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2015

Ducoat, J. (2015). Generalized rank weights : a duality statement. Contemporary mathematics, 632, 101-109.

<https://hdl.handle.net/10356/100886>

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Generalized rank weights: a duality statement

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The main part of this paper has been presented to the conference Finite Fields and its Applications, Magdeburg, Germany, Jul. 2013

Abstract

We consider linear codes over some fixed finite field extension $\mathbb{F}_{q^m}/\mathbb{F}_q$, where $\mathbb{F}_q$ is an arbitrary finite field. In [1], Gabidulin introduced rank metric codes, by endowing linear codes over $\mathbb{F}_{q^m}$ with a rank weight over $\mathbb{F}_q$ and studied their basic properties in analogy with linear codes and the classical Hamming distance. Inspired by the characterization of the security in wiretap II codes in terms of generalized Hamming weights by Wei [8], Kurihara *et al.* defined in [3] some generalized rank weights and showed their relevance for secure network coding. In this paper, we derive a statement for generalized rank weights of the dual code, completely analogous to Wei's one for generalized Hamming weights and we characterize the equality case of the r^{th} -generalized Singleton bound for the generalized rank weights, in terms of the rank weight of the dual code.

I. INTRODUCTION

Let q be the power of some prime number, let $m \geq 1$. We denote by $\mathbb{F}_q$ (resp. $\mathbb{F}_{q^m}$) the field (unique up to isomorphism) with q (resp. q^m) elements. Then $\mathbb{F}_{q^m}/\mathbb{F}_q$ is a field extension of degree m .

Let $n \geq 1$ and consider the vector space $\mathbb{F}_{q^m}^n$. Let $(u_1, \dots, u_m)$ be a basis of $\mathbb{F}_{q^m}$, seen as an m -dimensional vector space over $\mathbb{F}_q$. For every $x = [x_1, \dots, x_n] \in \mathbb{F}_{q^m}^n$, there exist some coefficients $x_{j,i} \in \mathbb{F}_q$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ such that, for every $1 \leq i \leq n$,

$$x_i = \sum_{j=1}^m x_{j,i} u_j.$$

We then set

$$\lambda(x) = [x_{i,j}] \in \text{Mat}_{n,m}(\mathbb{F}_q).$$

Let C be a linear code over $\mathbb{F}_{q^m}$ of length n (i.e. a vector subspace of $\mathbb{F}_{q^m}^n$). Gabidulin ([1] and also Roth independently, [5]) defined the rank of a vector $x \in \mathbb{F}_{q^m}^n$ (denoted by $\text{rk}(x)$) to be the rank of $\lambda(x)$, the rank distance between two codewords $x, y \in C$ to be $\text{rk}(x - y)$ and the rank weight of C by

$$d(\lambda(C)) = \min_{x \in C \setminus \{0\}} \text{rk}(x).$$

In [6], in the framework of linear network coding, Silva and Kschischang proposed the rank distance to characterize when wiretap network codes achieve perfect secrecy.

A natural question arose then, about the existence of generalized rank weights, in analogy with the generalized Hamming weights defined by Wei in [8], known to describe the equivocation of the eavesdropper for wiretap II codes.

A first step in this direction was given by Oggier and Sbouy [4] and was completed independently in [3], by Kurihara, Matsumoto and Uyematsu. We first introduce some tools. For every $x = [x_1, \dots, x_n] \in \mathbb{F}_{q^m}^n$, we denote by x^q the vector $[x_1^q, \dots, x_n^q]$. For every vector subspace $V \subset \mathbb{F}_{q^m}^n$, we set $V^q = \{x^q \mid x \in V\}$.

We then consider the set $\Gamma(\mathbb{F}_{q^m}^n) = \{V \subset \mathbb{F}_{q^m}^n \mid V^q = V\}$. For every vector subspace V of $\mathbb{F}_{q^m}^n$, we set

$$V^* = \sum_{j=0}^{m-1} V^{q^j}.$$

Then V^* is the smallest subspace containing V and belonging to $\Gamma(\mathbb{F}_{q^m}^n)$.

