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Coordination in vendor-buyer inventory systems: on price discounts, Stackelberg game and joint optimisation

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Abstract: We consider a version of the Stackelberg game that is used to model discount pricing decisions in vendor-buyer supply chains. The game consists of a *leader* who is selling a product to one (or more) *follower(s)* who in turn sell it to the ultimate consumers. We define *conditional strategy* for the leader as a strategy where the transfer price offered by the leader is conditional upon the specific decision taken by the follower. We show that the leader's optimal conditional strategy can achieve perfect coordination. We then discuss the application of the result to specific models for discount pricing decisions in vendor-buyer inventory systems and interpret its implications for these models.

Keywords: vendor-buyer inventory systems; coordination; Stackelberg game; quantity and volume discounts.

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1 Introduction

With a greater emphasis on supply chain management among leading-edge companies, firms have been paying more attention to the coordination of supply chain inventories. A fully integrated supply chain that manages the system in totality is the ideal. This however might not be possible due to various reasons. First, the firms (vendors, buyers etc.) in the supply chain may belong to different corporate entities and therefore

would be keen on maximising their own profits rather than that of the supply chain as a whole. Even if the entire supply chain belonged to a single corporate entity, optimisation of the inventories across the entire supply chain might be difficult because of the incentive structures for various divisions within the organisation. Therefore, it is desirable to have mechanisms in place that can effectively coordinate the supply chain, even when the vendors, buyers and other entities deal on an arms-length basis, minimising their own inventory related costs.

This paper addresses the problem of coordinating inventory and pricing decisions in a supply chain or distribution channel consisting of one vendor and one (or more buyers). Without any coordination or special pricing arrangements, the vendor and buyer will act independently and take decisions that maximise their respective profits; however, this might not be optimal for the whole system. By special coordination mechanisms such as price discounts, one of the entities in the supply chain can motivate the other entity or entities to change their inventory and pricing policies so that every member in the supply chain benefits. Of course, cooperative joint optimisation is the ideal situation, but as discussed this might not be possible because the members in the distribution channel might have different and sometimes conflicting objectives. Hence, it is necessary to identify mechanisms or arrangements where either the system as a whole improves its profit or at least some of the members improve with the other members not being worse off.

In the last two decades, the problem of coordinating and optimising channel inventory decisions through the use of price discounts has gathered the attention of many researchers. An early paper by Crowther (1964) demonstrated the use of price discounts for improving channel coordination. Subsequently, work has progressed in two directions. One line of work approaches the problem from the perspective of jointly optimising the *system profit* (which is the total profit of all the entities in the distribution channel) and thereafter devising appropriate mechanisms to share the profits. The second line of research uses a Stackelberg game framework to analyse the issue. The Stackelberg game consists of two (or more) players, with the leader making the initial move or decision and follower(s) responding to this move in a rational manner. With complete information assumed to be available to the leader, she can anticipate the follower's move corresponding to her decision. The leader can therefore construct the follower's response function for all possible decisions by the leader, and thereby determine the profit function. The leader will of course take the decision that maximises her profit. In the context of channel inventory coordination, the leader is the seller (or wholesaler) and the follower is the buyer (or retailer). The leader has to decide on the transfer price per unit of the product to be charged to the follower. The follower then decides her pricing and inventory policy, specifically the annual demand volume (or retail price) and the replenishment quantity in each order. The attractiveness of a model based on the Stackelberg game approach is that no cooperation between channel members is necessary; the transactions between the buyer and vendor are assumed to be of an arms-length nature. While cooperation between the distribution channel members is the ideal scenario, in reality due to the different stakeholders and different objectives for the channel members, this might not be possible.

Papers that use the joint optimisation perspective include Dada and Srikanth (1987), Kohli and Park (1989), Banerjee (1986a), Chakravarty and Martin (1988, 1989) and Abad (1994). Papers that adopt the Stackelberg game framework include Dolan (1978), Jeuland and Shugan (1983), Monahan (1984), Rosenblatt and Lee (1985), Lee and

Rosenblatt (1986), Lal and Staelin (1984), Kim and Hwang (1988) and Drezner and Wesolowsky (1989). Almost all the papers mentioned above consider fixed (i.e., inelastic) demand with a single buyer. Parlar and Wang (1994), Weng (1995) and Weng and Wong (1993) have studied the problem with demand that is price-elastic, but with a single buyer. Chen et al. (2001) have also considered price-elastic demands and with multiple buyers.

For the model with price-inelastic demand, the papers using the joint optimisation approach show how optimal profits for the whole system can be achieved using cooperative, coordination mechanisms. However, most papers using the Stackelberg game approach are silent on this issue. In fact, the papers in the two streams of thought seem to give the impression that with an arms-length relationship between the leader and follower as in the Stackelberg game framework, *perfect coordination* of the system cannot be achieved (By definition, perfect coordination is achieved when the system profit is equal to the profit that would be achieved under joint optimisation).

