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**PURE ASYMMETRIC QUANTUM MDS CODES
FROM CSS CONSTRUCTION:
A COMPLETE CHARACTERIZATION**

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Using the Calderbank-Shor-Steane (CSS) construction, pure q -ary asymmetric quantum error-correcting codes attaining the quantum Singleton bound are constructed. Such codes are called pure CSS asymmetric quantum maximum distance separable (AQMDS) codes. Assuming the validity of the classical MDS Conjecture, pure CSS AQMDS codes of all possible parameters are accounted for.

Keywords: asymmetric quantum codes, MDS codes, Singleton bound, Generalized Reed-Solomon codes, weight distribution

1. Introduction

The study of *asymmetric quantum codes* (AQCs) began when it was argued that, in many qubit systems, phase-flips (or Z-errors) occur more frequently than bit-flips (or X-errors) do [4, 8]. Steane first hinted the idea of adjusting the error-correction to the particular characteristics of the quantum channel in [13] and later, Wang *et al.* established a mathematical model of AQCs in the general qudit system in [14].

To date, the only known class of AQC's is given by the asymmetric version of the CSS construction [1, 14]. In this paper, the CSS construction is used to derive a class of pure^a AQC's attaining the quantum analogue of the Singleton bound. We call such optimal codes *asymmetric quantum maximum distance separable (AQMDS)* codes and if the codes are derived from the CSS construction, we call them CSS AQMDS codes.

Thus far, the only known AQMDS codes are pure CSS AQMDS and many results concerning these codes had been discussed in [6]. This paper provides a complete treatment of such codes by solving the remaining open problems. This enables us to provide a complete characterization. To be precise, assuming the validity of the MDS conjecture, pure CSS AQMDS codes of all possible parameters are constructed.

The paper is organized as follows. In Section 2, we discuss some preliminary concepts and results. In Sections 3 to 5, nested pairs of Generalized Reed-Solomon (GRS) codes and extended GRS codes are used to derive AQMDS codes of lengths up to $q + 2$. Section 6 presents an alternative view on the construction of AQMDS codes based on the weights of MDS codes. A summary is provided in Section 7.

2. Preliminaries

2.1. Classical linear MDS codes

Let q be a prime power and $\mathbb{F}_q$ the finite field having q elements. A *linear* $[n, k, d]_q$ -code C is a k -dimensional $\mathbb{F}_q$ -subspace of $\mathbb{F}_q^n$ with *minimum distance* $d := \min\{\text{wt}(\mathbf{v}) : \mathbf{v} \in C \setminus \{\mathbf{0}\}\}$, where $\text{wt}(\mathbf{v})$ denotes the *Hamming weight* of $\mathbf{v} \in \mathbb{F}_q^n$. Given two distinct linear codes C and D , $\text{wt}(C \setminus D)$ denotes $\min\{\text{wt}(\mathbf{u}) : \mathbf{u} \in C \setminus D\}$.

Every $[n, k, d]_q$ -code C satisfies the Singleton bound

$$d \leq n - k + 1,$$

and C is said to be *maximum distance separable (MDS)* if $d = n - k + 1$. Trivial families of MDS codes include the vector space $\mathbb{F}_q^n$, the codes equivalent to the $[n, 1, n]_q$ -repetition code and their duals $[n, n - 1, 2]_q$ for positive integers $n \geq 2$.

MDS codes which are not equivalent to the trivial ones are said to be *nontrivial*. Furthermore, we have the following conjecture which has been shown to be true when q is prime in [2].

Conjecture 1 (MDS Conjecture). If there is a nontrivial $[n, k, d]_q$ -MDS code, then $n \leq q + 1$, except when q is even and $k = 3$ or $k = q - 1$ in which case $n \leq q + 2$.

For $\mathbf{u} = (u_i)_{i=1}^n$ and $\mathbf{v} = (v_i)_{i=1}^n$, $\langle \mathbf{u}, \mathbf{v} \rangle_E := \sum_{i=1}^n u_i v_i$ is the *Euclidean inner product* of $\mathbf{u}$ and $\mathbf{v}$. With respect to this inner product, the *dual* $C^\perp$ of C is given by

$$C^\perp := \{\mathbf{u} \in \mathbb{F}_q^n : \langle \mathbf{u}, \mathbf{v} \rangle_E = 0 \text{ for all } \mathbf{v} \in C\}.$$

^aPurity in the CSS case is defined in Theorem 2

It is well known that $(C^\perp)^\perp = C$ and that the dual of an MDS code is MDS.

