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Analytical solutions to flexural vibration of slender piezoelectric multilayer cantilevers

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Short title

Analytical solutions to flexural vibration of piezoelectric multilayer cantilevers

Abstract

Modeling of vibration of piezoelectric cantilevers was often based on passive cantilevers of a homogeneous material. Although piezoelectric cantilevers and passive cantilevers share certain characteristics, this method caused confusion in incorporating the piezoelectric moment into the differential equation of motion. The extended Hamilton's principle is a fundamental approach to model flexural vibration of multilayer piezoelectric cantilevers. Previous works demonstrated derivation of differential equation of motion using this approach, however proper analytical solutions were not reported. This was partly due to the fact that the differential equation derived by the extended Hamilton's principle is a boundary-value problem with nonhomogeneous boundary conditions which cannot be solved by modal analysis. In the present study, an analytical solution to the boundary-value problem was obtained by transforming it into a new problem with homogeneous boundary conditions. After the transformation, modal analysis was used to solve the new boundary-value problem. The analytical solutions for unimorphs and bimorphs were verified with three-dimensional finite element analysis (FEA). Deflection profiles and frequency response functions under voltage, uniform pressure, and tip force were compared. Discrepancies between the analytical results and FEA results were within 3.5%. Following model validation, parametric studies were conducted to investigate effects of thickness of electrodes and piezoelectric layers, and the piezoelectric coupling coefficient d_{31} on the performance of piezoelectric cantilever actuators.

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1. Introduction

Piezoelectric cantilevers are widely used in applications such as ultrasonic motors [1], scanning probes of atomic force microscopes (AFM) [2], microwave switches [3], bandpass filters [4], and suspensions in hard disk drives [5]. Since many applications require controlled high-precision motion, piezoelectric cantilevers must be designed properly. Therefore, it is essential to have accurate analytical models which can predicate the performance of piezoelectric cantilevers.

Theoretical quasi-static behavior of piezoelectric cantilevers has been investigated extensively in the past decades. Steel *et al* [6] calculated tip deflection of a piezoelectric cantilever consisting of a lead zirconate titanate (PZT) layer bonded to a metal layer by minimizing the strain energy and piezoelectric energy. Smits and Choi [7] derived the constitutive equations of unimorphs by calculating the internal energy assuming thermodynamic equilibrium. Using the same approach, Smits *et al* [8] obtained the constitutive equations of bimorphs. In addition to the energy based methods, an alternative approach based on equilibrium of forces and moments and the strain compatibility condition was also used [9-11]. This approach was first proposed by Timoshenko [12] for analysis of bimetallic thermostats. Among those studies, earlier efforts [6-9] usually dealt with either unimorphs or bimorphs. Furthermore, as the effect of metal electrodes on structural properties was not significant in macroscale applications, to simplify the analysis, metal electrodes were often neglected in modeling. With the emergence of surface micromachining for micro-electro-mechanical systems (MEMS), miniaturized piezoelectric multilayer cantilevers became viable. Thus, more recent works [10, 11] expanded the formulations for unimorphs and bimorphs into a more general form for multimorphs with metal electrodes.

