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On the Most Energy-Efficient Input Functions and the Lowest Input Energy of Passive Linear Circuits

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A thesis submitted to the Nanyang Technological University
in partial fulfillment of the requirement for the degree of
Doctor of Philosophy

2018

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research, is free of plagiarised materials, and has not been submitted for a higher degree to any other University or Institution.

17 April 2019

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Supervisor Declaration Statement

I have reviewed the content and presentation style of this thesis and declare it is free of plagiarism and of sufficient grammatical clarity to be examined. To the best of my knowledge, the research and writing are those of the candidate except as acknowledged in the Author Attribution Statement. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

17 April 2019

A handwritten signature in black ink, reading "Eng Leong Tan". The signature is written in a cursive, flowing style.

Assoc. Prof. Tan Eng Leong

Authorship Attribution Statement

This thesis contains material from 3 papers published in the following peer-reviewed journal / from papers accepted at conferences in which I am listed as an author.

A part of Chapter 3 is published as R. Smunyahirun and E. L. Tan, "Derivation of the Most Energy-Efficient Source Functions by Using Calculus of Variations," in IEEE Transactions on Circuits and Systems I: Regular Papers, vol. 63, no. 4, pp. 494-502, April 2016.

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The contributions of the co-authors for all the manuscripts are as follows:

- Assoc. Prof. Tan Eng Leong suggested the materials area.
- I prepared the manuscript drafts.
- I performed all calculation and analysis.
- Assoc. Prof. Tan Eng Leong edited the manuscript drafts.

17 April 2019

Smunyahirun Radit

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Abstract

Energy-efficient has been an interested research for decades. For fundamentals circuits, resistors in a circuit always dissipate energy when they have current flow and it decreases energy efficiency. When the resistive circuits are in transient state, energy dissipation is usually inevitable. However, it can be minimized. Either energy dissipation or input energy are dependent with voltage and current functions in the circuits. Therefore, there are the most energy-efficient functions, the functions that minimize the input energy (or the energy dissipation). The functions can be obtained by applying calculus of variations, a powerful mathematics theory for functional optimization. Once the most energy-efficient functions are obtained, the lowest input energy can be calculated.

For a series RC circuit, one of the simplest first order circuit, the most energy-efficient input current function and input voltage function are derived and proved to be a constant function and a raised ramp function respectively. The optimization by using calculus of variations is expanded to any first order passive circuits. It turns out that the most energy-efficient function for general first order circuits is hyperbolic family. The lowest input energy is calculated and stated as corollaries. The corollaries are essentials to provide insight into transient duration analysis. The transient duration analysis is performed for three cases of increasing, decreasing and unchanging total stored energy. All these cases are analyzed comprehensively. In each case, it is further divided into two sub cases of (un)changing without and with zero crossing. The analysis reveals two important circuit behaviors which are speed and energy trade-off and energy flow. It is shown that the speed and energy trade-off may be violated under certain conditions. The energy flow of the circuit is discussed thoroughly and a zero-energy transient duration that is defined as a transient duration with total input energy being zero is derived. All the analysis can be summarized into a single digram called the *transient duration diagram*. The digram reveals important relationships between the circuit parameters in the transient duration analysis.

Not only first order circuit that are optimized, but also second order circuit. Series LC circuit and parallel LC circuit are chosen to be optimized because they are typical second order circuits. The most energy-efficient functions of the circuits can be categorized into three modes which are (i) hyperbolic mode, (ii) hyperbolic and linear mode and (iii) hyperbolic and trigonometric mode. Each mode are discussed and their transient responses are illustrated. The lowest input energy are evaluated and the transient duration analysis are done rigorously.

Finally, generalization of the most energy-efficient functions and the lowest input energy for any order circuits is drawn. A dummy function is introduced which is essential for generalization. The most energy-efficient input functions in terms of the dummy function are proposed as well as the lowest input energy. General formulas or general procedure for determining the most energy-efficient functions and the lowest input energy for any orders circuits is explained. Using of the general formulas is discussed. Applying the general formulas of RC chain circuit is explained thoroughly.

Chapter 1

Introduction

1.1 Motivation

Energy harvesting systems and ultra low power circuits targeted for internet of things (IoT) are topics that drive the research of circuit community recently. Sources of energy harvesting systems mostly come from environment where we have different systems for different types of sources. Commonly, there are photovoltaic (PV) system [1,2], piezoelectric system [3–5] and thermoelectric generator (TEG) system [6,7]. A critical issue of energy harvesting systems is energy efficiency because one often wants to extract as much energy as possible from the environment. Certainly, the energy is supplied to circuits that should be ultra low power since we do not have abundant energy. Ultra low power circuits have been a trend for a long time whether for IoT application or not, in analog [8–10], digital [11–13] or even magnetic circuits [14].

For fundamentals circuits, resistors in a circuit always dissipate energy when they have current flow. Frequently, such energy dissipation should be minimized. One way to reduce the energy dissipation is by reducing the resistance in the circuit as much as possible. However, there always are parasitic resistors and this is usually inevitable. For example, the circuit model of a supercapacitor is shown in Fig. 1.1 [15,16]. Supercapacitor is a promising device for extremely efficient and long life battery [15–20]. For battery application, it needs to be charged while parasitic resistors R_S and R_P will dissipate some energy. Therefore, all the energy from the source cannot be completely stored in a supercapacitor. If the energy dissipation is reduced, the efficiency of charging will be improved.

Figure 1.1: Circuit model of a supercapacitor.

A transformer is another example. Usually, it is used for energy transfer and it should not dissipate energy. However, the transformer dissipates some significant energy called core loss and this loss can be modeled as a resistor [21]. A flat display panel is also energy hungry and it can be modeled as a series R-C circuit in Fig. 1.2 [22]. If the energy dissipation through the resistor can be reduced, the flat display panel will consume less energy.

Figure 1.2: Series R-C circuit.

Another relevant example that concerns much about energy dissipation is adiabatic circuit. Adiabatic circuit is a logic circuit design that is invented for reducing energy dissipation [23–38]. Instead of using a constant voltage source for charging/discharging a capacitive load as in a conventional digital circuit, adiabatic circuit uses a ramp voltage source. However, it must sacrifice some speed in order to save some energy. Therefore, it is usually implemented for medium speed and ultra low power circuit applications [30, 31, 34, 36, 38].

Adiabatic logic can save only dynamic power loss while nowadays the devices become smaller and static power loss becomes a major problem. However, adiabatic logic is promising for some novel devices that have very low static power loss, i.e. Vertical Slit Field Effect Transistor (VeSFET) [39, 40], Carbon Nanotubes Field Effect Transistor (CNTFET) [41], Graphene Nano-Ribbon Field Effect Transistor (GNRFET) [42, 43], Magnetic Tunnel Junction (MJT) [44], etc. There is ongoing research for these devices and adiabatic could be an important circuit technique for emerging technologies [32].

To briefly explore the adiabatic circuit concept, let us consider a conventional digital circuit which can be modeled as a series R-C circuit in Fig. 1.2. Changing the output node from logic low to logic high can be represented by charging the capacitor with $v_I = V_{dd}$ [45].

Based on transient circuit analysis, we know that the voltage v_C will reach V_{dd} and the capacitor will be fully charged when time goes to infinity. In practice, we can assume that the capacitor is fully charged when v_C is 99% of its final value. The time that is required for charging the capacitor with a constant voltage is $T_S = -RC \ln(0.01) = 4.605RC$ which is called storage time. The dissipation energy through resistor is $0.5CV_{dd}^2$, while the stored energy in the capacitor is also $0.5CV_{dd}^2$ [46]. If an efficiency is defined as the ratio between the stored energy in the capacitor E_C and the input energy E_{in} ,

$$\text{Eff.} = \frac{E_C}{E_{in}}, \quad (1.1)$$

charging the capacitor with a constant voltage source will give only 50% efficiency, since half the input energy is the dissipative energy.

On the other hand, if the power supply is not a constant voltage source, the efficiency and the storage time will not be the same. For example, consider a two-step input voltage function and its capacitor voltage response for the series R-C circuit in Fig. 1.3. Initially, the capacitor is charged with a half of V_{dd} . When the voltage across the capacitor reaches 99% of $V_{dd}/2$, the input voltage changes to V_{dd} and the capacitor is charged further to V_{dd} . Since the duration for full charging with a constant voltage is approximately $4.605RC$, the total duration for charging the capacitor with the two-step input voltage function can be approximated as $2 \times 4.065RC = 9.21RC$. Charging by a ramp input voltage function with the same amount of time was also calculated. Efficiency comparison of charging the capacitor by two-step function and ramp function is given in Table 1.1. The energy dissipation and the input energy were also computed. All the values in this table are obtained from analytical solution by using general transient circuit analysis.

Figure 1.3: Two-step input voltage function and the capacitor voltage response for the circuit of Fig. 1.2.

Table 1.1: Efficiency comparison of charging the capacitor by two-step function and ramp function

	Two-step function	Ramp function
Storage time	$9.21RC$	$9.21RC$
Stored energy	$0.5CV_{dd}^2$	$0.5CV_{dd}^2$
Energy dissipation	$0.25CV_{dd}^2$	$0.1306CV_{dd}^2$
Input energy	$0.75CV_{dd}^2$	$0.6306CV_{dd}^2$
Efficiency	66.67%	79.29%

Notice that with the same storage time, the ramp function is better than the two-step function. But there is no guarantee that the ramp function is the best. By calculus of variations, one can prove that the ramp function is not the best function and also find the best function for charging the capacitor. Ref. [29] explained circuits' optimization with calculus of variations in numerical perspective. This dissertation presents in analytical perspective and focus on fundamental circuits rather than digital circuits for simplicity and generality of this theorem.

1.2 Major Contributions

This work focuses on applying calculus of variations for circuit optimization. Major contributions of this work are described as follows.

- For series RC circuit, the most energy-efficient input source functions, both current and voltage, are derived and proved to be a constant function and a raised ramp function respectively by applying calculus of variations.
- Arbitrary resistors network with a capacitor or an inductor are optimized as a generalization of a passive first order circuit optimization. The most energy-efficient input source functions and the lowest input energy are derived. The lowest input energy are generalized and stated as corollaries. These corollaries are discussed to provide insight into transient duration analysis.
- The transient duration analysis are performed for three cases of increasing, decreasing and unchanging total stored energy. In each case, it is further divided into two sub cases of (un)changing without and with zero crossing. The analysis reveals two important circuit behaviours which are speed and energy trade-off and energy flow. It shows that the speed and energy trade-off may be violated under certain conditions and these conditions are determined. The energy flow of the circuit are discussed thoroughly and a zero-energy transient duration that is defined as a

transient duration with total input energy being zero is derived. *Transient duration diagram* is proposed as a diagram that reveals important relationship between circuit parameters in transient duration analysis.

- Arbitrary resistors network with a series LC or a parallel LC are optimized as examples of a passive second order circuit optimization. The most energy-efficient input source functions and the lowest input energy are derived. The most energy-efficient functions are categorized into three modes which are (i) hyperbolic mode, (ii) hyperbolic and linear mode and (iii) hyperbolic and trigonometric mode. The lowest input energy are discussed to provide insight into transient duration analysis.
- Generalization of the most energy-efficient functions and the lowest input energy of passive linear circuits are derived. We propose a dummy function, an essential concept for generalization. Euler-Lagrange's differential equation for any orders of circuits is obtained in terms of the dummy function as well as the lowest input energy.
- RC chain circuit is optimized and the most energy-efficient functions are proved to be a summation of hyperbolic sine functions, a linear function and a constant.

1.3 Organization of the Dissertation

The dissertation is organized as follows:

- Chapter 2: Literature review. The chapter will be divided into two parts which are motivation and theory. In the first part, literature about the motivation which is energy-efficient will be reviewed in detail. The second part is a mathematical theory that will be applied in this work. The theory is called calculus of variations. It will be reviewed from fundamental to some advanced technique.
- Chapter 3: First order linear circuits optimization. Calculus of variations or more specifically, Euler-Lagrange's differential equation will be applied to derive analytically source functions that minimize energy dissipation in first order circuits during transient. Series RC circuit will be chosen to illustrate the applying of calculus of variations extensively. The theory will be extended for other first order linear circuits as well. Numerical results will be illustrated to depict the upper limit of speed-energy trade off and to justify the consistency with the analytical results. Some special circuit topologies that cannot be solved directly using standard calculus of variations will be treated.
- Chapter 4: The lowest input energy of passive first order circuit in transient state. We will generalize the using of calculus of variations with passive first-order circuits for optimization in this chapter. A general resistor network with a capacitor will be used for derivation of the most energy-efficient input source functions and the lowest input energy. The lowest input energy will be generalized and stated as corollaries. Transient duration analysis of the corollaries will be discussed comprehensively and it will be summarized into a single diagram called *transient duration diagram*.
- Chapter 5: The lowest input energy of series LC and parallel LC circuits in transient state. Calculus of variations, the powerful mathematics theory can be extended to second order circuits. According to fundamental circuits analysis, circuit behaviours of series LC and parallel LC circuits are the most interesting of second order circuit as we know that their unit step responses can be categorized into three modes which are overdamped, critical damped and underdamped. Therefore,

these two circuits will be chosen to be optimized thoughtfully. The lowest input energy and transient duration analysis will be explained.

- Chapter 6: Generalization of the most energy-efficient function and the lowest input energy of passive linear circuits. For higher order circuits or more generally, for any order circuits, the optimization will be generalization in this chapter. General formulas will be determined. There will be a few examples that illustrate on using the general formulas for determining the most energy-efficient functions and the lowest input energy. Results will be discussed and compared with other results in the previous chapters to show consistency. RC chain circuit will be optimized as an example of higher order circuit.
- Chapter 7: Conclusion and recommendations. All key concepts in the dissertation are concluded. Although the dissertation presents using calculus of variations for only passive linear circuits, recommendations for non-linear circuits or active circuits are drawn. For linear circuits, there are also a lot of things can be explore further such as constraint problems, the second variation, variable-endpoints conditions, corner conditions, multiple integrals and problem transformation.

Chapter 2

Literature Review

2.1 Introduction

In this chapter, literature review will be divided into two parts, motivation and theory. Section 2.2 will describe the motivation. The motivation is all about energy-efficient. Firstly, finding the minimum energy for generating and erasing information which leads to reversible and irreversible logic concept will be reviewed. Then, two methods of reducing energy dissipation in circuits which are ramp function excitation and resonant gate driver will be explained. Both of them will be discussed quantitatively and qualitatively. For digital circuits, there is a well-known technique called adiabatic switching that can reduce and recover some energy from the circuits. Some of adiabatic switching will be discussed.

The second part is a mathematical theory that will be applied in this work. The theory is called calculus of variations and will be explained in Section 2.3. Fundamental of the theory which is known as the Euler-Lagrange's differential equation will be reviewed. After that, some advance theories that will be used in the later chapters will be discussed. Some applications of calculus of variations in mathematics, physics and electrical circuits will be examined. Finally, Section 2.4 will draw a conclusion.

2.2 Motivation

2.2.1 Reversible and Irreversible Logic

In 1985, Bennett's and Landauer's concept was published in Scientific American [47]. They tried to understand and find the minimum energy for generating and erasing information. They discussed the energy limit of computation in many perspective such as mechanical, logical, biological and machinery. However, we will explain only mechanical and logical systems.

Consider a ball is released from a certain height above the ground. If the ground is frictionless, the ball will bounce up to an initial height and it will keep bounce up and down forever. Therefore, we can know an initial height by observing the maximum height that the ball bounce up from the ground. In logical terms, we can know the input (initial height) from the output (the maximum height that the ball bounce up). It is because the energy of the system is conserved. Every time that the ball bounce up to the maximum height, energy of the ball equal to the energy from the beginning. There is no energy dissipation.

However, if the ground has friction, the ball will bounce up less than the initial height and finally it will stop on the ground. In this case, we cannot know the initial height by observing the maximum height that the ball bounce up from the ground. We cannot know the input from the output. It can be said that we loss some information

because there is energy dissipation from friction. This phenomena leads to reversible and irreversible logic concept [48].

Reversible logic gate is the one that we can know the input by observing the output, i.e. inverter. On the other hand, irreversible logic gate is the one that we cannot know the input by observing the output, i.e. AND gate. If the output of AND gate is 1, we can know that the input will be 11. However, if the output of AND gate is 0, we cannot know that the input will be 00, 01 or 10. It can be said that we loss some information in form of energy dissipation. Notice that if the output bits is less than the input bits, the gate will be irreversible.

Nevertheless, AND gate can be modified to be reversible by adding some output bits called garbage bits. These bits will not be used in computation but if we erase these bits, we will lose some information for reversible process and there is energy dissipation for erasing. Table 2.1 shows the example of truth table for reversible AND gate. From this table, we see that we can know the inputs by observing the outputs.

Table 2.1: Reversible AND gate's truth table.

Normal bits			Garbage bits	
A	B	Output	C	D
1	1	1	1	0
1	0	0	1	0
0	1	0	0	1
0	0	0	0	0

Reversible logic gates are needed for energy conservation. Assume that the computational unit comprises only reversible logic gates, After it finished computing and the outputs were obtained, the computation can be reversed to the beginning. Consequently there will be no energy dissipation for a complete computation cycle as same as a ball bounce up to the initial height. Therefore, there is no minimum energy is required for computation.

Beside Ref. [47], there are several work that focus on low energy computation in CMOS technology published in 1980s [49, 50]. Their work are less general but more practical than Ref. [47]. In CMOS circuits, there are plenty of CMOS logic circuit styles but can be categorized in two main styles, dynamic style and static style. Static logic circuit dissipates energy only when the output is changed. Excluding of short circuit dissipation, sub-threshold leakage or other secondary effects and assume that transistors are switches, the energy dissipates through only transistors. Fig. 2.1 shows a conventional inverter static circuit with a capacitive load. When input is low, NMOS will turn off and PMOS will turn on to connect capacitive load with power supply. Power supply will charge a capacitor until the output node becomes V_{dd} or logic high. During charging, the voltage across source and drain of PMOS is $V_{dd} - v_{OUT}$ and the energy dissipation will be

$$\begin{aligned}
E_{dis} &= \int_0^\infty v(t)i(t)dt = \int_0^\infty (V_{dd} - v_{OUT})C\dot{v}_{OUT}dt \\
&= \int_0^{V_{dd}} C(V_{dd} - v_{OUT})dv_{OUT} = \frac{1}{2}CV_{dd}^2
\end{aligned} \tag{2.1}$$

where $\dot{v}_{OUT} = \frac{d}{dt}v_{OUT}$. Therefore, the total energy from supply to change the output from low to high is CV_{dd}^2 since the energy stored in the capacitor is also $0.5CV_{dd}^2$.

On the other hand, if the input is high and output is high, PMOS will turn off and NMOS will turn on to discharge the capacitor until the output voltage becomes zero.

Figure 2.1: Inverter static circuit with capacitive load.

All of energy stored in the capacitor will dissipate through the NMOS switch. For a complete cycle, the total energy dissipation is CV_{dd}^2 . Ref. [50] initiated some methods to reduce the energy dissipation during charging. One of them will be described here.

2.2.2 Ramp Function Excitation

For simplicity, the transistor switch is often modeled as a resistor when it is turned on. When capacitor is charged, CMOS circuit can be simplified as a simple series RC circuit in Fig. 2.2. $v_I(t) = V_{dd}$ for conventional logic circuits and the energy dissipation is $0.5CV_{dd}^2$ as illustrated in (2.1). However, if the voltage source function is not constant, the energy dissipation can be reduced. The ramp function in Fig. 2.3 was proposed to be a function that can reduce the energy dissipation. It reaches the final value V_F at a certain time xRC where x is a positive dimensionless quantity. For a special case, if $x = 0$, a ramp function will be a unit step function. A way to describe the rise time with a factor x of time constant RC is useful for analysis as we shall see in a moment.

Figure 2.2: Series RC circuit.

By using fundamental circuit analysis [51], the total energy dissipation is

$$E_{dis} = \left(\frac{1}{x} - \frac{1 - e^{-x}}{x^2} \right) CV_F^2. \quad (2.2)$$

If $x \gg 1$, the energy dissipation will be reduced to

$$E_{dis} \approx \frac{1}{x} CV_F^2 \quad (2.3)$$

as it was written in [50, 52].

This result revealed a way to reduce the energy dissipation. If the rise time is much longer than the time constant of the circuit ($x \rightarrow \infty$), the energy dissipation will approach to zero. However, the charging duration is longer. Assume that the charging duration

Figure 2.3: Ramp function.

which can be called a storage time T_S is the duration that voltage across the capacitor is 99% of its final value, the storage time can be expressed as

$$T_S = \left[\ln \left(\frac{1 - e^{-x}}{0.01x} \right) + x \right] RC. \quad (2.4)$$

If $x \gg 1$, the storage time can be approximated to

$$T_S \approx xRC. \quad (2.5)$$

It shows that there is a trade off between speed and energy dissipation. If the charging is slow, the energy dissipation will be small. The plot between energy dissipation and storage time without approximation is illustrated in Fig. 2.4.

Figure 2.4: The plot between storage time and energy dissipation when using the ramp function as an input.

In conventional logic circuit, voltage source abruptly changes to zero while discharging. Thus, all the stored energy in capacitor dissipates through a resistor. However, if the voltage source slowly changes from V_F to 0, a fraction of the stored energy can be recovered into the voltage source. From the same concept of charging, the ramp down function from V_F to 0 can be used and the analysis can be done in the same way as previous. The energy dissipation is the same as (2.2) and the discharging time is also the same as (2.4). For a complete cycle, the total energy dissipation can be approximated as

two times of the energy dissipation in (2.3). Therefore, using ramp voltage source can save the energy as much as the speed condition of the circuit allow.

The above method uses modified input source function technique. However, there is another method to save the energy for charging and discharging while using a constant voltage source but it needs an additional circuit element which is an inductor. This technique call be called as resonant gate driver or resonant power clock. The mechanism of this method was explained in [52] and was discussed briefly in [49, 50, 53–55].

2.2.3 Resonant Gate Driver

Let us consider R-L-C series circuit as shown in Fig. 2.5 which is a simple model for resonant power clock technique. The circuit is driven by constant voltage source of $V_F/2$ where V_F is the voltage across the capacitor in steady state. As usual, we assume that the initial voltage across the capacitor is zero and the initial current of inductor is also zero. The time response of the R-L-C series circuit which is the second order circuit can be categorized into three types: underdamped, critically damped, and overdamped. This technique is based on underdamped condition. Therefore, the three important parameters are

Figure 2.5: Capacitor charging by using resonant power clock technique scheme.

$$\alpha = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}. \quad (2.6)$$

The voltage across the capacitor and its derivative can be expressed as

$$v_C(t) = \frac{V_F}{2} - \frac{V_F}{2} e^{-\alpha t} \left(\cos \omega_d t + \frac{\alpha}{\omega_d} \sin \omega_d t \right) \quad (2.7)$$

and

$$\dot{v}_C(t) = V_F \frac{\omega_0^2}{2\omega_d} e^{-\alpha t} \sin \omega_d t \quad (2.8)$$

respectively. The local extrema of $v_C(t)$ respect to time can be found by solving $\dot{v}_C(t) = 0$ and the first local maximum which is also global maximum occurs at $t = \pi/\omega_d$. At this time, the circuit is disconnected with the voltage source. The voltage across the capacitor will be

$$v_C(\pi/\omega_d) = \frac{V_F}{2} (1 + e^{-\alpha\pi/\omega_d}). \quad (2.9)$$

As we can see, v_C at this time is not equal to V_F . However if the value of the exponential term is close to 1, v_C will be close to V_F . To satisfy this condition, $\alpha\pi/\omega_d$ must be small. From (2.6), the condition will be

$$L \gg \frac{\pi^2 + 1}{4} R^2 C. \quad (2.10)$$

The energy dissipation during charging can be found by manipulating the integral

$$\begin{aligned}
E_{dis} &= \int_0^{\pi/\omega_d} i_R^2 R dt = RC^2 \int_0^{\pi/\omega_d} \dot{v}_C^2(t) dt \\
&= \frac{1 - e^{-2\alpha\pi/\omega_d}}{8} CV_F^2 = \frac{1 - e^{-2\alpha T_S}}{8} CV_F^2.
\end{aligned} \tag{2.11}$$

where T_S is of course the storage time of this method which is equal to π/ω_d . As expected, (2.11) shows that if the storage time is large, the energy dissipation will be small. A trade off between speed and energy is held and we can adjust the storage time by choosing the value of an inductor. The larger inductor value, the longer storage time and the lower energy dissipation. Recall that the value of an inductor at least must be satisfied by (2.10).

For discharging, the process is the same as charging. Since the initial voltage of capacitor now is approximately V_F , when the switch is closed the energy will be recovered into voltage source. At the time $t = \pi/\omega_d$ the switch is opened and v_C is approximately equal to zero. The total energy dissipation during recovering is the same as (2.11). Therefore, the energy loss for a complete cycle is the twice of (2.11).

The advantage of resonant power clock technique over ramp function technique is using constant voltage source. However, the main disadvantages of this technique are using a large inductor and complicated switch controlling. While the ramp function technique does not need additional circuit elements or switch controlling. Nevertheless, it needs the ramp voltage source which is also rising the complexity. Both resonant power clock technique and ramp function technique save the energy but lose the circuit speed. If the storage time approaches to infinity, there will be no energy dissipation. It is similar to adiabatic process in thermodynamics [56]. Adiabatic process is the process that the system does not gain or lose the energy. If we charge the capacitor very slowly, the energy dissipation will be zero and our system, including voltage source is not gain or lose the energy. The energy is only transferred from source to capacitor. Therefore, these techniques call be called as adiabatic switching.

2.2.4 Adiabatic Switching

Adiabatic switching or adiabatic logic circuits is a family of digital circuit that does not use a constant voltage source for charging capacitive loads and can recover some energy from capacitive loads when they discharge. Actually, the ramp voltage source is very commonly used. There are many styles of adiabatic logic circuit. Some of them will be described here.

One of the styles that easy to understand is shown in Fig. 2.6. As always, the inverter is used to explain the mechanism of the circuit style. The ramp voltage source can also be called as the power clock source. At the beginning, the capacitive load is empty. The input signal comes and must settle before the circuit start computing. Assume that the input signal is low, therefore PMOS is turned on. Then the ramp voltage source start to ramp the voltage up to a certain level. This state call be called as energized state. If the rise time of the ramp voltage is slow, there will be a little of energy dissipation. The energy dissipation is inversely proportional to the storage time as it was explained above. When the circuit finishes charging, the output is high and the computation is complete. There is no static energy dissipation and this output is ready for the next stages.

When the latter stages do not need the information at the output anymore, the ramp voltage source ramps the voltage down to zero. This state call be called as de-energized state. Some of the stored energy in capacitive load is recovered into the ramp voltage source. For a complete cycle, there will be a little energy dissipation. It depends on rise

Figure 2.6: Pass transistor adiabatic inverter circuits.

time and fall time of the ramp voltage. If the rise time or fall time is longer, the energy dissipation will be lower but the circuit will be slow. Notice that PMOS must be turned on all the time until the de-energized state is complete. In contrast, if the input is high, PMOS will be turned off and the output will be low at all time. Therefore, there is no energy dissipation in that case.

The main disadvantage of pass transistors is threshold voltage drop [57]. If we consider the characteristic of PMOS rather than just model PMOS as a switch in Fig. 2.6, the output voltage will be V_{TP} after the circuit pass the de-energized state, where V_{TP} is a threshold voltage of PMOS. It causes bad logic low. Since the voltage across a capacitive is V_{TP} , there will be energy storage in capacitor and this stored energy will be forever. Therefore, we lose this amount of energy to store a capacitor and cannot recover. Unfortunately, if NMOS is used instead PMOS, the circuit in Fig. 2.6 will be a buffer circuit and this problem also cannot be solved. When NMOS is turned on and the circuit already pass the energized state, the output voltage will be $V_{DD} - V_{TN}$ where V_{TN} is threshold voltage of NMOS and V_{DD} is the NMOS gate voltage. It causes bad logic high. However, there is no remaining stored energy in capacitor that cannot be recovered like PMOS case.

A well-known technique that can solve threshold voltage drop issue is using transmission gate. Transmission gate is NMOS and PMOS that are parallel connected. The input signal of PMOS is the complement of the input signal of NMOS. Therefore, they will be turned on or turned off together. If the output has to be logic low, the NMOS will pull down the voltage to be zero. The voltage will not stuck at V_{TP} . Likewise, if the output has to be logic high, the PMOS will pull up the voltage to be V_{DD} and the voltage will not stuck at $V_{DD} - V_{TN}$. Another advantage of using transmission gate is reducing equivalence resistance of the switch. Because when it is turned on, both PMOS and NMOS are turned on together and conduct more current than only one transistor does. However, the transmission gate needs a complement signals and double number of transistors. Therefore, the logic circuit that use transmission gates should be capable of generate the complement signal as well. The inverter adiabatic circuit using transmission gate is shown in Fig. 2.7 [53].

The mechanism of a circuit in Fig. 2.7 is similar to the circuit in Fig. 2.6. Initially, all capacitors are empty. Assume that the input signal is logic high, then the bottom transmission gate is turned on while the top transmission gate is turned off. When the circuit is energized, the bottom capacitor is charged through the bottom transmission gate and $\overline{OUT}$ is high while the top capacitor is remained the same at logic low. After finished computing, the circuit is de-energized and the energy is recovered into the ramp voltage source via the bottom transmission gate. For a complete cycle, the energy dissipation occurs at the transmission gates that is tuned on which is in this case, the bottom transmission gate. The energy dissipation can be approximated as

$$E_{dis} = \frac{\xi RC_L}{T} C_L V_{dd}^2 \quad (2.12)$$

Figure 2.7: Inverter adiabatic CMOS using transmission gates.

where R is equivalent resistance of transmission gate, T is rise time of power clock, V_{dd} is the maximum voltage of power clock and ξ is the shape factor. If the power clock is ramp function, ξ will be 1 and (2.12) is the same as (2.3). Recall that the power clock is not necessary to be a ramp function and the shape factor will be different for other functions.

Adiabatic amplifier or adiabatic buffer in another style is shown in Fig. 2.8 [24, 54]. This circuit is very similar to the circuit in Fig. 2.7 except two additional parallel NMOS with capacitive loads. These transistors act like the pass transistors. Either P_1 or P_2 is turned on when the input signal comes to discharge the output node. For example, if IN is logic high, transmission gate T_1 and pass transistor P_2 will be turned on. When the circuit is energized, node OUT is charged through T_1 while node $\overline{OUT}$ is discharged through P_2 . Although node $\overline{OUT}$ should not have any charge, it can have charge due to any secondary effects such as a coupling between nodes. Therefore, pass transistor helps to discharge and guarantee that this node is absolutely zero voltage. Both P_1 and P_2 must be NMOS because they are used to pull down only. PMOS has threshold voltage drop problem for pulling down as described above. The analysis and optimization of the circuit in Fig. 2.8 was written in [24].

Figure 2.8: Adiabatic Amplifier.

There are many other adiabatic logic styles were proposed [27, 50, 58–63]. The concept can be expanded to memory, latch and flip-flop design [25, 53, 60, 64]. In system level, there are adiabatic registers, adiabatic full adder and etc [24, 25, 52, 64, 65]. Some of research work focused on energy analysis and generating power clock which is ramp function [25, 26, 28, 54, 66, 67]. However, all of the adiabatic logic circuit styles are based on the principle that was described above. Recall that we know that using ramp function is more energy-efficient than using the constant voltage source. Adiabatic switching relies

on this fact to charge and discharge the capacitive loads. But we also do not know that is there any function that is more energy-efficient than the ramp function or not. Ref. [29] postulated that the constant current source is the best function for charging a simple series RC circuit. They also proposed the algorithm to find the most energy-efficient function for charging capacitive loads through the transistors numerically. The algorithm is based on calculus of variations theory. Calculus of variations is a powerful theory that can be used to optimize the circuits, not only numerically but also analytically as we shall see in the rest chapters. However, in the next section, we will introduce calculus of variations and some classic examples in mathematics and physics problems for the better understanding in the theory.

2.3 Theory: Calculus of Variations

2.3.1 The Euler-Lagrange's Differential Equation

The fundamental of calculus of variations can be call as the Lagrangian method. The principle of the Lagrangian method can be described as follows [68].

Let us consider the integral

$$I[f(t)] = \int_{t_1}^{t_2} \mathcal{L}(t, f(t), \dot{f}(t)) dt \quad (2.13)$$

where $\dot{f}(t) = \frac{d}{dt}f(t)$ and $\mathcal{L}(t, f(t), \dot{f}(t))$ is named as Lagrangian function.

The integral I in (2.13) is called functional because it depends on the function $f(t)$. A particular $f(t)$ gives one value of I and I can have infinite number of values for all infinite number of functions $f(t)$. In other words, the function $f(t)$ is a variable for I . Therefore there can be some functions $f(t)$ that make I a stationary value, that is a local minimum, a local maximum or a saddle point.

To find these functions, the function must be defined at the boundaries of the integral, t_1 and t_2 . Assume that $f(t_1) = f_1$ and $f(t_2) = f_2$, then solving Euler-Lagrange differential equation,

$$\frac{\partial \mathcal{L}}{\partial f(t)} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{f}(t)} \right) = 0 \quad (2.14)$$

with sufficient boundary conditions (f_1 and f_2) will give the functions that make I a stationary value.

One classic example to illustrate the Lagrangian method is the shortest path between two points problem in Fig. 2.9. Given two arbitrary points in x-y plane marked as $P(x_1, y_1)$ and $Q(x_2, y_2)$ respectively, there are infinity number of paths between P and Q . By intuitiveness, the shortest path between them is a straight line. The Lagrangian method can be applied to prove this postulate.

Figure 2.9: An illustration of the shortest path between two points problem.

To that end, the infinitesimal length is $ds = \sqrt{dx^2 + dy^2}$ and total length between two points must be

$$S = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad (2.15a)$$

$$S = \int_{x_1}^{x_2} \sqrt{1 + [\dot{y}(x)]^2} dx. \quad (2.15b)$$

The integral S is functional. It depends on $y(x)$ which is the path between P and Q . By applying the Lagrangian method, the Lagrangian function is

$$\mathcal{L} = \sqrt{1 + [\dot{y}(x)]^2}. \quad (2.16)$$

Finding the shortest path in mathematical sense means finding $y(x)$ that give the minimum value of S . Therefore, we can find the shortest path by using Euler-Lagrange equation by substituting (2.16) in (4.5). The equation becomes

$$-\frac{d}{dx} \left(\frac{\dot{y}(x)}{\sqrt{1 + [\dot{y}(x)]^2}} \right) = 0. \quad (2.17)$$

By solving a simple differential equation in (2.17), the solution is

$$\dot{y}(x) = a \quad (2.18a)$$

$$y(x) = ax + b \quad (2.18b)$$

where a and b are constants.

Finally, a and b can be determined by applying two boundary conditions at $P(x_1, y_1)$ and $Q(x_2, y_2)$. Thus, it is clear that the shortest path is straight line from (2.18b) as expected.

Another famous problem that the Lagrangian method plays an important role is called brachistochrone problem. The problem is finding the path between two points that particle moves from starting point to end point in the shortest time. Actually it is physics problem rather than mathematics problem. For more concrete, let us consider a particle that rest above the ground at height h . There is an ending point at the ground far away from the starting point at distance l in horizontal. What is the path that connect between two points that makes the particle moves to the end point in the shortest time? Fig. 2.10 illustrates the problem.

Figure 2.10: An illustration of the brachistochrone problem.

From the energy conservation, the speed at a certain height is

$$\frac{ds}{dt} = \sqrt{2g(h - y)} \quad (2.19)$$

where $ds = \sqrt{dx^2 + dy^2}$ is infinitesimal length, g is gravity and t is time. By rearranging (2.19) with the same idea as (2.15b), the duration that the particle travels along the path is

$$T = \int_0^l \sqrt{\frac{1 + \dot{y}^2}{2g(h - y)}} dx. \quad (2.20)$$

As we can see, the path that provides the shortest time can be found by minimizing the integral in (2.20). Thus, the Lagrangian function is

$$\mathcal{L} = \sqrt{\frac{1 + \dot{y}^2}{2g(h - y)}}. \quad (2.21)$$

By substituting in (4.5), the equation becomes

$$\begin{aligned} \frac{1}{2(h - y)} \sqrt{\frac{1 + \dot{y}^2}{2g(h - y)}} - \frac{d}{dx} \left(\frac{\dot{y}}{\sqrt{2g(h - y)(1 + \dot{y}^2)}} \right) &= 0 \\ \frac{1}{2(h - y)} \sqrt{\frac{1 + \dot{y}^2}{h - y}} - \frac{\ddot{y}}{\sqrt{(h - y)(1 + \dot{y}^2)}} & \\ - \frac{\dot{y}^2}{2(h - y)\sqrt{(h - y)(1 + \dot{y}^2)}} + \frac{\dot{y}^2 \ddot{y}}{(1 + \dot{y}^2)\sqrt{(h - y)(1 + \dot{y}^2)}} &= 0 \\ \frac{1 + \dot{y}^2}{2(h - y)} - \ddot{y} - \frac{\dot{y}^2}{2(h - y)} + \frac{\dot{y}^2 \ddot{y}}{1 + \dot{y}^2} &= 0 \\ \frac{\ddot{y}}{1 + \dot{y}^2} &= \frac{1}{2(h - y)} \\ \int \frac{2\dot{y}}{1 + \dot{y}^2} d\dot{y} &= \int \frac{1}{h - y} dy \\ 1 + \dot{y}^2 &= \frac{A}{h - y} \end{aligned} \quad (2.22)$$

where A is a constant. By rearranging the terms, we get

$$\begin{aligned} \dot{y} &= \sqrt{\frac{A - (h - y)}{h - y}} \\ x &= \int_h^y \sqrt{\frac{h - y}{A - (h - y)}} dy = \int_0^{h-y} -\sqrt{\frac{u}{A - u}} du \end{aligned} \quad (2.23)$$

where $u = h - y$.

By introducing the parameter α as $\sin \alpha = \sqrt{u}/\sqrt{A}$, then

$$\begin{aligned} x &= - \int_{u=0}^{u=h-y} \tan \alpha (2A \sin \alpha \cos \alpha) d\alpha = - \int_{u=0}^{u=h-y} 2A \sin^2 \alpha d\alpha \\ x &= -A \left(\alpha - \frac{\sin 2\alpha}{2} \right) \Big|_{u=0}^{u=h-y} = -\frac{A}{2} (2\alpha - \sin 2\alpha). \end{aligned} \quad (2.24)$$

From the relationship $\sin \alpha = \sqrt{u}/\sqrt{A}$, y can also be expressed in the parametric form with α as

$$y = h - A \sin^2 \alpha = h - \frac{A}{2} (1 - \cos 2\alpha).$$

Rename the constant and the parameter as $-A/2 = R$ and $2\alpha = \theta$. The relationship of x and y in parametric form are

$$x = R(\theta - \sin \theta), \quad (2.25)$$

$$y = R(1 - \cos \theta) + h. \quad (2.26)$$

A constant R can be found by using a boundary condition $y(x = l) = 0$.

Although the procedure to solve brachistochrone problem involves non-linear differential equation and seems hopeless to solve analytically at the beginning, the solution can be found beautifully at the end. Actually it is well-known function family called cycloid. For more detail, cycloid curve is explained thoroughly in Appendix B.1.

2.3.2 Isoperimetric Problem

Furthermore, the optimization can be done even there are some conditions. The isoperimetric problem is one type of variational problems with subsidiary conditions. The problem can be stated as follows. Given the functional

$$I[y(t)] = \int_{t_1}^{t_2} F(t, y(t), \dot{y}(t)) dt \quad (2.27)$$

with the conditions

$$y(t_1) = y_1 \quad y(t_2) = y_2 \quad \text{and} \quad \int_{t_1}^{t_2} G(t, y(t), \dot{y}(t)) dt = K \quad (2.28)$$

where t_1, t_2, y_1, y_2 and K are constants. Finding the extrema of I is the same as finding the extrema of

$$\int_{t_1}^{t_2} F + \lambda G dt \quad (2.29)$$

where λ is a constant and sometimes it can be called as the Lagrange multiplier. Therefore, the Lagrangian function becomes

$$\mathcal{L} = F + \lambda G. \quad (2.30)$$

To illustrate this theory, let us consider a curve length l . This curve has two fixed end points at $(-a, 0)$ and $(a, 0)$. What is the function of the curve that enclose the largest area with the line $y = 0$? Assume that $l > 2a$. The problem is illustrated in Fig. 2.11.

Figure 2.11: An illustration of the the largest area with a constraint problem.

By intuitiveness, we know that the answer should be the arc of the circle. However, we can prove this assumption by using calculus of variations. The area under the curve is

$$A = \int_{-a}^a y(x) dx \quad (2.31)$$

with the conditions

$$y(-a) = 0, \quad y(a) = 0$$

and from (2.15b),

$$\int_{-a}^a \sqrt{1 + \dot{y}^2} dx = l.$$

This is the form of an isoperimetric problem, therefore

$$\mathcal{L} = y + \lambda \sqrt{1 + \dot{y}^2}. \quad (2.32)$$

where λ is a constant. By using the Euler-Lagrange differential equation in (4.5), the equation becomes

$$1 + \frac{d}{dx} \frac{\lambda \dot{y}}{\sqrt{1 + \dot{y}^2}} = 0. \quad (2.33)$$

Integrating respect to x gives

$$\begin{aligned} x + \frac{\lambda \dot{y}}{\sqrt{1 + \dot{y}^2}} &= k \\ \lambda^2 \dot{y}^2 &= (1 + \dot{y}^2)(x - k)^2 \\ \dot{y} &= \frac{x - k}{\sqrt{\lambda^2 - (x - k)^2}}. \end{aligned} \quad (2.34)$$

where k is a constant. Integrating again gives

$$\begin{aligned} \int dy &= \int \frac{x - k}{\sqrt{\lambda^2 - (x - k)^2}} dx \\ y - h &= \int \frac{d(x - k)^2}{2\sqrt{\lambda^2 - (x - k)^2}} = \sqrt{\lambda^2 - (x - k)^2} \\ (x - k)^2 + (y - h)^2 &= \lambda^2. \end{aligned} \quad (2.35)$$

where h is a constant. This is the equation of circle curve family as expected. All three constants can be determined by using three conditions that were described above.

2.3.3 Multiple Functions Problem

Beside single function optimization, multiple functions can also be applied. If the Lagrangian function comprises n functions,

$$\mathcal{L}(x, y_1(x), \dot{y}_1(x), y_2(x), \dot{y}_2(x), \dots, y_n(x), \dot{y}_n(x)),$$

there will be n Euler-Lagrange differential equations. Those equations are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial y_1(x)} - \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial \dot{y}_1(x)} \right) &= 0 \\ \frac{\partial \mathcal{L}}{\partial y_2(x)} - \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial \dot{y}_2(x)} \right) &= 0 \\ &\vdots \\ \frac{\partial \mathcal{L}}{\partial y_n(x)} - \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial \dot{y}_n(x)} \right) &= 0. \end{aligned} \quad (2.36)$$

All the functions can be found by solving all equations simultaneously.

The Lagrangian method is very common in classical mechanics because it is applied to the principle of stationary action or Hamilton's principle [69, 70]. Let us denote the kinetic energy as T and the potential energy as V . Then, the action integral is defined as

$$S = \int_{t_1}^{t_2} T - V dt \quad (2.37)$$

where t is time, t_1 and t_2 are initial time and final time respectively. The Hamilton's principle said that the action of the system occurs in the path that makes the action integral is a stationary value. In other words, the Lagrangian function is

$$\mathcal{L} = T - V. \quad (2.38)$$

By using the Lagrangian method, we can get the Newton's laws of motion equations of a system.

With this concept in classical mechanics, the Lagrangian method was adapted to circuit theory. We can get the circuit equations by using the Lagrangian method instead of using either Kirchhoff's current law or Kirchhoff's voltage law. The circuits can be called as the Euler-Lagrange system because their equations can be realized by using Euler-Lagrange differential equation [71]. In circuits, the kinetic energy and the potential energy in (2.38) are the energy storage in inductors and capacitors respectively. If the circuits comprise voltage sources, the Euler-Lagrange equations in (2.36) will be changed to

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1} \right) - \frac{\partial \mathcal{L}}{\partial q_1} &= a_1 \lambda + \mathcal{F}_1 \\ \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2} \right) - \frac{\partial \mathcal{L}}{\partial q_2} &= a_2 \lambda + \mathcal{F}_2 \\ &\vdots \\ \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_n} \right) - \frac{\partial \mathcal{L}}{\partial q_n} &= a_n \lambda + \mathcal{F}_n \end{aligned} \quad (2.39)$$

where $q_1, q_2, \dots, q_n$ are the charge functions for inductors and capacitors. $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n$ are the voltage source that in the same node corresponding to inductor or capacitor. λ is a constant and $a_1, a_2, \dots, a_n$ are the constants corresponding to the equation

$$a_1 \dot{q}_1 + a_2 \dot{q}_2 + \dots + a_n \dot{q}_n = 0. \quad (2.40)$$

Recall that the derivative of charge respect to time is the current function. Actually, the equation (2.40) is a Kirchhoff's current law.

For example, let us consider the LC circuit in Fig. 2.12. The circuit comprises two inductors, one capacitor and one voltage source. The procedure can be described as follows.

Figure 2.12: LC circuit that is illustrated as an Euler-Lagrange system.

