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2000

Sen, S., Sen, S., Sexton, J., & Adams, D. H. (2000). Geometric discretization scheme applied to the Abelian Chern-Simons theory. Physical Review E, 61(3), 3174-3185.

https://hdl.handle.net/10356/100977

https://doi.org/10.1103/PhysRevE.61.3174

Geometric discretization scheme applied to the Abelian Chern-Simons theory

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(Received 17 August 1999)

We give a detailed general description of a recent geometrical discretization scheme and illustrate, by explicit numerical calculation, the scheme's ability to capture topological features. The scheme is applied to the Abelian Chern-Simons theory and leads, after a necessary field doubling, to an expression for the discrete partition function in terms of untwisted Reidemeister torsion and of various triangulation-dependent factors. The discrete partition function is evaluated computationally for various triangulations of S^3 and of lens spaces. The results confirm that the discretization scheme is triangulation independent and coincides with the continuum partition function.

PACS number(s): 02.70.$-$c, 11.15.Ha, 02.40.Sf, 11.15.Tk

I. INTRODUCTION

A very useful way to regularize a quantum field theory is provided by the lattice formulation introduced by Wilson [1]. (See, for example, [2] for a detailed treatment.) However, this formulation has difficulty in capturing topological features of a field theory, for example, the topological θ term in QCD. It is therefore of interest to investigate alternative discretization schemes. In this paper we describe an alternative scheme that is applicable to antisymmetric tensor field theories including Abelian gauge theories and fermion field theory in the Kähler-Dirac framework [3], and which is well suited for capturing the topological features of such theories. It is based on developing analogies between the different types of fields and the way they appear in a quantum field theory with a corresponding list of discrete variables and operators. The method is valid for any arbitrary compact 3-manifold without boundary. We illustrate the scheme by applying it to the pure Abelian Chern-Simons gauge theory in three dimensions, and to a doubled version, the so-called Abelian BF gauge theory. In the latter case the topological features of the theory are completely reproduced by our discretization scheme, even before taking the continuum limit. We illustrate this by explicit numerical calculations of the discrete partition function when the space-time is S^3 or is a lens space, $L(p,1)$.

The discretization scheme involves several ingredients that are not used in the standard Wilson lattice formulation. These include a triangulation of the space-time (i.e., a decomposition into hypertetrahedra rather then hypercubes) and a mathematical tool called the Whitney map [4]. These have previously been used to discretize field theories in [5,6] where various convergence results (e.g., convergence of the discrete action to the continuum action) were established. Triangulations of space-time are also used in discretizations of other quantum field theories, for example, in quantum gravity (see [7] for a review). The thrust of the present work is quite different. We set up the discretization in such a way that the geometric structures of the continuum field theory are mirrored by analogous structures in the discrete formulation. As we will see, this requires a certain doubling of fields. With this doubling, the topological features of Abelian Chern-Simons theory are completely captured by the formulation. An alternate scheme of discretization proposed previously in [6], which does not involve the doubling of fields, fails to capture the topological features of the Abelian Chern-Simons theory as we will show numerically in Sec. V. A mathematical treatment of this discretization scheme has been given in [8,11]. A short version of some of the results has been published in [10,9].

Our aims in this paper are, first, to make the techniques and results of [8,11] accessible to a wider audience, and, secondly, to demonstrate a practical numerical implementation of the discretization scheme. Numerical implementations are developed for Abelian Chern-Simons theory defined on the three-dimensional sphere S^3 and on the lens spaces $L(p,1)$ for $p=1,\ 2$, and 3. Our discretization scheme is the only one which numerically reproduces the exact topological results for Abelian Chern-Simons theory [12].

This paper is organized as follows. In Sec. II the discretization scheme is described together with a summary of the topological results used to set it up. In Sec. III some features of the Abelian Chern-Simons theory on general 3-manifolds are reviewed. In Sec. IV the discretization scheme is applied to this theory. An expression for the partition function of the resulting discrete theory is derived in terms of the data specifying the triangulation of the space-time. In Sec. V the numerical evaluations of the partition functions corresponding to these triangulations are presented. In Sec. VI we summarize our conclusions.

II. DISCRETIZATION SCHEME

Let us start by considering a general quantum field theory defined on an arbitrary manifold M of dimension D. Suppose the theory has fields $\phi^p(\vec{x})$ where $\vec{x}\in M$, and where ϕ^p is a p-form (antisymmetric tensor field of degree p defined on M). The Lagrangian for the system involves the fields, the Laplacian operator, and possibly (as for the Chern-Simons theory) an antisymmetrized first-order differential operator.

In differential geometric terms, the theory is constructed using the following objects, which are defined on the manifold M: p-forms ϕ^p, which are generalized antisymmetric tensor fields, the exterior derivative $d:\phi^p\rightarrow\phi^{p+1}$, the Hodge star operator $*:\phi^p\rightarrow\phi^{D-p}$, which is required to define scalar products, and the wedge operator $\phi^p\wedge\phi^q=\phi^{p+q}$.

We want to construct discrete analogs of these objects. We begin by summarizing the basic properties of our operators of interest [13]. On a manifold M of dimension D, the operations $(\wedge,*,d)$ on p-forms $\phi^p (p=0,\ldots,D)$ satisfy the following:

(1) $\phi^p \wedge \phi^q = (-1)^{pq} \phi^q \wedge \phi^p$,
(2) $d(\phi^p \wedge \phi^q) = d\phi^p \wedge \phi^q + (-1)^p \phi^p \wedge d\phi^q$,
(3) $*\phi^p = \phi^{D-p}$,
(4) $** = (-1)^{Dp+1}$,
(5) $d^2 = 0$, $(d*)^2 = 0$,
(6) $d* = (-1)^{D(p+1)+1} * d*$, ($d*$ is the adjoint of d).

The following definitions will also be required: (1) The Laplacian on p-forms $\Delta_p = d_{p-1} d_p^* + d_{p+1}^* d_p$; (2) the inner product $\langle \phi_p, \phi_p{}' \rangle = \int_M \phi_p \wedge * \phi_p{}'$.

A few examples might now be helpful. First, consider QED in four dimensions. The gauge field $A = \phi^1$ is a 1-form. The electromagnetic field is a 2-form given by $F = dA$. The action for the gauge field in QED is given by

$$S(A) = (F,F) = \langle dA, dA \rangle = \int_M dA \wedge * dA. \tag{1}$$

Thus S involves the operators $*$, d, and the wedge product. Similarly, for Abelian Chern-Simons theory the gauge field A and electromagnetic field $F = dA$ are 1-forms and 2-forms, respectively, as for QED. The action for the theory is given by

$$S(A) = \int_M A \wedge dA = \langle A, * dA \rangle, \tag{2}$$

where the space-time M is three-dimensional. Note that in both cases we can think of the action $S(A)$ as a quadratic functional of the gauge field A.

We would like to discretize the fields of ϕ^p, the inner product $\langle \cdot , \cdot \rangle$, and the operators $(\wedge,*,d)$ such that discrete analogs of their continuum interrelationships hold. To do this it is necessary to introduce first a few basic ideas of discretization. We start by discretizing the manifold M. This involves replacing M by a collection of discrete objects, known as simplices, glued together. We need a few definitions [14].

First, for $p \geqslant 0$, a p-simplex $\sigma^{(p)} = [v_0, \ldots, v_p]$ is defined to be the convex hull in some Euclidean space $\mathbb{R}^D$ of a set of $p+1$ points $v_0, v_1, \ldots, v_p \in \mathbb{R}^m$. Here the vertices v_i are required to span a p-dimensional space. This requirement will hold so long as the equations $\Sigma_{i=0}^p \lambda_i v_i = 0$ and $\Sigma_{i=0}^p \lambda_i = 0$ admit only the trivial solution $\lambda_i = 0$ for $i = 0, \ldots, p$ for λ_i real.