Quasi-static analysis is only sufficient for applications at low frequencies far from resonances. If the piezoelectric cantilevers operate at higher frequencies that are near or beyond the fundamental resonance, dynamic analysis is needed to understand their behavior. As an extension of quasi-static analysis, previous works on dynamic analysis often adopted the same methods and assumptions as used in quasi-static piezoelectric analysis. Ajitsaria *et al* [13] solved the standard equations of motion for bending vibration of a bimorph and calculated the output voltage based on a constant radius of curvature. As pointed out by Erturk and Inman [14], although the assumption of a constant radius of curvature is valid for modeling of static behavior, beams in bending vibration do not have a constant radius of curvature throughout its length. Brissaud *et al* [15] calculated the static internal moment in a unimorph and incorporated the piezoelectric moment into the boundary conditions of the standard differential eigenvalue problem. By contrast, Erturk and Inman [16] used dynamic internal moment in the derivation of the electromechanical model for a unimorph piezoelectric energy harvester and included the piezoelectric moment in the equations of motion instead of the boundary conditions. Following a similar approach, Dietl *et al* [17] derived the equations of motion for a bimorph based on the Timoshenko beam theory. Although it was convenient and plausible to model the piezoelectric cantilevers after the passive cantilevers of a homogeneous material, this approach caused confusion regarding the role of the piezoelectric moment in the differential equations of motion [13-17]. Therefore, it is desirable to start with a more fundamental approach without relying on models of passive cantilevers as a basis. For this purpose, Tanaka [18] used the Hamilton's principle, a powerful variational principle of mechanics, to systematically derive the equations of motion, charge equation of electrostatics and boundary conditions for piezoelectric multilayer cantilevers. Tanaka did not consider the energy loss caused by non-conservative forces. Ballas [19] subsequently introduced frictional force to the system by using the extended Hamilton's principle. Both Tanaka and Ballas's derivations yielded the piezoelectric moment in the boundary conditions. This was consistent with the intuitive approach adopted by Brissaud *et al* [15]. The fourth-order partial differential equation of beam bending vibration is usually solved by modal analysis. However, with the piezoelectric moment in the boundary conditions, the boundary conditions become nonhomogeneous which causes difficulty for modal analysis [20]. Tanaka [18] did not solve the differential equations in his study. Ballas [19] attempted solution by modal analysis; however the piezoelectric moment was moved from the boundary conditions to the right hand of the differential equation by converting it as a load per unit length. Again, as in [16, 17], this method sacrificed rigor for

convenience without necessary proof. Up to the present time, proper analytical solutions to the differential equation of motion derived by the extended Hamilton's principle could not be found in open literature.

The aim of the present study was to model flexural vibration of piezoelectric multilayer cantilevers in a systematic and rigorous manner. The extended Hamilton's principle was used to derive the differential equations of motion. To solve the partial differential equations with nonhomogeneous boundary conditions by modal analysis, the boundary-value problem was transformed into a new problem with homogeneous boundary conditions. Modal analysis of the new problem would then lead to analytical solutions to flexural vibration of piezoelectric multilayer cantilevers. The analytical solutions were compared with three-dimensional (3D) finite-element-method (FEM) results. Finally, parametric studies were carried out to investigate effects of thickness of electrodes and piezoelectric layers, and the piezoelectric coupling coefficient d_{31} on the resonant frequencies and tip deflection of unimorphs and bimorphs.

2. Modeling

Figure 1(a) shows a piezoelectric multilayer cantilever subjected to time varying loadings such as uniform pressure $f(x, t)$, concentrated force $F_c(t)$ and tip moment $M(t)$. The piezoelectric layers are in parallel connection. The cantilever can operate as either an actuator or a sensor depending on whether voltage $V(t)$ is applied as a source or measured as a signal. Unimorphs (figure 1(b)) and bimorphs (figure 1(c)) are two most studied cases of piezoelectric multilayer cantilevers.

Figure 1. Piezoelectric multilayer cantilever: (a) general configuration; (b) unimorph; (c) bimorph.

2.1. Derivation of differential equations of motion

For a slender cantilever beam, stress components other than the axial stress are negligible. Thus, the standard three-dimensional piezoelectric constitutive equations [21] can be reduced to a one-dimensional form

$$\begin{bmatrix} S_1 \\ D_3 \end{bmatrix} = \begin{bmatrix} s_{11}^E & d_{31} \\ d_{31} & \varepsilon_{33}^T \end{bmatrix} \begin{bmatrix} T_1 \\ E_3 \end{bmatrix} \quad (1)$$

where S_1 is the axial strain in the x -direction, D_3 electric displacement in the z -direction, s_{11}^E elastic compliance at constant electric field, d_{31} piezoelectric coupling coefficient, ε_{33}^T permittivity at constant stress, T_1 axial stress in the x -direction, and E_3 electric field in the z -direction. The differential equations of motion will be derived using the extended Hamilton's principle which is expressed as [20]

$$\int_{t_1}^{t_2} (\delta T - \delta U + \delta W) dt = 0 \quad (2)$$

where T is the kinetic energy, U potential energy, W work done by external and nonconservative forces, and t_1, t_2 initial and final times. Inserting expressions for T, U and W into equation (2) and performing integration by parts with respect to time yield (more details about the derivation procedure can be found in [18-20])