Let $\mathbb{F}_q[X]_k$ denote the set of all polynomials of degree less than k in $\mathbb{F}_q[X]$. The set $\{1, x, \dots, x^{k-1}\}$ forms the standard basis for $\mathbb{F}_q[X]_k$ as a vector space over $\mathbb{F}_q$.

2.2. CSS construction and AQMDS codes

We begin with a formal definition of an AQC.

Definition 1. Let d_x and d_z be positive integers. A quantum code Q in $V_n = (\mathbb{C}^q)^{\otimes n}$ with dimension $K \geq 1$ is called an *asymmetric quantum code* with parameters $((n, K, d_z/d_x))_q$ or $[[n, k, d_z/d_x]]_q$, where $k = \log_q K$, if Q detects $d_x - 1$ qudits of bit-flips (or X -errors) and, at the same time, $d_z - 1$ qudits of phase-flips (or Z -errors).

The correspondence between pairs of classical linear codes and AQCs is given in [1, 14].

Theorem 2 (Standard CSS Construction for AQC). Let C_i be linear codes with parameters $[n, k_i, d_i]_q$ for $i = 1, 2$ with $C_1^\perp \subseteq C_2$. Let

$$\begin{aligned} d_z &:= \max\{\text{wt}(C_2 \setminus C_1^\perp), \text{wt}(C_1 \setminus C_2^\perp)\} \text{ and} \\ d_x &:= \min\{\text{wt}(C_2 \setminus C_1^\perp), \text{wt}(C_1 \setminus C_2^\perp)\}. \end{aligned} \quad (2.1)$$

Then there exists an AQC Q with parameters $[[n, k_1 + k_2 - n, d_z/d_x]]_q$. The code Q is said to be *pure* whenever $\{d_z, d_x\} = \{d_1, d_2\}$.

For a CSS AQC, the purity in Theorem 2 is equivalent to the general definition given in [14].

Furthermore, any CSS $[[n, k, d_z/d_x]]_q$ -AQC satisfies the following bound [12, Lem. 3.3],

$$k \leq n - d_x - d_z + 2. \quad (2.2)$$

This bound is conjectured to hold for all AQCs. A quantum code is said to be *asymmetric quantum MDS (AQMDS)* if it attains the equality in (2.2).

For our construction, the following result holds.

Lemma 1 ([14, Cor. 2.5]). *A pure CSS AQC is an asymmetric quantum MDS code if and only if both C_1 and C_2 in Theorem 2 are (classical) MDS codes.*

This means that constructing a pure q -ary CSS AQMDS code of a specific set of parameters is equivalent to finding a suitable corresponding nested pair of classical $\mathbb{F}_q$ -linear MDS codes.

Following Lemma 1, a CSS AQMDS code is said to be *trivial* if both C_1 and C_2 are trivial MDS codes.

From Lemma 1 and the MDS Conjecture, the following necessary condition for the existence of a nontrivial pure CSS AQMDS code is immediate.

Proposition 1. *Assuming the validity of the MDS Conjecture, every nontrivial pure q -ary CSS AQMDS code has length $n \leq q + 1$ if q is odd and $n \leq q + 2$ if q is even.*

Let Q be an AQC with parameters $[[n, k, d_z/d_x]]_q$. We usually require $k > 0$ (equivalently, $K = q^k > 1$) or for error detection purposes, $d_x \geq 2$. However, for completeness, we state the results for the two cases: first, when $d_x = 1$ and second, when $k = 0$.

Proposition 2. *Let n, k be positive integers such that $k \leq n - 1$. A pure CSS AQMDS code with parameters $[[n, k, d_z/1]]_q$ where $d_z = n - k + 1$ exists if and only if there exists an MDS code with parameters $[n, k, n - k + 1]_q$.*

Proof. We show only one direction. Let C be an MDS code with parameters $[n, k, n - k + 1]_q$. Apply Theorem 2 with $C_1 = C$ and $C_2 = \mathbb{F}_q^n$ to obtain the required AQMDS code. $\square$