A few examples might clarify the geometry. Consider $\sigma^{(0)} = [v_0]$. This is a point or 0-simplex. Next $\sigma^{(1)} = [v_0, v_1]$ is a line segment or 1-simplex. An orientation can be assigned by the ordering of the vertices, in which case $-\sigma^{(1)} = [v_1, v_0]$, for example. The faces of a 1-simplex are its vertices $[v_0]$ and $[v_1]$, which are 0-simplices. $\sigma^{(2)} = [v_0, v_1, v_2]$ is a triangle or 2-simplex. We note that an even permutation of the vertices has the same orientation as $\sigma^{(2)}$ while an odd permutation reverses it and will be written as $-\sigma^{(2)}$. The faces of a 2-simplex are its edges $[v_0, v_1]$, $[v_1, v_2]$, and $[v_2, v_0]$. Finally, $\sigma^{(3)} = [v_0, v_1, v_2, v_3]$ is a tetrahedron or 3-simplex (see Fig. 1). Its faces are the four triangles $[v_0, v_1, v_2]$, $[v_0, v_2, v_3]$, $[v_0, v_1, v_3]$, and $[v_1, v_2, v_3]$ that bound it.

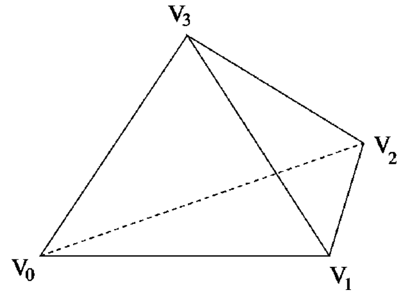

FIG. 1. A 3-simplex.

An important feature of our discretization scheme is that the original simplices are subdivided by using simplex barycenters. Geometrically the *barycenter* of a p-simplex $\sigma^{(p)}$ is the point that represents its ''center of mass.'' We denote the barycenter of $\sigma^{(p)} = [v_0, \ldots, v_p]$ as the point

$$\hat{\sigma}^{(p)} = \frac{1}{p+1} \sum_{i=0}^{p} v_i . \tag{3}$$

As an example, the barycenter of $\sigma^{(1)} = [v_0, v_1]$ is the midpoint of the line segment that joins the vertices v_0 and v_1.

We can now describe a particular way that a given manifold M can be discretized. Let S be a collection of simplices $\{\sigma_i^{(n)}\}$, $n = 0,1, \ldots, D$, with the property that the faces of the simplices that belong to S also belong to it. The elements of S glued together in the following way are known as a simplicial complex [14,15]; (1) $\sigma_i^{(n)} \cap \sigma_j^{(k)} = 0$ if $\sigma_i^{(n)}, \sigma_j^{(k)}$ have no common face; (2) $\sigma_i^{(n)} \cap \sigma_j^{(k)} \neq 0$ if $\sigma_i^{(n)}, \sigma_j^{(k)}$ have precisely one face in common, along which they are glued together. In many cases of interest (including all 3-manifolds and all differentiable manifolds [15]), M can be replaced by a complex K that it is topologically equivalent to. K is then said to be a triangulation of M (note that this triangulation is not unique). In this way of discretizing M, the building blocks are 0-,1-, . . . , D-dimensional objects, all of which are simplices, e.g., generalized oriented tetrahedra.

We now observe that the same manifold can be discretized in many different ways. In the discretization described, we used simplices. We could just as well have used generalized oriented cubes.

There is another method of discretizing a manifold which is the dual of the simplicial discretization just described. It associates with a simplicial complex K a dual complex $\hat{K}$. We proceed to describe this construction. We will see that the basic objects of the dual complex $\hat{K}$ are again 0-,1-,2-, . . . , D-dimensional objects, but this time they are no longer simplices. We illustrate the method by considering a manifold M that is a disk. This is a manifold with a boundary. We triangulate this by the simplicial complex K shown in Fig. 2.

Now consider the barycenters of the building blocks of the simplicial complex K. We have the list shown in Table I. Pictorially the simplicial complex K with its barycenters is shown in Fig. 3.

We can now construct the dual description of a triangulation K. Geometrically this utilizes a discrete analog of the

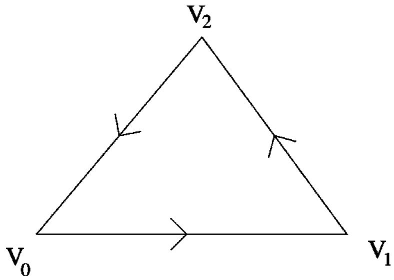

FIG. 2. Triangulation K of a disk.

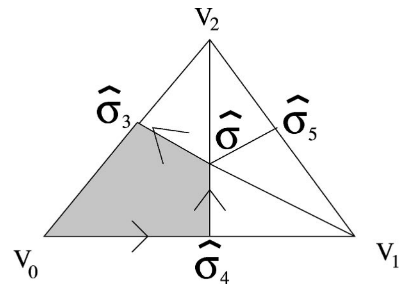

FIG. 3. Barycenters of K.

Hodge $*$ operator. Recall that the $*$ operator maps a p-form to a $(D-p)$-form, where D is the dimension of the manifold M on which the p-form is defined. In the dual geometrical decomposition of the manifold, we want to set up a correspondence between a p-dimensional object and a $(D-p)$-dimensional object.

This is done as follows. We first construct $(D-p)$-dimensional objects whose vertices are barycenters of a sequence of successively higher dimensional simplices, where each simplex is a face of the following one, in other words, $(D-p)$-dimensional objects of the form $\{\hat{\sigma}_p,\hat{\sigma}_{p+1},\ldots,\hat{\sigma}_D\}$, where σ_n is a face of σ_{n+1}. The orientations of these are set so as to be compatible with the manifold. Joining these objects together gives us the dual of σ_p.

Thus, for instance, the map $*$ acts on $[v_0]$ as follows:

$$*_K:[v_0]\rightarrow\epsilon_{01}[\hat{v}_0,\hat{\sigma}_1,\hat{\sigma}]\cup[\hat{v}_0,\hat{\sigma}_3,\hat{\sigma}]\epsilon_{03},$$

where the orientation of each of the small triangles has to be coherent with the orientation of the original triangle. This is shown in Fig. 3 and leads to mapping $[v_0]$ to the shaded two dimensional region. The orientations of the simplices are specified by arrows in the figure. Coherence of orientation means, for example, that the arrow of an edge agrees with the arrow of the triangle to which it belongs. Next we consider $[v_0,v_1]$. This is a 1-simplex and is to be mapped to a $[(2-1)=1]$-dimensional object. The map is defined as

$$*_K:[v_0,v_1]\rightarrow[\hat{\sigma}_1,\hat{\sigma}].$$

Again the orientation of $[\hat{\sigma}_1,\hat{\sigma}]$ has to be coherent with the orientation of the triangles already introduced when the map for $[v_0]$ was considered. Similarly,

$$*_K:[v_1,v_2]\rightarrow[\hat{\sigma}_2,\hat{\sigma}],$$

$$*_K:[v_2,v_0]\rightarrow[\hat{\sigma}_3,\hat{\sigma}],$$

and finally

$$*_K:[v_0,v_1,v_2]\rightarrow[\hat{\sigma}].$$

We then have the alternate discretization $\hat{K}$ for M shown in Fig. 4.

Note that when two edges are glued together they must have opposite orientations. We can now give the general rule for mapping an n-simplex $\sigma_n=[v_0,\ldots,v_n]$ to a $(D-n)$-dimensional object $[(D-n)$ cell$]$ as follows. We think of σ_n as an element of a simplicial complex K. We have

$$*_K:[v_0,\ldots,v_n]\rightarrow\cup[\hat{\sigma}_n,\hat{\sigma}_{n+1},\ldots,\hat{\sigma}_D],$$

where $\hat{\sigma}_{n+1}$ is the barycenter of an $(n+1)$-simplex that has σ_n as a face, $\hat{\sigma}_{n+2}$ is the barycenter of an $(n+2)$-simplex that has σ_{n+1} as a face, and so on. These objects have to be coherently oriented with respect to $[v_0,\ldots,v_n]$. The set of these cells constitutes the dual space $\hat{K}$ of K.

