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IMPROVED LOWER BOUNDS ON BOOK CROSSING NUMBERS OF COMPLETE GRAPHS*

E. DE KLERK[†], D. V. PASECHNIK[‡], AND G. SALAZAR[§]

Abstract. A book with k pages consists of a straight line (the *spine*) and k half-planes (the *pages*), such that the boundary of each page is the spine. If a graph is drawn on a book with k pages in such a way that the vertices lie on the spine, and each edge is contained in a page, the result is a k -page book drawing (or simply a k -page drawing). The k -page crossing number $\nu_k(G)$ of a graph G is the minimum number of crossings in a k -page drawing of G . In this paper we investigate the k -page crossing numbers of complete graphs. We use semidefinite programming techniques to give improved lower bounds on $\nu_k(K_n)$ for various values of k . We also use a maximum satisfiability reformulation to obtain a computer-aided calculation of the exact value of $\nu_k(K_n)$ for several values of k and n . Finally, we investigate the best construction known for drawing K_n in k pages, calculate the resulting number of crossings, and discuss this upper bound in light of the new results reported in this paper.

Key words. 2-page crossing number, book crossing number, semidefinite programming, maximum k -cut, Frieze–Jerrum maximum- k -cut bound, maximum satisfiability problem

AMS subject classifications. 90C22, 90C25, 05C10, 05C62, 57M15, 68R10

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1. Introduction. Motivated by applications to VLSI design, Chung, Leighton, and Rosenberg [7] studied embeddings of graphs in books. A *book* consists of a line (the *spine*) and $k \geq 1$ half-planes (the *pages*), such that the boundary of each page is the spine. In a *book embedding*, each edge is drawn on a single page, and no edge crossings are allowed. In a *book drawing* (or k -page drawing if the book has k pages), each edge is drawn on a single page, but edge crossings are allowed.

Obviously every fixed graph can be embedded in a book with sufficiently many pages. On the other hand, it is desirable to do so using as few pages as possible. Given a graph G , the minimum k such that G can be embedded in a k -page book is the *pagenumber* (or *book thickness*) of G [2, 7, 21]. Determining the pagenumber of an arbitrary graph is NP-Complete [7], but some results are known for particular families of graphs. For instance, it is not difficult to prove that the pagenumber of the complete graph K_n is $\lceil n/2 \rceil$. On the other hand, with few exceptions, the pagenumbers of the complete bipartite graphs $K_{m,n}$ are unknown (see [12, 26]). Yannakakis proved [38] that every planar graph can be embedded in four pages.

1.1. The k -page crossing number $\nu_k(G)$ of a graph G . When the number k of pages is fixed, the goal is to minimize the number of crossings in a k -page drawing of an input graph. The k -page crossing number $\nu_k(G)$ of a graph G is the minimum number of crossings in a k -page drawing of G .

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FIG. 1.1. A 3-page drawing of K_7 with 2 crossings.

Clearly, a graph G has $\nu_1(G) = 0$ if and only if it is outerplanar. Equivalent to 1-page drawings are *circular drawings*, in which the vertices are placed on a circle and all edges are drawn in its interior. In a similar vein, k -page drawings of $G = (V, E)$ can be alternatively viewed as a set of k circular drawings of graphs $G^{(i)} = (V, E^{(i)})$ ($i = 1, \dots, k$), where the edge sets $E^{(i)}$ form a k -partition of E . In other words, we assign each edge in E to exactly one of the k circular drawings. In Figure 1.1 we illustrate a 3-page drawing of K_7 .

Several computational approaches and heuristics for estimating (or determining) $\nu_1(G)$ and $\nu_2(G)$ have been devised (see, for instance, [18, 19, 20, 24, 25]). Few exact results or nontrivial bounds are known, and very little is known about $\nu_k(G)$ for $k > 2$.

Although the special cases $k = 1$ and 2 have received considerable attention, the only thorough investigation of $\nu_k(G)$ for arbitrary k is the work by Shahrokhi, et al. [32]. Shahrokhi et al. give general lower bounds for $\nu_k(G)$ for any graph G . They also give lower and upper bounds for $\nu_k(K_n)$, and they use their upper bounds for $\nu_k(K_n)$ to give general upper bounds for $\nu_k(G)$ for arbitrary graphs G .

As with every graph-theoretical parameter, there is a natural interest in computing (or at least estimating) the k -page crossing number of the complete graph K_n . In addition, estimates on $\nu_k(K_n)$ are an essential tool to derive bounds for $\nu_k(G)$ for other graphs G via the embedding method. This is the approach followed by Shahrokhi et al. in [32], where the constructions that yield their upper bounds for $\nu_k(K_n)$ are used to generate k -page drawings of dense graphs, whose number of crossings is within a constant factor of their k -page crossing number.

1.2. Structure of the rest of the paper. Our main contributions in this paper are improved lower bounds for the k -page crossing numbers of K_n . We also compute the exact value of $\nu_k(K_n)$ for several k and n . (No exact values were previously known for any n for any $k > 2$.)

In section 2 we survey the bounds and exact results known for $\nu_k(K_n)$. In section 3, we show how $\nu_k(K_n)$ may be obtained from the solution of a maximum- k -cut problem on a suitable graph or via the solution of a suitable weighted maximum satisfiability problem. These reformulations are used in section 4 to obtain previously unknown exact values and improved lower bounds on $\nu_k(K_n)$ for various values of k and n via computation. In section 5 we review the construction that gives the best upper bounds available for $\nu_k(K_n)$, calculate the resulting number of crossings, and analyze this upper bound in light of the new results obtained in the previous sections. Finally, in section 6 we present some concluding remarks and open questions.

2. k -page drawings of K_n : Exact results and bounds.

2.1. 1-page drawings of K_n . Calculating the 1-page crossing number of K_n is straightforward. Indeed, it is obvious that in every 1-page drawing of K_n , every four vertices define a crossing, and therefore $\nu_1(K_n) \geq \binom{n}{4}$. It is easy to give 1-page drawings with exactly $\binom{n}{4}$ crossings, and so the reverse inequality $\nu_1(K_n) \leq \binom{n}{4}$ follows.

It follows that the problem of calculating or estimating $\nu_k(K_n)$ is only of interest for $k \geq 2$.

2.2. 2-page drawings of K_n . We recall that the *crossing number* $\text{cr}(G)$ of a graph G is the minimum number of crossings in a drawing of G in the plane. Harary and Hill [16] described how to draw K_n in the plane with $Z_2(n)$ crossings, where

$$(2.1) \quad Z_2(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

No drawings of K_n with fewer than $Z_2(n)$ crossings are known, and to this date the Harary–Hill conjecture $\text{cr}(K_n) = Z_2(n)$ is still open. (It has been settled only for $n \leq 12$; see [27].)

The drawings given in [16] are not 2-page drawings, but it was later noticed that 2-page drawings with $Z_2(n)$ crossings do exist [3] (see also [15, 17]). This observation gave rise to the conjecture $\nu_2(K_n) = Z_2(n)$, popularized by Vrt’o [37].

