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## A Doubled Discretization of Abelian Chern-Simons Theory

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A new discretization of a doubled, i.e., BF, version of the pure Abelian Chern-Simons theory is presented. It reproduces the continuum expressions for the topological quantities of interest in the theory, namely, the partition function and correlation function of Wilson loops. Similarities with free spinor field theory are discussed, which are of interest in connection with lattice fermion doubling. [S0031-9007(97)03228-6]

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The Abelian Chern-Simons (CS) theory [1–3] is an important topological field theory in three dimensions. It provides the topological structure of topologically massive (Abelian) gauge theory [2] and, in the Euclidean metrics, provides a useful theoretical framework for the description of interesting phenomena in planar condensed matter physics as, for example, fractional statistics particles [4], the quantum Hall effect, and high  $T_c$  superconductivity [5]. It is also essentially the same as the weak coupling (large  $k$ ) limit of the non-Abelian CS gauge theory, a solvable yet highly nontrivial topological quantum field theory [3]. In this paper we describe a discretization of the Abelian CS theory which reproduces the topological quantities of interest after introducing a field doubling in the theory. This doubling leads to the Abelian Chern-Simons action being replaced by the action for the so-called Abelian BF gauge theory [(2) below], in which the correlation function of Wilson loops and partition function become the square and norm square, respectively, of what they originally were. Note that discretizing the theory is not the same as putting it on a lattice in the usual way. Instead, it involves using a lattice to construct a discrete *analog* of the theory which reproduces the key topological quantities and/or features, without having to take a continuum limit. A detailed version of this work [6] will be published elsewhere.

We take the spacetime to be Euclidean  $\mathbf{R}^3$  (the case of general 3-manifolds is dealt with in [6]). The Abelian CS

action for gauge field  $A = A_\mu dx^\mu$  can be written as

$$\begin{aligned} S(A) &= \lambda \int_{\mathbf{R}^3} dx \epsilon_{\mu\nu\rho} A_\mu \partial_\nu A_\rho \\ &= \lambda \int_{\mathbf{R}^3} A \wedge dA = \lambda \langle A, (*d)A \rangle, \end{aligned} \quad (1)$$

where  $\lambda$  is the coupling parameter,  $d$  is the exterior derivative,  $*$  is the Hodge star operator, and  $\langle \cdot, \cdot \rangle$  is the inner product in the space of 1-forms determined by Euclidean metric in  $\mathbf{R}^3$ . All the ingredients of the last expression in (1) have natural lattice analogs (as we will see explicitly below); however, the lattice analog of the operator  $*$  is the *duality* operator, which maps between cells of the lattice  $K$  and cells of the *dual* lattice  $\hat{K}$ . To accommodate this feature we introduce a new gauge field  $A'$  and consider a doubled version of the action (1),

$$\begin{aligned} \tilde{S}(A, A') &\equiv \lambda \left\langle \begin{pmatrix} A \\ A' \end{pmatrix}, \begin{pmatrix} 0 & *d \\ *d & 0 \end{pmatrix} \right\rangle \\ &= 2\lambda \int_{\mathbf{R}^3} A' \wedge dA. \end{aligned} \quad (2)$$

This is the action of the so-called Abelian BF gauge theory [7]. In this theory the correlation function of *framed* Wilson loops can be considered: A framed loop is a closed ribbon which we denote by  $(\gamma, \gamma')$  where  $\gamma$  and  $\gamma'$  are the two boundary loops of the ribbon. The Wilson correlation function of oriented framed loops  $(\gamma^{(1)}, \gamma^{(1)'})$ ,  $\dots$ ,  $(\gamma^{(r)}, \gamma^{(r)'})$  is

$$\langle (\gamma^{(1)}, \gamma^{(1)'}) \dots (\gamma^{(r)}, \gamma^{(r)'}) \rangle \equiv \tilde{Z}(\lambda)^{-1} \int_{\mathcal{A} \times \mathcal{A}} \mathcal{D}A \mathcal{D}A' \left[ \prod_{l=1}^r (e^{i \oint_{\gamma^{(l)}} A} e^{i \oint_{\gamma^{(l)'}} A'} \right] e^{i \tilde{S}(A, A')}. \quad (3)$$