Recall that C is a linear code over $\mathbb{F}_{q^m}$ of length n . Let k be its dimension. For every $1 \leq r \leq k$, a refinement of the definition proposed by Oggier and Sboui for the r^{th} -generalized rank weight in [4] is

$$d_r(\lambda(C)) = \min_{\substack{D \subset C \\ \dim D=r}} \max_{x \in D^*} \text{rk}(\lambda(x))$$

and the definition proposed by Kurihara, Matsumoto and Uyematsu in [3] is

$$\mathcal{M}_r(C) = \min_{\substack{V \in \Gamma(\mathbb{F}_{q^m}^n) \\ \dim(C \cap V) \geq r}} \dim V.$$

Notice that the D^* involved in the first definition means the smallest subspace containing D and stable by the q -power componentwise, as defined above.

We let the reader note that these two definitions are given in analogy with the r^{th} -generalized Hamming weight, defined as follows by Wei in [8] : for every $1 \leq r \leq k$,

$$d_r(C) = \min_{\substack{D \subset C \\ \dim D=r}} \max_{x \in D} |\text{Supp}(D)| = \min_{\substack{V \in \Lambda(\mathbb{F}_{q^m}^n) \\ \dim(V \cap C) \geq r}} \dim V,$$

where $\text{Supp}(D) = \{i \in \{1, \dots, n\} \mid \exists x = [x_1, \dots, x_n] \in D, x_i \neq 0\}$, $|\cdot|$ denotes the order of a set, and $\Lambda(\mathbb{F}_{q^m}^n)$ is the set of the vector subspaces of $\mathbb{F}_{q^m}^n$, generated by elements of the canonical basis. Note that the right equality is easy to check in that case.

Kurihara, Matsumoto and Uyematsu proved the following ([3], Lemma 11).

Proposition I.1.

$$\text{Let } n \leq m. \text{ For every } x \in \mathbb{F}_{q^m}^n, \dim(\langle x \rangle^*) = \text{rk}(\lambda(x)).$$

This immediately shows that $\mathcal{M}_1(C) = d(\lambda(C)) = d_1(\lambda(C))$. In Section II, we prove that $\mathcal{M}_r(C) = d_r(\lambda(C))$ for every $1 \leq r \leq k$ in the case where $n \leq m$.

In [3], Kurihara, Matsumoto and Uyematsu proved the following monotonicity property ([3], Lemma 9):

Theorem I.2. *We have $1 \leq \mathcal{M}_1(C) < \mathcal{M}_2(C) < \dots < \mathcal{M}_k(C) \leq n$.*

We also give in Section II a different proof of this statement. Note that the monotonicity property legitimates these two definitions as a suitable candidate for the notion of generalized rank weight.

In Section III, we continue the analogy with generalized Hamming weights, extending to generalized rank weights the statement that Wei proved in [8], Theorem 3. Let $C^\perp$ denote the dual code, that is to say the orthogonal vector subspace with respect to the usual bilinear form

$$\langle \cdot, \cdot \rangle : ([x_1, \dots, x_n], [y_1, \dots, y_n]) \mapsto \sum_{i=1}^n x_i y_i.$$

We then link the generalized rank weights of the dual code $C^\perp$ to the generalized rank weights of C :

Theorem I.3. *Let C be a linear code of dimension k over $\mathbb{F}_{q^m}$ and of length n . Then*

$$\{\mathcal{M}_r(C) \mid 1 \leq r \leq k\} = \{1, \dots, n\} \setminus \{n+1 - \mathcal{M}_r(C^\perp) \mid 1 \leq r \leq n-k\}.$$

As a consequence of this statement, we end this paper by deriving a characterization of the equality case in the r^{th} -generalized Singleton bound for the generalized rank weights ([3], Proposition 10), in terms of the rank weight of the dual code.

II. GENERAL PROPERTIES FOR THE GENERALIZED RANK WEIGHTS

The aim of this section is to prove that both previously proposed generalized weights are the same.

