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Sampling great circles at their rate of innovation

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ABSTRACT

In this work, we show that great circles, the intersection of a plane through the origin and a sphere centered at the origin, can be perfectly recovered at their rate of innovation. Specifically, we show that $4K(8K - 7) + 7$ samples are sufficient to perfectly recover K great circles, given an appropriate sampling scheme. Moreover, we argue that the number of samples can be reduced to $2K(4K - 1)$ while maintaining accurate results. This argument is supported by our numerical results. To improve the robustness to noise of our approach, we propose a modification that uses all the available information, instead of the critical amount. The increase in accuracy is demonstrated using numerical simulations.

Keywords: finite rate of innovation, great circles, spherical harmonics, sparse sampling, annihilating filter.

1. INTRODUCTION

Signals that are defined on the sphere appear in a wide range of applications, including acoustics,¹ astrophysics,² and medical imaging,³ to name only a few. Sampling theorems on the sphere dictate which of these can be recovered and under which conditions. The most common assumption is that signals on the sphere are bandlimited, meaning they can be described using a finite number of spectral coefficients. If this assumption holds, the spectral coefficients can be computed using a finite number of samples.^{4,5} Recently, McEwen et al.⁶ developed a sampling theorem that reduces the number of samples required by approximately half. For non-bandlimited signals on the sphere, there are very few guidelines to follow. The typical approach is to acquire as many samples as possible and reconstruct a lowpass approximation of the signal. This is obviously suboptimal.

In Cartesian space, Vetterli et al.⁷ showed that non-bandlimited parametric signals can be recovered by sampling at their rate of innovation, which corresponds to the number of degrees of freedom of the signal. This new approach was extended to a variety of signal classes and sampling schemes, see for example Refs. 8–10. In Ref. 11, we showed that signals defined on the two dimensional sphere can also be sampled efficiently, using methods similar to those proposed by Vetterli et al. Our work concentrated on orientations, essentially Diracs on the sphere, and Diracs integrated along the azimuth. In an effort to broaden the range of non-bandlimited signals that can be sampled and reconstructed, we extend our results to great circles here. In addition, we propose a new reconstruction algorithm for noisy signals with a finite rate of innovation on the sphere. This new algorithm uses all available spherical harmonics coefficients, instead of only a subset.

The remainder of the paper is organized as follows. In Section 2, we review the essential concept of Fourier analysis on the sphere and present certain results of our earlier work¹¹ which we will use later. In Section 3 we state sufficient conditions to recover great circles and described an algorithm to do so. In Section 4, we propose an alternative reconstruction algorithm which improves the robustness to noise. Section 5 describes and illustrates numerical simulations which validate our results and finally, Section 6 concludes the paper.

2. BACKGROUND

Before moving further, we clarify the notation used in this paper. We deal with functions defined on the 2 dimensional sphere in 3D Euclidean space, which we denote by $\mathbb{S}^2$. A point on the sphere can be identified by two angles, in this case the colatitude $\theta \in [0, \pi[$ and the azimuth $\varphi \in [0, 2\pi[$, or with a unit norm vector $\mathbf{u} \in \mathbb{R}^3$. In the following, we will interchangeably write $f(\theta, \varphi)$ and $f(\mathbf{u})$ where we assume both functions are equal if (θ, φ) and $\mathbf{u}$ correspond to the same point in $\mathbb{S}^2$. The conversion between the two notation is given by the common spherical to Cartesian formula

$$\mathbf{u} = [\sin \theta \cos \varphi \quad \sin \theta \sin \varphi \quad \cos \theta]^t$$

where $\mathbf{u}^t$ is the transpose of $\mathbf{u}$.

