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A two-dimensional co-rotational Timoshenko beam element with XFEM formulation

Xu Jin, C.K. LEE, K.H. TAN

School of Civil & Environmental Engineering, Nanyang Technological University, Singapore

xu0003in@e.ntu.edu.sg

Abstract

Pin connections and plastic hinges produce non-smooth displacement fields in beam structures. By using an appropriate extended finite element method (XFEM), non-smooth solutions can be obtained by regular coarse meshes, which do not necessarily conform to pins or plastic hinges. In this paper, a two-dimensional co-rotational beam element with XFEM formulation to simulate pin connections and plastic hinges is presented. Enrichments for the rotation and the deflection approximations are embedded in a co-rotational frame to capture the non-smoothness in both small and large deformations. Numerical examples on pin connections and plastic hinges demonstrate the accuracy and robustness of the present formulation.

Key words: extended finite element method, co-rotational formulation, shifted enrichment, Timoshenko beam, perfect pin, plastic hinge

1 Introduction

Three types of non-smoothness can be categorized [1] in engineering applications: 1) discontinuities with zero length, 2) discontinuities with extremely small length which can be idealised as zero length discontinuities, 3) discontinuities with finite length which lead to locally high gradients in deformation. Two kinds of non-smoothness can be found in a beam structure: perfect pins and plastic hinges. As shown in Fig. 1(a), a perfect pin connection produces a jump in rotation approximation and a kink in deflection approximation (type 2), while a plastic hinge (Fig. 1(b)) changes rotation and deflection sharply within a small length (type 3). Perfect pin connections are prescribed in a structure, while the locations and number of plastic hinges will be dependent on the loading conditions, the material and cross-section properties. In a bending-dominant frame with uniform cross-sections, plastic hinges are formed at locations of maximum bending moment. At a cross-section, when the principle stress at the extreme fibres reaches the yield strength, the cross-section begins to yield while the middle fibres remain

elastic. Hence, a kink of the stiffness of the cross-section appears. The rotation of the yielded cross-section shows a high gradient and the associated deflection shows a sharp change in its gradient within the length of the yielded zone. With increasing load, most of the fibres of the section will be yielded and the stiffness of the fully-yielded cross-section approaches zero. The kink in deflection and the high gradient in rotation become more obvious.

In a standard finite element (FE) formulation, if the locations of plastic hinges are not known in advance, their effects could only be captured by a fine mesh. Hence, a large amount of computational effort is needed to simulate the effects of plastic hinges in a simple frame made of beam-column elements. Furthermore, plastic hinges are common in progressive collapse analyses of structures, in which the deformation of each member is large and many plastic hinges could be formed before the structure collapses. Therefore, standard FE methods are not ideal to trace the behaviour of structures in progressive collapse analyses. Besides, in a standard FE model with internal pins, surrounding elements are required to conform to the internal pin.

The extended finite element method (XFEM) shows a great advantage in analyses on approximations of non-smooth solutions, since it is unnecessary to modify the surrounding elements to cater for non-smoothness in XFEM simulations. Special displacement functions, based on *a priori* knowledge of local discontinuities, are added into an XFEM approximation space so that a discontinuity inside an element can be captured. These special functions are called *enrichment functions*. The application of XFEM has been focused on non-smooth approximations such as the propagation of cracks [2-5], crack growth with frictional contact [6], cohesive crack growth [7-11], dislocations [12], interfaces of two different materials [13], delamination in shell structures [14], etc. More applications of XFEM can be found in [15].

Areias and Belytschko [16] applied XFEM formulation in a Mindlin-Reissner shell formulation to trace arbitrary crack propagations in shell structures with both geometrical and material nonlinearities. A modified enrichment function is used in their research due to the non-additive rotational degree of freedom (DOF) and a shift in location of a discontinuity is needed when it passes through an element node. Anahid and Khoei [17] applied XFEM formulation in elasto-plastic large deformation analyses, in which a Total Lagrangian formulation is used with

XFEM enrichment functions to capture the interface of two elasto-plastic materials. Khoei [18] extended Anahid and Khoei's work [17] from two-dimensional to three-dimensional.

In this article, a two-dimensional co-rotational Timoshenko beam element with XFEM formulation is developed. This element can be used to simulate perfect pin connections in a frame with a regular coarse mesh. It can also capture the formation of plastic hinges and trace the behaviour of a frame after a plastic mechanism is formed. A pin connection and a plastic hinge are introduced as two different types of non-smoothness in the present formulation. Since Timoshenko beam theory is used, rotation and deflection approximations are decoupled and they are separately enriched.

From the numerical point of view, the most critical difference in non-smoothness between an internal pin and a plastic hinge is that no strain energy is produced by bending strain in the former, while strain energy in the latter is of finite value. While a perfect pin connection is represented by a discontinuity with zero length, in order to capture the strain energy in a plastic hinge, the length of the plastic hinge *cannot* be idealised as zero. Therefore, discontinuous functions [2,13] and regularized discontinuous functions [19] are used as enrichments for a perfect pin and a plastic hinge, respectively.

This article is organized as follows: the formulation of a two-dimensional co-rotational beam element with elasto-plastic constitutive relationship is presented in Section 2. The XFEM formulation incorporated with co-rotational approach is derived in Section 3. In Section 4, some numerical examples are given. Finally, conclusions and discussions are given in Section 5.

2 Formulations for a co-rotational large deformation beam element with elasto-plastic constitutive relationship

In this section, a 3-node two-dimensional co-rotational Timoshenko beam element with perfect elasto-plastic material model is presented. Since the layered model is embedded in this element, the process of formation of a plastic hinge can be

captured. A co-rotational approach [20-22] is used in the present element so that it can be applied to large deformation but small strain cases.

2.1 The co-rotational frame

In the co-rotational approach, a large deformation is separated into two parts: rigid body motion and pure deformation. As shown in Fig. 2, (a) is the initial configuration, (c) is the deformed configuration and (b) is an intermediate configuration. The initial configuration (a) turns to an intermediate configuration (b) by a rigid body motion. From (b) to (c), there is pure deformation. It should be noted that the intermediate configuration is not physically unique. It depends on the definition of rigid body motion.

In order to express the rigid body motion of an element, two Cartesian coordinate systems are defined by using the co-rotational approach: the global coordinate system OXY , which is unique for the whole computational model; and the local coordinate system oxy , which co-rotates with the element. The local x and y axes are associated with unit vectors $\mathbf{e}_x$ and $\mathbf{e}_y$, respectively, in each element. In the present formulation, the element coordinate system is tied to the mid-node of the element, as shown in Fig. 2(b). The initial directions of the axes, ${}^0\mathbf{e}_x$ and ${}^0\mathbf{e}_y$ are defined as:

$${}^0\mathbf{e}_x = \frac{{}^0\mathbf{X}_3 - {}^0\mathbf{X}_1}{|{}^0\mathbf{X}_3 - {}^0\mathbf{X}_1|}, \quad {}^0\mathbf{e}_y = \left(-{}^0\mathbf{e}_{xY} \quad {}^0\mathbf{e}_{xX} \right)^T \quad (1)$$

In Eq. (1) ${}^0\mathbf{X}_1$ and ${}^0\mathbf{X}_3$ are respectively the initial position vectors of node 1 and node 3 in an element; ${}^0\mathbf{e}_{xX}$ and ${}^0\mathbf{e}_{xY}$ are the X- and Y-components of ${}^0\mathbf{e}_x$, respectively. ${}^0\mathbf{e}_x$ and ${}^0\mathbf{e}_y$ are the initial local axes before deformation.

The local axes in the deformed configuration can be found by

$$\begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{pmatrix} = \begin{bmatrix} \cos \psi_2 & -\sin \psi_2 \\ \sin \psi_2 & \cos \psi_2 \end{bmatrix} \begin{pmatrix} {}^0\mathbf{e}_x \\ {}^0\mathbf{e}_y \end{pmatrix} \quad (2)$$

where ψ_2 is the global rotational displacement of node 2. The translation and rotation of a local coordinate system is the rigid body motion.

