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SPARSE TENSOR GALERKIN DISCRETIZATION OF PARAMETRIC AND RANDOM PARABOLIC PDES—ANALYTIC REGULARITY AND GENERALIZED POLYNOMIAL CHAOS APPROXIMATION*

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Abstract. For initial boundary value problems of linear parabolic partial differential equations with random coefficients, we show analyticity of the solution with respect to the parameters and give an a priori error analysis for N -term generalized polynomial chaos approximations in a scale of Bochner spaces. The problem is reduced to a parametric family of deterministic initial boundary value problems on an infinite dimensional parameter space by Galerkin projection onto finitely supported polynomial systems in the parameter space. Uniform stability with respect to the support of the resulting coupled parabolic systems is established. Analyticity of the solution with respect to the countably many parameters is established, and a regularity result of the parametric solution is proved for both compatible as well as incompatible initial data and source terms. The present results imply convergence rates and stability of sparse, adaptive space-time tensor product Galerkin discretizations of these infinite dimensional, parametric problems in the parameter space recently proposed in [C. Schwab and C. J. Gittelsohn, *Acta Numer.*, 20 (2011), pp. 291–467; C. J. Gittelsohn, *Adaptive Galerkin Methods for Parametric and Stochastic Operator Equations*, Ph.D. thesis, ETH Zürich, 2011].

Key words. parabolic PDEs, random coefficients, best N -term approximation, analytic parameter dependence

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1. Introduction. The efficient numerical solution of parametric partial differential equations in high-dimensional parameter spaces has attracted considerable attention recently. We mention only the recent works [2, 9, 3] and the references therein for elliptic problems and, with particular relevance to the present paper, the recent work [10] for parabolic problems. In the present paper, we investigate the analytic regularity of a class of parametric parabolic problems. Such problems arise, for example, in the context of diffusion in random media when the medium’s permeability is a random field which is given, for example, as a Karhnen–Love expansion. Parametrizing the random input permeability in terms of the (countably many) coefficients in the Karhnen–Love expansion, the solution becomes, in turn, a deterministic function of these parameters with *analytic dependence on these input variables*. As we show here, the parametric solution admits a so-called *generalized polynomial chaos* (gpc) expansion with respect to these input variables, with deterministic coefficients which take values in the natural Lebesgue–Bochner spaces of deterministic parabolic problems. We also prove, following [4, 5] for elliptic problems, p -summability for some $0 < p < 1$ of the gpc coefficient sequences of the parametric solutions, in a scale of Lebesgue–

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Bochner spaces in the space-time cylinder, given p -summability and regularity of the input's Karhnen–Love expansion coefficients with the same value of p .

We also indicate consequences of this p -summability of gpc coefficients for convergence rates of a class of spectral approximations in the infinite dimensional stochastic parameter space: we show that this results in large, coupled systems of deterministic parabolic equations which are well-posed independently of the selection of active stochastic modes. The results in the present paper imply, in particular, convergence rates of adaptive Galerkin discretizations of the parametric, parabolic problem through finitely supported, tensorized polynomial representations through suitable adaptive algorithms as, for example, developed in [6, 11, 7, 8].

1.1. A class of random parabolic problems. For $0 < T < \infty$, we consider in the bounded time interval $I = (0, T)$ a class of parabolic initial boundary value problems with random coefficients. Throughout, we will consider a bounded Lipschitz domain $D \subset \mathbf{R}^d$ and the associated space-time cylinder $Q_T = I \times D$. In Q_T , we consider the random parabolic initial boundary value problem

$$(1.1) \quad \frac{\partial u}{\partial t} - \nabla \cdot (a(x, \omega) \nabla u) = g(t, x), \quad u|_{\partial D \times I} = 0, \quad u|_{t=0} = h(x).$$

At this stage, we assume the coefficient $a(x, \omega)$ to be a random field on a suitable probability space (Ω, Σ, P) taking values in $L^\infty(D)$. We assume in particular $a(x, \omega)$ to be independent of t (additional structural assumptions on the coefficient will be imposed shortly). The source term g and the initial data h are both assumed to be deterministic (this assumption could be relaxed without additional essential technical complications; for simplicity of exposition only we shall not pursue this here). We make the following assumption.

Assumption 1.1. There exist constants $0 < a_{\min} \leq a_{\max} < \infty$ so that

$$\forall x \in D \quad \forall \omega \in \Omega, \quad 0 < a_{\min} \leq a(x, \omega) \leq a_{\max}.$$

In view of the sparse tensor discretizations to be investigated, we consider a *space-time variational formulation* of problem (1.1). To state it, we denote $V = H_0^1(D)$ and $H = L^2(D)$ and identify H with its dual $H \simeq H'$. Then $V \subset H \simeq H \subset V' = H^{-1}(D)$. For the variational formulation of (1.1) we introduce the Bochner spaces

$$(1.2) \quad \mathcal{X} = L^2(I; V) \cap H^1(I; V') \quad \text{and} \quad \mathcal{Y} = L^2(I; V) \times H.$$

In $\mathcal{X}$ and $\mathcal{Y}$ norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$, respectively, are for $u \in \mathcal{X}$ and $v = (v_1, v_2) \in \mathcal{Y}$ given by

$$\|u\|_{\mathcal{X}} = (\|u\|_{L^2(I; V)}^2 + \|u\|_{H^1(I; V')}^2)^{1/2} \quad \text{and} \quad \|v\|_{\mathcal{Y}} = (\|v_1\|_{L^2(I; V)}^2 + \|v_2\|_H^2)^{1/2}.$$

Given a realization $\omega \in \Omega$, a *weak solution* of problem (1.1) is a function $u(\cdot, \cdot, \omega) \in \mathcal{X}$ such that

$$(1.3) \quad \int_I \left\langle \frac{du}{dt}, v_1 \right\rangle_H dt + \int_I \int_D a(x, \omega) \nabla u(t, x, \omega) \cdot \nabla v_1(t, x) dx dt + \langle u(0, \cdot, \omega), v_2 \rangle_H \\ = \int_I \langle g(t, \cdot), v_1 \rangle_H dt + \langle h, v_2 \rangle_H \quad \forall v \in \mathcal{Y}.$$

Here, $\langle \cdot \rangle_H$ denotes the inner product in H , extended to the dual pairing V', V . The following proposition from [12] guarantees its well-posedness for all $\omega \in \Omega$, under Assumption 1.1.

PROPOSITION 1.1. Assume that $g \in L^2(I, V')$, $h \in L^2(D)$ and that Assumption 1.1 holds. Then, for every $\omega \in \Omega$, the parabolic operator $B \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$ induced by (1.1) in the weak form (1.3) is an isomorphism: for given $(g, h) \in \mathcal{Y}'$ and every $\omega \in \Omega$, problem (1.3) has a unique solution $u(\cdot, \cdot, \omega)$ which satisfies the a priori estimate

$$(1.4) \quad \|u\|_{\mathcal{X}} \leq C (\|g\|_{L^2(I, V')} + \|h\|_{L^2(D)}),$$

where the constant C is bounded uniformly for all realizations.

Proof. The proof of Proposition 1.1 is based on showing that the operator $B \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$ satisfies an inf-sup condition on $\mathcal{X} \times \mathcal{Y}$. Inspecting the proof in [12] one verifies that, under Assumption 1.1, the inf-sup conditions only depend on T , $a_{\max}$, and $a_{\min}$ and therefore hold uniformly with respect to $\omega \in \Omega$. $\square$

In this paper, we assume that the coefficient a in (1.1) is characterized by a sequence of scalar random variables $(y_j)_{j \geq 1}$, i.e.,

$$(1.5) \quad a(\cdot, \omega) = \bar{a}(\cdot) + \sum_{j \geq 1} y_j(\omega) \psi_j(\cdot).$$

We assume in addition that the functions ψ_j are scaled in $L^\infty(D)$ such that $y_j : \Omega \rightarrow \mathbb{R}$, $j = 1, 2, \dots$, are independent random variables which are distributed identically and uniformly in $[-1, 1]$ and such that the range of the y_j is $[-1, 1]$.

Then all realizations of the random vector $y = (y_1, y_2, \dots)$ are supported in the cube $U = [-1, 1]^{\mathbb{N}}$. We interpret U as the unit ball in $\ell^\infty(\mathbb{N})$. Via the corresponding norm $\|y\|_{\ell^\infty(\mathbb{N})}$, open subsets of U are defined in the usual way, and we denote the σ -algebra of Borel subsets of U (in the topology of $\ell^\infty(\mathbb{N})$) by $\mathcal{B}(U)$.

We make the following assumption on ψ_j .

Assumption 1.2. With $\bar{a}_{\min} = \text{ess inf}_{x \in D} \bar{a}(x) > 0$ and some $\kappa > 0$, the functions $\bar{a}$ and ψ_j satisfy

$$\sum_{j \geq 1} \|\psi_j\|_{L^\infty(D)} \leq \frac{\kappa}{1 + \kappa} \bar{a}_{\min}.$$

This assumption implies that the fluctuations (resp., deviations) from mean of the random coefficient $a(x, \omega)$ in (1.5) are dominated by the mean field, i.e., that they are small with respect to the deterministic mean field. Assumption 1.1 is then satisfied by choosing

$$(1.6) \quad a_{\min} := \bar{a}_{\min} - \frac{\kappa}{1 + \kappa} \bar{a}_{\min} = \frac{1}{1 + \kappa} \bar{a}_{\min}.$$

1.2. Probability spaces. Using the structural assumption (1.5) on the random coefficient, the law of the random solution u of (1.1) takes the form of a parametric deterministic function of the (in general countably many components of) $y \in U$. The variational problem can be cast in the form of a parametric family of deterministic problems for y . In the next sections, we study sparse tensor discretizations of a variational problem for u as a function of (t, x, y) in $I \times D \times U$. To do so, we need to define probability measures on U .

Let Θ be the σ -algebra defined on U which is generated from the sets of the form $\prod_{j=1}^\infty S_j$, where the S_j are subintervals of $[-1, 1]$ and only a finite number of them are proper subsets of $[-1, 1]$. On Θ , we define a probability measure as the countable product measure of the uniform probability measure on $(-1, 1)$, i.e.,

$$d\rho(y) := \bigotimes_{j \geq 1} dy_j / 2.$$

Then (U, Θ, ρ) is a probability space. As the random parameters y_j are assumed to be distributed uniformly in $(-1, 1)$, for $S = \prod_{j=1}^{\infty} S_j$ we have

$$\rho(S) = \prod_{j=1}^{\infty} P\{\omega : y_j(\omega) \in S_j\}.$$

We introduce Bochner spaces $\underline{\mathcal{X}} = L^2(U, \rho; \mathcal{X})$ and $\underline{\mathcal{Y}} = L^2(U, \rho; \mathcal{Y})$ of strongly ρ -measurable mappings from U to $\mathcal{X}$ and to $\mathcal{Y}$, respectively. We note that these spaces are isomorphic to tensor product spaces, i.e., $\underline{\mathcal{X}} \simeq L^2(U, \rho) \otimes \mathcal{X}$, $\underline{\mathcal{Y}} \simeq L^2(U, \rho) \otimes \mathcal{Y}$.

1.3. Parametric deterministic parabolic problem. Consider the parametric family of deterministic parabolic problems: given a source term $g(t, x)$ and initial data $h(x)$, for $y \in U$, find $u(t, x, y)$ such that

$$(1.7) \quad \begin{aligned} \frac{\partial u}{\partial t}(t, x, y) - \nabla_x \cdot [a(x, y) \nabla_x u(t, x, y)] &= g(t, x) \text{ in } Q_T, \\ u(t, x, y)|_{\partial D \times I} &= 0, \quad u|_{t=0} = h(x), \end{aligned}$$

where, for every $y = (y_1, y_2, \dots) \in U$ we define the deterministic parametric coefficient

$$a(\cdot, y) = \bar{a}(\cdot) + \sum_{j=1}^{\infty} y_j \psi_j(\cdot).$$

For the weak formulation of (1.7), we follow (1.3) and define for $y \in U$ the parametric family of bilinear forms $U \ni y \rightarrow b(y; w, (v_1, v_2)) : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ by

$$(1.8) \quad \begin{aligned} b(y; w, (v_1, v_2)) &= \int_I \left\langle \frac{dw}{dt}, v_1(t, \cdot) \right\rangle_H dt \\ &+ \int_D \int_I a(x, y) \nabla w(t, x) \cdot \nabla v_1(t, x) dx dt + \langle w(0), v_2 \rangle_H. \end{aligned}$$

We also define the linear form

$$(1.9) \quad f(v) = \int_I \langle g(t), v_1(t) \rangle_H dt + \langle h, v_2 \rangle_H, \quad v = (v_1, v_2) \in \mathcal{Y}.$$

The variational formulation for (1.7) reads, given $f \in \mathcal{Y}'$, find $u(y) : U \ni y \rightarrow \mathcal{X}$ such that

$$(1.10) \quad b(y; u, v) = f(v) \quad \forall v = (v_1, v_2) \in \mathcal{Y}.$$

PROPOSITION 1.2. *For each parameter vector $y \in U$, the operator $B(y) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$ defined by $(B(y)w)(v) = b(y; w, v)$ is boundedly invertible. The norms of $B(y)$ and $B(y)^{-1}$ can be bounded uniformly by constants which only depend on $a_{\min}$, $a_{\max}$, T , and the spaces $\mathcal{X}$ and $\mathcal{Y}$. In particular, the solution u of the problem (1.10) is uniformly bounded in $\mathcal{X}$ for all $y \in U$.*

The proof of this theorem can be found in Appendix A of [12].