Buchheim and Zheng [4] proved that $\nu_2(K_n) = Z_2(n)$ for $n \leq 14$. Recently, de Klerk and Pasechnik [11] verified that $\nu_2(K_n) = Z_2(n)$ for $n \leq 18$ and $n = 20$ and 24 and used semidefinite programming techniques to give asymptotic estimates on $\nu_2(K_n)/Z_2(n)$. More recently, Abrego et al. proved that $\nu_2(K_n) = Z_2(n)$ for every n [1].

2.3. k -page drawings of K_n for $k \geq 3$: Upper bounds. Much less is known of the k -page crossing number $\nu_k(K_n)$ for $k > 2$.

As we mentioned above, Blažek and Koman [3] seemed to have been the first to construct 2-page drawings of K_n with $Z_2(n)$ crossings. In the same paper they briefly observed that their construction could be extended to k pages. Although no details were given, they gave the number of crossings obtained for the case $k = 3$ (see section 5.4).

Damiani, D’Antona, and Salemi proposed a way to draw K_n on k pages [8], using the adjacency matrix representation. (We call this the *DDS construction*.) They included a table with the resulting number of crossings for $n \leq 18$ and all $k \leq \lceil n/2 \rceil$ (recall that $\nu_k(K_n) = 0$ if $k > \lceil n/2 \rceil$). The exact number of crossings resulting from their construction was not explicitly given. (We have calculated this number; see Proposition 5.1.) The (table) results given in [8] coincide for the case $k = 3$ with the expression given by Blažek and Koman. Although Blažek and Koman did not explain in detail their proposed construction for $k > 2$, it is not difficult to fill out the details; by doing so, one can confirm that the (or, at least, one possible) natural way to generalize their construction for $k > 2$ yields precisely the DDS construction.

In [32], Shahrokhi et al. described a construction that draws K_n on k pages. This is also a natural generalization of the Blažek–Koman construction to $k > 2$ pages. Moreover, it agrees with the DDS construction whenever k divides n . (The DDS construction yields slightly better results for other values of k and n .) Based on their construction, Shahrokhi et al. gave the following general upper bound:

$$(2.2) \quad \nu_k(K_n) \leq \frac{2}{k^2} \left(1 - \frac{1}{2k}\right) \binom{n}{4} + \frac{n^3}{2k}.$$

In section 5 we give a detailed discussion on the DDS construction, including the calculation of the number of crossings that result by drawing K_n on k pages using this paradigm.

2.4. k -page drawings of K_n for $k \geq 3$: Lower bounds. Shahrokhi et al. proved in [32] that for every graph G and every positive integer k , one has $\nu_k(G) \geq m^3/37k^2n^2 - 27kn/37$. Following the derivation of this bound, it is easy to see that the factor $1/37$ in this expression can be improved, but only marginally so. Applying this bound to K_n , we obtain

$$\nu_k(G) \geq \frac{n(n-1)^3}{296k^2} - \frac{27kn}{37} = \frac{3}{37k^2} \binom{n}{4} + O(n^3).$$

This lower bound can be improved if n is sufficiently large compared to k , as follows. We recall that a k -planar drawing is similar to a k -page drawing but involves k unrestricted planar drawings. Formally, let $G = (V, E)$ be a graph. A k -planar drawing of G is a set of k planar drawings of graphs $G^{(i)} = (V, E^{(i)})$ ($i = 1, \dots, k$), where the edge sets $E^{(i)}$ form a k -partition of E . Loosely speaking, to obtain the k -planar drawing, we take the drawings of the graphs $G^{(i)}$ and (topologically) identify the k copies of each vertex. The k -planar crossing number $\text{cr}_k(G)$ of G is the minimum number of crossings in a k -planar drawing of G .

If k is even, then it is easy to obtain, from a k -page drawing of a graph G , a $k/2$ -planar drawing of G with the same number of crossings. Therefore $\nu_k(G) \geq \text{cr}_{k/2}(G)$ for every graph G and any positive even integer k . In [33], Shahrokhi et al. proved that for all $n \geq 2r^2 + 6r - 1$ and all $r \geq 1$,

$$\text{cr}_r(K_n) \geq \frac{1}{2(3r-1)^2} \binom{n}{4}.$$

From our previous observation, it follows that for all $n \geq 2(k/2)^2 + 6(k/2) - 1 = k^2/2 + 3k - 1$ and all even $k \geq 2$, $\nu_k(K_n) \geq \frac{2}{(3k-2)^2} \binom{n}{4}$. Obviously $\nu_{k-1}(G) \geq \nu_k(G)$ for any graph G and any integer $k \geq 2$, and so for any odd $k \geq 3$ and any $n \geq (k-1)^2/2 + 3(k-1) - 1 = k^2/2 + 2k - 7/2$ we have $\nu_k(K_n) \geq \nu_{k+1}(K_n) \geq \frac{2}{(3(k+1)-2)^2} \binom{n}{4} = \frac{2}{(3k+1)^2} \binom{n}{4}$. Now in their exhaustive investigation of biplanar (2-planar) crossing numbers [5, 6], Czaparka et al. prove the slightly better bound (for the 2-planar or *biplanar* crossing number) $\text{cr}_2(K_n) \geq n^4/952$. From this it follows that $\nu_4(K_n) \geq n^4/952$. Putting all these results together, we obtain the following lower bounds:

$$(2.3) \quad \nu_k(K_n) \geq \begin{cases} \frac{3}{119} \binom{n}{4} + O(n^3) & \text{if } k = 4; \\ \frac{2}{(3k-2)^2} \binom{n}{4} & \text{if } k \text{ is even, } k > 4, \text{ and } n \geq k^2/2 + 3k - 1; \\ \frac{2}{(3k+1)^2} \binom{n}{4} & \text{if } k \text{ is odd, and } n \geq k^2 + 2k - 7/2. \end{cases}$$

2.5. k -page drawings of K_n : Asymptotic lower and upper bounds. The following type of result is well known and easily shown; see, e.g., [29] or [31, Theorem 2].

Claim 2.1. For any integers $k > 0$ and $n > m \geq 4$,

$$\frac{\nu_k(K_n)}{\binom{n}{4}} \geq \frac{\nu_k(K_m)}{\binom{m}{4}}.$$

As a consequence, the sequence $\nu_k(K_n)/\binom{n}{4}$ is monotonically nondecreasing in n . Since it is also bounded from above by (2.2), the limit exists, and using (2.2) and (2.3) one has

$$(2.4) \quad \frac{3}{119} \leq \lim_{n \rightarrow \infty} \frac{\nu_4(K_n)}{\binom{n}{4}} \leq \frac{7}{64};$$

$$(2.5) \quad \frac{2}{(3k-2)^2} \leq \lim_{n \rightarrow \infty} \frac{\nu_k(K_n)}{\binom{n}{4}} \leq \frac{2}{k^2} \left(1 - \frac{1}{2k}\right) \text{ if } k \text{ is even, } k > 4;$$

$$(2.6) \quad \frac{2}{(3k+1)^2} \leq \lim_{n \rightarrow \infty} \frac{\nu_k(K_n)}{\binom{n}{4}} \leq \frac{2}{k^2} \left(1 - \frac{1}{2k}\right) \text{ if } k \text{ is odd, } k \geq 3.$$

3. Formulating $\nu_k(K_n)$ as a maximum k -cut or maximum satisfiability problem. We will show that $\nu_k(K_n)$ can be obtained by computing the maximum k -cut size in a certain graph $G_n = (V_n, E_n)$, say, which is a certain subgraph of the complement of the line graph of K_n . The same graph was used in [11] to investigate $\nu_2(K_n)$, and the general construction of graphs of this type is due to Buchheim and Zheng [4].