The objective of this paper is to show that perfect channel coordination can in fact be achieved under the Stackelberg game framework. We show that by using specific types of conditional pricing strategies, one can actually obtain system profits equal to the joint optimal profit! This implies that in the context of channel coordination through price discounts, the two perspectives adopted in literature earlier, namely, joint optimisation with allocation mechanisms for the excess profit and Stackelberg game framework with seller as the leader should actually lead to the same result. This conclusion has in fact been alluded to by Banerjee (1986b) for price-inelastic demands. He demonstrates the equivalence of his joint optimisation approach in Banerjee (1986a) with the Stackelberg game approach in Monahan (1984). However, there was no further work along this line. Both Parlar and Wang (1994) and Weng (1995) have shown that for the model with price-elastic demand, quantity discounts alone cannot achieve full channel coordination and the system profits with the best quantity discount decision might still be worse off than the system profits under joint/combined optimisation. However, as we point out in this paper, perfect coordination can be achieved by offering quantity and volume discounts simultaneously.

The rest of this paper is organised as follows. In the next section, we state our assumptions and derive the main result of the paper in general terms. We then apply this result and interpret it for specific models. For each of these specific models, we show how the optimal Stackelberg strategy can be easily derived. In Section 3, we interpret the results for a single-vendor, single buyer model with price-elastic but deterministic demand. We then consider the case where the demand rate is fixed in Section 4. In Section 5, we consider the single vendor, multi-buyer model and interpret how our result can be applied in this case. Finally a few concluding remarks are provided in Section 6.

2 Stackelberg game and system optimality

Before we present the main result of our paper in this section, we state the assumptions and introduce some notations. All the assumptions made here including the assumptions about the nature of the profit functions are applicable to most of the discount pricing models in distribution channels studied in the literature.

Let the decision vector for the leader be $\{\mathbf{y}, c\}$, where the vector $\mathbf{y}$ corresponds to inventory and ordering decisions for the leader, and c is the unit transfer price for the product sold to the follower. Let the decision vector for the follower be $\mathbf{x} = \{x_j\}$. For deriving our main result, we only consider one follower in the Stackelberg game, but in a later section show how the result can be interpreted for the case of multiple buyers.

Let $Z_F = \phi_F(\mathbf{x}, c)$ be the profit for the follower. The follower's profit is a function of her decision $\mathbf{x}$, and the transfer price c . We assume that ϕ_F is concave with respect to each x_j individually; i.e., $\partial^2 \phi_F / \partial x_j^2 \leq 0$, for all j . We also assume that $\phi_F(\mathbf{x}, c)$ can be written as the difference of a term that is dependent only on $\mathbf{x}$ and the transfer payment made to the leader; i.e.,

$$\phi_F(\mathbf{x}, c) = g_F(\mathbf{x}) - x_1 c. \quad (1)$$

The transfer payment is the product of the transfer price c and decision variable x_1 which is one of the components of the decision vector $\mathbf{x}$. In the discount pricing models we consider in the subsequent sections, x_1 is the demand realised by the follower.

Let $Z_L = \phi_L(\mathbf{x}, \mathbf{y}, c)$ be the profit for the leader. The leader's profit is a function of the transfer price c , as well the decision vectors for the leader and follower, $\mathbf{x}$ and $\mathbf{y}$. Again, we assume that ϕ_L can be separated in the following manner:

$$\phi_L(\mathbf{x}, \mathbf{y}, c) = x_1 c - g_L(\mathbf{x}, \mathbf{y})$$

where ' $x_1 c$ ' is the revenue or transfer payment received from buyer and $g_L(\mathbf{x}, \mathbf{y})$ is the cost term that is dependent only on $\mathbf{x}$ and $\mathbf{y}$. Since the leader will always optimise $\mathbf{y}$ for a given $\mathbf{x}$ to minimise the cost term, the variable $\mathbf{y}$ can be eliminated from the above expression and the leader's profit function can then be written as

$$\phi_L(\mathbf{x}, c) = x_1 c - g_L(\mathbf{x}). \quad (2)$$

It is assumed that for transfer price c greater than a threshold value c_L , the leader's profit is increasing in $\mathbf{x}$. The assumption of concavity of ϕ_F is satisfied for a wide range of demand functions (see Viswanathan and Wang, 2003). The profit functions ϕ_F and ϕ_L can be written in the manner given in equations (1) and (2), when the inventory holding cost is specified as a fixed dollar value per unit per year rather than as a percentage of the unit cost or the transfer price (see a more detailed discussion in Section 4).