2.2 Displacements in the global and the element coordinate systems

In the present standard beam element, three DOFs are defined at each node. The global displacement vector $\mathbf{U}$ of an element is expressed as:

$$\mathbf{U} = \{U_1, V_1, \psi_1 \mid U_2, V_2, \psi_2 \mid U_3, V_3, \psi_3\}^T \quad (3)$$

where U_i, V_i ($i = 1, 2, 3$) are the two translational DOFs in the global coordinate system. ψ_i ($i = 1, 2, 3$) are the rotational DOFs in the global X-Y coordinate system. The element displacement approximation $\mathbf{U}_h$ is of the form

$$\mathbf{U}_h = \begin{pmatrix} U_h \\ V_h \\ \psi_h \end{pmatrix} = \sum_{i=1}^3 N_i \begin{pmatrix} U_i \\ V_i \\ \psi_i \end{pmatrix} \quad (4)$$

where N_i are the Lagrangian shape functions for a 3-node beam element (Eqn. 5) defined in the natural coordinate system (Fig. 3). $\mathbf{U}_i = [U_i, V_i, \psi_i]$ is the nodal global displacement.

$$N_1 = -0.5\xi(1-\xi), \quad N_2 = (1-\xi)(1+\xi), \quad N_3 = 0.5\xi(1+\xi) \quad (5)$$

According to the definition of the local coordinate system in the present formulation, the local displacement of the mid-side node of each element is *zero during analyses*; therefore the local displacement of an element $\mathbf{u}$ can be expressed as:

$$\mathbf{u} = \{u_1, v_1, \theta_1 \mid u_3, v_3, \theta_3\}^T \quad (6)$$

where u_i and v_i ($i = 1, 3$) are the two translational DOFs in the local coordinate system. θ_i are the rotational DOFs in the local coordinate system, as shown in Fig. 4.

The local displacement approximation $\mathbf{u}_h$ is of the form:

$$\mathbf{u}_h = \begin{pmatrix} u_h \\ v_h \\ \theta_h \end{pmatrix} = N_1 \begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \end{pmatrix} + N_3 \begin{pmatrix} u_3 \\ v_3 \\ \theta_3 \end{pmatrix} \quad (7)$$

where N_i are given in Eq. (5). $\mathbf{u}_i = [u_i, v_i, \theta_i]^T$ is the nodal displacement for node i in the local coordinate system.

During analysis, the local displacement for each element is obtained by filtering out the rigid body motion from the global displacement. The relationship between the local nodal and the global nodal displacements can be expressed as [23]:

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \mathbf{R} \left(\begin{pmatrix} U_i \\ V_i \end{pmatrix} - \begin{pmatrix} U_2 \\ V_2 \end{pmatrix} \right) + (\mathbf{R} - {}^0\mathbf{R}) ({}^0\mathbf{X}_i - {}^0\mathbf{X}_2), \quad i = 1, 3 \quad (8)$$

$$\theta_i = \psi_i - \psi_2, \quad i = 1, 3 \quad (9)$$

where ${}^0\mathbf{X}_i$ is the position vector of node i in the initial configuration in the global coordinate system and $\mathbf{R} = [\mathbf{e}_x^T, \mathbf{e}_y^T]^T$ is the rotation matrix.

2.3 The stiffness matrix and the internal force vector

The strain in an arbitrary layer of an element $\boldsymbol{\varepsilon}_j$ can be decomposed into three parts: membrane strain ε_m , shear strain γ and bending strain (curvature) χ . They are expressed in Eq. (10) - (13).

$$\boldsymbol{\varepsilon}_j = \begin{Bmatrix} \varepsilon \\ \gamma \end{Bmatrix} = \begin{Bmatrix} \varepsilon_m \\ \gamma \end{Bmatrix} + z_j \begin{Bmatrix} \chi \\ 0 \end{Bmatrix} \quad (10)$$

$$\varepsilon_m = \frac{\partial u}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right] \quad (11)$$

$$\gamma = -\theta + \frac{\partial v}{\partial x} \quad (12)$$

$$\chi = -\frac{\partial \theta}{\partial x} \quad (13)$$

where z_j is the distance from the middle surface of the j^{th} layer to the reference surface, as shown in Fig. 5.

The potential energy Π is expressed as:

$$\Pi = \frac{1}{2} \sum_{j=1}^n \int (\varepsilon_m + z_j \chi) \mathbf{D}_{11} (\varepsilon_m + z_j \chi) dV + \frac{1}{2} \sum_{j=1}^n \int \gamma \mathbf{D}_{22} \gamma dV - W \quad (14)$$

where n is the total number of layers in an element; W is the work done by the external force; $\mathbf{D} = \text{diag}(D_{11}, D_{22})$ is the consistent material matrix. For those layers in which the material is still elastic:

$$\mathbf{D} = \begin{pmatrix} E & 0 \\ 0 & kG \end{pmatrix} \quad (15)$$

while for those layers in which the material has yielded:

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & kG \end{pmatrix} \quad (16)$$

In Eq. (15) and (16), E is the Young's modulus, G is the shear modulus and k is the shear correction factor for layered models and $k = 5/6$.

The internal force with respect to the local coordinate system $\mathbf{f}_L$ is expressed as:

$$\mathbf{f}_L = \sum_{j=1}^n \int (\mathbf{B}_m + z_j \mathbf{B}_b)^T \mathbf{D}_{11} (\boldsymbol{\varepsilon}_m + z_j \boldsymbol{\chi}) dV + \sum_{j=1}^n \int \mathbf{B}_\gamma^T \mathbf{D}_{22} \boldsymbol{\gamma} dV \quad (17)$$

where $\mathbf{B}_m$, $\mathbf{B}_b$ and $\mathbf{B}_\gamma$ are the strain-displacement matrices of the membrane part, bending part and shear part, respectively. Detailed expressions have been derived and are included in Appendix A. The stiffness matrix in the local coordinate system $\mathbf{K}_L$ is expressed as:

$$\mathbf{K}_L = \sum_{j=1}^n \int \left(\mathbf{B}_m^T \mathbf{D}_{11} \mathbf{B}_m + z_j^2 \mathbf{B}_b^T \mathbf{D}_{11} \mathbf{B}_b + \frac{\partial \mathbf{B}_m^T}{\partial \mathbf{u}} \mathbf{D}_{11} \boldsymbol{\varepsilon}_m \right) dV + \sum_{j=1}^n \int \mathbf{B}_\gamma^T \mathbf{D}_{22} \mathbf{B}_\gamma dV \quad (18)$$

The internal force vector with respect to the global coordinate system $\mathbf{f}_G$ is expressed as:

$$\mathbf{f}_G = \frac{\partial \Pi}{\partial \mathbf{U}^T} = \left(\frac{\partial \mathbf{u}}{\partial \mathbf{U}^T} \right)^T \frac{\partial \Pi}{\partial \mathbf{u}^T} = \mathbf{T}^T \mathbf{f}_L \quad (19)$$

where $\mathbf{T}$ is the transformation matrix. The stiffness matrix in the global coordinate system $\mathbf{K}_G$ is expressed as:

$$\mathbf{K}_G = \frac{\partial \mathbf{f}_G}{\partial \mathbf{U}^T} = \mathbf{T}^T \mathbf{K}_L \mathbf{T} + \frac{\partial \mathbf{T}^T}{\partial \mathbf{U}^T} \mathbf{f}_L \quad (20)$$

2.4 Elasto-plastic constitutive relationship

A perfect elasto-plastic constitutive relationship is employed in the present formulation. The modifications due to the elasto-plastic constitutive relationship are conducted in the *local coordinate frame only* and the co-rotational frame is not affected. In the present formulation, yield criteria is assumed to be a function of normal stresses, the direct stresses associated with flexure and tension-compression, but not of transverse shear stresses [24]. Therefore, the following uniaxial yield criterion is used:

$$\varepsilon = \varepsilon_e + \varepsilon_p \quad (21)$$

$$\sigma = \sigma_0 + E d \varepsilon_e \quad (22)$$

where ε is the total normal strain of a layer in a cross-section; ε_e is the elastic strain and ε_p is the plastic strain; σ_0 is the normal stress at the latest equilibrium state and E is the Young's modulus of the material.