To define the graph $G_n = (V_n, E_n)$, we consider a Hamiltonian cycle C_n with vertices $v_1, v_2, \dots, v_n$. Let V_n be the set of *chords* of the cycle, that is, the edges $v_i v_j$ with v_i and v_j at cyclic distance at least 2. Let us say that the chords $v_i v_j$ and $v_k v_\ell$ *overlap* if i, k, j, ℓ occur in this cyclic order as we traverse C_n , either in its natural or in its reverse direction. Finally, to define E_n , we let two chords $v_i v_j$ and $v_k v_\ell$ be adjacent if they overlap. Thus one has $|V_n| = \binom{n}{2} - n$, and it is easy to verify that $|E_n| = \binom{n}{4}$.

As Buchheim and Zheng observed in [4], the maximum k -cut problem for G_n is related to $\nu_k(K_n)$ as follows.

LEMMA 3.1 (see [4, Theorem 1]). *One has*

$$\nu_k(K_n) = |E_n| - \max\text{-}k\text{-cut}(G_n),$$

where $\max\text{-}k\text{-cut}(G_n)$ denotes the cardinality of a maximum k -cut in G_n .

As a consequence of this lemma, one may calculate $\nu_k(K_n)$ for fixed (in practice, sufficiently small) values of n by solving a maximum cut problem. This was done by Buchheim and Zheng [4] for $k = 2$ and $n \leq 13$ by solving the maximum cut problem with a branch-and-bound algorithm. Using the **BiqMac** solver [28], de Klerk and Pasechnik [11] computed the exact value of $\nu_2(K_n)$ for $n \leq 18$ and for $n \in \{20, 24\}$.

3.1. The Frieze–Jerrum max- k -cut bound. We follow the standard practice to use $\mathbb{R}^{p \times q}$ (respectively, $\mathbb{C}^{p \times q}$) to denote the space of $p \times q$ matrices over $\mathbb{R}$ (respectively, $\mathbb{C}$). For $\mathbf{A} \in \mathbb{R}^{p \times p}$, the notation $\mathbf{A} \succeq 0$ means that $\mathbf{A}$ is symmetric positive semidefinite, whereas for $\mathbf{A} \in \mathbb{C}^{p \times p}$, it means that $\mathbf{A}$ is Hermitian positive semidefinite.

Let G be a graph with p vertices, and let $\mathbf{L}$ be its Laplacian matrix. Frieze and Jerrum introduced the following semidefinite-programming-based upper bound on $\max\text{-}k\text{-cut}(G)$:

$$(3.1) \quad \mathcal{FJ}_k(G) := \max \left\{ \frac{k-1}{k} \text{trace}(\mathbf{LX}) \mid \mathbf{X} \succeq 0, X_{ii} = 1 \ (1 \leq i \leq p), \mathbf{X} \geq \frac{-1}{k-1} \mathbf{J} \right\},$$

where $\mathbf{J}$ denotes the all-ones matrix of order p .

For $k = 2$ this bound coincides with the maximum-cut bound of Goemans and Williamson [14].

The associated dual semidefinite program takes the form
(3.2)

$$\mathcal{FJ}_k(G) = \min_{\mathbf{w} \in \mathbb{R}^p, \mathbf{S} \succeq 0} \left\{ \sum_{i=1}^p w_i + \frac{1}{k-1} \text{trace}(\mathbf{JS}) \mid \text{Diag}(\mathbf{w}) - \frac{k-1}{2k} \mathbf{L} - \mathbf{S} \succeq 0 \right\},$$

where Diag is the operator that maps a p -vector to a $p \times p$ diagonal matrix in the obvious way.

3.2. The Frieze–Jerrum bound for G_n . Using the technique of symmetry reduction for semidefinite programming (see, e.g., [13]), one can simplify the dual problem (3.2) for the graphs G_n defined in section 3 by using that the automorphism group of G_n is isomorphic to the dihedral group D_n on n elements. We state the final expression as the following lemma.

LEMMA 3.2. *Let $n > 0$ be an odd integer and $d = \lfloor n/2 \rfloor$. One has*

$$\mathcal{FJ}_k(G_n) = \min_{y \in \mathbb{R}^{d-1}} n \sum_{i=2}^d y_i + \frac{n}{k-1} \text{trace}(JX^{(0)})$$

subject to

$$(3.3) \quad \text{Diag} \left(y - \frac{k-1}{2k} \text{val} \right) + \Lambda^{(m)} \succeq 0 \quad (0 \leq m \leq d),$$

where

$$\begin{aligned} \text{val}_i &= i(i-1) + 2(i-1)(d-i), \quad 2 \leq i \leq d, \\ \Lambda_{ij}^{(m)} &= \frac{k-1}{k} \sum_{t=\ell_{ij}+1}^{\ell_{ij}+i} e^{\frac{-2\pi m t \sqrt{-1}}{n}} - X_{ij}^{(0)} - 2 \sum_{t=1}^d X_{ij}^{(t)} e^{\frac{-2\pi m t \sqrt{-1}}{n}}, \quad 2 \leq i \leq j \leq d, \\ \ell_{ij} &= \begin{cases} d(i-j) \bmod n & \text{if } i \text{ and } j \text{ have the same parity,} \\ d(i-j) - j \bmod n & \text{otherwise,} \end{cases} \\ X^{(m)} &= (X^{(m)})^T \geq 0 \quad \text{for all } 0 \leq m \leq d. \end{aligned}$$

The proof of this is very similar to that of [11, Lemma 4]. For a detailed outline of the proof, we refer the interested reader to [9].

A few remarks on Lemma 3.2:

1. Note that we obtain a reduced semidefinite program with $d = \lfloor n/2 \rfloor$ linear matrix inequalities involving matrices of order $d-1$, as well as $d+1$ nonnegative matrix variables of order $d-1$. This should be compared to the original formulation (3.2) to obtain $\mathcal{FJ}_k(G_n)$, which involved a linear matrix inequality of order $n(d-1)$, as well as a nonnegative matrix variable of the same order.
2. Lemma 3.2 generalizes [11, Lemma 4] to include the case $k > 2$ but also refines it in the sense that the dihedral symmetry of the graph G_n is fully exploited. Indeed in [11, Lemma 4], only the cyclic part of $\text{Aut}(G_n)$ was used, leading to (complex) Hermitian linear matrix inequalities, as opposed to the real symmetric linear matrix inequalities of Lemma 3.2.