For the leader, we define two types of strategies. In a *pure strategy*, the leader just specifies a transfer price c unconditionally, and the follower is free to choose any decision $\mathbf{x} \geq \mathbf{0}$, where $\mathbf{0}$ is the zero vector. In this situation, the follower will optimise her decision $\mathbf{x}$. Let $\mathbf{x}^*(c)$ denote the follower's optimal decision for the price c . The equilibrium resulting from this strategy is the price c^* where the leader's profit is maximised. Let $\mathbf{x}^E = \mathbf{x}^*(c^*)$. The equilibrium point $(\mathbf{x}^E, c^*)$ is referred to in the literature as the initial equilibrium. Note that the initial equilibrium may be the result of a Stackelberg game model, or it may be the 'status-quo' based on other market conditions

that are exogenous to the Stackelberg game model. In a *conditional strategy* $(c^*, c^D, \mathbf{x}^D)$, the leader will specify two prices c^* and $c^D \leq c^*$, and a decision vector $\mathbf{x}^D > \mathbf{x}^E$. The price c^D applies if the follower takes a decision $\mathbf{x} \geq \mathbf{x}^D$. Otherwise the price c^* applies. Since,

- i both the leader and the follower strive to maximise their profits,
- ii the leader's profit is increasing in $\mathbf{x}$ for $c > c_L$
- iii the follower's profit is concave with each x_j ,

the conditional strategy $(c^*, c^D, \mathbf{x}^D)$ adopted by the leader will be such that $\phi_F(\mathbf{x}, c^D) \leq \phi_F(\mathbf{x}^D, c^D)$ for all $\mathbf{x} > \mathbf{x}^D$. Otherwise, the leader can find another conditional strategy that is better for both players. Therefore, the follower would choose the decision $\mathbf{x} = \mathbf{x}^D$ if she were to accept the conditional price c^D .

The total profit for the system can be written as

$$Z = Z_L + Z_F = g_F(\mathbf{x}) - g_L(\mathbf{x}). \quad (3)$$

Therefore, the system profit depends only on the follower's decision $\mathbf{x}$ and not on the transfer price. Let $\mathbf{x}^J$ be the value of $\mathbf{x}$ that maximises the system profit and let Z_J be the corresponding profit, i.e., Z_J is the optimal system profit obtained through joint optimisation, and $\mathbf{x}^J$ is the corresponding joint-optimal decision vector. We assume that¹ $\mathbf{x}^E \leq \mathbf{x}^J$.

At initial equilibrium, the profit for the leader is

$$Z_L^E = x_1^E c^* - g_L(\mathbf{x}^E)$$

and the profit for the follower is

$$Z_F^E = g_F(\mathbf{x}^E) - x_1^E c^*.$$

The system profit at initial equilibrium is

$$Z^E = Z_L^E + Z_F^E = g_F(\mathbf{x}^E) - g_L(\mathbf{x}^E) \leq Z^J = g_F(\mathbf{x}^J) - g_L(\mathbf{x}^J).$$

With a conditional strategy $(c^*, c^D, \mathbf{x}^D)$, the system profit depends on whether the conditional offer is attractive to the follower. The follower will accept the conditional transfer price c^D and adopt a decision $\mathbf{x} = \mathbf{x}^D$, only if her profit with this decision is better or at least not worse off than that at initial equilibrium. Therefore the system profit with the conditional strategy is

$$Z^C = \begin{cases} Z^E & \text{if } \phi_F(\mathbf{x}^E, c^*) > \phi_F(\mathbf{x}^D, c^D) \\ g_F(\mathbf{x}^D) - g_L(\mathbf{x}^D) & \text{otherwise} \end{cases}. \quad (4)$$

Of course, the leader's objective in adopting the conditional strategy is to improve her profits. If conditional offer is not accepted, then the profits for both the leader and the follower remain at that corresponding to the initial equilibrium. When the conditional offer is accepted, the leader's profit is $\phi_L(\mathbf{x}^D, c^D) = x_1^D c^D - g_L(\mathbf{x}^D)$. The leader will consider only strategies $(c^*, c^D, \mathbf{x}^D)$, where $\phi_L(\mathbf{x}^D, c^D) > \phi_L(\mathbf{x}^E, c^*)$. We are now ready to prove the main result of the paper.

Theorem 1: (i) There exists a conditional strategy $(c^*, c^J, \mathbf{x}^J)$, such that the system profit with this strategy Z^C is equal to the optimal system profit Z^J obtained through joint optimisation. (ii) Furthermore, $(c^*, c^J, \mathbf{x}^J)$, is the optimal conditional strategy for the leader.

Proof: If the conditional offer in the strategy $(c^*, c^J, \mathbf{x}^J)$ is attractive to the follower, then from equation (4), $Z^C = g_F(\mathbf{x}^J) - g_L(\mathbf{x}^J) = Z^J$. Therefore, to complete the proof of the theorem, we only need to show that

- a there exists a $c^J \leq c^*$ such that $\phi_F(\mathbf{x}^J, c^J) \geq \phi_F(\mathbf{x}^E, c^*)$
- b that this conditional strategy is optimal for the leader.