Any square integrable function on the sphere can be written as a sum of spherical harmonics. Explicitly, for $f(\mathbf{u}) \in L^2(\mathbb{S}^2)$ we have

$$f(\mathbf{u}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{f}_{\ell}^m Y_{\ell}^m(\mathbf{u})$$

where $Y_{\ell}^m(\mathbf{u})$ and $\hat{f}_{\ell}^m$ are respectively the spherical harmonic and spherical harmonic coefficients of degree ℓ and order m . The coefficients $\hat{f}_{\ell}^m$ can be computed from

$$\hat{f}_{\ell}^m = \int_{\mathbb{S}^2} f(\mathbf{u}) \bar{Y}_{\ell}^m(\mathbf{u}) d\mathbf{u} \quad (1)$$

where the overbar denotes conjugation. Sampling theorems on the sphere commonly assume that $f(\mathbf{u})$ has a bandlimit L , that is $\hat{f}_{\ell}^m = 0$ for $\ell \geq L$. In this case, the signal can be completely described by computing spherical harmonic coefficients up to the bandlimit. This is the approach taken in Refs. 4–6 where Eq. (1) is evaluated using exact quadrature rules. In contrast, we showed in Ref. 11 that orientations, defined as

$$f(\theta, \varphi) = \sum_{k=1}^K a_k \delta(\cos \theta - \cos \theta_k) \delta(\varphi - \varphi_k), \quad (2)$$

can be perfectly recovered even though they are not bandlimited. Using the definition in Eq. (1) and the signal model in Eq. (2), we showed that the spherical harmonic coefficients of a sum of orientations are given by

$$\hat{f}_{\ell}^m = \sum_{k=1}^K \bar{Y}_{\ell}^m(\theta_k, \varphi_k). \quad (3)$$

To sample and reconstruct a sum of orientations, we proposed a sampling scheme where $f(\theta, \varphi)$ is convolved with a sampling kernel $r(\mathbf{G})$ defined on the group $\mathbb{SO}(3)$ of all 3×3 orthogonal matrices with determinant one. Because of the spherical nature of the functions, the convolution operation is defined as

$$s(\mathbf{u}) = (f * r)(\mathbf{u}) = \int_{\mathbb{SO}(3)} f(\mathbf{G}^{-1}\mathbf{u}) r(\mathbf{G}) d\mathbf{G} \quad (4)$$

as proposed by Healy et al.¹² Square integrable functions defined on the rotation group can be expanded in terms of rotational harmonics $D_{\ell}^{mn}(\mathbf{G})$,^{13,14}

$$r(\mathbf{G}) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{8\pi^2} \sum_{m=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \hat{r}_{\ell}^{mn} \bar{D}_{\ell}^{mn}(\mathbf{G})$$

with $\mathbf{G} \in \mathbb{SO}(3)$ and where $\hat{r}_{\ell}^{mn}$ are the rotational harmonic coefficients of $r(\mathbf{G})$. While our earlier work shows that several sampling kernels can be used, in the interest of brevity we will consider only the simple ideal bandlimited kernel whose rotational harmonic coefficients are given by

$$\hat{r}_{\ell}^{mn} = \begin{cases} 1 & \text{if } m = n \text{ and } \ell < L \\ 0 & \text{otherwise} \end{cases}$$

for some bandlimit $L > 0$. Using this kernel, Eq. (4) can be reduced to

$$s(\mathbf{u}) = \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} \hat{f}_{\ell}^m Y_{\ell}^m(\mathbf{u}) \quad (5)$$

clearly indicating that $s(\mathbf{u})$ and $f(\mathbf{u})$ share the same spherical harmonic coefficients for $\ell < L$. Using the sampling theorem in Ref. 6, we showed that a subset of the spherical harmonic coefficients of $f(\mathbf{u})$ can be computed from the samples of $s(\mathbf{u})$. We then showed that these coefficients are sufficient to recover the parameters of $f(\mathbf{u})$ and hence $f(\mathbf{u})$ itself.

3. SAMPLING GREAT CIRCLES

In contrast to our earlier work,¹¹ we are interested in sampling great circles. As was stated earlier, great circles are formed by the intersection of a plane through the origin and a sphere centered at the origin. They are therefore completely defined by the orientation of the plane and the radius of the sphere. A plane can be described using its normal so it seems appropriate to describe great circles with orientations. The relation between the two notations, the great circles and the orientations, is the Funk transform^{15,16} also called Funk-Radon transform because of its relation to Radon's work¹⁷ or Funk-Minkowski transform. Explicitly, our signal of interest can be written as the Funk transform of orientations

$$\mathcal{R}f(\mathbf{u}) = \int_{\mathbb{S}^2} \delta(\mathbf{u}^t \mathbf{v}) f(\mathbf{v}) d\mathbf{v}.$$