3. The computation of $\mathcal{FJ}_k(G_n)$ is simpler in the case $k = 2$, since the bound then becomes the Goemans–Williamson maximum cut bound. Indeed, in [11], values of $\mathcal{FJ}_2(G_n)$ were reported for n close to 1,000. For $k > 2$, one is limited to more modest values: the largest value of n for which we will report computational results will be $n = 69$; see section 4. The difference in size of n that may be handled is primarily due to the nonnegative matrix variables $X^{(m)}$ ($0 \leq m \leq d$). These variables may be eliminated if $k = 2$, but not if $k > 2$.

3.3. A maximum satisfiability reformulation. It is well known that the maximum k -cut problem may be reformulated as a maximum satisfiability problem, and we will use this reformulation later on for computational purposes.

Consider a graph $G = (V, E)$ and a set of k colors (used to color the vertices V). We define the following logical variables:

$$x_i^j = \begin{cases} \text{TRUE} & \text{if vertex } i \text{ has color } j, \\ \text{FALSE} & \text{otherwise.} \end{cases}$$

Consider the clause

$$(3.4) \quad \neg x_i^p \vee \neg x_j^p \text{ if } (i, j) \in E$$

for each color $p = 1, \dots, k$. For a given edge, and a given color, this clause is satisfied if and only if the endpoints of the edge are not both colored using this color.

Moreover, each vertex should be assigned a color:

$$(3.5) \quad x_i^1 \vee \dots \vee x_i^k \quad (i \in V).$$

In order to obtain the maximum k -cut in G , we therefore need values of the logical variables that satisfy all the clauses (3.5) and as many of the clauses (3.4) as possible. This may be done by solving a weighted maximum satisfiability problem, where the weights of the satisfied clauses is maximized. In order to guarantee that the clauses (3.5) are all satisfied, we assign these clauses weight $k|E|$, while the clauses (3.4) are assigned weight 1.

Thus the cardinality of a maximum k -cut in $G = (V, E)$ coincides with the maximum weight of satisfied clauses in a truth assignment for the weighted logical formula:

$$(3.6) \quad \begin{aligned} &\neg x_i^p \vee \neg x_j^p \quad ((i, j) \in E, 1 \leq p \leq k), \\ &k|E| (x_i^1 \vee \dots \vee x_i^k) \quad (i \in V). \end{aligned}$$

We may now apply this idea to obtain $\nu_k(K_n)$ as follows.

LEMMA 3.3. *Consider the set of weighted clauses (3.7) for the graph $G = G_n = (V_n, E_n)$. Then $\nu_k(K_n)$ is the minimum weight of the unsatisfied clauses, taken over all possible truth assignments.*

Proof. The proof follows directly from Lemma 3.1. $\square$

4. Numerical results.

4.1. Exact computations. It is possible to compute $\nu_k(K_n)$ exactly using software for the weighted maximum satisfiability problem in Lemma 3.3. In Table 4.1 we show results obtained using the solver **Akmaxsat** by Kügel [22].

Each entry in Table 4.1 required at most 48 hours of computation on a laptop with a 2.5 GHz dual core processor and 4 GB RAM; the values that are missing from the table could not be computed using **Akmaxsat** within this time.

TABLE 4.1

Exact values of $\nu_k(K_n)$ for small values of k and n , as computed using the maximum satisfiability solver **Akmaxsat**. *The value $\nu_3(K_{15}) = 165$ was not determined using **Akmaxsat**; see Proposition 5.8.

$k \backslash n$	7	8	9	10	11	12	13	14	15
3	2	5	9	20	34	51	83	121	165*
4	0	0	3	7	12	18	34		
5	0	0	0	0	4	9			

We finally note that to our knowledge, prior to this work the exact value of $\nu_k(K_n)$ was not known for any n, k with $2 < k < \lceil n/2 \rceil$. (We recall that K_n can be drawn without crossings in $\lceil n/2 \rceil$ pages; thus $\nu_k(K_n) = 0$ for $k \geq \lceil n/2 \rceil$ and $\nu_k(K_n) > 0$ for $k < \lceil n/2 \rceil$.)

Some preliminary computation work was done using Sage [30].

4.2. Asymptotic bounds. As an example of the numerical results presented here, let $m = 69$ and $k = 10$. We computed $\mathcal{FJ}_{10}(G_{69}) \approx 856,520$, and using this value we get

$$\frac{\nu_{10}(K_{69})}{\binom{69}{4}} \geq \frac{\binom{69}{4} - \mathcal{FJ}_{10}(G_{69})}{\frac{69}{4}} \approx 9.2313 \times 10^{-3}.$$

Recall that for all $n > m \geq 4$,

$$\frac{\nu_k(K_n)}{\binom{n}{4}} \geq \frac{\nu_k(K_m)}{\binom{m}{4}}.$$

Thus it follows that

$$\frac{\nu_{10}(K_n)}{\binom{n}{4}} \geq \frac{\nu_{10}(K_{69})}{\frac{69}{4}} \geq 9.2313 \times 10^{-3} \quad (n > 69).$$

For $n > 69$, this is an improvement on the best previously known lower bound (from (2.3)), namely,

$$\frac{\nu_{10}(K_n)}{\binom{n}{4}} \geq \frac{2}{(3(10) - 2)^2} \approx 2.5510 \times 10^{-3}.$$

In Table 4.2 we give a systematic list of such improved bounds. Computation was done on a Dell Precision T7500 workstation with 92 GB of RAM using the semidefinite programming solver SDPT3 [35, 36] under MATLAB 7 together with the MATLAB package YALMIP [23].

Note that the values in the $m = 69$ column improve on the known lower bounds (2.3) in all cases for $n > 69$.

In Table 4.3 we summarize the best lower and upper bounds known for $\lim_{n \rightarrow \infty} \nu_k(K_n) / \binom{n}{4}$.

5. Drawing K_n in k pages: Conjectures and results. In this section we calculate the number $Z_k(n)$ of crossings that result by drawing K_n on k pages using the construction by Damiani, D'Antona, and Salemi [8] (we recall that we call this the *DDS construction*) and a generating function $G_k(z) := \sum_{n \geq 0} Z_k(n)z^n$ for it. This construction is a natural generalization of the construction by Blažek and

TABLE 4.2

Lower bounds for $\nu_k(K_n)/\binom{n}{4} \geq \mathcal{FJ}_k(G_m)/\binom{m}{4}$ for all $n > m$, $m \in \{39, 49, 59, 69\}$, and $k = 3, 4, \dots, 20$, and comparison with the previous best lower bounds on $\nu_k(K_n)/\binom{n}{4}$ (from (2.3)).