- Step 1: Use Kirchhoff's current law to get the constants, a_1, a_2 and a_3 .

$$\dot{q}_{L1} - \dot{q}_{C2} - \dot{q}_{L3} = 0. \quad (2.41)$$

Therefore,

$$a_1 = 1, \quad a_2 = -1, \quad \text{and} \quad a_3 = -1. \quad (2.42)$$

- Step 2: Determine the Lagrangian function from (2.38),

$$\mathcal{L} = \frac{1}{2}L_1\dot{q}_{L1}^2 + \frac{1}{2}L_3\dot{q}_{L3}^2 - \frac{1}{2C_2}q_{C2}^2. \quad (2.43)$$

- Step 3: Apply the Euler-Lagrange equation from (2.39),

$$\begin{aligned} L_1\ddot{q}_{L1} &= \lambda + E \\ \frac{1}{C_2}q_{C2} &= -\lambda \\ L_3\ddot{q}_{L3} &= -\lambda. \end{aligned} \quad (2.44)$$

- Step 4: Eliminate the constant λ . Three equations in (2.44) are reduced to two equations,

$$\begin{aligned} \ddot{q}_{L1} &= -\frac{1}{L_1C_2}q_{C2} + E \\ \ddot{q}_{L3} &= \frac{1}{L_3C_2}q_{C2}. \end{aligned} \quad (2.45)$$

- Step 5: Recall that $\dot{q}_{L1} = i_{L1}$, $\dot{q}_{L3} = i_{L3}$ and $q_{C2}/C_2 = v_{C2}$. From (2.45) and (2.41), the equation of the system can be expressed as

$$\begin{bmatrix} \dot{i}_{L1} \\ \dot{v}_{C2} \\ \dot{i}_{L3} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L_1} & 0 \\ \frac{1}{C_2} & 0 & -\frac{1}{C_2} \\ 0 & \frac{1}{L_3} & 0 \end{bmatrix} \begin{bmatrix} i_{L1} \\ v_{C2} \\ i_{L3} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} E. \quad (2.46)$$

As we can see, the system's equation can be derived without using Kirchhoff's voltage law. Further generalization and proofs can be found in [69, 72].

The Euler-Lagrange system realization has been extended to varied fields of engineering. For example, equivalent magnetic circuits [73, 74], L-C circuits [75], switching networks [71, 76], control theories in mechanical, electrical and electromechanical systems [72]. Recall that the Euler-Lagrange system realization is based on an action integral. We use calculus of variations to find the functions that make the action integral is a stationary value. Actually, the Euler-Lagrange system is not an optimization problem anymore. Calculus of variations is rarely applied to optimization problem directly. However, there are some work that use calculus of variations for numerical power clock optimization [29, 77].

Up to this point, parts of variational calculus were presented. From the beginning of using the Lagrangian method to find the extrema of the functional, then the isoperimetric problem was explained. Multiple functions can also be applied. The action integral was introduced and applying calculus of variations for Euler-Lagrange system was illustrated.

2.3.4 Higher Derivatives Euler-Lagrange's Differential Equation

Finally, we shall draw one more theory in variational calculus. That is functional depending on higher order derivatives. Or in other words, the Lagrangian function is

$$\mathcal{L}(x, y(x), \dot{y}(x), \ddot{y}(x), \dots, y^{(n)}(x))$$

where n is the highest order of derivative. Then, the corresponding Euler-Lagrange equation is

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial y(x)} - \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial \dot{y}(x)} \right) + \frac{d^2}{dx^2} \left(\frac{\partial \mathcal{L}}{\partial \ddot{y}(x)} \right) - \dots + (-1)^n \frac{d^n}{dx^n} \left(\frac{\partial \mathcal{L}}{\partial y^{(n)}(x)} \right) &= 0 \\ \frac{\partial \mathcal{L}}{\partial y(x)} + \sum_{i=1}^n (-1)^i \frac{d^i}{dx^i} \left(\frac{\partial \mathcal{L}}{\partial y^{(i)}(x)} \right) &= 0. \end{aligned} \quad (2.47)$$

If $n = 1$, (2.47) will reduce to (4.5).

2.4 Conclusion

Literature review about adiabatic switching and calculus of variations were proposed in this chapter. Adiabatic switching is a motivation while calculus of variations is a theory that will be use through out later chapters. Firstly, the analogy between the limitation of computation and physics was described. The concept of reversible and irreversible process in thermodynamics were mapped with reversible logic and irreversible logic. Adiabatic process concept in thermodynamics was also mapped with adiabatic switching in circuits. The principle of adiabatic switching was explained and some issues were proposed. Varied adiabatic logic circuit styles were introduced along with their advantages and disadvantages. Then the fundamentals of variational calculus were explained. From the elementary calculus of variations to some applications in mathematics, physics and circuits. All ingredients of variational calculus that are needed in the later chapters were demonstrated in this chapter completely.

Chapter 3

First Order Linear Circuits Optimization

3.1 Introduction

As we see in chapter 2, adiabatic switching commonly use ramp function as a power clock and the analysis of energy dissipation for charging series RC circuit is also described. In this chapter, Calculus of variations will be applied to derive analytically the source functions that minimize energy dissipation in circuits during transient. The most energy-efficient input current function and voltage source function for charging series RC circuit will be derived and proved to be a constant function and a raise ramp function respectively in Section 3.2. The technique of calculus of variations will be extended for other common first order linear circuits as well, whose most energy-efficient source functions and lowest energy dissipation expressions will be summarized in compact parameters involving speed and energy trade off limits in Section 3.3. Numerical results will be illustrated to depict the upper limit of speed-energy trade off and to justify the consistency with the analytical results. Section 3.4 will treat some special circuit topologies that cannot be solved directly using standard calculus of variations separately. These special topologies will be explained in graphical, mathematical and intuitive manner.

3.2 Series RC Circuit

The optimization problem should be stated clearly to avoid any misconception. Consider the series RC circuit in Fig. 3.1, initially there is no voltage across a capacitor. We want to charge a capacitor to a final voltage value V_F in certain amount of storage time T_F . What is the input voltage function $v_I(t)$ that causes the lowest total energy dissipation while charging? In other words, what is the most energy-efficient input voltage function?

Figure 3.1: R-C series circuit.

From Fig. 3.1, $i_R = i_C = C\dot{v}_C$ and total energy dissipation can be written as

$$E_R = \int_0^{T_F} (C\dot{v}_C)^2 R dt = RC^2 \int_0^{T_F} \dot{v}_C^2 dt. \quad (3.1)$$

Recall that $\dot{v}_C$ is a function of time and therefore E_R is a functional. By using the Lagrangian method to minimize energy dissipation in (3.1), the Lagrangian function is

$$\mathcal{L} = \dot{v}_C^2 \quad (3.2)$$

and apply Euler-Lagrange equation in (4.5) to obtain

$$-\frac{d}{dt}(2\dot{v}_C) = 0. \quad (3.3)$$

The solution of differential equation in (3.3) is

$$v_C(t) = at + b \quad (3.4)$$

where a and b are constants. With two boundary conditions $v_C(0) = 0$ and $v_C(T_F) = V_F$, the unique solution can be determined as

$$v_C(t) = \frac{V_F}{T_F} t \quad \text{for } t < T_F. \quad (3.5)$$

The voltage in (3.5) is the voltage across a capacitor while charging. For $t \geq T_F$, $v_C(t) = V_F$. Recall that this is not an input voltage function. Fortunately, we can find input voltage function that corresponding to $v_C(t)$. From usual circuit analysis,

$$v_I(t) = RC\dot{v}_C(t) + v_C(t) \quad (3.6a)$$

$$v_I(t) = \begin{cases} \frac{V_F}{T_F}(t + RC) & \text{for } t < T_F \\ V_F & \text{for } t \geq T_F. \end{cases} \quad (3.6b)$$

Moreover, the input current function can be determined as

$$i_I(t) = C\dot{v}_C(t) \quad (3.7a)$$

$$i_I(t) = \begin{cases} \frac{CV_F}{T_F} & \text{for } t < T_F \\ 0 & \text{for } t \geq T_F. \end{cases} \quad (3.7b)$$

Although this result is identical with the postulation in [29], the analysis is shown rigorously. The function in (3.6b) can be called as raised ramp function. The corresponding input voltage and current function is depicted in Fig. 3.2. Either voltage source or current source with the input function in (3.6b) or (3.7b) respectively are the most energy-efficient source functions for the series RC circuit. They cause the lowest total energy dissipation while charging a capacitor. Fig. 3.3 shows the most energy-efficient source functions and the capacitor voltage response.

This is a significant result. It gives the maximum efficiency for charging a capacitor in a series RC circuit. By calculating the efficiency with the same storage time for unit step input, the maximum efficiency is 69.72% while the unit step input gives 50.00% efficiency.

Although we can ensure that the input function in part (3.6b) - (3.7b) is the most energy-efficient input function by mathematical approach, we shall illustrate its efficiency

Figure 3.2: The most energy-efficient input source.

Figure 3.3: Time response of the most energy-efficient series RC circuit.

by comparing with some other functions. As expected, there are infinite number of input functions available. Therefore only a few functions could be chosen here. Since the most energy-efficient input function is a linear function with two specific parameters, i.e. slope and y-intercept, the comparison will be done among linear functions with different parameters.

The process of choosing the slope and the y-intercept is rather subtle. For convenience, let us denote the slope as a and the y-intercept as b :

$$v_I(t) = \begin{cases} at + b & \text{for } t < T_F \\ V_F & \text{for } t \geq T_F. \end{cases} \quad (3.8)$$

Either a or b can be chosen freely but not both because there is the storage time which is a constraint. For example, if a is freely chosen, b must be calculated by using a and T_F . Thus b can not be chosen arbitrarily. With tedious circuit analysis (B.2), the expression of b in terms of a and T_F can be found as

$$b = aRC + \frac{V_F - aT_F}{1 - e^{T_F/RC}}. \quad (3.9)$$

The efficiency of the input function in (3.8) was simulated. Fig. 3.4 depicts the efficiency of series R-C circuit with various slope of linear input voltage function a and storage time T_F while the y-intercept b is given by (3.9). In Fig. 3.4, there is a peak efficiency and this corresponds to the linear function in (3.6b) or (3.7b) which has $a = V_F/T_F$ and $b = V_F RC/T_F$. The values of parameters used for Fig. 3.4 are $R = 1\text{k}\Omega$, $C = 100\text{pF}$ and $V_F = 1\text{V}$.

Figure 3.4: Efficiency of series RC circuit with various slope of linear input voltage functions and storage time.

3.3 Other First Order Linear circuits

The calculus of variations above can be generalized for other first order linear circuits' optimization. However, the problems and conditions for optimization must be stated clearly and more general than those in Section 3.2. Firstly, let us consider the circuits in energy perspective. In linear first order circuits, there is only one energy storage element that is capacitor or inductor. Capacitor stores energy in voltage form and while inductor stores energy in current form. All other elements in circuit must be resistors which are energy dissipative elements. We assume either ideal current source or voltage source is energy source in circuit. For simplicity, the optimization problems will be set for the linear first order circuits with only one energy source. At this moment, the energy point of view for circuits' optimization problem is well defined.

Secondly, initial conditions of circuits must be specified. First order circuits need only one initial condition. In energy perspective, the initial condition refers to the initial energy in a circuit. If the circuit comprises a capacitor, the initial voltage of capacitor must be specified. In contrast, if the circuit comprises an inductor, the initial current of inductor must be specified. This is a general condition for solving first order circuits.

Thirdly, the final value of energy store in circuits must also be specified, that is the final voltage of a capacitor or the final current of an inductor. With these conditions, first order linear circuits can be optimized by following the procedure in Section 3.2. Two examples of using Lagrangian method for first order linear circuits optimization are shown below.

Example 1

Consider the circuit in Fig. 3.5. The circuit comprises two resistors and a capacitor. Initially the voltage across capacitor is zero. We want to charge the capacitor to the final voltage V_F in the duration T_F . What is the most energy-efficient source function? and what is the lowest energy dissipation?

We will divide the problem into two periods, transient and steady state. In transient or charging state, the energy dissipation is the integral of the total power dissipation through resistors. Since we want to minimize this integral, the Lagrangian function is the total power dissipation.

$$\mathcal{L} = P_{dis} = i_{RS}^2 R_S + \frac{v_{RP}^2}{R_P} = R_S \left(C \dot{v}_C + \frac{v_C}{R_P} \right)^2 + \frac{v_C^2}{R_P}. \quad (3.10)$$

For applying Euler-Lagrange differential equation, partial derivative of the Lagrangian

Figure 3.5: Series-Parallel R-C circuit.

function respect to v_C and $\dot{v}_C$ must be compute. Hence,

- $\frac{\partial \mathcal{L}}{\partial v_C} = \frac{2R_S}{R_P} \left(C\dot{v}_C + \frac{v_C}{R_P} \right) + \frac{2v_C}{R_P}.$
- $\frac{\partial \mathcal{L}}{\partial \dot{v}_C} = 2R_S C \left(C\dot{v}_C + \frac{v_C}{R_P} \right).$

By substituting in (4.5), we get

$$\frac{R_S}{R_P} \left(C\dot{v}_C + \frac{v_C}{R_P} \right) + \frac{v_C}{R_P} - R_S C \frac{d}{dt} \left(C\dot{v}_C + \frac{v_C}{R_P} \right) = 0.$$

After rearrangement,

$$\ddot{v}_C - \frac{R_S + R_P}{R_S R_P^2 C^2} v_C = 0. \quad (3.11)$$

Let us denote new parameter,

$$\tau = \sqrt{\frac{R_S R_P^2}{R_S + R_P}} C. \quad (3.12)$$

Then, the general solution of differential equation in (3.11) is

$$v_C(t) = A e^{-t/\tau} + B e^{t/\tau} \quad (3.13)$$

where A and B are constants. The constants can be determined by using two boundary conditions, $v_C(0) = 0$ and $v_C(T_F) = V_F$. After finding the constants and substituting in (3.13), the unique solution is

$$v_C(t) = \frac{\sinh(t/\tau)}{\sinh(T_F/\tau)} V_F. \quad (3.14)$$

The input voltage and current functions can be evaluated by using transient circuit analysis,

$$v_I(t) = R_S \left(C\dot{v}_C(t) + \frac{v_C(t)}{R_P} \right) + v_C(t) \quad (3.15)$$

and

$$i_I(t) = i_{RP}(t) + i_C(t) = \frac{v_C(t)}{R_P} + C\dot{v}_C(t). \quad (3.16)$$

After simplifying,

$$v_I(t) = \left(1 + \frac{R_S}{R_P} \right) \frac{V_F}{\sinh(T_F/\tau)} \left[\frac{1}{\alpha} \cosh(t/\tau) + \sinh(t/\tau) \right] \quad (3.17)$$

and

$$i_I(t) = \frac{V_F/R_P}{\sinh(T_F/\tau)} [\alpha \cosh(t/\tau) + \sinh(t/\tau)] \quad (3.18)$$

where $\alpha = \sqrt{1 + \frac{R_P}{R_S}}$.

Recall that the above analysis is in transient state. When the voltage across the capacitor reaches the final value V_F , the circuit enters the steady state. Next, we will calculate the input source functions in steady state.

In steady state, the voltage across the capacitor does not change that is $\dot{v}_C(t) = 0$ and $v_C(t) = V_F$. By substituting in (3.15) and (3.16), the voltage and current functions in steady state are

$$V_{SS} = V_F \left(1 + \frac{R_S}{R_P} \right) \quad (3.19)$$

and

$$I_{SS} = \frac{V_F}{R_P}. \quad (3.20)$$

Finally, the input source functions can be concluded as

- Voltage source function

$$v_I(t) = \begin{cases} \frac{V_F(1 + R_S/R_P)}{\sinh(T_F/\tau)} \left[\frac{1}{\alpha} \cosh(t/\tau) + \sinh(t/\tau) \right] & t < T_F \\ V_F \left(1 + \frac{R_S}{R_P} \right) & t \geq T_F \end{cases} \quad (3.21)$$

- Current source function

$$i_I(t) = \begin{cases} \frac{V_F/R_P}{\sinh(T_F/\tau)} [\alpha \cosh(t/\tau) + \sinh(t/\tau)] & t < T_F \\ \frac{V_F}{R_P} & t \geq T_F \end{cases} \quad (3.22)$$

The voltage and current source function in (3.21) and (3.22) are the most energy-efficient source functions. They give the lowest energy dissipation during transient. Notice that there is inevitable energy dissipation in steady state because there is a path that connect between source and ground and this path comprises resistors.

Furthermore, the lowest energy dissipation during transient can be determined but it is a little bit tedious. We shall simplify the power dissipation first and then integrate it to evaluate the lowest energy dissipation.

The expression of power dissipation is described in (3.10). When the most energy-efficient source function is applied, the voltage across the capacitor is the function in (3.14). With (3.16) and (3.18), the power dissipation can be written as

$$\begin{aligned} P_{dis} &= \frac{1}{R_P} \left(\frac{V_F}{\sinh(T_F/\tau)} \right)^2 \sinh^2(t/\tau) + R_S \left(\frac{V_F/R_P}{\sinh(T_F/\tau)} \right)^2 [\alpha \cosh(t/\tau) + \sinh(t/\tau)]^2 \\ P_{dis} &= \frac{R_S}{R_P} \left(\frac{V_F}{\sinh(T_F/\tau)} \right)^2 \left[\left(\frac{1}{R_S} + \frac{1}{R_P} \right) \sinh^2(t/\tau) \right. \\ &\quad \left. + \frac{2\alpha}{R_P} \cosh(t/\tau) \sinh(t/\tau) + \frac{\alpha^2}{R_P} \cosh^2(t/\tau) \right] \\ P_{dis} &= \frac{R_S}{R_P} \left(\frac{V_F}{\sinh(T_F/\tau)} \right)^2 \left[\frac{R_S + R_P}{R_S R_P} (\cosh^2(t/\tau) + \sinh^2(t/\tau)) + \frac{\alpha}{R_P} \sinh(2t/\tau) \right] \\ P_{dis} &= \frac{R_S}{R_P} \left(\frac{V_F}{\sinh(T_F/\tau)} \right)^2 \left[\frac{R_S + R_P}{R_S R_P} \cosh(2t/\tau) + \frac{\alpha}{R_P} \sinh(2t/\tau) \right]. \end{aligned} \quad (3.23)$$

Next, the lowest energy dissipation is the integral of (3.23) during transient.

$$\begin{aligned}
E_{dis} &= \int_0^{T_F} P_{dis} dt \\
E_{dis} &= \frac{R_S}{R_P} \left(\frac{V_F}{\sinh(T_F/\tau)} \right)^2 \left[\frac{\tau(R_S + R_P)}{2R_S R_P} \sinh(2T_F/\tau) + \frac{\tau\alpha}{2R_P} (\cosh(2T_F/\tau) - 1) \right] \\
E_{dis} &= \frac{R_S}{R_P} \left(\frac{V_F}{\sinh(T_F/\tau)} \right)^2 \left[\sqrt{\frac{R_S + R_P}{R_S}} C \cosh(T_F/\tau) \sinh(T_F/\tau) + C \sinh^2(T_F/\tau) \right] \\
E_{dis} &= \frac{R_S}{R_P} [\alpha \coth(T_F/\tau) + 1] C V_F^2. \tag{3.24}
\end{aligned}$$

In the example above, we presented the optimization of charging a capacitor in the circuit in Fig. 3.5. The most energy-efficient source function both in voltage and current were derived and the lowest energy dissipation was determined. In next example, we shall deal with charging an inductor in different circuit topology. The procedure is similar but some variables and expressions must be change.

Example 2

Consider the circuit in Fig. 3.6. The circuit comprises two resistors and an inductor. Initially the current through the inductor is zero. We want to charge the inductor to the final current I_F in the duration T_F . What is the most energy-efficient source function? and what is the lowest energy dissipation?

Figure 3.6: Parallel-Series R-L circuit.

As same as in example 1, we will divide the problem into two periods, transient and steady state. the Lagrangian function is

$$\mathcal{L} = P_{dis} = i_{RS}^2 R_S + \frac{v_{RP}^2}{R_P} = R_S i_L^2 + \frac{(L \dot{i}_L + R_S i_L)^2}{R_P}. \tag{3.25}$$

For applying Euler-Lagrange differential equation, partial derivative of the Lagrangian function respect to i_L and $\dot{i}_L$ must be compute. Hence,

- $\frac{\partial \mathcal{L}}{\partial i_L} = 2R_S i_L + 2R_S \frac{L \dot{i}_L + R_S i_L}{R_P}.$
- $\frac{\partial \mathcal{L}}{\partial \dot{i}_L} = 2L \frac{L \dot{i}_L + R_S i_L}{R_P}.$

By substituting in (4.5), we get

$$2R_S i_L + 2R_S \frac{L \dot{i}_L + R_S i_L}{R_P} - \frac{d}{dt} \left(2L \frac{L \dot{i}_L + R_S i_L}{R_P} \right) = 0.$$

After rearrangement,

$$\ddot{i}_L - \frac{R_S(R_S + R_P)}{L^2} i_L = 0. \tag{3.26}$$

Let us denote new parameter,

$$\tau = \frac{L}{\sqrt{R_S(R_S + R_P)}}. \quad (3.27)$$

Then, the general solution of differential equation in (3.26) is

$$i_L(t) = Ae^{-t/\tau} + Be^{t/\tau} \quad (3.28)$$

where A and B are constants. The constants can be determined by using two boundary conditions, $i_L(0) = 0$ and $i_L(T_F) = I_F$. After finding the constants and substituting in (3.28), the unique solution is

$$i_L(t) = \frac{\sinh(t/\tau)}{\sinh(T_F/\tau)} I_F. \quad (3.29)$$

The input voltage and current functions can be evaluated by using transient circuit analysis,

$$v_I(t) = L\dot{i}_L(t) + R_S i_L(t). \quad (3.30)$$

and

$$i_I(t) = i_{RP}(t) + i_L(t) = \frac{L\dot{i}_L(t) + R_S i_L(t)}{R_P} + i_L(t). \quad (3.31)$$

After simplifying,

$$v_I(t) = \frac{I_F R_S}{\sinh(T_F/\tau)} [\alpha \cosh(t/\tau) + \sinh(t/\tau)] \quad (3.32)$$

and

$$i_I(t) = \frac{I_F(1 + R_S/R_P)}{\sinh(T_F/\tau)} \left[\frac{1}{\alpha} \cosh(t/\tau) + \sinh(t/\tau) \right] \quad (3.33)$$

where $\alpha = \sqrt{1 + \frac{R_P}{R_S}}$.

In steady state, the current through the inductor does not change that is $\dot{i}_L(t) = 0$ and $i_L(t) = I_F$. By substituting in (3.30) and (3.31), the voltage and current functions in steady state are

$$V_{SS} = I_F R_S \quad (3.34)$$

and

$$I_{SS} = I_F \left(1 + \frac{R_S}{R_P} \right). \quad (3.35)$$

Finally, the input source functions can be concluded as

- Voltage source function

$$v_I(t) = \begin{cases} \frac{I_F R_S}{\sinh(T_F/\tau)} [\alpha \cosh(t/\tau) + \sinh(t/\tau)] & t < T_F \\ I_F R_S & t \geq T_F \end{cases} \quad (3.36)$$

- Current source function

$$i_I(t) = \begin{cases} \frac{I_F(1 + R_S/R_P)}{\sinh(T_F/\tau)} \left[\frac{1}{\alpha} \cosh(t/\tau) + \sinh(t/\tau) \right] & t < T_F \\ I_F \left(1 + \frac{R_S}{R_P} \right) & t \geq T_F \end{cases} \quad (3.37)$$

To find the lowest energy dissipation, the expression of power dissipation is described in (3.25). When the most energy-efficient source function is applied, the current through the inductor is the function in (3.29). With (3.30) and (3.32), the power dissipation can be written as

$$P_{dis} = \frac{1}{R_P} \left(\frac{I_F R_S}{\sinh(T_F/\tau)} \right)^2 [\alpha \cosh(t/\tau) + \sinh(t/\tau)]^2 + R_S \left(\frac{I_F}{\sinh(T_F/\tau)} \right)^2 \sinh^2(t/\tau)$$

$$P_{dis} = \frac{R_S}{R_P} \left(\frac{I_F}{\sinh(T_F/\tau)} \right)^2 [(R_S + R_P) \cosh(2t/\tau) + \alpha R_S \sinh(2t/\tau)]. \quad (3.38)$$

Next, the lowest energy dissipation is the integral of (3.38) during transient.

$$E_{dis} = \int_0^{T_F} P_{dis} dt$$

$$E_{dis} = \frac{R_S}{R_P} [\alpha \coth(T_F/\tau) + 1] L I_F^2. \quad (3.39)$$

With the same procedure as in two examples above, other first order linear circuits can be optimized. The most energy-efficient input voltage and current functions for common first order linear circuits and their lowest energy dissipation is shown in Table 3.1. Recall that the optimization needs two boundary conditions. All of the circuits in Table 3.1 satisfy the conditions $v_C(0) = 0$ and $v_C(T_F) = V_F$ for capacitor and $i_L(0) = 0$ and $i_L(T_F) = I_F$ for inductor. Table 3.1 shows six frequent circuits, their significant parameters, the most energy-efficient input source functions both voltage and current and also their lowest energy dissipation corresponding to their most energy-efficient input source functions. Notice that some of the circuits still dissipate energy while they are already in steady state. These circuits can be identified by using usual steady state analysis. The minimum energy dissipation that we concern is only that during transient.

Table 3.1: The most energy-efficient input voltage and current functions for common first order linear circuits and their lowest energy dissipation.

Circuit	Parameters	Input voltage function $v_I(t)$	The lowest energy dissipation
		Input current function $i_I(t)$
$\tau = R_S C$	$\frac{V_F}{T_F}(t + \tau) \quad t < T_F$ $V_F \quad t \geq T_F$	$\frac{\tau}{T_F} C V_F^2$
		$\frac{C V_F}{T_F} \quad t < T_F$ $0 \quad t \geq T_F$
$\tau = \frac{L}{R_P}$	$\frac{L I_F}{T_F} \quad t < T_F$ $0 \quad t \geq T_F$	$\frac{\tau}{T_F} L I_F^2$
		$\frac{I_F}{T_F}(t + \tau) \quad t < T_F$ $I_F \quad t \geq T_F$
$\tau = \sqrt{R_S(R_S + R_P)} C$ $\alpha = \sqrt{1 + \frac{R_P}{R_S}}$ $A_h = V_F \text{csch}\left(\frac{T_F}{\tau}\right)$	$A_h \left[\frac{1}{\alpha} \cosh\left(\frac{t}{\tau}\right) + \sinh\left(\frac{t}{\tau}\right) \right] \quad t < T_F$ $V_F \quad t \geq T_F$	$\frac{R_S}{R_P} \left[\coth\left(\frac{T_F}{\tau}\right) \alpha + 1 \right] C V_F^2$
		$\frac{A_h}{R_P} \left[\alpha \cosh\left(\frac{t}{\tau}\right) + \sinh\left(\frac{t}{\tau}\right) \right] \quad t < T_F$ $\frac{V_F}{R_P} \quad t \geq T_F$	

Table 3.1 (Continued): The most energy-efficient input voltage and current functions for common first order linear circuits and their lowest energy dissipation.

Circuit	Parameters	Input voltage function $v_I(t)$	The lowest energy dissipation
		Input current function $i_I(t)$
$\tau = \sqrt{\frac{R_S + R_P}{R_S R_P^2}} L$ $\alpha = \sqrt{1 + \frac{R_P}{R_S}}$ $A_h = I_F \text{csch}\left(\frac{T_F}{\tau}\right)$	$A_h R_S \left[\alpha \cosh\left(\frac{t}{\tau}\right) + \sinh\left(\frac{t}{\tau}\right) \right] \quad t < T_F$ $I_F R_S \quad t \geq T_F$	$\frac{R_S}{R_P} \left[\coth\left(\frac{T_F}{\tau}\right) \alpha + 1 \right] L I_F^2$
		$A_h \left[\frac{1}{\alpha} \cosh\left(\frac{t}{\tau}\right) + \sinh\left(\frac{t}{\tau}\right) \right] \quad t < T_F$ $I_F \quad t \geq T_F$
$\tau = \sqrt{\frac{R_S R_P^2}{R_S + R_P}} C$ $\alpha = \sqrt{1 + \frac{R_P}{R_S}}$ $A_h = V_F \text{csch}\left(\frac{T_F}{\tau}\right)$ $\beta = 1 + \frac{R_S}{R_P}$	$\beta A_h \left[\frac{1}{\alpha} \cosh\left(\frac{t}{\tau}\right) + \sinh\left(\frac{t}{\tau}\right) \right] \quad t < T_F$ $\beta V_F \quad t \geq T_F$	$\frac{R_S}{R_P} \left[\coth\left(\frac{T_F}{\tau}\right) \alpha + 1 \right] C V_F^2$
		$\frac{A_h}{R_P} \left[\alpha \cosh\left(\frac{t}{\tau}\right) + \sinh\left(\frac{t}{\tau}\right) \right] \quad t < T_F$ $\frac{V_F}{R_P} \quad t \geq T_F$
$\tau = \sqrt{\frac{1}{R_S(R_S + R_P)}} L$ $\alpha = \sqrt{1 + \frac{R_P}{R_S}}$ $A_h = I_F \text{csch}\left(\frac{T_F}{\tau}\right)$ $\beta = 1 + \frac{R_S}{R_P}$	$A_h R_S \left[\alpha \cosh\left(\frac{t}{\tau}\right) + \sinh\left(\frac{t}{\tau}\right) \right] \quad t < T_F$ $I_F R_S \quad t \geq T_F$	$\frac{R_S}{R_P} \left[\coth\left(\frac{T_F}{\tau}\right) \alpha + 1 \right] L I_F^2$
		$\beta A_h \left[\frac{1}{\alpha} \cosh\left(\frac{t}{\tau}\right) + \sinh\left(\frac{t}{\tau}\right) \right] \quad t < T_F$ $\beta I_F \quad t \geq T_F$	

There is a technique to get the parameter τ in Table 3.1. For the simple series RC circuit and simple RL parallel circuit, τ is an ordinary time constant $R_S C$ and L/R_P respectively. For other four circuits, τ can be obtained by the procedure as follows.

Let us consider series RC with a parallel resistor in Fig. 3.7. To obtain the parameter τ , we have to determine two intermediate parameters called open circuit resistance R_{OC} and short circuit resistance R_{SC} . R_{OC} is the resistance that is seen by the capacitor when the input is opened circuit and R_{SC} is the resistance that is seen by the capacitor when the input is shorted circuit. Fig. 4.3a and Fig. 4.3b illustrate the circuit topologies for determining R_{OC} and R_{SC} respectively. The opened circuit resistance and the shorted circuit resistance can be expressed as

$$R_{OC} = R_S + R_P \quad (3.40a)$$

$$R_{SC} = R_S. \quad (3.40b)$$

With these two intermediate parameters, τ can be determined by

$$\tau = \sqrt{R_{OC} R_{SC}} C. \quad (3.41)$$

By substituting (3.40) in (3.41), τ is the one in the third row of Table 3.1.

Figure 3.7: Series R-C with a parallel resistor.

Figure 3.8: The illustration of opened circuit resistance R_{OC} and shorted circuit resistance R_{SC} .

This method can also be applied to the circuit comprise an inductor. R_{OC} and R_{SC} are the same as explained above but the formula for determining τ in (3.41) must be changed to

$$\tau = \frac{L}{\sqrt{R_{OC} R_{SC}}}. \quad (3.42)$$

All the value of τ for each circuit topology are shown in Table 3.1. All of the α are the same and there are some similarities among circuits such as the pattern of input functions or the lowest energy dissipation.

Since this analysis is considering power dissipation through resistance, therefore the way of resistors were connected matters. It causes source transformation technique may

not be applied. For example, voltage source that is series connected with a resistor can be transformed to current source with a parallel resistor by using source transformation technique. However, the current source in the latter topology after using source transformation is not the most energy-efficient input source because the way that resistor is connected changes and the power dissipation function or the Lagrangian function is not the same as the topology before using source transformation.

The results from optimization in Table 3.1 are similar among the circuits. Therefore only one of them can be chosen for discussion. Numerical and graphical results for the circuit in the third row of Table 3.1 are selected. Fig. 3.9 shows the time response of series RC with parallel resistor when using the most energy-efficient input voltage function. The parameters of circuit are $C = 100\text{pF}$, $R_S = 10\text{k}\Omega$, $R_P = 1\text{M}\Omega$, $V_F = 1\text{V}$ and storage time $T_F = 10\tau = 15\mu\text{s}$. Fig. 3.10 illustrates efficiency versus storage time when applying the most energy-efficient voltage function and raised ramp voltage function. The raised ramp function is described in (3.6b) with $R = R_S$. As expected, the efficiency of the most energy-efficient voltage function is greater than the raised ramp voltage function. Recall that the raised ramp function is the most energy-efficient voltage source function for RC series circuit but it is not the most energy-efficient for series RC with parallel resistor circuit. Actually the efficiency of the most energy-efficient function is the upper limit. There is no other input source functions that can provide a higher efficiency beyond this limit. The efficiency of the raised ramp function is close to the limit at short storage time. In other words, the raised ramp function almost reaches the limit of speed-energy trade off for fast charging. However, if the charging is slow, the raised ramp function will not be very efficient.

Figure 3.9: Time response of series RC with parallel resistor when using the most energy-efficient input voltage function.

3.4 Special Circuit Topologies' Optimization

Although many circuits can be optimized by using calculus of variations, not all circuits can be optimized. In the same way as the extremal of functions, not all of them can be derived by using only differentiation. There are at least two circuit topologies that cannot be optimized: (1) a loop that has only capacitors and input sources and (2)

Figure 3.10: Efficiency versus storage time when applying the most energy-efficient voltage function and raised ramp voltage function.

a node that has only inductors and input sources. For example, parallel R-C circuit cannot be optimized by using calculus of variations. However, we can derive the most energy-efficient function by considering the limit of some circuit topologies.

Let us consider series RC with parallel resistor in Fig. 3.7, this will be a parallel RC circuit if $R_S = 0$. With $R_P = 1\text{k}\Omega$, $C = 100\text{pF}$, $T_F = 10\text{ns}$ and $V_F = 1\text{V}$, Fig. 3.11 shows the efficiency versus series resistance when applying the most energy-efficient source function in series R-C with parallel resistor circuit. The efficiency goes to 100% when R_S goes to 0. The most energy-efficient input voltage function and current function of series R-C with parallel resistor circuit when $R_S = 1000$, 100, 10 and $1\text{m}\Omega$ are illustrated in Fig. 3.12 and Fig. 3.13 respectively. When the resistance of R_S is low, the input voltage function appears to be a unit step function and the input current function appears to be an impulse at the time T_F .

Figure 3.11: Efficiency versus series resistance when applying the most energy-efficient source function in series R-C series with parallel resistor circuit.

Figure 3.12: The most energy-efficient input voltage function of series R-C series with parallel resistor circuit when $R_S = 1000, 100, 10$ and $1\text{m}\Omega$.

Figure 3.13: The most energy-efficient input current function of series R-C series with parallel resistor circuit when $R_S = 1000, 100, 10$ and $1\text{m}\Omega$.

By intuitiveness if there is no R_S , input voltage will directly apply across the capacitor. Therefore, the fastest way to charge the capacitor is using a unit step input voltage. Then the capacitor will be charged to steady state immediately and charging time is zero. However, according to specification, there is storage time T_F that defines the time when the capacitor is fully charged. Hence, the unit step input should be delayed for this amount of time to satisfy the storage time constraint. More explicitly, the input voltage function should be

$$v_I(t) = V_F u(t - T_F) \quad (3.43)$$

where $u(t)$ is unit step function.

With the unit step voltage, there will be no current before the storage time. At the storage time (or delay time) there will be huge current flow into the capacitor to charge it up in a very short period of time and the current in a resistor R_P will abruptly change from zero to the steady stage current, V_F/R_P . In mathematical expression the current

should read

$$i_I(t) = \frac{V_F}{R_P} u(t - T_F) + CV_F \delta(t - T_F) \quad (3.44)$$

where $\delta(t - T_F)$ is the Dirac delta function.

Finally, the energy dissipation during charging will be zero because there is no time needed for charging. At the onset of the unit step voltage, the capacitor will be charged immediately. Therefore, if there is no R_S , the best way to charge the capacitor is using the unit step input voltage. There will be no energy dissipation during charging but the current will be infinity at the storage time. These explanations are in accordance with the graphical plots in Fig. 3.11 - Fig. 3.13.

In mathematical approach, let us consider series RC with a parallel resistor in Fig. 3.7 when R_S approaches to zero. We shall determine three things: the lowest energy dissipation, the input voltage function and the input current function.

First, the lowest energy dissipation from Table 3.1 can be written in terms of R_S and R_P explicitly as

$$E_{dis} = \left[\sqrt{\frac{R_S}{R_P} \left(1 + \frac{R_S}{R_P} \right)} \coth \left(\frac{T_F}{C \sqrt{R_S(R_S + R_P)}} \right) + \frac{R_S}{R_P} \right] CV_F^2. \quad (3.45)$$

Taking the limit for R_S approaching zero will give

$$\lim_{R_S \rightarrow 0} E_{dis} = [0 \cdot \coth(\infty) + 0] CV_F^2 = 0. \quad (3.46)$$

Therefore the most energy-efficient input function will provide no dissipated energy.

To find the most energy-efficient input voltage and current functions, we have to deal with discontinuous functions in time domain such as unit step function or Dirac delta function. Laplace transform is the common technique to handle these functions. The input voltage function in Laplace domain can be found by definition of Laplace transform,

$$V_I(s) = L\{v_I(t)\} = \int_0^\infty v_I(t) e^{-st} dt \quad (3.47)$$

and upon rearrangement, the input function in Laplace domain is

$$V_I(s) = \frac{V_F R_S C}{(s/\beta)^2 - 1} \left[\frac{s}{\beta} \text{csch}(\beta T_F) - e^{-sT_F} \left(\frac{s}{\beta} \coth(\beta T_F) + 1 \right) \right] + \frac{V_F/\beta}{(s/\beta)^2 - 1} \left[\text{csch}(\beta T_F) - e^{-sT_F} \left(\coth(\beta T_F) + \frac{s}{\beta} \right) \right] + \frac{V_F}{s} e^{-sT_F} \quad (3.48)$$

where

$$\beta = \frac{1}{\tau} = \frac{1}{\sqrt{R_S(R_S + R_P)}C}. \quad (3.49)$$

Letting R_S approach zero as well as β approach infinity, the input function becomes

$$\begin{aligned} \lim_{\substack{\beta \rightarrow \infty \\ R_S \rightarrow 0}} V_I(s) &= \frac{0}{-1} [0 \cdot \text{csch}(\infty) - e^{-sT_F} (0 \cdot \coth(\infty) + 1)] \\ &\quad + \frac{0}{-1} [\text{csch}(\infty) - e^{-sT_F} (\coth(\infty) + 0)] + \frac{V_F}{s} e^{-sT_F} \\ \lim_{R_S \rightarrow 0} V_I(s) &= 0 \cdot [0 - e^{-sT_F}] + 0 \cdot 0 + \frac{V_F}{s} e^{-sT_F} \\ \lim_{R_S \rightarrow 0} V_I(s) &= \frac{V_F}{s} e^{-sT_F}. \end{aligned} \quad (3.50)$$

Finally, taking inverse Laplace transform of (3.50) gives

$$\lim_{R_S \rightarrow 0} v_I(t) = V_F \left[L^{-1} \left\{ \frac{1}{s} e^{-sT_F} \right\} \right] = V_F u(t - T_F) \quad (3.51)$$

where $u(t)$ is the unit step function.

By following the procedure in the previous, the input current function in Laplace domain is

$$\begin{aligned} I_I(s) = & \frac{(R_S + R_P)CV_F}{R_P[(s/\beta)^2 - 1]} \left[\frac{s}{\beta} \text{csch}(\beta T_F) - e^{-sT_F} \left(\frac{s}{\beta} \coth(\beta T_F) + 1 \right) \right] \\ & + \frac{V_F/\beta}{R_P[(s/\beta)^2 - 1]} \left[\text{csch}(\beta T_F) - e^{-sT_F} \left(\coth(\beta T_F) + \frac{s}{\beta} \right) \right] + \frac{V_F/R_P}{s} e^{-sT_F}. \end{aligned} \quad (3.52)$$

Taking R_S approach zero and β approach infinity, the current function becomes

$$\begin{aligned} \lim_{\substack{\beta \rightarrow \infty \\ R_S \rightarrow 0}} I_I(s) = & \frac{CV_F}{-1} [0 \cdot \text{csch}(\infty) - e^{-sT_F} (0 \cdot \coth(\infty) + 1)] \\ & + \frac{0}{-R_P} [\text{csch}(\infty) - e^{-sT_F} (\coth(\infty) + 0)] + \frac{V_F/R_P}{s} e^{-sT_F} \\ \lim_{R_S \rightarrow 0} I_I(s) = & -CV_F [0 - e^{-sT_F}] + 0 \cdot 0 + \frac{V_F/R_P}{s} e^{-sT_F} \\ \lim_{R_S \rightarrow 0} I_I(s) = & CV_F e^{-sT_F} + \frac{V_F/R_P}{s} e^{-sT_F}. \end{aligned} \quad (3.53)$$

Finally,

$$\begin{aligned} \lim_{R_S \rightarrow 0} i_I(t) = & L^{-1} \left\{ CV_F e^{-sT_F} + \frac{V_F/R_P}{s} e^{-sT_F} \right\} \\ \lim_{R_S \rightarrow 0} i_I(t) = & CV_F \delta(t - T_F) + \frac{V_F}{R_P} u(t - T_F) \end{aligned} \quad (3.54)$$

where $\delta(t)$ is the Dirac delta function.

3.5 Conclusion

In this chapter, Calculus of variations has been applied to derive analytically the source functions that minimize energy dissipation in circuits during transient. In particular, the most energy-efficient input current function and voltage source function for charging series RC circuit have been derived and proved to be a constant function and a raise ramp function respectively. The technique of calculus of variations has been extended for other common first order linear circuits as well, whose most energy-efficient source functions and lowest energy dissipation expressions have been summarized in compact parameters involving speed and energy trade off limits. Numerical results have been illustrated to depict the upper limit of speed-energy trade off and to justify the consistency with the analytical results. For some special circuit topologies that cannot be solved directly using standard calculus of variations, they have been treated separately and explained in graphical, mathematical and intuitive manner.

Chapter 4

The Lowest Input Energy of Passive First Order Circuits in Transient State

4.1 Introduction

In the previous chapter, the most energy-efficient input voltage function of the series RC circuit was proved to be the raised ramp function by using calculus of variations along with canonical topologies of first order circuit. However, the circuits that were explained are not quite general. Therefore, in this chapter, we will generalize the using of calculus of variations with passive first-order circuits for optimization.

Section 4.2 will use a general resistor network with a capacitor for derivation of the most energy-efficient input source functions and the lowest input energy. Then, the lowest input energy will be generalized and stated as corollaries. The most energy-efficient functions and the lowest input energy corollaries will be discussed which are essentials to provide insight into transient duration and circuit responses analysis. The transient duration analysis will be performed for three cases of increasing, decreasing and unchanging total stored energy. All these cases will be analyzed comprehensively in Section 4.3. In ease case, it will be further divided into two sub cases of (un)changing without and with zero crossing. The analysis in Section 4.3 reveals two important circuit behaviors which are speed and energy trade-off and energy flow. It will be shown that the speed and energy trade-off may be violated under certain conditions. The energy flow of the circuit will be discussed thoroughly and a zero-energy transient duration that is defined as a transient duration with total input energy being zero will be derived. Section 4.4 will summarize all the analysis into a single digram called the *transient duration digram*. The digram reveals important relationships between the circuit parameters in the transient duration analysis. Special cases of the lowest input energy for first-order circuits which are infinite open circuit resistance and zero shorted circuit resistance will be explained separately in Section 4.5. I-V characteristic of the most energy-efficient input source functions will be elaborated in Section 4.6. Finally, all concrete mathematical proofs can be found in Appendix.

4.2 The Most Energy-efficient Input Source Functions and the Lowest Input Energy

Typically, passive first order circuits consist of a resistors network with either one capacitor or one inductor [51]. We pick the circuit with a capacitor for explanations. The

Figure 4.1: A resistors network with a capacitor.

Figure 4.2: Two port network.

circuit with an inductor will be discussed later. Passive first order circuit with a capacitor can be simplified as the circuit in Fig. 4.1 where all resistors are collected in a single block that can be seen as a two port network. Although two port network can be described in many different parameters such as impedance parameters (**Z**-parameters) and admittance parameters (**Y**-parameters), useful parameters for our case are transmission parameters or **ABCD**-parameters. The transmission parameters are defined as

$$a_{11} = \left. \frac{v_1}{v_2} \right|_{i_2=0}, \quad a_{12} = \left. \frac{v_1}{i_2} \right|_{v_2=0}, \quad a_{21} = \left. \frac{i_1}{v_2} \right|_{i_2=0}, \quad a_{22} = \left. \frac{i_1}{i_2} \right|_{v_2=0}. \quad (4.1)$$

where v_1, v_2, i_1 and i_2 are corresponding to Fig. 4.2. v_1 is the voltage of the left port, i_1 is the current that enters the left port, v_2 is the voltage of the right port and i_2 is the current that leaves the right port. With the transmission parameters, the circuit in Fig. 4.1 can be expressed as

$$\begin{bmatrix} v_I \\ i_I \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} v_C \\ i_C \end{bmatrix} \quad (4.2)$$

where v_I is an input voltage, i_I is an input current, v_C is the voltage of the capacitor and i_C is the current of the capacitor.

The circuit will be in transient state if the voltage across the capacitor is changing, which is the state that we consider. Assume that the transient state starts at time $t = 0$, ends at time $t = T_F$ and the capacitor voltage changes from V_0 to V_F , boundary conditions of the transient state can be written as

$$v_C(0) = V_0 \quad \text{and} \quad v_C(T_F) = V_F. \quad (4.3)$$

There are infinity ways to change the capacitor voltage from V_0 to V_F . Input voltage function v_I can be a unit step function, a ramp function or any other functions with appropriate parameters that meet the boundary conditions. Different functions cause various energy dissipation and input energy. However, there is one special function that cause the lowest energy dissipation or in other words, the lowest input energy. This function is called the most energy-efficient input source function [78]. It can be derived by using calculus of variations as follows [68, 79].

Firstly, we write an input energy in terms of v_C . Since the input energy is a time integral of an input power $v_I i_I$ and the circuit can be described by the transmission

parameters in (4.2), the input energy can be written as

$$E_I = \int_0^{T_F} v_I i_I dt = \int_0^{T_F} (a_{11}v_C + a_{12}C\dot{v}_C)(a_{21}v_C + a_{22}C\dot{v}_C) dt \quad (4.4)$$

where the capacitor current $i_C = C\dot{v}_C$.