PROPOSITION 1.3. *There holds*

$$\|u(t, x, y) - u(t, x, y')\|_{\mathcal{X}} \leq C \|a(\cdot, y) - a(\cdot, y')\|_{L^\infty(D)} \quad \text{for every } y, y' \in U.$$

Proof. From the variational formulation (1.10), we find that the function $w = u(x, t, y) - u(x, t, y')$ satisfies the variational problem

$$(1.11) \quad \begin{aligned} & \int_I \left\langle \frac{dw}{dt}, v_1 \right\rangle dt + \int_I \int_D a(x, y) \nabla w \cdot \nabla v_1(t, x) dx dt + \int_D w(x, 0) v_2(x) dx \\ &= - \int_I \int_D (a(x, y) - a(x, y')) \nabla u(t, x, y') \cdot \nabla v_1(t, x) dx dt. \end{aligned}$$

From this we deduce that for every $y, y' \in U$ it holds

$$(1.12) \quad \begin{aligned} \|u(t, x, y) - u(t, x, y')\|_{\mathcal{X}} &\leq C \| (a(x, y) - a(x, y')) \cdot \nabla u(t, x, y) \|_{L^2(I; H)} \\ &\leq C \operatorname{ess\,sup}_x |a(x, y) - a(x, y')|. \quad \square \end{aligned}$$

PROPOSITION 1.4. *The map $u(\cdot, \cdot, y) : U \rightarrow \mathcal{X}$ is strongly measurable as an $\mathcal{X}$ -valued function.*

Proof. As the space $\mathcal{X}$ is a separable Hilbert space, strong measurability is implied by weak measurability and therefore it suffices to verify the latter. Let $h \in \mathcal{X}$ be arbitrary, fixed. We recall the $\mathcal{X}$ inner product

$$\langle u(y), h \rangle_{\mathcal{X}} = \langle u(y), h \rangle_{L^2(I, V)} + \langle u(y), h \rangle_{H^1(I, V')}.$$

To show that u is measurable as a Bochner function from U to $\mathcal{X}$, it is sufficient to show that, for every $h \in \mathcal{X}$, the real-valued function $U \ni y \mapsto \langle u(y), h \rangle_{\mathcal{X}}$ is measurable. To this end, we fix a real number a and show that the set $Y_a = \{y : \langle u(y), h \rangle_{\mathcal{X}} > a\}$ is in the σ -algebra defined on U . From Proposition 1.3 if $\langle u(y), h \rangle_{\mathcal{X}} > a$, then there is a positive number r such that if

$$(1.13) \quad \operatorname{ess\,sup}_x |a(x, y) - a(x, y')| < r,$$

then $\langle u(y'), h \rangle_{\mathcal{X}} > a$. We consider the set T_i of vectors $y \in U$ such that for $\bar{y} = (y_1, y_2, \dots, y_i, z_1, z_2, \dots)$, $\langle u(\bar{y}), h \rangle_{\mathcal{X}} > a$ for all $z_j \in [-1, 1]$, $j = 1, 2, \dots$. For each $y \in U$, from assumption (1.2),

$$|a(x, y) - a(x, \bar{y})| < r,$$

if i is large enough. Thus each vector $y \in Y_a$ belongs to a set T_i for some i . Let $R_i \subset [-1, 1]^i$ be the set of $t = (t_1, \dots, t_i)$ such that $(t_1, \dots, t_i, z_1, z_2, \dots) \in T_i$ for all $z_j \in [-1, 1]$ ($j = 1, 2, \dots$). From (1.12) and (1.13), R_i is an open set and thus can be represented as a countable union of open cubes. Thus T_i can be represented as a countable union of cubes of the form $\prod_{j \geq 1} S_j$, where S_j is an open interval in $(-1, 1)$ and $S_j = (-1, 1)$ when j is sufficiently large. Thus T_i is measurable and so is Y_a . $\square$

With the bilinear form $B(\cdot, \cdot) : \underline{\mathcal{X}} \times \underline{\mathcal{Y}} \rightarrow \mathbb{R}$ and the linear form $F(\cdot) : \underline{\mathcal{Y}} \rightarrow \mathbb{R}$ defined by

$$(1.14) \quad B(u, v) = \int_U b(y; u, v) \rho(dy) \quad \text{and} \quad F(v) = \int_U f(v) \rho(dy)$$

we consider the variational problem:

$$(1.15) \quad \text{find } u \in \underline{\mathcal{X}} \text{ such that } B(u, v) = F(v) \text{ for all } v \in \underline{\mathcal{Y}}.$$

PROPOSITION 1.5. *The problem (1.15) admits a unique solution $u \in \mathcal{X}$.*

Proof. The existence part is obvious. Moreover, from Proposition 1.2 the solution u of (1.10) belongs to $L^\infty(U, \mathcal{X})$ so $u \in L^2(U, \rho; \mathcal{X})$ is a solution of (1.15). Next we show the uniqueness of a solution $u \in L^2(U, \rho; \mathcal{X})$.

Let $v(t, x, y) = (v_1(t, x)w(y), v_2(t, x)w(y))$, where $w(y) \in L^2(U, \rho)$. Then

$$\int_U b(y; u, (v_1, v_2))w(y)\rho(dy) = \int_U f((v_1, v_2))w(y)\rho(dy).$$

As this holds for all $w(y) \in L^2(U, \rho)$, for ρ almost all y in U the function $u(y)$ satisfies

$$b(y; u, (v_1, v_2)) = f((v_1, v_2)).$$

As $\mathcal{Y}$ is separable, for ρ almost all $y \in U$ this holds for all (v_1, v_2) in a dense countable subset of $\mathcal{Y}$ so holds for all $(v_1, v_2) \in \mathcal{Y}$. For each $y \in U$, there is a unique function $u(t, x, y) \in \mathcal{X}$ that satisfies this equation which is uniformly bounded in $\mathcal{X}$ for all $y \in U$. This completes the proof. $\square$

With this result in hand, we recover the random solution $u(t, x, \omega)$ from the parametric solution by reinserting the random parameters.

THEOREM 1.1. *Under Assumptions 1.1, 1.2, for given $g \in L^2(I, V')$ and $h \in H$, the variational problem find $u \in L^2(\Omega, \mathcal{X})$ such that for every $v(t, x, \omega) = (v_1(t, x, \omega), v_2(t, x, \omega)) \in L^2(\Omega, \mathcal{Y})$ it holds*

$$\begin{aligned} (1.16) \quad & \mathbb{E} \left\{ \int_I \left\langle \frac{du}{dt}(t, \cdot, \omega), v_1(t, \cdot, \omega) \right\rangle_H dt \right\} + \mathbb{E} \left\{ \int_I \int_D a(x, \omega) \nabla u(t, x, \omega) \cdot \nabla v_1(t, x, \omega) dx dt \right\} \\ & + \mathbb{E} \left\{ \int_D u(0, x, \omega) v_2(x, \omega) dx \right\} \\ & = \mathbb{E} \left\{ \int_I \langle g(t, \cdot), v_1(t, \cdot, \omega) \rangle_H dt \right\} + \mathbb{E} \left\{ \int_D h(x) v_2(x, \omega) dx \right\} \end{aligned}$$

admits a unique solution which satisfies the a priori estimate

$$(1.17) \quad \|u\|_{L^2(\Omega, \mathcal{X})} \leq C (\|g\|_{L^2(I, V')} + \|h\|_H) .$$

2. Semidiscrete Galerkin approximation. We discretize the parametric parabolic problem (1.7) in the variational form (1.15) by Galerkin projection onto linear combinations of N polynomials of the parameters $y \in U$ with $\mathcal{X}$ -valued coefficients. We prove that this results in a coupled parabolic system of size N and establish its well-posedness regardless of the choice of particular N polynomials.

2.1. Polynomial spaces in U . Let $(L_n)_{n \geq 0}$ be the univariate Legendre polynomials normalized according to

$$(2.1) \quad \int_{-1}^1 |L_n(t)|^2 \frac{dt}{2} = 1.$$

Note that in this normalization, $L_0(t) = 1$. Let $\mathcal{F}$ be the countable set of sequences $\nu = (\nu_j)_{j \geq 1}$ of nonnegative integers such that only a finite number of ν_j are nonzero. For $\nu \in \mathcal{F}$, we introduce the tensorized Legendre polynomials

$$L_\nu(y) = \prod_{j \geq 1} L_{\nu_j}(y_j), \quad \nu \in \mathcal{F}.$$

The family L_ν forms a complete orthonormal system of $L^2(U, \rho)$. Therefore each function $u \in \underline{\mathcal{X}}$ can be represented as

$$(2.2) \quad u = \sum_{\nu \in \mathcal{F}} u_\nu L_\nu,$$

where the coefficients $u_\nu \in \mathcal{X}$ are defined by

$$u_\nu = \int_U u(\cdot, \cdot, y) L_\nu(y) d\rho(y) \in \mathcal{X};$$

the integral being understood as a Bochner integral of $\mathcal{X}$ -valued functions over U .

2.2. Well-posedness and quasi optimality. For every subset $\Lambda \subset \mathcal{F}$ of cardinality $N = \#\Lambda < \infty$ we define a space of $\mathcal{X}$ - and $\mathcal{Y}$ -valued polynomial expansions

$$\underline{\mathcal{X}}_\Lambda = \{u_\Lambda(t, x, y) = \sum_{\nu \in \Lambda} u_\nu(t, x) L_\nu(y) : u_\nu \in \mathcal{X}\} \subset \underline{\mathcal{X}}$$

and

$$\underline{\mathcal{Y}}_\Lambda = \{v_\Lambda(t, x, y) = \sum_{\nu \in \Lambda} v_\nu(t, x) L_\nu(y) : v_\nu \in \mathcal{Y}\} \subset \underline{\mathcal{Y}}.$$

In the Legendre basis $(L_\nu)_{\nu \in \mathcal{F}}$, we write

$$v_{1\Lambda}(t, x, y) = \sum_{\nu \in \Lambda} v_{1\nu}(t, x) L_\nu(y) \quad \text{and} \quad v_{2\Lambda}(x, y) = \sum_{\nu \in \Lambda} v_{2\nu}(x) L_\nu(y),$$

respectively, where $v_\nu = (v_{1\nu}, v_{2\nu}) \in \mathcal{Y}$ for all $\nu \in \mathcal{F}$. We consider the Galerkin approximation: find

$$(2.3) \quad u_\Lambda \in \underline{\mathcal{X}}_\Lambda : \quad B(u_\Lambda, v_\Lambda) = F(v_\Lambda) \quad \forall v_\Lambda \in \underline{\mathcal{Y}}_\Lambda.$$

THEOREM 2.1. *For any subset $\Lambda \subset \mathcal{F}$, problem (2.3) corresponds to a coupled system of $N = \#\Lambda$ many linear parabolic equations. Under Assumptions 1.1, 1.2, these systems are stable uniformly with respect to $\Lambda \subset \mathcal{F}$: for any $\Lambda \subset \mathcal{F}$, problem (2.3) admits a unique solution $u_\Lambda \in \underline{\mathcal{X}}_\Lambda$ which satisfies the a priori error bound*

$$\|u - u_\Lambda\|_{\underline{\mathcal{X}}} \leq c \left(\sum_{\nu \notin \Lambda} \|u_\nu\|_{\mathcal{X}}^2 \right)^{1/2}.$$

Here, $u_\nu \in \mathcal{X}$ are the Legendre coefficients of the solution of the parametric problem in (2.2) and c is independent of Λ .

Proof. To prove the uniform well-posedness of the coupled parabolic system resulting from the Galerkin discretization in U , we prove that the following inf-sup condition holds: there exist $\alpha, \beta > 0$ such that for any $\Lambda \subset \mathcal{F}$ it holds

$$(2.4) \quad \sup_{u_\Lambda \in \underline{\mathcal{X}}_\Lambda, v_\Lambda \in \underline{\mathcal{Y}}_\Lambda} \frac{|B(u_\Lambda, v_\Lambda)|}{\|u_\Lambda\|_{\underline{\mathcal{X}}} \|v_\Lambda\|_{\underline{\mathcal{Y}}}} \leq \alpha < \infty,$$

$$(2.5) \quad \inf_{0 \neq u_\Lambda \in \underline{\mathcal{X}}_\Lambda} \sup_{0 \neq v_\Lambda \in \underline{\mathcal{Y}}_\Lambda} \frac{|B(u_\Lambda, v_\Lambda)|}{\|u_\Lambda\|_{\underline{\mathcal{X}}} \|v_\Lambda\|_{\underline{\mathcal{Y}}}} \geq \beta > 0,$$

$$(2.6) \quad \forall 0 \neq v_\Lambda \in \underline{\mathcal{Y}}_\Lambda : \quad \sup_{0 \neq u_\Lambda \in \underline{\mathcal{X}}_\Lambda} |B(u_\Lambda, v_\Lambda)| > 0,$$

where the constants α, β are independent of $\Lambda \subset \mathcal{F}$ (a proof is provided in the Appendix).

The projected parametric deterministic parabolic problem (2.3) has a unique solution, and, in virtue of the independence of α, β of Λ , is well-posed and stable with stability bounds which are independent of the choice of $\Lambda \subset \mathcal{F}$. Hence, the error incurred by this projection is quasi optimal: there exists a constant $c > 0$ independent of the set $\Lambda \subset \mathcal{F}$ of “active” coefficients such that, with the constant β in (2.5),

$$\begin{aligned} \|u - u_\Lambda\|_{\underline{\mathcal{X}}} &\leq (1 + \beta^{-1}(\|g\|_{L^2(I, V')} + \|h\|_{L^2(D)})) \inf_{v_\Lambda \in \underline{\mathcal{X}}_\Lambda} \|u - v_\Lambda\|_{\underline{\mathcal{X}}} \\ &\leq c \left\| u - \sum_{\nu \in \Lambda} u_\nu L_\nu \right\|_{\underline{\mathcal{X}}} = c \left\| \sum_{\nu \notin \Lambda} u_\nu L_\nu \right\|_{\underline{\mathcal{X}}}. \end{aligned}$$

By the normalization (2.1) and Parseval’s equality, there holds

$$\left\| \sum_{\nu \notin \Lambda} u_\nu L_\nu \right\|_{\underline{\mathcal{X}}}^2 = \sum_{\nu \notin \Lambda} \|u_\nu\|_{\mathcal{X}}^2.$$

The conclusion then follows with $c = 1 + \beta^{-1}(\|g\|_{L^2(I, V')} + \|h\|_{L^2(D)})$. $\square$

3. Best N -term gpc approximations. Theorem 2.1 suggests we choose the set $\Lambda \subset \mathcal{F}$ as the set of the best N terms u_ν according to their $\mathcal{X}$ norms in the chaos expansion of the parametric solution. However, a priori, only bounds for u_ν in $\mathcal{X}$ are known. Therefore, one strategy will be to choose the set Λ according to these a priori bounds (this strategy was employed in [3] for the elliptic case). Alternatively, an optimal, adaptive Galerkin method will yield iteratively quasi-optimal sequences Λ_N of active indices. We now determine such a priori bounds. A best N -term convergence rate estimate where N denotes the number of terms in the chaos expansion will result from these bounds using the following lemma (see, e.g., [5] for a proof).

LEMMA 3.1. *Let $q \geq p > 0$ and let $\alpha = (\alpha_\nu)_{\nu \in \mathcal{F}}$ be a sequence in $l^p(\mathcal{F})$. If $\Lambda_N \subset \mathcal{F}$ is a set of indices corresponding to a set of N largest $|\alpha_\nu|$, then*

$$\left(\sum_{\nu \notin \Lambda_N} |\alpha_\nu|^q \right)^{1/q} \leq \|\alpha\|_{l^p(\mathcal{F})} N^{-\sigma}, \quad \text{where} \quad \sigma = \frac{1}{p} - \frac{1}{q}.$$

The convergence rate of *adaptive* spectral approximations such as (2.3) of the parabolic problem on the infinite dimensional parameter space U is determined by the summability of the Legendre coefficient sequence $(\|u_\nu\|_{\underline{\mathcal{X}}})_{\nu \in \mathcal{F}}$. We shall now prove that summability of this sequence is determined by that of the sequence $(\psi_j(x))_{j \in \mathbb{N}}$ in the input’s fluctuation expansion (1.5). Throughout, Assumptions 1.1 and 1.2 will be required to hold. To quantify sparsity in the random input’s expansion (1.5), we shall require the following holds.

Assumption 3.1. There exists $0 < p < 1$ such that

$$(3.1) \quad \sum_{j=1}^{\infty} \|\psi_j\|_{L^\infty(D)}^p < \infty.$$

Such a p exists, for example, for Karhunen–Loeve expansions of the random diffusion coefficient $a(x, \omega)$ provided that the covariance of $a(x, \omega)$ is sufficiently regular (see, e.g., [13] for details).