Setting

$$c^J = (g_F(\mathbf{x}^J) - g_F(\mathbf{x}^E) + x_1^E c^*) / x_1^J \quad (5)$$

will ensure that

$$\phi_F(\mathbf{x}^J, c^J) = \phi_F(\mathbf{x}^E, c^*). \quad (6)$$

Since $\mathbf{x}^E = \mathbf{x}^*(c^*)$ is the optimal strategy for the follower at price c^* , from (1)

$$g_F(\mathbf{x}^J) - g_F(\mathbf{x}^E) \leq c^* (x_1^J - x_1^E). \quad (7)$$

Combining equations (7) and (5), we get $c^J \leq c^*$.

The system profit with the above strategy is $Z^C = Z_L^C + Z_F^C = Z^J$. Since Z^J is the joint optimal system profit, Z_L^C will be the maximum possible profit for the leader, if Z_F^C is kept at its lowest possible value. As the profit for the follower can not be less than that at the initial equilibrium, the lowest possible value for Z_F^C is equal to Z_F^E . This is precisely the profit value achieved for the follower by setting c^J as in equation (5). Therefore, the conditional strategy $(c^*, c^J, \mathbf{x}^J)$ with c^J as determined by equation (5) achieves the optimal profit for the leader. $\square$

Note that perfect coordination would be achieved for any c^J in the following range:

$$(g_L(\mathbf{x}^J) - g_L(\mathbf{x}^E) + x_1^E c^*) / x_1^J \leq c^J \leq (g_F(\mathbf{x}^J) - g_F(\mathbf{x}^E) + x_1^E c^*) / x_1^J. \quad (8)$$

For values of c^J greater than right hand side of equation (8), the follower would be worse off than at initial equilibrium and therefore would not accept the conditional offer. For values of c^J smaller than left hand side of equation (8), the leader's profit would be lower than that at initial equilibrium, hence the leader would not consider values of the discounted transfer price to the left of the range in equation (8). For the range of values of c^J which is strictly within the range in equation (8), both the leader and follower would be better off than at initial equilibrium, and moreover perfect coordination would be achieved.

If only the leader had all the information about the follower's cost and demand parameters and not vice versa, and if the leader exerts the power in the channel, then follower would accept any transfer price that leaves her slightly better off than at initial equilibrium. Therefore a conditional strategy $(c^*, c^J - \delta, \mathbf{x}^J)$, would be an optimal strategy for the leader (where c^J is as given by equation (5) and δ is an infinitesimally small positive number). In this case the Stackelberg game approach is the appropriate model to

analyse the system. If however, the follower also has bargaining power in the supply chain and is privy to some of the leader's cost information, rather than just following the leader, she may resort to bargaining the transfer price offered in the conditional strategy. Therefore, the Stackelberg game may be followed by a bargaining game to determine actual transfer price. The actual transfer price would therefore depend on the channel power exerted by the two players. This shows that a joint optimisation approach with a coordination mechanism to share the additional profits (see references listed earlier such as Dada and Srikanth, 1987) would indeed be equivalent to the Stackelberg game approach in terms of the ultimate result achieved, especially if the bargaining power of the leader in the game is not absolute.

3 The single-vendor, single-buyer model with price-elastic demand

We now apply the result derived in the earlier section to the single-vendor, single-buyer, single product model with price-elastic demand. Versions of this model have been studied earlier by Parlar and Wang (1994), Weng (1995) and recently by Viswanathan and Wang (2003). We adopt the model parameters used in Viswanathan and Wang. The key notation and model assumptions are given below:

- c : The vendor's (leader's) transfer price or selling price per unit of the product sold to the buyer (follower)
- c_v : The vendor's own unit purchase cost from her supplier
- D : Demand rate for the product at the buyer. The demand is price elastic and therefore is a function of the price charged by the buyer
- p : Price charged by the buyer to its customers, $p = \psi(D)$, the price function which is the inverse of the demand function for the buyer
- $R(D)$: $= D\psi(D)$ is the buyer's dollar revenue corresponding to demand D . $R(D)$ is assumed to be concave and $R(0) = 0$. $R(D)$, $R'(D)$ and $R''(D)$ are continuous and differentiable in the range $D_l \leq D \leq D_u$, where the price function is defined. Also either $R'''(D) \leq 0$ (valid for linear demand function), or the demand function is Cobb-Douglas [$D = D_u e^{-\alpha p}$, or $\psi(D) = -(1/\alpha) \log(D/D_u)$], or constant price-sensitive [$D = \phi p^{-2}$]. The last assumption is required to prove that $\mathbf{x}^J \geq \mathbf{x}^E$ holds for this model
- K : The buyer's fixed ordering cost per order
- K_v : The vendor's fixed cost of processing one order placed by the buyer
- K_0 : The vendor's own fixed ordering cost per order
- h : Buyer's inventory holding cost rate
- h_v : The vendor's inventory holding cost rate
- Q : Buyer's replenishment order quantity.