In calculus of variations, this input energy is a functional which means its value depends on a variable function, not a variable value. E_I depends on the function v_C over a period of time $0 < t < T_F$, does not depend on just a value of v_C in a particular time. One function of v_C gives only one value of E_I . For example, if v_C is a linear function, E_I will be some value (a single number) and if v_C is a parabola function, E_I will be some another value. Thus, there could be a function of v_C that gives the minimum value of E_I and actually it does. This function v_C can be found by solving the Euler-Lagrange's differential equation,

$$\frac{\partial \mathcal{L}}{\partial v_C} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{v}_C} \right) = 0 \quad (4.5)$$

where $\mathcal{L}$ is called the Lagrangian function which is a function that is inside the integral.

According to (4.4), the Lagrangian function is

$$\mathcal{L} = (a_{11}v_C + a_{12}C\dot{v}_C)(a_{21}v_C + a_{22}C\dot{v}_C). \quad (4.6)$$

By substituting (4.6) into (4.5) and rearranging, the differential equation becomes

$$\ddot{v}_C - \frac{a_{11}a_{21}}{a_{12}a_{22}C^2}v_C = 0. \quad (4.7)$$

Notice that there is no $\dot{v}_C$ in the differential equation since they vanish when we arrange terms. Details can be found in Appendix B.3. Moreover, the differential equation can be simplified further by introducing new terms called open circuit resistance seen by output R_{OC} and short circuit resistance seen by output R_{SC} . Fig. 4.3 illustrates the resistances. They can be written in terms of **Z**-parameters and invert hybrid parameters or **G**-parameters. Additionally, the connections between **ABCD**-parameters with **Z**-parameters and **G**-parameters are known and R_{OC} and R_{SC} can be defined as

$$R_{OC} \equiv \left. \frac{v_2}{i_2} \right|_{i_1=0} = z_{22} = \frac{a_{22}}{a_{21}} \quad \text{and} \quad R_{SC} \equiv \left. \frac{v_2}{i_2} \right|_{v_1=0} = g_{22} = \frac{a_{12}}{a_{11}}. \quad (4.8)$$

Therefore, the differential equation is

$$\ddot{v}_C - \frac{v_C}{R_{OC}R_{SC}C^2} = 0. \quad (4.9)$$

(a) Open circuit resistance (R_{OC}) seen by the output. (b) Short circuit resistance (R_{SC}) seen by the output.

Figure 4.3: Circuit perspective of open circuit resistance (R_{OC}) and short circuit resistance (R_{SC}) seen by the output.

The differential equation in (4.9) is an ordinary differential equation and it has various ways to solve. A particular way of solving and rearranging terms is written in Appendix B.4. Although the solution can be expressed in time domain nicely, we prefer to express it in new terms which we called the *normalized time*. We define the normalized time x as

$$x(t) \equiv \frac{t - T_F/2}{\tau} \quad (4.10)$$

where

$$\tau \equiv \sqrt{R_{OC}R_{SC}}C = \sqrt{\frac{a_{22}a_{12}}{a_{21}a_{11}}}C \quad (4.11)$$

and the solution of (4.9) can be expressed with the normalized time as

$$v_C(t) = \frac{V_F \sinh(x + X_F) - V_0 \sinh(x - X_F)}{\sinh 2X_F} \quad (4.12)$$

where X_F is another parameter defined as

$$X_F \equiv x(T_F) = -x(0) = \frac{T_F}{2\tau}. \quad (4.13)$$

The solution in (4.12) is the function of capacitor voltage that causes the lowest energy dissipation when it change from V_0 to V_F in a duration of $0 < t < T_F$, or in a duration of the normalized time $-X_F < x < X_F$. This is not input functions. However, since we know v_C , we can determine either input voltage or input current by applying routine circuit analysis. By substituting (4.12) into (4.2), the input voltage and input current can be calculated as

$$v_I(t) = \frac{a_{11}}{k \sinh 2X_F} (V_F (k \sinh(x + X_F) + \cosh(x + X_F)) - V_0 (k \sinh(x - X_F) + \cosh(x - X_F))) \quad (4.14)$$

and

$$i_I(t) = \frac{a_{22}}{R_{OC} \sinh 2X_F} (V_F (\sinh(x + X_F) + k \cosh(x + X_F)) - V_0 (\sinh(x - X_F) + k \cosh(x - X_F))) \quad (4.15)$$

where

$$k \equiv \sqrt{\frac{R_{OC}}{R_{SC}}} = \sqrt{\frac{a_{22}a_{11}}{a_{21}a_{12}}}. \quad (4.16)$$

Determining these functions can be found in Appendix B.5.

The input voltage and input current function in (4.14) and (4.15) are the most energy-efficient input source functions. If these input functions are applied to the circuit, it will cause the lowest energy dissipation and of course, the lowest input energy. By substituting the functions into (4.4), the lowest input energy can be determined. The integral can be done numerically. However, there is an analytical result which is more useful for further behavior circuit analysis which is

$$E_I = \frac{C(V_F^2 - V_0^2)}{2(k^2 - 1)} \cdot \left(k + \frac{\tanh X_F}{\gamma} \right) \left(k + \frac{\gamma}{\tanh X_F} \right) \quad (4.17)$$

where γ is defined as

$$\gamma \equiv \frac{V_F - V_0}{V_F + V_0}. \quad (4.18)$$

Derivation of the lowest input energy can be found in Appendix B.6. Finally, we can apply (4.13) into (4.17) and state a corollary as

Corollary 1 For a resistors network with a capacitor C as shown in Fig 4.1, changing the capacitor voltage from V_0 to V_F within a duration of time T_F requires minimum input energy as

$$E_I = \frac{C(V_F^2 - V_0^2)}{2(k^2 - 1)} \cdot \left(k + \frac{\tanh(T_F/2\tau)}{\gamma} \right) \left(k + \frac{\gamma}{\tanh(T_F/2\tau)} \right) \quad (4.19)$$

where all parameters are defined in (4.8), (4.11), (4.16) and (5.34).

Recall that this corollary is derived by applying calculus of variations theory to first order circuit, specifically the circuit with a capacitor.

Figure 4.4: A resistors network with an inductor.

Nevertheless, there is another type of first order circuit which is the circuit with an inductor as shown in Fig. 4.4 that the theory of calculus of variations can be applied. Procedures of deriving the lowest input energy are the same as above when we derived the lowest input energy for capacitive circuits. Starting with writing the input energy in the integral form, indicating the Lagrangian function, applying the Euler-Lagrange's differential equation, solving the most energy-efficient input source functions and finally calculating the lowest input energy. Slight differences from capacitive circuit are the parameters τ and γ . Since the circuit will be in transient state if an inductor current is changing (voltage is changing for a capacitor), γ must be defined as

$$\gamma \equiv \frac{I_F - I_0}{I_F + I_0} \quad (4.20)$$

where I_F and I_0 are the inductor current at time $t = T_F$ and $t = 0$ respectively. The parameter τ also has to be changed which is

$$\tau \equiv \frac{L}{\sqrt{R_{OC}R_{SC}}}. \quad (4.21)$$

Then, another corollary can be stated as

Corollary 2 For a resistors network with an inductor L as shown in Fig 4.4, changing the inductor current from I_0 to I_F within a duration of time T_F requires minimum input energy as

$$E_I = \frac{L(I_F^2 - I_0^2)}{2(k^2 - 1)} \cdot \left(k + \frac{\tanh(T_F/2\tau)}{\gamma} \right) \left(k + \frac{\gamma}{\tanh(T_F/2\tau)} \right) \quad (4.22)$$

where all parameters are defined in (4.8), (4.16), (4.20) and (4.21).

Note that the corollaries have been proved rigorously by using calculus of variations theorem. The theorem guarantees that the lowest input energy is absolutely the lowest, without doubt. Therefore, the corollaries above are facts, not just results from an ordinary optimization.

Using of the corollary is illustrated as follows. Let us consider the circuit in Fig. 4.5 which is a circuit model of a supercapacitor [15,16,78]. If we want to change the capacitor

Figure 4.5: Circuit model of a supercapacitor.

voltage, for example, from $V_0 = 1\text{V}$ to $V_F = 2\text{V}$ within a duration of $T_F = 15\mu\text{s}$, we can determine the lowest input energy that requires for the transient state by using (4.19). Assume that $R_P = 1\text{M}\Omega$, $R_S = 10\text{k}\Omega$ and $C = 100\text{pF}$, the essential parameters that need to be determined first are the open circuit resistance and the short circuit resistance in (4.8),

$$R_{OC} = R_S + R_P, \quad R_{SC} = R_S.$$

Then, using (4.11), (4.16), (5.34) and (4.19), we get $E_I = 189.6\text{pJ}$. Thus, this is the lowest input energy that needs to deliver to the circuit for changing the voltage capacitor (corresponding to the given parameters) and a method of delivering is applying the most energy-efficient input source function in (4.14) or (4.15).

4.3 Analysis of Transient Duration and Stored Energy

Despite of using the corollaries calculate the lowest input energy directly, they reveal five important parameters, R_{OC} , R_{SC} , τ , k and γ . Without defining these parameters, the corollaries cannot be written compactly as we have already seen. Therefore, any special cases of parameters such as $k \rightarrow \infty$, $\gamma = 0$, $R_{SC} = 0$, etc. can be separately analyzed and studied for circuit behaviors.

It is commonly known that the stored energy in a capacitor is $CV^2/2$ and the stored energy in an inductor is $LI^2/2$. Therefore, if $V_F^2 \neq V_0^2$ or $I_F^2 \neq I_0^2$, the lowest input energy in corollary 1 and corollary 2 can be rewritten as a normalized lowest input energy u ,

$$u \equiv \frac{E_I}{\Delta E_{\text{stored}}} = \frac{1}{k^2 - 1} \cdot \left(k + \frac{\tanh(T_F/2\tau)}{\gamma} \right) \left(k + \frac{\gamma}{\tanh(T_F/2\tau)} \right) \quad (4.23)$$

where ΔE_{stored} is the total change of stored energy. According to the energy conservation, the input energy is a sum of the total change of stored energy and energy dissipation. Hence, we often wish that the normalized lowest input energy to be close to one as much as possible because this means minimum energy dissipation.

Intuitively, a circuit would subject to certain speed and energy trade-off and the minimum normalized lowest input energy should occur at $T_F \rightarrow \infty$ which means circuit response change is very slow. Under this condition, the normalized lowest input energy is

$$\lim_{T_F \rightarrow \infty} u = \frac{1}{k^2 - 1} \cdot (k + 1/\gamma)(k + \gamma). \quad (4.24)$$

The normalized lowest input energy in (4.24) is expected to be the minimum value because of the speed and energy trade-off. However, the speed and energy trade-off does not always hold. There are certain cases that fast response change needs less input energy than slow change. Moreover, an optimum transient duration can be found which is the transient duration that gives the minimum lowest input energy. All these will be analyzed in detail shortly. The analysis is performed for three cases of increasing total stored energy, decreasing total stored energy and unchanging total stored energy.

4.3.1 Increasing total stored energy

For a capacitive circuit, increasing total stored energy means that $V_F^2 > V_0^2$. The analysis can be further divided into two sub cases. The first case is where V_F and V_0 have the same sign. In this case, a capacitor voltage changes from V_0 to V_F without zero crossing. On the other hand, the second case is where V_F and V_0 have different signs. The capacitor voltage changes from the initial voltage V_0 to the final voltage V_F with zero crossing. These two cases will be analyzed separately since they have different circuit responses.

Increasing without zero crossing

This is the case where V_F and V_0 have the same sign and $|V_F| > |V_0|$. By the definition of γ in (5.34), it is equivalent to $0 < \gamma < 1$ and the final stored energy is more than the initial one. The normalized lowest input energy in (4.23) versus normalized time (T_F/τ) is plotted in Fig. 4.6 for $\gamma = 0.2$ and $k = 3, 4, 5, 7, 100$.

Figure 4.6: The normalized lowest input energy versus normalized time for $\gamma = 0.2$ and $k = 3, 4, 5, 7, 100$.

According to the figure, there is a minimum point and the transient duration at this point can be called as optimum transient duration. The optimum transient duration can be determined by differentiating E_I with respect to T_F and set it to zero. The result is

$$\frac{T_{F(\text{opt.})}}{\tau} = \ln \left(\frac{1 + |\gamma|}{1 - |\gamma|} \right). \quad (4.25)$$

For $0 < \gamma < 1$, the optimum transient duration can be simplified to

$$\frac{T_{F(\text{opt.})}}{\tau} = \ln \left(\frac{V_F}{V_0} \right) \quad (4.26)$$

and the optimum normalized lowest input energy is

$$u_{(\text{opt.})} = \frac{k + 1}{k - 1}. \quad (4.27)$$

Proof can be found in Appendix B.7.

The normalized lowest input energy in Fig. 4.6 reveals detailed speed and energy trade-off behavior. For transient duration that is shorter than the optimum transient duration, the speed and energy trade-off does hold. Slow response change requires less

input energy than fast change. However, for the transient duration that is longer than the optimum transient duration, the speed and energy trade-off does not hold. Slow response change actually requires more input energy than fast change. For very slow change or mathematically $T_F \rightarrow \infty$, the normalized lowest input energy diverges to a value that is governed by (4.24). Furthermore, the normalized lowest input energy will decrease if k increases but it is never less than one because the energy conservation is always true and the lowest input energy cannot be lower than the change of stored energy.

Recall that this is the case of increasing total stored energy or $0 < \gamma < 1$, the optimum transient duration and the optimum normalized lowest input energy in (4.26) and (4.27) are applicable for this case only. However, note that they are not applicable for all other cases, especially the optimum transient duration. If the signs of V_F and V_0 are different (zero crossing), the formula in (4.26) will not be applicable because the number inside the natural logarithm function is negative. If $V_F < V_0$, the optimum transient duration will be negative which is impossible and if $V_F = V_0$, the optimum transient duration will be zero which means that there is no transient duration at all. Based on all these arguments, we can see that the optimum transient duration formula itself implies that it is only applicable for increasing without zero crossing and the optimum normalized lowest input energy in (4.27) is applicable if and only if the optimum transient duration in (4.26) is applicable.

For transient responses illustrations, let us consider the circuit of Fig. 4.5 when $R_P = 1\text{M}\Omega$, $R_S = 10\text{k}\Omega$ and $C = 100\text{pF}$. Transient responses of the circuit are shown in Fig. 4.7 and Fig. 4.8 with the boundaries conditions of $V_0 = 1\text{ V}$, $V_F = 2\text{ V}$ and $T_F/\tau = 0.2, 0.6931, 2, 10$. Note that $T_F/\tau = 0.6931 = \ln 2$ is the optimum transient duration.

Increasing with zero crossing

This is the case where V_F and V_0 have different signs and $|V_F| > |V_0|$. By the definition of γ in (5.34), it is equivalent to $1 \leq \gamma < \infty$ and the final stored energy is more than the initial one. The normalized lowest input energy in (4.23) versus normalized time (T_F/τ) is plotted in Fig. 4.9 for $\gamma = 5$ and $k = 2, 3, 5, 10, 100$.

According to the figure, there is no optimum transient duration and optimum normalized lowest input energy. In this case, the speed and energy trade-off does hold. Long transient duration needs less input energy than short transient duration and the minimum lowest input energy occurs at $T_F \rightarrow \infty$ which is governed by (4.24). Transient responses of the circuit are shown in Fig. 4.10 and Fig. 4.11 with the boundaries conditions of $V_0 = -1\text{ V}$, $V_F = 2\text{ V}$ and $T_F/\tau = 0.2, 0.6931, 2, 10$.

4.3.2 Decreasing total stored energy

Similar to the above, the analysis for the case of decreasing total stored energy can be further divided into two sub cases which are decreasing without zero crossing and decreasing with zero crossing. We shall see that decreasing without zero crossing has the optimum transient duration and the optimum input energy like the case of increasing without zero crossing. However, in the decreasing case, there is another issue that needs to be considered which is energy flow.

For increasing total stored energy of a capacitive circuit, energy can flow only in one direction that is from a source to a capacitor, while some energy dissipates through a resistors' network. On the other hand, when the total stored energy decreases, the energy dissipates through a resistors' network and some may be transferred back to the source. Note however that this is not always the case. Under certain conditions, the source still needs to deliver energy to the circuit although the total stored energy of

(a) $T_F = 0.2\tau$

(b) $T_F = 0.6931\tau$

(c) $T_F = 2\tau$

(d) $T_F = 10\tau$

Figure 4.7: Voltage transient responses for $V_0 = 1$ V, $V_F = 2$ V and $T_F/\tau = 0.2, 0.6931, 2, 10$

(a) $T_F = 0.2\tau$

(b) $T_F = 0.6931\tau$

(c) $T_F = 2\tau$

(d) $T_F = 10\tau$

Figure 4.8: Current transient responses for $V_0 = 1$ V, $V_F = 2$ V and $T_F/\tau = 0.2, 0.6931, 2, 10$

Figure 4.9: The normalized lowest input energy versus normalized time for $\gamma = 5$ and $k = 2, 3, 5, 10, 100$.

the capacitor is reduced, so the energy from both the source and the capacitor dissipate through the resistors' network. There is also another case that the source neither receives nor delivers the energy. The energy dissipation through the resistors' network in this case is perfectly equal to the reduced energy of the capacitor. Nevertheless, while the circuit is in transient state, the source actually transfers some energy but in the end, it delivers and receives the energy equally. Therefore, it neither receives nor delivers net energy in total. Fig. 4.12 depicts energy flows of all three possible cases while the stored energy is decreasing.

Generally, energy flow depends on source functions and also constraints of V_0 , V_F and T_F . The most energy-efficient source function and the lowest input energy have been derived. For the given constraints of V_0 , V_F , and T_F , we can calculate the lowest input energy and determine the energy flow by using the corollary 1. If the lowest energy is negative, the source receives energy from the circuit as shown in Fig. 4.12a. If the lowest energy is positive, the energy must be transferred from the source to the circuit as shown in Fig. 4.12b. If the lowest energy is zero, the source neither receives nor delivers the energy as shown in Fig. 4.12c. Consequently, because the change of total stored energy ΔE_{stored} is always negative, the normalized lowest input energy is positive for the case of Fig. 4.12a, negative for the case of Fig. 4.12b and zero for the case of Fig. 4.12c.

Decreasing without zero crossing

This is the case where V_F and V_0 have the same sign and $|V_F| < |V_0|$. By the definition of γ in (5.34), it is equivalent to $-1 < \gamma < 0$ and the final stored energy is less than the initial one. The normalized lowest input energy in (4.23) versus normalized time (T_F/τ) is plotted in Fig. 4.13 for $\gamma = -0.2$ and $k = 3, 4, 5, 7, 100$.

According to the figure, there are optimum transient duration and optimum lowest input energy like the case of increasing without zero crossing. The optimum transient duration in terms of γ can be found in (4.25) and it can be simplified to

$$\frac{T_{F(\text{opt.})}}{\tau} = \ln \left(\frac{V_0}{V_F} \right). \quad (4.28)$$

Subsequently, the optimum normalized lowest input energy becomes

$$u_{(\text{opt.})} = \frac{k-1}{k+1}. \quad (4.29)$$

(a) $T_F = 0.2\tau$

(b) $T_F = 0.6931\tau$

(c) $T_F = 2\tau$

(d) $T_F = 10\tau$

Figure 4.10: Voltage transient responses for $V_0 = -1$ V, $V_F = 2$ V and $T_F/\tau = 0.2, 0.6931, 2, 10$

(a) $T_F = 0.2\tau$

(b) $T_F = 0.6931\tau$

(c) $T_F = 2\tau$

(d) $T_F = 10\tau$

Figure 4.11: Current transient responses for $V_0 = -1$ V, $V_F = 2$ V and $T_F/\tau = 0.2, 0.6931, 2, 10$

Figure 4.12: energy flows of all three possible cases while the total stored energy is decreasing.

Figure 4.13: The normalized lowest input energy versus normalized time for $\gamma = -0.2$ and $k = 3, 4, 5, 7, 100$.

Since k is always greater than one (see Appendix), the normalized lowest input energy is always greater than one. This means that the source always receives some energy at the optimum transient duration (Fig. 4.12a).

The normalized lowest input energy in Fig. 4.13 shows that the speed and energy trade-off does not hold as we have explained in the previous section. Moreover, if the transient duration is not the optimum, the normalized lowest input energy can be positive or negative. For $k = 7$ in Fig. 4.13, small values of transient duration give negative normalized lowest input energy and the source has to deliver some energy (Fig. 4.12b). For $k = 3$ or $k = 4$, both small and large values of transient duration give negative values of the normalized lowest input energy. Hence, the condition for receiving and delivering energy depends on not only the transient duration, but also the circuit parameters k and γ .

We shall call the transient durations with the total input energy being zero as *zero-energy* transient durations. When the circuit operates at zero-energy transient durations, the source neither delivers nor receives energy as depicted in Fig. 4.12c. Moreover, zero-energy transient durations distinguish between the durations for receiving and delivering energy of the source. From Fig. 4.13, the circuit can have one zero-energy transient duration or two zero-energy transient durations. It depends on the value of k and γ . Mathematically, the condition for having only one zero-energy transient duration is

$$k \geq -1/\gamma \quad (4.30)$$

and the zero-energy transient duration T_Z is given by

$$\frac{T_Z}{\tau} = \ln \left(\frac{k - \gamma}{k + \gamma} \right). \quad (4.31)$$

However, if the condition of having one zero-energy transient duration in (4.30) is not satisfied, there will be two zero-energy transient durations. One is the same as (4.31) and the other one is

$$\frac{T_Z}{\tau} = \ln \left(\frac{1 - k\gamma}{1 + k\gamma} \right). \quad (4.32)$$

They are easily derived by setting (4.23) to zero and solving for $T_F = T_Z$. The detail can be found in Appendix B.8.

For $\gamma = -0.2$, the condition for having only one zero-energy transient duration in (4.30) is satisfied when $k \geq 5$ which is consistent with Fig. 4.13. For $k = 3$ and 4, there are two zero-energy transient durations. For each $k = 5, 7$ and 100, there is only one transient duration. In fact for $k = 5$, the normalized lowest input energy approaches zero for a very long transient duration. Furthermore, the normalized lowest input energy will increase if k increases but it will never be greater than one because we cannot receive the energy more than the reduced stored energy.

Transient responses of the circuit are shown in Fig. 4.14 and Fig. 4.15 with the boundaries conditions of $V_0 = 2$ V, $V_F = 1$ V and $T_F/\tau = 0.2, 0.6931, 2, 10$. Note that $T_F/\tau = 0.6931 = \ln 2$ is the optimum transient duration.

Decreasing with zero crossing

This is the case where V_F and V_0 have different signs and $|V_F| < |V_0|$. By the definition of γ in (5.34), it is equivalent to $\infty < \gamma \leq 1$ and the final stored energy is less than the initial one. The normalized lowest input energy in (4.23) versus normalized time (T_F/τ) is plotted in Fig. 4.16 for $\gamma = -5$ and $k = 2, 3, 5, 10, 100$.

According to the figure, there is no optimum transient duration and no optimum normalized lowest input energy. The speed and energy trade-off does hold. The minimum lowest input energy occurs at $T_F \rightarrow \infty$ which is governed by (4.24). Nevertheless, the circuit can have one or no zero-energy transient duration. It depends on the circuit parameters. Since the speed and energy trade-off holds, transient duration that is longer than zero-energy transient duration gives positive value of the normalized lowest input energy as shown in Fig. 4.16. If the circuit has no zero-energy transient duration, the normalized lowest input energy will be negative for all transient durations. In the latter case, source must deliver some energy to the circuit for decreasing total stored energy. A condition for having one zero-energy transient duration is

$$k > -\gamma \quad (4.33)$$

and the zero-energy transient duration is the same as (4.31). It is important to know that even though we reduce the stored energy very slowly, we have to deliver some energy to the circuit if the condition in (4.33) is not satisfied. Transient responses of the circuit are shown in Fig. 4.17 and Fig. 4.18 with the boundaries conditions of $V_0 = 2$ V, $V_F = -1$ V and $T_F/\tau = 0.2, 0.6931, 2, 10$.

4.3.3 Unchanging total stored energy

The remaining case is where the circuit has neither increasing nor decreasing total stored energy. The final stored energy is equal to the initial one. Mathematically, $|V_F| = |V_0|$. According to the corollary 1, the input energy cannot be calculated directly because

(a) $T_F = 0.2\tau$

(b) $T_F = 0.6931\tau$

(c) $T_F = 2\tau$

(d) $T_F = 10\tau$

Figure 4.14: Voltage transient responses for $V_0 = 2$ V, $V_F = 1$ V and $T_F/\tau = 0.2, 0.6931, 2, 10$

(a) $T_F = 0.2\tau$

(b) $T_F = 0.6931\tau$

(c) $T_F = 2\tau$

(d) $T_F = 10\tau$

Figure 4.15: Current transient responses for $V_0 = 2$ V, $V_F = 1$ V and $T_F/\tau = 0.2, 0.6931, 2, 10$

Figure 4.16: The normalized lowest input energy versus normalized time for $\gamma = -5$ and $k = 2, 3, 5, 10, 100$.

$V_F^2 - V_0^2 = 0$ and γ can be 0 or ∞ . It causes the input energy to be in indeterminate (0/0) form. Therefore, the corollary must be rearranged by expanding γ and factoring $V_F^2 - V_0^2$ as

$$E_I = \frac{C}{2(k^2 - 1)} (k(V_F - V_0) + \tanh(T_F/2\tau)(V_F + V_0)) \cdot \left(k(V_F + V_0) + \frac{V_F - V_0}{\tanh(T_F/2\tau)} \right). \quad (4.34)$$

The analysis can be further divided into two sub cases which are $V_F = V_0$ and $V_F = -V_0$. In other words, they are unchanging without zero crossing and unchanging with zero crossing respectively.

Unchanging without zero crossing

In this case, $V_F = V_0$ or $\gamma = 0$ and the lowest input energy in (4.34) becomes

$$\frac{E_I}{1/2CV_0^2} = u_{un} = \frac{4k}{k^2 - 1} \tanh(T_F/2\tau). \quad (4.35)$$

where u_{un} is the normalized lowest input energy for the unchanging case. Notice that u_{un} is defined differently from u in (4.23) because the total change of stored energy is zero. However, we can write the lowest input energy in terms of the initial (or final) stored energy instead as in (4.35).

The normalized lowest input energy is directly proportional to hyperbolic tangent function of transient duration. Since hyperbolic tangent is an increasing function, the lowest input energy is also increasing. Therefore, the speed and energy trade-off is violated.

Nevertheless, because the final voltage and the initial voltage are the same, it is not the same as the previous cases where the capacitor voltage changes. In this case, we try not to change the capacitor voltage within the duration of time T_F . Therefore, longer duration of time requires more energy since it has more energy loss. The speed and energy trade-off is violated because T_F does not refer to speed of changing anymore, but it refers to the duration that we try not to change the total stored energy instead.

(a) $T_F = 0.2\tau$

(b) $T_F = 0.6931\tau$

(c) $T_F = 2\tau$

(d) $T_F = 10\tau$

Figure 4.17: Voltage transient responses for $V_0 = 2$ V, $V_F = -1$ V and $T_F/\tau = 0.2, 0.6931, 2, 10$

(a) $T_F = 0.2\tau$

(b) $T_F = 0.6931\tau$

(c) $T_F = 2\tau$

(d) $T_F = 10\tau$

Figure 4.18: Current transient responses for $V_0 = 2$ V, $V_F = -1$ V and $T_F/\tau = 0.2, 0.6931, 2, 10$

To further illustrate the arguments above, assume that we want the capacitor voltage at $t = 0$ and $t = T_F$ both to be V_0 . One way to do that is by keeping the capacitor voltage to be a constant at V_0 . From (4.2), since $i_C = 0$, the input voltage must be $v_I = a_{11}V_0$ and the input current must be $i_I = a_{21}V_0$. Hence, the input energy is $v_I i_I T_F = a_{11}a_{21}V_0^2 T_F$. After rearranging, we get

$$\frac{E_I}{1/2CV_0^2} = \frac{2k}{k^2 - 1} \cdot \frac{T_F}{\tau}. \quad (4.36)$$

Obviously, the input energy of constant voltage is proportional to the duration. A longer duration causes more input energy.

Recall that the voltage conditions are merely $v_C(0) = V_0$ and $v_C(T_F) = V_0$. The voltage between the time 0 and T_F is not necessary to be V_0 . By using the most energy-efficient input source function, the lowest input energy is (4.35). Fig. 4.19 shows the lowest input energy and the input energy of constant voltage for $k = 5$. The lowest input energy is always lower as expected. Transient responses of the circuit are shown in Fig. 4.20 and Fig. 4.21 with the boundaries conditions of $V_0 = 1$ V, $V_F = 1$ V and $T_F/\tau = 0.2, 0.6931, 2, 10$.

Figure 4.19: The normalized lowest input energy and the normalized input energy of constant voltage source versus normalized time for $k = 5$ and $\gamma = 0$.

Unchanging with zero crossing

The last case is unchanging with zero crossing where $V_F = -V_0$. It makes $\gamma = \infty$ if $V_0 > 0$ and $\gamma = -\infty$ if $V_0 < 0$. The lowest input energy in (4.34) becomes

$$\frac{E_I}{1/2CV_0^2} = u_{un} = \frac{4k}{k^2 - 1} \coth(T_F/2\tau). \quad (4.37)$$

It is directly proportional to hyperbolic cotangent function of transient duration. Since hyperbolic cotangent is a decreasing function, the lowest input energy is also decreasing. Therefore, the speed and energy trade-off does hold. Since this is unchanging total stored energy, but the capacitor voltage sign is changed, we can say that the lowest input energy in this case is the lowest input energy for changing the sign of the voltage. The lowest input energy approaches zero when T_F approaches infinity. Transient responses of the circuit are shown in Fig. 4.22 and Fig. 4.23 with the boundaries conditions of $V_0 = -1$ V, $V_F = 1$ V and $T_F/\tau = 0.2, 0.6931, 2, 10$.

(a) $T_F = 0.2\tau$

(b) $T_F = 0.6931\tau$

(c) $T_F = 2\tau$

(d) $T_F = 10\tau$

Figure 4.20: Voltage transient responses for $V_0 = 1$ V, $V_F = 1$ V and $T_F/\tau = 0.2, 0.6931, 2, 10$

(a) $T_F = 0.2\tau$

(b) $T_F = 0.6931\tau$

(c) $T_F = 2\tau$

(d) $T_F = 10\tau$

Figure 4.21: Current transient responses for $V_0 = 1$ V, $V_F = 1$ V and $T_F/\tau = 0.2, 0.6931, 2, 10$

(a) $T_F = 0.2\tau$

(b) $T_F = 0.6931\tau$

(c) $T_F = 2\tau$

(d) $T_F = 10\tau$

Figure 4.22: Voltage transient responses for $V_0 = -1$ V, $V_F = 1$ V and $T_F/\tau = 0.2, 0.6931, 2, 10$

(a) $T_F = 0.2\tau$

(b) $T_F = 0.6931\tau$

(c) $T_F = 2\tau$

(d) $T_F = 10\tau$

Figure 4.23: Current transient responses for $V_0 = -1$ V, $V_F = 1$ V and $T_F/\tau = 0.2, 0.6931, 2, 10$

4.4 Transient Duration Diagram

The analysis of transient duration and stored energy has been performed comprehensively in the previous section. All the analysis can be summarized in a single diagram called *transient duration diagram*. Such diagram describes all cases of increasing, decreasing and unchanging total stored energy in terms of k and γ . The transient duration diagram is shown in Fig. 4.24. There are nine different regions in the diagram as follows.

Figure 4.24: Transient duration diagram.

① $\gamma = \infty$

- Unchanging total stored energy with zero crossing and the initial voltage is positive, or mathematically, $V_F = -V_0$ and $V_0 > 0$.
- The normalized lowest input energy is simplified to (4.37) and always greater than one.
- No optimum transient duration, the speed and energy trade-off does hold.

② $1 \leq \gamma < \infty$

- Increasing total stored energy with zero crossing, $|V_F| > |V_0|$ and $V_F V_0 \leq 0$.
- The normalized lowest input energy is always greater than one, $u > 1$.
- No optimum transient duration, the speed and energy trade-off does hold.

③ $0 < \gamma < 1$

- Increasing total stored energy without zero crossing, $|V_F| > |V_0|$ and $V_F V_0 > 0$.
- The normalized lowest input energy is always greater than one, $u > 1$.
- The optimum transient duration is (4.26) and the optimum normalized lowest input energy is (4.27). The speed and energy trade-off is violated.

④ $\gamma = 0$

- Unchanging total stored energy without zero crossing, or mathematically, $V_F = V_0$.
- The normalized lowest input energy is simplified to (4.35) and $0 \leq u < 1$.
- The special case where we try not to change the capacitor voltage.

⑤ $-1 < \gamma < 0$ and $k < -1/\gamma$

- Decreasing total stored energy without zero crossing, $|V_F| < |V_0|$ and $V_F V_0 > 0$.
- The normalized lowest input energy is always lower than one, $u < 1$.
- The optimum transient duration is (4.28) and the optimum normalized lowest input energy is (4.29). The speed and energy trade-off is violated.
- There are two zero-energy transient durations which are (4.31) and (4.32).

⑥ $-1 < \gamma < 0$ and $k \geq -1/\gamma$

- Decreasing total stored energy without zero crossing, $|V_F| < |V_0|$ and $V_F V_0 > 0$.
- The normalized lowest input energy is always lower than one, $u < 1$.
- The optimum transient duration is (4.28) and the optimum normalized lowest input energy is (4.29). The speed and energy trade-off is violated.
- There is one zero-energy transient duration which is (4.31).

⑦ $-\infty < \gamma \leq -1$ and $k > -\gamma$

- Decreasing total stored energy with zero crossing, $|V_F| < |V_0|$ and $V_F V_0 \leq 0$.
- The normalized lowest input energy is always lower than 1, $u < 1$.
- No optimum transient duration, the speed and energy trade-off does hold.
- There is one zero-energy transient duration which is (4.31).

⑧ $-\infty < \gamma \leq -1$ and $k \leq -\gamma$

- Decreasing total stored energy with zero crossing, $|V_F| < |V_0|$ and $V_F V_0 \leq 0$.
- The normalized lowest input energy is always lower than one, $u < 1$.
- No optimum transient duration, the speed and energy trade-off does hold.
- There is no zero-energy transient duration.

⑨ $\gamma = -\infty$

- Unchanging total stored energy with zero crossing and the initial voltage is negative, or mathematically, $V_F = -V_0$ and $V_0 < 0$.
- The normalized lowest input energy is simplified to (4.37) and always greater than one.
- No optimum transient duration, the speed and energy trade-off does hold.

These regions are summarized in Table 4.1.

Table 4.1: Regions of transient duration diagram

Region	Section	γ	k	V_F and V_0	Normalized lowest input energy (4.23)	Optimum transient duration	Zero-energy transient duration
1	II C 2	$\gamma = \infty$	$k < \infty$	$V_F = -V_0$ and $V_0 > 0$	$u = \infty, u_{un} = \frac{4k}{k^2-1} \coth(T_F/2\tau)$	None	None
2	II A 2	$1 \leq \gamma < \infty$	$k < \infty$	$ V_F > V_0 $ and $V_F V_0 \leq 0$	$u > 1$	None	None
3	II A 1	$0 < \gamma < 1$	$k < \infty$	$ V_F > V_0 $ and $V_F V_0 > 0$	$u > 1$	$T_{F(\text{opt.})} = \tau \ln(V_F/V_0),$ $u_{(\text{opt.})} = (k+1)/(k-1)$	None
4	II C 1	$\gamma = 0$	$k < \infty$	$V_F = V_0$	$u = \pm\infty, u_{un} = \frac{4k}{k^2-1} \tanh(T_F/2\tau)$	None	None
5	II B 1	$-1 < \gamma < 0$	$k < -1/\gamma$	$ V_F < V_0 $ and $V_F V_0 > 0$	$u < 1$	$T_{F(\text{opt.})} = \tau \ln(V_0/V_F),$ $u_{(\text{opt.})} = (k-1)/(k+1)$	$T_Z = \tau \ln\left(\frac{k-\gamma}{k+\gamma}\right),$ $T_Z = \tau \ln\left(\frac{1-k\gamma}{1+k\gamma}\right)$
6	II B 1	$-1 < \gamma < 0$	$k \geq -1/\gamma$	$ V_F < V_0 $ and $V_F V_0 > 0$	$u < 1$	$T_{F(\text{opt.})} = \tau \ln(V_0/V_F),$ $u_{(\text{opt.})} = (k-1)/(k+1)$	$T_Z = \tau \ln\left(\frac{k-\gamma}{k+\gamma}\right)$
7	II B 2	$-\infty < \gamma \leq -1$	$k > -\gamma$	$ V_F < V_0 $ and $V_F V_0 \leq 0$	$u < 1$	None	$T_Z = \tau \ln\left(\frac{k-\gamma}{k+\gamma}\right)$
8	II B 2	$-\infty < \gamma \leq -1$	$k \leq -\gamma$	$ V_F < V_0 $ and $V_F V_0 \leq 0$	$u < 1$	None	None
9	II C 2	$\gamma = -\infty$	$k < \infty$	$V_F = -V_0$ and $V_0 < 0$	$u = -\infty, u_{un} = \frac{4k}{k^2-1} \coth(T_F/2\tau)$	None	None

The transient duration diagram also shows the relationships between k , γ and T_F and it can be applied to analyze circuits in other aspects, not limited to transient duration analysis. For example, if k and T_F are fixed, one can find the optimum γ by using (4.25). Or, if γ and T_F are fixed, one can find the k that makes T_F to be T_Z by using (4.31) and (4.32). It can be used in various ways, depending on applications or problems.

Although the transient duration analysis above has been performed only for capacitive circuit, it is applicable for inductive circuit too. The analysis of inductive circuit can be performed in a similar way, only with V_F changed to I_F and V_0 changed to I_0 . The transient duration diagram is also directly applicable without any exception because it depends only on k and γ that are general parameters for both circuits.

4.5 Special Cases of the Lowest Input Energy

General expression of the lowest input energy have been derived thoughtfully in the previous section. However, there are two special cases that are worthy for consideration. There are $R_{OC} = \infty$ and $R_{SC} = 0$. Because these two cases are actually familiar circuits, the circuits with only one resistor.

4.5.1 Infinite open circuit resistance

(a) Series resistor with a capacitor circuit. (b) Series resistor with an inductor circuit.

Figure 4.25: First-order infinite open circuit resistance circuits.

Infinite open circuit resistance circuits with a capacitor and an inductor are shown in Fig. 4.25. Actually, there are simple series R-L and series R-C circuits. By the definitions of R_{OC} and R_{SC} in (4.8), it is easily to see that $R_{OC} = \infty$ and $R_{SC} = R$ and the lowest input energy in the corollaries can be evaluated. However, the corollaries in (4.19) and (4.22) are in terms of k and τ which both are dependent on R_{OC} and R_{SC} . Although the corollaries can be expressed in terms of R_{OC} and R_{SC} instead of k and τ , it is more convenient to express τ in terms of k and R_{SC} and take limit $k \rightarrow \infty$ since $R_{OC} \rightarrow \infty$. From (4.16) and (4.11), we can see that

$$\frac{\tau}{k} = R_{SC}C \rightarrow \tau = kRC$$

since $R_{SC} = R$. The corollary 1 can be expressed as

$$E_I = \frac{C(V_F^2 - V_0^2)}{2(1 - 1/k^2)} \cdot \left(1 + \frac{\tanh(T_F/(2kRC))}{\gamma k}\right) \left(1 + \frac{\gamma/k}{\tanh(T_F/(2kRC))}\right).$$

By taking limit $k \rightarrow \infty$, we get

$$\lim_{k \rightarrow \infty} E_I = \frac{C(V_F^2 - V_0^2)}{2(1 - 0)} \cdot \left(1 + \frac{0}{\infty}\right) \left(1 + \frac{0}{0}\right).$$

Obviously, the last term is problematic because it is $0/0$. Nevertheless by using L'Hospital's rule, the last term becomes

$$\lim_{k \rightarrow \infty} \frac{\gamma/k}{\tanh(T_F/(2kRC))} = \lim_{k \rightarrow \infty} \frac{-\gamma/k^2}{\text{sech}^2(T_F/(2kRC)) \cdot (-T_F/(2k^2RC))} = \frac{2\gamma RC}{T_F}.$$

Finally, the lowest input energy of a capacitive circuit when $R_{OC} \rightarrow \infty$ is

$$\lim_{R_{OC} \rightarrow \infty} E_I = \frac{C(V_F^2 - V_0^2)}{2} \cdot \left(1 + \frac{2\gamma RC}{T_F}\right) = \frac{C(V_F^2 - V_0^2)}{2} + \left(\frac{RC}{T_F}\right) \cdot C(V_F - V_0)^2. \quad (4.38)$$

The first term of the right hand side is changed energy of the capacitor. Therefore, by energy conservation, it implies that the last term must be energy dissipation. The result of this special case is reported in [78]. Obviously, if T_F is larger, E_I will be smaller. The speed and energy trade-off is always held.

For inductive circuit, according to (4.16) and (4.21), we can see that

$$k\tau = \frac{L}{R_{SC}} \quad \rightarrow \quad \tau = \frac{L}{kR}$$

The corollary 2 can be expressed as

$$E_I = \frac{L(I_F^2 - I_0^2)}{2(1 - 1/k^2)} \cdot \left(1 + \frac{\tanh(T_F k R / 2L)}{\gamma k}\right) \left(1 + \frac{\gamma/k}{\tanh(T_F k R / 2L)}\right).$$

By taking limit $k \rightarrow \infty$, we get

$$\lim_{k \rightarrow \infty} E_I = \frac{L(I_F^2 - I_0^2)}{2(1 - 0)} \cdot \left(1 + \frac{1}{\infty}\right) \left(1 + \frac{0}{1}\right) = \frac{L(I_F^2 - I_0^2)}{2}. \quad (4.39)$$

However, the term on the right hand side is changed energy of the inductor. Therefore, it implies that the input energy is equal to the changed energy of the inductor and there is no energy dissipation. It seems to be impossible in practical and actually it is. By inspection the circuit of Fig. 4.25b, we can directly control the current of the inductor by using current source. Or in other words, we can directly control the energy of the inductor. The most energy-efficient way to change the current I_0 to I_F within the duration T_F is that when $t = 0$, we abruptly change the current from I_0 to 0 and then when $t = T_F$, we abruptly change the current from 0 to I_F . Therefore within the duration, there is no energy dissipation through the resistor because the current is zero and the input energy is equal to the changed energy of the inductor. Practically, this is impossible because when the current is changed abruptly, voltages in the circuit will be infinity because of the inductor. Nevertheless, infinity voltage is acceptable in theory as the result is shown in (4.39).

4.5.2 Zero shorted circuit resistance

(a) Parallel resistor with a capacitor circuit. (b) Parallel resistor with an inductor circuit.

Figure 4.26: First-order zero shorted circuit resistance circuits.

Shorted circuit resistance circuits with a capacitor and an inductor are shown in Fig. 4.26. Actually, there are simple parallel R-L and parallel R-C circuits. By the definitions of R_{OC} and R_{SC} in (4.8), it is easily to see that $R_{OC} = R$ and $R_{SC} = 0$ and the lowest input energy in the corollaries can be evaluated. When $R_{SC} = 0$, $k = \infty$ as

same as the infinity circuit resistance case. However, τ has to be written in different way. For capacitive circuit, according to (4.16) and (4.11), we get

$$k\tau = R_{OC}C \quad \rightarrow \quad \tau = \frac{RC}{k}$$

The corollary 1 can be expressed as

$$E_I = \frac{C(V_F^2 - V_0^2)}{2(1 - 1/k^2)} \cdot \left(1 + \frac{\tanh(T_F k / (2RC))}{\gamma k}\right) \left(1 + \frac{\gamma/k}{\tanh(T_F k / (2RC))}\right).$$

By taking limit $k \rightarrow \infty$, we get

$$\lim_{k \rightarrow \infty} E_I = \frac{C(V_F^2 - V_0^2)}{2(1 - 0)} \cdot \left(1 + \frac{1}{\infty}\right) \left(1 + \frac{0}{1}\right) = \frac{C(V_F^2 - V_0^2)}{2}. \quad (4.40)$$

The term on the right hand side is changed energy of the capacitor. Therefore, it implies that the input energy is equal to the changed energy of the capacitor and there is no energy dissipation. An argument for this case is the same as the series R-L circuit in the previous section. By inspection the circuit of Fig. 4.26a, we can directly control the voltage of the capacitor by using voltage source. Or in other words, we can directly control the energy of the capacitor. The most energy-efficient way to change the voltage V_0 to V_F within the duration T_F is that when $t = 0$, we abruptly change the voltage from V_0 to 0 and then when $t = T_F$, we abruptly change the voltage from 0 to V_F . Therefore within the duration, there is no energy dissipation through the resistor because the voltage is zero and the input energy is equal to the changed energy of the capacitor.

For inductive circuit,

$$\frac{\tau}{k} = \frac{L}{R_{OC}} \quad \rightarrow \quad \tau = \frac{kL}{R}.$$

The corollary 2 can be expressed as

$$E_I = \frac{L(I_F^2 - I_0^2)}{2(1 - 1/k^2)} \cdot \left(1 + \frac{\tanh(T_F R / (2kL))}{\gamma k}\right) \left(1 + \frac{\gamma/k}{\tanh(T_F R / (2kL))}\right).$$

By taking limit $k \rightarrow \infty$, we get

$$\lim_{k \rightarrow \infty} E_I = \frac{L(I_F^2 - I_0^2)}{2(1 - 0)} \cdot \left(1 + \frac{0}{\infty}\right) \left(1 + \frac{0}{0}\right).$$

Obviously, the last term is problematic because it is 0/0. Nevertheless by using L'Hospital's rule and following the same procedure as the previous section for the series R-C circuit, finally we get

$$\lim_{R_{OC} \rightarrow \infty} E_I = \frac{L(I_F^2 - I_0^2)}{2} \cdot \left(1 + \frac{2\gamma L}{RT_F}\right) = \frac{L(I_F^2 - I_0^2)}{2} + \left(\frac{L}{RT_F}\right) \cdot L(I_F - I_0)^2. \quad (4.41)$$

The first term of the right hand side is changed energy of the inductor. Therefore, by energy conservation, it implies that the last term must be energy dissipation. The result of this special case is reported in [78]. Obviously, if T_F is larger, E_I will be smaller. The speed and energy trade-off is always held.