Proposition II.1. *Let $n \leq m$. For every $1 \leq r \leq k$, $d_r(\lambda(C)) = \mathcal{M}_r(C)$.*

Proof. Let us first prove that $d_r(\lambda(C)) \leq \mathcal{M}_r(C)$. Let $V \in \Gamma(\mathbb{F}_{q^m}^n)$ such that $\dim(C \cap V) \geq r$. Let D be a subspace of $C \cap V$ of dimension r . For every $x \in D^*$, by Proposition I.1,

$$\dim(\langle x \rangle^*) = \text{rk}(\lambda(x)).$$

Since D^* is the smallest invariant subset containing D , then $D^* \subset V$, so $x \in V$ and since V is invariant by the elevation to the power q , we have $\langle x \rangle^* \subset V$, so $\dim(\langle x \rangle^*) \leq \dim V$. Hence, for every $x \in D^*$, $\text{rk}(\lambda(x)) \leq \dim V$, thus

$$\max_{x \in D^*} \text{rk}(\lambda(x)) \leq \dim V.$$

Therefore,

$$d_r(\lambda(C)) \leq \dim V.$$

Since this inequality is true for every invariant subspace V such that $\dim(V \cap C) \geq r$, we get that

$$d_r(\lambda(C)) \leq \mathcal{M}_r(C).$$

We now come to the converse inequality. It follows from the following lemma :

Lemma II.2. *Assume that $n \leq m$. Let $V \in \Gamma(\mathbb{F}_{q^m}^n)$. Then there exists $x \in V$ such that $V = \langle x \rangle^*$.*

Proof. Let l be the dimension of V . Then there exists some basis $(e_1, \dots, e_l)$ of V coming from $\mathbb{F}_q$ (i.e. every coefficient of the e_i belongs to $\mathbb{F}_q$, see [7], Lemma 1). Let $x \in V$ with coefficients $x_1, \dots, x_l$ when x is decomposed in the basis $(e_1, \dots, e_l)$ (these coefficients belong to $\mathbb{F}_{q^m}$). Assume that the family $(x_1, \dots, x_l)$ is free over $\mathbb{F}_q$. Then a vector $y = \sum_{i=1}^l y_i e_i$ in V belongs to $\langle x \rangle^*$ if and only if there exist some $\mu_0, \dots, \mu_{m-1} \in \mathbb{F}_{q^m}$ such that, for every $i = 1 \dots l$,

$$y_i = \sum_{j=0}^{m-1} \mu_j x_i^{q^j},$$

which is equivalent to

$$\begin{bmatrix} y_1 \\ \vdots \\ y_l \end{bmatrix} = \begin{bmatrix} x_1 & x_1^q & \cdots & x_1^{q^{m-1}} \\ \vdots & \vdots & & \vdots \\ x_l & x_l^q & \cdots & x_l^{q^{m-1}} \end{bmatrix} \begin{bmatrix} \mu_0 \\ \vdots \\ \mu_{m-1} \end{bmatrix}.$$

Since the family $(x_1, \dots, x_l)$ is free over $\mathbb{F}_q$, the matrix

$$\begin{bmatrix} x_1 & x_1^q & \cdots & x_1^{q^{m-1}} \\ \vdots & \vdots & & \vdots \\ x_l & x_l^q & \cdots & x_l^{q^{m-1}} \end{bmatrix}$$

has maximal rank l . Therefore, $\dim(\langle x^* \rangle) = l = \dim V$, which proves that $V = \langle x^* \rangle$. $\square$

This completes the proof of Proposition II.1. $\square$

We continue Section II by giving another proof of the monotonicity property, already stated by Kurihara, Matsumoto and Uyematsu ([3], Lemma 9). More precisely, we prove here the following proposition.

Proposition II.3. *Let C be a linear code of dimension k and length n over $\mathbb{F}_{q^m}$. Then, for every $1 < r \leq k$,*

$$(q^{mr} - 1)\mathcal{M}_{r-1}(C) \leq (q^{mr} - q^m)\mathcal{M}_r(C).$$

Proof. Let $1 < r \leq k$. Let t denote the quotient $\frac{q^{mr}-1}{q^m-1}$. It is well-known that t is the number of $(r-1)$ -dimensional subspaces in a vector space of dimension r over $\mathbb{F}_{q^m}$ (see for instance [2] Exercise 431).