$$\begin{aligned} \int_{t_1}^{t_2} (\delta T - \delta U + \delta W) dt = & \int_{t_1}^{t_2} dt \left\{ \int_0^l \left[-m \frac{\partial^2 w}{\partial t^2} - K \frac{\partial^4 w}{\partial x^4} + f(x, t) + F_c(t) \delta(x - x_c) \right] \delta w dx \right. \\ & + \sum_{i=1}^n b_i \int_{z_i}^{z_{i+1}} \int_0^l \frac{dD_{3,i}}{dz} \delta V dx dz + K \frac{\partial^3 w}{\partial x^3} \delta w \Big|_0^l - \left[K \frac{\partial^2 w}{\partial x^2} - M_P(t) - M(t) \right] \delta \frac{\partial w}{\partial x} \Big|_0^l \\ & \left. - \sum_{i=1}^n b_i \left[\int_0^l (\sigma_i + D_{3,i}) dx \delta V \Big|_{z_i}^{z_{i+1}} \right] \right\} = 0 \end{aligned} \quad (3)$$

where m is mass density per length, w deflection along the cantilever length, K bending stiffness, $\delta(x - x_c)$ a Dirac delta function, b_i width of the i -th layer, $M_P(t)$ piezoelectric moment and σ_i surface charge per unit area for the i -th layer. The mass density per length m is given by

$$m = \sum_{i=1}^n \rho_i (z_{i+1} - z_i) b_i \quad (4)$$

where ρ_i is density of the i -th layer. The bending stiffness K is

$$K = \sum_{i=1}^n \frac{1}{s_{11,i}} \int_{z_i}^{z_{i+1}} b_i (z - z_0)^2 dz \quad (5)$$

where the z -coordinate of the neutral plane z_0 is found to be

$$z_0 = \frac{\sum_{i=1}^n E_i b_i (z_{i+1}^2 - z_i^2)}{\sum_{i=1}^n 2E_i b_i (z_{i+1} - z_i)} \quad (6)$$

The piezoelectric moment $M_P(t)$ is written as

$$\begin{aligned} M_P(t) &= \sum_{i=1}^n \frac{b_i V_i d_{31,i}}{s_{11,i}^E} \int_{z_i}^{z_{i+1}} \frac{z_i - z_0}{z_{i+1} - z_i} dz \\ &= \sum_{i=1}^n \frac{b_i V_i d_{31,i}}{2s_{11,i}^E} (z_{i+1} + z_i - 2z_0) \end{aligned} \quad (7)$$

where V_i is the voltage applied to the i -th layer.

In equation (3), the virtual displacement δw is arbitrary. Hence, we must have

$$m \frac{\partial^2 w}{\partial t^2} + K \frac{\partial^4 w}{\partial x^4} = f(x, t) + F_c(t) \delta(x - x_c). \quad (8)$$

Similarly, we obtain the boundary conditions for the cantilevered beam as

$$w(0, t) = 0, \quad \frac{\partial w(0, t)}{\partial x} = 0, \quad K \frac{\partial^2 w(l, t)}{\partial x^2} = M_p(t) + M(t), \quad K \frac{\partial^3 w(l, t)}{\partial x^3} = 0 \quad (9)$$

where l is the length of the beam. As seen in equation (9), the two boundary conditions at the fixed end ($x = 0$) are homogeneous, while at the free end ($x = l$) one of the two boundary conditions is nonhomogeneous. The nonhomogeneous boundary condition shows that the converse piezoelectric effect induces uniform internal moment in the cantilever in the same manner as a tip moment. It shall be noted that both the internal piezoelectric moment $M_p(t)$ and external tip moment $M(t)$ entered the boundary conditions instead of the differential equations of motion, while in previous works [16, 17], the piezoelectric moment $M_p(t)$ was incorporated into the equations of motion as part of the total internal moment.

2.2. Transformation of the boundary-value problem

The solution to boundary-value problems consisting of nonhomogeneous differential equations with homogeneous boundary conditions is usually obtained by modal analysis. For boundary-value problems with nonhomogeneous boundary conditions, the common approach used in modal analysis does not work. However, a boundary-value problem with nonhomogeneous boundary conditions can be transformed into a problem with homogeneous boundary conditions which can be solved by modal analysis [20]. Hence we assume

$$w(x, t) = v(x, t) + h(x) [M_p(t) + M(t)] \quad (10)$$

where $v(x, t)$ is the solution to the transformed problem with homogeneous boundary conditions, and $h(x)$ a function to be found to satisfy the homogeneous boundary conditions of the new problem. Upon inserting equation (10), equation (8) becomes,

$$K \frac{\partial^4 v(x, t)}{\partial x^4} + m \frac{\partial^2 v(x, t)}{\partial t^2} = Q(x, t) \quad (11)$$

where $Q(x, t)$ is given by

$$Q(x, t) = f(x, t) + F_c(t) \delta(x - x_c) - mh(x) [\ddot{M}_p(t) + \ddot{M}(t)] - K \frac{d^4 h(x)}{dx^4} [M_p(t) + M(t)]. \quad (12)$$