3.1. Complex extension of the parametric problem. To estimate $\|u_\nu\|_{\mathcal{X}}$, we shall use tools from complex analysis and extend the parametric, deterministic problem (1.7) to parameter vectors taking values in the complex domain. To establish its well-posedness for such choices of the parameter vector, we fix $0 < K < 1$ such that

$$K \sum_{j=1}^{\infty} \|\psi_j\|_{L^\infty(D)} < \frac{a_{\min}}{8}.$$

We choose an integer $J_0 \in \mathbb{N}$ such that

$$\sum_{j>J_0} \|\psi_j\|_{L^\infty(D)} < \frac{a_{\min}K}{24(1+K)}.$$

With this choice of J_0 , we define the sets $E = \{1, 2, \dots, J_0\}$ and $F = \mathbb{N} \setminus E$. We then define, for a given $\nu \in \mathcal{F}$, its “restriction to $E \subset \mathbb{N}$ ”, i.e., $(\nu_E)_j := \nu_j$ if $j \in E$, and $(\nu_E)_j := 0$ otherwise, and likewise we define ν_F . Then

$$|\nu_F| = \sum_{j>J_0} |\nu_j|.$$

For each $\nu \in \mathcal{F}$ we define

$$(3.2) \quad r_m = K \text{ when } m \leq J_0 \text{ and } r_m = 1 + \frac{a_{\min}\nu_m}{4|\nu_F|\|\psi_m\|_{L^\infty(D)}} \text{ when } m > J_0,$$

where we adopt the convention that $\frac{|\nu_j|}{|\nu_F|} = 0$ if $|\nu_F| = 0$. For r_m as in (3.2), we consider the open discs

$$(3.3) \quad \mathcal{U}_m := \{z_m \in \mathbb{C} : |z_m| < 1 + r_m\} \subset \mathbb{C}, \quad m \in \mathbb{N}.$$

We extend the parametric deterministic problem (1.7) to parameter vectors z in the polydiscs

$$(3.4) \quad \mathcal{U} = \bigotimes_{m=1}^{\infty} \mathcal{U}_m \subset \mathbb{C}^{\mathbb{N}}.$$

To do so, we extend the parametric, deterministic coefficient function $a(x, y)$ in (1.5) to $z \in \mathcal{U}$ by

$$a(x, z) = \bar{a}(x) + \sum_{m=1}^{\infty} z_m \psi_m(x).$$

We show that this expression is meaningful for $z \in \mathcal{U}$: for almost every $x \in D$ it holds

$$\begin{aligned} |a(x, z)| &\leq \bar{a}(x) + \sum_{m=1}^{\infty} |\psi_m(x)|(1 + r_m) \\ &\leq \operatorname{ess\,sup}_{x \in D} |\bar{a}(x)| + \sum_{m=1}^{J_0} \|\psi_m\|_{L^\infty(D)}(1 + K) \\ &\quad + \sum_{m>J_0} \left(2 + \frac{a_{\min}\nu_m}{4|\nu_F|\|\psi_m\|_{L^\infty(D)}} \right) \|\psi_m\|_{L^\infty(D)} \\ &\leq \|\bar{a}\|_{L^\infty(D)} + 2 \sum_{m=1}^{\infty} \|\psi_m\|_{L^\infty(D)} + \frac{a_{\min}}{4}. \end{aligned}$$

For such $z \in \mathcal{U}$, we consider the following *deterministic, parametric, parabolic problem*

$$(3.5) \quad \frac{\partial u(t, x, z)}{\partial t} - \nabla \cdot (a(x, z) \nabla u(t, x, z)) = g(t, x), \quad u|_{\partial D} = 0, \quad u(0, x, z) = h(x).$$

For simplicity, we denote by $\mathcal{X}$ the space $L^2(I; V) \cap H^1(I; V')$ of complex-valued functions. The solutions of (3.5) will then be complex-valued functions of x and of t . Accordingly, we understand all Hilbert spaces as spaces of complex-valued functions and all inner products as sesquilinear forms.

LEMMA 3.2. *For each $z \in \mathcal{U}$, problem (3.5) has a unique solution in $\mathcal{X}$. There exists a positive constant C such that for all $z \in \mathcal{U}$*

$$(3.6) \quad \|u(\cdot, \cdot, z)\|_{\mathcal{X}} \leq C(\|g\|_{L^2(I; V')} + \|h\|_{L^2(D)}).$$

Proof. For every $z \in \mathcal{U}$ and almost every $x \in D$ it holds

$$\begin{aligned} \Re a &\geq \min \bar{a} - \sum_{m=1}^{\infty} \|\psi_m\|_{L^\infty(D)} (1 + r_m) \geq (\min \bar{a}) - \frac{\kappa}{\kappa + 1} (\min \bar{a}) \\ &\quad - K \sum_{m=1}^{J_0} \|\psi_m\|_{L^\infty(D)} - \sum_{m > J_0} \|\psi_m\|_{L^\infty(D)} - \sum_{m \geq J_0} \frac{a_{\min} \nu_m}{4|\nu_F| \|\psi_m\|_{L^\infty(D)}} \|\psi_m\|_{L^\infty(D)}. \end{aligned}$$

With the choice $a_{\min} = (\text{ess inf}_{x \in D} \bar{a}(x)) / (1 + \kappa)$, we have

$$\Re a \geq a_{\min} - \frac{a_{\min}}{8} - \frac{a_{\min}}{24} - \frac{a_{\min}}{4} > \frac{a_{\min}}{2}.$$

For each $z \in \mathcal{U}$, problem (3.5) thus has a unique solution; the proof of the a priori bound (3.6) is standard. $\square$

With each index $\nu \in \mathcal{F}$, we associate finite dimensional polydiscs

$$(3.7) \quad \begin{aligned} \mathcal{U}_\nu &:= \bigotimes_{m \in \text{supp}(\nu)} \mathcal{U}_m, \quad \text{where } \mathcal{U}_m \text{ is as in (3.3) and where} \\ \text{supp}(\nu) &:= \{j \in \mathbb{N} : \nu_j \neq 0\}. \end{aligned}$$

PROPOSITION 3.1. *For $\nu \in \mathcal{F}$ and $z \in \mathcal{U}$ with fixed z_k for all indices $k \notin \text{supp}(\nu)$, the map $u : \mathcal{U}_\nu \rightarrow \mathcal{X}$ is an analytic function taking values in the function space $\mathcal{X}$.*

Proof. For a fixed index $m \in \mathbb{N}$, we fix all coordinates z_k with $k \neq m$, and partition $z \in \mathbb{C}^{\mathbb{N}}$ as $z = (z_m, \bar{z}_m)$. Let $\delta \in \mathbb{C}$ with $|\delta| > 0$ sufficiently small. We show that the function $u(\cdot, \cdot, z)$ is strongly holomorphic in $\mathcal{U}$ as a $\mathcal{X}$ -valued function. To this end, it suffices to show complex differentiability, i.e., that there exists a function $v \in \mathcal{X}$ such that the difference quotient, with (complex) stepsize δ ,

$$\lim_{\delta \rightarrow 0} \left\| \frac{u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z)}{\delta} - v(\cdot, \cdot, z) \right\|_{\mathcal{X}} = 0$$

for all $z \in \mathcal{U}$.

As $\mathcal{U} \subset \mathbb{C}^{\mathbb{N}}$ is open, for $\delta \in \mathbb{C}$ with $|\delta| > 0$ sufficiently small, $(z_m + \delta, \bar{z}_m) \in \mathcal{U}$. For such δ , the difference quotient $v^\delta = \delta^{-1}(u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z_m, \bar{z}_m))$ is then well-defined by Lemma 3.2. By superposition it follows that the function v^δ is the unique solution of the parametric parabolic problem

$$\frac{\partial v^\delta}{\partial t} - \nabla \cdot (a(\cdot, z) \nabla v^\delta) = \nabla \cdot (\psi_m \nabla u(\cdot, \cdot, z_m + \delta, \bar{z}_m))$$

with the initial condition $v^\delta(0, \cdot, z_m + \delta, \bar{z}_m) = 0$. Let v be the weak solution of the equation

$$\frac{\partial v}{\partial t} - \nabla \cdot (a(\cdot, z) \nabla v) = \nabla \cdot (\psi_m \nabla u(\cdot, \cdot, z)),$$

with $v(0, \cdot, z) = 0$. Then

$$\frac{\partial(v^\delta - v)}{\partial t} - \nabla \cdot (a(\cdot, z) \nabla (v^\delta - v)) = \nabla \cdot (\psi_m \nabla (u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z))).$$

An argument similar to the proof of Proposition 1.3 shows that

$$\|u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z)\|_{\mathcal{X}} \leq C|\delta|.$$

Therefore,

$$\|\nabla \cdot (\psi_m \nabla (u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z)))\|_{L^2(I; V')} \leq c|\delta|.$$

Standard estimations for parabolic equations show that

$$\|v^\delta - v\|_{\mathcal{X}} \leq c|\delta|.$$

Thus v is the complex derivative of u with respect to z_m as a $\mathcal{X}$ -valued function. We conclude that for every $\nu \in \mathcal{F}$, the function u is strongly holomorphic as a function from $\mathcal{U}_\nu$ to $\mathcal{X}$. $\square$

3.2. Coefficient estimates. To analyze the p -summability of the coefficients u_ν in the gpc expansion (2.2), we require estimates on the norms of the Legendre coefficients in terms of size of the ψ_m .

PROPOSITION 3.2. *The following estimate holds:*

$$(3.8) \quad \|u_\nu\|_{\mathcal{X}} \leq C \left(\prod_{m \in \text{supp}(\nu)} \frac{2(1+K)}{K} \eta_m^{-\nu_m} \right),$$

where $\eta_m := r_m + \sqrt{1 + r_m^2}$ with r_m as in (3.2).

Proof. The proof follows that of Lemma A.3 in Bieri, Andreev, and Schwab [3]. For $\nu \in \mathcal{F}$, define $u_\nu \in \mathcal{X}$ by

$$(3.9) \quad u_\nu = \int_U u(y) L_\nu(y) \rho(dy),$$

where the integral is understood as the Bochner integral of $\mathcal{X}$ -valued functions. Let $S = \text{supp}(\nu)$ and $\bar{S} = \mathbb{N} \setminus S$. We then denote $\mathcal{U}_S = \otimes_{m \in S} \mathcal{U}_m$ and $\mathcal{U}_{\bar{S}} = \otimes_{m \in \bar{S}} \mathcal{U}_m$, and by $y_S = \{y_i, i \in S\}$ the extraction from y . Let $\mathcal{E}_m$ be the ellipse in $\mathcal{U}_m$ with foci at ± 1 and the sum of the semiaxes being η_m , and $\mathcal{E}_S = \prod_{m \in \text{supp}(\nu)} \mathcal{E}_m$. We can then write (3.9) as

$$u_\nu = \frac{1}{(2\pi i)^{|\nu|_0}} \int_U L_\nu(y) \oint_{\mathcal{E}_S} \frac{u(z_S, y_{\bar{S}})}{(z_S - y_S)^{\mathbf{1}}} dz_S d\rho(y).$$

For each $m \in \mathbb{N}$, let Γ_m be a copy of $[-1, 1]$ and $y_m \in \Gamma_m$. We denote $U_S = \prod_{m \in S} \Gamma_m$ and $U_{\bar{S}} = \prod_{m \in \bar{S}} \Gamma_m$. We then have

$$u_\nu = \frac{1}{(2\pi i)^{|\nu|_0}} \int_{U_S} \oint_{\mathcal{E}_S} u(z_S, y_{\bar{S}}) \int_{U_{\bar{S}}} \frac{L_\nu(y)}{(z_S - y_S)^{\mathbf{1}}} d\rho_S(y_S) dz_S d\rho_{\bar{S}}(y_{\bar{S}}).$$

To proceed further, we recall the definitions of the Legendre functions of the second kind,

$$Q_n(z) = \int_{[-1,1]} \frac{L_n(t)}{(z-t)} d\rho(t).$$

Let ν_S be the restriction of ν to S . We define

$$\mathcal{Q}_{\nu_S}(z_S) = \prod_{m \in \text{supp}(\nu)} Q_{\nu_m}(z_m).$$

Making the Joukovski transformation $z_m = \frac{1}{2}(w_m + w_m^{-1})$, the Legendre polynomials of the second kind are written as

$$Q_{\nu_m} \left(\frac{1}{2}(w_m + w_m^{-1}) \right) = \sum_{k=\nu_m+1}^{\infty} \frac{q_{\nu_m k}}{w_m^k}$$

with $|q_{\nu_m k}| \leq \pi$. Therefore

$$|\mathcal{Q}_{\nu_S}(z_S)| \leq \prod_{m \in S} \sum_{k=\nu_m+1}^{\infty} \frac{\pi}{\eta_m^k} = \prod_{m \in S} \pi \frac{\eta_m^{-\nu_m-1}}{1 - \eta_m^{-1}}.$$

We then have

$$\begin{aligned} \|u_\nu\|_{\mathcal{X}} &= \left\| \frac{1}{(2\pi i)^{|\nu|_0}} \int_{U_S} \oint_{\mathcal{E}_S} u(z_S, y_{\bar{S}}) \mathcal{Q}_{\nu_S}(z_S) dz_S d\rho_{\bar{S}}(y_S) \right\|_{\mathcal{X}} \\ &\leq \frac{1}{(2\pi)^{|\nu|_0}} \int_{U_S} \oint_{\mathcal{E}_S} \|u(z_S, y_{\bar{S}})\|_{\mathcal{X}} \mathcal{Q}_{\nu_S}(z_S) dz_S d\rho_{\bar{S}}(y_S) \\ &\leq \frac{1}{(2\pi)^{|\nu|_0}} \|u(z)\|_{L^\infty(\mathcal{E}_S \times U_S, \mathcal{X})} \max_{\mathcal{E}_S} |\mathcal{Q}_{\nu_S}| \prod_{m \in S} \text{Len}(\mathcal{E}_m) \\ &\leq \frac{1}{(2\pi)^{|\nu|_0}} \|u(z)\|_{L^\infty(\mathcal{E}_S \times \mathcal{U}_S, \mathcal{X})} \prod_{m \in S} \pi \frac{\eta_m^{-\nu_m-1}}{1 - \eta_m^{-1}} \text{Len}(\mathcal{E}_m) \\ &\leq C \prod_{m \in S} \frac{2(1+K)}{K} \eta_m^{-\nu_m}, \end{aligned}$$

as $\text{Len}(\mathcal{E}_m) \leq 4\eta_m$, $\eta_m \geq 1+K$, and $u(z)$ is uniformly bounded in $\mathcal{X}$. $\square$

To show the $\ell^p(\mathcal{F})$ summability of $\|u_\nu\|_{\mathcal{X}}$, we use the following proposition from [4].