4.6 I-V Characteristic of the Most Energy-efficient Input Source Functions

The most energy-efficient input source function is voltage or current input function that causes the lowest input energy. For capacitive circuit, the most energy-efficient input voltage and current function are (4.14) and (4.15) respectively. By following the same produce of determining the input function of capacitive circuit in Appendix B.5, the most energy-efficient input source function of inductive circuit can be determined as

$$v_I(t) = \frac{a_{11}R_{SC}}{\sinh 2X_F} (V_F (\sinh(x + X_F) + k \cosh(x + X_F)) - V_0 (\sinh(x - X_F) + k \cosh(x - X_F))) \quad (4.42)$$

and

$$i_I(t) = \frac{a_{22}}{k \sinh 2X_F} (I_F (k \sinh(x + X_F) + \cosh(x + X_F)) - I_0 (k \sinh(x - X_F) + \cosh(x - X_F))) \quad (4.43)$$

where x and X_F are defined in (B.75) and (4.13) respectively.

The most energy-efficient input source function above are expressed in time domain. However, they can be expressed as a relationship between voltage and current. For capacitive circuit, let us take a time derivative of v_I ,

$$\begin{aligned} \dot{v}_I(t) &= \frac{dv_I}{dx} \cdot \frac{dx}{dt} = \frac{dv_I}{dx} \cdot \frac{1}{\tau} = \frac{dv_I}{dx} \cdot \frac{k}{R_{OC}C} \\ C\dot{v}_I(t) &= \frac{a_{11}}{R_{OC} \sinh 2X_F} (V_F (\sinh(x + X_F) + k \cosh(x + X_F)) - V_0 (\sinh(x - X_F) + k \cosh(x - X_F))) \end{aligned}$$

Fortunately, we can see that

$$C\dot{v}_I(t) = \frac{a_{11}}{a_{22}} i_I(t)$$

or

$$\frac{i_I(t)}{a_{22}} = C \frac{\dot{v}_I(t)}{a_{11}}. \quad (4.44)$$

The equation above can be called as I-V characteristic of the most energy-efficient input source function of capacitive circuit. It is similar to the I-V characteristic of a capacitor where $i_C = C\dot{v}_C$. We just only change $i_C \rightarrow i_I/a_{22}$ and $v_C \rightarrow v_I/a_{11}$ to get the I-V characteristic of the most energy-efficient input source function.

This similarity can also be found in inductive circuit, let us take a time derivative of i_I in (4.43),

$$\begin{aligned} \dot{i}_I(t) &= \frac{di_I}{dx} \cdot \frac{dx}{dt} = \frac{di_I}{dx} \cdot \frac{1}{\tau} = \frac{di_I}{dx} \cdot \frac{kR_{SC}}{L} \\ L\dot{i}_I(t) &= \frac{a_{22}R_{SC}}{\sinh 2X_F} (I_F (\sinh(x + X_F) + k \cosh(x + X_F)) - I_0 (\sinh(x - X_F) + k \cosh(x - X_F))) \end{aligned}$$

According to (4.42), we found that

$$\frac{v_I(t)}{a_{11}} = L \frac{\dot{i}_I(t)}{a_{22}}. \quad (4.45)$$

The I-V characteristic of the most energy-efficient input source function of inductive circuit is similar to the I-V characteristic of an inductor where $v_L \rightarrow v_I/a_{11}$ and $i_L \rightarrow i_I/a_{22}$.

4.7 Conclusion

This chapter has presented the lowest input energy for first-order circuits in transient state. A resistors network with a capacitor has been chosen for derivation. The derivation is based on calculus of variations theory or more specifically, the Euler-Lagrange's differential equation. The most energy-efficient functions and the lowest input energy corollaries have been discussed which are essentials to provide insight into transient duration and circuit responses analysis. The transient duration analysis has been performed for three cases of increasing, decreasing and unchanging total stored energy. All these cases have been analyzed comprehensively. In each case, it has been further divided into two sub cases of (un)changing without and with zero crossing. The analysis has revealed two important circuit behaviours which are speed and energy trade-off and energy flow. It has been shown that the speed and energy trade-off may be violated under certain conditions. The energy flow of the circuit has been discussed thoroughly and a zero-energy transient duration that is defined as a transient duration with total input energy being zero has been derived. All the analysis can be summarized into a single digram called the *transient duration diagram*. The digram has revealed important relationships between the circuit parameters in the transient duration analysis. Special cases of the lowest input energy for first-order circuits which are infinite open circuit resistance and zero shorted circuit resistance have been explained quantitatively and qualitatively. Finally, I-V characteristic of the most energy-efficient input source functions have been noticed and elaborated.

Chapter 5

The Lowest Input Energy of Series LC and Parallel LC Circuits in Transient State

5.1 Introduction

Functional calculus or more specifically, the Euler-Lagrange's differential equation has been applied to first order circuits in the previous chapter. This powerful mathematics theory can be extended to second order circuits. There are several topologies that are second order circuits. The circuits can have two capacitors, two inductors or one capacitor and one inductor. According to fundamental circuits analysis, circuit behaviours of series LC and parallel LC circuits are the most interesting as we know that their unit step responses can be categorized into three modes which are overdamped, critical damped and underdamped. Therefore, these two circuits are chosen to be analyzed thoughtfully.

The chapter is organized as follows. The most energy-efficient functions of series LC circuits will be determined in Section 5.2. To determine the most energy-efficient functions, firstly boundaries conditions must be determined. General boundaries conditions and steady state boundaries conditions will be elaborated. Then, the Euler-Lagrange's differential equation for the series LC circuits will be derived. Solutions of the Euler-Lagrange's differential equation which are the most energy-efficient functions will be categorized into three modes which are (i) hyperbolic mode, (ii) hyperbolic and linear mode and (iii) hyperbolic and trigonometric mode. The lowest input energy and transient duration analysis of the lowest input energy for each mode will be explained in Section 5.3 including speed and energy trade-off, energy flow and optimum transient duration. Two special cases will be treated in Section 5.4. Section 5.5 will determine the most energy-efficient functions of parallel LC circuits and Section 5.6 will explain the lowest input energy and special cases of parallel LC circuits in a similar way to the series LC circuits. Mathematical details can be found in Appendix. Finally, Section 5.7 draws a conclusion.

5.2 The Most Energy-efficient Functions of Series LC Circuits

5.2.1 Boundary Conditions

Fig. 5.1 shows an arbitrary resistive topologies with a series LC. In the procedure of the optimization, boundary conditions must be given. These conditions are trickier than

Figure 5.1: Arbitrary resistive topologies with a series LC.

those first order circuits and we must determine them carefully. Two important rules are (i) the initial circuit energy must be given and (ii) the final circuit energy must be given. Because of the circuit consists of two storage elements which are the inductor and the capacitor, the initial and the final energy of the inductor and the capacitor must be given respectively. Since the energy of the inductor is in current form and the energy of the capacitor is in voltage form, therefore the initial and the final current of the inductor as well as the initial and the final voltage of the capacitor must be given. Let us assume that the time interval that we want to optimize is $0 < t < T_F$, where T_F is a transient duration, the conditions that must be given are

$$i_L(0^+) = I_0 \quad (5.1)$$

$$i_L(T_F^-) = I_F \quad (5.2)$$

$$v_C(0^+) = V_0 \quad (5.3)$$

$$v_C(T_F^-) = V_F \quad (5.4)$$

where I_0, I_F, V_0 and V_F are the initial inductor current, the final inductor current, the initial capacitor voltage and the final capacitor voltage respectively.

These conditions seem to be fine. However, they are not the conditions for steady state. It means that $i_L(0^-) \neq i_L(0^+)$, $i_L(T_F^-) \neq i_L(T_F^+)$, $v_C(0^-) \neq v_C(0^+)$ and $v_C(T_F^-) \neq v_C(T_F^+)$. Because of the inductor is in series with the capacitor, therefore $i_L = i_C = C\dot{v}_C$. If there is current flow through the inductor, the derivative of the voltage across the capacitor will not be zero. Consequently, the capacitor voltage is not steady and the circuit is not in the steady state. Obviously, if the inductor current is zero, the conditions are steady state boundary conditions and can be written as

$$i_L(0^-) = i_L(0^+) = C\dot{v}_C(0) = 0 \quad (5.5)$$

$$i_L(T_F^-) = i_L(T_F^+) = C\dot{v}_C(T_F) = 0 \quad (5.6)$$

$$v_C(0^-) = v_C(0^+) = v_C(0) = V_0 \quad (5.7)$$

$$v_C(T_F^-) = v_C(T_F^+) = v_C(T_F) = V_F. \quad (5.8)$$

They imply that for steady state boundary conditions, the initial circuit energy must only be stored in the capacitor as well the final circuit energy. There is no stored energy in the inductor at the initial state and also the final state. Otherwise, the circuit will be not in steady state after the final state or it will be not in steady state before the initial state.

Although it is quite tricky to determine the conditions, actually it is transient circuit analysis routine. Before doing transient analysis, initial conditions must be determined by using steady state analysis. For this case, before doing circuits' optimization, boundary conditions must be determined by using steady state analysis also. Unless the optimization does not concern about circuits response after the final state or before the

initial state, the general boundary conditions in (5.1) to (5.4) can be applied. However, the following optimization will mainly focus on steady state boundary conditions which are (5.5) to (5.8).

5.2.2 The Euler-Lagrange's Differential Equation

The most energy-efficient input source function is the function that cause the lowest input energy. In mathematical expression, it is the function that minimize the integral

$$E_I = \int_0^{T_F} v_I i_I dt. \quad (5.9)$$

By inspection the circuit of Fig. 5.1, $i_O = C\dot{v}_C$ and the output voltage v_O in terms of the voltage across the capacitor v_C is

$$v_O = v_L + v_C = L\dot{i}_L + v_C = \frac{1}{\omega^2}\ddot{v}_C + v_C \quad (5.10)$$

where

$$\omega \equiv 1/\sqrt{LC} \quad (5.11)$$

The relationship between the inputs and the outputs can be represented by the ABCD matrix of the resistive network and the Lagrangian function which is the input power is

$$\mathcal{L} = v_I i_I = (a_{11}v_O + a_{12}i_O)(a_{21}v_O + a_{22}i_O). \quad (5.12)$$

Because the output voltage in (5.10) involves the second derivative of v_C , the Lagrangian function also involves the second derivative of v_C . Therefore the Euler-Lagrange's differential equation for second order derivatives which is

$$\frac{\partial \mathcal{L}}{\partial v_C} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{v}_C} \right) + \frac{d^2}{dt^2} \left(\frac{\partial \mathcal{L}}{\partial \ddot{v}_C} \right) = 0 \quad (5.13)$$

must be applied.

By substituting (5.12) into (5.13), a differential equation for solving the most energy-efficient function can be determined. However, the differential equation can be simplified by using the transformation between the ABCD matrix and other matrices. Let us begin with the transformation between the ABCD matrix and the impedance matrix or Z matrix. An interested parameter is

$$z_{22} = \left. \frac{v_O}{i_O} \right|_{i_I=0} = \frac{a_{22}}{a_{21}} \equiv R_{OC} \quad (5.14)$$

which is the open circuit resistance seen by the output. Circuit perspective of R_{OC} is shown in Fig. 5.2a. The second transformation that help to simplify the differential equation is the the transformation between the ABCD matrix and the invert hybrid matrix or G matrix. An interested parameter is

$$g_{22} = \left. \frac{v_O}{i_O} \right|_{v_I=0} = \frac{a_{12}}{a_{11}} \equiv R_{SC} \quad (5.15)$$

which is the shorted circuit resistance seen by the output. Circuit perspective of R_{SC} is shown in Fig. 5.2b. Furthermore, if we define

$$Z \equiv \sqrt{\frac{L}{C}} \quad (5.16)$$

and a new parameter r as

$$r \equiv \frac{\sqrt{R_{OC}R_{SC}}}{2Z}, \quad (5.17)$$

we get the differential equation as

$$\ddot{v}_C + 2(1 - 2r^2)\omega^2\dot{v}_C + \omega^4v_C = 0. \quad (5.18)$$

Notice that r is always a real number and unitless. Full derivation of (5.18) can be found in Appendix B.9.

(a) Open circuit resistance (R_{OC}) seen by the output. (b) Shorted circuit resistance (R_{SC}) seen by the output.

Figure 5.2: Circuit perspective of open circuit resistance (R_{OC}) and shorted circuit resistance (R_{SC}) seen by the output.

The parameter r plays an important role in (5.18). It determines the functions of the solution. The functions can be different for different values of r . The values of r can be divided into four cases which are (i) $r > 1$, (ii) $r = 1$, (iii) $0 < r < 1$ and (iv) $r = 0$. For $r = 0$, it is a special case that we will treat it separately. For other cases, we called them as (i) hyperbolic mode, (ii) hyperbolic and linear mode and (iii) hyperbolic and trigonometric mode. Each case will be examined thoroughly.

5.2.3 The Most Energy-efficient Functions

Solving the fourth order differential equation in (5.18) can be done by determining roots of characteristic equation. By inspection, we can see that the roots is in form of $m\omega$ where m is the roots of

$$m^4 + 2(1 - 2r^2)m^2 + 1 = 0. \quad (5.19)$$

The roots can be determined in four mode as follows.

Hyperbolic Mode

For $r > 1$, the roots are

$$m_+\omega, -m_+\omega, m_-\omega, -m_-\omega \quad (5.20)$$

where

$$m_+ = r + \sqrt{r^2 - 1} \quad (5.21)$$

and

$$m_- = r - \sqrt{r^2 - 1}. \quad (5.22)$$

Obviously, a general solution of (5.18) must be

$$v_C(t) = K_1 \sinh m_+\omega t + K_2 \cosh m_+\omega t + K_3 \sinh m_-\omega t + K_4 \cosh m_-\omega t \quad (5.23)$$

where K_1, K_2, K_3 and K_4 are constants subjected to the boundaries conditions in (5.5) - (5.8).

Nevertheless, the constants cannot be evaluated robustly because the hyperbolic functions can be extremely large at $t = T_F$ and solving the constants can be an ill-conditioned problem. Therefore, the capacitor voltage function is rearranged and the constants are determined analytically. The analytical solution is lengthy and it is not important to show here. It can be found in Appendix B.10. The important thing is that the solution is a hyperbolic function family.

To determine the most energy-efficient input functions, both voltage and current, the unique solution of capacitor voltage is used with routine circuit analysis. For example, let us consider a circuit of Fig. 5.3, assume that the circuit elements are $R_1 = 79 \text{ k}\Omega$, $R_2 = 1 \text{ k}\Omega$, $L = 5 \text{ mH}$ and $C = 1 \text{ nF}$. By using the formula of Z and r in (5.16) and (5.17), we can calculate r to be equal to 2. Since $r > 1$, the solution is in hyperbolic mode.

Figure 5.3: Example circuit of series L-C with resistors network.

Fig. 5.4 and Fig. 5.5 show voltage and current transient responses for $V_0 = 1 \text{ V}$, $V_F = 2 \text{ V}$ and $\omega T_F = 1, 5, 10, 40$. The horizontal axes are normalized time which is the time multiply by ω and it is unitless. As shown in the figures, different transient durations T_F cause different shapes of responses although the solutions are all hyperbolic functions.

Hyperbolic and Linear Mode

For $r = 1$, the roots of (5.19) are

$$\omega, \omega, -\omega, -\omega. \quad (5.24)$$

Since there are repeat roots, the solution must have a linear function multiply with the hyperbolic functions. Concretely, the solution must be in form of

$$v_C(t) = K_1 \sinh \omega t + K_2 \cosh \omega t + K_3 t \sinh \omega t + K_4 t \cosh \omega t \quad (5.25)$$

where K_1, K_2, K_3 and K_4 are constants subjected to the boundaries conditions in (5.5) - (5.8). We can say that the solution is hyperbolic and linear mode. An analytical unique solution can be found in Appendix B.11.

Transient responses of the circuit in hyperbolic and linear mode are shown in Fig. 5.6 and Fig. 5.7. The boundaries conditions are $V_0 = 1 \text{ V}$, $V_F = 2 \text{ V}$ and $\omega T_F = 1, 5, 10, 40$. The circuit is a circuit in Fig. 5.3 where $R_1 = 19 \text{ k}\Omega$, $R_2 = 1 \text{ k}\Omega$, $L = 5 \text{ mH}$ and $C = 1 \text{ nF}$. By using the formula of Z and r in (5.16) and (5.17), we can calculate r to be equal to 1.

(a) $\omega T_F = 1$

(b) $\omega T_F = 5$

(c) $\omega T_F = 10$

(d) $\omega T_F = 40$

Figure 5.4: Voltage transient responses for $r = 2$, $V_0 = 1$ V, $V_F = 2$ V and $\omega T_F = 1, 5, 10, 40$

(a) $\omega T_F = 1$

(b) $\omega T_F = 5$

(c) $\omega T_F = 10$

(d) $\omega T_F = 40$

Figure 5.5: Current transient responses for $r = 2$, $V_0 = 1$ V, $V_F = 2$ V and $\omega T_F = 1, 5, 10, 40$

(a) $\omega T_F = 1$

(b) $\omega T_F = 5$

(c) $\omega T_F = 10$

(d) $\omega T_F = 40$

Figure 5.6: Voltage transient responses for $r = 1$, $V_0 = 1$ V, $V_F = 2$ V and $\omega T_F = 1, 5, 10, 40$

(a) $\omega T_F = 1$

(b) $\omega T_F = 5$

(c) $\omega T_F = 10$

(d) $\omega T_F = 40$

Figure 5.7: Current transient responses for $r = 1$, $V_0 = 1$ V, $V_F = 2$ V and $\omega T_F = 1, 5, 10, 40$

Hyperbolic and Trigonometric Mode

For $0 < r < 1$, the roots of (5.19) are

$$m_+\omega, -m_+\omega, m_-\omega, -m_-\omega \quad (5.26)$$

where

$$m_+ = r + j\sqrt{1-r^2} \quad (5.27)$$

and

$$m_- = r - j\sqrt{1-r^2}. \quad (5.28)$$

Obviously, a general solution can be written in hyperbolic functions as same as (5.23). However, the arguments in the hyperbolic functions are complex numbers. Therefore, the constants are also complex numbers. Nevertheless, the capacitor voltage function must be real number.

To obtain the capacitor voltage in forms of non-complex functions, we can use some hyperbolic and trigonometric identities. For example, let us consider

$$\begin{aligned} \sinh m_+\omega t &= \sinh(\omega t(r + j\sqrt{1-r^2})) \\ &= \sinh(r\omega t) \cosh(j\sqrt{1-r^2}\omega t) + \cosh(r\omega t) \sinh(j\sqrt{1-r^2}\omega t) \\ &= \sinh(r\omega t) \cos(\sqrt{1-r^2}\omega t) + j \cosh(r\omega t) \sin(\sqrt{1-r^2}\omega t). \end{aligned} \quad (5.29)$$

The hyperbolic sine with a complex argument can be expanded into hyperbolic and trigonometric functions. The hyperbolic cosine can be expanded in a similar way. Therefore, the general solution can be written as

$$\begin{aligned} v_C(t) &= K_1 \sinh(r\omega t) \sin(\sqrt{1-r^2}\omega t) + K_2 \sinh(r\omega t) \cos(\sqrt{1-r^2}\omega t) \\ &\quad + K_3 \cosh(r\omega t) \sin(\sqrt{1-r^2}\omega t) + K_4 \cosh(r\omega t) \cos(\sqrt{1-r^2}\omega t) \end{aligned} \quad (5.30)$$

where K_1, K_2, K_3 and K_4 are constants. It can be seen that the general solution is hyperbolic and trigonometric mode. For numerical calculation, all the formulas in Appendix B.10 can be applied with m_+ and m_- are complex numbers.

The circuit of Fig. 5.3 when $R_1 = 4 \text{ k}\Omega$, $R_2 = 1 \text{ k}\Omega$, $L = 5 \text{ mH}$ and $C = 1 \text{ nF}$ gives $r = 0.5$. Transient responses of the circuit in hyperbolic and trigonometric mode are shown in Fig. 5.8 and Fig. 5.9 with the boundaries conditions of $V_0 = 1 \text{ V}$, $V_F = 2 \text{ V}$ and $\omega T_F = 1, 5, 10, 40$.

5.3 The Lowest Input Energy and Transient Duration Analysis of Series LC Circuits

Once the most energy-efficient capacitor voltage function is determined by solving the differential equation in (5.18), the input voltage function and the input current function can be determined by using routine circuit analysis. Hence, the lowest input energy can be calculated by evaluating the integral in (5.9). Calculation of the integral gives us

$$E_I = \frac{C(V_F^2 - V_0^2)}{2(k^2 - 1)} \cdot \left(k^2 + 1 + \frac{k}{r} \cdot \frac{\ddot{v}_C(0)V_0 - \ddot{v}_C(T_F)V_F}{\omega^3(V_F^2 - V_0^2)} \right). \quad (5.31)$$

where k defined as

$$k \equiv \sqrt{\frac{R_{OC}}{R_{SC}}}. \quad (5.32)$$

(a) $\omega T_F = 1$

(b) $\omega T_F = 5$

(c) $\omega T_F = 10$

(d) $\omega T_F = 40$

Figure 5.8: Voltage transient responses for $r = 0.5$, $V_0 = 1$ V, $V_F = 2$ V and $\omega T_F = 1, 5, 10, 40$

(a) $\omega T_F = 1$

(b) $\omega T_F = 5$

(c) $\omega T_F = 10$

(d) $\omega T_F = 40$

Figure 5.9: Current transient responses for $r = 0.5$, $V_0 = 1$ V, $V_F = 2$ V and $\omega T_F = 1, 5, 10, 40$

Full calculation can be found in Appendix B.12. Since the lowest input energy depends on the capacitor voltage v_C , the lowest input energy for each mode is different. However, for hyperbolic mode ($r > 1$) and hyperbolic and linear mode ($r = 1$), transient duration analysis of the lowest input energy are similar and can be explained together. On the other hand, the lowest input energy in hyperbolic and trigonometric mode ($0 < r < 1$) is different from the others and will be explained separately.

5.3.1 Hyperbolic Mode and Hyperbolic and Linear Mode

For $r > 1$, a series LC circuit is in hyperbolic mode. By using an analytical solution in Appendix B.10 and the lowest input energy in (5.31), a normalized lowest input energy which is defined as a ratio between input energy and changed stored energy can be found as

$$u = \frac{E_I}{\Delta E_{\text{stored}}} = \frac{1}{k^2 - 1} \cdot \left[k^2 + 1 - 2k\sqrt{r^2 - 1} \left(\frac{\gamma}{m_- \tanh m_+ \omega T_F/2 - m_+ \tanh m_- \omega T_F/2} + \frac{1/\gamma}{m_- \coth m_+ \omega T_F/2 - m_+ \coth m_- \omega T_F/2} \right) \right] \quad (5.33)$$

where

$$\gamma \equiv \frac{V_F - V_0}{V_F + V_0}. \quad (5.34)$$

The calculation can be found in Appendix B.13.

For $r = 1$, the series LC circuit is in hyperbolic and linear mode. Similarly, by using the analytical results in Appendix B.11 and the lowest input energy in (5.31), the normalized lowest input energy can be obtained as

$$u = \frac{1}{k^2 - 1} \cdot \left[k^2 + 1 + k \left(\frac{\gamma(\cosh \omega T_F + 1)}{\sinh \omega T_F - \omega T_F} + \frac{1/\gamma(\cosh \omega T_F - 1)}{\sinh \omega T_F + \omega T_F} \right) \right] \quad (5.35)$$

The calculation can be found in Appendix B.14.

Although mathematical expressions of the normalized lowest input energy for both cases is different, its transient duration analysis is similar and can be analyzed together. Transient duration analysis is performed for three cases of increasing, decreasing and unchanging total stored energy. In each case, it is further divided into two sub cases of (un)changing without and with zero crossing.

For increasing total stored energy with zero crossing, $|V_F| > |V_0|$ and signs of V_F and V_0 are different. The condition is equivalent to $\gamma \geq 1$. The normalized lowest input energy is monotonic decreasing with transient duration. Speed and energy trade-off does hold and the normalized lowest input energy is minimum at $T_F = \infty$. For $r > 1$, according to (5.33), the minimum is

$$\lim_{T_F \rightarrow \infty} u = \frac{1}{k^2 - 1} \cdot \left[k^2 + 1 - \frac{2k\sqrt{r^2 - 1}}{m_- - m_+} \left(\gamma + \frac{1}{\gamma} \right) \right] = \frac{(k + \gamma)(k + 1/\gamma)}{k^2 - 1} \quad (5.36)$$

and for $r = 1$, by using (5.35), the minimum normalized lowest input energy is

$$\lim_{T_F \rightarrow \infty} u = \frac{1}{k^2 - 1} \cdot \left[k^2 + 1 + k \left(\gamma + \frac{1}{\gamma} \right) \right] = \frac{(k + \gamma)(k + 1/\gamma)}{k^2 - 1}. \quad (5.37)$$

For both cases, the minimum are the same and depends on γ and k . Fig 5.10 shows the normalized lowest input energy versus normalized transient duration (ωT_F) for $r = 1, 2$,

(a) $r = 2$

(b) $r = 1$

Figure 5.10: Normalized lowest input energy versus normalized transient duration for $\gamma = 3$ and $k = 3, 4, 5, 7, 10$

$\gamma = 3$ and $k = 3, 4, 5, 7, 10$. The normalized lowest input energy diverges to its minimum faster for $r = 1$.

For increasing total stored energy without zero crossing, $|V_F| > |V_0|$ and signs of V_F and V_0 are the same. The condition is equivalent to $0 < \gamma < 1$. There is an optimum point and the speed and energy trade-off is violated. The minimum lowest input energy at the optimum point is always lower than the lowest input energy at $T_F = \infty$. The optimum transient duration for $r > 1$ can be determined by solving

$$\left(\frac{m_+ \tanh m_- \omega T_{F(\text{opt.})}/2 - m_- \tanh m_+ \omega T_{F(\text{opt.})}/2}{m_+ \tanh m_+ \omega T_{F(\text{opt.})}/2 - m_- \tanh m_- \omega T_{F(\text{opt.})}/2} \right)^2 - \gamma^2 = 0 \quad (5.38)$$

numerically. For $r = 1$, the optimum transient duration can be determined by solving

$$\left(\frac{\sinh \omega T_{F(\text{opt.})} - \omega T_{F(\text{opt.})}}{\sinh \omega T_{F(\text{opt.})} + \omega T_{F(\text{opt.})}} \right)^2 - \gamma^2 = 0. \quad (5.39)$$

These two equations are derived by setting $du/dT_F = 0$. They have no solution if $\gamma \geq 1$, hence for $\gamma \geq 1$ there is no optimum transient duration and the normalized lowest input energy is monotonic decreasing as described above. Details of the equations can be found in Appendix B.15. Fig 5.11 shows the normalized lowest input energy versus normalized transient duration (ωT_F) for $r = 1, 2$, $\gamma = 0.5$ and $k = 3, 4, 5, 7, 10$.

(a) $r = 2$

(b) $r = 1$

Figure 5.11: Normalized lowest input energy versus normalized transient duration for $\gamma = 0.5$ and $k = 3, 4, 5, 7, 10$

For decreasing total stored energy without zero crossing, $|V_F| < |V_0|$ and signs of V_F and V_0 are the same. The condition is equivalent to $-1 < \gamma < 0$. There is an

optimum point and the speed and energy trade-off is violated as same as increasing total stored energy with zero crossing. The optimum transient duration can be found by solving (5.38) and (5.39) for $r > 1$ and $r = 1$ respectively. However, decreasing total stored energy gives negative changing total stored energy and the normalized lowest input energy u will be negative if the input energy is positive or the total input energy flows into the circuit. On the other hand, the normalized lowest input energy will be negative if the lowest input energy is negative or the total input energy actually flows into the source. Therefore, if $u > 0$, we receives some energy for the circuit. Fig 5.12 shows the normalized lowest input energy versus normalized transient duration (ωT_F) for $r = 1, 2$, $\gamma = -0.2$ and $k = 3, 4, 5, 7, 10$.

Figure 5.12: Normalized lowest input energy versus normalized transient duration for $\gamma = -0.2$ and $k = 3, 4, 5, 7, 10$

For decreasing total stored energy with zero crossing, $|V_F| < |V_0|$ and signs of V_F and V_0 are different. The condition is equivalent to $\gamma \leq -1$. The normalized lowest input energy is monotonic increasing with transient duration. Speed and energy trade-off does hold and the normalized lowest input energy is maximum at $T_F = \infty$. Since it is monotonic increasing, if u at $T_F = \infty$ is negative, it means that with any transient durations, the normalized lowest input energy is negative and we have to deliver some energy to the circuit in order to make the circuit decreases total stored energy. However, if u at $T_F = \infty$ is positive, we can receive some energy from the circuit if transient duration is long enough. According to (5.36) and (5.37) and because of $\gamma < 0$, conditions for $u > 0$ at $T_F = \infty$ are $k > -\gamma$ and $k > -1/\gamma$. Fig 5.13 shows the normalized lowest input energy versus normalized transient duration (ωT_F) for $r = 1, 2$, $\gamma = -5$ and $k = 3, 4, 5, 7, 10$.

For unchanging total stored energy with zero crossing, $V_F = -V_0$. The condition is equivalent to $\gamma = \pm\infty$. $\gamma = +\infty$ when $V_0 < 0$ and $\gamma = -\infty$ when $V_0 > 0$. Because $V_F = -V_0$, the lowest input energy in this case is actually the lowest input energy that the circuit requires for changing the capacitor voltage polarity. Change of total stored energy is absolutely zero. Therefore, the normalized lowest input energy must be defined differently. We define the normalized lowest input energy as a ratio between the lowest input energy and the total stored energy. For $r > 1$, the normalized lowest input energy is determined as

$$u_{un} = \frac{8k\sqrt{r^2 - 1}}{(k^2 - 1)(m_+ \tanh m_- \omega T_F / 2 - m_- \tanh m_+ \omega T_F / 2)} \quad (5.40)$$

and for $r = 1$, the normalized lowest input energy is

$$u_{un} = \frac{4k(\cosh \omega T_F + 1)}{(k^2 - 1)(\sinh \omega T_F - \omega T_F)}. \quad (5.41)$$

(a) $r = 2$

(b) $r = 1$

Figure 5.13: Normalized lowest input energy versus normalized transient duration for $\gamma = -5$ and $k = 3, 4, 5, 7, 10$

Full calculation can be found in Appendix B.13 and Appendix B.14 respectively. The normalized lowest input energy is monotonic increasing with transient duration. Speed and energy trade-off does hold and the normalized lowest input energy is maximum at $T_F = \infty$. Fig 5.14 shows the normalized lowest input energy versus normalized transient duration (ωT_F) for $r = 1, 2$, $\gamma = \infty$ and $k = 3, 4, 5, 7, 10$.

(a) $r = 2$

(b) $r = 1$

Figure 5.14: Normalized lowest input energy versus normalized transient duration for $\gamma = \infty$ and $k = 3, 4, 5, 7, 10$

Lastly, unchanging total stored energy without zero crossing, $V_F = V_0$. The condition is equivalent to $\gamma = 0$. Because the final voltage and the initial voltage are the same, it is not the same as the previous cases where the capacitor voltage changes. In this case, we try not to change the capacitor voltage within the duration of time T_F . The speed and energy trade-off is violated because T_F does not refer to speed of changing anymore, but it refers to the duration that we try not to change the total stored energy instead. The normalized lowest input energy can be found as

$$u_{un} = \frac{8k\sqrt{r^2 - 1}}{(k^2 - 1)(m_+ \coth m_- \omega T_F / 2 - m_- \coth m_+ \omega T_F / 2)} \quad (5.42)$$

and

$$u_{un} = \frac{4k(\cosh \omega T_F - 1)}{(k^2 - 1)(\sinh \omega T_F + \omega T_F)}. \quad (5.43)$$

for $r > 1$ and $r = 1$ respectively. Full calculation can be found in Appendix B.13 and Appendix B.14. Fig 5.15 shows the normalized lowest input energy versus normalized transient duration (ωT_F) for $r = 1, 2$, $\gamma = 0$ and $k = 3, 4, 5, 7, 10$.

Figure 5.15: Normalized lowest input energy versus normalized transient duration for $\gamma = 0$ and $k = 3, 4, 5, 7, 10$

5.3.2 Hyperbolic and Trigonometric Mode

For $0 < r < 1$, a series LC circuit is in hyperbolic and trigonometric mode. By using an analytical solution in Appendix B.10 and the lowest input energy in (5.31), the normalized lowest input energy can be found as same as (5.33). Because of $r < 1$, some terms in (5.33) are complex numbers. However, the normalized input energy is always a real number as expected. Because it is the ratio between the lowest input energy and the change of total stored energy. Moreover, by using hyperbolic identities that are similar with (5.29), we surely realize that the normalized lowest input energy involves both hyperbolic and trigonometric functions. Therefore, the normalized lowest input energy is neither monotonic decreasing nor increasing with transient duration, but it oscillates instead. There are multiple local maxima and local minima. The maxima and minima can be determined numerically by solving (5.38) and

$$\tanh^2 m_+ \omega T_{F(\text{opt.})}/2 - \tanh^2 m_- \omega T_{F(\text{opt.})}/2 = 0. \quad (5.44)$$

Fig 5.16 shows the normalized lowest input energy versus normalized transient duration (ωT_F) for $r = 0.2$ and $k = 3, 4, 5, 7, 10$.

5.4 Special Cases of Series LC Circuits

Analysis of the most energy-efficient function and the lowest input energy for series LC circuit has been discussed thoughtfully in the previous sections. However, there are two cases that we have not discussed yet, which are $r = 0$ and $r = \infty$. We treat these as special cases because the most energy-efficient function and their lowest input energy are not similar with the other cases. Importantly, with $r = 0$ or $r = \infty$, the k cannot be an arbitrary number, it must be ∞ only because these two parameters both depend on R_{OC} and R_{SC} .

To clarify, let us consider the case $r = 0$. According to (5.17) and we are considering series LC circuit, $r = 0$ if and only if $R_{OC} = 0$ or $R_{SC} = 0$. By using circuit theory, we know that $R_{OC} > R_{SC}$. Therefore, R_{OC} cannot be zero and we can conclude that $r = 0$ if and only if $R_{SC} = 0$. According to (5.32), $k = \infty$ when $R_{SC} = 0$. Another case is $r = \infty$. Because $R_{OC} > R_{SC}$, $r = \infty$ if and only if $R_{OC} = \infty$ and k is also equal to ∞ .

These two special cases can be depicted by circuit topologies easily. for $r = 0$, $R_{SC} = 0$, it means that there is only a parallel resistor in the circuit and for $r = \infty$, $R_{OC} = \infty$, it means that there is only a series resistor in the circuit. Fig 5.17 illustrates the circuit topologies when $r = 0$ and $r = \infty$. The most energy-efficient input functions

(a) $\gamma = \infty$

(b) $\gamma = 2$

(c) $\gamma = 0.6$

(d) $\gamma = 0$

(e) $\gamma = -0.8$

(f) $\gamma = -1.5$

Figure 5.16: Normalized lowest input energy versus normalized transient duration for $r = 0.2$ and $k = 3, 4, 5, 7, 10$

of $r = 0$ and $r = \infty$ are resonant mode and linear mode respectively. In both cases $k = \infty$. Each case is explained separately as follows.

5.4.1 Resonant Mode

With $r = 0$. The roots of (5.19) are

$$j\omega, j\omega, -j\omega, -j\omega. \quad (5.45)$$

Since there are repeat roots, the solution must have a linear function multiply with the trigonometric function. Concretely, the solution must be in form of

$$v_C(t) = K_1 \sin \omega t + K_2 \cos \omega t + K_3 t \sin \omega t + K_4 t \cos \omega t \quad (5.46)$$

where K_1, K_2, K_3 and K_4 are constants subjected to the boundaries conditions in (5.5) - (5.8). We can say that the solution is trigonometric and linear mode. An analytical unique solution can be found in Appendix B.11 by replacing ω with $j\omega$.

Figure 5.17: Circuit topologies when $r = 0$ and $r = \infty$

Unlike the other modes, although the capacitor voltage function is trigonometric function and linear function, the input voltage function is only trigonometric function. Because the circuit is in resonant mode. A concrete proof can be found in Appendix B.16. Therefore, the input voltage function is in form of

$$v_I(t) = A \cos(\omega t + \phi) \quad (5.47)$$

where A and ϕ are constants. Notice that hyperbolic and linear mode is not a resonant mode because the input voltage function is also the hyperbolic and linear function, it is not only the hyperbolic function.

The circuit of Fig. 5.3 when $R_1 = 5 \text{ k}\Omega$, $R_2 = 0$, $L = 5 \text{ mH}$ and $C = 1 \text{ nF}$ gives $r = 0$. Transient responses of the circuit in resonant mode are shown in Fig. 5.18 and Fig. 5.19 with the boundaries conditions of $V_0 = 1 \text{ V}$, $V_F = 2 \text{ V}$ and $\omega T_F = 1, 5, 10, 40$. Notice that in voltage responses, input voltage and series LC voltage are the same which is consistent when $R_2 = 0$.

The normalized lowest input energy can be determined by using (5.33) and substituting $r = 0$, $k = \infty$. Although $r = 0$ and $k = \infty$, its product can be determined as

$$kr = \sqrt{\frac{R_{OC}}{R_{SC}}} \cdot \frac{\sqrt{R_{OC}R_{SC}}}{2Z} = \frac{R_{OC}}{2Z}. \quad (5.48)$$

The normalized lowest input energy in resonant mode, both increasing and decreasing total stored energy can be obtained as

$$u = 1 + \frac{1}{kr} \left(\frac{\gamma(1 + \cos \omega T_F)}{\omega T_F - \sin \omega T_F} + \frac{1 - \cos \omega T_F}{\gamma(\omega T_F + \sin \omega T_F)} \right). \quad (5.49)$$

For unchanging total stored energy with zero crossing, $V_F = -V_0$ or $\gamma = \pm\infty$, the normalized lowest input energy is

$$u_{un} = \frac{4(1 + \cos \omega T_F)}{kr(\omega T_F - \sin \omega T_F)} \quad (5.50)$$

and for unchanging total stored energy without zero crossing, $V_F = V_0$ or $\gamma = 0$, the normalized lowest input energy is

$$u_{un} = \frac{4(1 - \cos \omega T_F)}{kr(\omega T_F + \sin \omega T_F)}. \quad (5.51)$$

Fig 5.20 shows the normalized lowest input energy versus normalized transient duration (ωT_F) for $r = 0$ and $kr = 3, 4, 5, 7, 10$. Calculation of the normalized lowest input energy in resonant mode can be found in Appendix B.17

(a) $\omega T_F = 1$

(b) $\omega T_F = 5$

(c) $\omega T_F = 10$

(d) $\omega T_F = 40$

Figure 5.18: Voltage transient responses for $r = 0$, $V_0 = 1$ V, $V_F = 2$ V and $\omega T_F = 1, 5, 10, 40$

(a) $\omega T_F = 1$

(b) $\omega T_F = 5$

(c) $\omega T_F = 10$

(d) $\omega T_F = 40$

Figure 5.19: Current transient responses for $r = 0$, $V_0 = 1$ V, $V_F = 2$ V and $\omega T_F = 1, 5, 10, 40$

(a) $\gamma = \infty$

(b) $\gamma = 2$

(c) $\gamma = 0.6$

(d) $\gamma = 0$

(e) $\gamma = -0.8$

(f) $\gamma = -1.5$

Figure 5.20: Normalized lowest input energy versus normalized transient duration for $r = 0$ and $kr = 3, 4, 5, 7, 10$

Interestingly, u_{un} can be zero for a certain transient duration. For $\gamma = \pm\infty$ or $V_F = -V_0$, $u_{un} = 0$ when $\cos \omega T_F = -1$ or $T_F = \pi/\omega, 3\pi/\omega, 5\pi/\omega, \dots$. It means that the circuit requires zero input energy for changing V_0 to $-V_0$. This is impossible for $r > 0$. However for $r = 0$, the circuit is in resonant mode. If the input voltage is zero, the capacitor voltage will oscillate between V_0 and $-V_0$ with resonant frequency. Therefore, the capacitor voltage can be $-V_0$ for every half period or mathematically, $T_F = \pi/\omega, 3\pi/\omega, 5\pi/\omega, \dots$. Hence, the circuit does not need input energy for changing V_0 to $-V_0$ for these transient durations. Fig 5.21 shows voltage transient responses of the circuit in Fig. 5.3 when $R_1 = 5 \text{ k}\Omega$, $R_2 = 0$, $L = 5 \text{ mH}$ and $C = 1 \text{ nF}$. The boundaries conditions are $V_0 = -1 \text{ V}$, $V_F = 1 \text{ V}$ and $\omega T_F = \pi, 3\pi$.

Similarly, for $\gamma = 0$ or $V_F = V_0$, u_{un} is zero when $\cos \omega T_F = 1$ or $T_F = 0, 2\pi/\omega, 4\pi/\omega, \dots$. If the input voltage is zero, the capacitor voltage will oscillate between V_0 and $-V_0$ with resonant frequency. Therefore, the capacitor voltage can be V_0 for every period and the circuit does not need input energy for these transient durations. Fig 5.22 shows voltage

(a) $\omega T_F = \pi$

(b) $\omega T_F = 3\pi$

Figure 5.21: Voltage transient responses for $r = 0$, $V_0 = -1$ V, $V_F = 1$ V and $\omega T_F = \pi, 3\pi$

transient responses of the circuit in Fig. 5.3 when $R_1 = 5$ k Ω , $R_2 = 0$, $L = 5$ mH and $C = 1$ nF. The boundaries conditions are $V_0 = 1$ V, $V_F = 1$ V and $\omega T_F = 2\pi, 4\pi$.

(a) $\omega T_F = 2\pi$

(b) $\omega T_F = 4\pi$

Figure 5.22: Voltage transient responses for $r = 0$, $V_0 = 1$ V, $V_F = 1$ V and $\omega T_F = 2\pi, 4\pi$

5.4.2 Linear Mode

The final case is $r = \infty$. According to (5.19), the second term is huge and all other can be neglected. Therefore, the equation becomes

$$-4r^2 m^2 = 0 \quad \rightarrow \quad m^2 = 0. \quad (5.52)$$

Roots of the equation are 0, 0 and the most energy-efficient capacitor voltage must be a linear function,

$$v_C(t) = K_1 t + K_2 \quad (5.53)$$

where K_1 and K_2 are constants which can be determined by the boundaries conditions. However, there are only two constants but we have four boundaries conditions. Therefore, some boundaries conditions cannot be met. If we satisfy the capacitor voltage conditions, $v_C(0) = V_0$ and $v_C(T_F) = V_F$, the inductor current conditions will not be satisfied and the most energy-efficient capacitor voltage function will be completely the same as a series RC circuit as well as its current function. Specifically, the most energy-efficient capacitor voltage function is

$$v_C(t) = \frac{V_F - V_0}{T_F} t + V_0. \quad (5.54)$$

However, the most energy-efficient input voltage function is not the same as the series RC circuit. Because the inductor current conditions are not satisfied, that are

$$i_L(0^-) = 0, \quad i_L(0^+) = C\dot{v}_C = \frac{C(V_F - V_0)}{T_F}, \quad \therefore i_L(0^-) \neq i_L(0^+) \quad (5.55)$$

and

$$i_L(T_F^+) = 0, \quad i_L(T_F^-) = C\dot{v}_C = \frac{C(V_F - V_0)}{T_F}, \quad \therefore i_L(T_F^-) \neq i_L(T_F^+), \quad (5.56)$$

the inductor current changes abruptly. Therefore, the inductor voltage which is proportional to the differentiation of the current will be infinite at $t = 0$ and $t = T_F$. Since the input voltage function is the sum of all voltages (voltage across R_{SC} , L and C in Fig. 5.17b) in the circuit, the input voltage will also be infinite at $t = 0$ and $t = T_F$.

To illustrate, let us apply the solution from hyperbolic mode, which is for $r > 1$ and use $r \gg 1$. By using a circuit of Fig. 5.3, assume that the circuit elements are $R_1 = 40 \text{ M}\Omega$, $R_2 = 5 \text{ k}\Omega$, $L = 5 \text{ mH}$ and $C = 1 \text{ nF}$. By using the formula of Z and r in (5.16) and (5.17), we can calculate r to be equal to 100 which is much greater than 1. Fig. 5.23a shows voltage transient responses for $V_0 = 1 \text{ V}$, $V_F = 2 \text{ V}$ and $\omega T_F = 5$. As shown in the figure, there are two spikes at $t = 0$ and $t = T_F$. Theoretically, if $r = \infty$, these two spikes will be infinite. Between the transient duration, the input voltage function and the capacitor voltage function are almost linear functions. Fig. 5.23b shows current transient responses for the same conditions. The current is almost a constant which is consistent with the explanation above.

Figure 5.23: Voltage and current transient responses for $r = 100$, $V_0 = 1 \text{ V}$, $V_F = 2 \text{ V}$ and $\omega T_F = 5$

The normalized lowest input energy can be determined by using (5.33) and substituting $r = \infty$, $k = \infty$. Although $r = \infty$ and $k = \infty$, their ratio can be obtained as

$$\frac{k}{r} = \sqrt{\frac{R_{OC}}{R_{SC}}} \cdot \frac{2Z}{\sqrt{R_{OC}R_{SC}}} = \frac{2Z}{R_{SC}} \quad (5.57)$$

and the normalized lowest input energy is

$$u = 1 + \frac{2\gamma R_{SC}C}{T_F} \quad (5.58)$$

which is exactly the same as the normalized lowest input energy of series RC circuit. Calculation can be found in Appendix B.18. In conclude, the most energy-efficient functions and the normalized lowest input energy in linear mode is almost the same as series RC circuit. The different is the spikes at the boundaries of the input voltage function.

5.5 The Most Energy-efficient Functions of Parallel LC Circuits

5.5.1 Boundary Conditions

Figure 5.24: Arbitrary resistive topologies with a parallel LC.