Similarly, the boundary conditions for the new problem can be obtained by substituting equation (10) into equation (9),

$$\begin{aligned} v(0, t) &= -h(0) [M_p(t) + M(t)], \quad \frac{\partial v(0, t)}{\partial x} = -h'(0) [M_p(t) + M(t)], \quad \frac{\partial^3 v(l, t)}{\partial x^3} = -h'''(l) [M_p(t) + M(t)], \\ K \frac{\partial^2 v(l, t)}{\partial x^2} &= [M_p(t) + M(t)] - Kh''(l) [M_p(t) + M(t)]. \end{aligned} \quad (13)$$

To render all boundary conditions of the new problem homogeneous, we must have

$$v(0, t) = 0, \quad \frac{\partial v(0, t)}{\partial x} = 0, \quad K \frac{\partial^2 v(l, t)}{\partial x^2} = 0, \quad \frac{\partial^3 v(l, t)}{\partial x^3} = 0. \quad (14)$$

Since $M_p(t)$ and $M(t)$ in equation (13) are not equal to zero, $h(x)$ and its derivatives must satisfy

$$h(0) = 0, \quad h'(0) = 0, \quad Kh''(l) = 1, \quad h'''(l) = 0. \quad (15)$$

$h(x)$ can be any arbitrary function satisfying equation (15). Assuming $h(x)$ is a polynomial and considering the term $\frac{d^4 h(x)}{dx^4}$ in equation (12), $h(x)$ must be of an order equal to or higher than fifth. For this reason, we assume

$$h(x) = Ax^5 + Bx^4. \quad (16)$$

Equation (16) satisfies $h(0) = 0$ and $h'(0) = 0$ in equation (15) regardless of coefficients A and B . Using the other two equations $h'''(l) = 0$ and $Kh''(l) = 1$ in equation (15), we can find the coefficients A and B

$$A = -\frac{1}{10Kl^3}, \quad B = \frac{1}{4Kl^2}. \quad (17)$$

Hence, $h(x)$ is

$$h(x) = \frac{1}{4Kl^2}x^4 - \frac{1}{10Kl^3}x^5. \quad (18)$$

Inserting $h(x)$ from equation (18) into $Q(x, t)$ in equation (12), the original boundary-value problem with nonhomogeneous boundary conditions (equations (8) and (9)) has now been transformed into a new boundary-value problem with homogeneous boundary conditions (equations (11), (12) and (14)) which can be solved by modal analysis.

2.3. Modal analysis

The differential equation of the eigenvalue problem is [20]

$$\frac{d^4 Y(x)}{dx^4} - \beta^4 Y(x) = 0, \quad \beta^4 = \frac{\omega^2 m}{K}, \quad 0 < x < l \quad (19)$$

and the boundary conditions are

$$Y(0, t) = 0, \quad \frac{\partial Y(0, t)}{\partial x} = 0, \quad \frac{\partial^3 Y(l, t)}{\partial x^3} = 0, \quad K \frac{\partial^2 Y(l, t)}{\partial x^2} = 0. \quad (20)$$

The eigenfunctions are given by [20]

$$Y_r(x) = A_r \left[(\sin \beta_r l - \sinh \beta_r l) (\sin \beta_r x - \sinh \beta_r x) + (\cos \beta_r l + \cosh \beta_r l) (\cos \beta_r x - \cosh \beta_r x) \right] \quad (21)$$

where A_r are constants and β_r can be obtained from the following characteristic equation

$$\cos \beta l \cosh \beta l = -1. \quad (22)$$

Moreover, the corresponding natural frequencies are

$$\omega_r = (\beta_r l)^2 \sqrt{\frac{K}{ml^4}}. \quad (23)$$

The modal equations are written as

$$\ddot{\eta}_r(t) + \omega_r^2 \eta_r(t) = N_r(t), \quad r = 1, 2, \dots, n \quad (24)$$

where $\eta_r(t)$ is the modal coordinate, and $N_r(t)$ the modal force. $N_r(t)$ is given by

$$N_r(t) = \int_0^l Y_r(x) Q(x, t) dx. \quad (25)$$

The modal coordinates are written as

$$\eta_r(t) = \frac{1}{\omega_r} \int_0^t N_r(\tau) \sin \omega_r(t - \tau) d\tau, \quad r = 1, 2, \dots \quad (26)$$