This can be formally evaluated using standard techniques [7] to obtain

$$\langle (\gamma^{(1)}, \gamma^{(1)'}) \dots (\gamma^{(r)}, \gamma^{(r)'}) \rangle = \exp \left[ \frac{-i}{2\lambda} \left( \sum_{l \neq m} L(\gamma^{(l)}, \gamma^{(m)}) + \sum_{l=1}^r L(\gamma^{(l)}, \gamma^{(l)'}) \right) \right], \quad (4)$$

where  $L(\gamma, \gamma')$  denotes the Gauss linking number of  $\gamma$  and  $\gamma'$ . The partition function of this theory,

$$\tilde{Z}(\lambda) \equiv \int_{\mathcal{A} \times \mathcal{A}} \mathcal{D}A \mathcal{D}A' e^{i \tilde{S}(A, A')}, \quad (5)$$

is also a quantity of topological interest. After compactifying the spacetime to  $S^3$  and imposing the covariant gauge-fixing condition

$$d^\dagger A = 0, \quad d^\dagger A' = 0 \quad (6)$$

(where  $d^\dagger$  is the adjoint of  $d$ ), the partition function can be formally evaluated as in [1] (see also [8]) to obtain

$$\begin{aligned} \tilde{Z}(\lambda) = & \det(\phi_0^\dagger \phi_0)^{-1} \det'(d_0^\dagger d_0) \\ & \times \det' \left[ \frac{-i\lambda}{\pi} \begin{pmatrix} 0 & *d \\ *d & 0 \end{pmatrix} \right]^{-1/2}, \end{aligned} \quad (7)$$

where we denote by  $d_q$  the restriction of  $d$  to the space  $\Omega^q(S^3)$  of  $q$ -forms, and  $\phi_0 : \mathbf{R} \rightarrow \Omega^0(S^3)$  maps  $r \in \mathbf{R}$  to the constant function equal to  $r$ . In this expression  $\det'(d_0^\dagger d_0)$  is the Faddeev-Popov determinant corresponding to (6) and  $\det(\phi_0^\dagger \phi_0)^{-1} [= V(S^3)^{-1}]$  is a ‘‘ghosts for ghosts’’ determinant which arises because constant gauge transformations act trivially on the gauge fields. The determinants in (7) are regularized via zeta regularization as in [1,8]. Using Hodge duality and the techniques of [1,8] we can rewrite (7) as

$$\tilde{Z}(\lambda) = \left( \frac{\lambda}{\pi} \right)^{-1} \tau_{RS}(S^3; d) \quad (8)$$

{the general phase factor of [[8], Eq. (6)] is trivial here since the operator in (2) has symmetric spectrum} where

$$\begin{aligned} \tau_{RS}(S^3; d) = & \det(\phi_0^\dagger \phi_0)^{-1/2} \det(\phi_3^\dagger \phi_3)^{1/2} \\ & \times \prod_{q=0}^2 \det'(d_q^\dagger d_q)^{(1/2)(-1)^q} \end{aligned} \quad (9)$$

is the Ray-Singer torsion of  $d$  [9] (see, in particular, section 3 of the second paper in [9]). [In (9)  $\phi_3 : \mathbf{R} \rightarrow \Omega^3(S^3)$  maps  $r \in \mathbf{R}$  to the harmonic 3-form  $\omega$  with  $\int_{S^3} \omega = r$  and we have used  $\det(\phi_3^\dagger \phi_3) = V(S^3)^{-1} = \det(\phi_0^\dagger \phi_0)^{-1}$  [6].] The Ray-Singer torsion is a topological invariant of  $S^3$ , i.e., it is independent of the metric on  $S^3$  used to construct  $*$  and  $\langle \cdot, \cdot \rangle$  in (2),  $d^\dagger$  in (6), and  $\phi_0^\dagger$  in (7). The physical significance of this is as follows: When compactifying  $\mathbf{R}^3$  to  $S^3 \approx \mathbf{R}^3 \cup \{\infty\}$  (e.g., via stereographic projection) the Euclidean metric on  $\mathbf{R}^3$  must be deformed towards infinity in order that it extend to a well-defined metric on  $S^3$ . The topological invariance of  $\tau_{RS}(S^3, d)$  means that the resulting partition function (8) is independent of how this deformation is carried out. In fact,  $\tau_{RS}(S^3, d) = 1$  (the argument for this will be given below) so  $\tilde{Z}(\lambda) = \pi/\lambda$ . If  $\mathbf{R}^3$  is compactified in a topologically more complicated way, leading to a general closed oriented 3-manifold  $M$ , then the preceding derivation of (8) continues to hold (with  $S^3$  replaced by  $M$ ) if  $H^1(M) = 0$  and can be generalized if  $H^1(M) \neq 0$