Once, the vendor sets the transfer price c , the decisions facing the buyer is to set the retail price (or demand volume D) and the replenishment quantity in each order. Therefore the decision vector for the buyer is $\mathbf{x} = \{D, Q\}$. The buyers' profit

$$Z_F = \phi_F(\mathbf{x}, c) = \phi_F(D, Q, c) = R(D) - Dc - KD/Q - Qh/2. \quad (9)$$

Therefore, $g_F(\mathbf{x}) = g_F(D, Q) = R(D) - KD/Q - Qh/2$. Clearly, ϕ_F is concave with respect to D and Q individually.

The vendor will first set the transfer price c , in response to which the buyer adopts the decision $\mathbf{x} = \{D, Q\}$. As argued in Rosenblatt and Lee (1985), for a given D and Q , the vendor's replenishment order quantity with its supplier will be an integer multiple of the buyer's replenishment quantity Q . The vendor has to only decide her order quantity multiple and therefore the decision vector $\mathbf{y} = \{n\}$, where n is order quantity multiple for vendor. Therefore, the vendor's profit for a given $\mathbf{x}$, $\mathbf{y}$ and c is

$$Z_L = \phi_L(\mathbf{x}, \mathbf{y}, c) = Dc - Dc_v - (K_v D/Q) - (K_0 D/nQ) - (Q(n-1)h_v/2) \quad (10)$$

and $g_L(\mathbf{x}, \mathbf{y}) = Dc_v + (K_v D/Q) + (K_0 D/nQ) + (Q(n-1)h_v/2)$. Clearly, for c greater than a certain threshold value, ϕ_L is increasing in D . In Theorem 2 in Appendix, we show that ϕ_L is increasing in Q . The system profit

$$Z = g_F(\mathbf{x}) - g_L(\mathbf{x}, \mathbf{y}) = R(D) - \frac{KD}{Q} - \frac{1}{2}Qh - Dc_v - \frac{K_v D}{Q} - \frac{K_0 D}{nQ} - \frac{1}{2}Q(n-1)h_v. \quad (11)$$

In Theorem 3 in the Appendix, it is shown that $\mathbf{x}^J \geq \mathbf{x}^E$ where $\mathbf{x}^J = \{D^J, Q^J\}$ is the joint optimal decision vector that maximises (11) above, and $\mathbf{x}^E = \{D^*, Q^*\}$ is the buyer's demand and order quantity at the initial Stackelberg equilibrium. Therefore all the assumptions and conditions laid out in the general model in Section 2 are satisfied for this model.

Theorem 1 implies that a conditional strategy of the form $(c^*, c^J, \mathbf{x}^J = \{D^J, Q^J\})$ will achieve perfect coordination. That is, a simultaneous offer of volume and quantity discount, with minimum annual volume of D^J , and minimum replenishment order quantity Q^J can achieve perfect coordination. The combined price discount for the two discount offers can be calculated using equation (5) and is

$$c^* - c^J = \frac{1}{D^J} \left(R(D^*) - R(D^J) - \sqrt{2KhD^*} + \frac{KD^J}{Q^J} + \frac{1}{2}Q^J h + c^*(D^J - D^*) \right).$$

In practice, even when the quantity and volume discounts are offered simultaneously, the price discount components for the quantity discount and volume discount might have to be specified separately. As argued and proved in Viswanathan and Wang (2003), this can always be done such that it is better for the buyer to accept both the discount offers.

Boyaci and Gallego (2002) have also provided a result similar to Theorem 1 for the model considered in this section. They have argued that perfect coordination can be achieved by inducing the buyer to make the joint optimal decision (that is demand volume of D^J and order quantity of Q^J). They had suggested several methods to achieve this. However, their suggestion involve either forcing the retailer to take the joint optimal decision by offering a single price discount, or having to specify a continuous transfer price function for different values of annual demand or charging a franchise fee and providing a quantity discount only for a order quantity exactly equal to Q^J , and not for a larger quantity. We have shown that perfect coordination can be achieved by offering a traditional type of quantity discount and volume discount independently and

simultaneously. Viswanathan and Wang (2003) had shown in their computational results that a combination of quantity and volume discounts achieved perfect coordination for all the problems that they tested; however they provided no analytical proof. This paper has provided an analytical proof for this result. Moreover our proof in Section 2 is more general and can be applied to more general models such as systems with shelf-space dependent demand considered by Wang and Gerchak (2001), and systems incorporating other decision variables (such as advertising dollars and promotion efforts etc.) that influence demand.