3 XFEM formulations

3.1 The enriched displacement field

In the present formulation, both of the translational and the rotational DOF are enriched. Therefore three additional DOFs per node are added to the nodal displacement vector. The global displacement vector for an enriched element is of the form:

$$\mathbf{U}_{\text{enr}} = \left\{ \mathbf{U}_{\text{enr}1}^T \quad \mathbf{U}_{\text{enr}2}^T \quad \mathbf{U}_{\text{enr}3}^T \right\}^T \quad (23)$$

In Eq. (23), $\mathbf{U}_{\text{enr}i}^T = (U_i, V_i, \psi_i, A_i, B_{U_i}, B_{V_i})$, $i = 1, 2, 3$. A_i , B_{U_i} and B_{V_i} are the additional DOFs corresponding to the nodal global displacement $\mathbf{U}_i$. The displacement approximation $\mathbf{U}_h = (U_h, V_h, \psi_h)^T$ in the global coordinate system is of the form:

$$\psi_h = \sum_{i \in I} N_i \psi_i + \sum_{i \in I^*} M_i A_i \quad (24)$$

$$\begin{pmatrix} U_h \\ V_h \end{pmatrix} = \sum_{i \in I} N_i \begin{pmatrix} U_i \\ V_i \end{pmatrix} + \sum_{i \in I^*} L_i \begin{pmatrix} B_{U_i} \\ B_{V_i} \end{pmatrix} \quad (25)$$

where M_i and L_i are the enriched interpolation functions for the rotational and the translational DOFs, respectively; I is the node set for the whole domain. I^* is the

node set of enriched nodes, which is a sub-set of I and $I^* \subseteq I$.

Two additional DOFs per node are added into the element displacement vector, so the local displacement vector for an enriched element $\mathbf{u}_{\text{enr}}$ is expressed as:

$$\mathbf{u}_{\text{enr}} = \{u_1, v_1, \theta_1, a_1, b_{v1} \mid a_2, b_{v2} \mid u_3, v_3, \theta_3, a_3, b_{v3}\}^T \quad (26)$$

The local displacement approximation $\mathbf{u}_h = (u_h, v_h, \theta_h)^T$ has the form

$$\theta_h = \sum_{i=1}^3 N_i \theta_i + \sum_{i=1}^3 M_i a_i \quad (27)$$

$$\begin{pmatrix} u_h \\ v_h \end{pmatrix} = \sum_{i=1}^3 N_i \begin{pmatrix} u_i \\ v_i \end{pmatrix} + \sum_{i=1}^3 L_i \begin{pmatrix} 0 \\ b_{vi} \end{pmatrix} \quad (28)$$

where a_i is the additional DOF for the nodal local non-smooth rotation for node i ; b_{vi} is the additional DOF for the nodal local non-smooth deflection for node i . The approximation of translational displacement along local x -axis is not enriched. However, in order to avoid ill-conditioning of the assembled stiffness matrix, a finite real number is placed in the diagonal cell corresponding to b_{ui} . Note that by keeping the unused translational enriched DOFs, the formulation of the element will become simpler and more consistent. It makes it much easier to establish the relationship between the local translational enriched DOFs and the global translational enriched DOFs in matrix form. Otherwise, a lengthy and tedious formulation would be derived. In addition, in some special cases, such formulation could lead to undesirable operations such as division by zero.

As co-rotational technique is used for the present beam element, a relationship between the global additional DOFs and the local additional DOFs should be established and the physical meaning of the additional DOFs should be clarified. In the XFEM formulation, a displacement approximation is a summation of a smooth component and a non-smooth component. The smooth approximation is constructed by interpolation of the nodal displacement $\mathbf{U}_{si} = [U_i, V_i, \psi_i]$ with shape function N_i which is smooth within the element domain and the non-smooth approximation is constructed by interpolation of the nodal global displacement $\mathbf{U}_{nsi} = [A_i, B_{Ui}, B_{Vi}]$ with some non-smooth functions M_i and L_i . Thus, the additional DOF can be regarded as the non-smooth nodal displacement. The relationship between the global additional DOFs and the local additional DOFs can be established in a similar way as the smooth nodal displacement

$$a_i = A_i \quad (29)$$

$$\begin{pmatrix} b_{ui} \\ b_{vi} \end{pmatrix} = \mathbf{R} \begin{pmatrix} B_{Ui} \\ B_{Vi} \end{pmatrix} \quad (30)$$

where $\mathbf{R}$ is the rotation matrix. It should be noted that, rigid body motion is measured by the motion of a local coordinate system which is defined by the nodal displacements at the mid-side node. On the other hand, shifted enrichments are adopted in the present formulation and the non-smooth displacement approximation has no contribution at each node. Hence, there is no rigid body motion in the global non-smooth approximation, and the co-rotational frame is independent of the global non-smooth approximation.

3.2 The enrichment functions

In an XFEM analysis, only those nodes supported by the elements cut by non-smoothness are enriched, and enrichments are localized by window functions [25]. Three kinds of elements are produced by the localization of enrichment: enriched elements, in which all their nodes are enriched; blending elements, in which only some of their nodes are enriched; standard elements, in which none of their nodes are enriched.

The formulation of blending element is one of the major concerns for XFEM formulation, since the partition of unity (PU) condition cannot be satisfied in blending elements and unwanted terms appear. An effective way to avoid these two issues is to use corrected XFEM formulations [26,27], in which the enrichment is multiplied by a ramp function in blending elements. Another approach is to shift the enrichments in enriched elements so that it vanishes at all nodes and is zero outside the domain of the enriched area [28,29]. The Kronecker- δ condition is another concern for XFEM formulation. A Lagrangian multiplier [23] is used in enriched elements to re-instate the Dirichlet boundary condition.

In the present formulation, shifted enrichments are adopted. Therefore, the above two issues resulting from blending elements will not affect the present XFEM formulation. Besides, the Kronecker- δ condition can be satisfied naturally at every node in the whole domain.

3.2.1 The enrichment functions for perfect pins

In the present formulation, shifted sign function [28] and abs-function [29] are used for the rotational and the translational DOF, respectively, since the rotational approximation shows a strong discontinuity and the deflection approximation shows a weak discontinuity in a perfect pin connection.

In the present study, the location of a perfect pin is prescribed in the natural coordinate system as ξ_0 ($-1 \leq \xi_0 \leq 1$). In the following derivation, the parametric location of the perfect pin is prescribed as 0.5, as shown in Fig. 6.

The shifted enrichment for the strong discontinuity (S_{pp}), as shown in Fig. 7, is of the form

$$S_{ppi} = H(\xi - \xi_0) - H(\xi_i - \xi_0) \quad (31)$$

where ξ_i is the natural coordinate of node i : $\xi_1 = -1$, $\xi_2 = 0$ and $\xi_3 = 1$. $H(\xi)$ is the sign function

$$H(\xi) = \begin{cases} 1 & \dots \xi > 0 \\ 0 & \dots \xi = 0 \\ -1 & \dots \xi < 0 \end{cases} \quad (32)$$

The Lagrangian shape functions, which are used as the window functions, are shown in Fig. 8, where element 1 and element 3 are the blending elements which are adjacent to the enriched element (element 2), $I^* = \{3, 4, 5\}$. Node 3, node 4 and node 5 are the enriched nodes, which are indicated by ■, the uneriched nodes are indicated by ● and the pin connection is indicated by ○.

The interpolation functions for the strong discontinuity are constructed by the multiplication of the Lagrangian shape functions and the strong discontinuous enrichment

$$M_{ppi} = N_i S_{ppi}, \quad i = 1, 2, 3 \quad (33)$$

The plot of the interpolation functions M_{ppi} is shown in Fig. 9.