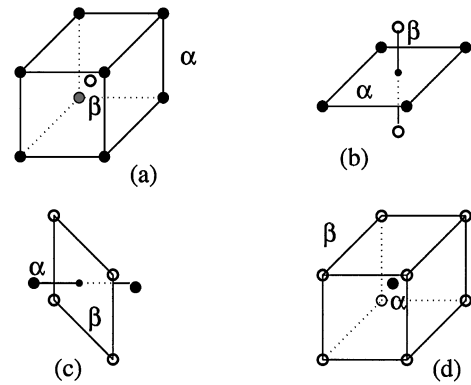


FIG. 1. Definition of the duality operator  $*^K$ :  $\alpha$  is a  $p$ -cell in  $K$  and  $\beta = *^K \alpha$  is the dual  $(3 - p)$ -cell in  $\hat{K}$ . [In (a)  $\beta$  is the point (vertex in  $\hat{K}$ ) at the center of the 3-cell  $\alpha$ , while in (d)  $\beta$  is the 3-cell in  $\hat{K}$  which has the point  $\alpha$  at its center.]

[6]. For example, if  $M$  is a lens space  $L(p, q)$  then  $\tau_{RS}[L(p, q), d] = 1/p$  and  $\tilde{Z}(\lambda) = \pi/p\lambda$ .

We will construct a discrete version of the doubled theory  $S(A, A')$  which reproduces the continuum expressions (4) and (8) for the correlation function of framed Wilson loops and partition function, respectively. Let  $K$  be a lattice decomposition of  $\mathbf{R}^3$  which, for convenience, we take to be cubic. It is well known [10,11] that the space  $\Omega^p$  of antisymmetric tensor fields of degree  $p$  (i.e.,  $p$ -forms) has a discrete analog, the space  $C^p(K)$  of  $p$ -cochains (i.e.,  $\mathbf{R}$ -valued functions on the  $p$ -cells of  $K$ ), in particular,  $C^1(K)$  is the analog of the space  $\mathcal{A} = \Omega^1$  of gauge fields. The space  $C_p(K)$  of  $p$ -chains (i.e., formal linear combinations over  $\mathbf{R}$  of oriented  $p$ -cells) has a canonical inner product  $\langle \cdot, \cdot \rangle$  defined by requiring that the  $p$ -cells be orthonormal; this allows one to identify  $C_p(K)$  with its dual space  $C^p(K)$  so we will speak only of  $C_p(K)$  in the following. The analog of  $d$  is the coboundary operator  $d^K : C_p(K) \rightarrow C_{p+1}(K)$ , i.e., the adjoint of the boundary operator  $\partial^K$ . The new feature of our discretization is that we also use the (co)chain spaces  $C_q(\hat{K})$  associated with the dual lattice  $\hat{K}$  (i.e., the cubic lattice whose vertices are the centers of the 3-cells of  $K$ ). The cells of  $K$  and  $\hat{K}$  are related by the duality operator  $*^K$ , defined in Fig. 1. An orientation for a  $p$ -cell  $\alpha$  determines an orientation for the dual  $(3 - p)$ -cell  $*^K \alpha$  by requiring that the product of the orientations of  $\alpha$  and  $*^K \alpha$  coincides with the standard orientation of  $\mathbf{R}^3$ . Thus the duality operator  $*^K$  determines a linear map  $*^K : C_p(K) \xrightarrow{\cong} C_{3-p}(\hat{K})$ ; this is the discrete analog of the Hodge star operator  $*$  in (1) and (2). Set  $*^{\hat{K}} \equiv (*^K)^\dagger = (*^K)^{-1}$ . The discrete theory is now constructed by