PROPOSITION 3.3. *Consider a sequence $b = (b_m)_{m \in \mathbb{N}}$, where $b_m \geq 0$. Then, for $0 < p < 1$,*

$$\left(\frac{|\nu|!}{\nu!} b^\nu \right)_{\nu \in \mathcal{F}} \in \ell^p(\mathcal{F}) \text{ iff (i) } \sum_{m \geq 1} b_m < 1 \text{ and (ii) } b \in \ell^p(\mathbb{N}).$$

PROPOSITION 3.4. *If Assumption 3.1 holds for some $0 < p < 1$, $\sum_{\nu \in \mathcal{F}} \|u_\nu\|_{\mathcal{X}}^p < \infty$ for this p .*

Proof. We have from the previous proposition that

$$\begin{aligned} \|u_\nu\|_{\mathcal{X}} &\leq C \prod_{m \in S} \frac{2(1+K)}{K} (1+r_m)^{-\nu_m} \\ &\leq C \left(\prod_{m \in E, \nu_m \neq 0} \frac{2(1+K)}{K} \eta^{\nu_m} \right) \left(\prod_{m \in F, \nu_m \neq 0} \frac{2(1+K)}{K} \left(\frac{4|\nu_F| \|\psi_m\|_{L^\infty(D)}}{a_{\min} \nu_m} \right)^{\nu_m} \right), \end{aligned}$$

where $\eta = 1/(1+K)$. Let $\mathcal{F}_E = \{\nu \in \mathcal{F} : \text{supp}(\nu) \subset E\}$ and $\mathcal{F}_F = \mathcal{F} \setminus E$. From this, we have

$$\sum_{\nu \in \mathcal{F}} \|u_\nu\|_{\mathcal{X}}^p \leq C A_E A_F,$$

where

$$A_E = \sum_{\nu \in \mathcal{F}_E} \prod_{m \in E, \nu_m \neq 0} \left(\frac{2(1+K)}{K} \right)^p \eta^{p\nu_m}$$

and

$$A_F = \sum_{\nu \in \mathcal{F}_F} \prod_{m \in F, \nu_m \neq 0} \left(\frac{2(1+K)}{K} \right)^p \left(\frac{4|\nu| \|\psi_m\|_{L^\infty(D)}}{a_{\min} \nu_m} \right)^{p\nu_m}.$$

We now show that both A_E and A_F are finite. For A_E , we have

$$A_E = \left(1 + \left(\frac{2(1+K)}{K} \right)^p \sum_{m \geq 1} \eta^{pm} \right)^{J_0},$$

which is finite because $\eta < 1$. For A_F , we note that for $\nu_m \neq 0$,

$$\frac{2(1+K)}{K} \leq \left(\frac{2(1+K)}{K} \right)^{\nu_m}.$$

Therefore

$$A_F \leq \sum_{\nu \in \mathcal{F}_F} \prod_{m \in F} \left(\frac{|\nu| d_m}{\nu_m} \right)^{p\nu_m},$$

where

$$d_m = \frac{8(1+K) \|\psi_m\|_{L^\infty(D)}}{K a_{\min}};$$

we use the convention that $0^0 = 1$. We now proceed as in [5]: from the Stirling estimate

$$\frac{n!e^n}{e\sqrt{n}} \leq n^n \leq \frac{n!e^n}{\sqrt{2\pi n}},$$

we infer $|\nu|^{|\nu|} \leq |\nu|! e^{|\nu|}$ and obtain

$$\prod_{m \in F} \nu_m^{\nu_m} \geq \frac{|\nu|! e^{|\nu|}}{\prod_{m \in F} \max\{1, e\sqrt{\nu_m}\}}.$$

Hence

$$A_F \leq \sum_{\nu \in \mathcal{F}_F} \left(\frac{|\nu|!}{\nu!} d^\nu \right)^p \left(\prod_{m \in F} \max\{1, e\sqrt{\nu_m}\} \right)^p \leq \sum_{\nu \in \mathcal{F}_F} \left(\frac{|\nu|!}{\nu!} \bar{d}^\nu \right)^p,$$

where $\bar{d}_m = ed_m$ and where and we have used the estimate $e\sqrt{n} \leq e^n$. From this, we have

$$\sum_{m \geq 1} \bar{d}_m \leq \sum_{m \in F} \frac{24(1+K)\|\psi_m\|_{L^\infty(D)}}{Ka_{\min}} \leq 1.$$

It is also obvious that

$$\|\bar{d}\|_{l^p(\mathbb{N})} < \infty.$$

From these estimates and from Proposition 3.3 we obtain the conclusion. $\square$

3.3. Best N -term convergence rates. With Lemma 3.1, we have from Proposition 3.4 and Theorem 2.1 the following best N -term approximation result.

THEOREM 3.3. *If Assumptions 1.1, 1.2, and 3.1 hold for some $0 < p < 1$, there is a sequence $(\Lambda_N)_{N \in \mathbb{N}} \subset \mathcal{F}$ of index sets with cardinality not exceeding N such that the solutions u_{Λ_N} of the Galerkin semidiscretized problems (2.3) satisfy*

$$\|u - u_{\Lambda_N}\|_{\mathcal{X}} \leq CN^{-\sigma}, \quad \sigma = \frac{1}{p} - \frac{1}{2}.$$

4. Space-time regularity of the parametric solutions. To obtain error bounds for space-time discretization schemes of the parametric equations, space-time regularity for the solution u of the parametric problem (1.15) is required. We will establish this in the following sections.

4.1. Compatible initial conditions. In this section, we derive a regularity estimate for the parametric solution in the case of a compatible initial condition h of the problem (1.1). To establish this, we first consider *compatible initial conditions*, in which case we assume that

$$(4.1) \quad h \in V \cap H^2(D), \quad g \in L^2(I; H) \cap H^1(I; V'), \quad g(0, \cdot) \in H.$$

Throughout this section, Assumptions 1.1, 1.2, and (4.1) are assumed to hold. Moreover, we impose an additional regularity assumption on the functions ψ_j .

Assumption 4.1. We assume that $\bar{a}(x) \in W^{1,\infty}(D)$ and that for $0 < p < 1$ as in Assumption 3.1

$$\sum_{j=1}^{\infty} \|\psi_j\|_{W^{1,\infty}(D)}^p < \infty.$$

We remark that in the ensuing arguments, we might weaken Assumption 4.1 to

$$\sum_{j=1}^{\infty} \|\psi_j\|_{W^{1,\infty}(D)}^q < \infty$$

for some $0 < p < q < 1$ which would imply corresponding weaker summability statements, of course. For simplicity, we do not elaborate on this and work under Assumption 4.1. With Assumption 4.1, we establish regularity for the solution u of (1.10) and the functions u_ν . We first show that under the compatibility conditions and Assumption 4.1, for almost every parameter vector y , the solution u belongs to the space

$$(4.2) \quad \mathcal{Z} = L^2(I; H^2(D)) \cap H^1(I; V) \cap H^2(I; V')$$

equipped with the norm

$$\|\circ\|_{\mathcal{Z}} = \left(\|\circ\|_{L^2(I;H^2(D))}^2 + \|\circ\|_{H^1(I;V)}^2 + \|\circ\|_{H^2(I;V')}^2 \right)^{1/2}.$$

PROPOSITION 4.1. *Assume that the domain D is convex. With the condition (4.1), under Assumption 4.1 it holds that $u(\cdot, \cdot, y) \in \mathcal{Z}$ for all $y \in U$ and $\|u\|_{\mathcal{Z}}$ is uniformly bounded for all $y \in U$.*

Proof. We note that due to the compatibility conditions (4.1), for every $y \in U$

$$\nabla \cdot (a(\cdot, y) \nabla h) = \nabla \cdot (\bar{a}(\cdot) \nabla h) + \sum_{j=1}^{\infty} y_j (\psi_j \Delta h + \nabla \psi_j \cdot \nabla h) \in H.$$

Following the standard results on regularity of the solutions for parabolic equations (as, e.g., in [15, Thm. 27.4]), we deduce that $u(\cdot, \cdot, y) \in H^1(I; V) \cap H^2(I; V')$. From

$$-\Delta u = \frac{1}{a} \left[\nabla a \cdot \nabla u + g - \frac{\partial u}{\partial t} \right] \in L^2(I; H)$$

and the convexity of D , we deduce that $u(\cdot, \cdot, y)$ is uniformly bounded in $L^2(I; H^2(D))$ with respect to y . $\square$

The following existence and uniqueness results for u in the norm of $\mathcal{Z}$ are parametric analogues to those in the norm of $\mathcal{X}$ in section 1.3.

PROPOSITION 4.2. *There exists a constant $C > 0$ such that*

$$\forall y, y' \in U, \quad \|u(\cdot, \cdot, y) - u(\cdot, \cdot, y')\|_{\mathcal{Z}} \leq C \|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)}.$$

Proof. For $y, y' \in U$, we define $w(t, x) = u(t, x, y) - u(t, x, y')$. Since, by assumption, $a(\cdot, y) \in W^{1,\infty}(D)$ for every $y \in U$, the function w is the solution of the parabolic problem

$$\begin{aligned} \frac{\partial w}{\partial t} - \nabla \cdot (a(x, y) \nabla w) &= \nabla \cdot (a(\cdot, y) - a(\cdot, y')) \cdot \nabla u(\cdot, t, y') \\ &\quad + (a(\cdot, y) - a(\cdot, y')) \Delta u(\cdot, t, y'), \\ w(t, x) &= 0 \text{ when } x \in \partial D, t > 0, \\ w(0, x) &= 0. \end{aligned}$$

In particular, the data g and h of this problem satisfy (4.1). We observe that for every $y' \in U$ we have $u(\cdot, \cdot, y') \in \mathcal{Z}$, and it holds that

$$\begin{aligned} \|\nabla(a(\cdot, y) - a(\cdot, y')) \cdot \nabla u(\cdot, \cdot, y') + (a(\cdot, y) - a(\cdot, y')) \Delta u(\cdot, \cdot, y')\|_{L^2(I; H) \cap H^1(I; V')} \\ \leq C \|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)}. \end{aligned}$$

The conclusion then follows. $\square$

PROPOSITION 4.3. *The map $u(\cdot, \cdot, y) : U \rightarrow \mathcal{Z}$ is strongly measurable as a Bochner function taking values in $\mathcal{Z}$.*

Proof. The proof of this proposition follows the lines of the proof of Proposition 1.4 except that here we use Proposition 4.2 in place of Proposition 1.3. $\square$

The above results show that $u \in L^2(U, \rho; \mathcal{Z})$. Therefore, for every $\nu \in \mathcal{F}$,

$$u_\nu = \int_U L_\nu(y) u(y) d\rho(y) \in \mathcal{Z}.$$

Next we establish a priori bounds for $\|u_\nu\|_{\mathcal{Z}}$. Let $\bar{K} < 1$ denote a positive number such that

$$\bar{K} \sum_{j=1}^{\infty} (\|\psi_j\|_{L^\infty(D)} + \|\nabla \psi_j\|_{L^\infty(D)}) < \frac{a_{\min}}{8}.$$

We again choose an integer $\bar{J}_0 \geq 1$ such that

$$\sum_{j > \bar{J}_0} (\|\psi_j\|_{L^\infty(D)} + \|\nabla \psi_j\|_{L^\infty(D)}) < \frac{a_{\min} \bar{K}}{24(1 + \bar{K})}.$$

Let $\bar{E} = \{1, 2, \dots, \bar{J}_0\}$ and $\bar{F} = \mathbb{N} \setminus \bar{E}$. We define

$$|\nu_{\bar{F}}| = \sum_{j > \bar{J}_0} |\nu_j|.$$

For each $\nu \in \mathcal{F}$ we define as before $\bar{r}_m = \bar{K}$ for $m \leq \bar{J}_0$ and for $m > \bar{J}_0$

$$(4.3) \quad \bar{r}_m = 1 + \frac{a_{\min} \nu_m}{4|\nu_{\bar{F}}|(\|\psi_m\|_{L^\infty(D)} + \|\nabla \psi_m\|_{L^\infty(D)})}$$

with the convention that $\frac{|\nu_j|}{|\nu_{\bar{F}}|} = 0$ if $|\nu_{\bar{F}}| = 0$. We then define the discs

$$(4.4) \quad \bar{\mathcal{U}}_m = \{z_m \in \mathbb{C} : |z_m| < 1 + \bar{r}_m\} \subset \mathbb{C}, \quad m \in \mathbb{N}$$

and the polydisc

$$(4.5) \quad \bar{\mathcal{U}} = \bigotimes_{m=1}^{\infty} \bar{\mathcal{U}}_m \subset \mathbb{C}^{\mathbb{N}}.$$

For $z = (z_m)_{m \geq 1} \in \bar{\mathcal{U}}$, we define

$$a(\cdot, z) := a_0 + \sum_{m=1}^{\infty} z_m \psi_m.$$

The sum converges in $L^\infty(D)$, uniformly in the parameter: for $z \in \bar{\mathcal{U}}$ and for a.e. $x \in D$ it holds

$$\begin{aligned} |a(x, z)| &\leq \max \bar{a}(x) + \sum_{m=1}^{\infty} \|\psi_m\|_{L^\infty(D)} (1 + \bar{r}_m) \\ &\leq \|\bar{a}\|_{L^\infty(D)} + 2 \sum_{m=1}^{\infty} \|\psi_m\|_{L^\infty(D)} + \frac{a_{\min}}{4}. \end{aligned}$$

This is proved as the analogous inequality in section 3.1. For $z \in \bar{\mathcal{U}}$, we again consider the problem (3.5).

PROPOSITION 4.4. *The problem (3.5) admits a unique solution, which is uniformly bounded in $\mathcal{Z}$ for all $z \in \bar{\mathcal{U}}$.*

Proof. For $z \in \bar{\mathcal{U}}$, by an argument analogous to what we did in section 3.1, we estimate

$$\Re a \geq \operatorname{ess\,inf}_{x \in D} a(x) - \sum_{m=1}^{\infty} \|\psi_m\|_{L^\infty(D)} (1 + \bar{r}_m) \geq \frac{a_{\min}}{2}.$$

The problem (3.5) thus has a unique solution under Assumptions 1.1, 1.2, 4.1 and (4.1). Furthermore for every $z \in \bar{\mathcal{U}}$ and a.e. $x \in D$,

$$\begin{aligned} |\nabla a(x, z)| &\leq \|\nabla \bar{a}\|_{L^\infty(D)} + \sum_{m=1}^{\infty} (1 + \bar{r}_m) \|\nabla \psi_m\|_{L^\infty(D)} \\ &\leq \|\nabla \bar{a}\|_{L^\infty(D)} + \sum_{m=1}^{\bar{J}_0} \|\nabla \psi_m\|_{L^\infty(D)} (1 + \bar{K}) \\ &\quad + \sum_{j > \bar{J}_0} \left(2 + \frac{a_{\min} \nu_j}{4 |\nu_F| \|\psi_j\|_{L^\infty(D)}} \right) \|\nabla \psi_j\|_{L^\infty(D)} \\ &\leq \|\nabla \bar{a}\|_{L^\infty(D)} + 2 \sum_{m=1}^{\infty} \|\nabla \psi_m\|_{L^\infty(D)} + \frac{a_{\min}}{4}. \end{aligned}$$

Therefore

$$g(0, \cdot) - \nabla \cdot (a(\cdot, z) \nabla h(\cdot)) \in H$$

for all $z \in \bar{\mathcal{U}}$ and its H norm is uniformly bounded. For $z \in \bar{\mathcal{U}}$ the solution of (3.5) is thus uniformly bounded in $\mathcal{Z}$. $\square$

For each index $\nu \in \mathcal{F}$ we define the polydiscs

$$(4.6) \quad \bar{\mathcal{U}}_\nu := \bigotimes_{j \in \text{supp}(\nu)} \bar{\mathcal{U}}_j.$$

PROPOSITION 4.5. *For $\nu \in \mathcal{F}$, fixing z_k where $k \notin \text{supp}(\nu)$, the map $u : \bar{\mathcal{U}}_\nu \rightarrow \mathcal{Z}$ is analytic as a $\mathcal{Z}$ -valued function.*

Proof. For $\nu \in \mathcal{F}$ and for an arbitrary, fixed index $m \in \text{supp}(\nu)$, we fix all coordinates z_k when $k \neq m$, and denote $z \in \bar{\mathcal{U}}_\nu \subset \mathbb{C}^{\mathbb{N}}$ as $z = (z_m, \bar{z}_m)$. Since $\bar{\mathcal{U}}_\nu$ is open, $(z_m + \delta, \bar{z}_m)$ belongs to $\bar{\mathcal{U}}_\nu$ for $\delta \in \mathbb{C}$ with $|\delta| > 0$ sufficiently small. We establish strong holomorphy of the parametric solution $u(\cdot, \cdot, z)$ as a Bochner function taking values in $\mathcal{Z}$ for $z \in \bar{\mathcal{U}}_\nu$ by verifying complex differentiability of it. To this end, we show that there exists a function $v \in \mathcal{Z}$ such that

$$\lim_{\delta \rightarrow 0} \left\| \frac{u(\cdot, \cdot, (z_m + \delta, \bar{z}_m)) - u(\cdot, \cdot, z)}{\delta} - v(\cdot, \cdot, z) \right\|_{\mathcal{Z}} = 0,$$

for all $z \in \mathcal{U}$. Let

$$v^\delta = \frac{u(\cdot, \cdot, (z_m + \delta, \bar{z}_m)) - u(\cdot, \cdot, (z_m, \bar{z}_m))}{\delta}.$$