The shifted weak discontinuous enrichment (F_{pp}) is expressed as

$$F_{pp} = \sum_{i=1}^3 N_i |\varphi_i| - \left| \sum_{i=1}^3 \varphi_i N_i \right| \quad (34)$$

In Eq. (34), φ_i is the value of the absolute-function for node i ($i = 1, 2, 3$). The enrichment for the shifted weak discontinuity F_{pp} is shown in Fig. 10.

The interpolation functions for weak discontinuity are constructed by the multiplication of the Lagrangian shape functions and the weak discontinuous enrichment

$$L_{ppi} = N_i F_{pp} \quad (35)$$

The plot of the interpolation function L_{ppi} is shown in Fig. 11.

3.2.2 The enrichment functions for plastic hinges

In the present formulation, the parametric location of the plastic hinge ($-1 \leq \xi_0 \leq 1$) and the parametric length of the plastic hinge ($0 \leq \omega_{ph} \leq 1$) are defined in the natural coordinate system as *a priori* information of the plastic hinge. An enriched element is divided into three sub-domains: the yield zone (the plastic hinge) and the two parts beside the yield zone. The parametric length of the yield zone (ω_{ph})

is dependent on the size of the enriched element (l_e) and the physical length of the plastic hinge (l_{ph}) such that

$$\omega_{ph} = \frac{l_{ph}}{l_e}, \quad 0 < l_{ph} < l_e \quad (36)$$

As shown in Fig. 12, the parametric length of the yield zone (ω_{ph}) is prescribed as 0.2 and the parametric location of the yield zone is prescribed as 0.5.

In the two elastic parts, the same enrichment functions used for a perfect pin connection are adopted for both rotational and translational displacement approximation. In the plastic part, a Hermite function is used to connect the two enrichment functions over the two elastic parts.

Regularized discontinuous enrichment is used in both rotational and translational displacement approximations for a plastic hinge. The enrichment S_{ph} for rotation approximation and the enrichment F_{ph} for translation approximation are expressed by

$$S_{ph}(\xi) = \begin{cases} H(\xi - \xi_0) - H(\xi_k - \xi_0) \cdots & \xi \geq \xi_0 + 0.5\omega_e \\ H_r & \cdots \xi_0 - 0.5\omega_e < \xi < \xi_0 + 0.5\omega_e \\ H(\xi - \xi_0) - H(\xi_k - \xi_0) \cdots & \xi \leq \xi_0 - 0.5\omega_e \end{cases} \quad (37)$$

$$F_{ph}(\xi) = \begin{cases} H_t & \cdots \xi_0 - 0.5\omega_e < \xi < \xi_0 + 0.5\omega_e \\ \sum_{k=1}^3 N_k |\varphi_k| - \left| \sum_{k=1}^3 \varphi_k N_k \right| & \cdots \text{otherwise} \end{cases} \quad (38)$$

The plots of S_{ph} and F_{ph} are shown in Fig. 13 and Fig. 14, respectively.

In Eq. (37) and (38), H_r and H_t are the Hermite functions for rotation and translation approximations in the plastic part, respectively. They are constructed within the parametric length of the yield zone by the expression

$$H_r = \mathbf{G}_r \cdot \mathbf{M} \cdot \mathbf{P} \quad (39)$$

$$H_t = \mathbf{G}_t \cdot \mathbf{M} \cdot \mathbf{P} \quad (40)$$

where $\mathbf{G}_r$ and $\mathbf{G}_t$ are the Hermite Geometry vectors for rotational and translational DOFs, respectively; $\mathbf{M}$ is the Hermite matrix and $\mathbf{P}$ is the cubic power basis vector.

$$\mathbf{G}_r = \begin{pmatrix} S_{ph}(\xi_1) & S_{ph}(\xi_2) & \frac{\partial S_{ph}(\xi_1)}{\partial \xi^*} & \frac{\partial S_{ph}(\xi_2)}{\partial \xi^*} \end{pmatrix} \quad (41)$$

$$\mathbf{G}_t = \begin{pmatrix} F_{ph}(\xi_1) & F_{ph}(\xi_2) & \frac{\partial F_{ph}(\xi_1)}{\partial \xi^*} & \frac{\partial F_{ph}(\xi_2)}{\partial \xi^*} \end{pmatrix} \quad (42)$$

$$\frac{\partial S_{ph}(\xi_i)}{\partial \xi^*} = \frac{\partial S_{ph}(\xi)}{\partial \xi} \frac{\partial \xi}{\partial \xi^*} \bigg|_{\xi=\xi_i} = \frac{\partial S_{ph}(\xi_i)}{\partial \xi} \omega_e, \quad i=1,2 \quad (43)$$

$$\frac{\partial F_{ph}(\xi_i)}{\partial \xi^*} = \frac{\partial F_{ph}(\xi)}{\partial \xi} \frac{\partial \xi}{\partial \xi^*} \bigg|_{\xi=\xi_i} = \frac{\partial F_{ph}(\xi_i)}{\partial \xi} \omega_e, \quad i=1,2 \quad (44)$$

$$\xi_1 = \xi_0 - 0.5\omega_e, \quad \xi_2 = \xi_0 + 0.5\omega_e \quad (45)$$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad (46)$$

$$\mathbf{P} = (1 \quad \xi^* \quad \xi^{*2} \quad \xi^{*3})^T \quad (47)$$

where $\xi^* \in [0,1]$ is the parent coordinate system for the yield zone.

The Lagrangian shape functions are used as window functions for the localization of the enrichment, as shown in Fig. 15, where element 1 and element 3 are the blending elements and element 2 is an enriched element so that $I^* = \{3, 4, 5\}$. The part in between the two dash lines is the yield zone. Node 3, 4 and 5 are the enriched nodes as indicated by ■. The unenriched nodes 1, 2, 6 and 7 are indicated by ●.

The interpolation functions for rotational and translational DOFs are constructed by multiplying the enrichments with the window functions.

$$M_{phi} = N_i S_{ph} \quad (48)$$

$$L_{phi} = N_i F_{ph} \quad (49)$$

The plots of the interpolation functions M_{phi} and L_{phi} are shown in Fig. 16 and Fig. 17, respectively.

3.3 The stiffness matrix and the internal force vector

Since a layered model is used in the present formulation, as shown in Fig. 5, the strain in the j^{th} layer in an enriched element $\mathbf{\epsilon}_{enrj}$ is given by Eq. (50), in which the three strain components ϵ_{menr} , γ_{enr} and χ_{enr} should be modified as follows:

$$\boldsymbol{\varepsilon}_{\text{enr}j} = \begin{Bmatrix} \varepsilon_{\text{menr}} \\ \gamma_{\text{enr}} \end{Bmatrix} + z_j \begin{Bmatrix} \chi_{\text{enr}} \\ 0 \end{Bmatrix} \quad (50)$$

$$\varepsilon_{\text{menr}} = \varepsilon_{\text{ms}} + \varepsilon_{\text{mns}} = \frac{\partial u}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial b_v}{\partial x} \right)^2 \right] \quad (51)$$

$$\gamma_{\text{enr}} = \gamma_s + \gamma_{\text{ns}} = -\theta - a + \frac{\partial v}{\partial x} + \frac{\partial b_v}{\partial x} \quad (52)$$

$$\chi_{\text{enr}} = \chi_s + \chi_{\text{ns}} = -\frac{\partial \theta}{\partial x} - \frac{\partial a}{\partial x} \quad (53)$$

where ε_{ms} , γ_s and χ_s are, respectively, the membrane strain, shear strain and curvature due to smooth displacement. ε_{mns} , γ_{ns} and χ_{ns} are the membrane strain, shear strain and curvature due to non-smooth displacement. The strain of an enriched element can be expressed in the matrix form:

$$\varepsilon_{\text{menr}} = \mathbf{B}_{\text{menr}} \mathbf{u}_{\text{enr}}, \quad \gamma_{\text{enr}} = \mathbf{B}_{\gamma\text{enr}} \mathbf{u}_{\text{enr}}, \quad \chi_{\text{enr}} = \mathbf{B}_{\text{benr}} \mathbf{u}_{\text{enr}} \quad (54)$$

where $\mathbf{u}_{\text{enr}}$ is the local displacement vector for the XFEM element, $\mathbf{B}_{\text{menr}}$, $\mathbf{B}_{\gamma\text{enr}}$, and $\mathbf{B}_{\text{benr}}$ are the strain-displacement matrices for the membrane, the shear and the bending parts in the enriched element. Detailed expressions of $\mathbf{B}_{\text{menr}}$, $\mathbf{B}_{\gamma\text{enr}}$, and $\mathbf{B}_{\text{benr}}$ are listed in Appendix A.