By this procedure, we claim, a discrete version of the Hodge star operation $*$ has been constructed. Let us explain. The Hodge $*$ operator involves forms. It maps p-forms in D dimensions to a $(D-p)$-form. The $*_K$ map involves not forms but geometrical objects. There is a simple correspondence relation between these two cases. Given a p-form ϕ_p and a p-dimensional geometrical space Σ_p, the p-form can be integrated over Σ_p to give a number. Thus Σ_p and ϕ_p are objects that can be paired. We can write this as a pairing

$$(\phi_p,\Sigma_p)=\int_{\Sigma_p}\phi_p.$$

In order to proceed, we need to introduce some more structure. We start by associating with a simplicial complex K

TABLE I. Elements of the simplicial complex K and their barycenters.

Geometrical object in K	Dimension	Corresponding barycenter
$\sigma_1^{(0)}=[v_0]$	0	$\hat{\sigma}_1^{(0)}=v_0$
$\sigma_2^{(0)}=[v_1]$	0	$\hat{\sigma}_2^{(0)}=v_0$
$\sigma_3^{(0)}=[v_2]$	0	$\hat{\sigma}_3^{(0)}=v_0$
$\sigma_1^{(1)}=[v_0,v_1]$	1	$\hat{\sigma}_1^{(1)}=\frac{1}{2}(v_0+v_1)=v_3$
$\sigma_2^{(1)}=[v_1,v_2]$	1	$\hat{\sigma}_1^{(1)}=\frac{1}{2}(v_1+v_2)=v_4$
$\sigma_3^{(1)}=[v_2,v_0]$	1	$\hat{\sigma}_1^{(1)}=\frac{1}{2}(v_2+v_0)=v_5$
$\sigma_1^{(2)}=[v_0,v_1,v_2]$	2	$\hat{\sigma}_1^{(2)}=\frac{1}{3}(v_0+v_1+v_2)=v_6$

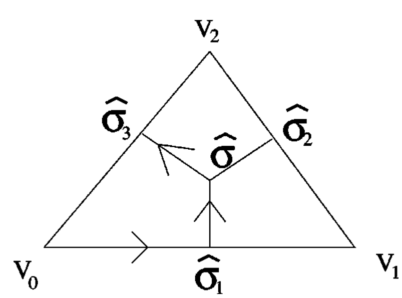

FIG. 4. Dual complex $\hat{K}$.

containing $\{\sigma_p^i\}$ $(i=1,\ldots,K_p;p=0,\ldots,D)$ a vector space consisting of finite linear combinations over the reals of the p-simplices it contains. This vector space is known as the space of p-chains, $C_p(K)$. For two elements $\sigma_p^i,\sigma_p^j \in C_p(K)$, a scalar product $(\sigma_p^i,\sigma_p^j)=\delta_j^i$ can be introduced. An oriented p-simplex changes sign under a change of orientation, i.e., if $\sigma_p=[v_0,\ldots,v_p]$ and τ is a permutation of the indices $[0,\ldots,p]$, then $[v_{\tau(0)},\ldots,v_{\tau(p)}]=(-1)^\tau[v_0,\ldots,v_p]$, with τ denoting the number of transpositions needed to bring $[v_{\tau(0)},\ldots,v_{\tau(p)}]$ to the order $[v_0,\ldots,v_p]$.

Given the vector space $C_p(K)$, the boundary operator ∂^K can be defined as

$$\partial^K : C_p(K) \to C_{p-1}(K).$$

It is the linear operator that maps an oriented p-simplex $\sigma^{(p)}$ to the sum of its $(p-1)$ faces with orientation induced by the orientation of σ^p. If $\sigma^p=[v_0,\ldots,v_p]$, then

$$\partial\sigma^p = \sum_{i=0}^{p} (-1)^i [v_0,\ldots,\hat{v}_i,\ldots v_p],$$

where $[v_0,\ldots,\hat{v}_i,\ldots,v_p]$ means that the vertex v_i has been omitted from σ^p to produce the face ''opposite'' to it.

Given that $C_p(K)$ is a vector space, it is possible to define a dual vector space $C^p(K)$, consisting of dual objects known as cochains; that is, we can take an element of $C_p(K)$ and an element of $C^p(K)$ to form a real number. Since the space $C_p(K)$ has a scalar product, namely, if $\sigma_p^i,\sigma_p^j\in C_p(K)$, then $(\sigma_p^i,\sigma_p^j)=\delta_{ij}$. We can use the scalar product to identify $C_p(K)\equiv C^p(K)$, so that we can consider oriented p-simplices as elements of $C^p(K)$ as well as $C_p(K)$. We can write our boundary operation as

$$([v_0,\ldots,\hat{v}_i,\ldots,v_p],\partial^K[v_0,\ldots,v_p])=(-1)^i.$$

This suggests introducing the adjoint operation d^K defined as

$$(d^K[v_0,\ldots,\hat{v}_i,\ldots,v_p],[v_0,\ldots,v_p]) = ([v_0,\ldots,\hat{v}_i,\ldots,v_p],\partial^K[v_0,\ldots,v_p]).$$

This is the coboundary operator that maps $C_p(K) \to C_{p+1}(K)$.

Indeed, we have

$$d^K[v_0,\ldots,v_p]=\sum_v [v,v_0,\ldots,v_p],$$

where the sum is over all vertices v such that $[v,v_0,\ldots,v_p]$ is a $(p+1)$ simplex.

The boundary operators ∂_K and the coboundary operator d_K have the property $\partial_K\partial_K=d_Kd_K=0$. Furthermore,

$$d_K : C_p \to C_{p+1},$$

$$\partial_K : C_p \to C_{p-1}.$$

These operators are the discrete analogues of the operators d and $(-1)^{D(p+1)+1}{*d*}=d^*$ which act on forms.

These operators could be defined only when a scalar product (''metric'') was introduced in the vector space C_p's. At this stage we have a discrete geometrical analogue of d, d^*, and $*$. We have also commented on the fact that the operation $*$ maps simplices into dual cells, i.e., not simplices. If the original simplicial system is described in terms of the union of the vector spaces of all p-chains, then the space into which elements of the vector space are mapped by $*$ is not contained within this space, unlike the situation for the Hodge star operation on forms. We will see that this difference leads inevitably to a doubling of the fields when discretization, preserving topological structures, is attempted.

We now need a way to relate a p-chain to a p-form. This together with a construction that linearly maps p-forms to p-simplices will allow us to translate expressions in continuum QFT to corresponding discrete geometrical objects. We start with the construction of the linear maps from p-chains to p-forms due to Whitney [4].

In order to define this map, we need to introduce barycentric coordinates associated with a given p-simplex σ^p. Regarding σ^p as an element of some $\mathbb{R}^N$, we introduce a set of real numbers $(\mu_0,\ldots,\mu_p)$ with the property

$$\mu_i \geqslant 0,$$

$$\sum_i \mu_i = 1.$$

A point $x\in\sigma^p$ can be written in terms of the vertices of σ^p and these real numbers as

$$x=\sum_{i=0}^{p} \mu_i v_i .$$

Note that if any set of $\mu_i=0$ then the vector x lies on a face of σ^p. One can think of x as the position of the center of mass of a collection of masses $(\mu_0,\ldots,\mu_p)$ located on the vertices $(v_0,\ldots,v_p)$, respectively. Setting $\mu_i=0$, for instance, means that the center of mass will be in the face opposite the vertex v_i. The Whitney map can now be defined. We have

$$W^K : C^p(K) \to \Phi^p(K),$$

where $\Phi^p(K)$ is a p-form. If $\sigma^p\in C^p(K)$ then

$$W[\sigma^p]=p!\sum_{i=0}^{p} (-1)^i \mu_i \, d\mu_0 \wedge \cdots \hat{d}\mu_i \wedge \cdots d\mu_p ,$$

where $\hat{d}\mu_i$ means this term is missing, and $(\mu_0,\cdots,\mu_p)$ are the barycentric coordinate functions of σ^p.

We next construct the linear map from p-forms to p-chains. This is known as the de Rham map. We have

$$A^K : \Phi^p(K) \to C^p(K),$$

defined by

$$\langle A^K(\Phi^p),\sigma^p\rangle = \int_{\sigma^p} \Phi^p,$$

for each oriented p-simplex $\in K$.