Therefore, we obtain the solution to the new problem

$$v(x, t) = \sum_{r=1}^{\infty} Y_r(x) \eta_r(t). \quad (27)$$

Finally, inserting equation (27) into equation (10) yields the solution to the original problem

$$w(x, t) = h(x) [M_P(t) + M(t)] + \sum_{r=1}^{\infty} Y_r(x) \eta_r(t). \quad (28)$$

2.4. Frequency response functions

The frequency response functions for input of voltage, concentrated force, tip moment and distributed force are derived based on equation (28). For voltage driving without external loading, we have

$$f(x, t) = 0, F_c(t) = 0, M(t) = 0. \quad (29)$$

The harmonic voltage excitation $V(t)$ can be expressed in complex form

$$V(t) = V_0 e^{i\omega t}. \quad (30)$$

Assuming the piezoelectric layers are in parallel connection, hence the piezoelectric moment $M_p(t)$ in equation (7) becomes

$$M_p(t) = \chi V_0 e^{i\omega t} \quad (31)$$

where χ is amplitude of the piezoelectric moment induced by unit voltage. According to equation (7), χ is given by

$$\chi = \sum_{i=1}^n \frac{b_i d_{31,i}}{2s_{11,i}^E} (z_{i+1} + z_i - 2z_0). \quad (32)$$

Similarly, we obtain the force $Q(x, t)$ from equation (12)

$$Q(x, t) = \chi V_0 \left[mh(x)\omega^2 - \left(\frac{6}{l^2} - \frac{12}{l^3} x \right) \right] e^{i\omega t} \quad (33)$$

and subsequently the modal coordinates $\eta_r(t)$ according to equation (26)

$$\eta_r(t) = \frac{\chi V_0 \int_0^l Y_r(x) \left[mh(x)\omega^2 - \left(\frac{6}{l^2} - \frac{12}{l^3} x \right) \right] dx}{\omega_r^2 - \omega^2} e^{i\omega t}. \quad (34)$$

Substituting equation (34) into equation (28) leads to the solution to flexural vibration under voltage driving. Finally, we obtain frequency response function for voltage driving $G_V(i\omega)$

$$G_V(i\omega) = \frac{w(i\omega)}{V(i\omega)} = \chi \left\{ \frac{1}{4Kl^2} x^4 - \frac{1}{10Kl^3} x^5 + \sum_{r=1}^{\infty} \frac{Y_r(x) \int_0^l Y_r(x) \left[m \left(\frac{1}{4Kl^2} x^4 - \frac{1}{10Kl^3} x^5 \right) \omega^2 - \left(\frac{6}{l^2} - \frac{12}{l^3} x \right) \right] dx}{\omega_r^2 - \omega^2} \right\}. \quad (35)$$

Following the same procedure, we obtain the frequency response functions for concentrated force $G_F(i\omega)$, tip moment $G_M(i\omega)$ and uniform distributed force $G_f(i\omega)$

$$G_F(i\omega) = \sum_{r=1}^{\infty} \frac{Y_r(x) Y_r(l)}{\omega_r^2 - \omega^2}, \quad (36)$$

$$G_M(i\omega) = \frac{1}{4Kl^2} x^4 - \frac{1}{10Kl^3} x^5 + \sum_{r=1}^{\infty} \frac{Y_r(x) \int_0^l Y_r(x) \left[m \left(\frac{1}{4Kl^2} x^4 - \frac{1}{10Kl^3} x^5 \right) \omega^2 - \left(\frac{6}{l^2} - \frac{12}{l^3} x \right) \right] dx}{\omega_r^2 - \omega^2}, \quad (37)$$

$$G_f(i\omega) = \sum_{r=1}^{\infty} \frac{Y_r(x) \int_0^l Y_r(x) dx}{\omega_r^2 - \omega^2}. \quad (38)$$

Comparing equations (35) and (37), we notice that the only difference between them is the constant χ . This shows that the excitation of voltage is equivalent to that of an external tip moment or uniform

internal moment along the beam length. This observation was used as a basis to derive the equations of motion in previous works [9-12].