$$(A, A') \in \mathcal{A} \times \mathcal{A} \rightarrow (a, a') \in C_1(K) \times C_1(\hat{K}), \quad (10)$$

$$\tilde{S}(A, A') \equiv \lambda \left\langle \begin{pmatrix} A \\ A' \end{pmatrix}, \begin{pmatrix} 0 & *d \\ *d & 0 \end{pmatrix} \begin{pmatrix} A \\ A' \end{pmatrix} \right\rangle \rightarrow \tilde{S}_K(a, a') \equiv \lambda \left\langle \begin{pmatrix} a \\ a' \end{pmatrix}, \begin{pmatrix} 0 & *^{\hat{K}} d^{\hat{K}} \\ *^K d^K & 0 \end{pmatrix} \begin{pmatrix} a \\ a' \end{pmatrix} \right\rangle. \quad (11)$$

The discrete action  $S_K(a, a')$  is invariant under  $a \rightarrow a + d^K b$ ,  $a' \rightarrow a' + d^{\hat{K}} b'$  for all  $(b, b') \in C_0(K) \times C_0(\hat{K})$  since  $d^K d^K = 0$  and  $d^{\hat{K}} d^{\hat{K}} = 0$ ; this is the discrete analog of the gauge invariance of the continuum theory.

Framed Wilson loops fit naturally into this discrete setup: The framed loops are taken to be ribbons  $(\gamma_K, \gamma_{\hat{K}})$  where one boundary loop  $\gamma_K$  is an edge loop in the lattice  $K$  and the other boundary loop  $\gamma_{\hat{K}}$  is an edge loop in the dual lattice  $\hat{K}$ . (It is always possible to find such a framing of an edge loop  $\gamma_K$  [6].) There is a natural discrete version

of line integrals,

$$\oint_{\gamma_K} A \rightarrow \langle \underline{\gamma}_K, a \rangle, \quad \oint_{\gamma_{\hat{K}}} A' \rightarrow \langle \underline{\gamma}_{\hat{K}}, a' \rangle, \quad (12)$$

where  $\underline{\gamma}_K \in C_1(K)$  denotes the sum of the 1-cells in  $K$  making up  $\gamma_K$ , and  $\underline{\gamma}_{\hat{K}} \in C_1(\hat{K})$  denotes the sum of the 1-cells in  $\hat{K}$  making up  $\gamma_{\hat{K}}$ . Then the correlation function of nonintersecting oriented framed edge loops  $(\gamma_K^{(1)}, \gamma_{\hat{K}}^{(1)}), \dots, (\gamma_K^{(r)}, \gamma_{\hat{K}}^{(r)})$  in the discrete theory is

$$\langle (\gamma_K^{(1)}, \gamma_{\hat{K}}^{(1)}), \dots, (\gamma_K^{(r)}, \gamma_{\hat{K}}^{(r)}) \rangle \equiv \tilde{Z}_K(\lambda)^{-1} \int_{C_1(K) \times C_1(\hat{K})} \mathcal{D}a \mathcal{D}a' \left[ \prod_{l=1}^r (e^{i\langle \underline{\gamma}_K^{(l)}, a \rangle} (e^{i\langle \underline{\gamma}_{\hat{K}}^{(l)}, a' \rangle}) \right] e^{iS_K(a, a')}. \quad (13)$$