Let D be an r -dimensional subspace of C such that $\mathcal{M}_r(C) = \dim D^*$. We enumerate by $D_1, \dots, D_t$ the list of all the $(r-1)$ -dimensional subspaces of D .

We want to show that

$$(q^{mr} - 1)\mathcal{M}_{r-1}(C) \leq (q^{mr} - q^m)\mathcal{M}_r(C),$$

i.e. that

$$(q^{mr} - 1)(\mathcal{M}_r(C) - \mathcal{M}_{r-1}(C)) \geq (q^m - 1)\mathcal{M}_r(C),$$

which is equivalent to

$$t(\mathcal{M}_r(C) - \mathcal{M}_{r-1}(C)) \geq \mathcal{M}_r(C).$$

Moreover, $\mathcal{M}_r(C) = \dim D^*$ and for every $1 \leq i \leq t$, $\dim D_i^* \geq \mathcal{M}_{r-1}(C)$, so it is enough to prove that

$$\sum_{i=1}^t (\dim D^* - \dim D_i^*) \geq \dim D^*. \quad (1)$$

Set $s = \dim D^*$. Since D^* belongs to $\Gamma(\mathbb{F}_{q^m}^n)$, we can find a basis $(e_1, \dots, e_s)$ of elements which have coordinates in $\mathbb{F}_q$ (see [7], Lemma 1). For $1 \leq j \leq s$, let V_j be the $(s-1)$ -dimensional subspace of D^* generated by the family $(e_1, \dots, \widehat{e_j}, \dots, e_s)$, where the $\widehat{e_j}$ means that the vector e_j is excluded from this family. These vector spaces V_j belong to $\Gamma(\mathbb{F}_{q^m}^n)$ (since they have a basis with coordinates in $\mathbb{F}_q$) and have dimension $s-1$.

Let $1 \leq j \leq s$ and consider the intersection $V_j \cap D$. Then $V_j \cap D \subsetneq D$ (otherwise it would contradict the minimality of $\dim D^*$). Since $D \not\subset V_j$, $\dim(V_j + D) > \dim V_j$, then $\dim(V_j + D) = \dim D^* = s$ and we have

$$\dim(V_j \cap D) = \dim V_j + \dim D - \dim(V_j + D) = s - 1 + \dim D - s = \dim D - 1 = r - 1.$$

Therefore, there exists $i_j \in \{1, \dots, t\}$ such that $D_{i_j} = V_j \cap D$. Here we catch the reader's attention on the fact that the i_j might be the same for different indices j . Up to reindexing the basis $(e_1, \dots, e_s)$ (and hence the subspaces $V_1, \dots, V_s$), we can assume that there exist some integers $t_1, \dots, t_s$ such that

$$\begin{aligned} &\text{for every } 1 \leq l \leq t_1, V_l \cap D = D_{i_{t_1}}, \\ &\text{for every } t_1 + 1 \leq l \leq t_2, V_l \cap D = D_{i_{t_2}}, \\ &\quad \dots \\ &\text{for every } t_s + 1 \leq l \leq t_s = s, V_l \cap D = D_{i_{t_s}}, \end{aligned}$$

with the subspaces $D_{i_{t_1}}, \dots, D_{i_{t_s}}$ two by two distinct.

Thus, we have, for every $1 \leq j \leq s$, $D_{i_{t_j}}^* \subset V_{t_{j-1}+1} \cap \dots \cap V_{t_j}$ (with the convention that $t_0 = 0$) and taking dimensions,

$$\dim D_{i_{t_j}}^* \leq s - (t_j - t_{j-1}).$$

Therefore,

$$\sum_{i=1}^s (\dim D^* - \dim D_{i_{t_j}}^*) \geq \sum_{i=1}^s (t_j - t_{j-1}) = t_s - t_0 = s.$$

Since we have the obvious inequality

$$\sum_{i=1}^s (\dim D^* - \dim D_{i_{t_j}}^*) \leq \sum_{i=1}^t (\dim D^* - \dim D_i^*),$$

Inequality (1) holds, which completes the proof of Proposition II.3. $\square$

As an immediate consequence of the monotonicity property (Theorem I.2), Kurihara, Matsumoto and Uyematsu stated that the generalized Singleton bounds hold for generalized rank weights ([3], Proposition 10).

