In the case of a Poisson point process with $j_\Phi^{(n)} = e^{-\nu(\Lambda)}$, $n \geq 0$, Proposition 5.1.4 recovers the covariance

$$\begin{aligned} c_{\Phi|\Xi=z_{1:m}}^{(2)}(A, B) &= q_d \nu(A \cap B) \\ &+ \sum_{z \in z_{1:m}} \frac{\int_{A \cap B} \tilde{l}_d(z|x) \nu(dx)}{l_c(z) + \int_\Lambda \tilde{l}_d(z|u) \nu(du)} - \sum_{z \in z_{1:m}} \frac{\int_A \tilde{l}_d(z|x) \nu(dx) \int_B \tilde{l}_d(z|y) \nu(dy)}{\left(l_c(z) + \int_\Lambda \tilde{l}_d(z|u) \nu(du)\right)^2}, \end{aligned}$$

see e.g. Proposition V.1(a) of [18], and the variance

$$\begin{aligned} c_{\Phi|\Xi=z_{1:m}}^{(2)}(\Lambda, \Lambda) &= q_d \nu(\Lambda) + \sum_{z \in z_{1:m}} \frac{\int_\Lambda \tilde{l}_d(z|x) \nu(dx)}{l_c(z) + \int_\Lambda \tilde{l}_d(z|u) \nu(du)} \left(1 - \frac{\int_\Lambda \tilde{l}_d(z|x) \nu(dx)}{\left(l_c(z) + \int_\Lambda \tilde{l}_d(z|u) \nu(du)\right)}\right), \end{aligned}$$

see also Exercise 4.3.4 in [3].

5.2 Determinantal PHD Filter

In this section we build a second-order PHD filter based on determinantal point processes. We show that approximate closed-form filter update expressions can be derived using approximation formulas stated in subsection 5.2.1 for the corrector terms $l_{z_{1:m}}^{(1)}$, $l_{z_{1:m}}^{(2)}$ and Janossy densities $j_\Phi^{(n)}$, when the underlying point process has low cross-correlations.

Determinantal Point Processes

We review the properties of determinantal point processes, see e.g. [4] and references therein for additional background. Let $\mathfrak{K}_\Psi$ be a Hermitian trace class operator on $L^2(\Lambda, \nu)$ with spectrum contained in $[0, 1)$ and kernel $K_\Psi : \Lambda \times \Lambda \rightarrow \mathbb{R}$. The

determinantal point process Ψ on Λ with kernel $K_\Psi(x, y)$, $x, y \in \Lambda$, is the point process defined by its Janossy densities

$$j_\Psi^{(0)} = \text{Det}(I_d - K_\Psi), \quad j_\Psi^{(1)}(x) = \text{Det}(I_d - K_\Psi) J_\Psi(x, x), \quad x \in \Lambda,$$

and

$$j_\Psi^{(n)}(x_1, \dots, x_n) := \text{Det}(I_d - K_\Psi) \det(J_\Psi(x_i, x_j))_{1 \leq i, j \leq n}, \quad (5.15)$$

$x_1, \dots, x_n \in \Lambda$, where $J_\Psi(x, y)$ is the kernel of the operator $\mathfrak{J}_\Psi$ defined by

$$\mathfrak{J}_\Psi := (I_d - \mathfrak{K}_\Psi)^{-1} \mathfrak{K}_\Psi, \quad (5.16)$$

and

$$\text{Det}(I_d - K_\Psi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det(K_\Psi(x_i, x_j))_{1 \leq i, j \leq n} \nu(dx_1) \cdots \nu(dx_n)$$

is the Fredholm determinant of the operator $I_d - K_\Psi$, with I_d denoting the identity operator.

The factorial moment densities $\rho_\Psi^{(n)}(x_1, \dots, x_n)$ of Ψ are linked to the kernel $K_\Psi(x, y)$ by the relation

$$\rho_\Psi^{(n)}(x_1, \dots, x_n) = \det(K_\Psi(x_i, x_j))_{1 \leq i, j \leq n},$$

$x_1, \dots, x_n \in \Lambda$, with $x_i \neq x_j$, $1 \leq i < j \leq n$, see e.g. Lemma 3.3 of [19] or Proposition 2.4 of [4]. In particular, we have

$$\mu_\Psi^{(1)}(x) = K_\Psi(x, x), \quad x \in \Lambda, \quad (5.17)$$

and

$$\rho_\Psi^{(2)}(x, y) = K_\Psi(x, x)K_\Psi(y, y) - (K_\Psi(x, y))^2, \quad (5.18)$$

$x, y \in \Lambda$, $x \neq y$, i.e.

$$(K_\Psi(x, y))^2 = \mu_\Psi^{(1)}(x)\mu_\Psi^{(1)}(y) - \rho_\Psi^{(2)}(x, y) \geq 0, \quad x, y \in \Lambda, \quad (5.19)$$

by letting $\rho_\Psi^{(2)}(x, x) := 0$, $x \in \Lambda$. The covariance of the determinantal process is then given by

$$\begin{aligned} c_\Psi^{(2)}(A, B) &= \int_{A \cap B} \mu_\Psi^{(1)}(x) \mu(dx) + \int_{A \times B} (\rho_\Psi^{(2)}(x, y) - \mu_\Psi^{(1)}(x)\mu_\Psi^{(1)}(y)) \nu(dx) \nu(dy) \\ &= \int_{A \cap B} K_\Psi(x, x) \nu(dx) - \int_{A \times B} (K_\Psi(x, y))^2 \nu(dx) \nu(dy), \end{aligned}$$

which shows that the determinantal point process Ψ is *negatively correlated*, since when $A \cap B = \emptyset$ we have

$$c_{\Psi}^{(2)}(A, B) = - \int_{A \times B} (K_{\Psi}(x, y))^2 \nu(dx) \nu(dy) \leq 0.$$

In the sequel we will restrict the class of determinantal kernels considered to a class of finite range interaction point processes, by enforcing the condition

$$J(x, y) = 0 \text{ for all } x, y \in \Lambda \text{ such that } |x - y| > \eta d(\Lambda), \quad (5.20)$$

cf. e.g. Proposition 3.9 in [7], where $d(\Lambda)$ is the diameter of Λ and $\eta \in (0, 1)$.

5.2.1 Janossy Density Approximation

Since the corrector terms $l_{z_{1:m}}^{(1)}(x)$, $l_{z_{1:m}}^{(1)}(x; z)$, $l_{z_{1:m}}^{(2)}(x, y)$, $l_{z_{1:m}}^{(2)}(x, y; z)$, $l_{z_{1:m}}^{(2)}(x, y; z, z')$ in (5.9), (5.12) and the kernel update formula (5.37) have no closed form expression in the determinantal setting, we propose to use the Janossy density approximations

$$j_{\Phi}^{(n)}(x_1, \dots, x_{r-1}, x, x_{r+1}, \dots, x_n) \simeq J_{\Phi}(x, x) j_{\Phi}^{(n-1)}(x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_n) \quad (5.21)$$

$n \geq 1$, which corresponds to a (Poisson) first order approximation, and

$$\begin{aligned} & j_{\Phi}^{(n)}(x_1, \dots, x_{r-1}, x, x_{r+1}, \dots, x_{p-1}, y, x_{p+1}, \dots, x_n) \\ & \simeq (J_{\Phi}(x, x) J_{\Phi}(y, y) - (J_{\Phi}(x, y))^2) j_{\Phi}^{(n-2)}(x_1, \dots, \hat{x}_r, \dots, \hat{x}_p, \dots, x_n), \end{aligned} \quad (5.22)$$

$n \geq 2$, which corresponds to a second order (determinant) approximation, obtained from (5.15) by assuming that the off-diagonal entries $J_{\Phi}(x_i, x_j)$, $i \neq j$, are small.

The Janossy approximation is specially relevant to α -determinantal Ginibre point processes (GPP) which approximate a Poisson point process when $\alpha \in [-1, 0)$ tends to 0, see [19].

Proposition 5.2.1. *Under (5.21) we have the first order Poisson approximations $l_{z_{1:m}}^{(1)}(x) \simeq J_{\Phi}(x, x)$, $m \geq 0$, and*

$$l_{z_{1:m}}^{(1)}(x; z) \simeq \frac{J_{\Phi}(x, x)}{l_c(z) + \int_{\Lambda} J_{\Phi}(u, u) \tilde{l}_d(z|u) \nu(du)},$$

$z \in z_{1:m}$, $x \in \Lambda$, $m \geq 1$.