$k \backslash m$	39	49	59	69	Lower bounds from (2.3)
3	1.4266×10^{-1}	1.4827×10^{-1}	1.5194×10^{-1}	1.5452×10^{-1}	2.0000×10^{-2}
4	7.4205×10^{-2}	7.9473×10^{-2}	8.2837×10^{-2}	8.5127×10^{-2}	2.5210×10^{-2}
5	4.2208×10^{-2}	4.6916×10^{-2}	5.0019×10^{-2}	5.2141×10^{-2}	7.8125×10^{-3}
6	2.5728×10^{-2}	2.9633×10^{-2}	3.2258×10^{-2}	3.4151×10^{-2}	7.8125×10^{-3}
7	1.6260×10^{-2}	1.9605×10^{-2}	2.1895×10^{-2}	2.3524×10^{-2}	4.1322×10^{-3}
8	1.0544×10^{-2}	1.3390×10^{-2}	1.5356×10^{-2}	1.6812×10^{-2}	4.1322×10^{-3}
9	6.9603×10^{-3}	9.3377×10^{-3}	1.1062×10^{-2}	1.2333×10^{-2}	2.5510×10^{-3}
10	4.6086×10^{-3}	6.6189×10^{-3}	8.1148×10^{-3}	9.2314×10^{-3}	2.5510×10^{-3}
11	3.0659×10^{-3}	4.7436×10^{-3}	6.0329×10^{-3}	7.0285×10^{-3}	1.7301×10^{-3}
12	2.0007×10^{-3}	3.4078×10^{-3}	4.5294×10^{-3}	5.3894×10^{-3}	1.7301×10^{-3}
13	1.2987×10^{-3}	2.4613×10^{-3}	3.4307×10^{-3}	4.2025×10^{-3}	1.2500×10^{-3}
14	8.2096×10^{-4}	1.7736×10^{-3}	2.6077×10^{-3}	3.2930×10^{-3}	1.2500×10^{-3}
15	4.7807×10^{-4}	1.2613×10^{-3}	1.9718×10^{-3}	2.5870×10^{-3}	9.4518×10^{-4}
16	2.6556×10^{-4}	8.9554×10^{-4}	1.5141×10^{-3}	2.0348×10^{-3}	9.4518×10^{-4}
17	1.3191×10^{-4}	6.2938×10^{-4}	1.1514×10^{-3}	1.6023×10^{-3}	7.3964×10^{-4}
18	5.2726×10^{-5}	4.2802×10^{-4}	8.5199×10^{-4}	1.2562×10^{-3}	7.3964×10^{-4}
19	8.8699×10^{-6}	2.7320×10^{-4}	6.3294×10^{-4}	9.8258×10^{-4}	5.9453×10^{-4}
20	0	1.7127×10^{-4}	4.7985×10^{-4}	7.7482×10^{-4}	5.9453×10^{-4}

TABLE 4.3

Summary of lower and upper bounds for $\lim_{n \rightarrow \infty} \nu_k(K_n)/\binom{n}{4}$. The second column gives the previously best lower bounds, as given in (2.4), (2.5), and (2.6). The third column presents the lower bounds we obtained by computing $\mathcal{FJ}_k(G_{69})$ for $k = 3, 4, \dots, 20$ (this is the fifth column of Table 4.2). In the fourth column we show the best upper bounds known, given by (2.2) (alternatively, using Observation 5.6 and that $\nu_k(K_n) \leq Z_k(n)$). Finally, in the fifth column we show the ratio between the values given in the third and fourth columns.

k	Previous lower bound on $\lim_{n \rightarrow \infty} \frac{\nu_k(K_n)}{\binom{n}{4}}$	Improved lower bound on $\lim_{n \rightarrow \infty} \frac{\nu_k(K_n)}{\binom{n}{4}}$	Best upper bound on $\lim_{n \rightarrow \infty} \frac{\nu_k(K_n)}{\binom{n}{4}}$	Quotient between lower and upper bound
3	2.0000×10^{-2}	1.5452×10^{-1}	1.8518×10^{-1}	0.8344
4	2.5210×10^{-2}	8.5127×10^{-2}	1.0937×10^{-1}	0.7783
5	7.8125×10^{-2}	5.2141×10^{-2}	7.2000×10^{-2}	0.7241
6	7.8125×10^{-3}	3.4151×10^{-2}	5.0925×10^{-2}	0.6706
7	4.1322×10^{-3}	2.3524×10^{-2}	3.7900×10^{-2}	0.6706
8	4.1322×10^{-3}	1.6812×10^{-2}	2.9296×10^{-2}	0.5738
9	2.5510×10^{-3}	1.2333×10^{-2}	2.3319×10^{-2}	0.5287
10	2.5510×10^{-3}	9.2314×10^{-3}	1.9000×10^{-2}	0.4858
11	1.7301×10^{-3}	7.0285×10^{-3}	1.5777×10^{-2}	0.4454
12	1.7301×10^{-3}	5.3894×10^{-3}	1.3310×10^{-2}	0.4049
13	1.2500×10^{-3}	4.2025×10^{-3}	1.1379×10^{-2}	0.3693
14	1.2500×10^{-3}	3.2930×10^{-3}	9.8396×10^{-3}	0.3346
15	9.4518×10^{-3}	2.5870×10^{-3}	8.5925×10^{-3}	0.3010
16	9.4518×10^{-4}	2.0348×10^{-3}	7.5683×10^{-3}	0.2688
17	7.3964×10^{-4}	1.6023×10^{-3}	6.7168×10^{-3}	0.2385
18	7.3964×10^{-4}	1.2562×10^{-3}	6.0013×10^{-3}	0.2093
19	5.9453×10^{-4}	9.8258×10^{-4}	5.3943×10^{-3}	0.1821
20	5.9453×10^{-4}	7.7482×10^{-4}	4.8750×10^{-3}	0.1589

FIG. 5.1. To draw K_{10} in 3 pages, we place the edges in $\mathcal{M}_{0,3} = M_0 \cup M_1 \cup M_2 \cup M_3$ in page 0 (left), the edges in $\mathcal{M}_{4,6} = M_4 \cup M_5 \cup M_6$ in page 1 (center), and the edges in $\mathcal{M}_{7,9} = M_7 \cup M_8 \cup M_9$ in page 2 (right).

Koman [3] (who fully described it for 2 pages, and briefly mentioned that it could be generalized to $k > 2$ pages) and a slight refinement of the construction by Shahrokhi et al. [32]. The description of the construction and calculation of $Z_k(n)$ and $G_k(z)$ is in section 5.1.

We calculate the exact value of $Z_k(n)$ for two reasons. First, the value $Z_k(n)$ was determined in neither [3] nor [8]; since no better (crossingwise) construction to draw K_n in k pages is known, this is a calculation worth doing. Second, for all values of k and n for which we now (that is, with the results reported in this paper) know the exact value of $\nu_k(K_n)$, we have $\nu_k(K_n) = Z_k(n)$. These confirmations, as well as an additional feature that we shall explain below (namely, Proposition 5.4, from which we will compute $\nu_3(K_{15})$), lend credibility to the conjecture $\nu_k(K_n) = Z_k(n)$, which we formally put forward in section 5.2. As we shall see, the value of $Z_k(n)$ depends on $n \bmod k$, and for each fixed k and each fixed $q \in \{0, 1, \dots, k-1\}$, there is a degree 4 polynomial $G_{q,k}(n)$ such that $Z_k(n) = G_{q,k}(n)$ for all n such that $n \bmod k = q$. In section 5.3 we explicitly give this polynomial for the case $n \bmod k = 0$. We finally present, in section 5.4, a slightly more detailed discussion and further results for the case $k = 3$.