Corollary II.4. *Keeping the notation above, let $1 \leq r \leq k$. Then, we have*

$$\mathcal{M}_r(C) \leq n - k + r.$$

We also remark here that it directly followed from the fact that for every $1 \leq r \leq k$, $\mathcal{M}_r(C)$ is always lower than or equal to the r^{th} -generalized Hamming weight.

Definition 1. *Keeping the notation above, we say that a linear code C of dimension k and length n over $\mathbb{F}_{q^m}$ is r^{th} -rank MRD (or in short r -MRD) if we have $\mathcal{M}_r(C) = n - k + r$.*

At the end of Section III, we give a characterization for a code to be r -MRD in terms of the (first) rank distance of its dual code $C^\perp$.

Note also that for (generalized) Hamming weights, a refinement of the (generalized) Singleton bound, called Griesmer bound holds (see for instance [2], Theorem 7.10.10). It is then natural to wonder whether such analogous bounds hold for the generalized rank weights. The answer is positive but due to the constraints on q , m and n , these bounds are exactly identical to the generalized Singleton bounds.

III. DUALITY AND GENERALIZED RANK WEIGHTS : PROOF OF THEOREM I.3

Recall that the dual (orthogonal) code of C , denoted by $C^\perp$, is defined as

$$C^\perp = \{x \in \mathbb{F}_{q^m}^n \mid \forall y \in C, \langle x, y \rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ is the bilinear form defined in Section I. We state the following lemma :

Lemma III.1. *Let $V \in \Gamma(\mathbb{F}_{q^m}^n)$. Then $V^\perp \in \Gamma(\mathbb{F}_{q^m}^n)$.*

Proof. Let $x \in V^\perp$. We need to show that $x^q \in V^\perp$. Then, let $y \in V$. Let us prove that $\langle x^q, y \rangle = 0$. Since $y = [y_1, \dots, y_n] \in V = V^q$, there exists some $z = [z_1, \dots, z_n] \in V$ such that $y = z^q$. Hence, we have

$$\begin{aligned} \sum_{1 \leq i \leq n} x_i^q y_i &= \sum_{1 \leq i \leq n} x_i^q z_i^q \\ &= \left(\sum_{1 \leq i \leq n} x_i z_i \right)^q \\ &= 0^q = 0, \end{aligned}$$

which completes the proof. $\square$

Let us recall the statement of Theorem I.3, which we are to prove here.

Theorem. *Let C be a linear code of dimension k over $\mathbb{F}_{q^m}$ and of length n . Then*

$$\{\mathcal{M}_r(C) \mid 1 \leq r \leq k\} = \{1, \dots, n\} \setminus \{n+1 - \mathcal{M}_r(C^\perp) \mid 1 \leq r \leq n-k\}.$$

Proof. We start with stating the following lemma :

Lemma III.2. *Let $1 \leq r \leq n-k$ and let $t = k + r - \mathcal{M}_r(C^\perp)$. Then,*

- 1) $\mathcal{M}_t(C) \leq n - \mathcal{M}_r(C^\perp)$;
- 2) for every $\Delta > 0$, $\mathcal{M}_{t+\Delta}(C) \neq n - \mathcal{M}_r(C^\perp) + 1$.