A formal evaluation analogous to the evaluation of (3) leading to (4) gives

$$\begin{aligned} \langle (\gamma_K^{(1)}, \gamma_{\hat{K}}^{(1)}), \dots, (\gamma_K^{(r)}, \gamma_{\hat{K}}^{(r)}) \rangle &= \exp \left[ \frac{-i}{4\lambda} \left\langle \left( \begin{array}{c} \underline{\gamma}_K^{(1)} + \dots + \underline{\gamma}_K^{(r)} \\ \underline{\gamma}_{\hat{K}}^{(1)} + \dots + \underline{\gamma}_{\hat{K}}^{(r)} \end{array} \right), \left( \begin{array}{cc} 0 & *^{\hat{K}} d^{\hat{K}} \\ *^K d^K & 0 \end{array} \right)^{-1} \left( \begin{array}{c} \underline{\gamma}_K^{(1)} + \dots + \underline{\gamma}_K^{(r)} \\ \underline{\gamma}_{\hat{K}}^{(1)} + \dots + \underline{\gamma}_{\hat{K}}^{(r)} \end{array} \right) \right\rangle \right] \\ &= \exp \left( \frac{-i}{2\lambda} \sum_{l,m=1}^r \langle \underline{\gamma}_K^{(l)}, (*^K d^K)^{-1} \underline{\gamma}_{\hat{K}}^{(m)} \rangle \right), \end{aligned} \quad (14)$$

where we have used  $(*^K d^K)^\dagger = *^{\hat{K}} d^{\hat{K}}$ . To show that this coincides with the continuum expression (4) we must show that for any oriented edge loop  $\gamma_K$  in  $K$  and oriented edge loop  $\gamma_{\hat{K}}$  in  $\hat{K}$ ,

$$\langle \underline{\gamma}_K, (*^K d^K)^{-1} \underline{\gamma}_{\hat{K}} \rangle = L(\gamma_K, \gamma_{\hat{K}}). \quad (15)$$

Then taking  $\gamma_K = \gamma_K^{(l)}$  and  $\gamma_{\hat{K}} = \gamma_{\hat{K}}^{(m)}$  in (15) and substituting in (14) reproduces the continuum expression (4). To derive (15) we recall that the linking number of  $\gamma_K$  and  $\gamma_{\hat{K}}$  can be characterized as follows. Let  $D$  be a surface in  $\mathbf{R}^3$  with  $\gamma_K$  as its boundary, and such that all intersections of  $D$  with  $\gamma_{\hat{K}}$  are transverse, then

$$L(\gamma_K, \gamma_{\hat{K}}) = \sum_{D \cap \gamma_{\hat{K}}} \pm 1, \quad (16)$$

where the sign of  $\pm 1$  for a given intersection of  $D$  and  $\gamma_{\hat{K}}$  is  $+$  if the product of the orientations of  $D$  (induced by the orientation of  $\gamma_K$ ) and  $\gamma_{\hat{K}}$  at the intersection coincides with the standard orientation of  $\mathbf{R}^3$ , and  $-$  otherwise. We now show that the left-hand side of (15) coincides with (16). First note that

$$\begin{aligned} \langle \underline{\gamma}_K, (*^K d^K)^{-1} \underline{\gamma}_{\hat{K}} \rangle &= \langle [(*^K d^K)^{-1}]^\dagger \underline{\gamma}_K, \underline{\gamma}_{\hat{K}} \rangle \\ &= \langle *^K (\partial^K)^{-1} \underline{\gamma}_K, \underline{\gamma}_{\hat{K}} \rangle. \end{aligned} \quad (17)$$

Choose a surface  $D_K$  in  $\mathbf{R}^3$  made up of a union of 2-cells of  $K$  and with  $\gamma_K$  as its boundary (illustrated in Fig. 2; such a choice is always possible [6]) and equip  $D_K$  with the orientation induced by  $\gamma_K$ . The formal sum

of the oriented 2-cells making up  $D_K$  is then an element  $\underline{D}_K \in C_2(K)$ , and  $\partial^K \underline{D}_K = \underline{\gamma}_K$ , so (17) gives

$$\langle \underline{\gamma}_K, (*^K d^K)^{-1} \underline{\gamma}_{\hat{K}} \rangle = \langle *^K \underline{D}_K, \underline{\gamma}_{\hat{K}} \rangle. \quad (18)$$

Now  $*^K \underline{D}_K \in C_1(\hat{K})$  is the sum of all the 1-cells in  $\hat{K}$  which are dual to the 2-cells making up  $D_K$ , as indicated in Fig. 2. Since  $\gamma_{\hat{K}}$  is an edge loop in the dual lattice  $\hat{K}$  all the 1-cells  $\beta$  making up  $\gamma_{\hat{K}}$  are duals of 2-cells  $\alpha$  in  $K$  as illustrated in Fig. 1(b) above. Hence intersections of  $\gamma_{\hat{K}}$  and  $D_K$  occur precisely when a 1-cell in  $\gamma_{\hat{K}}$  is the dual of a 2-cell in  $D_K$  (up to a sign), and it follows that the right-hand side of (18) equals (16) with  $D = D_K$ . This completes the derivation of (15), thereby showing that the Wilson correlation function (14) in the discrete theory reproduces the continuum expression (4) as claimed.