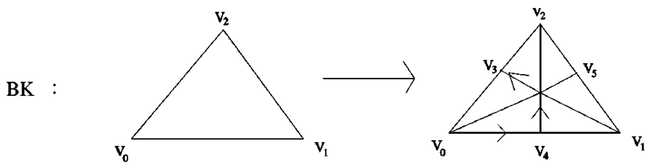

FIG. 5. Barycentric subdivision of K.

A discrete version of the wedge product can also be defined using the Whitney and de Rham maps such that $\wedge^K : C^p(K)\times C^q(K)\to C^{p+q}(K)$ as follows:

$$x\wedge^K y = A^K(W^K(x)\wedge W^K(y)).$$

It has many of the properties of the continuous wedge product in that it is skew-symmetric and obeys the Leibniz rule but it is nonassociative.

At this stage we have introduced all the building blocks necessary to discretize a system preserving geometrical structures. We summarize the properties of the maps introduced in the form of a theorem [4]:

(1) $A^K W^K =$ Identity.
(2) $dW^K = W^K d^K$, where $d:\phi^p\to\phi^{p+1}$.
(3) $\int_{|\beta|} W^K(\alpha)=\langle\alpha,\beta\rangle,\ \alpha,\beta\in K$.
(4) $d^K A^K = A^K d$.

This theorem shows how d^K can be considered as the discrete analog of d. We now show how $*^K$ can be considered as a discrete analog of $*$. For this we need barycentric subdivision.

We recall that given a simplicial complex $\{\sigma_i^p\}$, $i=1,\dots,K_p; p=0,\dots,D$. A set of points (vertices) could be assigned to each simplex, namely, $\hat\sigma_i^p$. These are the barycenters. These vertices, regarded as vertices of a simplex, subdivide the original simplices to give a finer triangulation of the original manifold. This is a barycentric subdivision map BK. Clearly the procedure can be repeated to give finer and finer subdivisions in which the simplices become "smaller." The procedure is illustrated for a triangle in Fig. 5.

Note that all the barycenters are present as vertices of the barycentric subdivision and that $*^K$ acting on simplices belonging to the simplicial complex K associated with $[v_0,v_1,v_2]$ leads to objects which are not, in general, simplices of $[v_0,v_1,v_2]$ but belong to a different space $\hat K$. However, both K and $\hat K$ are contained in the barycentric subdivision $B[v_0,v_1,v_2]$. This is a crucial observation. In order to construct the star map, two geometrically distinct spaces were introduced, the original simplicial decomposition K with its associated set of p-chains $C^p(K)$ and the dual cell decomposition $\hat K$ with its associated set of p-chains $C^p(\hat K)$. These spaces are distinct. However, both belong to the first barycentric subdivision of K. This allows the use of the $*^K$ operation if we think of K and $\hat K$ as elements of BK.

We proceed as follows. Let BK and $\hat K$ denote the barycentric subdivision and dual triangulation, and

$${}^K: C^p(K)\to C^{n-p}(\hat K).$$

However, $C^p(K)$ and $C^p(\hat K)$ are both contained in $C^p(BK)$ as we have seen. Let

$$W^{BK}: C^p(BK)\to\phi^p(M)$$

denote the Whitney map. Then we have for $x\in C^p(K)$, $y\in C^{n-p-1}(\hat K)$ [11]:

$$(1)\quad \langle *^K x,y\rangle=\frac{(n+1)!}{p!(n-p)!}\int_M W^{BK}(Bx)\wedge W^{BK}(By),$$

$$\langle *^{\hat K} y,x\rangle=\frac{(n+1)!}{p!(n-p)!}\int_M W^{BK}(By)\wedge W^{BK}(Bx);$$

$$(2)\quad \partial^K=(-1)^{np+1}*^{\hat K}d^{\hat K K}\quad\text{on}\quad C^p(K),$$

$$\partial^{\hat K}=(-1)^{nq+1}*^K d^K *^{\hat K}\quad\text{on}\quad C^q(\hat K).$$

These are the discrete analogues of the interrelationships between d, $d*$, $*$, and $\langle\cdot,\cdot\rangle$ in the continuum.

Note $K\neq\hat K$ and that properties of ∂_K, d_K analogous to those for differential forms only hold if $K,\hat K$ are both regarded as elements of BK. This feature of the discretization method is, as we shall see, crucial if we want to preserve topological properties of the original system. If a discretization method is introduced without the $*$ operation in it then as we shall see in Sec. V the topological properties of the partition function for the Abelian Chern-Simons gauge theory do not hold.

We proceed to apply these ideas to the Abelian Chern-Simons gauge theory on a compact 3-manifold M. First we summarize properties of the continuum field theory.

III. SCHWARZ'S TOPOLOGICAL FIELD THEORY AND THE RAY-SINGER TORSION

We begin our treatment of continuum field theory by describing Schwarz's method for evaluating the partition function of the Chern-Simons gauge theory on a 3-manifold M [16]. We assume that the first real homology group (to be defined shortly) of the manifold vanishes; this is done for the sake of simplicity. Schwarz's method is applicable for arbitrary compact 3-manifolds without boundary. Such manifolds are called "homology 3-spheres." The main examples we have in mind are the 3-sphere S^3 and the lens spaces $L(p,1)$, $p=1,2,\dots$ (for a definition and the basic properties of lens spaces see [17,18]). The fields of the theory are the 1-forms on M, i.e., $\omega\in\Omega^1(M)$. [In terms of a local coordinate system (X^μ) on M we have $\omega(x)=\omega_\mu(x)dx^\mu$.] The action is

$$S(\omega)=\int_M \omega\wedge d_1\omega=\int_M dx^1\,dx^2\,dx^3\,\epsilon^{\mu\nu\rho}\omega_\mu\,\partial_\nu\omega_\rho. \quad (4)$$

Here and in the following $\Omega^q(M)$ denotes the space of q-forms on M (i.e., the antisymmetric tensor fields of degree q) and $d_q:\Omega^q(M)\to\Omega^{q+1}(M)$, i.e., the exterior derivative. It has the property $d_q d_{q-1}=0$ so $\mathrm{Im}(d_{q-1})\subset\mathrm{Ker}(d_q)$, where $\mathrm{Im}(d_{q-1})$ is the image of the operator d_{q-1} while $\mathrm{Ker}(d_q)$ is the null space of the operator d_q and the cohomology spaces $H^q(M)$ are defined by

$$H^q(M)=\mathrm{Ker}(d_q)/\mathrm{Im}(d_{q-1}).$$

The $H^q(M)$ are Abelian groups that contain topological information about the manifold. The vanishing of $H^1(M)$, for instance, holds if the manifold is simply connected, that is, any loop in M can be smoothly deformed to any other loop in M [14]. Note that $\Omega^0(M)$ is the space of functions on M and since d_0 is the derivative $\mathrm{Ker}(d_0)$ consists of the constant functions, i.e.,

$$H^0(M)=\mathrm{Ker}(d_0)/0=\mathrm{Ker}(d_0)=\mathbb{R}.$$

Our requirement on M that $H^1(M)=0$ implies that

$$\mathrm{Im}(d_0)=\mathrm{Ker}(d_1).$$

A choice of metric on M determines an inner product in the spaces $\Omega^q(M)$ and allows the action (4) to be written as

$$S(\omega)=\lambda\langle\omega,(*d_1)\omega\rangle, \tag{5}$$

where $*$ is the Hodge star operator. (See [13] for background on this and other differential-geometric constructions.) Evaluation of the partition function of this action by Schwarz's method requires the introduction of the resolvent for $S(\omega)$. The partition function is defined as

$$Z(\lambda)=N\int d\omega\, e^{iS(\omega)}.$$

The main problem in evaluating $Z(\lambda)$ is to properly deal with the zeros of $S(\omega)$. These zero modes contain topological information regarding the manifold, as the space of zero modes is given by $\mathrm{Ker}(d_1)$, and hence should not be disgarded. Schwarz introduced an algebraic method (the resolvent method) for dealing with this problem. Although it is valid only for $S(\omega)$'s that are quadratic in ω, it can be used to analyze $S(\omega)$'s constructed on arbitrary compact manifolds without boundary. For systems of this type Schwarz's method is an algebraic analogue of the problem of gauge fixing. The advantage of the resolvent method is that it can be easily extended to deal with the process of discretization, as we will show.