By superposition, the function v^δ satisfies

$$\frac{\partial v^\delta}{\partial t} - \nabla \cdot (a(\cdot, z) \nabla v^\delta) = \nabla \cdot (\psi_m \nabla u(\cdot, \cdot, (z_m + \delta, \bar{z}_m))),$$

with the initial condition $v^\delta(0, \cdot, (z_m + \delta, \bar{z}_m)) = 0$. Let v denote the solution of the equation

$$\frac{\partial v}{\partial t} - \nabla \cdot (a(\cdot, z) \nabla v) = \nabla \cdot (\psi_m \nabla u(\cdot, \cdot, z)),$$

with $v(0, \cdot, z) = 0$. Then

$$\frac{\partial(v^\delta - v)}{\partial t} - \nabla \cdot (a(\cdot, z) \nabla (v^\delta - v)) = \nabla \cdot (\psi_m \nabla (u(\cdot, \cdot, (z_m + \delta, \bar{z}_m)) - u(\cdot, \cdot, z))).$$

An argument similar to the proof of Proposition 4.2 shows that

$$\|u(\cdot, \cdot, (z_m + \delta, \bar{z}_m)) - u(\cdot, \cdot, z)\|_{\mathcal{Z}} \leq C|\delta|.$$

Therefore

$$\|\nabla \cdot (\psi_m \nabla (u(\cdot, \cdot, (z_m + \delta, \bar{z}_m)) - u(\cdot, \cdot, z)))\|_{L^2(I; H) \cap H^1(I; V')} \leq C|\delta|.$$

Further,

$$\nabla \cdot (\psi_m \nabla (u(\cdot, \cdot, (z_m + \delta, \bar{z}_m)) - u(\cdot, \cdot, z))) = 0$$

at $t = 0$. Thus

$$\|v^\delta - v\|_{\mathcal{Z}} \leq C|\delta|.$$

This shows that v is the derivative of u as a $\mathcal{Z}$ -valued function. From Hartogs' theorem, we conclude that u is analytic as a function from $\bar{U}_\nu$ to $\mathcal{Z}$. $\square$

From the preceding arguments, we deduce the following result. Its proof is similar to that of Propositions 3.2 and 3.4.

THEOREM 4.1. *Assume that the domain D is convex. Under Assumptions 1.1, 1.2, 4.1, and for data g and h satisfying the compatibility conditions (4.1), the coefficients $u_\nu \in \mathcal{Z}$ satisfy the estimates*

$$(4.7) \quad \|u_\nu\|_{\mathcal{Z}} \leq C \prod_{m \in \text{supp}(\nu)} \frac{2(1 + \bar{K})}{\bar{K}} \bar{\eta}_m^{-\nu_m}, \quad \nu \in \mathcal{F},$$

where $\bar{\eta}_m = \bar{r}_m + \sqrt{1 + \bar{r}_m^2}$. Moreover, for the same constant $0 < p < 1$ as in Assumption 4.1, $\sum_{\nu \in \mathcal{F}} \|u_\nu\|_{\mathcal{Z}}^p$ is finite.

4.2. Incompatible initial conditions. We now consider the case where the initial condition does not satisfy the compatibility condition (4.1). We define the following spaces by interpolation: for $0 < \theta < 1$,

$$H^\theta = (H, V)_{\theta, 2}, \quad H^{1+\theta} = (V, V \cap H^2(D))_{\theta, 2}, \quad H^{-1+\theta} = (V', H)_{\theta, 2}.$$

We assume throughout this section that Assumption 4.1 holds. For each $y \in U$, we consider the parametric eigenvalue problem: find $\lambda(y) \in \mathbb{R}$ and $0 \neq \phi(y) \in V$ such that for all $\psi \in V$ it holds

$$(4.8) \quad \int_D a(x, y) \nabla \phi(y) \cdot \nabla \psi dx = \lambda(y) \int_D \phi(y) \psi dx \quad \forall \psi \in V.$$

By the spectral theorem, for every $y \in U$ there exists a countable family $(\lambda_i(y), \phi_i(y))_{i=1}^\infty$ of eigenpairs such that $\phi_i(y) \in V$ are an orthonormal basis of H . From the Poincaré inequality, $\lambda_i(y) \geq \bar{\lambda}$ for a positive constant $\bar{\lambda}$ which only depends on the constant $a_{\min}$ in Assumption 1.1 and the domain D , and is independent of y .

For each $t \in I$ and every $y \ni U$, we define the parametric evolution operator $T(y; t)$ in terms of the eigenfunctions $\phi_i(y)$ by

$$(4.9) \quad T(y; t)u = \sum_{i=1}^{\infty} e^{-\lambda_i(y)t} \langle u, \phi_i(y) \rangle_H \phi_i(y).$$

The solution of the parametric problem (1.10) can be represented as

$$(4.10) \quad u(t, \cdot, y) = T(y; t)h + \int_0^t T(y; t-s)g(s)ds, \quad 0 \leq t \leq T.$$

In this section, we assume $g : I \rightarrow H$ is strongly holomorphic on $[0, T]$ as an H -valued mapping.

PROPOSITION 4.6. *Assume that the domain D is convex. For $0 \leq \theta \leq 1$ and $0 \leq s \leq 1$, for every $l \in \mathbb{N}$ there exists $c > 0$ such that for every $y \in U$ and every $0 < t \leq T$ it holds*

$$\|T^{(l)}(y; t)\|_{H^\theta, H^{\pm 1+s}} \leq ct^{-l \mp 1/2 + \theta/2 - s/2}.$$

Proof. For every $y \in U$, we define on V the norm

$$\|\phi\|_{y;E}^2 = \int_D a(x, y) |\nabla \phi(x, y)|^2 dx, \quad y \in U.$$

As $a(x, y)$ satisfies Assumption 1.1, this is an equivalent norm in V , uniformly with respect to $y \in U$. Specifically, there are positive constants c_1 and c_2 which are independent of $y \in U$ such that

$$(4.11) \quad c_1 \|\cdot\|_V \leq \|\cdot\|_E \leq c_2 \|\cdot\|_V.$$

For $v \in H$ and every $y \in U$ and for every $i \in \mathbb{N}$, define $v_i(y) := \langle v, \phi_i(y) \rangle_H$. For all $y \in U$, we have

$$\|v\|_H^2 = \sum_{i=1}^{\infty} (v_i(y))^2.$$

From (4.9), we obtain

$$\|T^{(l)}(y; t)v\|_{y;E}^2 = \sum_{i=1}^{\infty} \lambda_i(y)^{2l+1} e^{-2\lambda_i(y)t} (v_i(y))^2.$$

For every $l \in \mathbb{N}$, the maximum value with respect to λ of $e^{-2\lambda t} \lambda^{2l+1}$ is bounded by $C(l)t^{-(2l+1)}$, $y \in U$. Thus we obtain that for every $l \in \mathbb{N}$, there exists $C(l) > 0$ such that for every $t > 0$ and $y \in U$ it holds

$$\|T^{(l)}(y; t)v\|_{y;E}^2 \leq C(l)t^{-(2l+1)} \sum_{i=1}^{\infty} (v_i(y))^2 = C(l)t^{-(2l+1)} \|v\|_H^2.$$

Therefore, there exists $c > 0$ such that for every $t > 0$ it holds

$$(4.12) \quad \|T^{(l)}(y; t)\|_{\mathcal{L}(H, V)} \leq ct^{-l-1/2}.$$

For $v \in V$ and for $y \in U$, we define

$$\|v\|_{y;E}^2 = \sum_{i=1}^{\infty} (v_i(y))^2 \lambda_i(y) .$$

For every $l \in \mathbb{N}$, there exists a constant $C(l) > 0$ such that for every $t > 0$, the supremum $\sup_{\lambda > 0} e^{-2\lambda t} \lambda^{2l}$ is $C(l)t^{-2l}$. This and (4.11) imply for every $l \in \mathbb{N}$ the uniform bound (w.r.t. $t > 0$ and $y \in U$)

$$(4.13) \quad \|T^{(l)}(y; t)\|_{\mathcal{L}(V,V)} \leq C(l)t^{-l} .$$

Similarly, by Parseval's equality for every $y \in U$ and every $t > 0$ there holds

$$\|T^{(l)}(y; t)v\|_H^2 = \sum_{i=1}^{\infty} \lambda_i^{2l} e^{-2\lambda_i t} v_i^2 .$$

Therefore, we also have

$$(4.14) \quad \|T^{(l)}(y; t)\|_{\mathcal{L}(H,H)} \leq C(l)t^{-l} .$$

Further, by Parseval's equality for every $y \in U$ and every $t > 0$

$$\|T^{(l)}(y; t)v\|_H^2 = \sum_{i=1}^{\infty} \lambda_i(y)^{2l-1} e^{-2\lambda_i t} v_i^2 \lambda_i(y) .$$

A similar argument shows that for every $l \geq 1$

$$(4.15) \quad \|T^{(l)}\|_{\mathcal{L}(V,H)} \leq C(l)t^{-l+1/2} .$$

We observe that for every $y \in U$, the function $w = T(y; t)v$ is the solution of the Cauchy problem

$$\frac{\partial w}{\partial t} - \nabla \cdot (a(x, y) \nabla w) = 0, \quad w(0, \cdot) = v .$$

Therefore, for every $y \in U$ and for every $l \in \mathbb{N}$,

$$w^{(l+1)}(t, \cdot) - \nabla \cdot (a(\cdot, y) \nabla w^{(l)}(t, \cdot)) = 0 ,$$

i.e.,

$$-\Delta w^{(l)}(t, \cdot) = -\frac{1}{a} w^{(l+1)}(t, \cdot) + \frac{1}{a} \nabla a \cdot \nabla w^{(l)}(t, \cdot) .$$

As the domain D was assumed to be convex, the H^2 regularity of the solution to the Poisson equation implies

$$\|w^{(l)}(t, \cdot)\|_{H^2(D)} \leq c(\|w^{(l+1)}(t, \cdot)\|_H + \|w^{(l)}(t, \cdot)\|_V)$$

which is bounded by $C(l)t^{-(l+1)}\|v\|_H$ and $C(l)t^{-(l+1/2)}\|v\|_V$. Thus, for every $l \in \mathbb{N}$, there exists $C(l) > 0$ such that for every $y \in U$ and $t > 0$ there holds

$$(4.16) \quad \|T^{(l)}(y; t)\|_{\mathcal{L}(H,H^2(D))} \leq C(l)t^{-(l+1)}$$

and

$$(4.17) \quad \|T^{(l)}(y; t)\|_{\mathcal{L}(V, H^2(D))} \leq C(l)t^{-(l+1/2)}.$$

From interpolation of Hilbert spaces using the real method (see, e.g., [14]), we deduce from (4.12), (4.13), (4.16), and (4.17) that for every $l \in \mathbb{N}$, there exists a constant $C(l) > 0$ such that

$$(4.18) \quad \forall y \in U, t > 0, \quad \|T^{(l)}(y; t)\|_{\mathcal{L}(H^\theta, H^{1+s})} \leq C(l)t^{-l-1/2+\theta/2-s/2}.$$

Now we consider $\|T^{(l)}(y; t)v\|_{V'}$. We have (with $\langle \cdot, \cdot \rangle$ denoting the $V \times V'$ duality pairing obtained by extending the H -inner product by continuity)

$$\begin{aligned} \|T^{(l)}(y; t)v\|_{V'} &= \sup_{\psi \in V} \langle T^{(l)}(y; t)v, \psi \rangle / \|\psi\|_V \\ &= \sup_{\psi \in V} a \left(\sum_{i=1}^{\infty} (-1)^l e^{-\lambda_i(y)t} (\lambda_i(y))^{l-1} v_i(y) \phi_i(y), \psi \right) / \|\psi\|_V \\ &\leq C \left(\sum_{i=1}^{\infty} (\lambda_i(y))^{2l-1} (v_i(y))^2 e^{-2\lambda_i(y)t} \right)^{1/2}. \end{aligned}$$

Therefore for $l \geq 1$ for all $y \in U$

$$(4.19) \quad \|T^{(l)}(y; t)\|_{\mathcal{L}(H, V')} \leq C(l)t^{-l+1/2}.$$

Similarly, we have that for $l \geq 1$

$$(4.20) \quad \|T^{(l)}(y; t)\|_{\mathcal{L}(V, V')} \leq C(l)t^{-l+1}.$$

From interpolation, we get from (4.14), (4.15), (4.19), and (4.20) that for $l \geq 1$

$$(4.21) \quad \|T^{(l)}(y; t)\|_{\mathcal{L}(H^\theta, H^{-1+s})} \leq C(l)t^{-l+1/2+\theta/2-s/2}. \quad \square$$

PROPOSITION 4.7. *Assume that $h \in H^\theta$ and the domain D is convex, for some $0 < \theta < 1$. Then for every $s < \theta$*

$$u(\cdot, \cdot, y) \in L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})$$

and there exists $C_1 > 0$ independent of y such that

$$\forall y \in U, \quad \|u(\cdot, \cdot, y)\|_{L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})} \leq C_1.$$

Proof. From Proposition 4.6, we have that

$$\|T(y; t)h\|_{H^{1+s}} \leq ct^{-1/2+\theta/2-s/2}.$$

Furthermore as $\|g(s)\|_H < c$ for all s

$$\left\| \int_0^t T(y; t-s)g(s)ds \right\|_{H^{1+s}} \leq c \int_0^t (t-s)^{-1/2-s/2} ds \leq ct^{1/2-s/2}.$$

Therefore for $0 < t \leq 1$,

$$(4.22) \quad \|u(t, \cdot, \cdot)\|_{H^{1+s}} \leq ct^{-1/2+\theta/2-s/2}.$$

From (4.10),

$$u'(t, \cdot, y) = T'(y; t)h + g(t) + \int_0^t T'(y; t-s)g(s)ds.$$

From Proposition 4.6,

$$\|T'(y; t)h\|_{H^{-1+s}} \leq ct^{-1/2+\theta/2-s/2}.$$

We also have

$$\left\| \int_0^t T'(y; t-s)g(s)ds \right\|_{H^{-1+s}} \leq c \int_0^t (t-s)^{-1/2-s/2}ds \leq ct^{1/2-s/2}.$$

Therefore for $0 < t \leq 1$,

$$(4.23) \quad \|u'(t, \cdot, \cdot)\|_{H^{-1+s}} \leq ct^{-1/2+\theta/2-s/2}.$$

This completes the proof. $\square$

PROPOSITION 4.8. Assume that the domain D is convex. For $s < \theta$, there exists a constant $C > 0$ such that $\forall y, y' \in U$,

$$\|u(\cdot, \cdot, y) - u(\cdot, \cdot, y')\|_{L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})} \leq C \|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)}.$$

Proof. Define $w(t, x, y, y') = u(t, x, y) - u(t, x, y')$ and

$$(4.24) \quad G(t, x) = \nabla(a(x, y) - a(x, y')) \cdot \nabla u(x, t, y') + (a(x, y) - a(x, y'))\Delta u(t, x, y').$$