In summary, for the model with price-elastic demand, quantity or volume discount alone may not always achieve perfect coordination, but a simultaneous offer of volume and quantity discount under a Stackelberg game framework can always achieve perfect coordination.

4 Model with fixed demand

We now consider the single-vendor, single-buyer model with price-inelastic or fixed demand. As mentioned in the introduction, this is the model that has been most widely studied. We use the same notation as in the previous section except that now the demand D is fixed. The retail price (buyer's selling price) p corresponding to this demand, and the vendor's initial selling price is provided as an input to the model. That is, the initial equilibrium demand D (and corresponding price p) and the transfer price c^* are provided exogenously rather than determined by a Stackelberg game.

As the demand is fixed, the only decision that the buyer has to make is the replenishment order quantity Q . Therefore, the decision vector for the buyer $\mathbf{x} = \{Q\}$. The buyers' profit

$$Z_F = \phi_F(\mathbf{x}, c) = \phi_F(Q, c) = pD - Dc - KD/Q - Qh/2, \text{ which is concave in } Q.$$

Therefore, $g_F(\mathbf{x}) = g_F(Q) = pD - KD/Q - Qh/2$.

The vendor's decision vector as before is $\mathbf{y} = \{n\}$, where n is the order quantity multiple for vendor. The vendor's profit for a given $\mathbf{x}$, $\mathbf{y}$ and c is

$$Z_L = \phi_L(\mathbf{x}, \mathbf{y}, c) = Dc - Dc_v - (K_v D/Q) - (K_0 D/nQ) - (Q(n-1)h_v/2).$$

Therefore, $g_L(\mathbf{x}, \mathbf{y}) = Dc_v + (K_v D/Q) + (K_0 D/nQ) + (Q(n-1)h_v/2)$. From Theorem 2, ϕ_L is increasing in Q . The system profit

$$Z = g_F(\mathbf{x}) - g_L(\mathbf{x}, \mathbf{y}) = D(p - c_v) - \frac{D}{Q} \left(K + K_v + \frac{K_0}{n} \right) - \frac{Q}{2} (h + (n-1)h_v). \quad (12)$$

At the initial equilibrium $\mathbf{x}^E = \{Q^*\}$, where $Q^* = \sqrt{2KD/h}$ is the economic order quantity for the buyer. Let the Q that maximises (12) be Q' . That is $\mathbf{x}^J = \{Q'\}$. In Theorem 4 in the Appendix, we show that $Q' \geq Q^*$. Hence all the conditions laid out in the general model are satisfied for this model also.

Theorem 1 for this model implies that a conditional strategy of the form $(c^*, c^J, \mathbf{x}^J = \{Q'\})$ will achieve perfect coordination. Therefore, for the model with fixed demand, a quantity discount scheme under the Stackelberg game framework can

achieve perfect coordination. The price discount to be offered can be calculated using equation (5) and is

$$c^* - c^j = \frac{1}{D} \left(\frac{KD}{Q^j} + \frac{1}{2} Q^j h - \sqrt{2KhD} \right).$$

Papers adopting the joint optimisation perspective show how coordination mechanisms can be designed to achieve joint optimality. Some authors such as Banerjee (1986b) have alluded to the similarity between the joint optimisation and Stackelberg game approach, implying that the Stackelberg game approach can achieve perfect coordination. However, to our knowledge, none of the papers have explicitly stated this.

Part of the reason that the above result may not be so obvious is that in many of papers, the holding cost is assumed to be proportional to the unit cost, rather than a fixed amount. In such a situation, the system profit and the joint optimal solution would be dependent on the transfer price. For instance, let $h = rc$ and $h_v = r_v c_v$. Then the vendor's profit, buyer's profit and the system profit are

$$\begin{aligned} Z_F &= \phi_F(\mathbf{x}, c) = \phi_F(Q, c) = pD - Dc - KD/Q - Qrc/2 \\ Z_L &= \phi_L(\mathbf{x}, \mathbf{y}, c) = Dc - Dc_v - (K_v D/Q) - (K_0 D/nQ) - (Q(n-1)r_v c_v/2) \\ Z &= D(p - c_v)D - (D/Q)(K + K_v + K_0/n) - (1/2)Q(rc - (n-1)r_v c_v). \end{aligned} \quad (13)$$

The profit for the vendor, buyer and the system can not be written in the manner done in equations (1)–(3), and therefore the general result developed in Section 2 is not applicable directly. We suggest two possible ways in which this problem can be overcome, and how one can obtain a conditional strategy (i.e., quantity discount scheme) for the vendor that results in optimal or near optimal system profits.