The resolvent is defined to be the following chain of maps:

$$0\to\mathbb{R}\to^{\phi_0}\Omega^0(M)\to^{d_0}\mathrm{Im}(d_0)=\mathrm{Ker}(d_1)\to\mathrm{Ker}(S)\to 0. \tag{6}$$

This chain of maps forms an exact sequence, that is, the image of a map is the kernel of the map that follows. With the help of the resolvent, Schwarz was able to show that the partition function for the theory was given by [16]

$$Z(\lambda)=e^{-(i\pi/4)\iota}\left(\frac{\lambda}{\pi}\right)^{-\zeta/2}\det{}'((*d_1)^2)^{-1/4}\det{}'(d_0^*d_0)^{1/2}\times\det(\phi_0^*\phi_0)^{-1/2}, \tag{7}$$

where ι is a nontopological geometry dependent function and ζ as shown in [11] is given by

$$\zeta=\dim H^0(M)-\dim H^1(M).$$

ι is part of a phase factor and thus the absolute value of the partition function $Z(\lambda)$ is a topological quantity. This will be our main concern [10].

For completeness we give a quick proof of this result, ignoring phase factors and constants.

Introducing a metric in the space of ω's allows us to write

$$Z(\lambda)=N\int_{\mathrm{Ker}\,d_1\oplus(\mathrm{Ker}\,d_1)^\perp}d\omega\, e^{iS(\omega)},$$
$$=\mathrm{Vol}(\mathrm{Ker}\,d_1)\cdot(\det d_1^*d_1)^{-1/4}N.$$

We proceed to rewrite $\mathrm{Vol}(\mathrm{Ker}\,d_1)$ using the exact sequence associated with $\mathrm{Ker}\,d_1$ and the manifold M. This procedure gives an expression for the partition function Z containing information about the spaces $\mathrm{Ker}\,d_1$ and $(\mathrm{Ker}\,d_1)^\perp$. Simply dropping $\mathrm{Vol}(\mathrm{Ker}\,d_1)$ leads to a loss of information. We have $\mathrm{Vol}(\mathrm{Ker}\,S)=\mathrm{Vol}(\mathrm{Ker}\ d_1)=\mathrm{Vol}(\mathrm{Im}\,d_0)$ by assumption (if $H_1(M)$ is nontrivial, this equation has to be modified [8]). Also,

$$d_0|_{(\mathrm{Ker}\,d_0)^\perp}:(\mathrm{Ker}\,d_0)^\perp\to(\mathrm{Im}\,d_0)\Rightarrow\mathrm{Vol}(\mathrm{Im}\,d_0)=|\det{}'d_0|\mathrm{Vol}(\mathrm{Ker}\,d_0)^\perp.$$

Note that

$$\Omega^0=\mathrm{Ker}\,d_0\oplus(\mathrm{Ker}\,d_0)^\perp$$

and

$$\phi_0:\mathcal{H}_0\to H_0$$

$$\mathrm{Vol}(H_0)=|\det\phi_0|\mathrm{Vol}(\mathcal{H}_0),$$

where $\mathcal{H}_0$ represents the space of harmonic 0-forms. Note that harmonic p-forms are solutions of $(d^*d+dd^*)\phi_p=0$. In this space the Hodge star operator is present and hence a scalar product and volume can be defined. The map ϕ_0 introduced relates the space of harmonic 0-forms to the space of de Rham cohomology H_0. By a theorem of Hodge this space of harmonic p-forms is isomorphic to the space of H_p [13]. The space of de Rham cohomology does not have a metric and hence we define the volume in this space with the help of the map ϕ_0. Therefore,

$$\mathrm{Vol}(\mathrm{Ker}\,d_0)^\perp=\mathrm{Vol}(\Omega_0)[\mathrm{Vol}(\mathrm{Ker}\,d_0)]^{-1}$$
$$=\mathrm{Vol}(\Omega_0)[\mathrm{Vol}(H_0)]^{-1}$$
$$=[\mathrm{Vol}(\Omega_0)](\det\phi_0)^{-1}[\mathrm{Vol}(\mathcal{H}_0)]^{-1},$$

so that finally we get

$$\mathrm{Vol}(\mathrm{Ker}\,S)=|\det{}'\ d_0||\det\phi_0|^{-1}\,\mathrm{Vol}(\Omega_0)(\mathrm{Vol}\ \mathcal{H},)^{-1}.$$

Choosing

$$N\,\mathrm{Vol}(\Omega_0)(\mathrm{Vol}\,\mathcal{H},)^{-1}=1,$$

we get

$$Z=\mathrm{Vol}(\mathrm{Ker}\,S)(\det\ d_1^*d_1)^{-1/4}$$
$$=|\det{}'\ d_0||\det\ d_1d^*|^{-1/4}|\det\phi_0|^{-1}.$$

A more careful calculation gives the determinant in Eq. (7). The quantities ι and ζ in Eq. (7) also need to be ζ-regularized—this was done in [11], where it was shown that the regularized ζ is given by

$$\zeta = \dim H^0(M) - \dim H^1(M).$$

So in the present case, where $H^0(M) \equiv \mathbb{R}$ and $H^1(M) = 0$, we have

$$\zeta = 1 - 0 = 1. \tag{8}$$

Using the formulas $d_1^* = *d_1^*$ and $** = 1$ (modulo a possible sign), we get $d_1 = *d_1^**$ and therefore (at $d_1)^2 = *d_1*d_1 = d_1^* d_1$, which gives

$$\det'((*d_1)^2) = \det'(d_1^* d_1). \tag{9}$$

Substituting Eqs. (8) and (9) in Eq. (6) we get

$$Z(\lambda) = e^{-(i\pi/4)\iota} \left(\frac{\lambda}{\pi}\right)^{-\zeta/2} \det'(d_1^* d_1)^{-1/4} \det'(d_0^* d_0)^{1/2} \times \det(\phi_0^* \phi_0)^{-1/2}. \tag{10}$$

We now rewrite the product of determinants in Eq. (10) in terms of the Ray-Singer torsion [20] of M. Since the Hodge star operator $*$ is unitary with $** = 1$ and $d_0^* = *d_2*$ (modulo a possible sign) we have

$$\begin{aligned} \det'(d_0^* d_0) &= \det'(*d_2*d_0) = \det'(*(*d_2*d_0)*) \\ &= \det'(d_2*d_0*) = \det'(d_2 d_2^*) \\ &= \det'(d_2^* d_2). \end{aligned}$$

It follows that

$$\begin{aligned} &\det'(d_0^* d_0)^{1/2} \det'(d_1^* d_1)^{-1/4} \\ &\quad = (\det'(d_0^* d_0)^{1/2} \det(d_1^* d_1)^{-1/2} \det'(d_2^* d_2)^{1/2})^{1/2}. \end{aligned} \tag{11}$$

It is possible to rewrite $\det(\phi_0^* \phi_0)$ using a standard result of manifold theory in a different form (see [21]). We start by noting that the integration map

$$\int_M : H^3(M) \to \mathbb{R}$$

is an isomorphism, i.e., for each $r \in \mathbb{R}$ there is a unique class $[\alpha] \in H^3(M)$ such that $\int \alpha = r$. [Note that the integration map is well defined on $H^3(M)$ since $\int_M \alpha + d\beta = \int_M \alpha$, i.e., $\int_M d\beta = 0$ by Stokes theorem.) Also from the definition

$$\begin{aligned} H^3(M) &= \Omega^3(M)/\mathrm{Im}(d_2) \\ &= [\mathrm{Im}(d_2) \oplus \mathrm{Im}(d_2)^\perp]/\mathrm{Im}(d_2) \\ &= \mathrm{Im}(d_2)^\perp, \end{aligned}$$

it follows that the map given by

$$\mathrm{Im}(d_2)^\perp \to \mathbb{R} \tag{12}$$

is also an isomorphism. Now define the map

$$\phi_3 : \mathbb{R} \to \mathrm{Im}(d_2)^\perp$$

to be the inverse of Eq. (12). Then using the properties of the Hodge star operator it can be shown that

$$\det(\phi_3^* \phi_3) = \det(\phi_0^* \phi_0)^{-1}.$$

It follows that

$$\det(\phi_0^* \phi_0)^{-1/2} = (\det(\phi_0^* \phi_0)^{-1/2} \det(\phi_3^* \phi_3)^{1/2})^{1/2}. \tag{13}$$

Substituting Eqs. (13) and (11) in Eq. (10) we get

$$|Z(\lambda)| = \left(\frac{\lambda}{\pi}\right)^{-1/2} \tau_{RS}(M)^{1/2}, \tag{14}$$

where

$$\begin{aligned} \tau_{RS}(M) &= \det(\phi_0^* \phi_0)^{-1/2} \det(\phi_3^* \phi_3)^{1/2} \det'(d_0^* d_0)^{1/2} \\ &\quad \times \det'(d_1^* d_1)^{-1/2} \det'(d_2^* d_2). \end{aligned} \tag{15}$$

This quantity $\tau_{RS}(M)$ is the Ray-Singer torsion of M [20]. It is a topological invariant of M, i.e., it is independent of the metric of M. Thus the modulus $|Z(\lambda)|$ of the partition function, given by Eqs. (13) and (14), is a topological invariant.