The internal force vector $\mathbf{f}_L$ is defined as

$$\mathbf{f}_L = (\mathbf{f}_{\text{sL}1}^T, \mathbf{f}_{\text{nsL}1}^T, \mathbf{f}_{\text{nsL}2}^T, \mathbf{f}_{\text{sL}3}^T, \mathbf{f}_{\text{nsL}3}^T)^T \quad (55)$$

where $\mathbf{f}_{\text{sL}i}$ ($i = 1, 3$) and $\mathbf{f}_{\text{nsL}j}$ ($j = 1, 2, 3$) are the internal force vectors corresponding to $\mathbf{u}_{\text{sL}i} = (u_i, v_i, \theta_i)^T$ and $\mathbf{u}_{\text{nsL}j} = (a_j, b_{uj}, b_{vj})^T$, respectively. The stiffness matrix of the enriched element $\mathbf{K}_{\text{enr}}$ is the Hessian matrix of the potential energy Π :

$$\mathbf{K}_{\text{enr}} = \sum_{j=1}^n \int (\mathbf{B}_{\text{menr}} + z_j \mathbf{B}_{\text{benr}})^T \mathbf{D}_{11} (\mathbf{B}_{\text{menr}} + z_j \mathbf{B}_{\text{benr}}) + \mathbf{B}_{\gamma\text{enr}}^T \mathbf{D}_{22} \mathbf{B}_{\gamma\text{enr}} dV \quad (56)$$

It should be mentioned that, as shown in Appendix B, the stiffness matrix of the enriched element $\mathbf{K}_{\text{enr}}$ is *symmetric*.

The internal force and the stiffness of the enriched element in the global coordinate system are the Jacobian matrix and the Hessian matrix of the potential energy Π with respect to the global displacement. By using chain rule, they can be obtained as:

$$\mathbf{f}_G = \frac{\partial \Pi}{\partial \mathbf{U}_{\text{enr}}^T} = \mathbf{T}^T \mathbf{f}_L \quad (57)$$

$$\mathbf{K}_G = \frac{\partial \mathbf{f}_G}{\partial \mathbf{U}_{\text{enr}}^T} = \mathbf{T}^T \mathbf{K}_L \mathbf{T} + \frac{\partial \mathbf{T}^T}{\partial \mathbf{U}_{\text{enr}}} \mathbf{f}_L \quad (58)$$

$$\mathbf{T} = \frac{\partial \mathbf{u}_{\text{enr}}}{\partial \mathbf{U}_{\text{enr}}^T} \quad (59)$$

where $\mathbf{T}_{\text{enr}}$ is the transformation matrix for an enriched element.

3.4 Implementation of the enriched elements

In the present XFEM elements, the integration of stiffness and internal force are conducted in sub-domains. For each sub-domain, the nine-point Gaussian integration scheme is employed.

If the location of a plastic hinge or a pin is located at the end of an element, its location would be shifted inwards the element by a parametric distance of 0.01, to ensure that the discontinuity is entirely located within one element only.

In the analyses incorporating plastic hinge formations, the enrichments are embedded into the element formulation. As the nonlinear analysis continues, the XFEM formulation will be activated when a section within an element is detected to be yielded. Therefore, at the beginning of the analysis, there is no need to determine where to introduce the enrichments into the analysis and the mapping of the stress, strain and plastic strain to the new Gaussian points from the previous Gaussian points can be avoided.

In the numerical simulation on a simple beam structure, a pin connection and a plastic hinge only affect one element in the numerical model. The additional DOFs can be regarded as extra internal variables and they could be eliminated in local coordinate systems by static condensation so that the transformation matrix $\mathbf{T}$ in Eq. (59) does not need any modification even when the XFEM formulation is embedded. However, the additional DOFs cannot be expressed explicitly in the global coordinate system if static condensation is employed. In the present formulation, the additional DOFs are kept in the global coordinate system so that the effects of the additional DOF can be shown explicitly.

4 Numerical examples

4.1 Beam bending examples with pin connection

In this section, two examples are tested with the same cross-section (UB section W12-12×12, 253.0 kg/m) and loading condition (a point load is applied at the middle of the beam), but with different boundary conditions and locations of the internal pin. The geometrical properties of UB section W12-12×12 (253.0 kg/m) are shown in Fig. 18: $D = 356.4$ mm, $B = 319.3$ mm, $T = 39.6$ mm, $t = 24.4$ mm. This cross-section is a Class 1 section according to Eurocode 3 [30], in which a full plastic hinge could be formed with large rotational capacity without local buckling. The area of the cross-section A is equal to 323 cm². The second moment of area I_{xx} is equal to 68494 cm⁴. The reference point load F_0 applied is 2.0262×10^5 kN. The material properties for the beams are: Young's modulus $E = 210$ GPa, shear modulus $G = 80.77$ GPa.

A pin is placed in the fixed ended beam in the two examples at different locations. The deformed configuration and rotation approximation are compared with the result from the ANSYS's beam3 element [31]. Master and slave DOF technique is used in the ANSYS analyses, which requires meshes to conform to the pin of the beams. Five elements with equal length are used in the XFEM analyses (1.2 m $\times$ 5) for each example. For both examples with the internal pin, two formulations are tested, viz. the formulation with both strong and weak discontinuities (S+W) and with strong discontinuity only (S).

4.1.1 Example 1: A fixed-fixed beam with an internal pin at the one-third point

In this example, a pin is located at the one-third point of the beam, as shown in Fig. 19, with the parametric location of the pin $\xi_0 = 0.33$. The equilibrium path (the deflection at the loading point (w) versus the loading factor (λ)) is shown in Fig. 20. The rotation and the deflection along the beam are shown in Fig. 21(a) and (b), respectively.

As shown in Fig. 21(a), the discontinuity in rotation cannot be captured if only the strong discontinuity is considered, while a combination of strong and weak enrichments can capture the behaviour of the discontinuity accurately.

4.1.2 Example 2: A fixed-fixed beam with an internal pin at the middle point

A pin is located at the middle of the beam, as shown in Fig. 22, with the parametric location of the pin $\xi_0 = 0.0$. The equilibrium path (the deflection at the loading point (w) versus the loading factor (λ)) is shown in Fig. 23. The rotation and the deflection along the beam are shown in Fig. 24(a) and (b), respectively.

In this example, as shown in Fig. 23 and Fig. 24, the formulation with both strong and weak discontinuous enrichments can capture the discontinuous phenomenon in rotation and deflection adequately. However, the formulation with strong discontinuity only can locate the jump in rotation but not the kink in deflection.

From the results obtained in Examples 1 and 2, it can be seen that the XFEM formulation worked well in the analyses involving internal pins if both strong and weak discontinuities are simultaneously added to the rotational and translational approximation fields.

4.2 Beam bending examples with plastic hinges

In the following three examples the UB section W12-12×12 used in Section 4.1 is re-used again. The plastic modulus and the elastic modulus of this UB section, with respect to the major principal axis are: $S_{xx} = 3844 \text{ cm}^3$ and $Z_{xx} = 4502 \text{ cm}^3$. The yield strength of steel is $\sigma_0 = 235 \text{ MPa}$. The maximum plastic moment is $M_u = Z_{xx} \times \sigma_0 = 1.058 \times 10^3 \text{ kNm}$.