Proof. By (5.10) and (5.21) we have

$$\begin{aligned}
& \Upsilon_{z_{1:m}}^{(1)}(x) \\
&= \sum_{S \subset \{1, \dots, m\}} \sum_{p=|S|}^{\infty} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \int_{\Lambda^p} j_{\Phi}^{(p+1)}(x_{1:p}, x) \prod_{i \in S} \tilde{l}_d(z_i | x_i) \nu(dx_{1:p}) \\
&\simeq J_{\Phi}(x, x) \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \int_{\Lambda^p} j_{\Phi}^{(p)}(x_{1:p}) \prod_{i \in S} \tilde{l}_d(z_i | x_i) \nu(dx_{1:p}) \\
&= J_{\Phi}(x, x) j_{\Xi}^{(m)}(z_1, \dots, z_m), \tag{5.23}
\end{aligned}$$

by (5.7), which yields the approximation $l_{z_{1:m}}^{(1)}(x) \simeq J_{\Phi}(x, x)$. On the other hand, for $r = 1, \dots, m$, using again (5.21) and (5.7) we have

$$\begin{aligned}
j_{\Xi}^{(m)}(z_1, \dots, z_m) &= \frac{\partial_{\delta_{z_1}}}{\partial g} \dots \frac{\partial_{\delta_{z_m}}}{\partial g} \mathcal{G}_{\Phi, \Xi}(\mathbf{1}, g)|_{g=0} \\
&= \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \int_{\Lambda^p} j_{\Phi}^{(p)}(y_{1:p}) \prod_{i \in S} \tilde{l}_d(z_i | y_i) \nu(dy_{1:p}) \\
&\simeq l_c(z_r) \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \setminus \{r\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \int_{\Lambda^p} j_{\Phi}^{(p)}(x_{1:p}) \prod_{i \in S} \tilde{l}_d(z_i | x_i) \nu(dx_{1:p}) \\
&\quad + \int_{\Lambda} J_{\Phi}(x_r, x_r) \tilde{l}_d(z_r | x_r) \nu(dx_r) \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p+1, r \in S}} \frac{q_d^{p+1-|S|}}{(p+1-|S|)!} \\
&\quad \times \prod_{j \notin S} l_c(z_j) \int_{\Lambda^p} j_{\Phi}^{(p)}(x_{1:p}) \prod_{i \in S \setminus \{r\}} \tilde{l}_d(z_i | x_i) \nu(dx_{1:p}) \\
&= l_c(z_r) \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \setminus \{r\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \int_{\Lambda^p} j_{\Phi}^{(p)}(x_{1:p}) \prod_{i \in S} \tilde{l}_d(z_i | x_i) \nu(dx_{1:p}) \\
&\quad + \int_{\Lambda} J_{\Phi}(u, u) \tilde{l}_d(z_r | u) \nu(du) \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \setminus \{r\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \\
&\quad \times \int_{\Lambda^p} j_{\Phi}^{(p)}(x_{1:p}) \prod_{i \in S \setminus \{r\}} \tilde{l}_d(z_i | x_i) \nu(dx_{1:p}) \\
&= \left(l_c(z_r) + \int_{\Lambda} J_{\Phi}(u, u) \tilde{l}_d(z_r | u) \nu(du) \right)
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \int_{\Lambda^p} j_{\Phi}^{(p)}(x_{1:p}) \prod_{i \in S} \tilde{l}_d(z_i|x_i) \nu(dx_{1:p}) \\
& = \left(l_c(z_r) + \int_{\Lambda} J_{\Phi}(u, u) \tilde{l}_d(z_r|u) \nu(du) \right) j_{\Xi}^{(m-1)}(z_1, \dots, z_{r-1}, z_{r+1}, \dots, z_m). \quad (5.24)
\end{aligned}$$

We conclude by taking $z_r = z$ and noting that by (5.9) and (5.23)-(5.24) we have

$$l_{z_{1:m}}^{(1)}(x; z) = \frac{\Upsilon_{z_{1:m} \setminus z}^{(1)}(x)}{j_{\Xi}^{(m)}(z_{1:m})} \simeq J_{\Phi}(x, x) \frac{j_{\Xi}^{(m-1)}(z_{1:m} \setminus z)}{j_{\Xi}^{(m)}(z_{1:m})}.$$

□

Proposition 5.2.2. *Under (5.21)-(5.22) we have the second order approximations*

$$\begin{aligned}
l_{z_{1:m}}^{(2)}(x, y) & \simeq J_{\Phi}(x, x) J_{\Phi}(y, y) - J_{\Phi}(x, y)^2, \\
l_{z_{1:m}}^{(2)}(x, y; z) & \simeq \frac{J_{\Phi}(x, x) J_{\Phi}(y, y) - J_{\Phi}(x, y)^2}{l_c(z) + \int_{\Lambda} J_{\Phi}(u, u) \tilde{l}_d(z|u) \nu(du)},
\end{aligned}$$

where $z \in z_{1:m}$, $x, y \in \Lambda$, $m \geq 1$, and

$$l_{z_{1:m}}^{(2)}(x, y; z, z') \simeq \frac{J_{\Phi}(x, x) J_{\Phi}(y, y) - J_{\Phi}(x, y)^2}{s_c(z) s_c(z') - \int_{\Lambda^2} J_{\Phi}(u, v)^2 \tilde{l}_d(z|u) \tilde{l}_d(z'|v) \nu(du) \nu(dv)},$$

for which $z, z' \in z_{1:m}$, $z \neq z'$, $x, y \in \Lambda$, $m \geq 2$, where

$$s_c(z) := l_c(z) + \int_{\Lambda} J_{\Phi}(v, v) \tilde{l}_d(z|v) \nu(dv), \quad z \in \Lambda.$$

Proof. By (5.14) and (5.22) we have

$$\begin{aligned}
\Upsilon_{z_{1:m}}^{(2)}(x, y) & = \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \int_{\Lambda^p} j_{\Phi}^{(p+2)}(x_{1:p}, x, y) \prod_{i \in S} \tilde{l}_d(z_i|x_i) \nu(dx_{1:p}) \\
& = (J_{\Phi}(x, x) J_{\Phi}(y, y) - J_{\Phi}(x, y)^2) \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \\
& \quad \times \int_{\Lambda^p} j_{\Phi}^{(p)}(x_{1:p}) \prod_{i \in S} \tilde{l}_d(z_i|x_i) \nu(dx_{1:p}) \quad (5.25)
\end{aligned}$$

$$= (J_{\Phi}(x, x) J_{\Phi}(y, y) - J_{\Phi}(x, y)^2) j_{\Xi}^{(m)}(z_{1:m}), \quad (5.26)$$