Indeed, if we let $f(\mathbf{v})$ be a sum of K orientations as defined in Eq. (2) then we have

$$\mathcal{R}f(\mathbf{u}) = \sum_{k=1}^K a_k \delta(\mathbf{u}^t \mathbf{v}_k)$$

where $\mathbf{v}_k = [\cos \phi_k \sin \theta_k \quad \sin \phi_k \sin \theta_k \quad \cos \theta_k]^t$. The function $\mathcal{R}f(\mathbf{u})$ takes the value a_k when $\mathbf{u}$ is perpendicular to $\mathbf{v}_k$ and zero everywhere else. Note that for every great circle, we can identify two orientations: $\mathbf{v}_k$ and $-\mathbf{v}_k$. Because we are interested in the great circle itself and not the underlying orientation, we limit the azimuth to the interval $[0, \pi[$ (corresponding to all the values of $\mathbf{v}_k$ with a positive x component) which removes this ambiguity.

As in our previous work,¹¹ we assume a sampling scheme where $\mathcal{R}f(\mathbf{u})$ is convolved with a bandlimited kernel $r(\mathbf{G})$ to yield the observed signal $s(\mathbf{u}) = (\mathcal{R}f * r)(\mathbf{u})$. Our objective is therefore to recover the parameters of the great circle given the samples $s_n = s(\mathbf{u}_n)$ for $n = 1, \dots, N$. Our proposed solution proceeds in two steps. First, we show that the parameters a_k and $\mathbf{v}_k$ for $k = 1, \dots, K$ can be recovered using a subset of the spherical harmonic coefficients of $\mathcal{R}f(\mathbf{u})$. Second, we show that these coefficients can be obtained from a finite number of samples s_n .

3.1 From spherical harmonic coefficients to parameters

To compute the spherical harmonics coefficients of $\mathcal{R}f(\mathbf{u})$ and relate them to the parameters, we make use of a corollary to the Funk-Hecke theorem. This corollary, which was proved by Descoteaux et al.,¹⁸ states that

$$\int_{\mathbb{S}^2} \delta(\mathbf{u}^t \mathbf{v}) Y_\ell^m(\mathbf{v}) d\mathbf{v} = 2\pi P_\ell(0) Y_\ell^m(\mathbf{u}) \quad (6)$$

where $P_\ell(0)$ is the Legendre polynomial of degree ℓ evaluated at 0,

$$P_\ell(0) = \begin{cases} 0 & \ell \text{ odd} \\ (-1)^{\ell/2} \frac{1 \cdot 3 \cdot 5 \cdots (\ell-1)}{2 \cdot 4 \cdot 6 \cdots \ell} & \ell \text{ even} \end{cases}.$$

Expanding $\mathcal{R}f(\mathbf{u})$ using spherical harmonics and using Eq. (6), we get

$$\begin{aligned} \mathcal{R}f(\mathbf{u}) &= \int_{\mathbb{S}^2} \delta(\mathbf{u}^t \mathbf{v}) \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{f}_\ell^m Y_\ell^m(\mathbf{v}) d\mathbf{v} \\ &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{f}_\ell^m \int_{\mathbb{S}^2} \delta(\mathbf{u}^t \mathbf{v}) Y_\ell^m(\mathbf{v}) d\mathbf{v} \\ &= 2\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{f}_\ell^m P_\ell(0) Y_\ell^m(\mathbf{u}). \end{aligned}$$

The spherical harmonics coefficients of $\mathcal{R}f(\mathbf{u})$, denoted by $\mathcal{R}\hat{f}_\ell^m$, are equal to those of $f(\mathbf{u})$ scaled by $2\pi P_\ell(0)$, that is

$$\mathcal{R}\hat{f}_\ell^m = 2\pi P_\ell(0) \hat{f}_\ell^m.$$

Note that $\mathcal{R}\hat{f}_\ell^m = 0$ for odd ℓ since $P_\ell(0) = 0$ for odd ℓ . As detailed in,¹¹ the spherical harmonic coefficients with equal degree and order can then be used to define

$$\begin{aligned} z_\ell &= \mathcal{R}\hat{f}_\ell^\ell / (N_\ell 2\pi P_\ell(0)) \\ &= \sum_{k=1}^K a_k \sin^\ell \theta_k \exp(-j\ell\varphi_k) \\ &= \sum_{k=1}^K a_k c_k^\ell \end{aligned} \quad (7)$$