Three cases are tested for each example in order to show the advantages of the proposed XFEM formulation.

Case one: five standard elements with equal length ($1.2 \text{ m} \times 5$).

Case two: five elements with equal length and with enrichments where necessary ($1.2 \text{ m} \times 5$). All the parametric lengths of the yielded zones (ω_{ph} in Eq. 36) are assumed to be 0.01.

Case three: thirty standard elements with equal length ($0.2 \text{ m} \times 30$).

Since in Case one a coarse mesh with standard elements is used, it is expected that it would predict a higher plastic load when comparing with predictions from Case three. Furthermore, since a very fine mesh is used in Case three, the results obtained could be considered as very close to the beam theory limit. The mesh scheme of Case two is the same as that in Case one. However, since the XFEM

formulation is embedded in Case two, the result from Case two should be closer to the beam theory limit.

4.2.1 Example 3: A pin-pin supported beam applied by a mid-point load

A pin-pin supported beam, as shown in Fig. 25, is tested in this example. The beam is applied by a point load ($F_0 = 500$ kN) at the mid-point. According to elementary beam theory, the initial yielding occurs when the load factor λ_0 is reached 1.20 and the ultimate load factor λ_u is 1.41. The deflection at the mid-point (w) versus the loading factor (λ) is plotted for the three cases as shown in Fig. 26. The deflection and the rotation fields when the deflection of the mid-point is equal to 0.08 m are shown in Fig. 27(a) and (b), respectively. From Figs. 26 and 27, it is obvious that the proposed XFEM outperformed the standard FE method with the predicted plastic load and deflected shape of the beam close to the beam theory limit.

4.2.2 Example 4: A clamped beam applied by a mid-point load

A fully clamped beam is tested, as shown in Fig. 28, with a point load ($F_0 = 500$ kN) applied at the mid-point. According to beam theory, the initial yielding occurs at when the load factor λ_0 is reached 2.41 and the ultimate load factor λ_u is 2.81. The mid-point (w) deflection versus the load factor (λ) is plotted for the three cases as shown in Fig. 29. The deflection and the rotation fields when the mid-span deflection is equal to 0.12m are shown in Fig. 30(a) and (b), respectively. Again, from Figs. 28 and 29, the proposed XFEM clearly shows its advantages when comparing with the standard FE method.

4.2.3 Example 5: A clamped beam applied by uniformly distributed loading (UDL)

As shown in Fig. 31, a fully clamped beam subjected to a UDL of intensity $q_0 = 300$ kN/m is tested. According to beam theory, initial yielding occurs when the load factor λ_0 is reached 1.00 and the ultimate load factor λ_u is 1.57. The mid-point deflection (w) versus load factor (λ) is plotted for the three cases as shown in Fig. 32. The deflection and the rotation fields corresponding to a mid-span deflection of 0.12m are shown in Fig. 33(a) and (b), respectively. Again, the XFEM is able to produce results close to the beam theory limit with a coarse mesh.

From Example 3 to Example 5, from the load factor versus w plots, it can be seen that the equilibrium path goes up in the late stage of the analyses. This could be due to the selection of the enrichment function in the yield zone. As the XFEM is a numerical method based on the *a priori* knowledge of the behaviour of a local discontinuity, the selection of the enrichment affects the predictions of the analyses. Unlike the cases of perfect pins, in which the lengths of the pins can be idealised as zero and the displacement approximation could be easily predicted, it is more difficult to approximate the width of a yield zone. The shape of the displacement approximation depends on many factors, such as the cross-section properties, the shape of the edges of the plastic hinge and the length of the plastic zone. In bending-dominant cases, extreme fibres yield much earlier than fibres in the middle part and the proportion the yielded part to the elastic part in the yield section varies with the cross-section shape. For example, this proportion for an I-section is higher than that for a rectangular section. As the beam is further loaded, the length of the yield zone increases, the neutral axis shifts upwards, some compressive extreme fibres are unloaded to tensile state, and the shape of a plastic hinge changes. As shown in Fig. 1b), where the shaded area is a plastic hinge, the shape of the edges of the plastic hinge on the both sides may not be symmetric, and it depends on the stiffness of the parts and the stress states on each fibre on both sides of the plastic hinge, which would make the approximation more complex.

4.3 A study on the parametric length of yield zone

In this section, Example 5 in Section 4.2 is reused. Four meshes are employed to investigate the influence of the *parametric length* of the yield zone (ω_{ph} Eq. (36)) at the mid-span of the beam on the numerical results. For the four meshes tested, as shown in Fig. 34, the physical length of the yielded zone is fixed ($l_{ph} = D = 0.356$ m). Therefore, the parametric length of the yield zone varies as the length of the element changes. Five elements are used in mesh 2, mesh 3 and mesh 4, and three elements are used in mesh 1. The lengths of the elements used in the meshes are listed in Table 2.

The results from the four meshes are shown in Fig. 35 and they are compared with the results obtained by using 1801 solid elements (SOLID 95) in ANSYS [31] which is close to the solid element limit.

It can be seen that the parametric length of the yield zone did not affect the results of mesh 1 to mesh 3, which are all close to the results obtained from a fine mesh scheme with standard beam elements. It should also be mentioned that ill-conditioning of the stiffness matrix was found in mesh 4 where the parametric length of the yield zone is 0.89 which was, in fact corresponding to a near failure of one element in practice. From this study, a value in between 0.01 and 0.6 is recommended for the parametric length of the yield zone in the present formulation.

4.4 Study on the physical length of yield zone

In this section, Example 5 in Section 4.2.3 is reused to study the effects of the physical length of the yield zone. In this case, the mesh scheme is fixed but different physical lengths of the yield zone (l_{ph} , Eq. (36)) at the mid-span of the beam are used. For all the six cases, 5 XFEM elements with equal length are used. The physical lengths of the yield zone are listed in Table 3.

The results from the four meshes are shown in Fig. 36, compared with the results from 1801 solid elements in ANSYS (SOLID 95) [31]. It can be seen that the results from Case 1 to Case 4 show little differences and are close to the fine mesh results by standard beam elements which can be regarded as the Timoshenko beam theory limit.

5 Conclusions and discussions

A two dimensional co-rotational beam element with an XFEM formulation is presented. Discontinuous enrichments for strong and weak discontinuities are adopted simultaneously to simulate perfect internal pin connections. Regularized discontinuous enrichments are employed to capture behaviour of plastic hinges. Since shifted enrichment functions are used in the present element, several advantages can be seen. Firstly, the PU condition can be satisfied in blending elements. Secondly, no unwanted term appears in blending elements. Thirdly, the discontinuous approximation vanishes at each node, so that the co-rotational frame is independent of the additional DOFs. Meanwhile, it is shown that discontinuous displacement has no contribution to rigid body motion. Finally, the Kronecker- δ condition can be satisfied in enriched elements, so that no special

treatment is needed when applying the Dirichlet boundary condition an enriched element.

In the present study, the Hermite functions, which are C_1 continuous polynomial functions, are adopted as the enrichment to simulate rotational and translational approximation in the yield zone. The Gaussian integration scheme is still valid for this enrichment and no special technique on numerical integration is required. However, this selection of the enrichment may not be sufficiently accurate for simulating beams or columns subjected to more complex loading and boundary conditions. Therefore, the selection of the enrichment in the yield zone requires further in-depth study.