3. Model validation

The analytical solutions were verified against a three-dimensional finite-element-method (FEM) model. A unimorph (figure 1(b)) and a bimorph (figure 1(c)) were studied for verification. The dimensions of the two cantilevers are listed in table 1. The material properties for the electrodes (gold), elastic layer (silicon) and piezoelectric zinc oxide (ZnO) layers are listed in tables 2 and 3. In this case, silicon was treated as an isotropic material in the FEM model. It could be replaced by any other isotropic materials, such as stainless steel.

Table 1. Dimensions of unimorph and bimorph.

	Length (μm)	Width (μm)	Thickness of piezoelectric layer(s) (μm)	Thickness of electrodes (nm)	Thickness of elastic layer (μm)
Unimorph	2000	200	1	100	20
Bimorph	2000	200	10	100	0

Table 2. Young's modulus and density of materials for electrodes, elastic layer and piezoelectric layers

Materials	Young's modulus (GPa)	Density (kg/m^3)
Electrodes (Au)	77.2	19300
Elastic layer (Si)	169	2329
Piezoelectric layers (ZnO)	NA	5750

Table 3. Compliance coefficients and piezoelectric constants of piezoelectric material (ZnO) [22].

Constants	Value	Constants	Value
s_{11} ($10^{-12} \text{ m}^2/\text{N}$)	11.12	s_{44} ($10^{-12} \text{ m}^2/\text{N}$)	42.47
s_{12} ($10^{-12} \text{ m}^2/\text{N}$)	-4.56	d_{31} (10^{-12} C/N)	-5.43
s_{13} ($10^{-12} \text{ m}^2/\text{N}$)	-1.4	d_{33} (10^{-12} C/N)	11.67
s_{33} ($10^{-12} \text{ m}^2/\text{N}$)	8.47	d_{15} (10^{-12} C/N)	-11.34

The commercial finite element program Ansys 14.5 was used for the FEM simulation. Element types SOLID5, SOLID45 and SHELL181 are used for the piezoelectric layers, elastic layer and electrodes, respectively. The FEM models are shown in figure 2. The unimorph and bimorph both have 20000

Figure 2. FEM model: (a) unimorph; (b) bimorph.

elements. All translational degrees of freedom are constrained at one end of the beams. The adjacent layers of the composite beams are bonded together by sharing nodes. For the unimorph, voltage is applied on the top electrode with the bottom electrode grounded, while for the bimorph voltage is applied on the top and bottom electrodes with the middle electrode grounded.

3.1. Quasi-static deflection profile

The quasi-static deflection profile depicts lengthwise vibration amplitude at a frequency much lower than the fundamental resonant frequency. The lengthwise vibration amplitude under quasi-static voltage excitation was obtained by setting the driving frequency ω in equation (35) to 0.2 kHz. Figure 3 shows that the tip vibration amplitudes of the unimorph are 80.91 nm and 80.67 nm for the analytical model and FEM model, respectively. The maximum error of the analytical result with respect to the FEM result is 0.24 nm at the tip as shown in figure 3(b). Similar results for the bimorph are shown in Figure 4. The tip vibration amplitudes of the bimorph are 158.5 nm and 159.6 nm for the analytical model and FEM model, respectively. The maximum absolute error of the analytical results with respect to the FEM results is 1.1 nm at the tip (figure 4(b)).

Figure 3. Vibration amplitude of unimorph subjected to unit voltage at 0.2 kHz : (a) vibration amplitude along the length; (b) error of analytical result with respect to FEM result.

Figure 4. Vibration amplitude of bimorph subjected to unit voltage at 0.2 kHz: (a) vibration amplitude along the length; (b) error of analytical result with respect to FEM result.