and for $r, u = 1, \dots, m$, using (5.21)-(5.22) and (5.7) we find

$$\begin{aligned}
j_{\Xi}^{(m)}(z_{1:m}) &= \sum_{n=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq n}} \prod_{j \notin S} l_c(z_j) \frac{q_d^{n-|S|}}{(n-|S|)!} \int_{\Lambda^n} j_{\Phi}^{(n)}(y_{1:n}) \prod_{i \in S} \tilde{l}_d(z_i | y_i) \nu(dy_{1:n}) \\
&\simeq l_c(z_r) l_c(z_u) \sum_{p=0}^{\infty} \sum_{\substack{S \subset \{1, \dots, m\} \setminus \{r, u\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} \prod_{j \notin S} l_c(z_j) \int_{\Lambda^p} j_{\Phi}^{(p)}(x_{1:p}) \prod_{i \in S} \tilde{l}_d(z_i | x_i) \nu(dx_{1:p}) \\
&\quad + l_c(z_r) \int_{\Lambda} J_{\Phi}(v, v) \tilde{l}_d(z_u | v) \nu(dv) \\
&\quad \times \sum_{p=0}^{\infty} \int_{\Lambda^p} \sum_{\substack{S \subset \{1, \dots, m\} \setminus \{r\} \\ |S| \leq p+1, u \in S}} \frac{q_d^{p+1-|S|}}{(p+1-|S|)!} j_{\Phi}^{(p)}(x_{1:p}) \prod_{j \notin S} l_c(z_j) \prod_{i \in S \setminus \{u\}} \tilde{l}_d(z_i | x_i) \nu(dx_{1:p}) \\
&\quad + l_c(z_u) \int_{\Lambda} J_{\Phi}(v, v) \tilde{l}_d(z_r | v) \nu(dv) \\
&\quad \times \sum_{p=0}^{\infty} \int_{\Lambda^p} \sum_{\substack{S \subset \{1, \dots, m\} \setminus \{u\} \\ |S| \leq p+1, r \in S}} \frac{q_d^{p+1-|S|}}{(p+1-|S|)!} j_{\Phi}^{(p)}(x_{1:p}) \prod_{j \notin S} l_c(z_j) \prod_{i \in S \setminus \{r\}} \tilde{l}_d(z_i | x_i) \nu(dx_{1:p}) \\
&\quad + \int_{\Lambda} (J_{\Phi}(x_r, x_r) J_{\Phi}(x_u, x_u) - J_{\Phi}(x_r, x_u)^2) \tilde{l}_d(z_r | x_r) \tilde{l}_d(z_u | x_u) \nu(dx_r) \nu(dx_u) \\
&\quad \times \sum_{p=0}^{\infty} \int_{\Lambda^p} \sum_{\substack{S \subset \{1, \dots, m\} \\ |S| \leq p+2, r \in S}} \frac{q_d^{p+2-|S|}}{(p+2-|S|)!} j_{\Phi}^{(p)}(x_{1:p}) \prod_{j \notin S} l_c(z_j) \prod_{i \in S \setminus \{r, u\}} \tilde{l}_d(z_i | x_i) \nu(dx_{1:p}) \\
&= l_c(z_r) l_c(z_u) \sum_{p=0}^{\infty} \int_{\Lambda^p} \sum_{\substack{S \subset \{1, \dots, m\} \setminus \{r, u\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} j_{\Phi}^{(p)}(x_{1:p}) \prod_{j \notin S} l_c(z_j) \prod_{i \in S} \tilde{l}_d(z_i | x_i) \nu(dx_{1:p}) \\
&\quad + l_c(z_r) \int_{\Lambda} J_{\Phi}(v, v) \tilde{l}_d(z_u | v) \nu(dv) \\
&\quad \times \sum_{p=0}^{\infty} \int_{\Lambda^p} \sum_{\substack{S \subset \{1, \dots, m\} \setminus \{r, u\} \\ |S| \leq p, u \in S}} \frac{q_d^{p-|S|}}{(p-|S|)!} j_{\Phi}^{(p)}(x_{1:p}) \prod_{j \notin S} l_c(z_j) \prod_{i \in S} \tilde{l}_d(z_i | x_i) \nu(dx_{1:p}) \\
&\quad + l_c(z_u) \int_{\Lambda} J_{\Phi}(v, v) \tilde{l}_d(z_r | v) \nu(dv) \\
&\quad \times \sum_{p=0}^{\infty} \int_{\Lambda^p} \sum_{\substack{S \subset \{1, \dots, m\} \setminus \{r, u\} \\ |S| \leq p, r \in S}} \frac{q_d^{p-|S|}}{(p-|S|)!} j_{\Phi}^{(p)}(x_{1:p}) \prod_{j \notin S} l_c(z_j) \prod_{i \in S} \tilde{l}_d(z_i | x_i) \nu(dx_{1:p}) \\
&\quad + \int_{\Lambda^2} (J_{\Phi}(u, u) J_{\Phi}(v, v) - J_{\Phi}(u, v)^2) \tilde{l}_d(z_r | u) \tilde{l}_d(z_u | v) \nu(du) \nu(dv)
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{p=0}^{\infty} \int_{\Lambda^p} \sum_{\substack{S \subset \{1, \dots, m\} \setminus \{r, u\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} j_{\Phi}^{(p)}(x_{1:p}) \prod_{j \notin S} l_c(z_j) \prod_{i \in S} \tilde{l}_d(z_i|x_i) \nu(dx_{1:p}) \\
& = \left(l_c(z_r) l_c(z_u) + l_c(z_r) \int_{\Lambda} J_{\Phi}(v, v) \tilde{l}_d(z_u|v) \nu(dv) + l_c(z_u) \int_{\Lambda} J_{\Phi}(v, v) \tilde{l}_d(z_r|v) \nu(dv) \right. \\
& \quad \left. + \int_{\Lambda^2} (J_{\Phi}(u, u) J_{\Phi}(v, v) - J_{\Phi}(u, v)^2) \tilde{l}_d(z_r|u) \tilde{l}_d(z_u|v) \nu(du) \nu(dv) \right) \\
& \quad \times \sum_{p=0}^{\infty} \int_{\Lambda^p} \sum_{\substack{S \subset \{1, \dots, m\} \setminus \{r, u\} \\ |S| \leq p}} \frac{q_d^{p-|S|}}{(p-|S|)!} j_{\Phi}^{(p)}(x_{1:p}) \prod_{j \notin S} l_c(z_j) \prod_{i \in S} \tilde{l}_d(z_i|x_i) \nu(dx_{1:p}) \\
& = \left(l_c(z_r) l_c(z_u) + l_c(z_r) \int_{\Lambda} J_{\Phi}(v, v) \tilde{l}_d(z_u|v) \nu(dv) + l_c(z_u) \int_{\Lambda} J_{\Phi}(v, v) \tilde{l}_d(z_r|v) \nu(dv) \right. \\
& \quad \left. + \int_{\Lambda^2} (J_{\Phi}(u, u) J_{\Phi}(v, v) - J_{\Phi}(u, v)^2) \tilde{l}_d(z_r|u) \tilde{l}_d(z_u|v) \nu(du) \nu(dv) \right) \\
& \quad \times j_{\Xi}^{(m-2)}(z_1, \dots, z_{r-1}, z_{r+1}, \dots, z_{u-1}, z_{u+1}, \dots, z_m) \\
& = \left(s_c(z_r) s_c(z_u) - \int_{\Lambda^2} J_{\Phi}(u, v)^2 \tilde{l}_d(z_r|u) \tilde{l}_d(z_u|v) \nu(du) \nu(dv) \right) \\
& \quad \times j_{\Xi}^{(m-2)}(z_1, \dots, z_{r-1}, z_{r+1}, \dots, z_{u-1}, z_{u+1}, \dots, z_m). \tag{5.27}
\end{aligned}$$

We conclude by taking $(z_r, z_u) = (z, z')$ and noting that by (5.13) and (5.26)-(5.27) we have

$$\begin{aligned}
l_{z_{1:m}}^{(2)}(x, y, z, z') &= \frac{\Upsilon_{z_{1:m} \setminus \{z, z'\}}^{(2)}(x, y)}{j_{\Xi}^{(m)}(z_{1:m})} \\
&\simeq (J_{\Phi}(x, x) J_{\Phi}(y, y) - J_{\Phi}(x, y)^2) \frac{j_{\Xi}^{(m-2)}(z_{1:m} \setminus \{z, z'\})}{j_{\Xi}^{(m)}(z_{1:m})} \\
&\simeq \frac{(J_{\Phi}(x, x) J_{\Phi}(y, y) - J_{\Phi}(x, y)^2)}{s_c(z) s_c(z') - \int_{\Lambda^2} J_{\Phi}(u, v)^2 \tilde{l}_d(z|u) \tilde{l}_d(z'|v) \nu(du) \nu(dv)},
\end{aligned}$$

$z, z' \in z_{1:m}$, $z \neq z'$, $m \geq 2$. Similarly, by (5.12) and (5.24), (5.26) we also have

$$\begin{aligned}
l_{z_{1:m}}^{(2)}(x, y, z) &= \frac{\Upsilon_{z_{1:m} \setminus z}^{(2)}(x, y)}{j_{\Xi}^{(m)}(z_{1:m})} \\
&\simeq (J_{\Phi}(x, x) J_{\Phi}(y, y) - J_{\Phi}(x, y)^2) \frac{j_{\Xi}^{(m-1)}(z_{1:m} \setminus z)}{j_{\Xi}^{(m)}(z_{1:m})} \\
&\simeq \frac{J_{\Phi}(x, x) J_{\Phi}(y, y) - J_{\Phi}(x, y)^2}{l_c(z) + \int_{\Lambda} J_{\Phi}(u, u) \tilde{l}_d(z|u) \nu(du)}, \quad z \in z_{1:m}, \quad m \geq 1.
\end{aligned}$$

□

As a consequence of (5.11) and Proposition 5.2.2, the second order conditional factorial moment density of Φ given that $\Xi = z_{1:m} = (z_1, \dots, z_m)$ will be approximated as

$$\begin{aligned} \rho_{\Phi|\Xi=z_{1:m}}^{(2)}(x, y) &\simeq q_d^2 (J_{\Phi}(x, x)J_{\Phi}(y, y) - J_{\Phi}(x, y)^2) \\ &+ q_d \sum_{z \in z_{1:m}} \frac{(J_{\Phi}(x, x)J_{\Phi}(y, y) - J_{\Phi}(x, y)^2)(\tilde{l}_d(z|x) + \tilde{l}_d(z|y))}{s_c(z)} \\ &+ \sum_{\substack{z, z' \in z_{1:m} \\ z \neq z'}} \frac{(J_{\Phi}(x, x)J_{\Phi}(y, y) - J_{\Phi}(x, y)^2)\tilde{l}_d(z|x)\tilde{l}_d(z'|y)}{s_c(z)s_c(z') - \int_{\Lambda^2} J_{\Phi}(u, v)^2 \tilde{l}_d(z|u)\tilde{l}_d(z'|v)\nu(du)\nu(dv)}, \end{aligned} \quad (5.28)$$

$m \geq 0$, with $\rho_{\Phi|\Xi=z_{1:m}}^{(2)}(x, x) := 0$, $x \in \Lambda$.