Before proving it, we first show that this lemma is enough to conclude. Lemma III.2 implies that for every $1 \leq r \leq n-k$ and for every $s \geq t$,

$$\mathcal{M}_s(C) \neq n+1 - \mathcal{M}_r(C^\perp).$$

Moreover, for every $s < t$, by the monotonicity property (Theorem I.2),

$$\mathcal{M}_s(C) < \mathcal{M}_t(C) < n+1 - \mathcal{M}_r(C^\perp),$$

hence

$$\{\mathcal{M}_s(C) \mid 1 \leq s \leq k\} \cap \{n+1 - \mathcal{M}_r(C^\perp) \mid 1 \leq r \leq n-k\} = \emptyset.$$

Furthermore, the cardinality of the union

$$\{\mathcal{M}_s(C) \mid 1 \leq s \leq k\} \cup \{n+1 - \mathcal{M}_r(C^\perp) \mid 1 \leq r \leq n-k\}$$

is equal to $k + n - k = n$ (thanks to the monotonicity property (Theorem I.2) again). Since now both sets are included in $\{1, \dots, n\}$, then

$$\{\mathcal{M}_s(C) \mid 1 \leq s \leq k\} \sqcup \{n+1 - \mathcal{M}_r(C^\perp) \mid 1 \leq r \leq n-k\} = \{1, \dots, n\},$$

which completes the proof of Theorem I.3.

Let us now prove Lemma III.2 :

Proof. Let $1 \leq r \leq n-k$.

- 1) We set $t = k + r - \mathcal{M}_r(C^\perp)$. We want to show that $\mathcal{M}_t(C) \leq n - \mathcal{M}_r(C^\perp)$. Let $V \in \Gamma(\mathbb{F}_{q^m}^n)$ such that $\dim(V \cap C^\perp) \geq r$ and $\dim V = \mathcal{M}_r(C^\perp)$. We have

$$\begin{aligned} \dim(V \cap C^\perp) &= \dim V + \dim C^\perp - \dim(V + C^\perp) \\ &= \mathcal{M}_r(C^\perp) + n - k - \dim((V^\perp \cap (C^\perp)^\perp)^\perp) \\ &= \mathcal{M}_r(C^\perp) + n - k - n + \dim(V^\perp \cap C) \\ &= \mathcal{M}_r(C^\perp) - k + \dim(V^\perp \cap C). \end{aligned}$$

Since $\dim(V \cap C^\perp) \geq r$, we get that

$$t = r + k - \mathcal{M}_r(C^\perp) \leq \dim(V^\perp \cap C).$$

Therefore,

$$n - \mathcal{M}_r(C^\perp) = n - \dim V = \dim(V^\perp) \geq \mathcal{M}_t(C)$$

(since $V \in \Gamma(\mathbb{F}_{q^m}^n)$, then $V^\perp \in \Gamma(\mathbb{F}_{q^m}^n)$ by Lemma III.1).

2) We make a proof by contradiction in assuming that there exists some $\Delta > 0$, such that

$$\mathcal{M}_{t+\Delta}(C) = n + 1 - \mathcal{M}_r(C^\perp).$$

Then there exists $V \in \Gamma(\mathbb{F}_{q^m}^n)$ such that $\dim(V \cap C) \geq t + \Delta$ and $\dim V = n + 1 - \mathcal{M}_r(C^\perp)$. We have

$$\dim(V \cap C) = \dim V + \dim C - \dim(V + C) = n + 1 - \mathcal{M}_r(C^\perp) + k - (n - \dim(V^\perp \cap C^\perp)).$$

Since $\dim(V \cap C) > t$, we get that

$$\begin{aligned} t &< 1 - \mathcal{M}_r(C^\perp) + k + \dim(V^\perp \cap C^\perp) \\ k + r - \mathcal{M}_r(C^\perp) &< 1 - \mathcal{M}_r(C^\perp) + k + \dim(V^\perp \cap C^\perp) \\ r - 1 &< \dim(V^\perp \cap C^\perp). \end{aligned}$$

Since $V^\perp \in \Gamma(\mathbb{F}_{q^m}^n)$ by Lemma III.1 and $\dim(V^\perp \cap C^\perp) \geq r$, we have

$$\dim V^\perp \geq \mathcal{M}_r(C^\perp).$$

However,

$$\dim V^\perp = n - \dim V = n - (n + 1 - \mathcal{M}_r(C^\perp)) = \mathcal{M}_r(C^\perp) - 1,$$

which contradicts the previous inequality.