where N_ℓ is a constant that depends on the normalization of the spherical harmonics and $c_k = \sin \theta_k \exp(-j\varphi_k)$. Note that because $\mathcal{R}\hat{f}_\ell^m = 0$ for odd ℓ , only the z_ℓ with even ℓ can be retrieved. These normalized samples z_ℓ have the sought after sum of exponential form meaning the values of c_k can be retrieved using the annihilating filter method. As described in Ref. 11, the approach must be slightly modified because only even powers can be retrieved. Without going into the details, we state that z_ℓ for $\ell = 0, 2, \dots, 4K - 2$ are sufficient to uniquely recover all the c_k . Noting the definition of c_k , we see that $\sin \theta_k$ and $-\varphi_k$ correspond to its magnitude and phase respectively. The values of a_k can be computed using the linear system of equations of Eq. (7) for $\ell = 0, 2, \dots, 2K$. The final step of the procedure is to recover the parameters θ_k , which cannot be uniquely deduced from $\sin \theta_k$. We consider the spherical harmonic coefficients whose order is one less than its degree, $m = \ell - 1$, and use them to define

$$\begin{aligned} w_\ell &= \mathcal{R}\hat{f}_\ell^{\ell-1} / (M_\ell 2\pi P_\ell(0)) \\ &= \sum_{k=1}^K a_k \cos \theta_k \sin^{\ell-1} \theta_k \exp(-j(\ell-1)\varphi_k) \\ &= \sum_{k=1}^K a_k \cos \theta_k c_k^{\ell-1} \end{aligned}$$

where M_ℓ is a constant that depends of the normalization of the spherical harmonics. For $\ell = 2, 4, \dots, 2K$, this linear system of equations uniquely defines the value of $\cos \theta_k$ and thus θ_k . In summary, to uniquely recover all the parameters of $\mathcal{R}f(\mathbf{u})$, we require $\mathcal{R}\hat{f}_\ell^\ell$ for $\ell = 0, 2, \dots, 4K - 2$ and $\mathcal{R}\hat{f}_\ell^{\ell-1}$ for $\ell = 2, 4, \dots, 2K$.

3.2 From samples to spherical harmonic coefficients

As previously discussed, we propose a sampling scheme where the signal of interest is filtered before being sampled, that is $s(\mathbf{u}) = (Rf * h)$. By comparing Eq. (5) we find that

$$s(\mathbf{u}) = \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} \mathcal{R}\hat{f}_\ell^m Y_\ell^m(\mathbf{u}). \quad (8)$$

A straightforward way to compute the coefficients $\mathcal{R}\hat{f}_\ell^m$ is to note that $s(\mathbf{u})$ is bandlimited and that it can be reconstructed using the sampling theorem of McEwen et al.⁶ using $N = (L-1)(2L-1) + 1$ samples. From Section 3.1, we know that the highest degree of the coefficient we need is $4K - 2$ meaning we need a bandlimit of at least $L = 4K - 1$. Therefore, $N = 4K(8K - 7) + 7$ samples are sufficient to recover the subset of the spherical harmonic coefficients we need to compute the parameters of the great circles. However, we argue that, in practice, an accurate reconstruction can still be achieved using fewer samples. Recall that the spherical harmonic coefficients $R\hat{f}_\ell^m$ are zero for odd ℓ , leaving only $L(L+1)/2$ coefficients to estimate. Let $\mathbf{Y}$ be the $N \times L(L+1)/2$ spherical harmonic matrix that projects the spherical harmonic coefficients $R\hat{f}_\ell^m$ on the samples s_n . The linear system of Eq. (8) can be written compactly as