5.1. The DDS construction: An upper bound $Z_k(n)$ for the k -page crossing number of K_n . The DDS construction was described in [8] in terms of the adjacency matrix. We have found it both more lively and more convenient (for our calculations) to follow the more geometrical viewpoint of Shahrokhi et al. to describe this construction, and this is the approach we follow below.

We draw K_n in k pages using the circular model. Label the vertices $0, 1, 2, \dots, n-1$ in the clockwise order in which they occur in the boundary of the circle. For $i = 0, 1, \dots, n-1$, let M_i be the set of edges whose endpoints have sum i (modulo n). Thus, M_i is a matching for each $i \in \{0, 1, \dots, n-1\}$. We note that each edge belongs to exactly one matching M_i . For $s, t \in \{0, 1, \dots, n-1\}$, $s < t$, let $\mathcal{M}_{s,t} := M_s \cup M_{s+1} \cup \dots \cup M_t$. Loosely speaking, $\mathcal{M}_{s,t}$ consists of the edges of $t-s+1$ “consecutive” matchings. In Figure 5.1 we illustrate the sets $\mathcal{M}_{0,3}$ (left), $\mathcal{M}_{4,6}$ (center), and $\mathcal{M}_{7,9}$ (right) for the case $n = 10$.

Let $p := \lfloor n/k \rfloor$ and $q := n \bmod k$ (thus $n = pk + q$). The DDS construction consists simply of distributing the edges of K_n into k pages $0, 1, \dots, k-1$ as follows:

1. for $0 \leq \ell < q$, place in page ℓ the edges in $\mathcal{M}_{\ell(p+1), \ell(p+1)+p}$ and
2. for $q \leq \ell < k$, place in page ℓ the edges in $\mathcal{M}_{\ell p+q, \ell p+q+(p-1)}$.

Thus, if $0 \leq \ell < q$, then page ℓ contains the edges of $p+1$ matchings, and if $q \leq \ell < k$, then page ℓ contains the edges of p matchings. Note that if k divides n

(that is, $q = 0$), then there is no ℓ such that $0 \leq \ell < q$, and so each page contains the edges of p matchings. In Figure 5.1 we illustrate the DDS construction for the case $k = 3, n = 10$.

We shall give $Z_k(n)$ in terms of a function F that we now define. First, let

$$f(r) := \frac{rn}{2} - \frac{r^2}{2} - \frac{n}{2} + \frac{1}{2},$$

and then let

$$(5.1) \quad F(r, n) := \sum_{\ell=0}^{r-1} (r - \ell) f(\ell) = -\frac{r^4}{24} + \frac{nr^3}{12} - \frac{nr^2}{4} + \frac{7r^2}{24} + \frac{nr}{6} - \frac{r}{4}.$$

PROPOSITION 5.1. *The number of crossings that result from drawing K_n in $k \geq 1$ pages using the DDS construction is*

$$Z_k(n) := (n \bmod k) \cdot F\left(\left\lfloor \frac{n}{k} \right\rfloor + 1, n\right) + (k - (n \bmod k)) \cdot F\left(\left\lfloor \frac{n}{k} \right\rfloor, n\right).$$

Thus K_n can be drawn in k pages with $Z_k(n)$ crossings, and so

$$\nu_k(K_n) \leq Z_k(n).$$

Although totally elementary, the proof of this result is somewhat long. For the proof, we refer the interested reader to [9].

Note that $Z_k(n)$ is a quasi polynomial of period k in n (cf., e.g., Stanley [34, section 4.4]). This implies that its generating function $G_k(z)$ is rational, i.e., the ratio of two polynomials in z , with the denominator having only k th roots of unity as roots. Using straightforward and standard techniques from [34], we can prove (see details in [9]) the following.

PROPOSITION 5.2. *For a fixed k , the generating function $G_k(z)$ for $Z_k(n)$ is*

$$G_k(z) := \sum_{n \geq 0} Z_k(n) z^n = z^{2k+1} \frac{(k-2)(1-z) + 1 - z^{k+1}}{(1-z)^3(1-z^k)^3}.$$

As a first application of this formula, one sees at once that $Z_k(n) = 0$ for $n \leq 2k$, as the first nonzero coefficient in the expansion of G_k into powers of z comes up for the $2k + 1$ th power.

5.2. A conjecture for the k -page crossing number of K_n . The DDS construction described in section 5.1 draws K_n in k pages with $Z_k(n)$ crossings. We conjecture the optimality of this construction.

Conjecture 5.3. For all positive integers k and n ,

$$\nu_k(K_n) = Z_k(n).$$

The naturality and aesthetical appeal of the construction, plus the fact that no construction to draw K_n in k pages with fewer crossings is known, seem good enough reasons to put forward this conjecture. Still, there is further evidence supporting the conjecture:

- The statement is true for $k \leq 2$. For $k = 1$ this is readily checked, and for $k = 2$ it follows since it has been recently verified that $\nu_2(K_n) = Z_2(n)$ [1].
- For all k, n for which we now know the exact value of $\nu_k(K_n)$ (Table 4.1, plus all k, n such that $k > \lceil n/2 \rceil$, for which it is known that $\nu_k(K_n) = 0$), we have $\nu_k(K_n) = Z_k(n)$.

There is yet another argument that supports Conjecture 5.3, at a somewhat (but not completely; see section 5.4) more speculative level. Recall that $Z_2(n) := \frac{1}{4} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor$. A well-known counting argument shows that for every positive integer r , $\nu_2(K_{2r-1}) = Z_2(2r-1)$ implies $\nu_2(K_{2r}) = Z_2(2r)$. This “odd implies even” phenomenon is used, for instance, to determine that the (usual) crossing number of K_{12} is $Z_2(12)$: this follows at once since the crossing number of K_{11} is $Z_2(11)$ [27]. An appealing feature of Conjecture 5.7 is that it implies a similar phenomenon for every k .

PROPOSITION 5.4. *For every positive integers k and r , one has $krZ_k(kr-1) = (kr-4)Z_k(kr)$ and*

$$\nu_k(K_{kr-1}) = Z_k(kr-1) \implies \nu_k(K_{kr}) = Z_k(kr).$$

Proof. The first claim follows from

$$Z_k(kr-1) = (k-1)F(r, kr-1) + F(r-1, kr-1) = \frac{kr-4}{r}F(r, kr) = \frac{kr-4}{kr}Z_k(kr)$$

after a long but routine manipulation.

Since $\nu_k(K_{kr}) \leq Z_k(kr)$ (cf. Proposition 5.1), we only need to prove the reverse inequality $\nu_k(K_{kr}) \geq Z_k(kr)$. Suppose that $\nu_k(K_{kr-1}) = Z_k(kr-1)$. Consider a k -page drawing D of K_{kr} with $\nu_k(K_{kr})$ crossings. This drawing contains kr drawings of K_{kr-1} , each of which has at least $\nu_k(K_{kr-1}) = Z_k(kr-1)$ crossings. It is easy to see that each crossing gets counted exactly $kr-4$ times, and so

$$\nu_k(K_{kr}) \geq \frac{krZ_k(kr-1)}{kr-4} = kF(r, kr) = Z_k(kr). \quad \square$$

In section 5.4 we will use Proposition 5.4 to prove that $\nu_3(K_{15}) = 165$ (cf. Proposition 5.8).