Proposition 5.2.3. *The (approximate) kernel update formula is given by*

$$\begin{aligned} K_{\Phi|\Xi=z_{1:m}}(x, y)^2 &\simeq q_d^2 J_{\Phi}(x, y)^2 + q_d J_{\Phi}(x, y)^2 \sum_{z \in z_{1:m}} \frac{(\tilde{l}_d(z|x) + \tilde{l}_d(z|y))}{s_c(z)} \\ &+ J_{\Phi}(x, x)J_{\Phi}(y, y) \sum_{z, z' \in z_{1:m}} \frac{\tilde{l}_d(z|x)\tilde{l}_d(z'|y)}{s_c(z)s_c(z')} \\ &+ \sum_{\substack{z, z' \in z_{1:m} \\ z \neq z'}} \frac{(J_{\Phi}(x, y)^2 - J_{\Phi}(x, x)J_{\Phi}(y, y))\tilde{l}_d(z|x)\tilde{l}_d(z'|y)}{s_c(z)s_c(z') - \int_{\Lambda^2} J_{\Phi}(u, v)^2 \tilde{l}_d(z|u)\tilde{l}_d(z'|v)\nu(du)\nu(dv)}, \end{aligned}$$

$m \geq 0$, $x, y \in \Lambda$.

Proof. By (5.8) and Proposition 5.2.1, we have the approximation

$$\begin{aligned} \mu_{\Phi|\Xi=z_{1:m}}^{(1)}(x) &= q_d l_{z_{1:m}}^{(1)}(x) + \sum_{z \in z_{1:m}} \tilde{l}_d(x|z) l_{z_{1:m}}^{(1)}(x; z) \\ &\simeq q_d J_{\Phi}(x, x) + \sum_{z \in z_{1:m}} \frac{J_{\Phi}(x, x)\tilde{l}_d(z|x)}{l_c(z) + \int_{\Lambda} \tilde{l}_d(z|u)J_{\Phi}(u, u)\nu(du)}, \quad m \geq 0, \end{aligned} \quad (5.29)$$

hence by (5.28) and (5.29) we find

$$\begin{aligned} \rho_{\Phi|\Xi=z_{1:m}}^{(2)}(x, y) - \mu_{\Phi|\Xi=z_{1:m}}^{(1)}(x)\mu_{\Phi|\Xi=z_{1:m}}^{(1)}(y) \\ \simeq -q_d J_{\Phi}(x, x)J_{\Phi}(y, y) \left(q_d + \sum_{z \in z_{1:m}} \frac{\tilde{l}_d(z|x) + \tilde{l}_d(z|y)}{s_c(z)} \right) \end{aligned}$$

$$\begin{aligned}
& - J_{\Phi}(x, x) J_{\Phi}(y, y) \sum_{z, z' \in z_{1:m}} \frac{\tilde{l}_d(z|x) \tilde{l}_d(z'|y)}{s_c(z) s_c(z')} \\
& + q_d^2 (J_{\Phi}(x, x) J_{\Phi}(y, y) - J_{\Phi}(x, y)^2) \\
& + q_d (J_{\Phi}(x, x) J_{\Phi}(y, y) - J_{\Phi}(x, y)^2) \sum_{z \in z_{1:m}} \frac{\tilde{l}_d(z|x) + \tilde{l}_d(z|y)}{s_c(z)} \\
& + \sum_{\substack{z, z' \in z_{1:m} \\ z \neq z'}} \frac{(J_{\Phi}(x, x) J_{\Phi}(y, y) - J_{\Phi}(x, y)^2) \tilde{l}_d(z|x) \tilde{l}_d(z'|y)}{s_c(z) s_c(z') - \int_{\Lambda^2} J_{\Phi}(u, v)^2 \tilde{l}_d(z|u) \tilde{l}_d(z'|v) \nu(du) \nu(dv)} \\
& = -q_d^2 J_{\Phi}(x, y)^2 - q_d J_{\Phi}(x, y)^2 \sum_{z \in z_{1:m}} \frac{\tilde{l}_d(z|x) + \tilde{l}_d(z|y)}{s_c(z)} \\
& - J_{\Phi}(x, x) J_{\Phi}(y, y) \sum_{z, z' \in z_{1:m}} \frac{\tilde{l}_d(z|x) \tilde{l}_d(z'|y)}{s_c(z) s_c(z')} \\
& + \sum_{\substack{z, z' \in z_{1:m} \\ z \neq z'}} \frac{(J_{\Phi}(x, x) J_{\Phi}(y, y) - J_{\Phi}(x, y)^2) \tilde{l}_d(z|x) \tilde{l}_d(z'|y)}{s_c(z) s_c(z') - \int_{\Lambda^2} J_{\Phi}(u, v)^2 \tilde{l}_d(z|u) \tilde{l}_d(z'|v) \nu(du) \nu(dv)},
\end{aligned}$$

where $m \geq 0$, and we conclude by (5.19), i.e.

$$(K_{\Phi|\Xi=z_{1:m}}(x, y))^2 = \mu_{\Phi|\Xi=z_{1:m}}^{(1)}(x) \mu_{\Phi|\Xi=z_{1:m}}^{(1)}(y) - \rho_{\Phi|\Xi=z_{1:m}}^{(2)}(x, y)$$

and (5.30). □

The next result, which provides an approximation formula for the posterior covariance of Proposition 5.1.4, is a consequence of Proposition 5.2.3 and (5.29).

Corollary 5.2.4. *Under (5.21)-(5.22) the posterior covariance of Φ given that $\Xi = z_{1:m} = (z_1, \dots, z_m)$ is approximated as*

$$\begin{aligned}
c_{\Phi|\Xi=z_{1:m}}^{(2)}(A, B) & \simeq q_d \int_{A \cap B} J_{\Phi}(x, x) \nu(dx) - q_d^2 \int_{A \times B} J_{\Phi}(x, y)^2 \nu(dx) \nu(dy) \\
& - q_d \sum_{z \in z_{1:m}} \frac{1}{s_c(z)} \int_{A \times B} J_{\Phi}(x, y)^2 (\tilde{l}_d(z|x) + \tilde{l}_d(z|y)) \nu(dx) \nu(dy) \\
& + \sum_{z \in z_{1:m}} \frac{1}{s_c(z)} \left(\int_{A \cap B} \tilde{l}_d(z|x) J_{\Phi}(x, x) \nu(dx) \right. \\
& \quad \left. - \frac{\int_A \tilde{l}_d(z|x) J_{\Phi}(x, x) \nu(dx) \int_B \tilde{l}_d(z|y) J_{\Phi}(y, y) \nu(dy)}{s_c(z)} \right)
\end{aligned} \tag{5.30}$$

$$+ \sum_{\substack{z, z' \in z_{1:m} \\ z \neq z'}} \frac{\int_{\Lambda^2} (J_{\Phi}(x, x)J_{\Phi}(y, y) - J_{\Phi}(x, y)^2) \tilde{l}_d(z|x) \tilde{l}_d(z'|y) \nu(dx) \nu(dy)}{s_c(z)s_c(z') - \int_{\Lambda^2} J_{\Phi}(u, v)^2 \tilde{l}_d(z|u) \tilde{l}_d(z'|v) \nu(du) \nu(dv)},$$

where $m \geq 0$.

5.2.2 Prediction Phase

The prediction point process Φ is constructed by branching the prior point process Ψ with a Bernoulli point process Φ_s with probability of survival $p_s(x)$ at the point x and spatial transition likelihood $l_s(y|x)$ from state x to state y , and probability generating functional

$$\mathcal{G}_{\Phi_s}(g | x) = 1 - p_s(x) + p_s(x) \int_{\Lambda} g(y) l_s(y|x) \nu(dy).$$

According to (5.2) the joint probability generating functional of the prediction point process Φ is given by

$$\begin{aligned} \mathcal{G}_{\Phi}(h) &= \mathcal{G}_{\Phi_b}(h) \mathcal{G}_{\Psi}(\mathcal{G}_{\Phi_s}(h | \cdot)) \\ &= \mathcal{G}_{\Phi_b}(h) \mathcal{G}_{\Psi} \left(1 - p_s(\cdot) + p_s(\cdot) \int_{\Lambda} h(y) l_s(y|\cdot) \nu(dy) \right), \end{aligned}$$

where $\mathcal{G}_{\Phi_b}$ is the Poisson birth point process of new targets. Using (5.17)-(5.18) and the prediction first and second order (factorial) moment densities

$$\begin{aligned} \mu_{\Phi}^{(1)}(x) &= \frac{\partial_{\delta_x} \mathcal{G}_{\Phi}(h) |_{h=1}}{\partial h} \\ &= \mu_{\Phi_b}^{(1)}(x) + \int_{\Lambda} p_s(y) l_s(x|y) \mu_{\Psi}^{(1)}(y) \nu(dy) \end{aligned}$$

and

$$\begin{aligned} \rho_{\Phi}^{(2)}(x, y) &= \int_{\Lambda^2} p_s(u) l_s(x|u) p_s(v) l_s(y|v) \rho_{\Psi}^{(2)}(u, v) \nu(du) \nu(dv) \\ &\quad + \mu_{\Phi_b}^{(1)}(y) \int_{\Lambda} p_s(u) l_s(x|u) \mu_{\Psi}^{(1)}(u) \nu(du) \\ &\quad + \mu_{\Phi_b}^{(1)}(x) \int_{\Lambda} p_s(v) l_s(y|v) \mu_{\Psi}^{(1)}(v) \nu(dv) + \rho_{\Phi_b}^{(2)}(x, y), \end{aligned}$$

obtained from (5.3) and (5.4), $x, y \in \Lambda$, $x \neq y$, we obtain the following proposition where for compactness of notation we let $\tilde{l}_s(y|x) := p_s(x) l_s(y|x)$.