Appendix A

The strain-displacement matrices for the membrane part $\mathbf{B}_m$, the shear part $\mathbf{B}_\gamma$ and the bending $\mathbf{B}_b$ for an unenriched element are expressed as:

$$\mathbf{B}_m = [\mathbf{B}_{m1} \quad \mathbf{B}_{m3}], \quad \mathbf{B}_\gamma = [\mathbf{B}_{\gamma1} \quad \mathbf{B}_{\gamma3}], \quad \mathbf{B}_b = [\mathbf{B}_{b1} \quad \mathbf{B}_{b3}] \quad (\text{A.1})$$

$$\mathbf{B}_{mi} = \left[\left(1 + \frac{\partial u}{\partial x} \right) \frac{\partial N_i}{\partial x} \quad \frac{\partial v_h}{\partial x} \frac{\partial N_i}{\partial x} \quad 0 \right], \quad i = 1, 3 \quad (\text{A.2})$$

$$\mathbf{B}_{\gamma i} = \left[0 \quad \frac{\partial N_i}{\partial x} \quad -N_i \right], \quad i = 1, 3 \quad (\text{A.3})$$

$$\mathbf{B}_{bi} = \left[0 \quad 0 \quad -\frac{\partial N_i}{\partial x} \right], \quad i = 1, 3 \quad (\text{A.4})$$

In an enriched element, the strain-displacement matrices for the membrane part $\mathbf{B}_{menr}$, the shear part $\mathbf{B}_{\gamma enr}$ and the bending part $\mathbf{B}_{benr}$ for an enriched element are expressed as:

$$\mathbf{B}_{menr} = [\mathbf{B}_{ms1}, \mathbf{B}_{mns1}, | \mathbf{B}_{mns2} | \mathbf{B}_{ms3}, \mathbf{B}_{mns3}] \quad (\text{A.5})$$

$$\mathbf{B}_{\gamma enr} = [\mathbf{B}_{\gamma s1}, \mathbf{B}_{\gamma ns1}, | \mathbf{B}_{\gamma ns2} | \mathbf{B}_{\gamma s3}, \mathbf{B}_{\gamma ns3}] \quad (\text{A.6})$$

$$\mathbf{B}_{benr} = [\mathbf{B}_{bs1}, \mathbf{B}_{bns1}, | \mathbf{B}_{bns2} | \mathbf{B}_{bs3}, \mathbf{B}_{bns3}] \quad (\text{A.7})$$

where $\mathbf{B}_{msi}$, $\mathbf{B}_{\gamma si}$ and $\mathbf{B}_{bsi}$ ($i = 1, 3$) are the sub-matrices corresponding to the smooth displacement. $\mathbf{B}_{mnsj}$, $\mathbf{B}_{\gamma nsj}$ and $\mathbf{B}_{bnsj}$ ($j = 1, 2, 3$) are corresponding to the non-smooth displacement. They can be expressed as:

$$\mathbf{B}_{msi} = [\mathbf{B}_{mi} \quad \mathbf{0}_{1 \times 3}], \quad i = 1, 3 \quad (\text{A.8})$$

$$\mathbf{B}_{\gamma si} = [\mathbf{B}_{\gamma i} \quad \mathbf{0}_{1 \times 3}], \quad i = 1, 3 \quad (\text{A.9})$$

$$\mathbf{B}_{bsi} = [\mathbf{B}_{bi} \quad \mathbf{0}_{1 \times 3}], \quad i = 1, 3 \quad (\text{A.10})$$

$$\mathbf{B}_{mnsj} = \begin{bmatrix} \mathbf{0}_{1 \times 3} & 0 & 0 & \frac{\partial v_h}{\partial x} \frac{\partial L_j}{\partial x} \end{bmatrix}, \quad j = 1, 3; \quad \mathbf{B}_{mns2} = \begin{bmatrix} 0 & 0 & \frac{\partial v_h}{\partial x} \frac{\partial L_2}{\partial x} \end{bmatrix} \quad (\text{A.11})$$

$$\mathbf{B}_{\gamma nsj} = \begin{bmatrix} \mathbf{0}_{1 \times 3} & -M_j & 0 & \frac{\partial L_j}{\partial x} \end{bmatrix}, \quad j = 1, 3; \quad \mathbf{B}_{\gamma ns2} = \begin{bmatrix} -M_2 & 0 & \frac{\partial L_2}{\partial x} \end{bmatrix} \quad (\text{A.12})$$

$$\mathbf{B}_{bnsj} = \begin{bmatrix} \mathbf{0}_{1 \times 3} & -\frac{\partial M_j}{\partial x} & 0 & 0 \end{bmatrix}, \quad j = 1, 3; \quad \mathbf{B}_{bns2} = \begin{bmatrix} -\frac{\partial M_2}{\partial x} & 0 & 0 \end{bmatrix} \quad (\text{A.13})$$

Appendix B

The potential energy in an enriched element is expressed as

$$\Pi = \frac{1}{2} \sum_{j=1}^n \int \left[(\varepsilon_{menr} + z_j \chi_{enr}) D_{11} (\varepsilon_{menr} + z_j \chi_{enr}) + \gamma_{enr} D_{22} \gamma_{enr} \right] dV - W \quad (\text{B.1})$$

According to the principle of stationary potential energy, the variation of the potential energy with respect to the enriched nodal displacement is zero:

$$\begin{aligned} \delta \Pi = & \sum_{j=1}^n \int \left[(\varepsilon_{ms} + z_j \chi_s) D_{11} (\mathbf{B}_{menr} + z_j \mathbf{B}_{benr}) + \gamma_s D_{22} \mathbf{B}_{\gamma enr} \right] dV \delta \mathbf{u}_{enr} \\ & + \sum_{j=1}^n \int \left[(\varepsilon_{mns} + z_j \chi_{ns}) D_{11} (\mathbf{B}_{menr} + z_j \mathbf{B}_{benr}) + \gamma_{ns} D_{22} \mathbf{B}_{\gamma enr} \right] dV \delta \mathbf{u}_{enr} - \mathbf{f}_L^T \delta \mathbf{u}_{enr} = 0 \quad (\text{B.2}) \end{aligned}$$

where $\mathbf{f}_L$ (Eq. (55)) is the internal force in the local coordinate systems. In Eq. (55), $\mathbf{f}_{sLi}$ ($i = 1, 3$) and $\mathbf{f}_{nsLj}$ ($j = 1, 2, 3$) can be expressed as

$$\mathbf{f}_{sLi} = \sum_{k=1}^n \int (\mathbf{B}_{menr} + z_k \mathbf{B}_{benr})^T D_{11} (\varepsilon_{ms} + z_k \chi_s) + \mathbf{B}_{\gamma enr}^T D_{22} \gamma_s dV \quad (\text{B.3})$$

$$\mathbf{f}_{nsLj} = \sum_{k=1}^n \int \left[(\mathbf{B}_{menr} + z_k \mathbf{B}_{benr})^T D_{11} (\varepsilon_{mns} + z_k \chi_{ns}) + \mathbf{B}_{\gamma enr}^T D_{22} \gamma_{ns} \right] dV \quad (\text{B.4})$$

The stiffness matrix of an enriched element $\mathbf{K}_{enr}$ can be expressed as

$$\mathbf{K}_{enr} = [\mathbf{K}_{ij, enr}] = \begin{bmatrix} \mathbf{K}_{ij, s-s} & \mathbf{K}_{ij, s-ns} \\ \mathbf{K}_{ij, ns-s} & \mathbf{K}_{ij, ns-ns} \end{bmatrix}, \quad (i, j = 1, 2, 3) \quad (\text{B.5})$$

where $\mathbf{K}_{ij,s-s}$, $\mathbf{K}_{ij,s-ns}$, $\mathbf{K}_{ij,ns-s}$ and $\mathbf{K}_{ij,ns-ns}$ can be expressed as

$$\mathbf{K}_{ij,s-s} = \sum_{k=1}^n \int (\mathbf{B}_{msi} + z_k \mathbf{B}_{bsi})^T \mathbf{D}_{11} (\mathbf{B}_{msj} + z_k \mathbf{B}_{bsj}) + \mathbf{B}_{ysi}^T \mathbf{D}_{22} \mathbf{B}_{ysj} dV \quad (\text{B.6})$$

$$\mathbf{K}_{ij,s-ns} = \sum_{k=1}^n \int (\mathbf{B}_{msi} + z_k \mathbf{B}_{bsi})^T \mathbf{D}_{11} (\mathbf{B}_{mnsj} + z_k \mathbf{B}_{bnsj}) + \mathbf{B}_{ysi}^T \mathbf{D}_{22} \mathbf{B}_{ynsj} dV \quad (\text{B.7})$$

$$\mathbf{K}_{ij,ns-s} = \sum_{k=1}^n \int (\mathbf{B}_{mnsi} + z_k \mathbf{B}_{bnsi})^T \mathbf{D}_{11} (\mathbf{B}_{msj} + z_k \mathbf{B}_{bsj}) + \mathbf{B}_{ynsi}^T \mathbf{D}_{22} \mathbf{B}_{ynsj} dV \quad (\text{B.8})$$

$$\mathbf{K}_{ij,ns-ns} = \sum_{k=1}^n \int (\mathbf{B}_{mnsi} + z_k \mathbf{B}_{bnsi})^T \mathbf{D}_{11} (\mathbf{B}_{mnsj} + z_k \mathbf{B}_{bnsj}) + \mathbf{B}_{ynsi}^T \mathbf{D}_{22} \mathbf{B}_{ynsj} dV \quad (\text{B.9})$$