In the first approach, one can determine the buyer's replenishment order quantity $Q^j(c)$ that optimises (13) for a given transfer price c . Corresponding to this order quantity, one can determine that transfer price c^q that will ensure that the buyer's profit is the same as that at initial equilibrium. If $c^q = c$, then $(c^*, c, Q^j(c))$ is the optimal conditional strategy or quantity discount scheme. One can search for the transfer price c in a valid range to determine the optimal discount scheme. However, the problem with this approach is that there may be more than one such value of the transfer price c , or even possibly none at all. If there are more than one such value of c , then from among these, the one that results in the highest system profit should be chosen. If there are no value of c such that $c^q = c$, then one might have to be happy with a satisfactory solution.

The specification/determination of the proportional holding cost rate itself has assumptions behind it. In reality, holding cost consists of two components, the warehousing and material handling related cost and the interest cost. The interest cost is determined based on the cost of capital for the firm, and the warehousing and other related cost are calculated based on the average level of inventory and allocation of the fixed warehousing cost to this inventory. Clearly, the warehousing related costs are not based or dependent on the transfer price. The interest cost calculation itself involves assumptions about how the cost of capital of the firm is calculated. In other words, the proportional holding cost rate is itself an approximate number, and therefore the vendor can use a rate based on her unit purchase cost i.e., $h = r_F c_v$. If the whole system belonged to one entity, then the proportional holding cost rate would in fact be defined with respect to the vendor's purchase cost rather than the transfer price. Therefore, one

could obtain the joint optimal Q' using $h = r_F c_v$, and thereafter find the transfer price c such that the buyer is not worse off than at initial equilibrium. This quantity discount scheme will result in system profits that are close to the optimal (at least from the vendor's perspective).

In summary, for the single-buyer, single-vendor model with fixed demand, a quantity discount scheme under a Stackelberg game framework can achieve perfect coordination. However, when the buyer's holding cost rate is defined as a proportion of the transfer price, then perfect coordination may not always be achieved in a theoretical sense.

5 The single-vendor, multiple-buyer model

We finally interpret our result for the single-vendor, multi-buyer model. A more detailed discussion of this model is provided in an unabridged working paper. We just summarise our key results here. Applying the result in Section 2 for the single-vendor, multi-buyer, supply chain with fixed demand, we show that a separate buyer-specific quantity discount offer for each buyer under a Stackelberg game framework will achieve perfect coordination. When the demand is price-elastic and there are multiple buyers, a separate buyer-specific simultaneous offer of quantity and volume discount for each buyer will achieve perfect coordination. However, offering different price discounts for different buyers might violate fair trade laws such as the Robinson-Patman Act (see Stern et al., 1996). But it might be justified in some cases based on the different cost parameters for the buyers. For the multi-buyer model, development of a perfect coordination mechanism that does not violate free trade laws is still an open research issue.

6 Concluding remarks

The main contribution of this paper is to show that perfect coordination of inventories or optimal system profits can be achieved in distribution channels under an arms-length, non-cooperative approach such as the Stackelberg game framework, through the use of conditional strategies. For models with fixed demand, the optimal quantity discount scheme can achieve this. For models with price elastic demand, a simultaneous offer of quantity and volume discounts is required to achieve this. However, for the model with multiple buyers, perfect coordination can be achieved only by offering buyer-specific quantity/volume discounts and such conditional strategies might possibly violate fair trade laws.

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Notes

¹As we show in the subsequent sections, for the each of the specific models that we consider, the main assumptions viz.

- i concavity of ϕ_F with respect to each x_j
 - ii ability to partition the profit functions for leader and follower as shown in equations (1) and (2)
 - iii the leader's profit increasing in $\mathbf{x}$, for $c > c_L$
 - iv $\mathbf{x}^E \leq \mathbf{x}^J$,
- are valid.

Appendix

Theorem 2: For the single-vendor, single-buyer model with price elastic demand considered in section 3, the vendor's profit given by equation (10) is increasing in the buyer's order quantity Q .

Proof: For a fixed D , and fixed Q , the value of n that maximises (10) is given by the smallest n^* that satisfies

$$n^*(n^*-1) < (2K_0D/Q^2h_v) \leq n^*(n^*+1). \quad (14)$$

The value of n in equation (10) determined by expression (14) above is optimal only for a particular range of Q , and as Q increases beyond this range, n will decrease. Let $\tilde{n}$ be the optimal value of n for Q in the range $Q_l \leq Q \leq Q_u$. At the transition point Q_u , (14) will be satisfied for $\tilde{n}-1$ with equality on the right hand side and both $\tilde{n}$ and $\tilde{n}-1$ will result in same value of ϕ_L . That is $\tilde{n}(\tilde{n}-1) = 2K_0D/Q_u^2h_v$. Therefore,

$$K_0D/\tilde{n}Q_u^2 = (\tilde{n}-1)h_v/2. \quad (15)$$

Now for $Q_l \leq Q \leq Q_u$,

$$\frac{dZ_L}{dQ} = K_vD/Q^2 + K_0D/\tilde{n}Q^2 - (\tilde{n}-1)h_v/2. \quad (16)$$