Parallel LC circuits can be optimized, determined the most energy-efficient functions and the lowest input energy similarly with the series LC circuits. Fig. 5.24 shows an arbitrary resistive topologies with a parallel LC. In the procedure of the optimization, boundary conditions must be given. Specifically, the initial circuit energy and the final circuit energy must be given. Because $v_C = v_L$, if there is voltage across the capacitor, the derivative of current of the inductor will not be zero and the circuit will not be in steady state. Therefore, steady state conditions must be

$$v_C(0^-) = v_L(0^+) = L\dot{i}_L(0) = 0 \quad (5.59)$$

$$v_C(T_F^-) = v_L(T_F^+) = L\dot{i}_L(T_F) = 0 \quad (5.60)$$

$$i_L(0^-) = i_L(0^+) = i_L(0) = I_0 \quad (5.61)$$

$$i_L(T_F^-) = i_L(T_F^+) = i_L(T_F) = I_F. \quad (5.62)$$

They imply that for steady state boundary conditions, the initial circuit energy must only be stored in the inductor as well the final circuit energy. There is no stored energy in the capacitor at the initial state and also the final state.

5.5.2 The Euler-Lagrange's Differential Equation and The Most Energy-efficient Functions

By inspection the circuit of Fig. 5.24, $v_O = L\dot{i}_L$ and the output current i_O in terms of the current through the inductor i_L is

$$i_O = i_C + i_L = C\dot{v}_C + i_L = \frac{1}{\omega^2}\ddot{i}_L + i_L \quad (5.63)$$

The relationship between the inputs and the outputs can be represented by the ABCD matrix of the resistive network and the Lagrangian function which is the input power is (5.12). Because the output current in (5.63) involves the second derivative of i_L , the Lagrangian function also involves the second derivative of i_L . Therefore the Euler-Lagrange's differential equation for second order derivatives which is

$$\frac{\partial \mathcal{L}}{\partial i_L} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{i}_L} \right) + \frac{d^2}{dt^2} \left(\frac{\partial \mathcal{L}}{\partial \ddot{i}_L} \right) = 0 \quad (5.64)$$

must be applied.

Manipulating the Euler-Lagrange's differential equation of parallel LC circuits can be done similarly with series LC circuits as elaborated in Appendix B.9. The differential equation is obtained as

$$\ddot{\ddot{i}}_L + 2(1 - 2r_L^2)\omega^2\ddot{i}_L + \omega^4 i_L = 0 \quad (5.65)$$

where

$$r_L = \frac{Z}{2\sqrt{R_{OC}R_{SC}}}. \quad (5.66)$$

It is similar with (5.18) where $v_C \rightarrow i_L$ and $r \rightarrow r_L$. Recall that the most energy-efficient functions are obtained from the differential equation. Therefore, the most energy-efficient functions of parallel LC circuits are also similar with series LC circuits. It can be divided into three modes, (i) hyperbolic mode for $r_L > 1$, (ii) hyperbolic and linear mode for $r_L = 1$ and (iii) hyperbolic and trigonometric mode for $0 < r_L < 1$. The most energy-efficient capacitor voltage functions for series LC circuits are similar with the most energy-efficient inductor current functions for parallel LC circuits as well as the input voltage functions for series LC circuits are similar with the input current functions for parallel LC circuits. Moreover, the most energy-efficient input current functions for series LC circuits are also similar with the most energy-efficient voltage functions for parallel LC circuits. Therefore, transient responses that are illustrated in the previous sections are similar with transient responses for parallel LC circuits while voltage responses and current responses must be interchanged.

5.6 The Lowest Input Energy and Special Cases of Parallel LC Circuits

Once the most energy-efficient inductor current function is determined by solving the differential equation in (5.65), the input voltage function and the input current function can be determined by using routine circuit analysis. Hence, the lowest input energy can be calculated similarly with series LC circuits as shown in Appendix B.12. Calculation gives us

$$E_I = \frac{L(I_F^2 - I_0^2)}{2(k^2 - 1)} \cdot \left(k^2 + 1 + \frac{k}{r_L} \cdot \frac{\ddot{\ddot{i}}_L(0)I_0 - \ddot{\ddot{i}}_L(T_F)I_F}{\omega^3(I_F^2 - I_0^2)} \right). \quad (5.67)$$

which is totally similar with (5.31). By using

$$\gamma_L = \frac{I_F - I_0}{I_F + I_0} \quad (5.68)$$

instead of γ , the transient duration analysis of the lowest input energy of parallel LC circuits can be done similarly with the transient duration analysis of the lowest input energy of series LC circuits.

Two special cases of series LC circuits are $r = 0$ and $r = \infty$. They are corresponding to resonant mode and linear mode respectively. Theirs circuit topology are shown in Fig. 5.17. For parallel LC circuits, $r_L = 0$ and $r_L = \infty$ are two special cases and they are also corresponding to resonant mode and linear mode respectively. However, since the definition of r and r_L are slightly different, theirs circuit topology and theirs meaning are also different. For $r_L = 0$, it means that $R_{OC} = \infty$ and for $r_L = \infty$, it means that $R_{SC} = 0$. The circuit topologies for these two special cases are shown in Fig. 5.25.

Although the circuits are different for $r = 0$ and $r_L = 0$ as well as $r = \infty$ and $r_L = \infty$, theirs transient responses and theirs lowest input energy are similar because the important parameters are r_L and r which are the same values.

Figure 5.25: Circuit topologies when $r_L = 0$ and $r_L = \infty$

5.7 Conclusion

In this chapter, the lowest input energy and the most energy-efficient functions of series LC circuits and parallel LC circuits in transient state have been presented. The series LC circuit has been analyzed comprehensively and the parallel LC circuit has been shown that it can be analyzed in a similar manner. Firstly, boundaries conditions which are essential for optimization, have been elaborated and distinguished between general boundaries conditions and steady state boundaries conditions. The Euler-Lagrange's differential equation for the series LC circuit has been derived. Solutions of the Euler-Lagrange's differential equation which are the most energy-efficient functions have been categorized into three modes which are (i) hyperbolic mode, (ii) hyperbolic and linear mode and (iii) hyperbolic and trigonometric mode. Each mode has been discussed and their transient responses have been illustrated. Once the most energy-efficient functions have been determined, the lowest input energy has been evaluated. Transient duration analysis of the lowest input energy for each mode has been explained thoroughly. Speed and energy trade-off, energy flow and optimum transient duration have been analyzed both graphically and mathematically. There are two special cases which are resonant mode and linear mode. Both cases have been discussed clearly in details, be it transient responses or the lowest input energy.

Chapter 6

Generalization of the Most Energy-efficient Functions and the Lowest Input Energy of Passive Linear Circuits

6.1 Introduction

A series RC circuit which is the simplest first order circuit has been optimized in chapter 3. The optimize has been generalized for any first order circuits and it has been explained thoroughly in chapter 4. Furthermore, second order circuits which are series LC and parallel LC have been optimized in chapter 5. For higher order circuits or more generally, for any order circuits, the optimization will be generalized in this chapter.

Firstly, we will introduce a dummy function which is an essential function for generalization in Section 6.2. Briefly, the dummy function is a function that vanishing a complementary solutions of ODE. Therefore, a reviews of complementary and particular solutions of ODE will be discussed for deep understanding of the dummy function. Section 6.3 will generalize formulas for determining the most energy-efficient functions. Once the most energy-efficient functions are determined, the lowest input energy can also be determined as explained in Section 6.4. There will be a few examples that illustrate on using the general formulas for determining the most energy-efficient functions and the lowest input energy in Section 6.5. The results will be discussed and compared with other results in the previous chapters to show consistency. Lastly, Section 6.6 will apply the general formula to RC chain circuit which is a higher order circuit. Conclusion will be draw in Section 6.7.

6.2 Dummy Function

6.2.1 Reviews of Complementary and Particular Solutions of ODE

In chapter 3, the most energy-efficient functions of a simple series RC circuit have been derived. Let us consider this circuit again to review about complementary and particular solutions of ODE. The series RC circuit is shown in Fig. 6.1. The differential equation of the circuit can be written easily as

$$v_I(t) = v_C(t) + RC\dot{v}_C(t). \quad (6.1)$$

where all variables are their usual meaning as in the previous chapters.

Figure 6.1: A series RC circuit.

We have already known that the most energy-efficient input voltage function for the series RC circuit is the raised ramp function. We obtained the input voltage function by using the most energy-efficient capacitor voltage function that we derived from the Euler-Lagrange's differential equation. However, we have never demonstrated or calculated the capacitor voltage function by using the most energy-efficient input voltage function. In other words, assume that we know the input voltage, what is the capacitor voltage response?

We know that the most energy-efficient input voltage function is the raised ramp function or mathematically, a linear function with specific constants according to the boundaries conditions. However, let us consider a general input voltage function that is a linear function,

$$v_I(t) = at + b \quad (6.2)$$

where a and b are constants. By solving (6.1) The capacitor voltage response will be

$$v_C(t) = at + (b - aRC) - (b - aRC)e^{-t/RC} \quad (6.3)$$

if $v_C(0) = 0$. The calculation can be found in Appendix B.2.

The capacitor voltage can be separated into two parts, the first part, $at + (b - aRC)$ is a particular solution of ODE. The particular solution is a solution of a nonhomogeneous equation. In this case, the input voltage function is a linear function. Therefore, the particular solution must be a linear function.

The second part is a complementary solution which is $(b - aRC)e^{-t/RC}$. The complementary solution is a solution of a homogeneous equation. For first order differential equation, the complementary solution must be an exponential function.

For arbitrary constants a and b , the capacitor voltage comprises of the complementary and homogeneous solutions. However, it is easily to see that if $b/a = RC$, the complementary solution will vanish and the capacitor voltage will be only the particular solution. For the most energy-efficient input voltage function, b/a is exactly equal to RC (see Section 3.2) and the capacitor voltage is a linear function which is only the particular solution. Moreover, according to the previous chapters, we are capable to notice that all of the most energy-efficient input functions cause the complementary solution vanishes. Therefore, the input functions must have a property that make the complementary solution vanishes. One way to obtain this property is the input functions must be written in terms of dummy function as follows.

6.2.2 Input Functions with Dummy Function

For the series RC circuit, if the input voltage function is in terms of a dummy function $f(t)$ as $v_I(t) = f(t) + RC\dot{f}(t)$, the differential equation will be

$$f(t) + RC\dot{f}(t) = v_C(t) + RC\dot{v}_C(t). \quad (6.4)$$

Intuitively, if the initial condition of $f(t)$ is the same as $v_C(t)$ or more specifically $f(0) = v_C(0)$, the solution of the ODE is $v_C(t) = f(t)$. The solution is equal to the dummy function. Therefore, the complementary solution vanishes.

The concept of using the dummy function can be considered in frequency domain. For series RC circuit, the input impedance of the circuit can be written as

$$Z(s) = \frac{V_I(s)}{I_I(s)} = R \frac{s + 1/RC}{s}. \quad (6.5)$$

For any input voltage functions, the input current function in frequency domain can be written as

$$I_I(s) = \frac{V_I(s)}{Z(s)} = \frac{sV_I(s)}{R(s + 1/RC)}. \quad (6.6)$$

Normally, according to the input current function in (6.6), the input current in time domain must have an exponential term because of $s + 1/RC$ in the frequency domain. We can roughly say that the $s + 1/RC$ term causes the complementary solution. However, by using the dummy function, the input voltage function can be written as $F(s) + sRCF(s)$ or $RCF(s)(s + 1/RC)$ where $F(s)$ is the Laplace transform of $f(t)$. Then, the input current function is

$$I_I(s) = sRC \cdot \frac{F(s)}{R}. \quad (6.7)$$

It is easily to see that the complementary solution vanishes.

The idea of the dummy function can be generalized and we can safely propose that for input impedance,

$$Z(s) = \frac{\sum_{i=0}^{\infty} a_i s^i}{\sum_{i=0}^{\infty} b_i s^i} \quad (6.8)$$

where $a_0, b_0, a_1, b_1 \dots$ are constants, the most energy-efficient input voltage function is in form of

$$v_I(t) = \sum_{i=0}^{\infty} a_i f^{(i)}(t). \quad (6.9)$$

An input current function can be obtained analogically with the input voltage function as

$$i_I(t) = \sum_{i=0}^{\infty} b_i f^{(i)}(t). \quad (6.10)$$

Now, the optimization is different from the previous chapters. In this chapter, it is not necessary to know actual circuit topology, only input impedance is needed and of course, boundaries conditions. But the boundaries conditions are now the initial and final conditions of the input function or more specifically, the initial and final conditions of the dummy function instead of the boundaries conditions of circuit elements. To avoid any misconceptions, let us state the optimization problem clearly: what is the most energy-efficient input functions for changing the input voltage function from $v_I(0^-) = V_{I0}$ to $v_I(T_F^+) = V_{IF}$ or $i_I(0^-) = I_{I0}$ to $i_I(T_F^+) = I_{IF}$ of a circuit that has the input impedance described in (6.8)?

The first step to determine the most energy-efficient input voltage function has already explained which is writing the input voltage function in form of the dummy function as (6.9). The later step is solving the dummy function by using calculus of variations. But before solving, the boundaries conditions of the dummy function must be determined.

6.2.3 Boundaries Conditions of the Dummy Function

According to higher derivatives Euler-Lagrange's differential equation, if the highest derivatives of a variable function in Lagrangian function is N , boundaries conditions for 0 to $N - 1$ derivatives of the variable function must be determined. At initial state and final state, the circuit should be in steady state as well as the input voltage. Therefore, all the derivatives of the dummy function are equal to zero,

$$f^{(1)}(0) = f^{(1)}(T_F) = f^{(2)}(0) = f^{(2)}(T_F) = \dots = f^{(N-1)}(0) = f^{(N-1)}(T_F) = 0. \quad (6.11)$$

If $v_I(0^-) = V_{I0}$ and $v_I(T_F^+) = V_{IF}$, the dummy function at the boundaries must be equal to

$$f(0) = \frac{V_{I0}}{a_0} \quad (6.12)$$

and

$$f(T_F) = \frac{V_{IF}}{a_0}. \quad (6.13)$$

There are some circuits that at the steady state, the input voltage must be zero. In this cases, a_0 in (6.8) or (6.9) is zero and the boundaries conditions above are not applicable. However, for these kinds of circuit, the boundaries conditions of the circuit must be determined by the input current and the boundaries conditions must be

$$f(0) = \frac{I_{I0}}{b_0} \quad (6.14)$$

and

$$f(T_F) = \frac{I_{IF}}{b_0}. \quad (6.15)$$

In practical, a_0 and b_0 cannot be zero at the same time.

Notice that solving the Euler-Lagrange's differential equation does not need the N derivatives values of the dummy function at the boundaries. Therefore, $f^{(N)}(0^-) \neq f^{(N)}(0^+)$ and $f^{(N)}(T_F^-) \neq f^{(N)}(T_F^+)$ and because of the input voltage is the sum of the dummy function derivatives including $f^{(N)}$ with weights, the input voltage function must have discontinuities at $t = 0$ and $t = T_F$. Mathematically, $v_I(0^-) \neq v_I(0^+)$ or $v_I(0^+) \neq V_{I0}$ as well as $v_I(T_F^-) \neq v_I(T_F^+)$ or $v_I(T_F^-) \neq V_{IF}$. The explanation of the discontinuity is also applicable for the input current function.

6.3 Generalization of the Most Energy-efficient Functions

To determined to most energy-efficient functions, the dummy function must be determined. According to (6.9), Laplace transform of $v_I(t)$ is

$$V_I(s) = \sum_{i=0}^{\infty} a_i s^i F(s). \quad (6.16)$$

With the input impedance in (6.8), the input current can be determined as

$$I_I(s) = \sum_{i=0}^{\infty} b_i s^i F(s) \quad (6.17)$$

or in time domain, it is the same as (6.10).

Because we want to minimize the input energy which is an integration of an input power, Lagrangian function is the input power $\mathcal{L} = v_I i_I$. General Euler-Lagrange's differential equation for higher derivatives must be applied.

$$\sum_{n=0}^{\infty} (-1)^n \frac{d^n}{dt^n} \left(\frac{\partial \mathcal{L}}{\partial f^{(n)}} \right) = 0 \quad (6.18)$$

To manipulate this differential equation, firstly let us consider

$$\frac{\partial \mathcal{L}}{\partial f^{(n)}} = v_I \frac{\partial i_I}{\partial f^{(n)}} + i_I \frac{\partial v_I}{\partial f^{(n)}}.$$

By substituting (6.9) and (6.10), we get

$$\frac{\partial \mathcal{L}}{\partial f^{(n)}} = v_I b_n + i_I a_n = b_n \sum_{i=0}^{\infty} a_i f^{(i)}(t) + a_n \sum_{i=0}^{\infty} b_i f^{(i)}(t).$$

Total derivative respect to t must be apply to this partial derivative n times. Only the dummy function f can be differentiated. The dummy function has been already differentiated i times, therefore applying total derivative n times give us

$$\frac{d^n}{dt^n} \left(\frac{\partial \mathcal{L}}{\partial f^{(n)}} \right) = b_n \sum_{i=0}^{\infty} a_i f^{(i+n)}(t) + a_n \sum_{i=0}^{\infty} b_i f^{(i+n)}(t)$$

and the Euler-Lagrange's differential equation becomes

$$\sum_{n=0}^{\infty} (-1)^n \left(b_n \sum_{i=0}^{\infty} a_i f^{(i+n)}(t) + a_n \sum_{i=0}^{\infty} b_i f^{(i+n)}(t) \right) = 0.$$

For more convenient, let us denote a new index of iteration $k = i + n$. When $i = 0$, $k = n$ and when $i = \infty$, $k = \infty$. By pulling the summation to the outside, the differential equation is

$$\sum_{n=0}^{\infty} \sum_{k=n}^{\infty} (-1)^n f^{(k)}(t) (a_n b_{k-n} + b_n a_{k-n}) = 0. \quad (6.19)$$

Although the differential equation in (6.19) seems to be compact enough, it is not convenient because the dummy function is inside the double sum. If we expand the summation, it is hard to realize the differential equation in a traditional form such as $f(t) + \dots f^{(1)}(t) + \dots f^{(2)}(t) + \dots = 0$. However, if the double summation is reordered, the differential equation in the traditional form can be realized.

Fig. 6.2 illustrates domain of summation of (6.19). The inner iterates from the line $k = n$ to ∞ and the outer sum iterates from $k = 0$ to ∞ . The iteration is depicted in Fig. 6.2a. The arrows represent the direction of the inner sum which are parallel with the k axis.

To reorder the summation, we change the direction of the inner sum as depicted in Fig. 6.2b. The direction of the inner sum has changed to be parallel with the n axis and the iteration of the inner sum is $n = 0$ to $n = k$. While the outer sum must be $k = 0$ to $n = \infty$. Therefore, Euler-Lagrange's differential equation in (6.19) is written as

$$\sum_{k=0}^{\infty} f^{(k)}(t) \sum_{n=0}^k (-1)^n (a_n b_{k-n} + b_n a_{k-n}) = 0. \quad (6.20)$$

Notice that we can pull out the term $f^{(k)}(t)$ from the inner summation because it does not involve the iterator n .

Figure 6.2: Domain of summation of (6.19).

The differential equation in (6.20) is now in the traditional form. Coefficient of each derivative is the inner sum. Obviously, this is a linear ODE and can be solved numerically once the coefficients are determined. Although the coefficients or the inner sum seem to be compact enough, it can be reduced further.

Let us consider the inner sum which can distribute as

$$\sum_{n=0}^k (-1)^n (a_n b_{k-n} + b_n a_{k-n}) = \sum_{n=0}^k (-1)^n a_n b_{k-n} + \sum_{n=0}^k (-1)^n b_n a_{k-n}.$$

For the second term, let us introduce a temporary iterator $l = k - n$. When $n = 0$, $l = k$ and when $n = k$, $l = 0$. By substituting this iterator, the summation becomes

$$\sum_{n=0}^k (-1)^n (a_n b_{k-n} + b_n a_{k-n}) = \sum_{n=0}^k (-1)^n a_n b_{k-n} + \sum_{l=0}^k (-1)^{k-l} b_{k-l} a_l$$

or

$$\sum_{n=0}^k (-1)^n (a_n b_{k-n} + b_n a_{k-n}) = \sum_{n=0}^k (-1)^n a_n b_{k-n} + (-1)^k \sum_{l=0}^k (-1)^l a_l b_{k-l}.$$

We can see that the sum of the first term with the iterator n and the sum of the second term with the iterator l is the same summation. Therefore, it can be rearranged to

$$\sum_{n=0}^k (-1)^n (a_n b_{k-n} + b_n a_{k-n}) = \sum_{n=0}^k (-1)^n a_n b_{k-n} (1 + (-1)^k).$$

If k is an odd number, $1 + (-1)^k = 0$ and if k is an even number, $1 + (-1)^k = 2$. Hence, the summation above is equal to 0 if k is odd. For k is even, we can write $k = 2p$ where $p \in \{0, 1, 2, \dots\}$ and the summation becomes

$$\sum_{n=0}^{2p} (-1)^n (a_n b_{2p-n} + b_n a_{2p-n}) = 2 \sum_{n=0}^{2p} (-1)^n a_n b_{2p-n}.$$

By substituting this result into (6.20), the Euler-Lagrange's differential equation is

$$\sum_{p=0}^{\infty} f^{(2p)}(t) \sum_{n=0}^{2p} (-1)^n a_n b_{2p-n} = 0. \quad (6.21)$$

The differential equation involves only even order of derivatives. Recall that the inner sum is only a coefficient. Therefore, we define the coefficient of each order of derivative as

$$c_p \equiv \sum_{n=0}^{2p} (-1)^n a_n b_{2p-n} \quad (6.22)$$

and the differential equation is

$$\sum_{p=0}^{\infty} c_p f^{(2p)}(t) = 0 \quad (6.23)$$

By solving the differential equation in (6.23), the most energy-efficient dummy function can be found. Once the dummy function is determined, the input voltage function or the input current function can be calculated by using (6.9) and (6.10) respectively.

Although the sum in (6.23) seems to be infinite, it is finite for any N^{th} order circuits. For N^{th} order circuit, $a_{N+1} = b_{N+1} = a_{N+2} = b_{N+2} = \dots = 0$ and c_p is equal to zero for $p > N$. Because of the definition of c_p in (6.22). If $p > N$, there will be n which is less than $2p$ that cause $n > N$ or $2p - n > N$. Therefore, either a_n or b_{2p-n} is zero and the sum in (6.22) will be zero. Moreover, we can say that for N^{th} order circuit, the Euler-Lagrange's differential equation is

$$\sum_{p=0}^N c_p f^{(2p)}(t) = 0 \quad (6.24)$$

Notice that the highest derivative in the differential equation is $2N$ and solving this equation needs $2N$ boundaries conditions. These boundaries conditions are determined in the previous sections.

6.4 Generalization of the Lowest Input Energy

Not only the most energy-efficient functions can be generalized by utilizing the dummy function, but also the lowest input energy. The input energy can be calculated straightforwardly as

$$\begin{aligned} E_I &= \int_0^{T_F} v_I(t) i_I(t) dt = \int_0^{T_F} \sum_{i=0}^{\infty} a_i f^{(i)}(t) \cdot \sum_{j=0}^{\infty} b_j f^{(j)}(t) dt \\ &= \int_0^{T_F} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i b_j f^{(i)}(t) f^{(j)}(t) dt. \end{aligned} \quad (6.25)$$

Although the double summation with the iterators i and j seems to be convenient, it is not quite useful. We shall denote another iterator $k = i + j$ and substitute $j = k - i$ into (6.25). Domain of summation of (6.25) is illustrated in Fig. 6.3. Originally, direction of the summation can be viewed as in Fig. 6.3a. However, by using the iterator k , the direction of summation is diagonal as Fig. 6.3b and the input energy can be written as

$$E_I = \sum_{k=0}^{\infty} \sum_{i=0}^k a_i b_{k-i} \int_0^{T_F} f^{(i)}(t) f^{(k-i)}(t) dt. \quad (6.26)$$

Next, we calculate the integral by dividing into two cases which are k is an odd number and k is an even number. For k is odd, it can be written as $k = 2p + 1$ where $p \in \{0, 1, 2, \dots\}$ and the integral becomes

$$\int_0^{T_F} f^{(i)}(t) f^{(2p+1-i)}(t) dt.$$

Figure 6.3: Domain of summation of (6.25).

Let us assume that $i < 2p + 1 - i$, by using integration by parts, we get

$$\int_0^{T_F} f^{(i)}(t) f^{(2p+1-i)}(t) dt = \left[f^{(i)}(t) f^{(2p-i)}(t) \right]_0^{T_F} - \int_0^{T_F} f^{(i+1)}(t) f^{(2p-i)}(t) dt.$$

According to boundaries conditions in (6.11), all derivatives from 1 to $N - 1$ must be zero for a N^{th} -order circuit. Moreover, although the summation in (6.26) seems to sum for all i and k , it sums only for $i \leq N$ and $k \leq 2N$ because for the other values, the $a_i b_{k-i}$ is zero. Therefore, for the integral, we also can only consider for $i \leq N$ and $2p + 1 - i \leq N$. With all these conditions, the first term of the right hand side is always zero except for $i = 0$ and $p = 0$ where $f^{(0)} \neq 0$. For the exceptional case, we will calculate it later. But for the other cases, the integral should be

$$\int_0^{T_F} f^{(i)}(t) f^{(2p+1-i)}(t) dt = - \int_0^{T_F} f^{(i+1)}(t) f^{(2p-i)}(t) dt.$$

By applying the integration by parts one more time, we get

$$\int_0^{T_F} f^{(i)}(t) f^{(2p+1-i)}(t) dt = - \left[f^{(i+1)}(t) f^{(2p-i-1)}(t) \right]_0^{T_F} + \int_0^{T_F} f^{(i+2)}(t) f^{(2p-i-1)}(t) dt.$$

With the boundaries conditions, the non-integral term is zero. The integration by parts can be done $p - i$ times and the result is

$$\begin{aligned} \int_0^{T_F} f^{(i)}(t) f^{(2p+1-i)}(t) dt &= (-1)^{p-i} \int_0^{T_F} f^{(i+p-i)}(t) f^{(2p+1-i-p+i)}(t) dt \\ &= (-1)^{p-i} \int_0^{T_F} f^{(p)}(t) f^{(p+1)}(t) dt. \end{aligned}$$

The right hand side can be calculated easily as

$$\begin{aligned} \int_0^{T_F} f^{(i)}(t) f^{(2p+1-i)}(t) dt &= (-1)^{p-i} \int_0^{T_F} f^{(p)}(t) d \left(f^{(p)}(t) \right) \\ &= \left[\frac{(f^{(p)}(t))^2}{2} \right]_0^{T_F} = 0 \quad (6.27) \end{aligned}$$

except $p = 0$. For the exceptional case, $i = 0$ and $p = 0$, we can calculate the integral directly as

$$\int_0^{T_F} f^{(0)}(t) f^{(1)}(t) dt = \int_0^{T_F} f^{(0)}(t) d\left(f^{(0)}(t)\right) = \left[\frac{(f^{(0)}(t))^2}{2} \right]_0^{T_F} = \frac{f^2(T_F) - f^2(0)}{2} \quad (6.28)$$

Although the calculation above is only for $i < 2p + 1 - i$, it can be applied for $i > 2p + 1 - i$ by introducing a new parameter $i' = 2p + 1 - i$ and by rearranging, we get $i = 2p + 1 - i'$. Therefore, in this case we have $2p + 1 - i' > i'$ which is the same condition as above. Hence, the result of the calculation must be the same.

In conclude, the calculation above give us that for k is odd, the integration is zero except for $p = 0$ and $i = 0$ which the result is given in (6.28). By substituting this result into (6.26), we obtain

$$E_I = \frac{(a_0 b_1 + a_1 b_0)(f^2(T_F) - f^2(0))}{2} + \sum_{p=0}^{\infty} \sum_{i=0}^{2p} a_i b_{2p-i} \int_0^{T_F} f^{(i)}(t) f^{(2p-i)}(t) dt. \quad (6.29)$$

Notice that only the summation for k is even or $k = 2p$ is left to be calculated.

For the remaining integral, we also can calculate by using the integration by parts as

$$\int_0^{T_F} f^{(i)}(t) f^{(2p-i)}(t) dt = \left[f^{(i-1)}(t) f^{(2p-i)}(t) \right]_0^{T_F} - \int_0^{T_F} f^{(i-1)}(t) f^{(2p-i+1)}(t) dt$$

where i or $2p - i$ are not equal to zero. (If i is zero and $2p - i$ is not equal to zero, we can define $i' = 2p - i$ and the calculation will be the same way as follows for i or $2p - i$ are not equal to zero). Again, we know that the non-integral is zero because of the boundaries conditions. The integral by parts can be done $i - 1$ times to get

$$\int_0^{T_F} f^{(i)}(t) f^{(2p-i)}(t) dt = (-1)^{i-1} \int_0^{T_F} f^{(1)}(t) f^{(2p-1)}(t) dt.$$

By applying the integration by parts one more time, we obtain

$$\int_0^{T_F} f^{(i)}(t) f^{(2p-i)}(t) dt = (-1)^{i-1} \left[f^{(0)}(t) f^{(2p-1)}(t) \right]_0^{T_F} + (-1)^i \int_0^{T_F} f^{(0)}(t) f^{(2p)}(t) dt.$$

Notice that for this expression, the non-integral term is not zero because neither $f(t)$ nor $f^{(2p-1)}(t)$ is always zero at the boundaries.

In case of $i = 0$ and $2p - i = 0$ or $p = 0$, the integral do not need any the integration by parts. By substituting this result into (6.29), the input energy becomes

$$\begin{aligned} E_I = \frac{(a_0 b_1 + a_1 b_0)(f^2(T_F) - f^2(0))}{2} - \sum_{p=1}^{\infty} \sum_{i=0}^{2p} a_i b_{2p-i} (-1)^i \left[f^{(0)}(t) f^{(2p-1)}(t) \right]_0^{T_F} \\ + \sum_{p=0}^{\infty} \sum_{i=0}^{2p} a_i b_{2p-i} (-1)^i \int_0^{T_F} f^{(0)}(t) f^{(2p)}(t) dt. \end{aligned}$$

Nevertheless, by using the definition of c_p in (6.22), the input energy is reduced to

$$\begin{aligned} E_I = \frac{(a_0 b_1 + a_1 b_0)(f^2(T_F) - f^2(0))}{2} - \sum_{p=1}^{\infty} c_p \left[f^{(0)}(t) f^{(2p-1)}(t) \right]_0^{T_F} \\ + \int_0^{T_F} f^{(0)}(t) \sum_{p=0}^{\infty} c_p f^{(2p)}(t) dt. \end{aligned}$$

Notice that the summation inside the integral is Euler-Lagrange's differential equation in (6.23) and it is equal to zero. Therefore, the lowest input energy can be derived as

$$E_I = \frac{(a_0b_1 + a_1b_0)(f^2(T_F) - f^2(0))}{2} + \sum_{p=1}^{\infty} c_p(f(0)f^{(2p-1)}(0) - f(T_F)f^{(2p-1)}(T_F)) \quad (6.30)$$

Moreover, from the boundaries conditions, $f^{(2p-1)} = 0$ for $2p-1 < N$ or $p < (N+1)/2$ and $c_p = 0$ for $p > N$. Hence, the boundaries of summation can be changed to

$$E_I = \frac{(a_0b_1 + a_1b_0)(f^2(T_F) - f^2(0))}{2} + \sum_{p=\lceil \frac{N+1}{2} \rceil}^N c_p(f(0)f^{(2p-1)}(0) - f(T_F)f^{(2p-1)}(T_F)) \quad (6.31)$$

where $\lceil x \rceil$ is the ceiling function. In conclude, if the most energy-efficient function is applied to N^{th} -order circuit, it will cause the lowest input energy as described in (6.31).

6.5 Applying General Formulas of the Most Energy-efficient Functions and the Lowest Input Energy

In the previous sections, general formulas or general methods for determining the most energy-efficient functions and the lowest input energy have been derived. In this section, several examples of applying the general formulas will be described and compared with results in the other chapters. Consistency of using general formulas and using the Euler-Lagrange's differential equation directly will be shown.

Generally, procedure of determining the most energy-efficient function and the lowest input energy can be concluded as follows.

- Step 0: The input impedance of the circuit and the boundaries conditions must be known.
- Step 1: Write the input impedance in form of

$$Z(s) = \frac{\sum_{i=0}^{\infty} a_i s^i}{\sum_{i=0}^{\infty} b_i s^i} \quad (6.32)$$

and determine the coefficients c_p by using the formula

$$c_p = \sum_{n=0}^{2p} (-1)^n a_n b_{2p-n} \quad (6.33)$$

- Step 2: For N^{th} -order circuit, solve the differential equation

$$\sum_{p=0}^N c_p f^{(2p)}(t) = 0 \quad (6.34)$$

to obtain a general solution of the dummy function

- Step 3: Apply the boundaries conditions to obtain a unique solution of the dummy function
- Step 4: Apply the most energy-efficient input voltage function formula,

$$v_I(t) = \sum_{i=0}^{\infty} a_i f^{(i)}(t). \quad (6.35)$$

to obtain the most energy-efficient input voltage function and apply the most energy-efficient input current function formula,

$$i_I(t) = \sum_{i=0}^{\infty} b_i f^{(i)}(t). \quad (6.36)$$

to obtain the most energy-efficient input current function

- Step 5: Apply the lowest input energy formula,

$$E_I = \frac{(a_0 b_1 + a_1 b_0)(f^2(T_F) - f^2(0))}{2} + \sum_{p=\lceil \frac{N+1}{2} \rceil}^N c_p (f(0)f^{(2p-1)}(0) - f(T_F)f^{(2p-1)}(T_F)) \quad (6.37)$$

to obtain the lowest input energy

Examples of using the procedure are described as follows.

6.5.1 Example 1: Series RC Circuit

Let us start with the simplest one, the series RC circuits. The circuit is shown in Fig. 6.1. Assume that we want to change the input voltage from V_{I0} to V_{IF} within a transient duration T_F , what is the most energy-efficient input voltage function and what is the lowest input energy in transient state?

Firstly, the input impedance can be easily determined as

$$Z(s) = R + \frac{1}{sC} = \frac{RCs + 1}{Cs}. \quad (6.38)$$

By comparing with (6.32), we realize that $a_0 = 1$, $a_1 = RC$, $b_0 = 0$ and $b_1 = C$. By using the c_p formula in (6.33), we get

$$c_0 = a_0 b_0 = 0, \quad c_1 = -(a_0 b_2 + a_1 b_1 + a_2 b_0) = -RC^2.$$

According to (6.34), the differential equation becomes

$$f^{(2)}(t)(-RC^2) = 0. \quad (6.39)$$

Obviously, general solution of the differential equation is

$$f(t) = K_1 t + K_0 \quad (6.40)$$

where K_0 and K_1 are constants.

Because of $a_0 = 1$, the boundaries conditions from (6.12) and (6.13) can be obtained as $f(0) = V_{I0}$ and $f(T_F) = V_{IF}$. A unique solution must be

$$f(t) = \frac{V_{IF} - V_{I0}}{T_F} t + V_{I0}. \quad (6.41)$$

From (6.35), the most energy-efficient input voltage function is

$$v_I(t) = \frac{V_{IF} - V_{I0}}{T_F}t + V_{I0} + \frac{(V_{IF} - V_{I0})RC}{T_F} \quad (6.42)$$

and from (6.36), the most energy-efficient input current function is

$$i_I(t) = \frac{(V_{IF} - V_{I0})C}{T_F}. \quad (6.43)$$

To compare these results with the previous results in chapter 3, note that V_{IF} and V_{I0} in these results are not V_F and V_0 in the previous results. V_{IF} and V_{I0} are the initial and the final voltage of the input voltage while V_F and V_0 are the initial and the final voltage of the capacitor voltage. However, for the series RC circuit at steady state, $V_{IF} = V_F$ and $V_{I0} = V_0$. Moreover, the results in chapter 3 are the results when $V_{I0} = V_0 = 0$. Therefore, for $V_{I0} = 0$, the most energy-efficient input voltage function becomes

$$v_I(t) = \frac{V_{IF}}{T_F}t + \frac{V_{IF}RC}{T_F}$$

and the most energy-efficient input current function becomes

$$i_I(t) = \frac{V_{IF}C}{T_F}.$$

which are consistent with the previous results in (3.6b) and (3.7b). For the lowest input energy, we can apply (6.37) to obtain

$$\begin{aligned} E_I &= \frac{C(V_{IF}^2 - V_{I0}^2)}{2} + c_1 \left[V_{I0} \left(\frac{V_{IF} - V_{I0}}{T_F} \right) - V_{IF} \left(\frac{V_{IF} - V_{I0}}{T_F} \right) \right] \\ &= \frac{C(V_{IF}^2 - V_{I0}^2)}{2} + \frac{RC^2(V_{IF} - V_{I0})^2}{T_F} \end{aligned} \quad (6.44)$$

which is consistent with (4.38) in chapter 4.

6.5.2 Example 2: Parallel RL Circuit

Another passive first order circuit that the optimization is very similar to the series RC circuit is parallel RL circuit. The circuit is shown in Fig. 6.4. However, recall that at steady state, the input voltage must be zero. Therefore, for the parallel RL circuit, the optimization is done for changing the input current from I_{I0} to I_{IF} within a duration T_F .

Figure 6.4: A parallel RL circuit

Firstly, the input impedance can be realized as

$$Z(s) = \frac{sLR}{sL + R} \quad (6.45)$$

Hence, $a_0 = 0$, $a_1 = LR$, $b_0 = R$ and $b_1 = L$. c_p can be calculated as

$$c_0 = a_0 b_0 = 0, \quad c_1 = -(a_0 b_2 + a_1 b_1 + a_2 b_0) = -RL^2$$

and the differential equation becomes

$$f^{(2)}(t)(-RL^2) = 0. \quad (6.46)$$

General solution of the differential equation is

$$f(t) = K_1 t + K_0 \quad (6.47)$$

where K_0 and K_1 are constants.

Unlike the series RC circuit, boundaries conditions for the dummy function can be determined by using the boundaries conditions of the input current instead of the input voltage. According to (6.14) and (6.15), because of $b_0 = R$,

$$f(0) = \frac{I_{I0}}{R}, \quad f(T_F) = \frac{I_{IF}}{R}.$$

A unique solution must be

$$f(t) = \frac{I_{IF} - I_{I0}}{T_F R} t + \frac{I_{I0}}{R}. \quad (6.48)$$

From (6.35), the most energy-efficient input voltage function is

$$v_I(t) = \frac{(I_{IF} - I_{I0})L}{T_F} \quad (6.49)$$

and from (6.36), the most energy-efficient input current function is

$$i_I(t) = \frac{I_{IF} - I_{I0}}{T_F} t + I_{I0} + \frac{(I_{IF} - I_{I0})L}{T_F R} \quad (6.50)$$

For the lowest input energy, we can apply (6.37) to obtain

$$\begin{aligned} E_I &= \frac{L(I_{IF}^2 - I_{I0}^2)}{2} + c_1 \left[\frac{I_{I0}}{R} \left(\frac{I_{IF} - I_{I0}}{T_F R} \right) - \frac{I_{IF}}{R} \left(\frac{I_{IF} - I_{I0}}{T_F R} \right) \right] \\ &= \frac{L(I_{IF}^2 - I_{I0}^2)}{2} + \frac{L^2(I_{IF} - I_{I0})^2}{T_F R} \end{aligned} \quad (6.51)$$

which is consistent with (4.41) in chapter 4. Recall that $I_{IF} = I_F$ and $I_{I0} = I_0$ at steady state.

6.5.3 Example 3: Parallel RC Circuit

In chapter 3, parallel RC circuit has been optimized separately as a special case because we can control the capacitor voltage directly and the most energy-efficient input functions involve unit step function and delta function. By using the general formulas that we have derived in this chapter, the most energy-efficient input functions and the lowest input energy can be obtained easily. The parallel RC circuit is shown in Fig. 6.5.

The input impedance is

$$Z(s) = \frac{R/sC}{R + 1/sC} = \frac{R}{sRC + 1}. \quad (6.52)$$

The only c_p that does not vanish is $c_0 = R$. Therefore, the differential equation is

$$f(t)R = 0 \quad (6.53)$$

Figure 6.5: A parallel RC circuit

and the solution is $f(t) = 0$ within a transient duration.

However, if we assume that the input voltage must change from V_{I0} to V_{IF} within a transient duration T_F , there will be boundaries conditions $f(0) = V_{I0}/R$ and $f(T_F) = V_{IF}/R$ which are not applicable for the solution $f(t) = 0$. In this case, the boundaries conditions must be defined carefully as $f(0^-) = V_{I0}/R$ and $f(T_F^+) = V_{IF}/R$ and the dummy function becomes

$$f(t) = \frac{1}{R}(V_{I0} - V_{I0}u(t) + (V_{IF} - V_{I0})u(t - T_F)) \quad (6.54)$$

where $u(t)$ is a unit step function.

The most energy-efficient input voltage and input current function can be calculated by using the formulas in (6.35) and (6.36). We obtain

$$v_I(t) = V_{I0} - V_{I0}u(t) + (V_{IF} - V_{I0})u(t - T_F) \quad (6.55)$$

and

$$i_I(t) = \frac{1}{R}(V_{I0} - V_{I0}u(t) + (V_{IF} - V_{I0})u(t - T_F)) + C(-V_{I0}\delta(t) + (V_{IF} - V_{I0})\delta(t - T_F)) \quad (6.56)$$

where $\delta(t)$ is the Dirac delta function. These input functions are consistent with (3.43) and (3.44) when $V_{I0} = 0$. (Notice that $V_{IF} = V_F$ for this circuit.)

For the lowest input energy, recall that we have derived the formulas in (6.37) by integration the input power. Hence the boundaries conditions in the formulas actually are $f(0^+)$, $f(T_F^-)$ and they are zeroes as well as $f^{(1)}(0)$ and $f^{(1)}(T_F)$. Therefore, the lowest input energy is zero which is also consistent with an explanation in section 3.4.

6.5.4 Example 4: A Resistors Network With a Capacitor

For more general, let us consider a resistors network with a capacitor that we have explored thoroughly in chapter 4. The circuit is shown in Fig. 6.6. The circuit equation system can be described in transmission parameters or **ABCD**-parameters as in (4.2). By using Laplace transform, the circuit equation system is

$$\begin{bmatrix} V_I(s) \\ I_I(s) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} V_C(s) \\ sCV_C(s) \end{bmatrix} \quad (6.57)$$

where $V_C(s)$ is Laplace transform of the capacitor voltage $v_C(t)$.

According to the definition of an input impedance which is the ratio between the input voltage and the input current, the input impedance of the circuit can be written as

$$Z(s) = \frac{sa_{12}C + a_{11}}{sa_{22}C + a_{21}}. \quad (6.58)$$

The c_p are

$$c_0 = a_0b_0 = a_{11}a_{21}, \quad c_1 = -(a_0b_2 + a_1b_1 + a_2b_0) = -a_{12}a_{22}C^2$$

Figure 6.6: A resistors network with a capacitor.

and the differential equation must be

$$-a_{12}a_{22}C^2f^{(2)}(t) + a_{11}a_{21}f(t) = 0$$

or

$$f^{(2)}(t) - \frac{a_{11}a_{21}}{a_{12}a_{22}C^2}f(t) = 0 \quad (6.59)$$

for $a_{12}a_{22} \neq 0$.

Notice that the differential equation in (6.59) and the differential equation in (4.7) are the same except that the functions are the dummy function and the capacitor voltage respectively. However, if we write the input voltage function in terms of the dummy function according to (6.35) and the input voltage function in terms of the capacitor voltage by using KVL and KCL laws, we will get

$$v_I(t) = a_0f(t) + a_1f^{(1)}(t) = a_{11}f(t) + a_{12}Cf^{(1)}(t)$$

and

$$v_I(t) = a_{11}v_C(t) + a_{12}C\dot{v}_C(t)$$

respectively. Obviously, the dummy function is actually the capacitor voltage function. It is not always be but for this circuit it does. Therefore, the most energy-efficient functions can be solved as they have shown in section 4.2.

The lowest input energy can be determined by applying the formula in (6.37). Recall that the dummy function is actually the most energy-efficient capacitor voltage which is described in (4.12). The lowest input energy can be calculated as

$$E_I = \frac{C(a_{11}a_{22} + a_{12}a_{21})(f(T_F)^2 - f(0)^2)}{2} - a_{12}a_{22}C^2(f(0)f^{(1)}(0) - f(T_F)f^{(1)}(T_F)).$$

By applying the identity of the transmission parameters, $a_{11}a_{22} - a_{12}a_{21} = 1$, the boundaries conditions $f(0) = V_0$ and $f(T_F) = V_F$, the lowest input energy becomes

$$E_I = \frac{C(1 + 2a_{12}a_{21})(V_F^2 - V_0^2)}{2} + a_{12}a_{22}C^2(V_Ff^{(1)}(T_F) - V_0f^{(1)}(0))$$

which is consistent with (B.35) in Appendix B.6. Hence, the lowest input energy after manipulating must be the expression in (4.19).

6.5.5 Example 5: A Resistors Network With a Series LC

According to the previous examples, the results from the general formulas are consistent with the results from using Euler-Lagrange's differential equation. However, the examples are only first-order circuit. We shall apply the general formulas for second order circuit which is a resistors network with a series LC circuit in this example. The circuit is shown in Fig. 6.7.

Figure 6.7: A resistors network with a series LC.