The function $G \in L^2(I; V')$. Therefore w is the weak solution of the problem

$$(4.25) \quad \frac{\partial w}{\partial t} - \nabla \cdot (a(x, y)\nabla w) = G(t, x) \quad w(t, x) = 0 \text{ when } x \in \partial D, \quad w(0, x) = 0.$$

From (4.22), we deduce that for all t and for every $y, y' \in U$ it holds

$$\|G(t, \cdot)\|_{H^{-1+s}} \leq c \|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)} t^{-1/2+\theta/2-s/2}.$$

Note that $G \notin L^2(I; H)$. However, for every $0 < t_0 < t$ and for every $y, y' \in U$ we have

$$w(t, \cdot, y, y') = T(y; t-t_0)w(t_0) + \int_{t_0}^t T(y; t-r)G(r)dr.$$

Therefore

$$\begin{aligned} \|w(t, \cdot, y, y')\|_{H^{1+s}} &\leq \|T(y; t-t_0)w(t_0)\|_{H^{1+s}} \\ &\quad + c \|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)} \int_0^t (t-r)^{-1/2-s/2} r^{-1+\theta/2} dr. \end{aligned}$$

Passing to the limit $t_0 \rightarrow 0$, as $\|w(t_0)\|_H \rightarrow 0$, from Proposition 4.6 we obtain

$$\|w(t, \cdot, y, y')\|_{H^{1+s}} \leq c \|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)} t^{-1/2+\theta/2-s/2}.$$

Similarly,

$$\begin{aligned} \|w'(t, \cdot, y, y')\|_{H^{-1+s}} &\leq \|T'(y; t-t_0)w(t_0)\|_{H^{-1+s}} + \|G(t)\|_{H^{-1+s}} \\ &\quad + c \|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)} \int_0^t (t-r)^{-1/2-s/2} r^{-1+\theta/2} dr. \end{aligned}$$

Letting $t_0 \rightarrow 0$, we get

$$\|w'(t, \cdot, y, y')\|_{H^{-1+s}} \leq c \|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)} t^{-1/2+\theta/2-s/2}.$$

This completes the proof. $\square$

PROPOSITION 4.9. *Assume that $h \in H^\theta$ and the domain D is convex. Then, for $s < \theta$, the map $u(\cdot, \cdot, y) : U \rightarrow L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})$ is strongly measurable as a Bochner function.*

Proof. The proof is by exactly the same argument as the proof of Proposition 1.4. We use Proposition 4.8 in place of Proposition 1.3. $\square$

Thus we conclude that $u(\cdot, \cdot, y) \in L^2(U, \rho; L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s}))$. Therefore the function

$$u_\nu(\cdot, \cdot) = \int_U L_\nu(y) u(\cdot, \cdot, y) d\rho(y) \in L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s}).$$

To describe the analyticity of the parametric solutions, we employ the (open) polycylinder $\bar{U} \subset \mathbb{C}^{\mathbb{N}}$ defined in (4.6). For $z \in \bar{U}$, we consider again problem (3.5). For each index $\nu \in \mathcal{F}$, we also recall the domain $\bar{U}_\nu$ defined in (4.6).

PROPOSITION 4.10. *Assume that the domain D is convex. Given $\nu \in \mathcal{F}$, for every $z \in \bar{U}_\nu \subset \mathbb{C}^{\mathbb{N}}$ with fixed coordinates z_i where $i \notin \text{supp}(\nu)$, the map $u : \bar{U}_\nu \rightarrow L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})$ is analytic.*

Proof. First, we show that $\|u(\cdot, \cdot, z)\|_{L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})}$ is uniformly bounded for all $z \in \bar{U}_\nu$. We write $u = u_1 + u_2$, where

$$(4.26) \quad \frac{\partial u_1}{\partial t} - \nabla \cdot (a(x, z) \nabla u_1) = 0, \quad u_1|_{\partial D \times I} = 0, \quad u_1|_{t=0} = h(x),$$

and

$$(4.27) \quad \frac{\partial u_2}{\partial t} - \nabla \cdot (a(x, z) \nabla u_2) = g(t, x), \quad u_2|_{\partial D \times I} = 0, \quad u_2|_{t=0} = 0.$$

Let $T_1(z)$ be the map which associates $h \in H$ to $u_1 \in L^2(I; V) \cap H^1(I; H^{-1})$. We have the estimate $\|T_1(z)\|_{H \rightarrow H^1(I; H^{-1}) \cap L^2(I; V)} \leq c$, where the bound c only depends on $a_{\min}$, $a_{\max}$, and T , but is independent of z . When $h \in H^2 \cap V$, i.e., when the initial condition is compatible, it follows that $\|T_1(z)\|_{H^2 \cap V \rightarrow H^1(I; V)} \leq c$. Using interpolation, we deduce the bound $\|T_1(z)\|_{V \rightarrow H^1(I; H)} \leq c$. From

$$-\Delta u_1 = \frac{1}{a} \left(-\frac{\partial u_1}{\partial t} + \nabla a \cdot \nabla u_1 \right),$$

and the convexity of the domain D , we deduce that $T_1(z)$ is a linear map from V to $L^2(I; H^2 \cap V) \cap H^1(I; H)$, uniformly bounded for all z . From interpolation, $\|T_1(z)\|_{H^\theta \rightarrow L^2(I; H^{1+\theta}) \cap H^1(I; H^{-1+\theta})} \leq c$ where the bound c only depends on $a_{\min}$, $a_{\max}$, and T .

Equation (4.27) has homogeneous initial data, and g is analytic from I to H so that $\|u_2\|_{L^2(I; H^2 \cap V) \cap H^1(I; V)}$ is uniformly bounded for all $z \in \bar{U}_\nu$.

For a fixed index m , we fix all the coordinates z_k when $k \neq m$, and partition $z \in \bar{U}_\nu \subset \mathbb{C}^{\mathbb{N}}$ as $z = (z_m, \bar{z}_m)$. We establish strong complex differentiability of the analytic continuation of the parametric solution by a difference quotient argument. To this end, in the remainder of the proof we let $\delta \in \mathbb{C}$ with $|\delta| > 0$ sufficiently small

so that the point $(z_m + \delta, \bar{z}_m) \in \bar{\mathcal{U}}_\nu$. We show that there exists a function $v \in \mathcal{Z}$ such that for all $0 < s \leq \theta$

$$\lim_{\delta \rightarrow 0} \left\| \frac{u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z)}{\delta} - v(\cdot, \cdot, z) \right\|_{L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})} = 0$$

for all $z \in \bar{\mathcal{U}}_\nu$. We define the difference quotient

$$v_1^\delta = \frac{u_1(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u_1(\cdot, \cdot, z_m, \bar{z}_m)}{\delta}.$$

The function v_1^δ satisfies

$$(4.28) \quad \frac{\partial v_1^\delta}{\partial t} - \nabla \cdot (a(\cdot, z) \nabla v_1^\delta) = \nabla \cdot (\psi_m \nabla u_1(\cdot, \cdot, z_m + \delta, \bar{z}_m))$$

with the homogeneous initial condition $v_1^\delta(0, \cdot, z_m + \delta, \bar{z}_m) = 0$. Let v_1 satisfy the equation

$$\frac{\partial v_1}{\partial t} - \nabla \cdot (a(\cdot, z) \nabla v_1) = \nabla \cdot (\psi_m \nabla u_1(\cdot, \cdot, z))$$

with homogeneous initial data $v_1(0, \cdot, z) = 0$. Then

$$(4.29) \quad \frac{\partial(v_1^\delta - v_1)}{\partial t} - \nabla \cdot (a(\cdot, z) \nabla (v_1^\delta - v_1)) = \nabla \cdot (\psi_m \nabla (u_1(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u_1(\cdot, \cdot, z))) .$$

When $h \in H$, from (4.28), we have

$$(4.30) \quad \begin{aligned} \|u_1(z_m + \delta, \bar{z}_m) - u_1(z)\|_{L^2(I; V) \cap H^1(I; H^{-1})} &\leq c|\delta| \|\nabla \cdot (\psi_m \nabla u_1(z_m + \delta, \bar{z}_m))\|_{L^2(I; V')} \\ &\leq c|\delta| \|h\|_H . \end{aligned}$$

Therefore, from (4.29),

$$(4.31) \quad \|v_1^\delta - v_1\|_{L^2(I; V) \cap H^1(I; V')} \leq c|\delta| \|h\|_H .$$

Next, we consider the case where $h \in V \cap H^2$. As $\|u_1((z_m + \delta, \bar{z}_m))\|_{H^1(I; V)} \leq c\|h\|_{V \cap H^2}$, we deduce from (4.28) that

$$(4.32) \quad \begin{aligned} &\|u_1(z_m + \delta, \bar{z}_m) - u_1(z)\|_{H^1(I; V)} \\ &\leq c|\delta| \|\nabla \cdot (\psi_m \nabla u_1(z_m + \delta, \bar{z}_m))\|_{H^1(I; V')} + c|\delta| \|\nabla \cdot (\psi_m \nabla h)\|_H \\ &\leq c|\delta| \|h\|_{H^2 \cap V} . \end{aligned}$$

Therefore the $H^1(I, V')$ norm of the right-hand side in (4.29) is bounded by $c|\delta| \|h\|_{H^2 \cap V}$. We then deduce that

$$\|v_1^\delta - v_1\|_{H^1(I; V)} \leq c|\delta| \|h\|_{H^2 \cap V} .$$

By interpolation we find that

$$(4.33) \quad \|v_1^\delta - v_1\|_{H^1(I; H)} \leq c|\delta| \|h\|_V .$$

By interpolation, we deduce from (4.30) and (4.32) that

$$\|u_1(z_m + \delta, \bar{z}_m) - u_1(z)\|_{H^1(I; H)} \leq c|\delta| \|h\|_V .$$

We note further that when $h \in V$, $\|u_1(z)\|_{L^2(I; V \cap H^2)} \leq c\|h\|_V$. Thus, we deduce from (4.28) that

$$\|u_1(z_m + \delta, \bar{z}_m) - u_1(z)\|_{L^2(I; V \cap H^2)} \leq c|\delta|\|h\|_V.$$

Therefore the $L^2(I; H)$ norm of the right-hand side of (4.29) is bounded by $c|\delta|\|h\|_V$ for $\delta \in \mathbb{C}$ with $|\delta| > 0$ sufficiently small. From (4.33), we have

$$(4.34) \quad \|v_1^\delta - v_1\|_{L^2(I; V \cap H^2) \cap H^1(I; H)} \leq c|\delta|\|h\|_V.$$

From (4.31) and (4.34), we deduce from interpolation that

$$\|v_1^\delta - v_1\|_{L^2(I; H^{1+\theta}) \cap H^1(I; H^{-1+\theta})} \leq c|\delta|\|h\|_{H^\theta} \text{ for } 0 < \theta < 1.$$

Similarly, we define

$$v_2^\delta = \frac{u_2(z_m + \delta, \bar{z}_m) - u_2(z)}{\delta},$$

and v_2 as the solution of

$$\frac{\partial v_2}{\partial t} - \nabla \cdot (a(\cdot, z) \nabla v_2) = \nabla \cdot (\psi_m \nabla u_2(z))$$

with zero initial condition. As $u_2 \in H^1(I, V) \cap L^2(I; V \cap H^2)$,

$$\|u_2(z_m + \delta, \bar{z}_m) - u_2(z)\|_{H^1(I; V) \cap L^2(I; H^2)} \leq c|\delta|.$$

We then deduce that $\|v_2^\delta - v_2\|_{L^2(I; V \cap H^2) \cap H^1(I; V)} \leq c|\delta|$. This shows that $v = v_1 + v_2$ is the derivative of u with respect to z_m as an $L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})$ -valued function. From Hartogs' theorem, we conclude that u is analytic as a function from $\bar{\mathcal{U}}_\nu$ to $L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})$. $\square$

PROPOSITION 4.11. *Assume that $h \in H^\theta$. For any interval (a, b) such that $0 < a < b < \min(1, T)$ and for every $y \in U$ it holds the bounds*

$$(4.35) \quad \|u(\cdot, \cdot, y)\|_{H^l((a,b); V)}^2 \leq C(l) \int_a^b t^{-2l-1+\theta} dt$$

and

$$(4.36) \quad \|u(\cdot, \cdot, y)\|_{H^l((a,b); V')}^2 \leq C(l) \int_a^b t^{-2l+1+\theta} dt,$$

where $C(l)$ does not depend on a , b , and on y .

Proof. From (4.12) and (4.13),

$$(4.37) \quad \|T^{(l)}(y; t)\|_{\mathcal{L}(H^\theta, V)} \leq C(l) t^{-l-1/2+\theta/2}.$$

From (4.10) we obtain for every $y \in U$

$$(4.38) \quad u^{(l)}(t, \cdot, y) = T^{(l)}(y; t)h + \sum_{i=0}^{l-1} T^{(i)}(y; t)g^{(l-1-i)}(0) + \int_0^t T(y; r)g^{(l)}(t-r)dr.$$

Then

$$\|T^{(l)}(y; t)h\|_V \leq C(l) t^{-l-1/2+\theta/2}.$$

Further

$$\left\| \sum_{i=0}^{l-1} T^{(i)}(y; t) g^{(l-1-i)}(0) \right\|_V \leq c \sum_{i=0}^{l-1} t^{-i-1/2} \leq ct^{-l+1/2}$$

and

$$\left\| \int_0^t T(y; r) g^{(l)}(t-r) dr \right\|_V \leq c \int_0^t r^{-1/2} dr \leq ct^{1/2}.$$

From these bounds we deduce

$$\|u(\cdot, \cdot, y)\|_{H^l((a,b);V)}^2 \leq C(l) \int_a^b t^{-2l-1+\theta} dt.$$

The proof for $\|u(\cdot, \cdot, y)\|_{H^l((a,b);V')}^2$ is similar. $\square$

PROPOSITION 4.12. Assume that the domain D is convex and $h \in H^\theta$. For all $y, y' \in U$,

$$\|u(\cdot, \cdot, y) - u(\cdot, \cdot, y')\|_{H^l((a,b);V)}^2 \leq C(l) \int_a^b t^{-2l-1+\theta} dt \|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)}^2.$$

Proof. For $t_0 < t$, the solution w of (4.25) is written as

$$(4.39) \quad w(t, \cdot) = T(y; t - t_0)w(t_0) + \int_0^{t-t_0} T(y; r)G(t-r)dr.$$

Therefore,

$$(4.40) \quad \begin{aligned} w^{(l)}(t, \cdot, y) &= T^{(l)}(y; t - t_0)w(t_0) + \sum_{i=0}^{l-1} T^{(i)}(y; t - t_0)G^{(l-1-i)}(t_0) \\ &\quad + \int_0^{t-t_0} T(y; r)G^{(l)}(t-r)dr. \end{aligned}$$

From (4.38) and (4.18) we get for every $l \in \mathbb{N}$ the existence of a constant c (depending on l) such that for every $y \in U$ and for every $t > 0$ it holds

$$\|u^{(l)}(t, \cdot, y)\|_V \leq ct^{-l-1/2+\theta/2} + c \sum_{i=0}^{l-1} t^{-i-1/2} + c \int_0^t r^{-1/2} dr \leq C(l)t^{-l-1/2+\theta/2}.$$

From (4.38) and (4.21) then, again with a constant $c > 0$ depending on l , for every $t > 0$ and every $y \in U$,

$$\|u^{(l)}(t, \cdot, y)\|_H \leq ct^{-l+\theta/2} + c \sum_{i=0}^{l-1} t^{-i} + c \int_0^t dr \leq C(l)t^{-l+\theta/2}.$$