From equation (15), the second and thirds terms in equation (16) are equal at $Q = Q_u$. Since $Q < Q_u$, the second term is larger than the third term in equation (16) and therefore $dZ_L/dQ > 0$. The same result can be proved in a similar manner for every range of Q , corresponding to different values of n . This implies that for fixed D , ϕ_L increases as Q increases, and this completes the proof of the proposition. $\square$

Theorem 3: For the single-vendor, single-buyer model with price elastic demand considered in Section 3, let the buyer's decision corresponding to the initial equilibrium be $\mathbf{x}^E = \{D^*, Q^*\}$, and let the decision vector corresponding to the joint optimal solution be $\mathbf{x}^J = \{D^J, Q^J\}$. Then $\mathbf{x}^J \geq \mathbf{x}^E$.

Proof: We prove in Theorem 4 that for a fixed D , $Q^J \geq Q^*$. Also, it can be shown that the order quantity Q that maximises (9) and (11) increases as D increases. Therefore, we need only prove that $D^J \geq D^*$. Optimising the system profit in equation (11) with respect to n and Q , we can write it as

$$Z(D) = R(D) - Dc_v - \sqrt{2LD} \quad (17)$$

where $L = K + K_v + (K_0/n)(h - h_v + nh_v)$ and n is the smallest n^* that satisfies

$$n^*(n^* - 1) \leq ((h - h_v)K_0)/(h_v(K + K_v)) \leq n^*(n^* + 1). \quad (18)$$

The buyer's profit at the initial equilibrium price c^* similarly can be written as

$$Z_F(D) = R(D) - Dc^* - \sqrt{2KhD}. \quad (19)$$

Since $D = 0$ (and correspondingly zero profits for vendor and buyer) is a demand decision choice available, the buyer's profit at the initial equilibrium demand D^* will not be more than the system profit at the same demand. Using equations (17) and (19), this condition can be written as

$$D^*C_v + \sqrt{2LD^*} \leq D^*c^* + \sqrt{2KhD^*}.$$

As $L > Kh$, we have

$$D^*C_v + \sqrt{\frac{LD^*}{2}} \leq D^*c^* + \sqrt{\frac{KhD^*}{2}}. \quad (20)$$

As proved in Viswanathan and Wang (2003), for demand functions with $R'''(D) < 0$ (e.g., linear demand function), and for the Cobb-Douglas [$\psi(D) = -(1/\alpha)\log(D/D_u)$], and constant price-sensitive [$D = \phi p^{-2}$] demand functions, $Z'(D) \geq 0$ and $Z'_F(D) \geq 0$ only in at most one contiguous range of D . If $Z'_F(D) < 0$ for all $D > 0$, then clearly $D^J \geq D^* = 0$. Therefore, we need only prove that $D^J \geq D^*$ for the case where $D^* > 0$.

If $D^* > 0$, then since D^* is the optimal demand for the buyer at the initial equilibrium transfer price c^* , $Z'_F(D^*) = 0$. To prove that $D^J \geq D^*$, we need only show that $Z'(D^*) \geq Z'_F(D^*)$, which we do below.

$$\begin{aligned} Z'(D^*) - Z'_F(D^*) &= -c_v - \sqrt{\frac{L}{2D^*}} + c^* + \sqrt{\frac{Kh}{2D^*}} \\ &= \frac{1}{D^*} \left(-D^*c_v - \sqrt{\frac{LD^*}{2}} + D^*c^* + \sqrt{\frac{KhD^*}{2}} \right) \geq 0, \end{aligned}$$

from equation (20). □

Theorem 4: For the single-vendor, single-buyer model with fixed demand, let the buyer's optimal replenishment order quantity at initial equilibrium be Q^* , and let the buyer's replenishment order quantity corresponding to the joint optimal solution be Q^J . Then $Q^J \geq Q^*$.

Proof: $Q^* = \sqrt{2KD/h}$, Optimising equation (12) we get

$$Q^J = \sqrt{\frac{2D(K + K_v + K_0/n)}{h + (n-1)h_v}}$$

where, n is given by the smallest n^* that satisfies equation (18). From equation (18) we get,

$$\frac{K_0/n}{(n-1)h_v} \geq \frac{K + K_v}{h - h_v} \geq \frac{K + K_v}{h}.$$

Therefore,

$$\frac{K_0/n}{(n-1)h_v} \geq \frac{K + K_v + K_0/n}{h + (n-1)h_v} \geq \frac{K + K_v}{h} \geq \frac{K}{h}.$$

Hence $Q^J \geq Q^*$. □