By using routine circuit analysis, we know that $V_O(s) = 1/\omega^2 \cdot s^2 V_C(s) + V_C(s)$ and $I_O(s) = sC V_C(s)$ where $\omega = 1/\sqrt{LC}$. By applying the transmission parameters, the input impedance can be written as

$$Z(s) = \frac{V_I(s)}{I_I(s)} = \frac{a_{11}V_O(s) + a_{12}I_O(s)}{a_{21}V_O(s) + a_{22}I_O(s)} = \frac{a_{11}/\omega^2 s^2 + a_{12}Cs + a_{11}}{a_{21}/\omega^2 s^2 + a_{22}Cs + a_{21}}. \quad (6.60)$$

The c_p can be calculated as

$$c_0 = a_{11}a_{21}, \quad c_1 = 2a_{11}a_{21}/\omega^2 - a_{12}a_{22}C^2, \quad c_2 = a_{11}a_{21}/\omega^4 \quad (6.61)$$

and Euler-Lagrange's differential equation becomes

$$\frac{a_{11}a_{21}}{\omega^4} f^{(4)}(t) + \left(\frac{2a_{11}a_{21}}{\omega^2} - a_{12}a_{22}C^2 \right) f^{(2)}(t) + a_{11}a_{21}f(t) = 0. \quad (6.62)$$

The differential equation in (6.62) and the differential equation in (B.71) are the same except that the functions are the dummy function and the capacitor voltage respectively. As same as the previous exmple, if we write the input voltage function in terms of the dummy function according to (6.35) and the input voltage function in terms of the capacitor voltage by using KVL and KCL laws, we will get

$$v_I(t) = a_0 f(t) + a_1 f^{(1)}(t) + a_2 f^{(2)}(t) = a_{11}f(t) + a_{12}Cf^{(1)}(t) + a_{11}/\omega^2 f^{(2)}(t)$$

and

$$v_I(t) = a_{11}v_C(t) + a_{12}C\dot{v}_C(t) + a_{11}/\omega^2 \ddot{v}_C(t)$$

respectively. The dummy function is actually the capacitor voltage function. Therefore, the most energy-efficient functions can be solved as they have shown in section 5.2.

By applying (6.37), the lowest input energy can be calculated as

$$E_I = \frac{C(a_{11}a_{22} + a_{12}a_{21})(f(T_F)^2 - f(0)^2)}{2} + c_2(f(0)f^{(3)}(0) - f(T_F)f^{(3)}(T_F)).$$

Notice that the lower boundary of summation is $\lceil (2+1)/2 \rceil = 2$. Therefore, the summation involves only the term that $p = 2$. By applying the boundaries conditions and c_2 , we obtain

$$E_I = \frac{C(a_{11}a_{22} + a_{12}a_{21})(V_F^2 - V_0^2)}{2} + \frac{a_{11}a_{21}}{\omega^4}(V_0 f^{(3)}(0) - V_F f^{(3)}(T_F))$$

which is consistent with (B.125) in Appendix B.12. Hence, the lowest input energy after manipulating must be the expression in (5.31).

6.5.6 Example 6: Parallel RL With a Series Capacitor

Although the second order circuits which are resistors network with series and parallel LC have been discussed thoroughly, other second order circuits have not been discussed yet. Therefore, we will explore another second order circuit which is parallel RL with a series capacitor in this example. The circuit is shown in Fig. 6.8.

Figure 6.8: Parallel RL with a series capacitor.

As always, we calculate the input impedance as

$$Z(s) = \frac{1}{sC} + \frac{sLR}{sL + R} = \frac{RLCs^2 + Ls + R}{LCs^2 + RCs} \quad (6.63)$$

The c_p are

$$c_0 = 0, \quad c_1 = -RLC + RLC = 0, \quad c_2 = RL^2C^2. \quad (6.64)$$

Surprisingly, both c_0 and c_1 are zeroes and the differential equation is only

$$RL^2C^2 f^{(4)}(t) = 0. \quad (6.65)$$

General solution must be

$$f(t) = K_3 t^3 + K_2 t^2 + K_1 t + K_0 \quad (6.66)$$

where K_0, K_1, K_2 and K_3 are constants. As we can see, the most energy-efficient functions for second order circuits are not always exponential functions. For the parallel RL with a series capacitor, the most energy-efficient function is a cubic function.

If we want to change the input voltage from V_{I0} to V_{IF} within transient duration T_F , the boundaries conditions will be $f(0) = V_{I0}/R$, $f(T_F) = V_{IF}/R$, $f^{(1)}(0) = 0$ and $f^{(1)}(T_F) = 0$. By applying these four boundaries conditions, a unique solution can be determined as

$$f(t) = -\frac{2(V_{IF} - V_{I0})}{RT_F^3} t^3 + \frac{3(V_{IF} - V_{I0})}{RT_F^2} t^2 + \frac{V_{I0}}{R}. \quad (6.67)$$

Notice that the term $K_1 t$ vanishes because of the boundaries conditions. Once the dummy function is determined, the most energy-efficient input voltage and input current functions can be determined easily by using (6.35) and (6.36) respectively.

The lowest input energy can be calculated by applying the formula in (6.37).

$$E_I = \frac{R^2 C (V_{IF}^2/R^2 - V_{I0}^2/R^2)}{2} + c_2 \left(\frac{V_{I0}}{R} f^{(3)}(0) - \frac{V_{IF}}{R} f^{(3)}(T_F) \right)$$

or

$$E_I = \frac{C(V_{IF}^2 - V_{I0}^2)}{2} + \frac{12L^2C^2(V_{IF} - V_{I0})^2}{RT_F^3} \quad (6.68)$$

Interestingly, although the circuit in this example is the second order circuit which tend to be difficult, the results are only the cubic functions and it is much easier than the series LC or the parallel LC that we have explored in the previous chapter.

Furthermore, Fig. 6.8 can be considered as a series RC circuit with a parallel inductor additional to the resistor. Recall that in example 1, the lowest input energy is (6.44) and the most energy-efficient input voltage function is a linear function. However, if we intentionally add the parallel inductor with the resistor in the series RC circuit, the circuit will be Fig. 6.8. By changing the input voltage with the most energy-efficient input voltage function which is the cubic function, the lowest input energy can be determined as (6.68). Notice that the boundaries conditions of the series RC circuit in example 1 and the *augmented* inductor series RC circuit in this example is the same.

Therefore, their lowest input energy can be compared and values of the inductor that can lower the lowest input energy for a certain transient duration can be found easily. Let us denote the condition that the lowest input energy in the augmented inductor series RC circuit must be lower than the lowest input energy in the ordinary RC circuit. From (6.44) and (6.68), the condition is

$$\begin{aligned} \frac{C(V_{IF}^2 - V_{I0}^2)}{2} + \frac{12RL^2C^2(V_{IF} - V_{I0})^2}{T_F^3} &< \frac{C(V_{IF}^2 - V_{I0}^2)}{2} + \frac{RC^2(V_{IF} - V_{I0})^2}{T_F} \\ \frac{12L^2}{RT_F^3} &< \frac{R}{T_F} \\ L &< \frac{RT_F}{2\sqrt{3}}. \end{aligned} \quad (6.69)$$

This condition is useful for design the circuit. For example, if we want to charge a capacitor with a series resistance, we can either use the most energy-efficient input voltage source function for the series RC circuit which is the linear function or we can add the augmented inductor that correspond to the condition in (6.69) and apply the most energy-efficient input voltage source function which is the cubic function. Of course, the latter case is more efficient but it needs one more circuit element which is the inductor.

6.6 Application For Higher Order Circuit: RC Chain Circuit

General formulas can be apply for any orders of circuits, not only first order and second order that we have extensively explained in the examples. To demonstrate, A RC chain circuit is chosen. The RC chain circuit is a common circuit model for wires, switches, transmission gates in digital circuits, flat panel displays [22, 80–85], etc. Its timing response and its energy dissipation are crucial performances for circuit systems. The analysis of energy dissipation in RC chain networks and some other common networks can be found in [26, 86–89]. It is commonly known that the circuit will dissipate half of the input energy if a unit step input voltage is applied. However, if other input functions are applied, the energy dissipation will be different and we may be able to save some energy. We shall apply the general formulas to derive the most energy-efficient functions and the lowest input energy.

6.6.1 Determining Input Impedance

Before applying Euler-Lagrange's differential equation, the input impedance must be determined. A traditional RC chain circuit is shown in Fig. 6.9. It has N RC networks where each network consists of one resistor and one capacitor. Assume that all the resistance and capacitance in the chain are equal to R and C respectively, the voltage at

node n where $n \leq N + 1$ can be written as

$$v_n(t) = \sum_{j=0}^{n-1} \binom{n-1+j}{2j} \tau^j v_1^{(j)}(t) \quad (6.70)$$

and the current that flows from node $n + 1$ to node n can be written as

$$i_n(t) = \sum_{j=1}^n \frac{1}{R} \binom{n-1+j}{2j-1} \tau^j v_1^{(j)}(t) \quad (6.71)$$

where $\tau = RC$ [86, 90].

Figure 6.9: A traditional RC chain circuit.

Although derivation of the voltage function and the current function in (6.70) and (6.71) is not important for deriving the most energy-efficient functions or the lowest input energy, we will derive them here. The derivation can be done by mathematical induction and also concrete mathematical proving. The mathematical induction will be explained here while the concrete mathematical proving is explained in Appendix B.19.

Assume that all resistance and capacitance are equal. From Kirchhoff's current law (KCL), the current in network n can be written as

$$i_n = \sum_{j=1}^n C v_j^{(1)} \quad \text{for } n \in [1, N] \quad (6.72)$$

where n, N must be positive integers. Subsequently, from Kirchhoff's voltage law (KVL), the voltage at node n can be written as

$$v_n = \begin{cases} v_1 & \text{for } n = 1 \\ v_{n-1} + i_{n-1}R & \text{for } n \in (1, N + 1] \end{cases} \quad (6.73)$$

However, expressing the currents and voltages in terms of v_1 and its derivative only is needed for optimization. Therefore by combining (6.73) with (6.72), for example if $n = 2$, the voltage can be written as

$$\begin{aligned} v_2 &= v_1 + i_1 R \\ &= v_1 + RC v_1^{(1)} \end{aligned}$$

and the current will be

$$\begin{aligned} i_2 &= C v_1^{(1)} + C v_2^{(1)} = C v_1^{(1)} + C(v_1^{(1)} + RC v_1^{(2)}) \\ &= C(2v_1^{(1)} + RC v_1^{(2)}). \end{aligned}$$

If $n = 3$, the voltage will be

$$\begin{aligned} v_3 &= v_2 + i_2 R = v_1 + RCv_1^{(1)} + (2Cv_1^{(1)} + RC^2v_1^{(2)})R \\ &= v_1 + 3RCv_1^{(1)} + (RC)^2v_1^{(2)} \end{aligned}$$

and the current will be

$$\begin{aligned} i_3 &= Cv_1^{(1)} + Cv_2^{(1)} + Cv_3^{(1)} = C(2v_1^{(1)} + RCv_1^{(2)}) + C(v_1^{(1)} + 3RCv_1^{(2)} + (RC)^2v_1^{(3)}) \\ &= C(3v_1^{(1)} + 4RCv_1^{(2)} + (RC)^2v_1^{(3)}). \end{aligned}$$

By repeating this procedure, the expressions of the voltage for $n \in [1, 5]$ can be given as follows:

$$\begin{aligned} v_1 &= v_1 \\ v_2 &= v_1 + \tau v_1^{(1)} \\ v_3 &= v_1 + 3\tau v_1^{(1)} + \tau^2 v_1^{(2)} \\ v_4 &= v_1 + 6\tau v_1^{(1)} + 5\tau^2 v_1^{(2)} + \tau^3 v_1^{(3)} \\ v_5 &= v_1 + 10\tau v_1^{(1)} + 15\tau^2 v_1^{(2)} + 7\tau^3 v_1^{(3)} + \tau^4 v_1^{(4)} \end{aligned} \tag{6.74}$$

where $\tau = RC$. The current can also be given as

$$\begin{aligned} i_1 &= Cv_1^{(1)} \\ i_2 &= C(2v_1^{(1)} + \tau v_1^{(2)}) \\ i_3 &= C(3v_1^{(1)} + 4\tau v_1^{(2)} + \tau^2 v_1^{(3)}) \\ i_4 &= C(4v_1^{(1)} + 10\tau v_1^{(2)} + 6\tau^2 v_1^{(3)} + \tau^3 v_1^{(4)}) \\ i_5 &= C(5v_1^{(1)} + 20\tau v_1^{(2)} + 21\tau^2 v_1^{(3)} + 8\tau^3 v_1^{(4)} + \tau^4 v_1^{(5)}) \end{aligned} \tag{6.75}$$

and so on.

Surprisingly, all the coefficient numbers of every term in (6.74) and (6.75) can be found inductively by inspecting the right-justified Pascal's triangle which is shown in Fig. 6.10. This well-known triangle can be written in the numeric form in Fig. 6.10a as well as binomial coefficient form in Fig. 6.10b [91]. The coefficients of the voltage and the current for an arbitrary n can be found by reading the right-justified Pascal's triangle diagonally as shown in Fig. 6.10a. According to the figure, we can see that all coefficients in (6.74) and (6.75) are perfectly match with all numbers in the triangle. Therefore, the expression of v_n and i_n can be found inductively by using the right-justified Pascal's triangle as described.

Moreover, the expression of v_n and i_n can be written compactly by using a binomial coefficient notation as shown in Fig. 6.10b. For example, (6.74) can be expressed in binomial coefficient notation as

$$\begin{aligned} v_1 &= \binom{0}{0} v_1 \\ v_2 &= \binom{1}{0} v_1 + \binom{2}{2} v_1 \tau v_1^{(1)} \\ v_3 &= \binom{2}{0} v_1 + \binom{3}{2} v_1 \tau v_1^{(1)} + \binom{4}{4} v_1 \tau^2 v_1^{(2)} \\ v_4 &= \binom{3}{0} v_1 + \binom{4}{2} v_1 \tau v_1^{(1)} + \binom{5}{4} v_1 \tau^2 v_1^{(2)} + \binom{6}{6} v_1 \tau^3 v_1^{(3)} \\ v_5 &= \binom{4}{0} v_1 + \binom{5}{2} v_1 \tau v_1^{(1)} + \binom{6}{4} v_1 \tau^2 v_1^{(2)} + \binom{7}{6} v_1 \tau^3 v_1^{(3)} + \binom{8}{8} v_1 \tau^4 v_1^{(4)}. \end{aligned} \tag{6.76}$$

Figure 6.10: The right-justified Pascal's triangle.

By applying a suitable index, the voltage of the RC chain circuit can be written as

$$v_n = \sum_{j=0}^{n-1} \binom{n-1+j}{2j} \tau^j v_1^{(j)}. \quad (6.77)$$

Subsequently, the current of the RC chain circuit can also be written compactly as

$$i_n = C \sum_{j=1}^n \binom{n-1+j}{2j-1} \tau^{j-1} v_1^{(j)}.$$

or

$$i_n = \frac{1}{R} \sum_{j=1}^n \binom{n-1+j}{2j-1} \tau^j v_1^{(j)}. \quad (6.78)$$

Similar equations of the RC chain circuit are reported in [86].

Actually, the induction can also be done on the left-justified Pascal's triangle with the same manner as the right-justified one. However, the index of binomial coefficients will be different because it will be read backward. For example,

$$v_4 = \binom{3}{3} v_1 + \binom{4}{2} v_1 \tau v_1^{(1)} + \binom{5}{1} v_1 \tau^2 v_1^{(2)} + \binom{6}{0} v_1 \tau^3 v_1^{(3)}$$

which is different from (6.76) although the numerical values are the same. This is because of a symmetry of Pascal's triangle. Therefore, the voltage and the current of the RC chain circuit can also be written as

$$v_n = \sum_{j=0}^{n-1} \binom{n-1+j}{n-1-j} \tau^j v_1^{(j)}$$

and

$$i_n = \frac{1}{R} \sum_{j=1}^n \binom{n-1+j}{n-j} \tau^j v_1^{(j)}.$$

These seem to be different from (6.77) and (6.78). Nevertheless, there is one binomial coefficient property states that

$$\binom{n}{k} = \binom{n}{n-k} \quad \text{for } k \leq n$$

which can be applied to prove that they are the same.

From Fig. 6.9, it is obvious that $v_I(t) = v_{N+1}(t)$ and $i_I(t) = i_N(t)$. Notice that the voltage and the current expressions in (6.70) and (6.71) are written in form of the dummy function in (6.35) and (6.36) respectively where $v_1(t)$ is the dummy function. Hence, the input impedance can be determined as

$$Z(s) = \frac{\sum_{j=0}^N \binom{N+j}{2j} \tau^j s^j}{\sum_{j=1}^N \frac{1}{R} \binom{N-1+j}{2j-1} \tau^j s^j}. \quad (6.79)$$

6.6.2 Solving for the Most Energy-efficient Functions

An ill-conditioned problem of determining unique solution

According to (6.33) and (6.34), a few expansions of the differential equation for different values of N are given as follows:

$$\begin{aligned} -\tau^2 f^{(2)}(t) &= 0 \quad \text{for } N = 1 \\ \tau^4 f^{(4)}(t) - 5\tau^2 f^{(2)}(t) &= 0 \quad \text{for } N = 2 \\ -\tau^6 f^{(6)}(t) + 11\tau^4 f^{(4)}(t) - 14\tau^2 f^{(2)}(t) &= 0 \quad \text{for } N = 3 \end{aligned} \quad (6.80)$$

The general solution must be

$$f(t) = \sum_{i=1}^{N-1} K_{ie+} e^{m_i t / \tau} + \sum_{i=1}^{N-1} K_{ie-} e^{-m_i t / \tau} + K_{le} t + K_{ce} \quad (6.81)$$

where K_{ie+} , K_{ie-} , K_{le} and K_{ce} are constants. m_i/τ is i^{th} positive root of characteristic equation. Several values of m_i and N are shown in Table 6.1. Notice that if $N = 1$, there will be no exponential function and the solution will be the same as a series RC circuit.

Table 6.1: Several values of m_i and N

N	m_1	m_2	m_3	m_4
1	-	-	-	-
2	2.2361	-	-	-
3	3.0873	1.2120	-	-
4	3.4540	2.1243	0.7465	-
5	3.6392	2.6908	1.5041	0.5035

If we want to change the input voltage from V_{I0} to V_{IF} within a transient duration T_F , the boundaries conditions can be written systematically as

$$\left[f_1^{(0)}(0^+), f_1^{(1)}(0^+), \dots, f_1^{(N-1)}(0^+) \right]^T = [V_{I0}, 0, \dots, 0]^T, \quad (6.82)$$

$$\left[f_1^{(0)}(T_F^-), f_1^{(1)}(T_F^-), \dots, f_1^{(N-1)}(T_F^-) \right]^T = [V_{IF}, 0, \dots, 0]^T. \quad (6.83)$$

To determine a unique solution, all the constants in (6.81) must be found. From the boundary conditions in (6.82) and (6.83), it can be seen that the general solution of $f_1^{(0)}, f_1^{(1)}, \dots, f_1^{(N-1)}$ are necessary for finding those constants. According to the general solution in (6.81), the derivatives are

$$\begin{aligned}
f_1^{(0)}(t) &= \sum_{i=1}^{N-1} K_{ie+} e^{m_i t/\tau} + \sum_{i=1}^{N-1} K_{ie-} e^{-m_i t/\tau} + K_{le} t + K_{ce} \\
f_1^{(1)}(t) &= \sum_{i=1}^{N-1} K_{ie+} (m_i/\tau) e^{m_i t/\tau} + \sum_{i=1}^{N-1} K_{ie-} (-m_i/\tau) e^{-m_i t/\tau} + K_{le} \\
f_1^{(2)}(t) &= \sum_{i=1}^{N-1} K_{ie+} (m_i/\tau)^2 e^{m_i t/\tau} + \sum_{i=1}^{N-1} K_{ie-} (-m_i/\tau)^2 e^{-m_i t/\tau} \\
&\vdots \\
f_1^{(N-1)}(t) &= \sum_{i=1}^{N-1} K_{ie+} (m_i/\tau)^{N-1} e^{m_i t/\tau} + \sum_{i=1}^{N-1} K_{ie-} (-m_i/\tau)^{N-1} e^{-m_i t/\tau}
\end{aligned}$$

By applying the steady state boundary conditions, an equation system for solving all the constants is

$$\begin{bmatrix}
1 & 1 & \dots & 0 & 1 \\
m_1/\tau & -m_1/\tau & \dots & 1 & 0 \\
(m_1/\tau)^2 & (-m_1/\tau)^2 & \dots & 0 & 0 \\
(m_1/\tau)^3 & (-m_1/\tau)^3 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(m_1/\tau)^{N-1} & (-m_1/\tau)^{N-1} & \dots & 0 & 0 \\
e^{m_1 T_F/\tau} & e^{-m_1 T_F/\tau} & \dots & T_F & 1 \\
(m_1/\tau) e^{m_1 T_F/\tau} & (-m_1/\tau) e^{-m_1 T_F/\tau} & \dots & 1 & 0 \\
(m_1/\tau)^2 e^{m_1 T_F/\tau} & (-m_1/\tau)^2 e^{-m_1 T_F/\tau} & \dots & 0 & 0 \\
(m_1/\tau)^3 e^{m_1 T_F/\tau} & (-m_1/\tau)^3 e^{-m_1 T_F/\tau} & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(m_1/\tau)^{N-1} e^{m_1 T_F/\tau} & (-m_1/\tau)^{N-1} e^{-m_1 T_F/\tau} & \dots & 0 & 0
\end{bmatrix}
\begin{bmatrix}
K_{1e+} \\
K_{1e-} \\
K_{2e+} \\
K_{2e-} \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
K_{(N-1)e+} \\
K_{(N-1)e-} \\
K_{le} \\
K_{ce}
\end{bmatrix}
=
\begin{bmatrix}
V_{I0} \\
0 \\
0 \\
\vdots \\
\vdots \\
0 \\
V_{IF} \\
0 \\
0 \\
\vdots \\
\vdots \\
\vdots \\
0
\end{bmatrix}
\quad (6.84)$$

This equation is an ordinary linear equation system in form of

$$\mathbf{Ax} = \mathbf{b} \quad (6.85)$$

which can be solve numerically [92].

Unfortunately, the equation system is almost impossible to solve because it can be an ill-conditioned problem easily. The problem that is an ill-conditioned problem is the equation system that the condition number of $\mathbf{A}$ is much greater than unity and the condition number is defined as

$$\kappa(\mathbf{A}) = \|\mathbf{A}\| \cdot \|\mathbf{A}^{-1}\| \quad (6.86)$$

where $\|\cdot\|$ is the norm [92]. Obviously, the condition number depends on circuit parameters. For example, in Fig. 6.11, the second norm condition number with $N = 2$ is plotted versus a storage time T_F in a unit of the time constant RC . The resistance and the capacitance are $1\text{k}\Omega$ and 1pF respectively. It shows that the condition number is in the order 10^{19} which is much greater than unity and also increase with the storage time. If

Figure 6.11: The second norm condition number with $N = 2$ versus a storage time T_F in a unit of the time constant RC .

$N = 3$, the condition number will be approximately 10^{55} which is enormous. Evidently, the equation system in (6.84) cannot be solved practically. Therefore, we must simplify the equation and make it solvable with an ordinary numerical method.

A solution of the ill-conditioned problem

The constants in (6.81) can be changed into many different forms as long as they are constants and independent to each others. Therefore, we will change those constant as follows.

$$\begin{aligned} K_{ie+} &= K_{in+} e^{-m_i T_F / 2\tau}, \\ K_{ie-} &= K_{in-} e^{m_i T_F / 2\tau}, \\ K_{le} &= K_{ln} / \tau, \\ K_{ce} &= K_c - K_{ln} T_F / 2\tau. \end{aligned}$$

Consequently, the general solution becomes

$$f(t) = \sum_{i=1}^{N-1} K_{in+} e^{m_i(t-T_F/2)/\tau} + \sum_{i=1}^{N-1} K_{in-} e^{-m_i(t-T_F/2)/\tau} + K_{ln} \frac{t - T_F/2}{\tau} + K_c.$$

For simplicity, let us define a normalized time as

$$x \equiv \frac{t - T_F/2}{\tau}. \quad (6.87)$$

Then, the general solution is

$$f(x) = \sum_{i=1}^{N-1} K_{in+} e^{m_i x} + \sum_{i=1}^{N-1} K_{in-} e^{-m_i x} + K_{ln} x + K_c. \quad (6.88)$$

With this form, time derivatives of $f(t)$ can be calculated by using the chain rule as

$$f^{(n)}(t) = \frac{d^n f(x)}{dx^n} \cdot \frac{dx}{dt^n} = \tau^{-n} f^{(n)}(x). \quad (6.89)$$

Moreover, let us define a boundary normalized time as

$$X_F \equiv \frac{T_F}{2\tau}. \quad (6.90)$$

We can see that at the initial condition ($t = 0$) and final condition ($t = T_F$), the normalized time in (6.87) are equal to $-X_F$ and X_F respectively. At the boundaries, time derivatives of $f(t)$ are equal to zero. According to (6.89), the boundary conditions in a function of time, (6.82) and (6.83) can be transformed to a function of normalized time as

$$[f^{(0)}(-X_F), f^{(1)}(-X_F), \dots, f^{(N-1)}(-X_F)]^T = [V_{I0}, 0, \dots, 0]^T \quad (6.91)$$

$$[f^{(0)}(X_F), f^{(1)}(X_F), \dots, f^{(N-1)}(X_F)]^T = [V_{IF}, 0, \dots, 0]^T. \quad (6.92)$$

From the general solution in (6.88), its derivative is

$$f^{(n)}(x) = \begin{cases} \sum_{i=1}^{N-1} K_{in+} e^{m_i x} + \sum_{i=1}^{N-1} K_{in-} e^{-m_i x} + K_{ln} x + K_c & \text{for } n = 0 \\ \sum_{i=1}^{N-1} K_{in+} m_i e^{m_i x} - \sum_{i=1}^{N-1} K_{in-} m_i e^{-m_i x} + K_{ln} & \text{for } n = 1 \\ \sum_{i=1}^{N-1} K_{in+} m_i^n e^{m_i x} + (-1)^n \sum_{i=1}^{N-1} K_{in-} m_i^n e^{-m_i x} & \text{for } n \geq 2 \end{cases} \quad (6.93)$$

By applying the boundary conditions from (6.91) and (6.92) into (6.93), there will be $2N$ of equations for solving $2N$ of unknown constants which are $K_{1n+}, K_{1n-}, \dots, K_{ln}, K_c$. Of course, all the constants can be solved theoretically but it will be an ill-conditioned problem as described earlier. Interestingly, we can determine that $K_{in-} = -K_{in+}$. The explanation is explained as follows.

Let us consider the first derivatives boundary conditions,

$$\begin{aligned} f^{(1)}(-X_F) &= \sum_{i=1}^{N-1} K_{in+} m_i e^{m_i X_F} - \sum_{i=1}^{N-1} K_{in-} m_i e^{-m_i X_F} + K_{ln} = 0, \\ f^{(1)}(X_F) &= \sum_{i=1}^{N-1} K_{in+} m_i e^{-m_i X_F} - \sum_{i=1}^{N-1} K_{in-} m_i e^{m_i X_F} + K_{ln} = 0. \end{aligned}$$

By subtracting with each other we get

$$\sum_{i=1}^{N-1} m_i (K_{in+} + K_{in-}) (e^{m_i X_F} - e^{-m_i X_F}) = 0.$$

The equation convinces that K_{in-} should be equal to $-K_{in+}$. However, only this equation cannot guarantee that it must be because we still have $2N - 2$ (K_c and K_{ln} vanish) unknown constants in only one equation and those constants can be any numbers. $K_{in-} = -K_{in+}$ is just one possibility of the solutions. To obtain more equations, let us do an addition for the even order derivatives and do a subtraction for the odd order derivatives. For example, by adding the second derivatives boundary conditions together, we get

$$f^{(2)}(-X_F) + f^{(2)}(X_F) = \sum_{i=1}^{N-1} m_i^2 (K_{in+} + K_{in-}) (e^{m_i X_F} + e^{-m_i X_F}) = 0.$$

For the third derivatives boundary conditions. we do a subtraction to get

$$f^{(3)}(-X_F) - f^{(3)}(X_F) = \sum_{i=1}^{N-1} m_i^3 (K_{in+} + K_{in-}) (e^{m_i X_F} - e^{-m_i X_F}) = 0.$$

By repeating this procedure until it reaches the highest derivatives boundary conditions which is $N - 1$, we will have $N - 1$ equations to solve $2N - 2$ unknown constants. These equations are

$$\sum_{i=1}^{N-1} m_i^n (K_{in+} + K_{in-}) (e^{m_i X_F} + (-1)^n e^{-m_i X_F}) = 0 \quad \text{for } n \in [1, N - 1]. \quad (6.94)$$

Obviously, there are not enough equations for solving all the constants. However, let us assume that K_{in+} are not an unknown constants and we want to solve K_{in-} in terms of K_{in+} . With this assumption, we will have only $(2N - 2)/2 = N - 1$ unknown constants which are K_{in-} to be solved with $N - 1$ equations. Since the number of the unknown constants are now equal to the number of the equations, all K_{in-} can be determined uniquely in terms of K_{in+} . By inspection, we can see that if $K_{in-} = -K_{in+}$, they will satisfy (6.94) and this is actually a unique solution of (6.94). Therefore, the statement that $K_{in-} = -K_{in+}$ has been proved.

By substituting this important result into (6.88) and define a new constant $K_{in} = 2K_{in+}$, the general solution in term of normalized time becomes

$$f(x) = \sum_{i=1}^{N-1} K_{in} \sinh m_i x + K_{ln} x + K_c$$

and the constants can be normalized as

$$f(x) = \sum_{i=1}^{N-1} K_i \frac{\sinh m_i x}{\sinh m_i X_F} + K_l \frac{x}{X_F} + K_c \quad (6.95)$$

where $K_{in} = K_i / \sinh m_i X_F$ and $K_{ln} = K_l / X_F$.

The general solution in (6.95) seems to be easier to solve the constants than the general solution in (6.81) and it can be simplified further. The constant K_c can be determined easily by adding the initial and final conditions of $f(x)$ together,

$$f(X_F) + f(-X_F) = 2K_c = V_{IF} + V_{I0} \quad \rightarrow \quad \therefore K_c = \frac{V_{IF} + V_{I0}}{2}. \quad (6.96)$$

Furthermore, by subtraction, we get

$$f(X_F) - f(-X_F) = \sum_{i=1}^{N-1} 2K_i + 2K_l = V_{IF} - V_{I0} \quad \rightarrow \quad \therefore K_l = \frac{V_{IF} - V_{I0}}{2} - \sum_{i=1}^{N-1} K_i. \quad (6.97)$$

Visibly, the constant K_l in (6.97) cannot be determined explicitly as same as K_c in (6.96). However, if all other constants are determined, K_l can be calculated by using (6.97).

Determining all other constants K_i can be done by solving the derivatives of f in (6.95) at the boundary condition must be equal to zero.

$$f^{(1)}(X_F) = 0 = \sum_{i=1}^{N-1} K_i m_i \coth m_i X_F + K_l \frac{1}{X_F} \quad (6.98)$$

and

$$f^{(n)}(X_F) = 0 = \begin{cases} \sum_{i=1}^{N-1} K_i m_i^n & \text{for } 1 < \text{even } n \leq N-1 \\ \sum_{i=1}^{N-1} K_i m_i^n \coth m_i X_F & \text{for } 1 < \text{odd } n \leq N-1 \end{cases}. \quad (6.99)$$

By substituting K_l from (6.97) into (6.98) and multiplying the entire equation by X_F , it yields

$$\sum_{i=1}^{N-1} K_i (1 - m_i X_F \coth m_i X_F) = \frac{V_{IF} - V_{I0}}{2}. \quad (6.100)$$

Furthermore, the constants K_i can be solved systematically by using (6.99) and (6.100) as

$$\begin{bmatrix} 1 - m_1 X_F \coth m_1 X_F & \dots & 1 - m_{N-1} X_F \coth m_{N-1} X_F \\ m_1^2 & \dots & m_{N-1}^2 \\ m_1^3 \coth m_1 X_F & \dots & m_{N-1}^3 \coth m_{N-1} X_F \\ m_1^4 & \dots & m_{N-1}^4 \\ m_1^5 \coth m_1 X_F & \dots & m_{N-1}^5 \coth m_{N-1} X_F \\ \vdots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \\ \vdots \\ K_{N-1} \end{bmatrix} = \begin{bmatrix} \frac{V_{IF} - V_{I0}}{2} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (6.101)$$

If (6.101) can be solved without an ill-conditioned problem, the constant K_l in (6.97) can be determined and finally the unique solution can be found.

Figure 6.12: The second norm condition number of (6.101) with $N = 3$ versus a storage time T_F in a unit of the time constant RC .

To illustrate that (6.101) can avoid the ill-conditioned problem better than (6.84), the condition numbers are compared. Fig. 6.11 shows the second norm condition number of (6.84) when $N = 2$ while the condition number of (6.101) is a unity because it has only one variable which is K_1 . For $N = 3$ the condition number of (6.84) is approximately 10^{55}

while the condition number of (6.101) is shown in Fig. 6.12. According to the figure, the condition number strongly depends on the storage time. Nevertheless, it is much lower than 10^{55} . Unfortunately, (6.101) can be the ill-conditioned problem if it is in extremely cases which are huge number of N , huge number of T_F and tiny number of T_F . However, it is obviously more advantageous than (6.84).

In summary, The ill-conditioned problem in (6.84) has been solved by using several techniques. First, the constants in (6.81) have been transformed and the normalized time x has been defined in (6.87). The general solution in terms of the normalized time has been written in (6.88). Next, some of steady state boundary conditions have been applied to prove that $K_{in-} = -K_{in+}$ and this reduces the number of unknown constants from $2N$ to $N + 1$. By substituting this result into the general solution (6.88), the general solution can be written in a function of sine hyperbolic and it has been posted in (6.95). The constant K_c in (6.95) can be determined explicitly in (6.96). The constants $K_1, K_2, \dots, K_{N-1}$ can be determined by solving (6.101) and finally K_l can be determined by using (6.97). Notice that we use $N + 1$ boundary conditions to solve $N + 1$ unknown constants $K_1, K_2, \dots, K_{N-1}, K_l$ and K_c while originally we have $2N$ boundary conditions in (6.82) and (6.83). It seems that we have not applied the initial conditions except $f(-X_F) = V_{I0}$ but actually we have already applied them to prove that $K_{in-} = -K_{in+}$. Therefore, all the boundary conditions have been used and the constants from (6.96), (6.97) and (6.101) will satisfy all the boundary conditions.

As an example, Fig. 6.13 shows the timing response of all voltages and the most energy-efficient input voltage function when the parameters are $N = 15$, $R = 1\text{k}\Omega$, $C = 1\text{pF}$, $V_0 = 0\text{V}$, $V_F = 1\text{V}$ and $T_F = 200\text{ns}$. Once the dummy function is determined, the lowest input energy can be calculated numerically by using (6.37).

Figure 6.13: Timing response of all voltages and the most energy-efficient input voltage function when the parameters are $N = 15$, $R = 1\text{k}\Omega$, $C = 1\text{pF}$, $V_{I0} = 0\text{V}$, $V_{IF} = 1\text{V}$ and $T_F = 200\text{ns}$.

6.7 Conclusion

In this chapter, generalization of the most energy-efficient functions and the lowest input energy of passive linear circuits has been explained. Firstly, we have introduced a dummy function which is an essential function for generalization. Briefly, the dummy function is a function that vanishing a complementary solutions of ODE. Therefore, a reviews of complementary and particular solutions of ODE have been discussed for deep understanding of the dummy function. Secondly, the most energy-efficient input functions have been proposed in form of the dummy function. Generalization of the most energy-efficient functions has been explained and we have obtained Euler-Lagrange's differential equation for any orders of circuits. However, the differential equation does not involve the input functions directly, it involves the dummy function instead and the most energy-efficient input functions can be obtained once the dummy function is solved. Since the dummy function is the function that it must be solved, boundaries conditions of the dummy function have been examined clearly. The lowest input energy in terms of the dummy function has been derived. There are a few examples that illustrate on using the general formulas for determining the most energy-efficient functions and the lowest input energy. The results have been discussed and compared with other results in the previous chapters to show consistency. Lastly, the general formula has been applied for RC chain circuit which is a higher order circuit.

For the RC chain circuit, general solution of the differential equation has been presented in terms of exponential functions, a linear function and a constant. Unfortunately, determining the unique solution can be an ill-conditioned problem easily. Therefore, a solution of the ill-conditioned problem has been proposed. The general solution has been manipulated and rearranged in terms of the hyperbolic sine functions, a linear function and a constant. Using this form, the ill-conditioned problem can be partially solved with the condition numbers improved. Then, the unique solution has been determined. The procedure to determine the most energy-efficient input source functions has been provided and the timing response of the circuit has been shown.

Chapter 7

Conclusion and Recommendations

7.1 Conclusion

There are five main chapters in the dissertation: (i) literature review, (ii) first order linear circuits optimization, (iii) the lowest input energy of passive first order circuit in transient state, (iv) the lowest input energy of series LC and parallel LC circuits in transient state and (v) generalization of the most energy-efficient function and the lowest input energy of passive linear circuits. Each chapter has its own key concepts and or contributions as follows.

Chapter 2: literature review. The chapter has been separated into two parts which are motivation and theory. The motivation is all about energy-efficient. Concept about reversible and irreversible logic has been reviewed, from mechanical perspective to electrical perspective. Two methods of reducing energy dissipation in circuits during charging which are ramp function excitation and resonant gate driver have been discussed both quantitatively and qualitatively. For logic circuits, a famous technique for reducing energy dissipation is adiabatic switching and it has been reviewed extensively whether it be pass transistor, transmission gate, inverter circuits or amplifier.

The theory is calculus of variations. One of the most powerful mathematical theory for functional optimization. The fundamental of calculus of variation is the Lagrangian method or the Euler-Lagrange's differential equation has been discussed. Shortest path problem and brachistochrone problem have been explained to illustrate on applying the Euler-Lagrange's differential equation. Some advanced theories in calculus of variations which are isoperimetric problem, multiple functions problem and higher derivatives Euler-Lagrange's differential equation have been discussed along with some applications in mathematics, physics and circuits.

Chapter 3: first order linear circuits optimization. A series RC circuit, the simplest circuit for applying calculus of variations for optimization has been optimize rigorously. Euler-Lagrange's differential equation has been applied to derive the source function that minimize energy dissipation in the circuit during transient analytically. The most energy-efficient input current function and input voltage function have been derived to be a constant function and a raised ramp function respectively. They causes the lowest total energy dissipation while charging a capacitor and they give the maximum efficiency for a series RC circuit where the efficiency has been defined as the ratio between the stored energy in the capacitor and the input energy. The maximum efficiency is 69.72% while the unit step input voltage gives 50.00% efficiency. Although the functions that we obtained from the Euler-Lagrange's differential equation are guaranteed by mathematical theory that they are the most energy-efficient function, numerical comparison with other functions has been shown.

Calculus of variations has been applied for other first order linear circuits. Two

examples which are series-parallel RC circuit and parallel-series RL circuit have been optimized rigorously to illustrate on using calculus of variations for circuits optimization. The most energy-efficient function family of these two circuits is hyperbolic function. For common first order linear circuits, the most energy-efficient input voltage and current functions and their lowest energy dissipation expressions have been summarized in compact parameters and have been shown in a table. Numerical and graphical results have been shown. Speed and energy trade off limit has been explained and the upper limit of the trade off has been explored. For some special circuit that cannot be solved directly using standard calculus of variations, they have been treated separately and explained in graphical, mathematical and intuitive manner. Unit step function and Dirac delta function must be used for these special circuits.

Chapter 4: the lowest input energy of passive first order circuit in transient state. Generalization of using of calculus of variations with passive first order circuits for optimization can be found in this chapter. An arbitrary resistors connection has been group as a general resistors two port network. A resistors network with a capacitor has been picked for explanation while the circuit with an inductor can be optimized in a similar manner. By describing the two port network with the transmission parameters or **ABCD**-parameters, the general optimization can be done easily. However, there are some parameters that we have defined for simplicity. Those are

- R_{OC} : open circuit resistance seen by the output of the two port network
- R_{SC} : shorted circuit resistance seen by the output of the two port network
- $\tau = \sqrt{R_{OC}R_{SC}}C$ for a first order capacitive circuit and $\tau = L/\sqrt{R_{OC}R_{SC}}$ for a first order inductive circuit
- $k = \sqrt{R_{OC}/R_{SC}}$, the square root of the ratio of the open circuit resistance and shorted circuit resistance
- $\gamma = (V_F - V_0)/(V_F + V_0)$ for a first order capacitive circuit and $\gamma = (I_F - I_0)/(I_F + I_0)$ for a first order inductive circuit.

With these parameters, we have stated two corollaries as

Corollary 1: For a resistors network with a capacitor C as shown in Fig 7.1, changing the capacitor voltage from V_0 to V_F within a duration of time T_F requires minimum input energy as

$$E_I = \frac{C(V_F^2 - V_0^2)}{2(k^2 - 1)} \cdot \left(k + \frac{\tanh(T_F/2\tau)}{\gamma} \right) \left(k + \frac{\gamma}{\tanh(T_F/2\tau)} \right) \quad (7.1)$$

and

Corollary 2: For a resistors network with an inductor L as shown in Fig 7.2, changing the inductor current from I_0 to I_F within a duration of time T_F requires minimum input energy as

$$E_I = \frac{L(I_F^2 - I_0^2)}{2(k^2 - 1)} \cdot \left(k + \frac{\tanh(T_F/2\tau)}{\gamma} \right) \left(k + \frac{\gamma}{\tanh(T_F/2\tau)} \right) \quad (7.2)$$

Analysis of transient duration has been performed on the corollaries for three cases of increasing, decreasing and unchanging total stored energy. All these cases have been analyzed comprehensively. In each case, it has been further divided into two sub cases of (un)changing without and with zero crossing. The analysis has revealed two important circuit behaviours which are speed and energy trade-off and energy flow. It has been shown that the speed and energy trade-off may be violated under certain conditions.

Figure 7.1: A resistors network with a capacitor.

Figure 7.2: A resistors network with an inductor.

Optimum transient duration has been determined. The energy flow of the circuit has been discussed thoroughly and a zero-energy transient duration that is defined as a transient duration with total input energy being zero has been derived. All the analysis can be summarized into a single digram called the *transient duration diagram*. The digram has revealed important relationships between the circuit parameters in the transient duration analysis. Special cases of the lowest input energy for first-order circuits which are infinite open circuit resistance and zero shorted circuit resistance have been explained quantitatively and qualitatively. I-V characteristic of the most energy-efficient input source functions have been noticed and elaborated.

Chapter 5: the lowest input energy of series LC and parallel LC circuits in transient state. Second order circuits which are series LC and parallel LC have been optimized in this chapter. The optimization has been done step by step from assuming boundaries conditions, the Euler-Lagrange's differential equation formation, solving the most energy-efficient functions and calculating the lowest input energy. As well as the first order circuit optimization, we have defined some important parameters as follows.

- $\omega = 1/\sqrt{LC}$
- $Z = \sqrt{L/C}$
- $r = \sqrt{R_{OC}R_{SC}}/(2Z)$ for series LC circuit and $r = Z/(2\sqrt{R_{OC}R_{SC}})$ for parallel LC circuit

The optimization is more sophisticated than the first order circuit and it has been categorized into three modes which are (i) hyperbolic mode, (ii) hyperbolic and linear mode and (iii) hyperbolic and trigonometric mode. The mode depends on value of r . If $r > 1$, it will be hyperbolic mode. If $r = 1$, it will be hyperbolic and linear mode and if $0 < r < 1$, it will be hyperbolic and trigonometric mode. There are two special cases which are $r = 0$ and $r = \infty$. For $r = 0$, it is a resonant mode and for $r = \infty$, it is a linear mode. Each mode has been discussed and their transient responses have been illustrated. Once the most energy-efficient functions have been determined, the lowest input energy has been evaluated. Transient duration analysis of the lowest input energy for each mode has been explained thoroughly. Speed and energy trade-off, energy flow and optimum transient duration have been analyzed both graphically and mathematically.

Chapter 6: generalization of the most energy-efficient functions and the lowest input energy of passive linear circuits. For first order and second order circuits, the optimization are straightforward. However, to generalize of using calculus of variations for any orders

of passive circuits, the dummy function has been introduced. It is an essential function for generalization. Briefly, the dummy function is a function that vanishing a complementary solutions of ODE. the most energy-efficient input functions have been proposed in form of the dummy function. Extensive proof of generalization has been shown and the lowest input energy has been derived in terms of the dummy function. Nevertheless, using of general formulas or procedure of determining the most energy-efficient function and the lowest input energy has been concluded and we will repeat it here as follows.

- Step 0: The input impedance of the circuit and the boundaries conditions must be known.
- Step 1: Write the input impedance in form of

$$Z(s) = \frac{\sum_{i=0}^{\infty} a_i s^i}{\sum_{i=0}^{\infty} b_i s^i} \quad (7.3)$$

and determine the coefficients c_p by using the formula

$$c_p = \sum_{n=0}^{2p} (-1)^n a_n b_{2p-n} \quad (7.4)$$

- Step 2: For N^{th} -order circuit, solve the differential equation

$$\sum_{p=0}^N c_p f^{(2p)}(t) = 0 \quad (7.5)$$

to obtain a general solution of the dummy function

- Step 3: Apply the boundaries conditions to obtain a unique solution of the dummy function
- Step 4: Apply the most energy-efficient input voltage function formula,

$$v_I(t) = \sum_{i=0}^{\infty} a_i f^{(i)}(t). \quad (7.6)$$

to obtain the most energy-efficient input voltage function and apply the most energy-efficient input current function formula,

$$i_I(t) = \sum_{i=0}^{\infty} b_i f^{(i)}(t). \quad (7.7)$$

to obtain the most energy-efficient input current function

- Step 5: Apply the lowest input energy formula,

$$E_I = \frac{(a_0 b_1 + a_1 b_0)(f^2(T_F) - f^2(0))}{2} + \sum_{p=\lceil \frac{N+1}{2} \rceil}^N c_p (f(0)f^{(2p-1)}(0) - f(T_F)f^{(2p-1)}(T_F)) \quad (7.8)$$

to obtain the lowest input energy

Examples of using the procedure have been described in the chapter.

7.2 Recommendations

The dissertation has presented that calculus of variations can be applied as a functional optimization for any order of passive linear circuits. However, the author believe that it can be expanded to non-linear circuits or active circuits. Calculus of variations itself also has a lot of things to explore and it can be useful for optimization such as

- Constraint problems. In the dissertation, only the isoperimetric problems have been used. Recall that the isoperimetric problems have the constraints equations in form of (2.28). However, there are another types of the constraint problems such as holonomic, nonholonomic and inequality constraints.
- The second variation. The optimization is done on the first variation while the second variation will help us to realize that which types of the solutions are, local minimum, local maximum or saddle point.
- Variable-endpoints conditions. If the initial point and the final point of the integral are not fixed, the optimization can also be done by using the Lagrangian method along with variable-endpoints conditions.
- Corner conditions. With this conditions, the optimized functions with corners can be found.
- Multiple integrals.
- Problem transformation. The Lagrangian method can be transformed to another method called *Hamiltonian* method or *Canonical* method. These methods are dual but sometime one method is easier than another.