$$\mathbf{s} = \mathbf{Y}\mathcal{R}\hat{\mathbf{f}} \quad (9)$$

where $\mathbf{s}$ and $\mathcal{R}\hat{\mathbf{f}}$ are column vectors that contain the samples s_n and spherical harmonic coefficients $R\hat{f}_\ell^m$, respectively. Assuming $N \geq L(L+1)/2$ and sample locations that produce a full column rank matrix $\mathbf{Y}$, the system yields a unique solution $\mathcal{R}\hat{\mathbf{f}} = \mathbf{Y}^\dagger \mathbf{s}$ where $\cdot^\dagger$ stands for the Moore-Penrose pseudoinverse. Note that there are no theoretical guaranties that, for an arbitrary number of samples N , it is possible to build a full rank matrix $\mathbf{Y}$. In practice however, well-conditioned matrices can be produced using sample locations that are uniformly distributed on the sphere. Assuming a full rank matrix $\mathbf{Y}$, the coefficients $\mathcal{R}\hat{f}_\ell^\ell$ for $\ell = 0, 2, \dots, 4K-2$ and $\mathcal{R}\hat{f}_\ell^{\ell-1}$ for $\ell = 2, 4, \dots, 2K$ can be estimated using $N = L(L+1)/2 = 2K(4K-1)$ samples. This proposition will be verified though numerical simulation in Section 5.

4. SAMPLING GREAT CIRCLES IN THE PRESENCE OF NOISE

To sample and reconstruct great circles, the algorithm described in Section 3 makes use of the specific sum of exponential form of the spherical harmonic coefficients with equal degree and order. This allows us to use the critical number of spherical harmonic coefficients, but does not makes use of all the available information. If we consider the ideal bandlimited kernel, we can compute $L(L+1)/2$ non-zero spherical harmonic coefficients of $f(\mathbf{u})$ but we only use $3K$ in the reconstruction process. To improve the robustness to noise of the algorithm, it is desirable to use all of the available information, which is what we propose here.

4.1 Using all available information

Consider the inner product of the spherical harmonic coefficients of two functions, $s(\mathbf{u})$ and $g(\mathbf{u})$, given by

$$\mathcal{R}\hat{\mathbf{f}} \cdot \hat{\mathbf{g}} = \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} \hat{g}_\ell^m \bar{\hat{f}}_\ell^m$$

where $y(\mathbf{u}) = \mathcal{R}f(\mathbf{u})$ is a sum of K great circles and $g(\mathbf{u})$ is bandlimited at L . More specifically, we define $g_\ell(\mathbf{u}) = \sin^\ell(\theta) \exp(j\ell\varphi)$, a function whose non-zero spherical harmonics coefficients are of degree ℓ . Using the property in Eq. (3) and rearranging the order of the summations, we find that

$$\begin{aligned} \hat{\mathbf{y}} \cdot \hat{\mathbf{g}}_\ell &= \sum_{m=-\ell}^{\ell} \hat{g}_\ell^m \bar{\hat{y}}_\ell^m \\ &= \sum_{m=-\ell}^{\ell} \hat{g}_\ell^m 2\pi P_\ell(0) \bar{\hat{f}}_\ell^m \\ &= 2\pi P_\ell(0) \sum_{m=-\ell}^{\ell} \hat{g}_\ell^m \sum_{k=1}^K a_k Y_\ell^m(\theta_k, \varphi_k) \\ &= 2\pi P_\ell(0) \sum_{k=1}^K a_k \sum_{m=-\ell}^{\ell} \hat{g}_\ell^m Y_\ell^m(\theta_k, \varphi_k) \\ &= 2\pi P_\ell(0) \sum_{k=1}^K a_k g(\theta_k, \varphi_k). \end{aligned}$$

In words, the result of the inner product $\hat{\mathbf{y}} \cdot \hat{\mathbf{g}}_\ell$, up to a known scale factor, is a linear combination of $g_\ell(\theta, \varphi)$ evaluated at the location of the orientations. As previously discussed, $P_\ell(0) = 0$ for odd ℓ meaning we have

$$\hat{\mathbf{y}} \cdot \hat{\mathbf{g}}_\ell = 2\pi P_\ell(0) \sum_{k=1}^K a_k \sin^\ell(\theta_k) \exp(j\ell\varphi_k) = 2\pi P_\ell(0) \sum_{k=1}^K a_k c_k^\ell = \begin{cases} 2\pi P_\ell(0) z_\ell & \text{even } \ell \\ 0 & \text{odd } \ell \end{cases}. \quad (10)$$

The values of z_ℓ recovered in Eq. (10) are identical to the ones recovered using Eq. (7). Upon closer inspection, we find that the spherical harmonics coefficients of $g_\ell(\theta, \varphi)$ are all zero except $\hat{g}_\ell^\ell$ whose value is $1/N_\ell$. Therefore, by computing $\hat{\mathbf{y}} \cdot \hat{\mathbf{g}}_\ell$, we are automatically selecting and scaling the spherical harmonics coefficients with equal