Proposition 5.2.5. *Assume that the prior point process Ψ is a determinantal point process with kernel $K_\Psi(x, y)$. Then the prediction first and second order moment densities of Φ are given by*

$$\mu_\Phi^{(1)}(x) = \mu_{\Phi_b}^{(1)}(x) + \int_\Lambda \tilde{l}_s(x|u) K_\Psi(u, u) \nu(du), \quad x \in \Lambda, \quad (5.31)$$

and

$$\begin{aligned} \rho_\Phi^{(2)}(x, y) &= \int_{\Lambda^2} \tilde{l}_s(x|u) \tilde{l}_s(y|v) (K_\Psi(u, u) K_\Psi(v, v) - (K_\Psi(u, v))^2) \nu(du) \nu(dv) \\ &\quad + \mu_{\Phi_b}^{(1)}(y) \int_\Lambda \tilde{l}_s(x|u) K_\Psi(u, u) \nu(du) \\ &\quad + \mu_{\Phi_b}^{(1)}(x) \int_\Lambda \tilde{l}_s(y|v) K_\Psi(v, v) \nu(dv) + \rho_{\Phi_b}^{(2)}(x, y), \end{aligned} \quad (5.32)$$

where $x, y \in \Lambda$, $x \neq y$.

From Proposition 5.2.5 we can model the prediction point process Φ as a determinantal process with prediction kernel K_Φ , whose diagonal entries are given by

$$K_\Phi(x, x) = \mu_\Phi^{(1)}(x) = \mu_{\Phi_b}^{(1)}(x) + \int_\Lambda \tilde{l}_s(x|y) K_\Psi(y, y) \nu(dy),$$

and whose nondiagonal entries satisfy

$$K_\Phi(x, y) = \sqrt{K_\Phi(x, x) K_\Phi(y, y) - \rho_\Phi^{(2)}(x, y)}, \quad x, y \in \Lambda,$$

from Relation (5.18), with $\rho_\Phi^{(2)}(x, x) := 0$, $x \in \Lambda$. The prediction Janossy kernel $J_\Phi(x, y)$ of the operator $\mathfrak{J}_\Phi$ is then computed by the formula

$$\mathfrak{J}_\Phi = (I_d - \mathfrak{K}_\Phi)^{-1} \mathfrak{K}_\Phi,$$

see (5.16).

5.2.3 Determinantal Approximation to Posterior Phase

Proposition 5.1.2 and Relation (5.17) show that the diagonal values of the posterior kernel $K_{\Phi|\Xi=z_{1:m}}$ are given by

$$K_{\Phi|\Xi=z_{1:m}}(x, x) = \mu_{\Phi|\Xi=z_{1:m}}^{(1)}(x)$$

$$= q_d l_{z_{1:m}}^{(1)}(x) + \sum_{z \in z_{1:m}} \tilde{l}_d(z|x) l_{z_{1:m}}^{(1)}(x; z), \quad x \in \Lambda, \quad m \geq 0.$$

Based on the approximation of the corrector term $l_{z_{1:m}}^{(1)}(x; z)$ stated in Proposition 5.2.1 we will estimate the first order posterior moment density as

$$K_{\Phi|\Xi=z_{1:m}}(x, x) = \mu_{\Phi|\Xi=z_{1:m}}^{(1)}(x) \quad (5.33)$$

$$\simeq q_d K_{\Phi}(x, x) + \sum_{z \in z_{1:m}} \frac{J_{\Phi}(x, x) \tilde{l}_d(z|x)}{l_c(z) + \int_{\Lambda} \tilde{l}_d(z|u) J_{\Phi}(u, u) \nu(du)}, \quad (5.34)$$

where we choose to approximate $l_{z_{1:m}}^{(1)}(x) \simeq \mu_{\Phi}^{(1)}(x) = K_{\Phi}(x, x)$ for consistency with the standard Poisson PHD filter. The estimate (5.33) allows us to locate the targets by maximizing $K_{\Phi|\Xi=z_{1:m}}(x, x) = \mu_{\Phi|\Xi=z_{1:m}}^{(1)}(x)$ over x , and to estimate the number of targets as

$$\gamma_{\Phi|\Xi=z_{1:m}} := \int_{\Lambda} K_{\Phi|\Xi=z_{1:m}}(x, x) \nu(dx), \quad x \in \Lambda. \quad (5.35)$$

In addition, as a consequence of Relation (5.19), i.e.

$$(K_{\Phi|\Xi=z_{1:m}}(x, y))^2 = \mu_{\Phi|\Xi=z_{1:m}}^{(1)}(x) \mu_{\Phi|\Xi=z_{1:m}}^{(1)}(y) - \rho_{\Phi|\Xi=z_{1:m}}^{(2)}(x, y),$$

$x, y \in \Lambda$, the cross-diagonal entries of the posterior kernel $K_{\Phi|\Xi=z_{1:m}}(x, y)$ are estimated from Relation (5.18) as

$$K_{\Phi|\Xi=z_{1:m}}(x, y) = \sqrt{K_{\Phi|\Xi=z_{1:m}}(x, x) K_{\Phi|\Xi=z_{1:m}}(y, y) - \rho_{\Phi|\Xi=z_{1:m}}^{(2)}(x, y)}, \quad (5.36)$$

$x, y \in \Lambda$, which rewrites by Propositions 5.1.2-5.1.3 as

$$\begin{aligned} & (K_{\Phi|\Xi=z_{1:m}}(x, y))^2 \\ &= q_d^2 (l_{z_{1:m}}^{(1)}(x) l_{z_{1:m}}^{(1)}(y) - l_{z_{1:m}}^{(2)}(x, y)) \\ &\quad - q_d \sum_{z \in z_{1:m}} (\tilde{l}_d(z|x) + \tilde{l}_d(z|y)) l_{z_{1:m}}^{(2)}(x, y; z) \\ &\quad + q_d \sum_{z \in z_{1:m}} \left(\tilde{l}_d(z|y) l_{z_{1:m}}^{(1)}(x) l_{z_{1:m}}^{(1)}(y; z) + \tilde{l}_d(z|x) l_{z_{1:m}}^{(1)}(y) l_{z_{1:m}}^{(1)}(x; z) \right) \\ &\quad + \sum_{z \in z_{1:m}} \tilde{l}_d(z|x) \tilde{l}_d(z|y) l_{z_{1:m}}^{(1)}(x; z) l_{z_{1:m}}^{(1)}(y; z) \\ &\quad + \sum_{\substack{z, z' \in z_{1:m} \\ z \neq z'}} \tilde{l}_d(z|x) \tilde{l}_d(z'|y) (l_{z_{1:m}}^{(1)}(x; z) l_{z_{1:m}}^{(1)}(y; z') - l_{z_{1:m}}^{(2)}(x, y, z, z')), \end{aligned} \quad (5.37)$$

$x, y \in \Lambda$, $m \geq 0$. In practice we will estimate the posterior kernel $K_{\Phi|\Xi=z_{1:m}}(x, y)$ in (5.36) using the approximations (5.33) and

$$\begin{aligned} \rho_{\Phi|\Xi=z_{1:m}}^{(2)}(x, y) &\simeq q_d^2 (J_{\Phi}(x, x)J_{\Phi}(y, y) - J_{\Phi}(x, y)^2) \\ &+ q_d \sum_{z \in z_{1:m}} \frac{(J_{\Phi}(x, x)J_{\Phi}(y, y) - J_{\Phi}(x, y)^2)(\tilde{l}_d(z|x) + \tilde{l}_d(z|y))}{s_c(z)} \\ &+ \sum_{\substack{z, z' \in z_{1:m} \\ z \neq z'}} \frac{(J_{\Phi}(x, x)J_{\Phi}(y, y) - J_{\Phi}(x, y)^2)\tilde{l}_d(z|x)\tilde{l}_d(z'|y)}{s_c(z)s_c(z') - \int_{\Lambda^2} J_{\Phi}(u, v)^2 \tilde{l}_d(z|u)\tilde{l}_d(z'|v)\nu(du)\nu(dv)} \end{aligned} \quad (5.38)$$

obtained from (5.28) in subsection 5.2.1, with $\rho_{\Phi|\Xi=z_{1:m}}^{(2)}(x, x) := 0$, $x \in \Lambda$, see Proposition 5.2.3. After completing the update step we move to the next prediction step by taking $K_{\Psi}(x, y) := K_{\Phi|\Xi=z_{1:m}}(x, y)$, $x, y \in \Lambda$.