3.2. Frequency response functions

Setting $x = l$ and varying ω in equations (35), (36) and (38), frequency responses of the tip deflection to harmonic excitations of voltage, tip force and uniform distributed force can be obtained. Figures 5, 6, and 7 present the frequency responses of the tip deflection to the three harmonic excitations, respectively. The input amplitudes for voltage and uniform distributed force were 1 V and 1 N/m², respectively. To limit tip deflection of the cantilevers to a realistic level, the input amplitude for tip force was set to 1 μN instead of 1 N. As seen in figures 5, 6 and 7, the resonant frequency for a certain mode remains the same under different excitations. Resonant frequencies of the first three modes are listed in table 4. For the unimorph, the maximum percentage of error is 1% for the third mode, while for the bimorph 1.4% also for the third mode. Tip deflections at quasi-static state up to 0.4 kHz are shown in the insets of figures 5, 6 and 7. Table 5 lists the amplitudes of tip deflections at 0.2 kHz. The percentages of error under excitation of voltage are the smallest, 0.3% for the unimorph and 0.7% for the bimorph, respectively. By contrast, the biggest percentages of error, 1.6% for the unimorph and 3.4% for the bimorph, occur under excitation of distributed force.

The analytical resonant frequencies are all smaller than those found by FEM (table 4). Contrary to resonant frequencies, the amplitudes of analytical tip deflections are all greater than the corresponding FEM results except for the bimorph under excitation of voltage (table 5). Lower resonant frequencies and larger tip deflections comply with the well-known trade-off between the two parameters. The discrepancy between the analytical model and 3D FEM model is attributed to assumptions of the analytical model. The one-dimensional analytical model assumes axial stress and considers only beam modes. In comparison, the 3D FEM model is able to simulate all modes which also include plate deformations such as twisting and cupping that are not associated with flexural vibration [23]. Those plate modes occupy elastic energy thus reducing tip deflections.

It can be found from tables 4 and 5 that the analytical solutions of the unimorph are more accurate than those of the bimorph. The lower accuracy of the analytical model for the bimorph is because of its structural stiffness dominated by the two piezoelectric layers (10 μm each). In addition to omission of modes that are not related to flexural vibration, the analytical model simplifies the anisotropic piezoelectric material into an isotropic material. The latter part also causes modeling errors to bending stiffness and hence resonant frequencies and tip deflection. Different to the bimorph, the unimorph has an elastic layer (20 μm) much thicker than the piezoelectric layer (1 μm). The bending stiffness of the

unimorph is mainly dependent on the elastic layer. Therefore, simplifying the anisotropic piezoelectric material into an isotropic material causes less error to analytical results of the unimorph.

Figure 5. Frequency response to voltage excitation: (a) unimorph; (b) bimorph.

Figure 6. Frequency response to excitation of tip force: (a) unimorph; (b) bimorph.

Figure 7. Frequency response to excitation of uniform distributed force: (a) unimorph; (b) bimorph.

Table 4. Resonant frequencies of the first three modes.

		Resonant frequency of 1 st mode (kHz)	Resonant frequency of 2 nd mode (kHz)	Resonant frequency of 3 rd mode (kHz)
Unimorph	Analytical	6.58	41.21	115.4
	FEM	6.64	41.52	116.6
	Error (%)	-0.9	-0.7	-1
Bimorph	Analytical	3.18	19.93	55.81
	FEM	3.22	20.14	56.6
	Error (%)	-1.2	-1	-1.4

Table 5. Amplitude of tip deflection under excitations at quasi-static state (0.2 kHz).

		Tip deflection subjected to voltage (nm)	Tip deflection subjected to tip force (nm)	Tip deflection subjected to distributed force (nm)
Unimorph	Analytical	80.91	107.3	16.13
	FEM	80.67	106.1	15.87
	Error (%)	0.3	1.1	1.6
Bimorph	Analytical	158.5	214.1	32.17
	FEM	159.6	208.3	31.1
	Error (%)	-0.7	2.8	3.4

The above results of quasi-static deflection profile and frequency response to typical excitations verified that the discrepancies between the analytical model and three-dimensional FEM model were fairly small. This proves validity of the analytical solutions which were based on the major assumption of one-dimensional axial stress, derivations of the differential equations of motion, transformation of the boundary-value problem and modal analysis.

4. Parametric study

After the analytical solutions were verified, a parametric study was conducted to study the effects of thickness of the electrodes and piezoelectric layer, and the piezoelectric coupling coefficient d_{31} on the performance of piezoelectric cantilever actuators. Equations (23) and (35) were used to obtain the resonant frequencies and the tip deflection at the quasi-static state (0.2 kHz) of the unimorph and bimorph.