Furthermore, all the results that we have obtained, all the formulas that we have derived can be used or at least a baseline for some advanced optimization such as wireless charging or vehicle charging. The efficiency of the charging systems or circuits could be improved by using this optimization. Moreover, not only the most energy-efficient input source functions can be derived, more efficiency circuit can be derived as well. The augmented inductor circuit in chapter 6 is an example. It comes along with the design equation. For the circuit applications, in spite of finding the most energy-efficient input source functions, design the circuits can be considered. Finally, although the most energy-efficient input source functions can be derived analytically, the functions can be difficult to generate. For example, the most energy-efficient input source function for almost all of first order circuits is hyperbolic family. Therefore, sub-optimal source functions can be considered based on the most energy-efficient functions that we obtained.

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Appendices

Appendix A

Author Publications

- R. Smunyahirun and E. L. Tan, "Most energy-efficient input voltage function for RC delay line," 2018 IEEE International Symposium on Electromagnetic Compatibility and 2018 IEEE Asia-Pacific Symposium on Electromagnetic Compatibility (EMC/APEMC), Singapore, 2018, pp. 1022-1026.
- R. Smunyahirun and E. L. Tan, "Optimum lowest input energy for first-order circuits in transient state," 2017 14th International Conference on Electrical Engineering/Electronics, Computer, Telecommunications and Information Technology (ECTI-CON), Phuket, 2017, pp. 143-146.
- R. Smunyahirun and E. L. Tan, "Derivation of the Most Energy-Efficient Source Functions by Using Calculus of Variations," in IEEE Transactions on Circuits and Systems I: Regular Papers, vol. 63, no. 4, pp. 494-502, April 2016.

Appendix B

Mathematics Proofs

B.1 Cycloid Curve

There are many perspective to derive the cycloid curve equation. One of them is given here.

Let us consider a ball radius R rolling on the ground in Fig.B.1. A ball rolls forward with a constant angular speed ω without slipping. A point P is the point on the rim of a ball. At the beginning, a point P is at the origin. While the ball moves, a point P creates a trajectory and this trajectory is called cycloid. To formulate the equation, the velocity at any time t is the combination of translation velocity and rotation velocity. Therefore, in horizontal direction,

$$\frac{dx}{dt} = \omega R - \omega R \cos \omega t. \quad (\text{B.1})$$

By integration, the horizontal position of point P at any time t is

$$x(t) = R\omega t - R \sin \omega t. \quad (\text{B.2})$$

The vertical component of velocity of point P is

$$\frac{dy}{dt} = \omega R \sin \omega t \quad (\text{B.3})$$

and by integration, the vertical position of point P at any time t can be expressed as

$$y(t) = R(1 - \cos \omega t). \quad (\text{B.4})$$

By introducing a parameter $\theta = \omega t$ into (B.2) and (B.4), the parametric form of cycloid can be written as

$$x(\theta) = R(\theta - \sin \theta), \quad (\text{B.5})$$

$$y(\theta) = R(1 - \cos \theta). \quad (\text{B.6})$$

Finally, the parametric equation can be generalized by moving the ground to the line $y = h$ and the starting point of P can be at some point (k, h) where k and h are some constants. Therefore, the equations of the cycloid family are

$$x(\theta) = R(\theta - \sin \theta) + k, \quad (\text{B.7})$$

$$y(\theta) = R(1 - \cos \theta) + h. \quad (\text{B.8})$$

Figure B.1: Cycloid trajectory.

B.2 Efficiency Analysis of A Series RC Circuit

In chapter 3, Section 3.2, there is the efficiency comparison of linear functions with various slope and storage time. The analysis of the efficiency is provided below.

Let us consider the series R-C circuit in Fig. B.2. The input voltage function is a linear function during transient,

$$v_I(t) = \begin{cases} at + b & \text{for } t < T_F \\ V_F & \text{for } t \geq T_F. \end{cases} \quad (\text{B.9})$$

where a and b are constants, T_F is storage time and V_F is a final voltage of the capacitor. Either a or b can be chosen freely but not both because there is the storage time which is a constraint. If the slope a is varied, the y-intercept b must be calculated from a and T_F .

Figure B.2: series RC circuit.

To find the expression of b , first we have to do a usual circuit analysis.

$$\begin{aligned} v_I(t) &= RC\dot{v}_C(t) + v_C(t) = at + b \\ \dot{v}_C(t) + \frac{1}{RC}v_C(t) &= \frac{a}{RC}t + \frac{b}{RC}. \end{aligned}$$

By multiplying both sides by the integrating factor $e^{t/RC}$,

$$\begin{aligned} e^{t/RC}\frac{dv_C}{dt} + v_C\frac{e^{t/RC}}{RC} &= \frac{a}{RC}te^{t/RC} + \frac{b}{RC}e^{t/RC} \\ e^{t/RC}\frac{dv_C}{dt} + v_C\frac{de^{t/RC}}{dt} &= \frac{a}{RC}te^{t/RC} + \frac{b}{RC}e^{t/RC} \\ \frac{d}{dt}(v_Ce^{t/RC}) &= \frac{a}{RC}te^{t/RC} + \frac{b}{RC}e^{t/RC} \end{aligned}$$

By integrating both sides,

$$v_C(t)e^{t/RC} - v_C(0)(1) = \int_0^t \frac{a}{RC}te^{t/RC} + \frac{b}{RC}e^{t/RC} dt \quad (B.10)$$

Since the initial condition is $v_C(0) = 0$,

$$\begin{aligned} v_C(t) &= e^{-t/RC} \int_0^t \frac{a}{RC}te^{t/RC} + \frac{b}{RC}e^{t/RC} dt \\ v_C(t) &= e^{-t/RC} \left(b(e^{t/RC} - 1) + ate^{t/RC} - \int_0^t ae^{t/RC} dt \right) \\ v_C(t) &= b(1 - e^{-t/RC}) + at - aRC(1 - e^{-t/RC}) \\ v_C(t) &= (b - aRC)(1 - e^{-t/RC}) + at. \end{aligned} \quad (B.11)$$

The equation (B.11) is a function of the voltage across the capacitor during transient when it is applied by the ramp function. Recall that there is another condition that is $v_C(T_F) = V_F$. By applying this condition to (B.11), we get

$$b = aRC + \frac{V_F - aT_F}{1 - e^{-T_F/RC}} \quad (B.12)$$

which is the same as (3.9). Note that if $a = V_F/T_F$ then $b = V_FRC/T_F$ which correspond to the most energy-efficient input voltage function for the series R-C circuit. By substituting the constant b from (B.12) back into (B.11), the voltage across the capacitor becomes

$$v_C(t) = \frac{V_F - aT_F}{1 - e^{-T_F/RC}}(1 - e^{-t/RC}) + at. \quad (B.13)$$

Next step is finding the total energy dissipation. First of all, we have to find the derivative of the voltage across the capacitor. From (B.11), the derivative can be written as

$$\dot{v}_C(t) = \frac{V_F - aT_F}{RC(1 - e^{-T_F/RC})}e^{-t/RC} + a. \quad (B.14)$$

Then, the energy dissipation can be calculated as follows.

$$\begin{aligned} E_{dis} &= \int_0^{T_F} i_R^2 R dt = RC^2 \int_0^{T_F} \dot{v}_C^2 dt \\ E_{dis} &= RC^2 \int_0^{T_F} \left[\frac{V_F - aT_F}{RC(1 - e^{-T_F/RC})}e^{-t/RC} + a \right]^2 dt \\ E_{dis} &= RC^2 \left[\frac{RC}{2} \left(\frac{V_F - aT_F}{RC(1 - e^{-T_F/RC})} \right)^2 (1 - e^{-2T_F/RC}) \right. \\ &\quad \left. + 2aRC \frac{V_F - aT_F}{RC(1 - e^{-T_F/RC})}(1 - e^{-T_F/RC}) + a^2T_F \right] \\ E_{dis} &= RC^2 \left[\frac{(V_F - aT_F)^2(1 + e^{-T_F/RC})}{2RC(1 - e^{-T_F/RC})} + 2aV_F - a^2T_F \right] \\ E_{dis} &= RC^2 \left[\frac{(V_F - aT_F)^2}{2RC} \coth \left(\frac{T_F}{2RC} \right) + 2aV_F - a^2T_F \right]. \end{aligned} \quad (B.15)$$

Finally, the efficiency is

$$\text{Eff.} = \frac{0.5CV_F^2}{0.5CV_F^2 + E_{dis}} \quad (B.16)$$

where E_{dis} is determined by (B.15). The plot of (B.16) is shown in Fig. 3.4.

B.3 Manipulating the Euler-Lagrange's Differential Equation of a Capacitor - Resistors Circuit

From Section 4.2, the Lagrangian function in (4.6) is substituted into (4.5). Let us consider the first term, we get

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial v_C} &= (a_{11}v_C + a_{12}C\dot{v}_C)a_{21} + (a_{21}v_C + a_{22}C\dot{v}_C)a_{11} \\ &= 2a_{11}a_{21}v_C + a_{12}a_{21}C\dot{v}_C + a_{11}a_{22}C\dot{v}_C\end{aligned}$$

and the second term,

$$\begin{aligned}\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{v}_C} \right) &= (a_{11}\dot{v}_C + a_{12}C\ddot{v}_C)a_{22}C + (a_{21}\dot{v}_C + a_{22}C\ddot{v}_C)a_{12}C \\ &= a_{12}a_{21}C\dot{v}_C + a_{11}a_{22}C\dot{v}_C + 2a_{12}a_{22}\ddot{v}_C.\end{aligned}$$

By subtracting together, it is obviously that the $\dot{v}_C$ terms vanish and the Euler-Lagrange's differential equation becomes

$$\frac{\partial \mathcal{L}}{\partial v_C} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{v}_C} \right) = 2a_{11}a_{21}v_C - 2a_{12}a_{22}\ddot{v}_C = 0 \quad \text{or} \quad \ddot{v}_C - \frac{a_{11}a_{21}}{a_{12}a_{22}C^2}v_C = 0 \quad (\text{B.17})$$

which is the equation in (4.7).

B.4 Solving the Euler-Lagrange's Differential Equation of a Capacitor - Resistors Circuit

From the differential equation in (4.9), we can guess that the solution is a function of e^{st} . By substituting this guess into (4.9), we found that $s = \pm 1/\tau$ where τ is defined as (4.11). Therefore, the solution must be a linear combination of all possible function,

$$v_C(t) = K_1 e^{t/\tau} + K_2 e^{-t/\tau} \quad (\text{B.18})$$

where K_1 and K_2 are constants. Since they are arbitrary constants, we can change them as $K_1 = K'_1 e^{-T_F/2\tau}$ and $K_2 = K'_2 e^{T_F/2\tau}$ and the solution is

$$v_C(t) = K'_1 e^{(t-T_F/2)/\tau} + K'_2 e^{-(t-T_F/2)/\tau} = K'_1 e^x + K'_2 e^{-x} \quad (\text{B.19})$$

where x is the normalized time defined in (B.75). Moreover, let us change these constants into two new constants as $K'_1 = (K''_1 + K''_2)/2$ and $K'_2 = (K''_1 - K''_2)/2$ then the solution is changed to

$$v_C(t) = \frac{K''_1 + K''_2}{2} \cdot e^x + \frac{K''_1 - K''_2}{2} \cdot e^{-x} = K''_1 \cosh x + K''_2 \sinh x. \quad (\text{B.20})$$

By applying the boundary conditions from (4.3),

$$v_C(0) = V_0 = K''_1 \cosh X_F - K''_2 \sinh X_F \quad \text{and} \quad v_C(T_F) = V_F = K''_1 \cosh X_F + K''_2 \sinh X_F. \quad (\text{B.21})$$

These constants can be determined easily, $K''_1 = (V_F + V_0)/2 \cosh X_F$ and $K''_2 = (V_F - V_0)/2 \sinh X_F$. Therefore, the solution is

$$v_C(t) = \frac{V_F + V_0}{2} \cdot \frac{\cosh x}{\cosh X_F} + \frac{V_F - V_0}{2} \cdot \frac{\sinh x}{\sinh X_F} \quad (\text{B.22})$$

where X_F is defined as in (4.13). Furthermore, by applying some hyperbolic identities, we get

$$v_C(t) = \frac{V_F(\sinh x \cosh X_F + \cosh x \sinh X_F) - V_0(\sinh x \cosh X_F - \cosh x \sinh X_F)}{2 \sinh X_F \cosh X_F} \quad (\text{B.23})$$

or

$$v_C(t) = \frac{V_F \sinh(x + X_F) - V_0 \sinh(x - X_F)}{\sinh 2X_F}. \quad (\text{B.24})$$

B.5 Determining the Most Energy-efficient Source Function

For input voltage function, From (4.2), we get

$$\begin{aligned} v_I(t) &= a_{11}v_C(t) + a_{12}i_C(t) = a_{11}v_C(t) + a_{12}C\dot{v}_C(t) \\ &= a_{11} \left(v_C(t) + R_{SC}C \cdot \frac{dx}{dt} \cdot \frac{dv_C(t)}{dx} \right) \end{aligned} \quad (\text{B.25})$$

where $R_{SC} = a_{12}/a_{11}$ as it has been defined in (4.3). By using (4.12), (4.11) and (B.75), the input voltage function becomes

$$\begin{aligned} v_I(t) &= a_{11} \left(\frac{V_F \sinh(x + X_F) - V_0 \sinh(x - X_F)}{\sinh 2X_F} \right. \\ &\quad \left. + \sqrt{\frac{R_{SC}}{R_{OC}}} \cdot \frac{V_F \cosh(x + X_F) - V_0 \cosh(x - X_F)}{\sinh 2X_F} \right) \end{aligned} \quad (\text{B.26})$$

or in terms of k that are defined in (4.16),

$$\begin{aligned} v_I(t) &= \frac{a_{11}}{k \sinh 2X_F} (V_F (k \sinh(x + X_F) + \cosh(x + X_F)) \\ &\quad - V_0 (k \sinh(x - X_F) + \cosh(x - X_F))) \end{aligned} \quad (\text{B.27})$$

For input current function, From (4.2), we get

$$\begin{aligned} i_I(t) &= a_{21}v_C(t) + a_{22}i_C(t) = a_{21}v_C(t) + a_{22}C\dot{v}_C(t) \\ &= a_{22} \left(\frac{1}{R_{OC}}v_C(t) + C \cdot \frac{dx}{dt} \cdot \frac{dv_C(t)}{dx} \right) \end{aligned} \quad (\text{B.28})$$

where $R_{SC} = a_{22}/a_{21}$ as it has been defined in (4.3). By using (4.12), (4.11) and (B.75), the input current function becomes

$$\begin{aligned} i_I(t) &= \frac{a_{22}}{R_{OC}} \cdot \left(\frac{V_F \sinh(x + X_F) - V_0 \sinh(x - X_F)}{\sinh 2X_F} \right. \\ &\quad \left. + \sqrt{\frac{R_{OC}}{R_{SC}}} \cdot \frac{V_F \cosh(x + X_F) - V_0 \cosh(x - X_F)}{\sinh 2X_F} \right) \end{aligned} \quad (\text{B.29})$$

or in terms of k ,

$$\begin{aligned} i_I(t) &= \frac{a_{22}}{R_{OC} \sinh 2X_F} (V_F (\sinh(x + X_F) + k \cosh(x + X_F)) \\ &\quad - V_0 (\sinh(x - X_F) + k \cosh(x - X_F))) \end{aligned} \quad (\text{B.30})$$

B.6 Determining the Lowest Input Energy

From (4.4), the input energy can be written as

$$E_I = \int_0^{T_F} a_{11}a_{21}v_C^2 dt + \int_0^{T_F} a_{12}a_{22}C^2\dot{v}_C^2 dt + \int_0^{T_F} (a_{12}a_{21} + a_{11}a_{22})Cv_C\dot{v}_C dt \quad (\text{B.31})$$

Let us consider the last term first, by using chain rule and the reciprocal property of transmission parameters, $a_{11}a_{22} - a_{12}a_{21} = 1$, we get

$$\begin{aligned} \int_0^{T_F} (a_{12}a_{21} + a_{11}a_{22})Cv_C\dot{v}_C dt &= (1 + 2a_{12}a_{21})C \int_0^{T_F} v_C dv_C \\ &= \left(\frac{1}{2} + a_{12}a_{21}\right) C(V_F^2 - V_0^2). \end{aligned} \quad (\text{B.32})$$

Furthermore, let us consider the second term of integral of input energy, by using the integration by part technique, we get

$$\begin{aligned} \int_0^{T_F} a_{12}a_{22}C^2\dot{v}_C^2 dt &= a_{12}a_{22}C^2 \left(\dot{v}_C v_C \Big|_{t=0}^{t=T_F} - \int_0^{T_F} v_C \ddot{v}_C dt \right) \\ &= a_{12}a_{22}C^2 \left(\dot{v}_C(T_F)V_F - \dot{v}_C(0)V_0 - \int_0^{T_F} v_C \ddot{v}_C dt \right). \end{aligned} \quad (\text{B.33})$$

By substituting these integration back, we obtain

$$\begin{aligned} E_I &= a_{12}a_{22}C^2 \left(\dot{v}_C(T_F)V_F - \dot{v}_C(0)V_0 + \int_0^{T_F} v_C \left(\frac{a_{11}a_{21}}{a_{12}a_{22}C^2} v_C - \ddot{v}_C \right) dt \right) \\ &\quad + \left(\frac{1}{2} + a_{12}a_{21}\right) C(V_F^2 - V_0^2). \end{aligned} \quad (\text{B.34})$$

Recall that in our derivation above, it is just rearranging terms and not involve any theory about the lowest energy. The expression of input energy above is true for any input functions.

However, the function inside the integral is the Euler-Lagrange's differential equation in (4.7) and it is zero if the circuit is applied by the most energy-efficient source functions. Therefore, the lowest input energy is

$$E_I = a_{12}a_{22}C^2 (\dot{v}_C(T_F)V_F - \dot{v}_C(0)V_0) + \left(\frac{1}{2} + a_{12}a_{21}\right) C(V_F^2 - V_0^2). \quad (\text{B.35})$$

According to (4.12), the time derivative of v_C is

$$\dot{v}_C(t) = \frac{V_F \cosh(x + X_F) - V_0 \cosh(x - X_F)}{\tau \sinh 2X_F} \quad (\text{B.36})$$

and its value at the boundaries must be

$$\dot{v}_C(0) = \dot{v}_C(x = -X_F) = \frac{V_F - V_0 \cosh 2X_F}{\tau \sinh 2X_F} \quad (\text{B.37})$$

and

$$\dot{v}_C(T_F) = \dot{v}_C(x = X_F) = \frac{V_F \cosh 2X_F - V_0}{\tau \sinh 2X_F}. \quad (\text{B.38})$$

By substituting back into the lowest input energy expression, we get

$$E_I = \frac{a_{12}a_{22}C^2}{\tau \sinh 2X_F} ((V_F^2 + V_0^2) \cosh 2X_F - 2V_0V_F) + \left(\frac{1}{2} + a_{12}a_{21} \right) C(V_F^2 - V_0^2). \quad (\text{B.39})$$

By applying the expression of τ in term of transmission parameters from (4.11), the lowest input energy can be written as

$$E_I = \frac{\sqrt{a_{11}a_{22}a_{12}a_{21}}}{\sinh 2X_F} C((V_F^2 + V_0^2) \cosh 2X_F - 2V_0V_F) + \left(\frac{1}{2} + a_{12}a_{21} \right) C(V_F^2 - V_0^2). \quad (\text{B.40})$$

Follow by using the reciprocal property, the lowest input energy becomes

$$E_I = \frac{\sqrt{(1 + a_{12}a_{21})a_{12}a_{21}}}{\sinh 2X_F} C((V_F^2 + V_0^2) \cosh 2X_F - 2V_0V_F) + \left(\frac{1}{2} + a_{12}a_{21} \right) C(V_F^2 - V_0^2). \quad (\text{B.41})$$

Now, the expression of the lowest input energy involves the transmission parameters only $a_{12}a_{21}$ and fortunately, they can be replaced by k . According to the definition of k in (4.16) with the reciprocal property, we get

$$k = \sqrt{\frac{1 + a_{12}a_{21}}{a_{12}a_{21}}} \rightarrow a_{12}a_{21} = \frac{1}{k^2 - 1}. \quad (\text{B.42})$$

Note that k is always greater than one since $1 + a_{12}a_{21} > a_{12}a_{21}$ and the lowest input energy becomes

$$E_I = \frac{k}{(k^2 - 1) \sinh 2X_F} C((V_F^2 + V_0^2) \cosh 2X_F - 2V_0V_F) + \left(\frac{1}{2} + \frac{1}{k^2 - 1} \right) C(V_F^2 - V_0^2). \quad (\text{B.43})$$

Although we found the expression of the lowest input energy, it is quite ugly and seems not to be useful for further analysis. Fortunately, the input energy can be simplified further by introducing new parameter called γ which is defined in (5.34). Let us replace some terms of E_I with γ as

$$\begin{aligned} V_F^2 + V_0^2 &= \frac{(V_F + V_0)^2 + (V_F - V_0)^2}{2} = \frac{\frac{V_F + V_0}{V_F - V_0} + \frac{V_F - V_0}{V_F + V_0}}{2} \cdot (V_F + V_0)(V_F - V_0) \\ &= \frac{1/\gamma + \gamma}{2} (V_F^2 - V_0^2) \end{aligned} \quad (\text{B.44})$$

and

$$\begin{aligned} 2V_0V_F &= \frac{(V_F + V_0)^2 - (V_F - V_0)^2}{2} = \frac{\frac{V_F + V_0}{V_F - V_0} - \frac{V_F - V_0}{V_F + V_0}}{2} \cdot (V_F + V_0)(V_F - V_0) \\ &= \frac{1/\gamma - \gamma}{2} (V_F^2 - V_0^2). \end{aligned} \quad (\text{B.45})$$

Therefore,

$$\begin{aligned} E_I &= \frac{C(V_F^2 - V_0^2)}{2} \cdot \left(\frac{k(1/\gamma + \gamma) \cosh 2X_F - k(1/\gamma - \gamma)}{(k^2 - 1) \sinh 2X_F} + \frac{2}{k^2 - 1} + 1 \right) \\ &= \frac{C(V_F^2 - V_0^2)}{2(k^2 - 1)} \cdot \left(\frac{(\cosh 2X_F - 1)k/\gamma + (\cosh 2X_F + 1)k\gamma}{\sinh 2X_F} + k^2 + 1 \right). \end{aligned}$$

From hyperbolic identities

$$\tanh \frac{x}{2} = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}, \quad (\text{B.46})$$

we get

$$E_I = \frac{C(V_F^2 - V_0^2)}{2(k^2 - 1)} \cdot \left(k \left(\frac{\tanh X_F}{\gamma} + \frac{\gamma}{\tanh X_F} \right) + k^2 + 1 \right) \quad (\text{B.47})$$

which can be simplified to

$$E_I = \frac{C(V_F^2 - V_0^2)}{2(k^2 - 1)} \cdot \left(k + \frac{\tanh X_F}{\gamma} \right) \left(k + \frac{\gamma}{\tanh X_F} \right). \quad (\text{B.48})$$

That is the expression of the lowest input energy in (4.17).

B.7 Determining Optimum Transient Duration and Optimum Lowest Input Energy

From the corollary 1, the lowest input energy can be rearranged as

$$E_I = \frac{C(V_F^2 - V_0^2)}{2(k^2 - 1)} \cdot \left(k^2 + \left(\frac{\tanh(T_F/2\tau)}{\gamma} + \gamma \coth(T_F/2\tau) \right) k + 1 \right). \quad (\text{B.49})$$

The optimum transient duration can be found by differentiating E_I respect to T_F and equate to zero which is

$$\frac{1}{\gamma \cosh^2(T_{F(\text{opt.})}/2\tau)} - \frac{\gamma}{\sinh^2(T_{F(\text{opt.})}/2\tau)} = 0 \quad (\text{B.50})$$

where $T_{F(\text{opt.})}$ is the optimum transient duration. By multiplying the equation by $\gamma \sinh^2(T_{F(\text{opt.})}/2\tau)$, we get

$$\tanh^2(T_{F(\text{opt.})}/2\tau) - \gamma^2 = 0 \quad \rightarrow \quad \tanh(T_{F(\text{opt.})}/2\tau) = \pm \gamma \quad (\text{B.51})$$

However, $T_{F(\text{opt.})}/2\tau$ is always positive and it causes $\tanh(T_{F(\text{opt.})}/2\tau)$ to be positive. Therefore, the tan-hyperbolic function will be equal to γ when γ is positive and it will be equal to $-\gamma$ when γ is negative. Mathematically, we can write

$$\tanh(T_{F(\text{opt.})}/2\tau) = |\gamma|. \quad (\text{B.52})$$

Furthermore, the maximum value of the tan-hyperbolic function is 1. Therefore, the optimum transient duration exists when $|\gamma| < 1$ only. This condition can be analyzed in terms of V_F and V_0 as

$$-1 < \frac{V_F - V_0}{V_F + V_0} < 1 \quad (\text{B.53})$$

If $V_F + V_0 > 0$, the inequality can be separated as

$$\begin{array}{ll} -V_F - V_0 < V_F - V_0 & \text{and} \quad V_F - V_0 < V_F + V_0 \\ V_F > 0 & \text{and} \quad V_0 > 0. \end{array}$$

It means that the condition is satisfied when both $V_F > 0$ and $V_0 > 0$.

Another case is $V_F + V_0 < 0$, the inequality becomes

$$\begin{array}{ll} -V_F - V_0 > V_F - V_0 & \text{and} \quad V_F - V_0 > V_F + V_0 \\ V_F < 0 & \text{and} \quad V_0 < 0. \end{array}$$

In other words, the condition is satisfied when both $V_F < 0$ and $V_0 < 0$. By combining these two cases above, we can conclude that the inequality in (B.53) is satisfied when both V_F and V_0 are the same sign. Either they are both positive or negative.

According to (B.52), determining the optimum transient duration must be considered in two cases, the first case is when $0 < \gamma < 1$ and the other case is when $-1 < \gamma < 0$. These conditions can be found in terms of V_F and V_0 . Firstly, let us consider the case when $0 < \gamma < 1$, the relationship between V_F and V_0 can be found as

$$\frac{V_F - V_0}{V_F + V_0} > 0 \quad \rightarrow \quad \frac{V_F/V_0 - 1}{V_F/V_0 + 1} > 0 \quad \rightarrow \quad V_F/V_0 > 1. \quad (\text{B.54})$$

Recall that V_F and V_0 must be the same sign to satisfy the condition in (B.53), therefore the denominator is always positive and we can conclude that $\gamma > 0$ when $V_F/V_0 > 1$. For $\gamma < 0$, a condition of V_F and V_0 can be found similarly with $\gamma > 0$ as

$$\frac{V_F - V_0}{V_F + V_0} < 0 \quad \rightarrow \quad \frac{V_F/V_0 - 1}{V_F/V_0 + 1} < 0 \quad \rightarrow \quad V_F/V_0 < 1 \quad \rightarrow \quad V_0/V_F > 1. \quad (\text{B.55})$$

By expanding $\tanh(T_{F(\text{opt.})}/2\tau)$ into the exponential functions, we have

$$\frac{e^{T_{F(\text{opt.})}/\tau} - 1}{e^{T_{F(\text{opt.})}/\tau} + 1} = \gamma = \frac{V_F - V_0}{V_F + V_0} = \frac{V_F/V_0 - 1}{V_F/V_0 + 1} \quad \rightarrow \quad e^{T_{F(\text{opt.})}/\tau} = V_F/V_0 \quad \rightarrow$$

$$T_{F(\text{opt.})} = \ln(V_F/V_0)\tau. \quad (\text{B.56})$$

for $\gamma > 0$ or $V_F/V_0 > 1$ and

$$\frac{e^{T_{F(\text{opt.})}/\tau} - 1}{e^{T_{F(\text{opt.})}/\tau} + 1} = -\gamma = \frac{V_0 - V_F}{V_0 + V_F} = \frac{V_0/V_F - 1}{V_0/V_F + 1} \quad \rightarrow \quad e^{T_{F(\text{opt.})}/\tau} = V_0/V_F \quad \rightarrow$$

$$T_{F(\text{opt.})} = \ln(V_0/V_F)\tau. \quad (\text{B.57})$$

for $\gamma < 0$ or $V_0/V_F > 1$. In summary, the optimum transient duration exists when V_F and V_0 are the same sign. If $V_F/V_0 > 1$, the optimum transient duration is (B.56). If $V_0/V_F > 1$, the optimum transient duration is (B.57).

Moreover, the optimum transient duration can be written in terms of γ . According to (B.52), we can solve for $T_{F(\text{opt.})}$ by expanding the tan-hyperbolic function as

$$\frac{e^{T_{F(\text{opt.})}/\tau} - 1}{e^{T_{F(\text{opt.})}/\tau} + 1} = |\gamma| \quad \rightarrow \quad e^{T_{F(\text{opt.})}/\tau} - 1 = |\gamma|e^{T_{F(\text{opt.})}/\tau} + |\gamma| \quad \rightarrow$$

$$e^{T_{F(\text{opt.})}/\tau} = \frac{1 + |\gamma|}{1 - |\gamma|}. \quad (\text{B.58})$$

Therefore,

$$T_{F(\text{opt.})} = \ln\left(\frac{1 + |\gamma|}{1 - |\gamma|}\right) \tau \quad (\text{B.59})$$

for $|\gamma| < 1$.

After the optimum transient duration has been calculated, optimum lowest input energy can be determined easily. By substituting (B.52) into the corollary 1, the optimum lowest input energy is

$$E_{I(\text{opt.})} = \frac{C(V_F^2 - V_0^2)}{2(k^2 - 1)} \cdot \left(k + \frac{|\gamma|}{\gamma}\right) \left(k + \frac{\gamma}{|\gamma|}\right) \quad (\text{B.60})$$

If $\gamma > 0$ or in other words, $V_F/V_0 > 1$, the optimum lowest input energy is equal to

$$E_{I(\text{opt.})} = \frac{C(V_F^2 - V_0^2)}{2(k^2 - 1)} \cdot (k + 1)^2 = \frac{C(k + 1)}{2(k - 1)}(V_F^2 - V_0^2) \quad (\text{B.61})$$

and if $\gamma < 0$ or $V_0/V_F > 1$, the optimum lowest input energy is equal to

$$E_{I(\text{opt.})} = \frac{C(V_F^2 - V_0^2)}{2(k^2 - 1)} \cdot (k - 1)^2 = \frac{C(k - 1)}{2(k + 1)}(V_F^2 - V_0^2). \quad (\text{B.62})$$

B.8 Determining Zero Transient Durations

From the corollary 1, the lowest input energy will be zero if

$$k + \frac{\tanh(T_Z/2\tau)}{\gamma} = 0 \quad \text{or} \quad k + \frac{\gamma}{\tanh(T_Z/2\tau)} = 0 \quad (\text{B.63})$$

where T_Z is zero transient duration. By solving these two equations, we get

$$\begin{aligned} \tanh(T_Z/2\tau) &= -k\gamma & \text{or} & & \tanh(T_Z/2\tau) &= -\frac{\gamma}{k} \\ \frac{e^{T_Z/\tau} - 1}{e^{T_Z/\tau} + 1} &= -k\gamma & \text{or} & & \frac{e^{T_Z/\tau} - 1}{e^{T_Z/\tau} + 1} &= -\frac{\gamma}{k} \\ e^{T_Z/\tau} - 1 &= -k\gamma e^{T_Z/\tau} - k\gamma & \text{or} & & ke^{T_Z/\tau} - k &= -\gamma e^{T_Z/\tau} - \gamma \\ (1 + k\gamma)e^{T_Z/\tau} &= 1 - k\gamma & \text{or} & & (k + \gamma)e^{T_Z/\tau} &= k - \gamma \\ T_Z &= \tau \ln \left(\frac{1 - k\gamma}{1 + k\gamma} \right) & \text{or} & & T_Z &= \tau \ln \left(\frac{k - \gamma}{k + \gamma} \right). \end{aligned}$$

Although we did not assume anything before we derive T_Z , there is a condition of γ . Firstly, if $\gamma > 0$, both terms in natural logarithm functions will be less than 1 and T_Z will be negative which is impossible. If $\gamma = 0$, T_Z will be infinity. Therefore, the condition of γ is $\gamma < 0$. From Appendix B.7, $\gamma < 0$ means the circuit discharges as expected because the lowest input energy can be zero only if the capacitor discharges some energy.

Moreover, the zero transient duration is applicable only when the number in the natural logarithm is greater than 1. If $\gamma < 0$, the first zero transient duration (on the left) is applicable when $1 + k\gamma > 0$ or

$$k < -1/\gamma. \quad (\text{B.64})$$

For the second zero transient duration (on the right), it is applicable when $k + \gamma > 0$ or

$$k > -\gamma. \quad (\text{B.65})$$

In case of discharging without crossing zero, from Appendix B.7, $-1 < \gamma < 0$. Since k is always greater than 1, the second condition is always satisfied and the second zero transient duration is always applicable. Therefore, there are one or two zero transient durations in this case. However, for discharging with crossing zero, $\gamma < -1$ and the first condition is never satisfied. Therefore, there is one or no zero transient duration in this case.

B.9 Manipulating the Euler-Lagrange's Differential Equation of a Series LC circuits

From (5.12), the Lagrangian function can be written in terms of v_C , $\dot{v}_C$ and $\ddot{v}_C$ as

$$\begin{aligned}\mathcal{L} &= (a_{11}v_O + a_{12}i_O)(a_{21}v_O + a_{22}i_O) \\ &= a_{11}a_{21}v_O^2 + (a_{12}a_{21} + a_{11}a_{22})v_Oi_O + a_{12}a_{22}i_O^2 \\ &= a_{11}a_{21}(\ddot{v}_C/\omega^2 + v_C)^2 + (a_{12}a_{21} + a_{11}a_{22})(\ddot{v}_C/\omega^2 + v_C)C\dot{v}_C + a_{12}a_{22}C^2\dot{v}_C^2. \quad (\text{B.66})\end{aligned}$$

The Lagrangian function must be substituted into the Euler-Lagrange's differential equation,

$$\frac{\partial \mathcal{L}}{\partial v_C} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{v}_C} \right) + \frac{d^2}{dt^2} \left(\frac{\partial \mathcal{L}}{\partial \ddot{v}_C} \right) = 0. \quad (\text{B.67})$$

For simplicity, first let us calculate these terms separately,

$$\frac{\partial \mathcal{L}}{\partial v_C} = 2a_{11}a_{21}(\ddot{v}_C/\omega^2 + v_C) + (a_{12}a_{21} + a_{11}a_{22})C\dot{v}_C, \quad (\text{B.68})$$

$$-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{v}_C} \right) = -(a_{12}a_{21} + a_{11}a_{22})(\ddot{v}_C/\omega^2 + \dot{v}_C)C - 2a_{12}a_{22}C^2\ddot{v}_C, \quad (\text{B.69})$$

$$\frac{d^2}{dt^2} \left(\frac{\partial \mathcal{L}}{\partial \ddot{v}_C} \right) = \frac{2a_{11}a_{21}}{\omega^2}(\ddot{\ddot{v}}_C/\omega^2 + \ddot{v}_C) + \frac{a_{12}a_{21} + a_{11}a_{22}}{\omega^2}\ddot{\ddot{v}}_CC. \quad (\text{B.70})$$

Then, by following the Euler-Lagrange's differential equation, sum of these terms must equal to zero.

$$\frac{2a_{11}a_{21}}{\omega^4}\ddot{\ddot{v}}_C + \left(\frac{4a_{11}a_{21}}{\omega^2} - 2a_{12}a_{22}C^2 \right) \ddot{v}_C + 2a_{11}a_{21}v_C = 0 \quad (\text{B.71})$$

$$\ddot{\ddot{v}}_C + \left(2 - \frac{a_{12}a_{22}}{a_{11}a_{21}}\omega^2C^2 \right) \omega^2\ddot{v}_C + \omega^4v_C = 0. \quad (\text{B.72})$$

Notice that the first and the third derivatives of v_C vanish. By the definitions of R_{OC} and R_{SC} in (5.2a) and (5.2b), the differential equation becomes

$$\ddot{\ddot{v}}_C + 2 \left(1 - \frac{R_{OC}R_{SC}}{2L/C} \right) \omega^2\ddot{v}_C + \omega^4v_C = 0. \quad (\text{B.73})$$

Finally, by the definitions of Z and r in (5.16) and (5.17), we get

$$\ddot{\ddot{v}}_C + 2(1 - 2r^2)\omega^2\ddot{v}_C + \omega^4v_C = 0. \quad (\text{B.74})$$

B.10 Unique Solution of Capacitor Voltage in Hyperbolic Mode

For symmetry, let us define new variable x as

$$x \equiv t - \frac{T_F}{2}. \quad (\text{B.75})$$

Therefore, at the boundaries $t = 0$ and $t = T_F$, $x = -T_F/2$ and $x = T_F/2$ respectively. Moreover, since $dx/dt = 1$, the variable t in the differential equation of (5.18) can be replaced with the variable x and the general solution can be written as

$$v_C(x) = K'_1 \sinh m_+\omega x + K'_2 \cosh m_+\omega x + K'_3 \sinh m_-\omega x + K'_4 \cosh m_-\omega x \quad (\text{B.76})$$

where m_+ and m_- are defined in (5.21) and (5.22). All constants K'_1 , K'_2 , K'_3 and K'_4 can be determined by using all boundaries conditions.

Nevertheless, the hyperbolic function can be huge and solving all the constants subjected to the boundaries conditions can be an ill-conditioned problem. Therefore, the unique solution must be rearranged by changing the constants. Let us change the constants as follows.

$$K'_1 \rightarrow \frac{K_1}{m_+ \cosh m_+ \omega T_F/2} \quad (\text{B.77})$$

$$K'_2 \rightarrow \frac{K_2}{m_+ \sinh m_+ \omega T_F/2} \quad (\text{B.78})$$

$$K'_3 \rightarrow \frac{K_3}{m_- \cosh m_- \omega T_F/2} \quad (\text{B.79})$$

$$K'_4 \rightarrow \frac{K_4}{m_- \sinh m_- \omega T_F/2} \quad (\text{B.80})$$

where K_1, K_2, K_3 and K_4 are constants that must be determined.

With these constants, the capacitor voltage and its first derivative are

$$v_C(x) = \frac{K_1}{m_+} \cdot \frac{\sinh m_+ \omega x}{\cosh m_+ \omega T_F/2} + \frac{K_2}{m_+} \cdot \frac{\cosh m_+ \omega x}{\sinh m_+ \omega T_F/2} + \frac{K_3}{m_-} \cdot \frac{\sinh m_- \omega x}{\cosh m_- \omega T_F/2} + \frac{K_4}{m_-} \cdot \frac{\cosh m_- \omega x}{\sinh m_- \omega T_F/2} \quad (\text{B.81})$$

and

$$\dot{v}_C(x) = K_1 \cdot \frac{\omega \cosh m_+ \omega x}{\cosh m_+ \omega T_F/2} + K_2 \cdot \frac{\omega \sinh m_+ \omega x}{\sinh m_+ \omega T_F/2} + K_3 \cdot \frac{\omega \cosh m_- \omega x}{\cosh m_- \omega T_F/2} + K_4 \cdot \frac{\omega \sinh m_- \omega x}{\sinh m_- \omega T_F/2} \quad (\text{B.82})$$

respectively. At the boundaries, $x = T_F/2$ and $x = -T_F/2$, the first derivative must be equal to zero according to (5.5) and (5.6). Therefore, we get

$$K_1 + K_2 + K_3 + K_4 = 0 \quad (\text{B.83})$$

and

$$K_1 - K_2 + K_3 - K_4 = 0. \quad (\text{B.84})$$

By adding and subtracting these two equations together, we obtain

$$K_3 = -K_1 \quad \text{and} \quad K_4 = -K_2. \quad (\text{B.85})$$

By substituting back into $v_C(x)$, we get

$$v_C(x) = \frac{K_1}{m_+} \cdot \frac{\sinh m_+ \omega x}{\cosh m_+ \omega T_F/2} + \frac{K_2}{m_+} \cdot \frac{\cosh m_+ \omega x}{\sinh m_+ \omega T_F/2} - \frac{K_1}{m_-} \cdot \frac{\sinh m_- \omega x}{\cosh m_- \omega T_F/2} - \frac{K_2}{m_-} \cdot \frac{\cosh m_- \omega x}{\sinh m_- \omega T_F/2}. \quad (\text{B.86})$$

Next, by applying the boundaries conditions (5.7) and (5.8), we have

$$K_1 \left(\frac{\tanh m_+ \omega T_F/2}{m_+} - \frac{\tanh m_- \omega T_F/2}{m_-} \right) + K_2 \left(\frac{\coth m_+ \omega T_F/2}{m_+} - \frac{\coth m_- \omega T_F/2}{m_-} \right) = V_F \quad (\text{B.87})$$

and

$$-K_1 \left(\frac{\tanh m_+ \omega T_F/2}{m_+} - \frac{\tanh m_- \omega T_F/2}{m_-} \right) + K_2 \left(\frac{\coth m_+ \omega T_F/2}{m_+} - \frac{\coth m_- \omega T_F/2}{m_-} \right) = V_0. \quad (\text{B.88})$$

By solving these two equations, we get

$$K_1 = \frac{(V_F - V_0)/2}{m_- \tanh m_+ \omega T_F/2 - m_+ \tanh m_- \omega T_F/2} \quad (\text{B.89})$$

and

$$K_2 = \frac{(V_F + V_0)/2}{m_- \coth m_+ \omega T_F/2 - m_+ \coth m_- \omega T_F/2}. \quad (\text{B.90})$$

Therefore, an analytical solution of capacitor voltage in hyperbolic mode is (B.86) where x , K_1 and K_2 are (B.75), (B.89) and (B.90) respectively.

B.11 Unique Solution of Capacitor Voltage in Hyperbolic and Linear Mode

From a definition of x in (B.75), a general solution of capacitor voltage in hyperbolic and linear mode is

$$v_C(x) = K'_1 \sinh \omega x + K'_2 \cosh \omega x + K'_3 x \sinh \omega x + K'_4 x \cosh \omega x \quad (\text{B.91})$$

where K'_1 , K'_2 , K'_3 and K'_4 can be determined by using all boundaries conditions. Let us change the constants as follows.

$$K'_1 \rightarrow \frac{K_1}{\cosh \omega T_F/2} \quad (\text{B.92})$$

$$K'_2 \rightarrow \frac{K_2}{\sinh \omega T_F/2} \quad (\text{B.93})$$

$$K'_3 \rightarrow \frac{K_3 \omega / \cosh \omega T_F/2}{\omega T_F/2 + \tanh \omega T_F/2} \quad (\text{B.94})$$

$$K'_4 \rightarrow \frac{K_4 \omega / \sinh \omega T_F/2}{\omega T_F/2 + \coth \omega T_F/2} \quad (\text{B.95})$$

where K_1, K_2, K_3 and K_4 are constants that must be determined.

With these constants, the capacitor voltage and its first derivative are

$$v_C(x) = K_1 \cdot \frac{\sinh \omega x}{\cosh \omega T_F/2} + K_2 \cdot \frac{\cosh \omega x}{\sinh \omega T_F/2} + \frac{K_3}{\omega T_F/2 + \tanh \omega T_F/2} \cdot \frac{\omega x \sinh \omega x}{\cosh \omega T_F/2} + \frac{K_4}{\omega T_F/2 + \coth \omega T_F/2} \cdot \frac{\omega x \cosh \omega x}{\sinh \omega T_F/2} \quad (\text{B.96})$$

and

$$\begin{aligned} \dot{v}_C(x) = & K_1 \cdot \frac{\omega \cosh \omega x}{\cosh \omega T_F/2} + K_2 \cdot \frac{\omega \sinh \omega x}{\sinh \omega T_F/2} \\ & + \frac{K_3 \omega}{\omega T_F/2 + \tanh \omega T_F/2} \cdot \left(\frac{\omega x \cosh \omega x}{\cosh \omega T_F/2} + \frac{\sinh \omega x}{\cosh \omega T_F/2} \right) \\ & + \frac{K_4 \omega}{\omega T_F/2 + \coth \omega T_F/2} \cdot \left(\frac{\omega x \sinh \omega x}{\sinh \omega T_F/2} + \frac{\cosh \omega x}{\sinh \omega T_F/2} \right) \end{aligned} \quad (\text{B.97})$$

respectively. At the boundaries, $x = T_F/2$ and $x = -T_F/2$, the first derivative must be equal to zero according to (5.5) and (5.6). Therefore, we get

$$K_1 + K_2 + K_3 + K_4 = 0 \quad (\text{B.98})$$

and

$$K_1 - K_2 - K_3 + K_4 = 0. \quad (\text{B.99})$$

By adding and subtracting these two equations together, we obtain

$$K_3 = -K_2 \quad \text{and} \quad K_4 = -K_1. \quad (\text{B.100})$$

By substituting back into $v_C(x)$, we get

$$\begin{aligned} v_C(x) = & K_1 \cdot \frac{\sinh \omega x}{\cosh \omega T_F/2} + K_2 \cdot \frac{\cosh \omega x}{\sinh \omega T_F/2} \\ & - \frac{K_2}{\omega T_F/2 + \tanh \omega T_F/2} \cdot \frac{\omega x \sinh \omega x}{\cosh \omega T_F/2} - \frac{K_1}{\omega T_F/2 + \coth \omega T_F/2} \cdot \frac{\omega x \cosh \omega x}{\sinh \omega T_F/2} \end{aligned} \quad (\text{B.101})$$

Finally, by applying the boundaries conditions (5.7) and (5.8), we obtain

$$K_1 = \frac{(V_F - V_0)}{2} \cdot \frac{\omega T_F/2 + \coth \omega T_F/2}{1 - \omega T_F/2(\coth \omega T_F/2 - \tanh \omega T_F/2)} \quad (\text{B.102})$$

and

$$K_2 = \frac{(V_F + V_0)}{2} \cdot \frac{\omega T_F/2 + \tanh \omega T_F/2}{1 + \omega T_F/2(\coth \omega T_F/2 - \tanh \omega T_F/2)}. \quad (\text{B.103})$$

Therefore, an analytical solution of capacitor voltage in hyperbolic and linear mode is (B.101) where x , K_1 and K_2 are (B.75), (B.102) and (B.103) respectively.