It can be seen that the sub-matrix $\mathbf{K}_{ij,s-s}$ is the same as the stiffness matrix for an ordinary element, in which $\mathbf{K}_{ij,s-s} = \mathbf{K}_{ji,s-s}^T$ and $\mathbf{K}_{ii,s-s}$ is symmetric. Since $\mathbf{K}_{ij,s-ns} = \mathbf{K}_{ji,ns-s}^T$, $\mathbf{K}_{ij,ns-ns} = \mathbf{K}_{ji,ns-ns}^T$ and $\mathbf{K}_{ii,ns-ns}$ is symmetric. Thus, the stiffness matrix of an enriched element $\mathbf{K}_{\text{enr}}$ is *symmetric*.

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Fig. 1 Two kinds of non-smoothness in a beam

Fig. 2 Co-rotational approach for a 2D beam element

Fig. 3 The natural coordinate system

Fig. 4 The local coordinate system

Fig. 5 The layered model

(a) An internal pin inside an element

(b) Definition of the pin in the natural coordinate system

Fig. 6 The location of a pin

(a) S_{pp1} and S_{pp2} (b) S_{pp3} **Fig. 7** The enrichment function S_{ppi} for the rotational DOFs in pin connection **Fig. 8** The window functions for an enriched element with a perfect pin

Fig. 9 The interpolation shape function M_{ppi}

Fig. 10 The enrichment F_{pp} for translational DOFs in a pin connection

Fig. 11 The interpolation function L_{ppi}

(a) A plastic hinge inside an element

(b) Definition of plastic hinge in the natural coordinate system

Fig. 12 An element with a plastic hinge

Fig. 13 The enrichment function S_{ph} for the rotational DOFs in a plastic hinge

Fig. 14 The enrichment function F_{ph} for the translational DOF in a plastic hinge

Fig. 15 The window functions for enriched element with a plastic hinge

Fig. 16 The interpolation functions M_{phi}

Fig. 17 The interpolation functions L_{ϕ}

Fig. 18 The cross-section for the examples

Fig. 19 A fixed-fixed beam with a pin at the one-third point

Fig. 20 Equilibrium paths obtained for Example 1

(a) Rotation

Fig. 21 The displacement fields for Example 1

Fig. 22 A fixed-fixed beam with a pin at the middle point

Fig. 23 Equilibrium paths obtained for Example 2

(a) The rotation field

(b) The deflection field

Fig. 24 Displacement fields for Example 2 **Fig. 25** A pin-pin supported beam applied by a mid-point load (elasto-plastic)

Fig. 26 Equilibrium paths obtained for Example 3

(a) The deflection field

(b) The rotation field

Fig. 27 Displacement fields for Example 3 **Fig. 28** A clamped beam applied by a mid-point load (elasto-plastic) **Fig. 29** Equilibrium paths obtained for Example 4

(a) The deflection field

(b) The rotation field

Fig. 30 Displacement fields for Example 4

Fig. 31 A clamped beam applied by UDL (elasto-plastic)

Fig. 32 Equilibrium paths obtained for Example 5

(a) The deflection field

(b) The rotation field

Fig. 33 Displacement fields for Example 5

Fig. 34 The four meshes used in the study of effects of different parametric lengths of the yield zone

Fig. 35 Comparison of results obtained from meshes with different parametric lengths of the yield zone

Fig. 36. Comparison of results obtained from meshes with different physical lengths

Table 1 Symbols

A_b, a_i	Additional DOF for rotation	$\mathbf{U}_i, \mathbf{u}_i$	Displacement vector for a node in the global and the local coordinate systems
B_b, b_i	Additional DOF for translation	$\mathbf{U}^{\text{enr}}, \mathbf{u}^{\text{enr}}$	Displacement vector for an enriched element in the global and the local coordinate systems
$\mathbf{B}$	Strain-displacement matrix	$\mathbf{U}, \mathbf{u}$	Displacement vector for an element in the global and the local coordinate systems
$\mathbf{D}$	Material matrix	$\mathbf{U}_h, \mathbf{u}_h$	Displacement approximation in the global and the local coordinate systems
$\mathbf{e}_x, \mathbf{e}_y$	The x-axis and the y-axis of the local coordinate system	V	Volume of an element
E	Young's modulus	W	Work done by the external force
$\mathbf{f}$	Internal force vector	V_i, v_i	Translational DOF along the Y-axis and the y-axis
$F_{\text{ph}}, F_{\text{pp}}$	Enrichment for translational DOF	$\mathbf{X}$	The position vector
G	Shear modulus	z_j	Distance between the mid-surface of the j^{th} layer and the reference surface of the element
G_r, G_t	Hermite Geometric vector	γ	Shear strain
H	Sign function	$\boldsymbol{\varepsilon}$	Strain vector
H_r, H_t	Enrichment in the yielded zone	ε_m	Membrane strain
I	The nodal set of the whole domain	θ_i	Rotational DOF in the local coordinate system
I^*	The nodal set containing all the enriched nodes	λ	Loading factor
k	Shear correction factor (=5/6)	ζ	Natural coordinate of the element
$\mathbf{K}$	Stiffness matrix	ζ_0	Parametric location of a perfect pin or a plastic hinge
l_e	Physical length of an element	ζ_1, ζ_2	Parametric locations of the two edges of a plastic hinge
l_{ph}	The physical length of a plastic	σ	stress

	hinge		
$L_{ph,i}$ $L_{pp,i}$	Interpolation functions for additional translational DOF	σ_0	Yield strength
$M_{ph,i}$ $M_{pp,i}$	Interpolation functions for additional rotational DOF	φ	Distance from a point to the location of a perfect pin or the centre of a plastic hinge
N_i	The Lagrangian shape functions	χ	Bending strain
$\mathbf{P}$	The cubic power basis vector	ψ_i	Rotational DOF in the global coordinate system
$\mathbf{R}$	The rotation matrix	ω_{ph}	Parametric length of a plastic hinge
S_{ph}, S_{pp}	Enrichments for rotational DOF	ω_e	Parametric length of an element (=2.0)
$\mathbf{T}$	Transformation matrix	Π	Strain energy
U_i, u_i	Translational DOFs along the X-axis and the x-axis		

Table 2

The four meshes employed to study the effect of parametric length of the yield zone

	Length of each element	ω_{ph}
Mesh 1	2 m + 2 m + 2 m	0.178
Mesh 2	1.2 m + 1.2 m + 1.2 m + 1.2 m + 1.2 m	0.29667
Mesh 3	1.2 m + 1.5 m + 0.6 m + 1.5 m + 1.2 m	0.59333
Mesh 4	1.2 m + 1.6 m + 0.4 m + 1.6 m + 1.2 m	0.89

Table 3

Six cases with different physical lengths of the yield zone

	$l_{ph}(m)$	ω_{ph}	l_{ph}/D
Case 1	0.012	0.01	0.034
Case 2	0.06	0.05	0.168
Case 3	0.12	0.1	0.337
Case 4	0.6	0.5	1.684

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89x42mm (300 x 300 DPI)

For Peer Review

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