We are now ready to construct a discrete version of the preceding topological field theory which reproduces the continuum expression for the partition function where subdivision invariance is the discrete property corresponding to topological invariance. We will see that in order to do this it is crucial that there is an analogue of the Hodge star operator in the discrete theory. As we will see in the next section, this requires a field doubling. Therefore we consider a doubled version of the preceding theory, with the fields ω_1 and ω_2 in $\Omega^1(M)$ and with the action functional (5) changed by

$$\begin{aligned} S(\omega) &= \lambda \langle w, (*d_1) w \rangle \to \tilde{S}(\omega_1, \omega_2) \\ &= \lambda \left\langle \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \begin{pmatrix} 0 & *d_1 \\ *d_1 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \right\rangle. \end{aligned} \tag{16}$$

The reason for this specific choice of action $\tilde{S}(\omega_1, \omega_2)$ for the doubled theory will become clear in the next section. An obvious generalization of the preceding, with

$$T \to \tilde{T} = \begin{pmatrix} 0 & *d_1 \\ *d_1 & 0 \end{pmatrix},$$

shows that the partition function of the doubled theory

$$\tilde{Z}(\lambda) = \int_{\Omega^1(M) \times \Omega^1(M)} \mathcal{D}\omega_1 \, \mathcal{D}\omega_2 \, e^{-\lambda \tilde{S}(\omega_1, \omega_2)}$$

can be evaluated to obtain the square of (15),

$$\tilde{Z}(\lambda) = |Z(\lambda)|^2 = \left(\frac{\lambda}{\pi}\right)^{-1} \tau_{RS}(M). \tag{17}$$

Note that there is no phase factor here. This is because the quantity $\iota = d_+ + d_-$ for the action $\tilde{S}$ in Eq. (16) vanishes since

$$\tilde{T}=\begin{pmatrix} 0 & *d_1 \\ *d_1 & 0 \end{pmatrix}$$

has a symmetric spectrum.

IV. DISCRETE VERSION OF THE TOPOLOGICAL FIELD THEORY

We proceed to construct a discrete version of Abelian Chern-Simons gauge theory. The Whitney map enables the Abelian Chern-Simons theory to be discretized by replacing the gauge field (1-form) $A\in\Omega^1(M)$ by the discrete analogue, a 1-cochain $x\in C^1(K)$.

The most immediate way to do this is to construct the action S_K of the discrete theory by

$$\lambda S_K(x)=\lambda S(W^K(x))=\lambda\int_M dW^K(x)\wedge W^K(x).$$

This can be shown to coincide with the discrete action for the Abelian Chern-Simons theory introduced in [6]. This prescription fails, however, in the sense that the resulting partition function $Z_K(\lambda)$ is not a topological invariant, i.e., is not independent of K, and does not reproduce the continuum expression for the partition function. We demonstrate this by considering the resolvent for S_K obtained in an analogous way to the resolvent of the continuum action S described in the previous section. Let $T_K:C^1(K)\to C^1(K)$ denote the self-adjoint operator on $C^1(K)$ determined by

$$S_K(x)=\int_M dW^K(x)\wedge W^K(x)=\langle T_K x,x\rangle.$$

Then

$$\mathrm{Ker}(T_K)\subset \mathrm{Ker}(d_1^K).$$

Since for $x\in\mathrm{Ker}(d_1^K)$ we have

$$\langle T_K x,x\rangle=\int_M dW^K(x)\wedge W^K(x) \tag{18}$$

$$=\int_M W^K(d_1^K x)\wedge W^K(x). \tag{19}$$

Thus the discrete analogue of the resolvent (6) is a resolvent for S_K:

$$0\to\mathbb{R}\to^{\phi_0}\to\Omega^0(M)\to^{d_0^K}\mathrm{Ker}(d_1^K)\subseteq\mathrm{Ker}(T_K)=\mathrm{Ker}(S_K)\to 0.$$

The resulting partition function is the discrete analogue of the partition function $Z(\lambda)$:

$$Z_K(\lambda)=\det{}'((\phi_0^K)^*\phi_0^K)^{-1/2}\det{}'((d_0^K)^*d_0^K)^{1/2}$$
$$\times\det{}'\left(-\frac{i\lambda}{\pi}T_K\right)^{-1/2}.$$

In [11] the following formula for T_K was obtained:

$$T_K[v_0,v_1]=\frac{1}{6}\sum\,[v_2,v_3]$$

where the sum is over all 1-simplices $[v_2,v_3]$ such that $[v_0,v_1,v_2,v_3]$ is a 3-simplex with orientation compatible with the orientation of M.

It is possible to show [8] that $\det((\phi_0^K)^*\phi_0^K)=N_0^K=\dim C^{(K)}=$ the number of vertices of K. Then the failure of the discretization prescription can be demonstrated by showing that the quantity

$$|Z(\lambda)|^2=\frac{1}{N_0^K}\det{}'(\partial_1^K d_0^K)\det{}'(T_K^2)^{-1/2}$$

is not independent of K.

A discrete version of the doubled topological field theory with action (16) has been constructed in [11] in such a way that the expression (17) for the continuum partition function is reproduced. We briefly describe this in the following.

The discretization prescription is

$$(\omega_1,\omega_2)\in\Omega^1(M)\times\Omega^1(M)\to(x,y)\in C^1(K)\times C^1(\hat{K}), \tag{20}$$

$$S(\omega)=\lambda\left\langle\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix},\begin{pmatrix}0 & *d_1\\ *d_1 & 0\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right\rangle\to\tilde{S}_K(x,y)$$
$$=\lambda\left\langle\begin{pmatrix}x\\ y\end{pmatrix},\begin{pmatrix}0 & *^K d^K\\ *^{\hat{K}}d^{\hat{K}} & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\right\rangle, \tag{21}$$

where K is the simplicial complex triangulating M, $\hat{K}$ is its dual, and $C^q(K)$, $C^p(\hat{K})$, d^K, and $d^{\hat{K}}$ are as described in the previous section. The analogue of the Hodge star operator is the duality operator $*^K$. This is a map $*^K:C^q(K)\to C^{z-q}(\hat{K})$ [and $*^{\hat{K}}:C^p(\hat{K})\to C^{z-p}(K)$] which explains the need for field doubling and the expression (21) for the discrete action $\tilde{S}_K(x,y)$. There is a natural choice of resolvent for $\tilde{S}_K(x,y)$, analogous to the resolvent (6) in the continuum case. It is

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{R} & \to & C^0(K) & \to^{d_0^K} & \mathrm{Ker}\,(d^K) & \to & 0,\\
\oplus & & \oplus & & \oplus & & \oplus & & \oplus,\\
0 & \to & \mathbb{R} & \to & C^0(\hat{K}) & \to^{d_0^{\hat{K}}} & \mathrm{Ker}\,(d^{\hat{K}}) & \to & 0.
\end{array}$$

The partition function is

$$\tilde{Z}_K(\lambda)=\int_{C^1(K)\times C^1(\hat{K})}\mathcal{D}x\,\mathcal{D}y\,e^{-\tilde{S}_K(x,y)}. \tag{22}$$

Evaluating this by Schwarz's method with the resolvent above leads to

$$\tilde{Z}_K(\lambda)=\left(\frac{\lambda}{\pi}\right)^{-1+N_0^K-N_1^K}\det((\phi_0^K)^*\phi_0^K)^{-1/2}\det{}'((d_0^K)^*d_0^K)^{1/2}$$
$$\times\det{}'((d_1^K)^*d_1^K)^{-1/4}\det((\phi_0^{\hat{K}})^*\phi_0^{\hat{K}})^{-1/2}$$
$$\times\det{}'((d_0^{\hat{K}})^*d_0^{\hat{K}})^{1/2}\det{}'((d_1^{\hat{K}})^*d_1^{\hat{K}})^{-1/4}.$$

There is no phase factor in Eq. (22) since ζ vanishes just as in Eq. (17). We have also used the fact that $\zeta=1-N_0^K+N_1^K$, which is shown in [11].