Figure 1. Magnitude of the spherical harmonic coefficients of $\gamma_2(\mathbf{u})$ where $\mathbf{G}$ is the identity matrix in (a) and where $\mathbf{G}$ represents a rotation of $\pi/2$ and $\pi/4$ about the y -axis in (b) and (c), respectively.

degree and order as we did in Section 3. However, we are still using only a single coefficient per degree to compute the value of z_ℓ . To use all the coefficients, the energy of $g_\ell(\theta, \varphi)$ needs to be spread on more than just a single coefficient. This can be achieved by rotating the function $g_\ell(\theta, \varphi)$. Define $\gamma_\ell(\mathbf{u}) = g_\ell(\mathbf{G}^{-1}\mathbf{u})$ where the dependence of $\gamma_\ell(\mathbf{u})$ on $\mathbf{G}$ is left implicit to simplify the notation. As illustrated in Figure 1, the value of $\mathbf{G}$ can be used to modify the energy spread of the spherical harmonics coefficients of $\gamma_\ell(\mathbf{u})$.

Replacing $\hat{\mathbf{g}}_\ell$ by $\hat{\gamma}_\ell$ in Eq. (10) yields

$$\hat{\mathbf{y}} \cdot \hat{\gamma}_\ell = \begin{cases} 2\pi P_\ell(0)z'_\ell & \text{even } \ell \\ 0 & \text{odd } \ell \end{cases} \quad (11)$$

where z'_ℓ represent the same information as z_ℓ , but in a rotated coordinate system determined by $\mathbf{G}$. By computing the value of $\hat{\mathbf{y}} \cdot \hat{\gamma}_\ell$ for $\ell = 0, 2, \dots, 4K - 2$ and using the annihilating filter method as in Section 3, we can recover the values of φ'_k and a_k which are also in a rotated coordinate system. To recover the values of θ'_k , we proceed in a similar manner and compute

$$\hat{\mathbf{y}} \cdot \hat{\beta}_\ell = \begin{cases} 2\pi P_\ell(0)w'_\ell & \text{even } \ell \\ 0 & \text{odd } \ell \end{cases} \quad (12)$$

where $\beta_\ell(\theta, \varphi) = \cos \theta \sin^\ell \theta \exp(j\ell\varphi)$ and w'_ℓ is a rotated version of w_ℓ . From θ'_k , φ'_k , and a_k we can return to

Figure 2. Illustration of the reconstruction process for a signal composed of a single great circle. The original signal is displayed in (a) and its filtered version in (b) along with the $N = 8$ sample locations. The recovered signal is shown in (c), the parameters recovered to machine precision.

the original coordinate system using $\mathbf{u}_k = \mathbf{G}\mathbf{u}'_k$ with

$$\mathbf{u}'_k = [\sin \theta'_k \cos \varphi'_k \quad \sin \theta'_k \sin \varphi'_k \quad \cos \theta'_k]^t.$$

4.2 Selecting the rotation matrix

The selection of the rotation matrix $\mathbf{G}$ will have an important impact on the quality of the reconstruction. Instinctively, the best matrix is the one that distributes the power of $\gamma_\ell(\mathbf{u})$ evenly across all coefficients of degree ℓ . However, our empirical evidence indicates that this is not the case. Instead, the ideal matrix $\mathbf{G}$ varied for each simulation. What we propose is to compute the parameters θ_k , φ_k , and a_k for a set of pre-selected rotation matrices $\mathbf{G}$ and then choose the solution that minimizes

$$\frac{1}{N} \sum_{n=1}^N \|s_n - \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} \sum_{k=1}^K a_k \bar{Y}_\ell^m(\theta_k, \varphi_k) Y_\ell^m(\theta_n, \varphi_n)\|_2 = \|\mathbf{s} - \mathbf{Y}\mathbf{f}\|_2.$$

This approach increases the complexity of the reconstruction algorithm linearly with the number of matrices $\mathbf{G}$ tested, but provides a significant improvement in performance as, showed in Section 5.