5.3 Sequential Monte Carlo Implementation

We implement the DPP and PPP filters using the sequential Monte Carlo (or particle filtering) method as in [9], together with the roughening method of [10], which allow us to estimate otherwise intractable integrals using discretized particle summations. Here we restate the ground truth and measurement dynamics of *nearly constant turn-rate* as in Subsection 3.2.4, in which we adopted the motion trajectories for our implementation.

Ground Truth and Measurement Dynamics

Our ground truth dynamics follows the *nearly constant turn-rate* motion dynamics for the sequential Monte Carlo implementation of our Determinantal PHD filter. The state of each target at time t is given by $\mathbf{x}_t = (x_t, \dot{x}_t, y_t, \dot{y}_t, \theta_t)^\top$, where x_t, y_t are the cartesian coordinates, $\dot{x}_t, \dot{y}_t$ are the respective velocities, and θ_t is the turn rate. As in (3.36), the location of every target at time $t + 1$ is given by

$$\mathbf{x}_{t+1} = \mathbf{F}(\theta_t)\mathbf{x}_t + \mathbf{G}\mathbf{v}_t,$$

where $\mathbf{v}_t = (v_x, v_y, v_\theta)^\top$ is a zero-mean acceleration noise distributed according to the zero-mean Gaussian noise

$$\mathbf{v}_t \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{v_x}^2 & 0 & 0 \\ 0 & \sigma_{v_y}^2 & 0 \\ 0 & 0 & \sigma_{v_\theta}^2 \end{pmatrix}\right),$$

and

$$\mathbf{F} = \begin{pmatrix} 1 & \sin(\tau\theta_t)/\theta_t & 0 & (\cos(\tau\theta_t) - 1)/\theta_t & 0 \\ 0 & \cos(\tau\theta_t) & 0 & -\sin(\tau\theta_t) & 0 \\ 0 & (\cos(\tau\theta_t) - 1)/\theta_t & 1 & \sin(\tau\theta_t)/\theta_t & 0 \\ 0 & \sin(\tau\theta_t) & 0 & \cos(\tau\theta_t) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{G} = \begin{pmatrix} \tau^2/2 & 0 & 0 \\ \tau & 0 & 0 \\ 0 & \tau^2/2 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau \end{pmatrix},$$

with τ the time sampling period.

The measurement vector of each target at time t is written as $\mathbf{m}_t = (m_{r_t}, m_{\omega_t})^\top$ using the range and bearing components m_{r_t} and m_{ω_t} . As in (3.38), the measurement generated by every target at time $t + 1$ is then given by

$$\mathbf{m}_{t+1} = \mathbf{p}_{t+1} + \mathbf{w}_{t+1},$$

where $\mathbf{p}_{t+1} = \begin{pmatrix} \sqrt{x_{t+1}^2 + y_{t+1}^2} \\ \arctan(y_{t+1}/x_{t+1}) \end{pmatrix}$ and the measurement noise vector $\mathbf{w}_{t+1} = (w_{r_{t+1}}, w_{\omega_{t+1}})^\top$ is distributed according to the zero-mean Gaussian noise

$$\mathbf{w}_{t+1} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{w_r}^2 & 0 \\ 0 & \sigma_{w_\omega}^2 \end{pmatrix}\right).$$

The target birth process Φ_b follows a time-dependent Poisson point process with intensity proportional to (5.35). The spatial transition likelihood kernel $l_s(\mathbf{y}|\mathbf{x})$ from state $\mathbf{x}$ to state $\mathbf{y}$ follows the multivariate Gaussian distribution of the *nearly constant turn-rate* motion dynamics (3.36), and the likelihood kernel $\tilde{l}_d(\mathbf{z}|\mathbf{x})$ from a target state $\mathbf{x}$ to a measurement $\mathbf{z}$ follows the multivariate Gaussian distribution (3.38).

In addition, the model generates measurement information from every target with a constant probability of detection p_d , and the spatially distributed clutter measurement points are generated according to a Poisson point process with uniform density $l_c(z)$.

5.3.1 Algorithm

Initialization ($t = 0$)

The state dynamics of the initial set of $N_{\Phi_b,0}$ particles is sampled according to a uniform distribution on the state space Λ . The diagonal entries of the prior discretized determinantal kernel $K_{\Phi,0}$ at time $t = 0$ are initialized to $\gamma_{\Phi_b,0}/N_{\Phi_b,0}$ with $\gamma_{\Phi_b,0}$ a prior intensity value. The nondiagonal entries are initialized to $\alpha\gamma_{\Phi_b,0}/N_{\Phi_b,0}$, where α is a prior correlation coefficient, except for those which are set to zero according to Condition (5.20) with the matrix index threshold $\eta = 10\%$.

Using (5.16) we then compute the discretized Janossy kernel $J_{\Phi,0}$ which is needed for the evaluation of the posterior determinantal kernel $\tilde{K}_{\Phi|\Xi,0}$. Letting P_p denote the number of resampled particles per target and $\gamma_0 := \sum_{i=1}^{N_{\Phi_b,0}} K_{\Phi|\Xi,0}(\mathbf{x}_i, \mathbf{x}_i)$, we resample $\tilde{N}_0 := P_p \times \lfloor \gamma_0 \rfloor$ particles $\{\mathbf{x}_i\}_{i=1}^{\tilde{N}_0}$ that better describe the target locations as in [10], by maximizing the diagonal entries of $\tilde{K}_{\Phi|\Xi,0}$ over Λ . Those particles are then used to initialize the post-resampling determinantal kernel K_0 and to compute the post-resampling Janossy kernel J_0 in order to estimate the updated kernel $\tilde{K}_{\Phi|\Xi}$ from (5.33), (5.36) and (5.38). The updated kernel $\tilde{K}_{\Phi|\Xi}$ is then set as the prior kernel K_Ψ of the next time step.

Algorithm ($t \geq 1$)

The general algorithm proceeds to compute the prediction state transition dynamics $\{\mathbf{x}_{t+1|t}^{(i)}\}_{i=1}^{\tilde{N}_t}$ using (3.36), followed by the computation of the prediction determinantal state transition kernel $K_{\Phi,t+1|t}$ using (5.31) and (5.32). Letting P_b denote the number of particles per birth target and $\gamma_{\Phi_b,t+1} := \sum_{i=1}^{\tilde{N}_t} K_{\Psi,t+1}(\mathbf{x}_i, \mathbf{x}_i)\nu(\{x_i\})$, we sample the state dynamics of $N_{\Phi_b,t+1} := P_b \times \lfloor \gamma_{\Phi_b,t+1} \rfloor$ particles for the target birth process Φ_b .

The discretized prediction determinantal kernel $K_{\Phi,t+1|t}$ is then extended to incorporate the set of additional $N_{\Phi_b,t+1}$ particles by assigning the diagonal entries corresponding to these new particles to $\gamma_{\Phi_b,t+1}/N_{\Phi_b,t+1}$ and the nondiagonal entries to $\alpha\gamma_{\Phi_b,t+1}/N_{\Phi_b,t+1}$, and by setting all other new entries to 0 according to Condition (5.20) with the matrix index threshold $\eta := 10\%$. Thereafter, we compute the discretized Janossy kernel $J_{\Phi,t+1|t}$ using (5.16) and then the discretized posterior determinantal kernel $K_{\Phi|\Xi,t+1}$ using (5.33), (5.36) and (5.38). Next, letting $\gamma_{t+1} := \sum_{i=1}^{N_{t+1}} K_{\Phi|\Xi,t+1}(\mathbf{x}_i, \mathbf{x}_i)$ we resample $\tilde{N}_{t+1} := P_p \times \lfloor \gamma_{t+1} \rfloor$ particles with state

Initialization (time $t = 0$)