The partition function in this discrete theory is

$$\tilde{Z}_K(\lambda) \equiv \int_{C_1(K) \times C_1(\hat{K})} \mathcal{D}a \mathcal{D}a' e^{iS_K(a, a')}. \quad (19)$$

As before we compactify the spacetime to  $S^3$ ; taking  $K$  to be a lattice decomposition for  $S^3$  the analog of the gauge-fixing condition (6) is

$$\partial^K a = 0, \quad \partial^{\hat{K}} a' = 0, \quad (20)$$

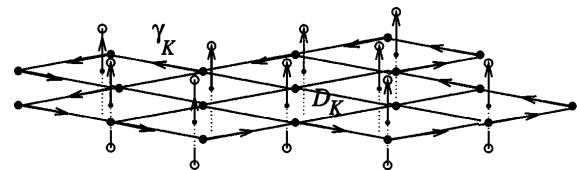


FIG. 2.  $\gamma_K$  is the boundary of the surface  $D_K$  made up of 2-cells of  $K$ . The vertical line segments are the duals of the 2-cells making up  $D_K$ .

and a formal evaluation of (19) analogous to the one leading to (7) gives

$$\tilde{Z}(\lambda) = \det[(\phi_0^K)^\dagger \phi_0^K]^{-1/2} \det[(\phi_0^{\hat{K}})^\dagger \phi_0^{\hat{K}}]^{-1/2} \det'(\partial_1^K d_0^K)^{1/2} \det'(\partial_1^{\hat{K}} d_0^{\hat{K}})^{1/2} \det' \left[ \frac{-i\lambda}{\pi} \begin{pmatrix} 0 & *^{\hat{K}} d_1^{\hat{K}} \\ *^K d_1^K & 0 \end{pmatrix} \right]^{-1/2}. \quad (21)$$

Here  $\phi_0^K : \mathbf{R} \rightarrow C_0(K)$  and  $\phi_0^{\hat{K}} : \mathbf{R} \rightarrow C_0(\hat{K})$  are natural discrete analogs of  $\phi_0$ ;  $\det[(\phi_0^K)^\dagger \phi_0^K] = N_0^K$  and  $\det[(\phi_0^{\hat{K}})^\dagger \phi_0^{\hat{K}}] = N_0^{\hat{K}}$  where  $N_p^K \equiv \dim C_p(K)$ ,  $N_q^{\hat{K}} = \dim C_q(\hat{K})$  [6]. There is a natural discrete analog  $\phi_3^K$  of  $\phi_3$  with  $\det[(\phi_3^K)^\dagger \phi_3^K] = 1/N_3^K = 1/N_0^{\hat{K}} = \det[(\phi_0^{\hat{K}})^\dagger \phi_0^{\hat{K}}]^{-1}$ ; using this and the properties of  $*^K$  [ $\partial_q^K = (-1)^q *^K d_{3-q}^K$ ] we can rewrite (21) as [6]

$$\tilde{Z}_K(\lambda) = \left( \frac{\lambda}{\pi} \right)^{-1+N_0^K-N_1^K} \tau(S^3; K, d^K), \quad (22)$$

where

$$\tau(S^3; K, d^K) = \det[(\phi_0^K)^\dagger \phi_0^K]^{-1/2} \det[(\phi_3^K)^\dagger \phi_3^K]^{1/2} \prod_{q=0}^2 \det'(\partial_{q+1}^K d_q^K)^{(1/2)(-1)^q} \quad (23)$$