Now rewrite the determinant involving $\hat{K}$ objects in terms of determinants of K objects. Modulo a possible sign $\pm$ we have the formulas [11]

$$(^{\hat{K}})^{-1}=(*^{\hat{K}})^{*}=*^{K}, \tag{23}$$

$$(*^{K})^{-1}=(*^{K})^{*}=*^{\hat{K}}, \tag{24}$$

$$(d_q^K)^{*}=*^{\hat{K}}d^{\hat{K}}_{n-q-1}*^{K}, \tag{25}$$

$$(d_p^{\hat{K}})^{*}=*^{K}d^{K}_{n-p-1}*^{\hat{K}}. \tag{26}$$

(The $\pm$ signs are omitted because they will all cancel out in the following calculation.) Now,

$$\det{}'((d_0^{\hat{K}})^{*}d_0^{\hat{K}})=\det{}'(*^{K}d_2^{K}*^{\hat{K}}d_0^{\hat{K}}) \tag{27}$$

$$=\det{}'(*^{\hat{K}}(*^{K}d_2^{K}*^{\hat{K}}d_0^{\hat{K}})*^{K}) \tag{28}$$

$$=\det{}'(d_2^{K}*^{\hat{K}}*^{K}) \tag{29}$$

$$=\det{}'(d_2^{K}(d_2^{K})^{*}) \tag{30}$$

$$=\det((d_2^{K})^{*}d_2^{K}), \tag{31}$$

and

$$\det{}'((d_1^{\hat{K}})^{*}d_1^{\hat{K}})=\det{}'(*^{K}d_1^{K}*^{\hat{K}}d_1^{\hat{K}}), \tag{32}$$

$$=\det{}'(*^{\hat{K}}(*^{K}d_1^{K}*^{\hat{K}}d_1^{\hat{K}})*^{K}) \tag{33}$$

$$=\det{}'(d_1^{K}*^{\hat{K}}*^{K}), \tag{34}$$

$$=\det{}'(d_1^{K}(d_1^{K})^{*}) \tag{35}$$

$$=\det((d_1^{K})^{*}d_1^{K}). \tag{36}$$

The integration map (12) has a discrete analog

$$\mathrm{Ker}\,(d_2^K)^{\perp}\rightarrow\mathbb{R},$$

$$a\rightarrow\langle a,[M]\rangle, \tag{37}$$

where $[M]\in C_3(K)$, the orientation cycle of M, i.e., the sum of all 3-simplices of K, oriented so that their orientations are compatible with the orientation of M. [Note that $a\in\mathrm{Ker}(d_2^K)^{\perp}\subset C^3(K)$ can be evaluated on any element $\sigma\in C_3(K)$ to get a real number $\langle a,\sigma\rangle\in\mathbb{R}$.] Define the map

$$\phi_3^K:\mathbb{R}\rightarrow\mathrm{Ker}(d_2^K)^{\perp} \tag{38}$$

to be the inverse of Eq. (37). Then using the properties of $*^{K}$ and $*^{\hat{K}}$, it can be shown that

$$\det((\phi_3^K)^{*}\phi_3^K)=\det((\phi_0^{\hat{K}})^{*}\phi_0^{\hat{K}})^{-1}. \tag{39}$$

Now using Eqs. (39), (31), and (36) we can rewrite Eq. (22) as

$$\tilde{Z}_K(\lambda)=\left(\frac{\lambda}{\pi}\right)^{-1+N_0^K-N_1^K}\tau_K(M), \tag{40}$$

where

$$\tau_K(M)=\det((\phi_0^K)^{*}\phi_0^K)^{-1/2}\det((\phi_3^K)^{*}\phi_3^K)^{1/2}\times\prod_{q=0}^{2}\det{}'((d_q^K)^{*}d_q^K)^{-1/2(-1)^q}. \tag{41}$$

This quantity $\tau_K(M)$ is the R torsion of the triangulation K of M. It is a combinatorial invariant of M, i.e., it is independent of the choice of triangulation K [22–24].

This is the untwisted torsion of M; more generally the torsion can be "twisted" by a representation of $\pi_1(M)$. The factors involving the determinants $\det'((d_q^K)^{*}d_q^K)$ constitute the usual Reidemeister torsion of M [19]. When these are put together with the factors involving $\det((\phi_i^K)\phi_i^K)$, $i=0,3$, as in Eq. (41), we get the R torsion "as a function of the cohomology" introduced and shown to be triangulation independent in [20].

The expression (41) for $\tau_K(M)$ is analogous to the expression (15) for the R torsion $\tau_{RS}(M)$, and in fact it has been shown [22,23] that these torsions are equal,

$$\tau_K(M)=\tau_{RS}(M).$$

It follows that the partition function (40) of the discrete theory coincides with the partition function (15) of the continuum theory, except for the K-dependent quantities N_0^K and N_1^K appearing in Eq. (40). These quantities can be removed by a suitable K-dependent renormalization of the coupling parameter λ.

It is possible to show that [8]

$$\det((\phi_0^K)^{*}\phi_0^K)=N_0^K,$$

$$\det((\phi_3^K)^{*}\phi_3^K)=\frac{1}{N_3^K}.$$

We will use this result in our numerical work.

V. NUMERICAL RESULTS

We are now in a position to proceed to numerically evaluate the discrete expressions for the torsion obtained. This allows us to check the underlying theoretical ideas by numerically verifying that the discrete expressions agree with expected analytic results. It also allows us to check that the results obtained are subdivision invariant. The subdivision invariance of torsion is demonstrated by showing that if any simplex of the triangulation is subdivided, the value of the

TABLE II. Results for T.

Complex	X_1	New	$\frac{X_1\times X_1}{\text{new}}$	N_3	N_0	Z^4
5-5	125	15625	1	5	5	0.04
8-6	1152	2985984	0.444	8	6	0.0123
9-6	2304	15116540	0.351	9	6	0.00975

TABLE III. R torsion for S^3.

Complex	X_1	X_2	X_3	$\frac{X_1 \times X_3}{X_2}$	N_3	N_0	T
$s3$	625	15625	625	25	5	5	1
$a1s3$	5184	2.985984×10^6	2.7648×10^4	48	8	6	1
$a2s3$	7776	1.511654×10^7	1.04976×10^5	54	9	6	1
$a3s3$	5184	2.985984×10^6	2.7648×10^4	48	8	6	1
$a4s3$	625	15625	625	25	5	5	1

torsion does not change. This is what is meant by topological invariance in the discrete setting. The expected analytic result for torsion for a lens space $L(p,1)$ is (see [20])

$$T(L(p,1)) = \frac{1}{p}.$$

Thus $T(S^3) = T(L(1,1)) = 1$.

We also show, numerically, that the discrete expression for the Chern-Simons partition function obtained without using the $*$ operator is not a topological invariant. This shows very clearly the importance of the doubling construction method used in the discretization method, for capturing topological information.