Set $\gamma_{\Phi_b,0} \in \mathbb{R}^+$, $P_p \in \mathbb{N}$, $\eta \in (0,1)$, and $\alpha \in \mathbb{R}^+$.
 Sample $N_{\Phi_b,0}$ particle state dynamics $\{\mathbf{x}_i\}_{i=1}^{N_{\Phi_b,0}}$ for initial birth process Φ_b uniformly distributed within state space Λ .
 Initialize the (prior) determinantal kernel $K_{\Phi,0}$.
for $1 \leq i, j \leq N_{\Phi_b,0}$ **do**
 $K_{\Phi,0}(\mathbf{x}_i, \mathbf{x}_i) := \gamma_{\Phi_b,0}/N_{\Phi_b,0}$
 if $i \neq j$ **and** $|i - j| \leq \eta P_b$ **then**
 $K_{\Phi,0}(\mathbf{x}_i, \mathbf{x}_j) := \alpha \gamma_{\Phi_b,0}/N_{\Phi_b,0}$
 end if
end for
 Compute the Janossy kernel $J_{\Phi,0} := (I - K_{\Phi,0})^{-1}K_{\Phi,0}$.
 Compute the posterior determinantal kernel

$K_{\Phi|\Xi,0}$ using (5.33), (5.36) and (5.38).
 Perform resampling as in [10] to obtain the particle state dynamics $\{\mathbf{x}_i\}_{i=1}^{\tilde{N}_0}$ where $\tilde{N}_0 := P_p \times \lfloor \gamma_0 \rfloor$ and $\gamma_0 := \sum_{i=1}^{N_{\Phi_b,0}} K_{\Phi|\Xi,0}(\mathbf{x}_i, \mathbf{x}_i)$.
 Initialize the post-resampling determinantal kernel K_0 as follows:
for $1 \leq i, j \leq \tilde{N}_0$ **do**
 $K_0(\mathbf{x}_i, \mathbf{x}_i) := \gamma_0/\tilde{N}_0$
 if $i \neq j$ **and** $|i - j| \leq \eta P_p$ **then**
 $K_0(\mathbf{x}_i, \mathbf{x}_j) := \alpha \gamma_0/\tilde{N}_0$
 end if
end for
 Compute the post-resampling Janossy kernel $J_0 = (I - K_0)^{-1}K_0$.
 Compute the posterior determinantal kernel $\tilde{K}_{\Phi|\Xi,0}$ using (5.33), (5.36) and (5.38).
 Estimate the number of targets as $\gamma_{\Phi|\Xi,0} := \sum_{i=1}^{\tilde{N}_0} \tilde{K}_{\Phi|\Xi,0}(\mathbf{x}_i, \mathbf{x}_i)$.

dynamics $\{\mathbf{x}_i\}_{i=1}^{\tilde{N}_{t+1}}$ as in [10], by maximizing the diagonal entries of $K_{\Phi|\Xi,t+1}$ over Λ .

Those particles are then used to initialize the post-resampling determinantal kernel K_{t+1} by setting diagonal entries to $\gamma_{t+1}/\tilde{N}_{t+1}$ and nondiagonal entries to $\alpha \gamma_{t+1}/\tilde{N}_{t+1}$, except for those which are set to zero according to Condition (5.20) with $\eta = 10\%$. Finally, we recompute the post-resampling Janossy kernel J_{t+1} and the posterior determinantal kernel $\tilde{K}_{\Phi|\Xi,t+1}$ using (5.33), (5.36) and (5.38).

5.3.2 Numerical Simulation Results

Numerical Simulation 1

In Figure 5.1 we assess the spooky effect (see [6]) of our DPP filter, following the approach applied in [18] to second order PHD filters. As in [18], our tracking scenario consists of two disjoint square domains A and B of size 100 m by 100 m, which are located $50\sqrt{2}$ m diagonally apart. In each domain, 10 targets are initialized and their state dynamics are centrally distributed at the first time step. The runtime of the experiment is set at 50 with 100 Monte Carlo (MC) runs, the targets survive throughout ($p_s = 1$), and their trajectories remain within the observation domains.

DPP Filter (time $t + 1 \geq 1$)

Compute the (prediction) state transition dynamics $\{\mathbf{x}_{t+1|t}^{(i)}\}_{i=1}^{\tilde{N}_t}$ from $\{\mathbf{x}_t^{(i)}\}_{i=1}^{\tilde{N}_t}$ based on (3.36).

Compute the (prediction) determinantal state transition kernel $K_{\Phi,t+1|t}$ from $K_{\Psi,t+1} := \tilde{K}_{\Phi|\Xi,t}$ using (5.31) and (5.32).

Sample $N_{\Phi_b,t+1}$ new particle state dynamics for birth process Φ_b at time $t + 1$ uniformly distributed within state space Λ to generate $\{\mathbf{x}^{(i)}\}_{i=1}^{N_{\Phi_b,t+1}}$ where $N_{\Phi_b,t+1} := P_b \times \lfloor \gamma_{\Phi_b,t+1} \rfloor$ and $\gamma_{\Phi_b,t+1} := \sum_{i=1}^{\tilde{N}_t} K_{\Psi,t+1}(\mathbf{x}_i, \mathbf{x}_i)$.

Extend the (prediction) determinantal kernel $K_{\Phi,t+1|t}$ to dimension $N_{t+1} := \tilde{N}_t + N_{\Phi_b,t+1}$ with state dynamics $\{\mathbf{x}_i\}_{i=1}^{N_{t+1}} := \{\mathbf{x}_{t+1|t}^{(i)}\}_{i=1}^{\tilde{N}_t} \cup \{\mathbf{x}^{(i)}\}_{i=1}^{N_{\Phi_b,t+1}}$, where the following indexes are allocated to new particles.

for $\tilde{N}_t + 1 \leq i, j \leq N_{t+1}$ **do**

$K_{\Phi,t+1|t}(\mathbf{x}_i, \mathbf{x}_i) := \gamma_{\Phi_b,t+1}/N_{\Phi_b,t+1}$

if $i \neq j$ **and** $|i - j| \leq \eta P_b$ **then**

$K_{\Phi,t+1|t}(\mathbf{x}_i, \mathbf{x}_j) := \alpha \gamma_{\Phi_b,t+1}/N_{\Phi_b,t+1}$

end if

end for

for $1 \leq i, j \leq N_{t+1}$ **do**

if $i \leq \tilde{N}_t$ & $j \geq \tilde{N}_t + 1$ **then**

$K_{\Phi,t+1|t}(\mathbf{x}_i, \mathbf{x}_j) := 0$

$K_{\Phi,t+1|t}(\mathbf{x}_j, \mathbf{x}_i) := 0$

end if

end for

Compute the Janossy kernel

$J_{\Phi,t+1|t} :=$

$(I - K_{\Phi,t+1|t})^{-1} K_{\Phi,t+1|t}$.

Compute the posterior determinantal kernel $K_{\Phi|\Xi,t+1}$ using (5.33), (5.36) and (5.38).

Perform resampling as in [10] to obtain the particle state dynamics $\{\mathbf{x}_i\}_{i=1}^{\tilde{N}_{t+1}}$ where $\tilde{N}_{t+1} := P_p \times \lfloor \gamma_{t+1} \rfloor$ (capped at 1000 particles) and $\gamma_{t+1} := \sum_{i=1}^{\tilde{N}_{t+1}} K_{\Phi|\Xi,t+1}(\mathbf{x}_i, \mathbf{x}_i)$.

Initialize the post-resampling determinantal kernel K_{t+1} by

for $1 \leq i, j \leq \tilde{N}_{t+1}$ **do**

$K_{t+1}(\mathbf{x}_i, \mathbf{x}_i) := \gamma_{t+1}/\tilde{N}_{t+1}$

if $i \neq j$ **and** $|i - j| \leq \eta P_p$ **then**

$K_{t+1}(\mathbf{x}_i, \mathbf{x}_j) :=$

$\alpha \gamma_{t+1}/\tilde{N}_{t+1}$

end if

end for

Compute the post-resampling Janossy kernel

$J_{t+1} = (I - K_{t+1})^{-1} K_{t+1}$.

Compute the posterior determinantal kernel $\tilde{K}_{\Phi|\Xi,t+1}$ using (5.33), (5.36) and (5.38).

Estimate the number of targets as

$\gamma_{\Phi|\Xi,t+1} :=$

$\sum_{i=1}^{\tilde{N}_{t+1}} \tilde{K}_{\Phi|\Xi,t+1}(\mathbf{x}_i, \mathbf{x}_i)$.

In Figure 5.1 we take the spatial standard deviations (s.d.) $\sigma_{v_x} = \sigma_{v_y} = 1.0$ m/s², turn-rate noise s.d. $\sigma_{v_\theta} = \pi/180$ rad/s, bearing distribution s.d. $\sigma_{w_\omega} = \pi/180$ rad, and range distribution s.d. $\sigma_{w_r} = \sqrt{2}$ m. In a similar setting to [18], all targets in domain B are compelled to be miss-detected in every cycle of 10 time steps. We use a constant probability of detection $p_d = 0.9$ and clutter count at 5 in each measurement space. At initialization in Figure 5.1 we set $N_{\Phi_b,0} = 800$, $\gamma_{\Phi_b,0} = 2$ and $\alpha = 4$. The DPP filter implementation uses $P_p = 30$ resampled particles per target, and $P_b = 10$ particles per birth target.