5.3. Explicit estimates for $Z_k(n)$. It is clear that for each fixed k and $q \in \{0, 1, \dots, k-1\}$, there exists a polynomial $G_{k,q}(n)$ such that $G_{k,q}(n) = Z_k(n)$ for all n such that $n \bmod k = q$. For the case $q = 0$, a routine manipulation yields the following.

Observation 5.5. If k divides n , then

$$Z_k(n) = \left(\left(\frac{1}{12k^2} \right) \left(1 - \frac{1}{2k} \right) \right) n^4 + \left(-\frac{1}{4k} \right) n^3 + \left(\frac{7}{24k} + \frac{1}{6} \right) n^2 + \left(-\frac{1}{4} \right) n.$$

We recall (see section 2.5) that $\nu_k(K_n) / \binom{n}{4} \geq \nu_k(K_m) / \binom{m}{4}$, whenever $n > m \geq 4$. Using this and Observation 5.5, we obtain the following asymptotic general estimate for $Z_k(n)$.

Observation 5.6. For each positive integer k ,

$$Z_k(n) = \left(\left(\frac{1}{12k^2} \right) \left(1 - \frac{1}{2k} \right) \right) n^4 + O(n^3).$$

We note that the upper bound (2.2), given by Shahrokhi et al. [32], is in line with this last observation. (This was expected, since the DDS construction and the construction in [32] agree whenever k divides n .)

TABLE 5.1
MAX- k -CUT approximation guarantees for $3 \leq k \leq 10$

k :	3	4	5	6	7	8	9	10
α_k	0.836008	0.857487	0.876610	0.891543	0.903259	0.912664	0.920367	0.926788

5.4. Drawing K_n in 3 pages: Further results. A long but straightforward manipulation shows that

$$(5.2) \quad Z_3(n) = \begin{cases} \frac{(n-6)(n-3)n(5n-9)}{648} & \text{if } n \equiv 0 \pmod{3}; \\ \frac{(n-4)(n-1)(5n^2-29n+30)}{648} & \text{if } n \equiv 1 \pmod{3}; \\ \frac{(n-2)(n-3)(n-5)(5n-4)}{648} & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

We remark that this coincides with the number of crossings given by Blažek and Koman in [3], where they briefly mentioned that their construction for 2 pages could be generalized to $k > 2$ pages and reported the expression in (5.2) for the number of crossings obtained by drawing K_n in 3 pages (no further details were given).

As we have already observed, $\nu_3(K_n) = Z_3(n)$ for all 8 values of $n \geq 7$ for which we have calculated $\nu_3(K_n)$ (Table 4.1). This evidence gives special credence to the case $k = 3$ of Conjecture 5.3.

Conjecture 5.7 (3-page crossing number of K_n). *For every positive integer n ,*

$$\nu_3(K_n) = Z_3(n).$$

As an additional support for this conjecture, we note that our calculations reported in section 4 confirm that $\nu_3(K_n)$ is reasonably close to $Z_3(n)$, at least asymptotically. Indeed, from Table 4.3 we have $\lim_{n \rightarrow \infty} \nu_3(K_n)/\binom{n}{4} \geq 0.15452$. This implies that $\lim_{n \rightarrow \infty} \nu_3(K_n)/n^4 \geq 0.006438$. Now from (5.2) we have $\lim_{n \rightarrow \infty} Z_3(n)/n^4 = 5/648$. These results yield

$$\lim_{n \rightarrow \infty} \frac{\nu_3(K_n)}{Z_3(n)} > \frac{0.006438}{5/648} \approx 0.8344.$$

We finally show that, as hinted above, Proposition 5.4 (the generalization to $k > 2$ pages of the “odd implies even” phenomenon for $k = 2$) is not only a speculative curiosity: we use this statement to determine the exact value of $\nu_3(K_{15})$.

PROPOSITION 5.8. $\nu_3(K_{15}) = 165$.

Proof. It follows from Proposition 5.4, using (from our calculation, reported in Table 4.1) that $\nu_3(K_{14}) = Z_3(14)$. $\square$

6. Concluding remarks and open questions. De Klerk, Pasechnik, and Warners [10] proved the lower bounds α_k on the ratio $\frac{\max\text{-}k\text{-cut}(G)}{\mathcal{FJ}(G)}$ given in Table 5.1. These lower bounds may be used to obtain upper bounds on $\nu_k(K_n)$ ($3 \leq k \leq 10$), namely,

$$\nu_k(K_n) \leq |E_n| - \alpha_k \mathcal{FJ}(G_n),$$

but these bounds seem weaker than the upper bounds given by the best known drawings, based on computations for $3 \leq k \leq 10$ and $n \leq 69$.

Limits of the type (2.6) are of independent interest if one replaces the k -page crossing number by the rectilinear crossing number. (The rectilinear crossing number

of a graph is the minimum number of edge crossings in a drawing of the graph in the plane if all edges are drawn by straight lines.) Indeed, for the rectilinear crossing number $\overline{\text{cr}}(K_n)$, the limit

$$\lim_{n \rightarrow \infty} \frac{\overline{\text{cr}}(K_n)}{\binom{n}{4}}$$

is related to the Sylvester four-point problem in geometric probability as follows. Consider an open set $R \in \mathbb{R}^2$ with finite area. Denote by $q(R)$ the probability that the convex hull of four points in R , drawn uniformly at random, is a convex quadrilateral (as opposed to a line or triangle). Scheinerman and Wilf [31] showed that

$$\inf_R q(R) = \lim_{n \rightarrow \infty} \frac{\overline{\text{cr}}(K_n)}{\binom{n}{4}},$$

where the infimum is taken over all open sets R in the plane with finite area.

It remains an interesting question whether these limits also have alternative interpretations if one replaces the rectilinear crossing number by other notions of crossing numbers, like the k -page crossing number.