This completes the proof of Lemma III.2 and that of Theorem I.3. $\square$

We can then derive from Theorem I.3 the following characterization of the r -MRD codes in terms of the rank weight of the dual code :

Corollary III.3. *Keeping notation as in Theorem I.3, for every $1 \leq r \leq k$, the code C is r -MRD if and only if $\mathcal{M}_r(C^\perp) \geq k - r + 2$.*

Proof. Let $1 \leq r \leq k$. Assume first that $\mathcal{M}_r(C) = n - k + r$. By monotonicity property (Theorem I.2), for all $r \leq s \leq k$, we have $\mathcal{M}_s(C) = n - k + s$. Hence, for all $r \leq s \leq k$,

$$n + 1 - \mathcal{M}_s(C) = n + 1 - (n - k + s) = k - s + 1$$

and by Theorem I.3,

$$\{1, 2, \dots, k - r + 1\} \subseteq \{1, \dots, n\} \setminus \{\mathcal{M}_t(C^\perp) \mid 1 \leq t \leq n - k\}.$$

It implies that $d(\lambda(C^\perp)) = \mathcal{M}_1(C^\perp) \geq k - r + 2$.

Conversely, assume that $d(\lambda(C^\perp)) \geq k - r + 2$. By monotonicity property (Theorem I.2), it means that

$$\{1, \dots, k - r + 1\} \cap \{\mathcal{M}_t(C^\perp) \mid 1 \leq t \leq n - k\} = \emptyset$$

and Theorem I.3 implies that

$$\{1, \dots, k - r + 1\} \subseteq \{n + 1 - \mathcal{M}_s(C) \mid 1 \leq s \leq k\}.$$

Finally, again by the monotonicity property (Theorem I.2), we obtain that

$$\mathcal{M}_k(C) = n, \mathcal{M}_{k-1}(C) = n - 1, \dots, \mathcal{M}_r(C) = n + 1 - (k - r + 1) = n - k + r$$

which proves that C is r -MRD. $\square$

ACKNOWLEDGMENTS

This research was supported by the Singapore National Research Foundation under Research Grant NRF-RF2009-07. The author would like to warmly thank Prof. F. Oggier for introducing him to this nice topic and for her meaningful advice and careful read-through of the paper.

REFERENCES

- [1] E. M. Gabidulin, "Theory of codes with maximum rank distance," *Probl. Inf. Transm.*, vol. 21, no. 1, pp. 1–12, 1985.
- [2] W. C. Huffman and V. Pless, "Fundamentals of Error Correcting Codes," *Cambridge University Press*, 2003, ISBN 0-521-78280-5.
- [3] J. Kurihara, R. Matsumoto and T. Uyematsu, "Relative generalized rank weight of linear codes and its applications to network coding," preprint, available at <http://arxiv.org/pdf/1301.5482v1.pdf>.
- [4] F. Oggier and A. Sboui, "On the existence of generalized rank weights," in *Proc. 2012 Int. Symp. Information Theory and Its Applications*, Honolulu, Hawaii, USA, Oct. 2012, pp. 406–410.
- [5] R. M. Roth, "Maximum-Rank Array Codes and their Application to Crisscross Error Correction," *IEEE Transactions on Information Theory*, vol. 37, no. 2, pp. 328–336, 1991.
- [6] D. Silva and F. R. Kschischang, "Universal secure network coding via rank-metric codes," *IEEE Trans. Inf. Theory*, vol. 57, no. 2, pp. 1124–1135, Feb. 2011. [Online]. Available: [arXiv:0809.3546 \[cs.IT\]](http://arxiv.org/abs/0809.3546)
- [7] H. Stichtenoth, "On the dimension of subfield subcodes," *IEEE Trans. Inf. Theory*, vol. 36, no. 1, pp. 90–93, 1990.
- [8] V. K. Wei, "Generalized Hamming weights for linear codes," *IEEE Trans. Inf. Theory*, vol. 37, no. 5, pp. 1412–1418, May 1991.