B.12 Calculation of the Lowest Input Energy of a Series LC circuits

A function inside the integral of the input energy in (5.9) actually is the Lagrangian function in (B.66). However, a full expansion of the Lagrangian function is quite long, so we define each terms of the expansion as

$$f_1 = a_{11}a_{21}\ddot{v}_C^2/\omega^4 \quad (\text{B.104})$$

$$f_2 = 2a_{11}a_{21}v_C\ddot{v}_C/\omega^2 \quad (\text{B.105})$$

$$f_3 = a_{11}a_{21}v_C^2 \quad (\text{B.106})$$

$$f_4 = (a_{12}a_{21} + a_{11}a_{22})C\dot{v}_C\ddot{v}_C/\omega^2 \quad (\text{B.107})$$

$$f_5 = (a_{12}a_{21} + a_{11}a_{22})Cv_C\dot{v}_C \quad (\text{B.108})$$

$$f_6 = a_{12}a_{22}C^2\dot{v}_C^2 \quad (\text{B.109})$$

for tidy writing only. It is easily to see that

$$E_I = \int_0^{T_F} \mathcal{L} dt = \int_0^{T_F} (f_1 + f_2 + f_3 + f_4 + f_5 + f_6) dt. \quad (\text{B.110})$$

We will calculate each terms separately as follows.

$$\int_0^{T_F} f_1 dt = a_{11}a_{21}/\omega^4 \int_0^{T_F} \ddot{v}_C \cdot \ddot{v}_C dt. \quad (\text{B.111})$$

By using the integration by part technique, we get

$$\int_0^{T_F} f_1 dt = a_{11}a_{21}/\omega^4 \left(\dot{v}_C \ddot{v}_C \Big|_0^{T_F} - \int_0^{T_F} \dot{v}_C \ddot{\ddot{v}}_C dt \right). \quad (\text{B.112})$$

By applying the boundaries conditions from (5.5) and (5.6), the first term in the parenthesis is zero. For the second term, we apply the integration by part technique again and get

$$\int_0^{T_F} f_1 dt = a_{11}a_{21}/\omega^4 \left(-v_C \ddot{\ddot{v}}_C \Big|_0^{T_F} + \int_0^{T_F} v_C \ddot{\ddot{\ddot{v}}}_C dt \right). \quad (\text{B.113})$$

With the boundaries conditions from (5.7) and (5.8), the integration becomes

$$\int_0^{T_F} f_1 dt = a_{11}a_{21}/\omega^4 \left(\ddot{\ddot{v}}_C(0)V_0 - \ddot{\ddot{v}}_C(T_F)V_F + \int_0^{T_F} v_C \ddot{\ddot{\ddot{v}}}_C dt \right). \quad (\text{B.114})$$

Next, let us calculate the integration of f_6 , the integral by part can be used again and we get

$$\int_0^{T_F} f_6 dt = a_{12}a_{22}C^2 \left(v_C \dot{v}_C \Big|_0^{T_F} - \int_0^{T_F} v_C \ddot{v}_C dt \right). \quad (\text{B.115})$$

By substituting the boundaries conditions, we conclude that

$$\int_0^{T_F} f_6 dt = -a_{12}a_{22}C^2 \int_0^{T_F} v_C \ddot{v}_C dt. \quad (\text{B.116})$$

Let us come back for the integration of f_2 , the integral can be calculated as

$$\int_0^{T_F} f_2 dt = 2a_{11}a_{21}/\omega^2 \left(v_C \dot{v}_C \Big|_0^{T_F} - \int_0^{T_F} \dot{v}_C^2 dt \right). \quad (\text{B.117})$$

The integration of $\dot{v}_C^2$ is similar with the integration of f_6 . According to (B.116), we can conclude that

$$\int_0^{T_F} f_2 dt = 2a_{11}a_{21}/\omega^2 \int_0^{T_F} v_C \ddot{v}_C dt. \quad (\text{B.118})$$

For f_4 , the integration can be done by applying chain rule. Therefore,

$$\int_0^{T_F} f_4 dt = (a_{12}a_{21} + a_{11}a_{22})C/\omega^2 \int_0^{T_F} \dot{v}_C d\dot{v}_C = (a_{12}a_{21} + a_{11}a_{22})C/\omega^2 \cdot \frac{\dot{v}_C^2}{2} \Big|_0^{T_F} = 0. \quad (\text{B.119})$$

The integration of f_5 also can be done by applying chain rule,

$$\int_0^{T_F} f_5 dt = (a_{12}a_{21} + a_{11}a_{22})C \int_0^{T_F} v_C dv_C = (a_{12}a_{21} + a_{11}a_{22})C \cdot \frac{v_C^2}{2} \Big|_0^{T_F}. \quad (\text{B.120})$$

By applying the boundaries conditions, the integration of f_5 can be written as

$$\int_0^{T_F} f_5 dt = (a_{12}a_{21} + a_{11}a_{22})C \cdot \frac{V_F^2 - V_0^2}{2}. \quad (\text{B.121})$$

Lastly for f_3 , there is nothing to simplify but we can write it as

$$\int_0^{T_F} f_3 dt = a_{11}a_{21} \int_0^{T_F} v_C \cdot v_C dt. \quad (\text{B.122})$$

By combining (B.114), (B.116), (B.118), (B.119), (B.121) and (B.122), the input energy can be written as

$$\begin{aligned} E_I = & (a_{12}a_{21} + a_{11}a_{22})C \cdot \frac{V_F^2 - V_0^2}{2} + \frac{a_{11}a_{21}}{\omega^4} (\ddot{v}_C(0)V_0 - \ddot{v}_C(T_F)V_F) \\ & + \int_0^{T_F} \left[\frac{a_{11}a_{21}}{\omega^4} \cdot \ddot{v}_C + \left(\frac{2a_{11}a_{21}}{\omega^2} - a_{12}a_{22}C^2 \right) \ddot{v}_C + a_{11}a_{21}v_C \right] v_C dt. \end{aligned} \quad (\text{B.123})$$

Notice that the input energy in (B.123) is true for any functions of v_C subjected to the boundaries conditions in (5.5) to (5.8). It is not the lowest input energy since we have not applied any special functions yet.

Nevertheless, the function inside the integral can be factorized by $a_{11}a_{21}/\omega^4$ and it is

$$\frac{a_{11}a_{21}}{\omega^4} \int_0^{T_F} \left[\ddot{v}_C + \left(2 - \frac{a_{12}a_{22}}{a_{11}a_{21}} \omega^2 C^2 \right) \omega^2 \ddot{v}_C + \omega^4 v_C \right] v_C dt. \quad (\text{B.124})$$

Surprisingly, the term in the integral is exactly the same as the left hand side of the Euler-Lagrange's equation in (B.67) and it is equal to zero for all t if v_C is the most energy-efficient function. Therefore, the input energy in (B.123) is the lowest input energy if the integral term is zero,

$$E_I = (a_{12}a_{21} + a_{11}a_{22})C \cdot \frac{V_F^2 - V_0^2}{2} + \frac{a_{11}a_{21}}{\omega^4} (\ddot{v}_C(0)V_0 - \ddot{v}_C(T_F)V_F). \quad (\text{B.125})$$

Furthermore, by the reciprocal property of transmission parameters and the definition of Z in (5.16), the lowest input energy can be rearranged as

$$E_I = \frac{C(V_F^2 - V_0^2)}{2} \cdot \left(1 + 2a_{12}a_{21} + 2a_{11}a_{21}Z \cdot \frac{\ddot{v}_C(0)V_0 - \ddot{v}_C(T_F)V_F}{\omega^3(V_F^2 - V_0^2)} \right). \quad (\text{B.126})$$

All transmission parameters can be substituted by the parameters k and r . By the definition of k in (5.32), we can see that

$$k = \sqrt{\frac{R_{OC}}{R_{SC}}} = \sqrt{\frac{a_{11}a_{22}}{a_{12}a_{21}}} = \sqrt{\frac{1 + a_{12}a_{21}}{a_{12}a_{21}}} \rightarrow a_{12}a_{21} = \frac{1}{k^2 - 1}. \quad (\text{B.127})$$

Note that k is always greater than 1. For the term $a_{11}a_{21}$, we can write it as

$$a_{11}a_{21} = \frac{a_{11}}{a_{12}} \cdot a_{12}a_{21} = \frac{1}{R_{SC}(k^2 - 1)}. \quad (\text{B.128})$$

The lowest input energy becomes

$$E_I = \frac{C(V_F^2 - V_0^2)}{2} \cdot \left(1 + \frac{2}{k^2 - 1} + \frac{2Z/R_{SC}}{k^2 - 1} \cdot \frac{\ddot{v}_C(0)V_0 - \ddot{v}_C(T_F)V_F}{\omega^3(V_F^2 - V_0^2)} \right). \quad (\text{B.129})$$

By the definition of r in (5.17), it is easily to see that $k/r = 2Z/R_{SC}$. Finally, the lowest input energy can be written as

$$E_I = \frac{C(V_F^2 - V_0^2)}{2(k^2 - 1)} \cdot \left(k^2 + 1 + \frac{k}{r} \cdot \frac{\ddot{v}_C(0)V_0 - \ddot{v}_C(T_F)V_F}{\omega^3(V_F^2 - V_0^2)} \right). \quad (\text{B.130})$$

B.13 Normalized Lowest Input Energy of Series LC Circuits in Hyperbolic Mode

From (5.31) and (B.86), we can see that to determine the lowest input energy, the third derivative of v_C at the boundaries must be firstly determined.

$$\begin{aligned} \ddot{v}_C(t=0) &= \ddot{v}_C(x=-T_F/2) = \omega^3(K_1m_+^2 - K_2m_+^2 - K_1m_-^2 + K_2m_-^2) \\ &= \omega^3(m_+^2 - m_-^2)(K_1 - K_2) = 4r\sqrt{r^2 - 1}(K_1 - K_2)\omega^3 \end{aligned} \quad (\text{B.131})$$

and

$$\begin{aligned} \ddot{v}_C(t=T_F) &= \ddot{v}_C(x=T_F/2) = \omega^3(K_1m_+^2 + K_2m_+^2 - K_1m_-^2 - K_2m_-^2) \\ &= \omega^3(m_+^2 - m_-^2)(K_1 + K_2) = 4r\sqrt{r^2 - 1}(K_1 + K_2)\omega^3. \end{aligned} \quad (\text{B.132})$$

Therefore,

$$\begin{aligned} \frac{\ddot{v}_C(0)V_0 - \ddot{v}_C(T_F)V_F}{\omega^3(V_F^2 - V_0^2)} &= -4r\sqrt{r^2 - 1} \cdot \frac{K_1(V_F - V_0) + K_2(V_F + V_0)}{V_F^2 - V_0^2} \\ &= -2r\sqrt{r^2 - 1} \left(\frac{(V_F - V_0)/(V_F + V_0)}{m_- \tanh m_+ \omega T_F/2 - m_+ \tanh m_- \omega T_F/2} \right. \\ &\quad \left. + \frac{(V_F + V_0)/(V_F - V_0)}{m_- \coth m_+ \omega T_F/2 - m_+ \coth m_- \omega T_F/2} \right). \end{aligned} \quad (\text{B.133})$$

If $|V_F| \neq |V_0|$, a normalized lowest input energy can be written as

$$u = \frac{1}{k^2 - 1} \cdot \left[k^2 + 1 - 2k\sqrt{r^2 - 1} \left(\frac{\gamma}{m_- \tanh m_+ \omega T_F/2 - m_+ \tanh m_- \omega T_F/2} + \frac{1/\gamma}{m_- \coth m_+ \omega T_F/2 - m_+ \coth m_- \omega T_F/2} \right) \right]. \quad (\text{B.134})$$

where γ is defined in (5.34).

If $|V_F| = |V_0|$, $V_F^2 - V_0^2 = 0$. The stored energy does not change. The lowest input energy from (5.31) can be rearranged and calculated as

$$E_I = \frac{C}{2(k^2 - 1)} \cdot \frac{k}{r} \cdot \frac{\ddot{v}_C(0)V_0 - \ddot{v}_C(T_F)V_F}{\omega^3}. \quad (\text{B.135})$$

For $V_F = -V_0$, the lowest input energy is

$$E_I = \frac{C}{2(k^2 - 1)} \cdot \frac{k}{r} \cdot \left(-4r\sqrt{r^2 - 1} \cdot \frac{(-2V_0)^2}{m_- \tanh m_+ \omega T_F/2 - m_+ \tanh m_- \omega T_F/2} \right). \quad (\text{B.136})$$

The normalized lowest input energy for unchanging case is defined as the ratio between the lowest input energy and the stored energy and it can be determined as

$$u_{un} = \frac{8k\sqrt{r^2 - 1}}{(k^2 - 1)(m_+ \tanh m_- \omega T_F/2 - m_- \tanh m_+ \omega T_F/2)}. \quad (\text{B.137})$$

For $V_F = V_0$, the lowest input energy is

$$E_I = \frac{C}{2(k^2 - 1)} \cdot \frac{k}{r} \cdot \left(-4r\sqrt{r^2 - 1} \cdot \frac{(2V_0)^2}{m_- \coth m_+ \omega T_F/2 - m_+ \coth m_- \omega T_F/2} \right). \quad (\text{B.138})$$

and the normalized lowest input energy becomes

$$u_{un} = \frac{8k\sqrt{r^2 - 1}}{(k^2 - 1)(m_+ \coth m_- \omega T_F/2 - m_- \coth m_+ \omega T_F/2)}. \quad (\text{B.139})$$

B.14 Normalized Lowest Input Energy of Series LC Circuits in Hyperbolic and Linear Mode

From (5.31) and (B.101), we can see that to determine the lowest input energy, the third derivative of v_C at the boundaries must be firstly determined.

$$\begin{aligned} \ddot{v}_C(t=0) &= \ddot{v}_C(x = -T_F/2) = \\ &K_1 - K_2 + \frac{K_2(\omega T_F/2 + 3 \tanh \omega T_F/2)}{\omega T_F/2 + \tanh \omega T_F/2} - \frac{K_1(\omega T_F/2 + 3 \coth \omega T_F/2)}{\omega T_F/2 + \coth \omega T_F/2} \\ \ddot{v}_C(t=0) &= -K_1 \frac{2 \coth \omega T_F/2}{\omega T_F/2 + \coth \omega T_F/2} + K_2 \frac{2 \tanh \omega T_F/2}{\omega T_F/2 + \tanh \omega T_F/2} \end{aligned} \quad (\text{B.140})$$

and

$$\begin{aligned} \ddot{v}_C(t=T_F) &= \ddot{v}_C(x = T_F/2) = \\ &K_1 + K_2 - \frac{K_2(\omega T_F/2 + 3 \tanh \omega T_F/2)}{\omega T_F/2 + \tanh \omega T_F/2} - \frac{K_1(\omega T_F/2 + 3 \coth \omega T_F/2)}{\omega T_F/2 + \coth \omega T_F/2} \\ \ddot{v}_C(t=T_F) &= -K_1 \frac{2 \coth \omega T_F/2}{\omega T_F/2 + \coth \omega T_F/2} - K_2 \frac{2 \tanh \omega T_F/2}{\omega T_F/2 + \tanh \omega T_F/2} \end{aligned} \quad (\text{B.141})$$

Therefore,

$$\begin{aligned} \frac{\ddot{v}_C(0)V_0 - \ddot{v}_C(T_F)V_F}{\omega^3} &= \frac{2K_1(V_F - V_0) \coth \omega T_F/2}{\omega T_F/2 + \coth \omega T_F/2} + \frac{2K_2(V_F + V_0) \tanh \omega T_F/2}{\omega T_F/2 + \tanh \omega T_F/2} \\ &= \frac{(V_F - V_0)^2 \coth \omega T_F/2}{1 - \omega T_F/2(\coth \omega T_F/2 - \tanh \omega T_F/2)} \\ &\quad + \frac{(V_F + V_0)^2 \tanh \omega T_F/2}{1 + \omega T_F/2(\coth \omega T_F/2 - \tanh \omega T_F/2)}. \end{aligned} \quad (\text{B.142})$$

By applying hyperbolic identities, the term can be reduced to

$$\begin{aligned} \frac{\ddot{v}_C(0)V_0 - \ddot{v}_C(T_F)V_F}{\omega^3} &= \\ &\frac{(V_F - V_0)^2(\cosh \omega T_F + 1)}{\sinh \omega T_F - \omega T_F} + \frac{(V_F + V_0)^2(\cosh \omega T_F - 1)}{\sinh \omega T_F + \omega T_F}. \end{aligned} \quad (\text{B.143})$$

If $|V_F| \neq |V_0|$, a normalized lowest input energy can be written as

$$u = \frac{1}{k^2 - 1} \cdot \left[k^2 + 1 + k \left(\frac{\gamma(\cosh \omega T_F + 1)}{\sinh \omega T_F - \omega T_F} + \frac{1/\gamma(\cosh \omega T_F - 1)}{\sinh \omega T_F + \omega T_F} \right) \right]. \quad (\text{B.144})$$

where γ is defined in (5.34).

If $|V_F| = |V_0|$, $V_F^2 - V_0^2 = 0$. The stored energy does not change. The lowest input energy from (5.31) can be rearranged and calculated as

$$E_I = \frac{C}{2(k^2 - 1)} \cdot k \cdot \frac{\ddot{v}_C(0)V_0 - \ddot{v}_C(T_F)V_F}{\omega^3}. \quad (\text{B.145})$$

For $V_F = -V_0$, the lowest input energy is

$$E_I = \frac{C}{2(k^2 - 1)} \cdot k \cdot \frac{(-2V_0)^2(\cosh \omega T_F + 1)}{\sinh \omega T_F - \omega T_F}. \quad (\text{B.146})$$

The normalized lowest input energy for unchanging case is defined as the ratio between the lowest input energy and the stored energy and it can be determined as

$$u_{un} = \frac{4k(\cosh \omega T_F + 1)}{(k^2 - 1)(\sinh \omega T_F - \omega T_F)}. \quad (\text{B.147})$$

For $V_F = V_0$, the lowest input energy is

$$E_I = \frac{C}{2(k^2 - 1)} \cdot k \cdot \frac{(2V_0)^2(\cosh \omega T_F - 1)}{\sinh \omega T_F + \omega T_F}. \quad (\text{B.148})$$

and the normalized lowest input energy becomes

$$u_{un} = \frac{4k(\cosh \omega T_F - 1)}{(k^2 - 1)(\sinh \omega T_F + \omega T_F)}. \quad (\text{B.149})$$

B.15 Solving for Optimum Transient Duration of Series LC Circuit

Optimum transient duration can be determined by setting $du/dT_F = 0$. For $r > 1$ and $0 < r < 1$, by differentiating u and setting it to be zero, we get

$$\begin{aligned} & -\frac{1}{\gamma} \frac{1 - \coth^2 m_+ \omega T_{F(\text{opt.})}/2 - 1 + \coth^2 m_- \omega T_{F(\text{opt.})}/2}{(m_- \coth m_+ \omega T_{F(\text{opt.})}/2 - m_+ \coth m_- \omega T_{F(\text{opt.})}/2)^2} \\ & - \gamma \frac{1 - \tanh^2 m_+ \omega T_{F(\text{opt.})}/2 - 1 + \tanh^2 m_- \omega T_{F(\text{opt.})}/2}{(m_- \tanh m_+ \omega T_{F(\text{opt.})}/2 - m_+ \tanh m_- \omega T_{F(\text{opt.})}/2)^2} = 0. \end{aligned} \quad (\text{B.150})$$

After tedious rearrangement,

$$\begin{aligned} & -\frac{1}{\gamma} \frac{\tanh^2 m_+ \omega T_{F(\text{opt.})}/2 - \tanh^2 m_- \omega T_{F(\text{opt.})}/2}{(m_- \tanh m_+ \omega T_{F(\text{opt.})}/2 - m_+ \tanh m_- \omega T_{F(\text{opt.})}/2)^2} \\ & + \gamma \frac{\tanh^2 m_+ \omega T_{F(\text{opt.})}/2 - \tanh^2 m_- \omega T_{F(\text{opt.})}/2}{(m_- \tanh m_+ \omega T_{F(\text{opt.})}/2 - m_+ \tanh m_- \omega T_{F(\text{opt.})}/2)^2} = 0. \end{aligned} \quad (\text{B.151})$$

The equation can be satisfied if

$$\tanh^2 m_+ \omega T_{F(\text{opt.})}/2 - \tanh^2 m_- \omega T_{F(\text{opt.})}/2 = 0. \quad (\text{B.152})$$

For $r > 1$, tan-hyperbolic function is a real number and obviously there is no $T_{F(\text{opt.})}$ that can satisfied this condition. However, for $0 < r < 1$, tan-hyperbolic function is a complex number and trigonometric function involves. The condition can satisfied with multiple values of $T_{F(\text{opt.})}$.

Although the condition in (B.152) cannot be satisfied, (B.151) can also be satisfied by another condition as

$$\left(\frac{m_+ \tanh m_- \omega T_{F(\text{opt.})}/2 - m_- \tanh m_+ \omega T_{F(\text{opt.})}/2}{m_+ \tanh m_+ \omega T_{F(\text{opt.})}/2 - m_- \tanh m_- \omega T_{F(\text{opt.})}/2} \right)^2 - \gamma^2 = 0. \quad (\text{B.153})$$

This condition seems to be always has a solution. However, for $r > 1$, it will has a solution only if $-1 < \gamma < 1$. To clarify, for any real numbers x , we know that

$$m_+ \tanh m_+ x > m_+ \tanh m_- x \quad (\text{B.154})$$

and

$$m_- \tanh m_- x < m_- \tanh m_+ x. \quad (\text{B.155})$$

Therefore,

$$m_+ \tanh m_+ x - m_- \tanh m_- x > m_+ \tanh m_- x - m_- \tanh m_+ x. \quad (\text{B.156})$$

Since $m_+ \tanh m_+ x - m_- \tanh m_- x > 0$ (because $m_+ > m_-$), the inequality can be rearrange to

$$\frac{m_+ \tanh m_- x - m_- \tanh m_+ x}{m_+ \tanh m_+ x - m_- \tanh m_- x} < 1 \quad (\text{B.157})$$

and it leads to

$$0 < \left(\frac{m_+ \tanh m_- \omega T_{F(\text{opt.})}/2 - m_- \tanh m_+ \omega T_{F(\text{opt.})}/2}{m_+ \tanh m_+ \omega T_{F(\text{opt.})}/2 - m_- \tanh m_- \omega T_{F(\text{opt.})}/2} \right)^2 < 1 \quad (\text{B.158})$$

According to this inequality, the condition in (B.153) can be satisfied if and only if $0 < \gamma^2 < 1$ or $-1 < \gamma < 1$.

For $r = 1$, by differentiating u and setting it to be zero, we get

$$\begin{aligned} & \gamma \frac{\sinh^2 \omega T_{F(\text{opt.})} - \omega T_{F(\text{opt.})} \sinh \omega T_{F(\text{opt.})} - \cosh^2 \omega T_{F(\text{opt.})} + 1}{(\sinh \omega T_{F(\text{opt.})} - \omega T_{F(\text{opt.})})^2} \\ & + \frac{1}{\gamma} \frac{\sinh^2 \omega T_{F(\text{opt.})} + \omega T_{F(\text{opt.})} \sinh \omega T_{F(\text{opt.})} - \cosh^2 \omega T_{F(\text{opt.})} + 1}{(\sinh \omega T_{F(\text{opt.})} + \omega T_{F(\text{opt.})})^2} = 0. \end{aligned} \quad (\text{B.159})$$

After tedious arrangement,

$$\left(\frac{\sinh \omega T_{F(\text{opt.})} - \omega T_{F(\text{opt.})}}{\sinh \omega T_{F(\text{opt.})} + \omega T_{F(\text{opt.})}} \right)^2 - \gamma^2 = 0. \quad (\text{B.160})$$

It is easily to see that this equation has a solution if and only if $-1 < \gamma < 1$ because $\sinh \omega T_{F(\text{opt.})} - \omega T_{F(\text{opt.})} < \sinh \omega T_{F(\text{opt.})} + \omega T_{F(\text{opt.})}$.

B.16 The Most Energy-efficient Input Voltage Function of Series LC Circuit in Resonant Mode

For $r = 0$, the most energy-efficient capacitor voltage function is a trigonometric and a linear mode as

$$v_C(t) = K_1 \sin \omega t + K_2 \cos \omega t + K_3 t \sin \omega t + K_4 t \cos \omega t. \quad (\text{B.161})$$

Furthermore, $r = 0$ implies $R_{SC} = 0$ from the definition of r in (5.17). R_{OC} cannot be zero because by circuit theory, R_{OC} must be greater than R_{SC} . From the definition of R_{SC} in (5.2b), $a_{12} = 0$. Therefore, the input voltage function is

$$v_I(t) = a_{11}v_O(t) = a_{11} \left(\frac{1}{\omega^2} \ddot{v}_C(t) + v_C(t) \right). \quad (\text{B.162})$$

The second derivative of v_C can be calculated as

$$\ddot{v}_C = \omega^2 \left(-K_1 \sin \omega t - K_2 \cos \omega t + K_3 \left(-t \sin \omega t + \frac{2}{\omega} \cos \omega t \right) + K_4 \left(-t \cos \omega t - \frac{2}{\omega} \sin \omega t \right) \right). \quad (\text{B.163})$$

Therefore, the input function is in form of

$$v_I(t) = \frac{2a_{11}K_3}{\omega} \cos \omega t - \frac{2a_{11}K_4}{\omega} \sin \omega t \quad (\text{B.164})$$

or

$$v_I(t) = A \cos(\omega t + \phi) \quad (\text{B.165})$$

where A and ϕ are constants.

B.17 Normalized Lowest Input Energy of Series LC Circuits in Resonant Mode

From (5.33), the normalized lowest input energy in resonant mode can be calculated by substituting $r = 0$ and $k = \infty$. However, it will be an intermediate form. Therefore, we have to rearrange and calculate it carefully. Although $r = 0$ and $k = \infty$, its product can be determined as

$$rk = \sqrt{\frac{R_{OC}}{R_{SC}}} \cdot \frac{\sqrt{R_{OC}R_{SC}}}{2Z} = \frac{R_{OC}}{2Z}. \quad (\text{B.166})$$

Therefore, k can be expressed in terms of r .

Let us consider one term in (5.33) that is intermediate form when $r = 0$ and $k = \infty$,

$$\frac{2k\sqrt{r^2 - 1}}{k^2 - 1} \cdot \frac{\gamma}{m_- \tanh m_+ \omega T_F/2 - m_+ \tanh m_- \omega T_F/2}. \quad (\text{B.167})$$

By substituting k in terms of r , we get

$$\frac{2\gamma\sqrt{r^2 - 1}}{R_{OC}/2Z(1 - 4Z^2r^2/R_{OC}^2)} \cdot \frac{r}{m_- \tanh m_+ \omega T_F/2 - m_+ \tanh m_- \omega T_F/2}. \quad (\text{B.168})$$

Because of $m_+ = r + \sqrt{r^2 - 1}$ and $m_- = r - \sqrt{r^2 - 1}$, by substituting $r = 0$, the term becomes

$$\frac{4Z\gamma j}{R_{OC}} \cdot \frac{0}{0}. \quad (\text{B.169})$$

The latter fraction is intermediate form. Therefore, we apply L'Hospital's rule to the fraction,

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{r}{m_- \tanh m_+ \omega T_F/2 - m_+ \tanh m_- \omega T_F/2} &= \\ \lim_{r \rightarrow 0} \frac{1}{\frac{\omega T_F/2}{\sqrt{r^2 - 1} \cosh^2 m_+ \omega T_F/2} - \frac{m_- \tanh m_+ \omega T_F/2}{\sqrt{r^2 - 1}} + \frac{\omega T_F/2}{\sqrt{r^2 - 1} \cosh^2 m_- \omega T_F/2} - \frac{m_+ \tanh m_- \omega T_F/2}{\sqrt{r^2 - 1}}} &= \\ = \frac{j}{\frac{\omega T_F}{\cos^2 \omega T_F/2} - 2 \tan \omega T_F/2} = \frac{j \cos^2 \omega T_F/2}{\omega T_F - \sin \omega T_F} = \frac{j(1 + \cos \omega T_F)/2}{\omega T_F - \sin \omega T_F}. \end{aligned} \quad (\text{B.170})$$

Therefore, the whole term becomes

$$\frac{4Z\gamma j}{R_{OC}} \cdot \frac{j(1 + \cos \omega T_F)/2}{\omega T_F - \sin \omega T_F} = \frac{-2Z\gamma(1 + \cos \omega T_F)}{R_{OC}(\omega T_F - \sin \omega T_F)}. \quad (\text{B.171})$$

By using similar procedure with other intermediate terms, we can calculate the normalized lowest input energy from (5.33) as

$$u = 1 + \frac{2Z}{R_{OC}} \left(\frac{\gamma(1 + \cos \omega T_F)}{\omega T_F - \sin \omega T_F} + \frac{1 - \cos \omega T_F}{\gamma(\omega T_F + \sin \omega T_F)} \right). \quad (\text{B.172})$$

According to (B.172), the lowest input energy can be written as

$$E_I = \frac{1}{2}C(V_F - V_0)^2 + \frac{C}{2kr} \left(\frac{(V_F - V_0)^2(1 + \cos \omega T_F)}{\omega T_F - \sin \omega T_F} + \frac{(V_F + V_0)^2(1 - \cos \omega T_F)}{\omega T_F + \sin \omega T_F} \right) \quad (\text{B.173})$$

If $V_F = -V_0$ or $\gamma = \infty$, the normalized lowest input energy for unchanging total stored energy is

$$u_{un} = \frac{4(1 + \cos \omega T_F)}{kr(\omega T_F - \sin \omega T_F)} \quad (\text{B.174})$$

and if $V_F = V_0$ or $\gamma = 0$, the normalized lowest input energy is

$$u_{un} = \frac{4(1 - \cos \omega T_F)}{kr(\omega T_F + \sin \omega T_F)}. \quad (\text{B.175})$$

B.18 Normalized Lowest Input Energy of Series LC Circuits in Linear Mode

From (5.33), the normalized lowest input energy in resonant mode can be calculated by substituting $r = \infty$ and $k = \infty$. However, it will be an intermediate form. Therefore, we have to rearrange and calculate it carefully. Although $r = \infty$ and $k = \infty$, their ratio can be determined as

$$\frac{k}{r} = \sqrt{\frac{R_{OC}}{R_{SC}}} \cdot \frac{2Z}{\sqrt{R_{OC}R_{SC}}} = \frac{2Z}{R_{SC}} \quad (\text{B.176})$$

Therefore, k can be expressed in terms of r .

Let us consider one term in (5.33) that is intermediate form when $r = \infty$ and $k = \infty$,

$$\frac{2k\sqrt{r^2 - 1}}{k^2 - 1} = \frac{2\sqrt{r^2 - 1}}{1/k - k} = \frac{2\sqrt{r^2 - 1}}{\frac{R_{SC}}{2Zr} - \frac{2Zr}{R_{SC}}} = \frac{2r\sqrt{r^2 - 1}}{\frac{R_{SC}}{2Z} - \frac{2Zr^2}{R_{SC}}}. \quad (\text{B.177})$$

By substituting $r = \infty$, this term becomes

$$- \frac{R_{SC}}{Z} \quad (\text{B.178})$$

Another term that is problematic is

$$\frac{\gamma}{m_- \tanh m_+ \omega T_F/2 - m_+ \tanh m_- \omega T_F/2} + \frac{1/\gamma}{m_- \coth m_+ \omega T_F/2 - m_+ \coth m_- \omega T_F/2} \quad (\text{B.179})$$

Since $r = \infty$, $m_+ = \infty$ and $m_- = 0$. If we substitute directly, we will get

$$\frac{\gamma}{0 \cdot 1 - \infty \cdot 0} + \frac{1/\gamma}{0 \cdot 1 - \infty \cdot \infty}. \quad (\text{B.180})$$

The denominator of $1/\gamma$ term is infinity and this term is zero. However, the denominator of γ term is $-\infty \cdot 0$ which is problematic. Therefore, we consider this term separately. Let us apply L'Hospital's rule.

$$\begin{aligned} \lim_{r \rightarrow \infty} m_+ \tanh m_- \omega T_F / 2 &= \lim_{r \rightarrow \infty} \frac{\tanh m_- \omega T_F / 2}{1/m_+} \\ &= \lim_{r \rightarrow \infty} \frac{-m_- / \sqrt{r^2 - 1} \cdot (\omega T_F / 2) / \cosh^2 m_- \omega T_F / 2}{-1/m_+^2 \cdot m_+ / \sqrt{r^2 - 1}} = m_+ m_- \omega T_F / 2 = \omega T_F / 2. \end{aligned} \quad (\text{B.181})$$

Therefore, the normalized lowest input energy is

$$u = 1 + \frac{2R_{SC}\gamma}{Z\omega T_F} = 1 + \frac{2R_{SC}C\gamma}{T_F}. \quad (\text{B.182})$$

B.19 Obtaining The Equations of RC Chain Circuit by Mathematical Proving

Beyond mathematical induction, the voltage and the current can also be found by mathematical proving [90]. However, it requires more mathematical corollaries. Therefore, some useful corollaries will be given here before deriving the voltage and the current equation.

Firstly, let us carefully define the binomial coefficient. Normally, the binomial coefficient is defined as

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \quad \text{for } k \leq n \quad (\text{B.183})$$

which k and n must be the non-negative integers. Moreover, the binomial coefficient can be calculated by a recursive formula,

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1} \quad (\text{B.184})$$

which is the formula for computing the Pascal's triangle. One interesting property of the binomial coefficient is that when $k = 0$ the binomial coefficient is equal to

$$\binom{n}{0} = \frac{n!}{(n-0)!0!} = 1. \quad (\text{B.185})$$

Another property is

$$\binom{n}{n} = \frac{n!}{(n-n)!n!} = 1. \quad (\text{B.186})$$

From these properties, number one can be rewritten as the binomial coefficient with an arbitrary value of the non-negative integer n .

Nevertheless, the definition of the binomial coefficient can be extended further by using the gamma function which is defined as [93]

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx. \quad (\text{B.187})$$

If n is a positive integer, the integration will be equal to $(n - 1)!$. Therefore, the gamma function when n is a positive integer is equal to

$$\Gamma(n) = (n - 1)! \quad (\text{B.188})$$

While the rest values of n , the gamma function can be evaluated by calculating the improper integral numerically.

By substituting (B.188) into (B.183), the binomial coefficient can be defined by the gamma function as

$$\binom{n}{k} = \frac{\Gamma(n + 1)}{\Gamma(n - k + 1)\Gamma(k + 1)} \quad (\text{B.189})$$

Since the gamma function is defined over the complex numbers except the non-positive integers, now the binomial coefficient is not limited to the non-negative integers. But it is extended to those gamma domain as well. Graphical illustration of the gamma function is shown in Fig. B.3.

Figure B.3: Graphical illustration of the gamma function.

Although the gamma function is undefined at the non-positive integers, the values approach positive infinity or negative infinity, cf. Fig. B.3. Therefore, if n and k are the non-negative integers and $k > n$, the argument of $\Gamma(n - k + 1)$ in (B.189) will be the non-positive integer while the argument of $\Gamma(n + 1)$ and $\Gamma(k + 1)$ will be the positive integers. Consequently,

$$\binom{n}{k} = 0 \quad \text{for } k > n. \quad (\text{B.190})$$

This property is essential for proving the voltage and the current expressions.

Another corollary that will be used for deriving the voltage and the current expressions is

$$\sum_{r=0}^n \binom{r}{k} = \binom{n+1}{k+1}$$

for k and n are the non-negative integers [94, 95]. From (B.190), if $r < k$ then the binomial coefficient will be zero and the summation can be rewritten as

$$\sum_{r=0}^n \binom{r}{k} = \sum_{r=k}^n \binom{r}{k} = \binom{n+1}{k+1}. \quad (\text{B.191})$$

Furthermore, let us consider

$$\sum_{r=0}^n \binom{r+m}{k}.$$

where m is the non-negative integer and $m \leq k$. The result of this summation can be found by changing the index of summation and applying the corollary in (B.191).

$$\sum_{r=0}^n \binom{r+m}{k} = \sum_{j=m}^{n+m} \binom{j}{k} = \sum_{j=k}^{n+m} \binom{j}{k} = \binom{n+m+1}{k+1} \quad \text{for } m \leq k. \quad (\text{B.192})$$

These corollaries will often be applied in the prove.

Now, we are ready to proof the voltage and current function expressions. From Fig. 6.9, although the voltage can be expressed as in (6.73), it can also be expressed as

$$v_n = v_1 + R \sum_{k_1=1}^{n-1} i_{k_1} \quad \text{for } n \in [1, N+1].$$

If $n = 1$, the upper limit of the summation will be zero and it causes empty sum which is equal to zero by definition. Therefore $v_n = v_1$ for $n = 1$ as expected. By substituting the current from (6.72), the voltage is read

$$v_n = v_1 + \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \tau v_{k_2}^{(1)}. \quad (\text{B.193})$$

By differentiating (B.193), $v_{k_2}^{(1)}$ can be evaluated as

$$v_{k_2}^{(1)} = v_1^{(1)} + \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} \tau v_{k_4}^{(2)}.$$

By substituting back into (B.193) and considering only the double summation term, it yields

$$\begin{aligned} \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \tau v_{k_2}^{(1)} &= \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \tau \left(v_1^{(1)} + \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} \tau v_{k_4}^{(2)} \right) \\ &= \tau v_1^{(1)} \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} 1 + \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} \tau^2 v_{k_4}^{(2)} \end{aligned} \quad (\text{B.194})$$

The double summation of $\tau^2 v_{k_4}^{(2)}$ in the last term of (B.194) can be calculated with the same procedure above and (B.194) itself can imply that

$$\sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} \tau^2 v_{k_4}^{(2)} = \tau^2 v_1^{(2)} \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} 1 + \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} \sum_{k_5=1}^{k_4-1} \sum_{k_6=1}^{k_5} \tau^3 v_{k_6}^{(3)}.$$

By substituting back into (B.194), it yields

$$\begin{aligned} \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \tau v_{k_2}^{(1)} &= \tau v_1^{(1)} \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} 1 + \tau^2 v_1^{(2)} \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} 1 \\ &\quad + \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} \sum_{k_5=1}^{k_4-1} \sum_{k_6=1}^{k_5} \tau^3 v_{k_6}^{(3)}. \end{aligned}$$

The recursive procedure can be done repeatedly. A few terms are given as follows:

$$\begin{aligned} \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \tau v_{k_2}^{(1)} &= \tau v_1^{(1)} \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} 1 \\ &\quad + \tau^2 v_1^{(2)} \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} 1 \\ &\quad + \tau^3 v_1^{(3)} \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} \sum_{k_5=1}^{k_4-1} \sum_{k_6=1}^{k_5} 1 \\ &\quad + \tau^4 v_1^{(4)} \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} \sum_{k_5=1}^{k_4-1} \sum_{k_6=1}^{k_5} \sum_{k_7=1}^{k_6-1} \sum_{k_8=1}^{k_7} 1 + \dots \end{aligned}$$

or

$$\sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \tau v_{k_2}^{(1)} = \sum_{j=1}^{\infty} \tau^j v_1^{(j)} \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} \dots \sum_{k_{2j-1}=1}^{k_{2j-2}-1} \sum_{k_{2j}=1}^{k_{2j-1}} 1. \quad (\text{B.195})$$

Notice that for the term $\tau^j v_1^{(j)}$ where j is the positive integer, there are $2j$ summation of 1. The term can be written as

$$\underbrace{\sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} \dots \sum_{k_{2j-1}=1}^{k_{2j-2}-1} \sum_{k_{2j}=1}^{k_{2j-1}} 1}_{2j}. \quad (\text{B.196})$$

Let us consider the two deepest summation first:

$$\sum_{k_{2j-1}=1}^{k_{2j-2}-1} \sum_{k_{2j}=1}^{k_{2j-1}} 1 = \sum_{k_{2j-1}=1}^{k_{2j-2}-1} k_{2j-1} = \sum_{k_{2j-1}=1}^{k_{2j-2}-1} \binom{k_{2j-1}}{1}$$

where we have rewritten k_{2j-1} in term of the binomial coefficient. Because by applying the corollary in (B.191), the result of the summation can be found as

$$\sum_{k_{2j-1}=1}^{k_{2j-2}-1} \sum_{k_{2j}=1}^{k_{2j-1}} 1 = \sum_{k_{2j-1}=1}^{k_{2j-2}-1} \binom{k_{2j-1}}{1} = \binom{k_{2j-2}}{2}.$$

By substituting back into (B.196), the term is read

$$\underbrace{\sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} \dots \sum_{k_{2j-3}=1}^{k_{2j-4}-1} \sum_{k_{2j-2}=1}^{k_{2j-3}} \binom{k_{2j-2}}{2}}_{2j-2}. \quad (\text{B.197})$$

Let us consider another two summation:

$$\sum_{k_{2j-3}=1}^{k_{2j-4}-1} \sum_{k_{2j-2}=1}^{k_{2j-3}} \binom{k_{2j-2}}{2} = \sum_{k_{2j-3}=1}^{k_{2j-4}-1} \binom{k_{2j-3}+1}{3}$$

where the corollary in (B.191) has been applied already. Next, by applying the corollary in (B.192), the result of the summation can be found as

$$\sum_{k_{2j-3}=1}^{k_{2j-4}-1} \sum_{k_{2j-2}=1}^{k_{2j-3}} \binom{k_{2j-2}}{2} = \sum_{k_{2j-3}=1}^{k_{2j-4}-1} \binom{k_{2j-3}+1}{3} = \binom{k_{2j-4}+1}{4}.$$

Consequently, by substituting back into (B.197), the term is read

$$\underbrace{\sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} \cdots \sum_{k_{2j-5}=1}^{k_{2j-6}-1} \sum_{k_{2j-4}=1}^{k_{2j-5}}}_{2j-4} \binom{k_{2j-4}+1}{4}. \quad (\text{B.198})$$

Another two summation can also be calculated by applying (B.192) as

$$\sum_{k_{2j-5}=1}^{k_{2j-6}-1} \sum_{k_{2j-4}=1}^{k_{2j-5}} \binom{k_{2j-4}+1}{4} = \sum_{k_{2j-5}=1}^{k_{2j-6}-1} \binom{k_{2j-4}+2}{5} = \binom{k_{2j-6}+2}{6}$$

and the term can be reduced to

$$\underbrace{\sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} \cdots \sum_{k_{2j-7}=1}^{k_{2j-8}-1} \sum_{k_{2j-6}=1}^{k_{2j-7}}}_{2j-6} \binom{k_{2j-6}+2}{6}. \quad (\text{B.199})$$

The procedure can be done repeatedly until the summation vanishes. Nevertheless, by inspecting (B.196) to (B.199), the term can be expressed with the remaining summation as

$$\underbrace{\sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} \cdots \sum_{k_{2j-2m-1}=1}^{k_{2j-2m-2}-1} \sum_{k_{2j-2m}=1}^{k_{2j-2m-1}}}_{2j-2m} \binom{k_{2j-2m}+m-1}{2m}$$

where m is the non-negative integer. The summation will vanishes if $m = j$ and the result of the summation will be

$$\binom{k_0+j-1}{2j}.$$

By inspecting the corresponding indexes of the summation, $k_0 = n$. Finally, the result of the summation is

$$\underbrace{\sum_{k_1=1}^{n-1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2-1} \sum_{k_4=1}^{k_3} \cdots \sum_{k_{2j-1}=1}^{k_{2j-2}-1} \sum_{k_{2j}=1}^{k_{2j-1}}}_{2j} 1 = \binom{n+j-1}{2j}. \quad (\text{B.200})$$

The voltage expression can be obtained by substituting (B.200) into (B.195) and (B.193),

$$v_n = v_1 + \sum_{j=1}^{\infty} \tau^j v_1^{(j)} \binom{n+j-1}{2j}.$$

From the corollary in (B.190), the term in the summation will be zero for $n + j - 1 < 2j$ or $j > n - 1$. Therefore, the upper index of summation can be replaced with $n - 1$. Furthermore, from the binomial property in (B.185), the coefficient of v_1 which is 1 can be expressed in the binomial coefficient term as

$$v_n = \binom{n-1}{0} \tau^0 v_1^{(0)} + \sum_{j=1}^{n-1} \tau^j v_1^{(j)} \binom{n+j-1}{2j}.$$

With this replacement, the first term can be collected into the summation as described in (6.77). Hence, the expression of the voltage has been proved.

Once the voltage equation has been obtained, the current equation can be determined more easily. From Fig. 6.9, the current in the network n can be written as

$$i_n = \frac{v_{n+1} - v_n}{R}$$

By applying the voltage equation from (6.73), it yields

$$i_n = \frac{1}{R} \left(\sum_{j=0}^n \binom{n+j}{2j} \tau^j v_1^{(j)} - \sum_{j=0}^{n-1} \binom{n-1+j}{2j} \tau^j v_1^{(j)} \right)$$

Follow by applying the recursive formula of the binomial coefficient in (B.184),

$$i_n = \frac{1}{R} \left(\binom{2n}{2n} \tau^n v_1^{(n)} + \sum_{j=0}^{n-1} \binom{n-1+j}{2j-1} \tau^j v_1^{(j)} \right)$$

or

$$i_n = \frac{1}{R} \left(\binom{2n-1}{2n-1} \tau^n v_1^{(n)} + \sum_{j=0}^{n-1} \binom{n-1+j}{2j-1} \tau^j v_1^{(j)} \right) = \frac{1}{R} \sum_{j=0}^n \binom{n-1+j}{2j-1} \tau^j v_1^{(j)}$$

where the property in (B.186) is applied and the first term is collected into the summation. Furthermore, when $j = 0$, the binomial coefficient is zero because the lower argument is negative. Therefore, the boundary of the summation can be changed to

$$i_n = \frac{1}{R} \sum_{j=1}^n \binom{n-1+j}{2j-1} \tau^j v_1^{(j)}.$$

Hence, the current expression in (6.78) has been proved.