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REFERENCES

- [1] B. M. ÁBREGO, O. AICHHOLZER, S. FERNÁNDEZ-MERCHANT, P. RAMOS, AND G. SALAZAR, *The 2-page crossing number of K_n* , Proceedings of the 28th ACM Symposium on Computational Geometry (SoCG'12), Chapel Hill, NC, USA, 2012, pp. 397–403.
- [2] F. BERNHART AND P. C. KAINEN, *The book thickness of a graph*, J. Combin. Theory Ser. B, 27 (1979), pp. 320–331.
- [3] J. BLAŽEK AND M. KOMAN, *A minimal problem concerning complete plane graphs*, in 1964 Theory of Graphs and Its Applications Czechoslovak Academy of Sciences, Prague, 1963, pp. 113–117.
- [4] C. BUCHHEIM AND L. ZHENG, *Fixed Linear Crossing Minimization by Reduction to the Maximum Cut Problem*, Lecture Notes in Comput. Sci., 4112, 2006, pp. 507–516.
- [5] E. CZABARKA, O. SÝKORA, L. A. SZÉKELY, AND I. VRT'Ó, *Biplanar crossing numbers. I. A survey of results and problems*, in More Sets, Graphs and Numbers, Bolyai Soc. Math. Stud. 15, Springer, Berlin, 2006, pp. 57–77.
- [6] E. CZABARKA, O. SÝKORA, L. A. SZÉKELY, AND I. VRT'Ó, *Biplanar crossing numbers. II. Comparing crossing numbers and biplanar crossing numbers using the probabilistic method*, Random Structures Algorithms, 33 (2008), pp. 480–496.
- [7] F. R. K. CHUNG, F. T. LEIGHTON, AND A. L. ROSENBERG, *Embedding graphs in books: A layout problem with applications to VLSI design*, SIAM J. Algebraic Discrete Methods, 8 (1987), pp. 33–58.
- [8] E. DAMIANI, O. D'ANTONA, AND P. SALEMI, *An upper bound to the crossing number of the complete graph drawn on the pages of a book*, J. Combin. Inform. System Sci., 19 (1994), pp. 75–84.
- [9] E. DE KLERK, D. V. PASECHNIK, AND G. SALAZAR, *Improved Lower Bounds on Book Crossing Numbers of Complete Graphs*, manuscript, arXiv:1207.5701 [math.CO], 2012.
- [10] E. DE KLERK, D. V. PASECHNIK, AND J. P. WARNERS, *On approximate graph colouring and MAX- k -CUT algorithms based on the ϑ -function*, J. Comb. Optim., 8 (2004), pp. 267–294.
- [11] E. DE KLERK AND D. V. PASECHNIK, *Improved lower bounds for the 2-page crossing numbers of $K_{m,n}$ and K_n via semidefinite programming*, SIAM J. Optim., 22 (2012), pp. 581–595.
- [12] H. ENOMOTO, T. NAKAMIGAWA, AND K. OTA, *On the pagenumber of complete bipartite graphs*, J. Combin. Theory Ser. B, 71 (1997), pp. 111–120.
- [13] K. GATERMANN AND P. A. PARRILO, *Symmetry groups, semidefinite programs, and sums of squares*, J. Pure Appl. Algebra, 192 (2004), pp. 95–128.

- [14] M. X. GOEMANS AND D. P. WILLIAMSON, *Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming*, J. ACM, 42 (1995), pp. 1115–1145.
- [15] R. K. GUY, T. JENKYN, AND J. SCHAEER, *The toroidal crossing number of the complete graph*, J. Combin. Theory, 4 (1968), pp. 376–390.
- [16] F. HARARY AND A. HILL, *On the number of crossings in a complete graph*, Proc. Edinb. Math. Soc. (2), 13 (1962/1963), pp. 333–338.
- [17] H. HARBORTH, *Special numbers of crossings for complete graphs*, Discrete Math., 244 (2002), pp. 95–102.
- [18] H. HE, A. SÁLÁGEAN, AND E. MÄKINEN, *One- and two-page crossing numbers for some types of graphs*, Int. J. Comput. Math., 87 (2010), pp. 1667–1679.
- [19] H. HE, O. SÝKORA, AND E. MÄKINEN, *Genetic algorithms for the 2-page drawing problem of graphs*, J. Heuristics, 13 (2007), pp. 77–93.
- [20] H. HE, O. SÝKORA, AND I. VRT’O, *Minimisation Heuristics for 2-page Drawings*, Electron. Notes Discrete Math., 22 (2005), pp. 527–534.
- [21] P. C. KAINEN, *An introduction to topological graph theory*, in Graphs and Combinatorics, R. Bari and F. Harary, eds., Springer, New York, 1974, pp. 76–108.
- [22] A. KÜGEL, *Improved exact solver for the weighted Max-SAT problem*, in Proceedings of POS-10. Pragmatics of SAT (Daniel Le Berre, ed.), EasyChair Proceedings in Computing 8, 2012, pp. 15–27.
- [23] J. LÖFBERG, *YALMIP: A toolbox for modeling and optimization in MATLAB*, Proceedings of the CACSD Conference, Taipei, Taiwan, 2004; also available online from <http://users.isy.liu.se/johanl/yalmip/>.
- [24] S. MASUDA, K. NAKAJIMA, T. KASHIWABARA, AND T. FUJISAWA, *On the NP-completeness of a computer network layout problem*, in Proceedings of the 1987 IEEE International Symposium on Circuits and Systems, IEEE Press, New York, 1987, pp. 292–295.
- [25] S. MASUDA, K. NAKAJIMA, T. KASHIWABARA, AND T. FUJISAWA, *Crossing minimization in linear embeddings of graphs*, IEEE Trans. Comput., 39 (1990), pp. 124–127.
- [26] D. J. MUDER, M. L. WEAVER, AND D. B. WEST, *Pagenumber of complete bipartite graphs*, J. Graph Theory, 12 (1988), pp. 469–489.
- [27] S. PAN AND R. B. RICHTER, *The crossing number of K_{11} is 100*, J. Graph Theory, 56 (2007), pp. 128–134.
- [28] F. RENDL, G. RINALDI, AND A. WIEGELE, *Solving Max-Cut to optimality by intersecting semidefinite and polyhedral relaxations*, Math. Program., 121 (2010), pp. 307–335.
- [29] R. B. RICHTER AND C. THOMASSEN, *Relations between crossing numbers of complete and complete bipartite graphs*, Amer. Math. Monthly, 104 (1997), pp. 131–137.
- [30] W. STEIN ET AL., *Sage: Open Source Mathematics Software*, version 5.0, The Sage Group, <http://www.sagemath.org> 2012.
- [31] E. R. SCHEINERMAN AND H. S. WILF, *The rectilinear crossing number of a complete graph and Sylvester’s “four point problem,”* Amer. Math. Monthly, 101 (1994), pp. 939–943.
- [32] F. SHAHROKHI, O. SÝKORA, L. A. SZÉKELY, AND I. VRT’O, *The book crossing number of a graph*, J. Graph Theory 21 (1996), pp. 413–424.
- [33] F. SHAHROKHI, O. SYKORA, L. A. SZÉKELY, AND I. VRT’O, *On k -planar crossing numbers*, Discrete Appl. Math., 155 (2007), pp. 1106–1115.
- [34] R. STANLEY, *Enumerative Combinatorics*, vol. 1, 2nd ed., Cambridge University Press, New York, 2011; also available online from <http://www-math.mit.edu/~rstan/ec/ec1/>.
- [35] K. C. TOH, M. J. TODD, AND R. H. TUTUNCU, *SDPT3—A MATLAB software package for semidefinite programming*, Optim. Methods Softw., 11 (1999), pp. 545–581.
- [36] R. H. TUTUNCU, K. C. TOH, AND M. J. TODD, *Solving semidefinite-quadratic-linear programs using SDPT3*, Math. Program. Ser. B, 95 (2003), pp. 189–217.
- [37] I. VRT’O, *Two special crossing number problems for complete graphs*, in Abstracts of the 6th Slovenian International Conference on Graph Theory, Bled’07.
- [38] M. YANNAKAKIS, *Embedding planar graphs in four pages*, J. Comput. System Sci., 38 (1989), pp. 36–67.