In order to proceed, we need to efficiently triangulate the spaces S^3 and $L(p,q)$. First we triangulate S^3. We do this by considering a four-dimensional simplex $[v_0,v_1,v_2,v_3,v_4]$ and observing that the boundary of this object is precisely the triangulation K of S^3 that we require. Next we turn to spaces $L(p,q)$, which we need to triangulate in order to proceed. An efficient triangulation of this space has been constructed by Brehm and Swiatkowski [25]. We use this procedure for our computations [26].

We can now summarize our numerical results. The R torsion for a simplicial complex K, with dual cell complex $\hat{K}$, for either S^3 or $L(p,1)$ involves evaluating

$$T = \sqrt{\frac{1}{N_0 N_3} \det(\partial_1 d_0) \det(\partial_2 d_1)^{-1} \det(\partial_3 d_2)},$$

where N_i are the numbers of i-simplices in K, $\partial_1, \partial_2, \partial_3$ are boundary operators on K and d_0, d_1, d_2 are coboundary operators on K. Note that, in the discrete setting, these operators can be expressed as matrices. We do this by using the vertices, edges, faces, and tetrahedra of our complex as a basis list. Any p-simplex in the complex can then be expressed in terms of this. Since we know to what d maps the various basis list elements, we can set the coefficients of its matrix representation. When we say $\det(\partial_1 d_0)$, for example, we simply mean the determinant of the matrix that results from multiplying the matrices corresponding to the operators ∂_1 and d_0.

If our complex consisted of just one triangle $[0,1,2]$, the basis list would be

$$\begin{array}{ll} 0 & [0], \\ 1 & [1], \\ 2 & [2], \\ 3 & [0,1], \\ 4 & [0,2], \\ 5 & [1,2], \\ 6 & [0,1,2]. \end{array}$$

We know that $d[0] = [1,0] + [2,0]$. This can be expressed in terms of our basis list as d acting on basis element 0 going to $-3-4$. So the matrix d for this complex is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}.$$

Note that the last column is zero since d has nothing to map a 2-simplex to and that the first three rows are zero

TABLE IV. R torsion for S^3.

Complex	X_1	X_2	X_3	$\frac{X_1 \times X_3}{X_2}$	N_3	N_0	T
$s3$	625	1.562×10^4	625	25	5	5	1
$1s3$	5184	2.985984×10^6	2.7648×10^4	48	8	6	1
$2s3$	3.9448×10^4	5.2166×10^8	1.018325×10^6	77	11	7	1
$3s3$	2.79936×10^5	8.418024×10^{10}	3.36798×10^7	112	14	8	1
$4s3$	1.876833×10^6	1.266709×10^{13}	1.032626×10^9	153	17	9	1
$5s3$	1.270325×10^7	2.037398×10^{15}	3.207653×10^{10}	200	20	10	1

TABLE V. R torsion for lens spaces.

Complex	X_1	X_2	X_3	$\frac{X_1 \times X_3}{X_2}$	N_3	N_0	T
$L(2,1)$	1.062937×10^{10}	3.618662×10^{29}	3.74484×10^{21}	110	40	11	1/2
$L(3,1)$	9.108×10^{12}	1.143589×10^{43}	1.08817×10^{32}	86.666	60	13	1/3
$L(4,1)$	1.1027×10^{16}	4.0468×10^{58}	2.89×10^{44}	78.7487	84	15	1/4
$L(5,1)$	1.790617×10^{19}	1.761432×10^{76}	7.49187×10^{58}	76.16	112	17	1/5

since nothing is mapped to 0-simplices. If we act on 0 with this matrix we get $-3-4$, as expected, and $d^2=0$.

If we do not use $\hat{K}$ then from Sec. IV,

$$\hat{T}=|Z(1)|^2=\frac{1}{N_0^K}(\det{}'\,\partial_1^K d_0^K)(\det T_K^2)^{-1/2}.$$

We evaluated the quantity $\hat{T}$ numerically for various triangulations K of S^3 and found the results shown in Table II for the change of $\hat{T}$ under subdivision where $m-n$ corresponds to a triangulation of S^3 with n vertices and m tetrahedra and where $X_i=\det\partial_i d_{i-1}$.

It is clear that $\hat{T}$ for S^3 is not subdivision invariant. We next evaluate T for S^3 and for $L(p,1)$ and check that it is indeed subdivision invariant and agrees with the analytic calculations for $L(p,1)$, with $p=2$, 3, 4, and 5. These results are shown in Tables III–V.

As a check on the numerical method we also count the number of zero modes of the Laplacian operator on the different p-chain spaces. These numbers give the dimension of the homology groups and are shown in Table VI.

A. S^3

As a further check, a systematic way of carrying out subdivision known as the Alexander moves [27] was used to study the subdivision invariance of the torsion. There are four such moves in three dimensions. They are best explained by example. We have

$$(1)\quad [0,1,2,3,4]\rightarrow[x,1,2,3,4]+[0,x,2,3,4],$$

$$(2)\quad [0,1,2,3,4]\rightarrow[x,1,2,3,4]+[0,x,2,3,4]+[0,1,x,3,4],$$

$$(3)\quad [0,1,2,3,4]\rightarrow[x,1,2,3,4]+[0,x,2,3,4]+[0,1,x,3,4]+[0,1,2,x,4],$$

$$(4)\quad [0,1,2,3,4]\rightarrow[x,1,2,3,4]+[0,x,2,3,4]+[0,1,x,3,4]+[0,1,2,x,4]+[0,1,2,3,x].$$

These are all natural operations in that the first (nth) move corresponds to adding a vertex splitting the 1-simplex (n-simplex) $[0,1]$ ($[0,1,\ldots,n]$) and connecting it to all the vertices, resulting in two $(n+1)$ tetrahedra.

The torsion T, in terms of its component determinants, and the way they change under the Alexander moves is exhibited in Table III, where $X_i=\det\partial_i d_{i-1}$ and $a2s3$ means the complex that resulted after the type 2 Alexander moves were performed on the S^3. Its clear from this that

$$T=\frac{1}{N_3N_0}\det(\partial_1 d_0)(\det\partial_2 d_1)^{-1}\det(\partial_3 d_2)$$

is subdivision invariant and thus a topological invariant of the manifold.

As a final check we tried several other subdivisions. We took a given triangulation and barycentric subdivided one or more (n) of its faces to get $1s3,2s3,\ldots,(ns3)$. The results are in Table IV.

B. Lens spaces

We conclude with the results for the lens spaces. The results are shown in Table V. As can be seen, these agree extremely well with the known analytic result $T(L(p,1))=1/p$.

VI. CONCLUSIONS

The method of discretization introduced works extremely well. The main point of the method is to construct discrete analogs for the set $(\Omega^p,d,\wedge,*)$. Previous work in this direction has neglected the Hodge star operator * [5,29]. We have thus demonstrated that the Hodge star operator plays a vital role in the construction of topological invariant objects from field theory. We were able to construct an expression for the partition function that is correct even as far as overall normalization is concerned. Mathematically, the equivalence between the Ray-Singer torsion and the combinatorial torsion of Reidemeister was proved independently in 1976 by Cheeger [22] and Müller [23]. It is nice to see the result emerge in a direct manner by a formal process of discretization. On the way we also had to double the original system so that K, the triangulation, and $\hat{K}$, its dual, are both present. If this doubling and the reason for it are overlooked, then the topological information present in the discretization is lost, as our numerical results demonstrated.

It is clear that the geometry motivated discretization method introduced is very general and that it can be used to analyze a wide variety of physical systems. In the approach outlined we have captured topological features. In applica-

TABLE VI. Zero modes for S^3.

p	No. of zero modes of Δ_p	Dim H^p
0	1	1
1	0	0
2	0	0
3	1	1

tions, it is also very important to capture geometrical features of a problem. We are currently investigating this aspect of our approach.

A limitation of the method is that there is no simple generalization to deal with non-Abelian theories. The discrete analogies of $d,d^{\dagger}$ were linear: there is no natural discrete analog of $d_A := d + A$, with A a Lie algebra valued 1-form.

ACKNOWLEDGMENTS

The work of Siddhartha Sen was part of a project supported by Enterprise Ireland. D.A. would like to acknowledge support from Enterprise Ireland and Hitachi. Samik Sen and Siddhartha Sen would like to thank D. Birmingham for explaining the triangulation method of Ref. [25].

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