is the R-torsion of  $d^K$  [9]. The R-torsion is a combinatorial invariant of  $S^3$ ; i.e., it is the same for all choices of lattice  $K$  for  $S^3$  (including noncubic, e.g., tetrahedral, lattices). Thus when compactifying the spacetime  $\mathbf{R}^3$  to  $S^3$  the resulting expression (22) for the partition function in the discrete theory is independent of how the lattice  $K$  for  $\mathbf{R}^3$  is modified to obtain a lattice decomposition of  $S^3$ , except for the exponent of  $\lambda/\pi$  in (22). A straightforward calculation using the tetrahedral lattice for  $S^3$  obtained by identifying  $S^3$  with the boundary of the standard 4-simplex in  $\mathbf{R}^4$  gives  $\tau(S^3; K, d^K) = 1$ . Thus  $\tilde{Z}_K(\lambda) = (\pi/\lambda)^{1-N_0^K+N_1^K}$  in the present case. As in the continuum case, the derivation of (22) continues to hold for general closed oriented 3-manifold  $M$  with  $H^1(M) = 0$  and can be generalized if  $H^1(M) \neq 0$  [6]. A deep mathematical result, proved independently by Cheeger and Müller [12], states that R-torsion and Ray-Singer torsion are equal; in particular,  $\tau(M; K, d^K) = \tau_{RS}(M, d)$  [so  $\tau_{RS}(S^3, d) = 1$  as mentioned earlier]. It follows that the partition function  $\tilde{Z}_K(\lambda)$  of the discrete theory reproduces the continuum partition function  $\tilde{Z}(\lambda)$  when  $\lambda = \pi$ , and also when  $\lambda \neq \pi$  after a lattice-dependent renormalization of  $\lambda$  in the discrete theory.

The results of this paper are of interest in connection with lattice fermion doubling. From (1) and (2) we see that the Lagrangians  $\mathcal{L}_{CS}$  and  $\mathcal{L}_{BF}$  of the Abelian CS and BF theories can be written in an analogous way to the Lagrangian  $\psi^* \gamma^\mu \partial_\mu \psi$  for a free spinor field,

$$\mathcal{L}_{CS} = A^\dagger e^\mu \partial_\mu A, \quad \mathcal{L}_{BF} = \tilde{A}^\dagger \tilde{e}^\mu \partial_\mu \tilde{A}, \quad (24)$$

where  $A = (A_\mu)$  and  $\tilde{A} = (A, A')$  are considered as a 3-vector and 6-vector,  $A^\dagger$  and  $\tilde{A}^\dagger$  are their transposes,  $e^\mu$  is a  $3 \times 3$  matrix [ $(e^\mu)_{\nu\rho} = \epsilon_{\nu\mu\rho}$ ], and  $\tilde{e}^\mu$  is a  $6 \times 6$  matrix. If we formulate the Abelian CS and BF theories on a spacetime lattice in the same way as for a spinor field theory and calculate the momentum space propagator in the standard way we find a “doubling” of exactly the same kind as for spinor fields on the lattice (described, e.g., in Chap. 5 of [13]). Thus, on the one hand, when the

Abelian CS or BF theory is put on the lattice in the same way as a spinor theory an analog of “fermion doubling” appears, while, on the other hand, the discretization of the Abelian BF theory described in this paper successfully reproduces continuum quantities.

The doubled, i.e., BF, version of the Abelian CS theory has the following analog of chiral invariance. The  $6 \times 6$  matrix  $\tilde{C}$  defined by  $\tilde{C}(A, A') = (A, -A')$  satisfies the chirality conditions  $\tilde{C}^2 = I$  and  $\tilde{C}\tilde{e}^\mu = -\tilde{e}^\mu\tilde{C}$ . Thus  $\tilde{C}$  is analogous to  $\gamma^5$  in spinor theory, and the Abelian BF Lagrangian in (24) has chiral invariance under  $\tilde{A} \rightarrow e^{\alpha\tilde{C}}\tilde{A}$  ( $\alpha \in \mathbf{R}$ ). The original field  $A$  and new field  $A'$  then have positive and negative chirality, respectively, in analogy with spinors of positive and negative chirality.

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