5. NUMERICAL SIMULATIONS

To validate our results, several numerical simulations were performed. We first considered noiseless signals generated with the ideal bandlimited kernel. The samples were computed using

$$s_n = s(\theta_n, \varphi_n) = \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} \mathcal{R} \hat{f}_\ell^m Y_\ell^m(\theta_n, \varphi_n)$$

where the sample location (θ_n, φ_n) with $n = 1, \dots, N$ were computed numerically using the electrostatic repulsion model¹⁹ which yields approximately evenly distributed sample locations. The parameters θ_k , φ_k were generated randomly from a uniform distribution on the interval $[0, \pi[$ and the values of a_k were generated with a normal distribution. Examples of reconstructions are illustrated in Figures 2 and 3 using $N = L(L+1)/2$ samples. In both cases, the reconstructed signal is visually indistinguishable from the original signal.

To get an objective measure of the performance, we computed the angular error using

$$\alpha = \frac{1}{K} \sum_{k=1}^K \cos^{-1}(|\mathbf{u}_k^t \mathbf{u}'_k|)$$

Figure 3. Illustration of the reconstruction process for a signal composed of 3 great circles. The original signal is displayed in (a) and its filtered version in (b) along with the $N = 66$ sample locations. The recovered signal is shown in (c), the parameters recovered to machine precision.

Figure 4. The angular error α as a function of the number of samples used for signals with $K = 2, 3, 4$, and 5 . The bandlimit of the ideal bandlimited kernel was set to $L = 4K - 1$ and the number of samples varied from $L(L + 1)/2 - 10$ to $L(L + 1)/2 + 10$.

where $\mathbf{u}_k$ and $\mathbf{u}'_k$ are the true and recovered orientations, respectively. The average angular error over 100 repetitions for $K = 1, \dots, 5$ is illustrated in Figure 4 as a function of N . For each value of K , a sharp drop in angular error is observed when the number of samples satisfies $N \geq L(L + 1)/2$. When this limit is reached, the spherical harmonic coefficients of the sum of great circles are estimated precisely and our reconstruction algorithm yields accurate parameters. These results support our suggestion that $N = L(L + 1)/2 = 2K(4K - 1)$ are sufficient to accurately sample and reconstruct a sum of K great circles.

To test our algorithms robustness to noise, further tests were performed using noisy samples. In this case, the samples are given by

$$\tilde{s}_n = s(\theta_n, \varphi_n) + \epsilon_n$$

where ϵ_n is taken from a zero-mean normal distribution whose standard deviation is selected as a function of the desired signal to noise ratio. The sample locations were again generated using the electrostatic repulsion model. A comparison between the performance of SFRI (the algorithm of Section 3) and SFRI_G (the algorithm of Section 4) is illustrated in Figure 5. For SFRI_G, we selected 10 rotation matrices that represented every

Figure 5. The angular error α as a function of the signal to noise ratio in dB for a sum of great circles with $K = 2$. The bandlimit of the ideal bandlimited kernel and the number of samples are $L = 4K - 1 = 7$ and $N = 36$, respectively. The light gray dots are the results of individual simulations using SFRI_G .

combination of rotation of $0, \pi/4, \pi/2$, and $3\pi/4$ about the y and z axes. For both algorithms, the angular error decreases as the signal to noise ratio increases. Over the entire range of signal to noise ratios tested, SFRI_G maintained a lower mean angular error than SFRI. At 16dB, the angular error of SFRI_G is 3.9° , less than half that of SFRI (8.1°). However, the improvement in accuracy comes at the cost of efficiency; SFRI_G computed the solution an order of magnitude more slowly than SFRI.

6. CONCLUSION

In this work, we extended the class of signals that can be sampled and reconstructed at their rate of innovation to include great circles, the intersection between a sphere and a plane. We showed that great circles can be exactly recovered using $4K(8K - 7) + 7$ samples. However, we suggest that, in practice, accurate reconstruction still occurs when $N = 2K(4K - 1)$ noiseless samples are available. To substantiate our claim, we performed several numerical simulations and our results show that $N = 2K(4K - 1)$ are sufficient to recover a sum of K great circles accurately. To improve the robustness of our reconstruction algorithm to noise, we proposed modification to our method that allows the use of all available information, namely all spherical harmonic coefficients. Our numerical simulations show that these modifications can halve the angular error when the signal to noise ratio is above 15dB.

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