Figure (5.1a) shows the estimated intensities in domains A and B, where domain A is unaffected by the rapid drop in the intensity of domain B. And hence, Fig-

(a) Target counts estimates.

(b) Correlation estimates.

Figure 5.1: MC estimates with $p_d = 0.9$ and 5 clutter points per domain with $\alpha = 4.0$.

ure (5.1a) exhibits no presence of spooky effect in the Determinantal PHD filter, which is similarly concluded for the Poisson PHD filter assessment in [18]. The correlation estimates in Figure (5.1b) show a negative correlation due to the determinantal point process nature, and led to a drop in negative correlation during the compelled miss-detection at each 10-steps cycle.

Numerical Simulation 2

In Figure 5.2 we compare the robustness of the DPP and PPP filter when both filters are subjected to sudden death in the number of targets in a single domain of size 100m by 100m, beginning with 15 targets at the first time step.

(a) Graph with $P_p = 50$ and $P_b = 40$.(b) Graph with $P_p = 50$ and $P_b = 60$.Figure 5.2: MC estimates from 15 to 5 targets, $p_d = 0.9$ and 1 to 0.3 clutter points.

Figure (5.2b) uses 300 Monte Carlo runs, while Figure (5.2a) relies on 200 Monte Carlo runs. The runtime of each Monte Carlo run spans from time $t = 0$ to time $t = 15$, and the probability of survival is $p_s := 1$. The initial 15 targets are maintained until time $t = 9$ when 10 random targets are forced to die and the remaining 5 targets survive until the end of the time interval. In Figure 5.2 we take the spatial standard deviations (s.d.) $\sigma_{v_x} = \sigma_{v_y} = 1.0 \text{ m/s}^2$, turn-rate noise s.d. $\sigma_{v_\theta} = \pi/180 \text{ rad/s}$, with bearing and range distribution s.d. $\sigma_{w_\omega} = \pi/180 \text{ rad}$, $\sigma_{w_r} = \sqrt{2} \text{ m}$ as in Figure 5.1, with probability of detection $p_d = 0.95$, clutter count at 1 up to time $t = 9$ and then at 0.06 afterwards for Figure (5.2b), and clutter count at 1 up to time $t = 9$ and then at 0.3 afterwards for Figure (5.2a). At initialization in Figure 5.2, we set $N_{\Phi_b,0} = 6000$ and $\gamma_{\Phi_b,0} = 0.2$. Both our DPP and PPP filter implementations use $P_p = 50$ resampled particles per target in Figure 5.2, $P_b = 60$ particles per birth target in Figure (5.2b), and $P_b = 40$ particles per birth target in Figure (5.2a).

Numerical Simulation 3

In Figure 5.3 we compare the robustness and performance recovery of the DPP and PPP filters when subjected to a rapid birth in the number of targets in a single domain of size 100 m by 100 m. The experiment starts with a single target which survives throughout the 45 time steps, without birth of new targets from time $t = 0$ to time $t = 9$. At time $t = 10$, 9 new targets are born centrally distributed within the target space and survive through the remaining time steps.

(a) Graph with $P_p = 40$ and $P_b = 9$.

(b) Graph with $P_p = 50$ and $P_b = 15$.

Figure 5.3: MC estimates from 0 to 10 targets, $p_d = 0.9$ and 0 to 5 clutter points.

Each Monte Carlo run spans 45 time steps, with 400 and 100 Monte Carlo runs in the experiments of Figures (5.3a) and (5.3b) respectively. In Figure 5.3 the

spatial standard deviations (s.d.) $\sigma_{v_x} = \sigma_{v_y} = 1.0 \text{ m/s}^2$, turn-rate noise s.d. $\sigma_{v_\theta} = \pi/180 \text{ rad/s}$, and bearing and range s.d. $\sigma_{w_\omega} = \pi/180 \text{ rad}$, $\sigma_{w_r} = \sqrt{2} \text{ m}$ are the same as in Figure 5.2. The model generates measurement information from each target with a constant probability of detection $p_d = 0.90$, clutter count at 0 up to time $t = 9$ and then at 5 afterwards for Figure (5.3a), and clutter count at 0.05 up to time $t = 9$ and then at 5 afterwards for Figure (5.3b). We set $N_{\Phi_b,0} = 300$ and $\gamma_{\Phi_b,0} = 0.2$ at initialization in Figure 5.3. Both DPP and PPP filter implementations use $P_b = 40$ and $P_b = 50$ resampled particles per target in Figure (5.3a) and Figure (5.3b) respectively. For the target birth process we set $P_b = 9$ and $P_b = 15$ particles per birth target in Figure (5.3a) and Figure (5.3b) respectively.

Chapter 6

Conclusion and Future Work

In this thesis, we have constructed a second order PHD filter based on Determinantal Point Processes (DPPs), which are able to capture some level of interactions among configuration points in a domain space using *nearly constant turn rate* motion trajectories. Our approach relies on posterior moment formulas for the estimation of a general hidden point process after a thinning operation and a superposition with a Poisson Point Process (PPP), and on suitable approximation formulas of the determinantal point process hypothesis. We have derived closed-form first and second order moment expressions for the Determinantal PHD filtering equations. In our Determinantal PHD filter, the knowledge of first and second order moments of the prediction and posterior filtering equations suffices, in principle, to update the respective Janossy densities that characterize the underlying determinantal point process. The repulsive properties of determinantal point processes also apply to the modeling of negative correlation between distinct observation domains. Monte Carlo simulations with correlation estimates are provided.

In Chapter 2, we have provided a review of some basic operations and properties that are associated with discrete random variables and spatial point processes. In Section 2.1, we have described the notions of probability generating functions (PGFs), factorial moments and branching processes which forms the fundamental basis to the construction of the integer-valued PHD filter. In Section 2.2, we stated some useful properties about spatial point processes, probability generating functionals (PGFls) and its associated relationship with factorial moment and Janossy densities. The concept of branching of point processes and some operations that involve general point processes have been introduced as well. These key concepts and

properties associated with spatial point processes are the foundations to construct the spatial point process PHD filter.

In Chapter 3, we showed computations incorporating Janossy densities within associated PGFLs, and by properties linked to the characteristic PGFL of Poisson point process (PPP), we derived closed-form first and second order factorial moment densities of the Poisson PHD filtering equations. For the *nearly constant turn rate* motion trajectories, we extended the motion dynamics to incorporate targets interaction among targets. We implemented the Poisson PHD filtering algorithm using Sequential Monte Carlo (SMC) method and presented performance comparison plot to the outputs of independent and interactive ground truth motion dynamics among multiple targets in target space.

In Chapter 4, we established the important properties related to determinantal point processes, branched determinantal point processes and branched point processes that allow for the construction of the Determinantal PHD filter. We derived the factorial moments and Janossy densities of determinantal point processes and branched determinantal point processes from the associated DPP PGFLs that differ from the convention PGFL expressions of general point processes. In addition, the relationship between the factorial moments and Janossy densities of DPPs and branched DPPs are established. The derivations of the marginal first and second order factorial moment densities evaluated using the branched DPP PGFL expressions are presented. Finally, we derived combinatorial expressions of the marginal and joint Janossy densities using PGFL of general branched point processes.

In Chapter 5, we derived general formulas for the computation of the first and second moments of a posterior point process Φ given an observation process Ξ of targets constructed by branching. In addition, general expressions for the conditional Janossy densities and posterior covariance were derived. We established the modeling of the posterior distribution of branched point processes into the context of multi-object tracking problem. In Section 5.2, we constructed a second order PHD filter based on determinantal point processes, with the computation of the prediction kernel $K_\Phi(x, y)$ and of the updated kernel $K_{\Phi|\Xi}(x, y)$. In Section 5.3, we have presented an algorithm using Sequential Monte Carlo (SMC) method to implement the Determinantal PHD filter, for which the algorithm relies on closed-form filter update expressions obtained from approximation formulas for corrector

terms and Janossy densities. In addition, we have successfully implemented and obtained numerical simulations of the Determinantal PHD filter employed under specific scenarios, and the results have been shown to exhibit better performance when compared under the Poisson setting.

Future Work

Future development work regarding the adaptation of the theory for stability and the approximation of branching distribution flows based on [35] for the Determinantal PHD filter would be explored. In particular, we would consider exploring the uniform convergence properties w.r.t the time parameter, for which such properties have been investigated in a general class of stochastic filtering algorithms, including sequential Monte carlo type models. The context of Bernoulli and PHD filter for stability has been explored in [35], it is therefore of importance that such properties in stability be investigated for the DPP PHD filter. In addition, other future development effort would also include the application of the Bayesian low-rank factorization method introduced in [36].

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