From

$$-\Delta u^{(l)}(t, \cdot, y) = \frac{1}{a}g^{(l)}(t) + \frac{1}{a}\nabla a \cdot \nabla u^{(l)}(t, \cdot, y) - \frac{1}{a}u^{(l+1)}(t, \cdot, y)$$

and the convexity of the domain D , we deduce that

$$\|u^{(l)}(t, \cdot, y)\|_{H^2(D)} \leq C(l)t^{-l-1+\theta/2}.$$

Thus for all i , $\|G^{(i)}(t)\|_H \leq C(i)t^{-i-1+\theta/2}\|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)}$. With $t_0 = 0$ we get from (4.39) the bound

$$\|w(t, \cdot, y)\|_H \leq c \int_0^t (t-r)^{-1+\theta/2} dr \|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)} \leq ct^{\theta/2}.$$

From (4.40) we obtain for every $y \in U$ the estimate

$$\begin{aligned} \|w^{(l)}(t, \cdot, y)\|_V &\leq C(l) \left((t-t_0)^{-l-1/2} t_0^{\theta/2} + c \sum_{i=0}^{l-1} (t-t_0)^{-i-1/2} t_0^{-l+i+\theta/2} \right. \\ &\quad \left. + \int_0^{t-t_0} r^{-1/2} (t-r)^{-l-1+\theta/2} dr \right) \|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)}. \end{aligned}$$

Let $t_0 = t/2$, we deduce that

$$\|w^{(l)}(t, \cdot, y)\|_V \leq C(l)t^{-l-1/2+\theta/2}\|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)}.$$

This completes the proof. $\square$

PROPOSITION 4.13. *Assume that the domain D is convex. For every $l \in \mathbb{N}_0$, and for every $0 < a < b < \infty$, the function $u : U \rightarrow H^l((a, b); V)$ is measurable as a Bochner function. Moreover, for every $\nu \in \mathcal{F}$, and for $z \in \bar{\mathcal{U}}_\nu$ with fixed coordinates z_i where $i \notin \text{supp}(\nu)$, the map $u : \bar{\mathcal{U}}_\nu \rightarrow H^l((a, b); V)$ is analytic.*

Proof. The first assertion is proved similarly to Proposition 1.4; here, however, we use Proposition 4.12 in place of Proposition 1.3.

We now show that $\|u(z)\|_{H^l((a,b);V)}$ is uniformly bounded for all $z \in \bar{\mathcal{U}}_\nu$. We choose the constants $0 < s_0 < s'_0 < s_1 < s'_1 < \dots < s_{l-1} < s'_{l-1} < 1$. As $\|u(\cdot, \cdot, z)\|_{L^2(I;V)} \leq c$, there is a $t_0 \in (0, s_0 a)$ such that $\|u(t_0, \cdot, z)\|_V \leq c/(s_0 a)^{1/2}$. From the proof of Proposition 4.10, we deduce that $\|u(\cdot, \cdot, z)\|_{L^2(t_0, T; V \cap H^2)} \leq c/(s_0 a)^{1/2}$ so there is $t'_0 \in (s_0 a, s'_0 a)$ such that $\|u(t'_0, \cdot, z)\|_{V \cap H^2} \leq c/(s_0^{1/2}(s'_0 - s_0)^{1/2} a)$. Therefore $\|u(\cdot, \cdot, z)\|_{H^1(t'_0, T; V)} \leq c/(s_0^{1/2}(s'_0 - s_0)^{1/2} a)$. Thus there is a constant $t_1 \in (s'_0 a, s_1 a)$ such that $\|\frac{du}{dt}(t_1, \cdot, z)\|_V \leq c/(s_0^{1/2}(s'_0 - s_0)^{1/2}(s_1 - s'_0)^{1/2} a^{3/2})$ which implies $\|\frac{du}{dt}(\cdot, \cdot, z)\|_{L^2(t_1, T; V \cap H^2)} \leq c/(s_0^{1/2}(s'_0 - s_0)^{1/2}(s_1 - s'_0)^{1/2} a^{3/2})$, and $\|\frac{du}{dt}(\cdot, \cdot, z)\|_{H^1(t'_1, T; V)} \leq c/(s_0^{1/2}(s'_0 - s_0)^{1/2}(s_1 - s'_0)^{1/2}(s'_1 - s_1)^{1/2} a^2)$ for $t'_1 \in (s_1 a, s'_1 a)$. Continuing this process we deduce that $\|\frac{d^l u}{dt^l}(\cdot, \cdot, z)\|_{L^2(t'_{l-1}, T; V)}$ is uniformly bounded for all $z \in \mathcal{U}$.

To show analyticity of the mapping $u : U \rightarrow H^l((a, b); V)$, we show again complex differentiability by a difference quotient argument: we fix all z_k where $k \neq m$ and show that there exists a function $v \in H^l((a, b); V)$ such that

$$\lim_{\delta \rightarrow 0} \left\| \frac{u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z)}{\delta} - v(\cdot, \cdot, z) \right\|_{H^l((a,b);V)} = 0$$

for all $z \in \mathcal{U}$, and for $\delta \in \mathbb{C}$ with $|\delta| > 0$ sufficiently small. Let

$$v_\delta = \frac{u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z_m, \bar{z}_m)}{\delta}.$$

The function v^δ satisfies

$$(4.41) \quad \frac{\partial v^\delta}{\partial t} - \nabla \cdot (a(\cdot, \cdot, z) \nabla v^\delta) = \nabla \cdot (\psi_m \nabla u(\cdot, \cdot, z_m + \delta, \bar{z}_m)),$$

with the initial condition $v^\delta(0, \cdot, z_m + \delta, \bar{z}_m) = 0$. Let v satisfy the equation

$$(4.42) \quad \frac{\partial v}{\partial t} - \nabla \cdot (a(\cdot, \cdot, z) \nabla v) = \nabla \cdot (\psi_m \nabla u(\cdot, \cdot, z)),$$

with $v(0, \cdot, z) = 0$. Then

$$(4.43) \quad \frac{\partial(v^\delta - v)}{\partial t} - \nabla \cdot (a(\cdot, \cdot, z) \nabla(v^\delta - v)) = \nabla \cdot (\psi_m \nabla(u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z))).$$

For all z , we can choose $\bar{t}_0 \in (s_0 a/2, s_0 a)$ such that $\|u(\cdot, \cdot, z)\|_{L^2(\bar{t}_0, T; H^2 \cap V) \cap H^1(\bar{t}_0, T; V)} \leq c$ and such that $\|u(\bar{t}_0, \cdot, z)\|_{H^2 \cap V} \leq c$, where the constant c does not depend on z . Further, as $\|u(\cdot, \cdot, z)\|_{L^2(I, V)}$ is uniformly bounded, $\|v^\delta\|_{L^2(I, V)}$ is uniformly bounded so we can choose $\bar{t}_0$ such that $\|v^\delta(\bar{t}_0, \cdot)\|_V \leq c$, where $c > 0$ is independent of z . By interpolation, when the initial condition is in V and the right-hand side vanishes, then the solution is in $L^2(\bar{t}_0, T; V \cap H^2)$. The right-hand side of (4.41) satisfies the compatibility condition on $(\bar{t}_0, T)$. By superposition, we deduce that $\|v^\delta\|_{L^2(\bar{t}_0, T; V \cap H^2)} \leq c$. We can therefore choose a constant $\bar{t}'_0 \in (s_0 a, (s_0 + s_1)a/2)$ such that $\|v^\delta(\bar{t}'_0, \cdot)\|_{V \cap H^2} \leq c$ and such that $\|\nabla \cdot (\psi_m \nabla u(\bar{t}'_0, \cdot, z_m + \delta, \bar{z}_m))\|_H \leq c$, where c is independent of z , so $\|v^\delta\|_{H^1(\bar{t}'_0, T; V) \cap L^2(\bar{t}'_0, T; V \cap H^2)} \leq c$. Thus

$$\|\nabla \cdot (\psi_m \nabla(u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z)))\|_{H^1(\bar{t}'_0, T; H^{-1}) \cap L^2(\bar{t}'_0, T; H)} \leq c|\delta|.$$

On the other hand, as $\|u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z)\|_{L^2(I; V)} \leq c\delta$, we deduce that $\|v^\delta - v\|_{L^2(I; V)} \leq c|\delta|$ so we can choose $\bar{t}'_0$ such that $\|v^\delta(\bar{t}'_0, \cdot) - v(\bar{t}'_0, \cdot)\|_V \leq c|\delta|$. Further, we choose $\bar{t}'_0$ so that

$$\|\nabla \cdot (\psi_m \nabla(u(\bar{t}'_0, \cdot, z_m + \delta, \bar{z}_m) - u(\bar{t}'_0, \cdot, z)))\|_H \leq c|\delta|,$$

so that the compatibility conditions hold. We then deduce that $\|v^\delta - v\|_{L^2(\bar{t}'_0, T; V \cap H^2)} \leq c\delta$. We can moreover choose $\bar{t}_1 \in ((s_0 + s_1)a/2, s_1 a/2)$ so that $\|v^\delta(\bar{t}_1, \cdot) - v(\bar{t}_1, \cdot)\|_{V \cap H^2} \leq c|\delta|$ and such that

$$\|\nabla \cdot (\psi_m \nabla(u(\bar{t}_1, \cdot, z_m + \delta, \bar{z}_m) - u(\bar{t}_1, \cdot, z)))\|_H \leq c|\delta|.$$

Therefore $\|v^\delta - v\|_{H^1(\bar{t}_1, T; V)} \leq c\delta$. For higher derivatives with respect to t , this shows that v is the derivative of u as an $H^1((a, b); V)$ -valued function. From Hartogs' theorem, we conclude that u is analytic as a function from $\bar{U}_\nu$ to $H^1((a, b); V)$. $\square$

PROPOSITION 4.14. *There exists a constant $c > 0$ such that for all constants $k \in (0, 1)$ and for all $y \in U$ it holds*

$$\|u(\cdot, \cdot, y) - u(k, \cdot, y)\|_{L^2((0, k); V) \cap H^1((0, k); V')} \leq ck^{\theta/2}.$$

Proof. The result follows from estimates (4.22) and (4.23). $\square$

PROPOSITION 4.15. *Fixing $k \in (0, 1)$, $u(k, \cdot, \cdot)$ as a map from U to $L^2((0, k); V) \cap H^1((0, k); V')$ is measurable.*

Proof. From Proposition 4.12 we obtain

$$\|u(k, \cdot, y) - u(k, \cdot, y')\|_V \leq c\|a(\cdot, y) - a(\cdot, y')\|_{W^{1, \infty}(D)}.$$

The remaining part of the proof of this proposition is similar to that of Proposition 1.4. $\square$

PROPOSITION 4.16. *Assume that the domain D is convex and $h \in H^\theta$. Fixing the coordinates z_i , where $i \notin \text{supp}(\nu)$, the map $u(\cdot, \cdot, \cdot) - u(k, \cdot, \cdot) : \bar{U}_\nu \rightarrow L^2((0, k); V) \cap H^1((0, k); V')$ is analytic. Further there is a constant c that is independent of z such that $\|u(\cdot, \cdot, z) - u(k, \cdot, z)\|_{L^2((0, k); V) \cap H^1((0, k); V')} \leq ck^{\theta/2}$.*

Proof. We first show that for all z , $\|u(\cdot, \cdot, z) - u(k, \cdot, z)\|_{L^2((0, k); V) \cap H^1((0, k); V')} \leq ck^\theta$ where c can be chosen independently of z . First we consider the case where $h \in H$. For u_1 in (4.26), $\|u_1\|_{L^1(I; V)} \leq c\|h\|_H$ so there is a constant $t_1 \in (0, k/4)$ such that $\|u_1(t_1)\|_V \leq c\|h\|_H/k^{1/2}$. By an interpolation argument as in the proof of Proposition 4.10, we have $\|u_1\|_{L^2(t_1, T; H^2 \cap V)} \leq c\|u_1(t_1)\|_V \leq c\|h\|_H/k^{1/2}$. Thus we can find $t_2 \in (k/4, k/2)$ such that $\|u_1(t_2)\|_{H^2 \cap V} \leq c\|h\|_H/k$. From this we deduce that $\|u_1\|_{H^1(t_2, T; V)} \leq c\|h\|_H/k$. As $\|u_1\|_{L^2(k/2, 3k/4; V)} \leq c\|h\|_H$, there exists $t_3 \in (k/2, 3k/4)$ such that $\|u_1(t_3)\|_V \leq c\|h\|_H/k^{1/2}$. We then have

$$\begin{aligned} \|u_1(k) - u_1(t_3)\|_V &\leq \int_{t_3}^k \left\| \frac{\partial u_1}{\partial t}(t, \cdot, z) \right\|_V dt \leq (k - t_3)^{1/2} \left(\int_{t_3}^k \left\| \frac{\partial u_1}{\partial t}(t, \cdot, z) \right\|_V^2 dt \right)^{1/2} \\ &\leq ck^{1/2} \|h\|_H \frac{1}{k} \leq c\|h\|_H \frac{1}{k^{1/2}}. \end{aligned}$$

From this we deduce $\|u_1(k)\|_V \leq c\|h\|_H k^{-1/2}$. The solution u_2 of (4.27) is in $H^1(I; V)$ so the proof for u_2 is trivial. We then have

$$\|u(\cdot, \cdot, z) - u(k, \cdot, z)\|_{L^2((0, k); V)} \leq c\|h\|_H.$$

When $h \in V \cap H^2$, as u is uniformly bounded in $H^1(I; V) \cap H^2(I; V')$ we have

$$\begin{aligned} \|u(s, \cdot, z) - u(k, \cdot, z)\|_V &\leq \int_s^k \left\| \frac{\partial u}{\partial t}(s, \cdot, z) \right\|_V dt \leq k^{1/2} \left(\int_0^k \left\| \frac{\partial u}{\partial t}(s, \cdot, z) \right\|_V^2 dt \right)^{1/2} \\ &\leq ck^{1/2} \|h\|_{V \cap H^2}. \end{aligned}$$

Therefore,

$$\|u(\cdot, \cdot, z) - u(k, \cdot, z)\|_{L^2((0, k); V)} \leq ck\|h\|_{V \cap H^2}.$$

Using interpolation, we deduce that when $h \in V$,

$$\|u(\cdot, \cdot, z) - u(k, \cdot, z)\|_{L^2((0, k); V)} \leq ck^{1/2} \|h\|_V,$$

and when $h \in H^\theta$,

$$\|u(\cdot, \cdot, z) - u(k, \cdot, z)\|_{L^2((0, k); V)} \leq ck^{\theta/2} \|h\|_{H^\theta}.$$

The proof for the $H^1((0, k); V')$ norm is similar.

The analyticity of u as an $L^2((0, k); V) \cap H^1((0, k); V')$ -valued function of the parameters z_k is established in Proposition 3.1. The analyticity of $u(k, \cdot, \cdot)$ for fixed $k > 0$, as a function taking values in V , follows from Proposition 4.13. $\square$

As $u_\nu \in H^l((a, b), V)$, $u_\nu(k, \cdot)$ is uniquely determined and given by

$$u_\nu(k, \cdot) = \int_U u(k, \cdot, y) L_\nu(y) d\rho(y)$$

with the integral to be understood as a Bochner integral over U of V -valued functions. We then deduce the following results.