4.1. Effect of electrode thickness

The electrode thickness for the unimorph and bimorph was varied from 100 nm to 500 nm. The other parameters shown in table 1 remained constant. Figure 8 shows the effect of electrode thickness on the resonant frequencies and tip deflection. Although all the resonant frequencies of the unimorph and bimorph decrease with thicker electrodes, only the third resonant frequency of the unimorph decreases noticeably with only minimal change for the rest. The negligible influence of the electrodes on the resonant frequencies is because of the thinness of the electrodes compared to silicon (unimorph) and ZnO (bimorph). The decreasing nature of the resonant frequencies is attributed to the lower Young's modulus and higher density of the electrode material (gold) compared with silicon and ZnO (tables 2 and 3). As shown in equation (23), the resonant frequencies are related to the bending stiffness K and mass density m . Although both K and m increase with thicker electrodes, m increases more than K causing the resonant frequencies to decrease. With a greater bending stiffness, the tip deflection for unit voltage decreases for

both the unimorph and bimorph (figure 8). By contrast with the resonant frequencies, the tip deflection is much more sensitive to change in the electrode thickness. It is interesting to note that there is no trade-off effect of the electrode thickness on the resonant frequency and tip deflection. Therefore, to achieve optimal performance, thinner electrodes are preferred.

Figure 8. Effect of electrode thickness on resonant frequencies and amplitude of tip deflection under unit voltage at 0.2 kHz: (a) unimorph; (b) bimorph.

4.2. Effect of ZnO thickness

Contrary to the electrode thickness, the ZnO thickness has a trade-off effect on the resonant frequencies and tip deflection. As seen in figure 9, the resonant frequencies of both the unimorph and bimorph increase with thicker ZnO layers, while the tip deflection decreases. As explained in the previous section on the effect of electrode thickness, changing the ZnO thickness can affect both the bending stiffness K and the mass density of the cantilevers m . The tip deflection only depends on K , while the resonant frequencies on both K and m . Increasing the ZnO thickness leads to a higher K , thus a smaller tip deflection. However, the combined effects of increasing the ZnO thickness (higher K and greater m) are higher resonant frequencies. Therefore, choosing a ZnO thickness is not as straightforward as the electrode thickness. It must be determined by taking into consideration the specific requirements on both the resonant frequencies and tip deflection.

Figure 9. Effect of ZnO thickness on resonant frequencies and amplitude of tip deflection under unit voltage at 0.2 kHz: (a) unimorph; (b) bimorph.

4.3. Effect of d_{31}

Under quasi-static voltage excitation (1 V at 0.2 kHz), the effect of d_{31} on tip vibration amplitude was analyzed using equation (35). Figure 10 shows that the tip vibration amplitudes of the unimorph and bimorph are proportional to d_{31} . This is due to the fact that the tip vibration is proportional to the piezoelectric moment χ (equation (35)) and the piezoelectric moment χ proportional to d_{31} (equation (32)). It also can be seen that the tip vibration amplitude of the bimorph are larger and increases faster than that of the unimorph. Therefore, for applications of large deflection, bimorphs are preferred over unimorphs. It can be seen from equation (23) that resonant frequencies are not a function of d_{31} . As a result, varying d_{31} changes tip deflection but not resonant frequencies.

Figure 10. Tip vibration amplitude at 0.2 kHz versus d_{31} .

5. Conclusions

The constitutive equations for flexural vibration of multilayer piezoelectric cantilevers were derived by the extended Hamilton's principle. The piezoelectric moment was found in the boundary conditions alongside the external tip moment. The frequency response functions showed that the effect of the piezoelectric moment is equivalent to that of the external tip moment. The analytical solutions were verified with three-dimensional FEM analysis. Deflection profiles and frequency response functions under voltage, tip force and distributed force were compared. Discrepancies between the analytical and FEM results were within 3.5%.

A parametric study was conducted to study the effects of thickness of electrodes and piezoelectric layers, and d_{31} on the performance of cantilever actuators. The tip deflection was found to decrease with the increased thickness of electrodes and piezoelectric layers, while the resonant frequencies could either increase or decrease depending on the material properties of the electrodes, piezoelectric layers and substrate. Compared to the dual effects of thickness, d_{31} has the advantage of changing tip deflection while keeping resonant frequencies unaffected.

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