PROPOSITION 4.17. *Assume that the domain D is convex and that the initial data $h \in H^\theta$. Assume that $0 < s < \theta$. Then the coefficient sequences*

$$\|u_\nu\|_{L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})}, \quad \|u_\nu\|_{H^1((a,b); V)},$$

$$\text{and } \|u_\nu(\cdot, \cdot, y) - u_\nu(k, \cdot, y)\|_{L^2((0,k); V) \cap H^1((0,k); V')}$$

are in $l^p(\mathcal{F})$.

Proof. An argument similar to the proof of Proposition 3.2 shows that for every $\nu \in \mathcal{F}$ there exists a constant $\mathcal{B}(\nu)$ such that

$$\|u_\nu\|_{L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})} \leq c\mathcal{B}(\nu)\|u\|_{L^\infty(\bar{U}; L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s}))},$$

and, for every $l \in \mathbb{N}$, there exists a constant $C(l) > 0$ such that for every $a > 0$

$$\|u_\nu\|_{H^1((a,b); V)}^2 \leq \mathcal{B}(\nu)^2 C(l) a^{-2l},$$

and there exists a constant $C > 0$ such that for every $k > 0$

$$\|u_\nu(\cdot, \cdot) - u_\nu(k, \cdot)\|_{L^2((0,k); V) \cap H^1((0,k); V')} \leq C\mathcal{B}(\nu)k^{\theta/2},$$

where we denote

$$(4.44) \quad \mathcal{B}(\nu) := \prod_{m \in \text{supp}(\nu)} \frac{2(1 + \bar{K})}{\bar{K}} \bar{\eta}_m^{-\nu_m}$$

with $\bar{\eta}_m$ as in (4.7). Proceeding as in the proof of Proposition 3.4, we obtain that for $0 < p < 1$ as in Assumption 4.1, $\sum_{\nu \in \mathcal{F}} (\mathcal{B}(\nu))^p$ is finite. $\square$

Appendix. In this appendix, we prove the inf-sup conditions (2.4), (2.5), and (2.6) for the semidiscretized parametric problems. Specifically, we show that under Assumptions 1.1, 1.2, there exist constants $0 < \beta \leq \alpha < \infty$ such that for *any* subset $\Lambda \subset \mathcal{F}$ it holds

$$(A) \quad \sup_{u_\Lambda \in \underline{\mathcal{X}}_\Lambda, v_\Lambda \in \underline{\mathcal{Y}}_\Lambda} \frac{|B(u_\Lambda, v_\Lambda)|}{\|u_\Lambda\|_{\underline{\mathcal{X}}} \|v_\Lambda\|_{\underline{\mathcal{Y}}}} \leq \alpha < \infty,$$

$$(B) \quad \inf_{0 \neq u_\Lambda \in \underline{\mathcal{X}}_\Lambda} \sup_{0 \neq v_\Lambda \in \underline{\mathcal{Y}}_\Lambda} \frac{|B(u_\Lambda, v_\Lambda)|}{\|u_\Lambda\|_{\underline{\mathcal{X}}} \|v_\Lambda\|_{\underline{\mathcal{Y}}}} \geq \beta > 0,$$

$$(C) \quad \forall 0 \neq v_\Lambda \in \underline{\mathcal{Y}}_\Lambda, \quad \sup_{0 \neq u_\Lambda \in \underline{\mathcal{X}}_\Lambda} |B(u_\Lambda, v_\Lambda)| > 0.$$

We note that

$$\|u_\Lambda\|_{\underline{\mathcal{X}}}^2 = \sum_{\nu \in \Lambda} (\|u_\nu\|_{L^2(I; V)}^2 + \|u_\nu\|_{H^1(I; V')}^2) \quad \text{and} \quad \|v_\Lambda\|_{\underline{\mathcal{Y}}}^2 = \sum_{\nu \in \Lambda} (\|v_{1\nu}\|_{L^2(I; V)}^2 + \|v_{2\nu}\|_{H'}^2).$$

First we show the continuity condition (A). We have

$$\begin{aligned}
& |B(u_\Lambda, v_\Lambda)| \\
& \leq \int_U \left\{ \int_I \left(\left\| \frac{du_\Lambda}{dt}(t, \cdot, y) \right\|_{V'} \|v_{1\Lambda}(t, \cdot, y)\|_V + C \|\nabla u_\Lambda(t, \cdot, y)\|_H \|\nabla v_{1\Lambda}(t, \cdot, y)\|_H \right) dt \right. \\
& \quad \left. + \|u_\Lambda(0, \cdot, y)\|_H \|v_{2\Lambda}(\cdot, y)\|_H \right\} d\rho(y) \\
& \leq \int_U \left\{ \int_I \left(\left\| \frac{du_\Lambda(t, \cdot, y)}{dt} \right\|_{V'} + C \|u_\Lambda(t, \cdot, y)\|_V \right) \|v_{1\Lambda}\|_V dt \right. \\
& \quad \left. + M \|u_\Lambda(\cdot, \cdot, y)\|_{\mathcal{X}} \cdot \|v_{2\Lambda}(\cdot, y)\|_H \right\} d\rho(y) \\
& \leq \left\{ \int_U \int_I \left(\left\| \frac{du_\Lambda(t, \cdot, y)}{dt} \right\|_{V'} + C \|u_\Lambda(t, \cdot, y)\|_V \right)^2 dt d\rho(y) \right\}^{1/2} \\
& \quad \cdot \left\{ \int_U \int_I \|v_{1\Lambda}(t, \cdot, y)\|_V^2 dt d\rho(y) \right\}^{1/2} + M \|u_\Lambda\|_{L^2(U; \mathcal{X})} \cdot \|v_{2\Lambda}\|_{L^2(U, H)} \\
& \leq C \|u_\Lambda\|_{\underline{\mathcal{X}}} \cdot \|v\|_{\underline{\mathcal{Y}}},
\end{aligned}$$

where we have used that the “initial value trace operator” from $\mathcal{X}$ is bounded, i.e., the constant

$$M = \sup_{0 \neq w \in \mathcal{X}} \frac{\|w(0)\|_H}{\|w\|_{\mathcal{X}}}$$

is finite. To show the inf-sup condition (B), we note that

$$\int_U \int_I \left\langle \frac{du_\Lambda}{dt}, v_{1\Lambda} \right\rangle_H dt = \sum_{\nu \in \Lambda} \int_I \left\langle \frac{du_\nu(t, \cdot)}{dt}, v_{1\nu} \right\rangle_H dt$$

and

$$\begin{aligned}
& \int_U \int_I \int_D a(x, y) \nabla u_\Lambda(t, x, y) \cdot \nabla v_{1\Lambda}(t, x, y) dt dx d\rho(y) \\
\text{(A.1)} \quad & = \sum_{\nu \in \Lambda, \mu \in \Lambda} \int_I \int_D \left(\int_U a(x, y) L_\nu(y) L_\mu(y) d\rho(y) \right) \nabla u_\mu(t, x) \cdot \nabla v_{1\nu}(t, x) dx dt \\
& = \sum_{\nu \in \Lambda, \mu \in \Lambda} \int_I \int_D A^{\nu\mu}(x) \nabla u_\mu(t, x) \cdot \nabla v_{1\nu}(t, x) dx dt,
\end{aligned}$$

where

$$A^{\nu\mu}(x) = \int_U a(x, y) L_\nu(y) L_\mu(y) d\rho(y)$$

and

$$\int_U \langle u_\Lambda(0), v_{2\Lambda} \rangle_H d\rho(y) = \sum_{\nu} \langle u_\nu(0, \cdot), v_{2\nu} \rangle_H.$$

Then

$$B(u_\Lambda, v_\Lambda) = \sum_{\nu \in \Lambda} \int_I \left\langle \frac{du_\nu(t, \cdot)}{dt}, v_{\nu 1} \right\rangle_H dt \\ + \sum_{\nu \in \Lambda, \mu \in \Lambda} \int_I \int_D A^{\nu\mu}(x) \nabla u_\mu(t, x) \cdot \nabla v_{1\nu}(t, x) dx dt + \sum_{\nu \in \Lambda} \langle u_\nu(0, \cdot), v_{2\nu} \rangle_H.$$

To show the ellipticity of $A^{\nu\mu}(x)$, we observe that for any array of vectors $\xi^\nu \in \mathbf{R}^d$ it holds (with implied summation over repeated indices)

$$A^{\nu\mu}(x) \xi_i^\nu \xi_i^\mu = \int_U a(x, y) (L_\nu(y) \xi_i^\nu) (L_\mu(y) \xi_i^\mu) d\rho(y) \geq a_{\min} \sum_{i=1}^d \sum_{\nu \in \Lambda} (\xi_i^\nu)^2.$$

Next, to establish the inf-sup condition, we follow the approach of Babuska and Janik [1] and consider the operator

$$\mathbf{A} : [V]^N \rightarrow [V]^N$$

defined by (with the convention of summation over repeated indices)

$$(\mathbf{A}w)^\nu = -\nabla(A^{\nu\mu}(x) \nabla w_\mu(x)),$$

for each vector $\mathbf{w} = \{(w_\nu(x)) : \nu \in \Lambda\} \in [V]^N$. Let λ_i ($i = 1, 2, \dots$) be the eigenvalues of this operator. The corresponding eigenvectors are $\mathbf{w}_1, \mathbf{w}_2, \dots$. For each $\phi \in [V]^N$,

$$\int_D A^{\nu\mu} \nabla \mathbf{w}_{i\mu}(x) \cdot \nabla \phi_\nu(x) dx = \lambda_i \int_D \mathbf{w}_{i\nu}(x) \phi_\nu(x) dx,$$

where summation is taken over μ and ν . Denote the vector $\mathbf{u}_\Lambda = \{(u_\nu(t, x))\}$. Then

$$\mathbf{u}_\Lambda(t, x) = \sum_{i=1}^{\infty} a_i(t) \mathbf{w}_i(x).$$

We choose $\mathbf{w}_i$ as an orthonormal base of $[H]^N$. We denote by

$$\mathbf{v}_1(x) = \{(v_{1\nu}(x)), \nu \in \Lambda\} = \sum_{i=1}^{\infty} b_i(t) \mathbf{w}_i.$$

Then

$$(a_{\max})^{-1} \int_0^T \sum_{i=1}^{\infty} \lambda_i (a_i(t))^2 dt \leq \|\mathbf{u}\|_{L^2(I; [V]^N)}^2 \leq (a_{\min})^{-1} \int_0^T \sum_{i=1}^{\infty} \lambda_i (a_i(t))^2 dt$$

and

$$a_{\min} \int_0^T \sum_{i=1}^{\infty} \frac{(\dot{a}_i(t))^2}{\lambda_i} dt \leq \left\| \frac{d\mathbf{u}_\Lambda}{dt} \right\|_{L^2(I; [V]^N)}^2 \leq a_{\max} \int_0^T \sum_{i=1}^{\infty} \frac{(\dot{a}_i(t))^2}{\lambda_i} dt.$$

Thus

$$\int_0^T \left(\sum_{i=1}^{\infty} \frac{(\dot{a}_i(t))^2}{\lambda_i} + \lambda_i (a_i(t))^2 \right) dt$$

is equivalent to $\|\mathbf{u}_\Lambda\|_{[\mathcal{X}]^N}^2 = \|u_\Lambda\|_{\underline{\mathcal{X}}}^2$. We expand $\mathbf{v}_2(x) = \{(v_{2\nu}(x)), \nu \in \Lambda\}$ in $\mathbf{w}_i$ and write

$$\mathbf{v}_2(x) = \sum_{i=1}^{\infty} c_i \mathbf{w}_i(x) .$$

Then the expression

$$\int_0^T \sum_{i=1}^{\infty} \lambda_i (b_i(t))^2 dt + \sum_{i=1}^{\infty} c_i^2$$

is equivalent to $\|v_\Lambda\|_{\underline{\mathcal{Y}}}^2$ and we have

$$B(u_\Lambda, v_\Lambda) = \int_I \sum_{i=1}^{\infty} \lambda_i^{1/2} b_i(t) \left[\frac{\dot{a}_i(t)}{\lambda_i^{1/2}} + \lambda_i^{1/2} a_i(t) \right] dt + \sum_{i=1}^{\infty} a_i(0) c_i .$$

We choose $b_i(t) \in L^2(0, T)$ such that

$$\lambda_i^{1/2} b_i(t) = \frac{\dot{a}_i(t)}{\lambda_i^{1/2}} + \lambda_i^{1/2} a_i(t), \quad \text{and} \quad c_i = a_i(0) .$$

The bilinear form then becomes

$$\begin{aligned} & \int_I \sum_{i=1}^{\infty} \left[\frac{\dot{a}_i(t)}{\lambda_i^{1/2}} + \lambda_i^{1/2} a_i(t) \right]^2 dt + \sum_{i=1}^{\infty} [a_i(0)]^2 \\ &= \int_I \sum_{i=1}^{\infty} \left[\left(\frac{\dot{a}_i(t)}{\lambda_i^{1/2}} \right)^2 + (\lambda_i^{1/2} a_i(t))^2 \right] dt + \sum_{i=1}^{\infty} (a_i(T))^2 \\ &\geq \int_I \sum_{i=1}^{\infty} \left[\left(\frac{\dot{a}_i(t)}{\lambda_i^{1/2}} \right)^2 + (\lambda_i^{1/2} a_i(t))^2 \right] dt \geq C_1 \|u_\Lambda\|_{\underline{\mathcal{X}}}^2 . \end{aligned}$$

With the above choice of $b_i(t)$, we also have

$$\begin{aligned} \|v_{1\Lambda}\|_{L^2(U, \rho; L^2(I, V))}^2 &\leq (a_{\min})^{-1} \sum_{i=1}^{\infty} \int_I \lambda_i (b_i(t))^2 dt \\ (A.2) \quad &\leq (a_{\min})^{-1} 2 \sum_{i=1}^{\infty} \int_I \left[\left(\frac{\dot{a}_i(t)}{\lambda_i^{1/2}} \right)^2 + (\lambda_i^{1/2} a_i(t))^2 \right] dt \\ &\leq C_2 \|u_\Lambda\|_{\underline{\mathcal{X}}}^2 . \end{aligned}$$

We also have

$$\|v_{2\Lambda}\|_{L^2(U, \rho; H)}^2 = \|u_\Lambda(0, \cdot, \cdot)\|_{L^2(U, \rho; H)}^2 \leq M^2 \|u_\Lambda\|_{\underline{\mathcal{X}}}^2 .$$

Thus

$$\|v_\Lambda\|_{\underline{\mathcal{Y}}}^2 \leq (C_2 + M^2) \|u_\Lambda\|_{\underline{\mathcal{X}}}^2 .$$

Therefore, in this case

$$B(u_\Lambda, v_\Lambda) \geq C_1(C_2 + M^2)^{-1/2} \|u_\Lambda\|_{\underline{\mathcal{X}}} \|v_\Lambda\|_{\underline{\mathcal{Y}}}.$$

We now show condition (C). We again use

$$B(u_\Lambda, v_\Lambda) = \int_I \left(\sum_{i=1}^{\infty} \left[\frac{\dot{a}_i(t)}{\lambda_i^{1/2}} + \lambda_i^{1/2} a_i(t) \right] \lambda_i^{1/2} b_i(t) \right) dt + \sum_{i=1}^{\infty} a_i(0) c_i.$$

For each v_Λ , we choose $a_i(t)$ such that

$$\frac{\dot{a}_i(t)}{\lambda_i^{1/2}} + \lambda_i^{1/2} a_i(t) = \lambda_i^{1/2} b_i(t), \quad a_i(0) = c_i.$$

It then follows that $B(u_\Lambda, v_\Lambda) > 0$.

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