Proposition 3. *Let n, k be positive integers such that $k \leq n - 1$. A pure CSS AQMDS code with parameters $[[n, 0, d_z/d_x]]_q$ where $\{d_z, d_x\} = \{n - k + 1, k + 1\}$ exists if and only if there exists an MDS code with parameters $[n, k, n - k + 1]_q$.*

Proof. Again, we show one direction. Let C be an MDS code with parameters $[n, k, n - k + 1]_q$ and let $C_1^\perp = C_2 = C$. Following [3], assume that a quantum code with $K = 1$ is pure and hence, there exists an AQMDS with parameters $[[n, 0, d_z/d_x]]_q$ where $\{d_z, d_x\} = \{n - k + 1, k + 1\}$. $\square$

In the subsequent sections, pure CSS AQMDS codes with $k \geq 1$ and $d_x \geq 2$ are studied.

3. AQMDS Codes of length $n \leq q$

Let us recall some basic results on GRS codes (see [7, Sect. 5.3]). Choose n distinct elements $\alpha_1, \alpha_2, \dots, \alpha_n$ in $\mathbb{F}_q$ and define $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_n)$. Let $\mathbf{v} := (v_1, v_2, \dots, v_n)$, where $v_1, v_2, \dots, v_n$ are nonzero elements in $\mathbb{F}_q$. Then, given $\boldsymbol{\alpha}$ and $\mathbf{v}$, a GRS code of length $n \leq q$ and dimension $k \leq n$ is defined as

$$\mathcal{GRS}_{n,k}(\boldsymbol{\alpha}, \mathbf{v}) := \{(v_1 f(\alpha_1), \dots, v_n f(\alpha_n)) : f(X) \in \mathbb{F}_q[X]_k\}.$$

Since $\mathbb{F}_q[X]_k \subset \mathbb{F}_q[X]_{k+1}$ for fixed $n, \mathbf{v}$, and $\boldsymbol{\alpha}$, it follows immediately that

$$\mathcal{GRS}_{n,k}(\boldsymbol{\alpha}, \mathbf{v}) \subset \mathcal{GRS}_{n,k+1}(\boldsymbol{\alpha}, \mathbf{v}). \quad (3.1)$$

Based on the standard basis for $\mathbb{F}_q[X]_k$, a generator matrix G for $\mathcal{GRS}_{n,k}(\alpha, \mathbf{v})$ is given by

$$G = \begin{pmatrix} v_1 & v_2 & \dots & v_n \\ v_1\alpha_1 & v_2\alpha_2 & \dots & v_n\alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ v_1\alpha_1^{k-1} & v_2\alpha_2^{k-1} & \dots & v_n\alpha_n^{k-1} \end{pmatrix} \quad (3.2)$$

and $\mathcal{GRS}_{n,k}(\alpha, \mathbf{v})$ is an MDS code with parameters $[n, k, n - k + 1]_q$. Hence, the following result gives a construction of an AQMDS code of length $n \leq q$.

Theorem 3. Let $q \geq 3$. Let n, k and j be positive integers such that $n \leq q$, $k \leq n - 2$ and $j \leq n - k - 1$. Then there exists a nontrivial AQMDS code with parameters $[[n, j, d_z/d_x]]_q$ where $\{d_z, d_x\} = \{n - k - j + 1, k + 1\}$.

Proof. Apply Theorem 2 with $C_1^\perp = (\mathcal{GRS}_{n,k}(\alpha, \mathbf{v})) \subset C_2 = \mathcal{GRS}_{n,k+j}(\alpha, \mathbf{v})$. $\square$

4. AQMDS Codes of Length $n = q + 1$

Let $\alpha_1, \alpha_2, \dots, \alpha_q$ be distinct elements in $\mathbb{F}_q$ and $v_1, v_2, \dots, v_{q+1}$ be nonzero elements in $\mathbb{F}_q$. Let $k \leq q$ and consider the code E given by

$$E := \left\{ (v_1 f(\alpha_1), \dots, v_q f(\alpha_q), v_{q+1} f_{k-1}) : f(X) = \sum_{i=0}^{k-1} f_i X^i \in \mathbb{F}_q[X]_k \right\}.$$

Let $\mathbf{x} = (0, \dots, 0, v_{q+1})$ and G be as in (3.2) with $n = q$. Then $G_E := (G | \mathbf{x}^T)$ is a generator matrix of E . The code E is an extended GRS code with parameters $[q + 1, k, q - k + 2]_q$ (see [7, Sect. 5.3]).

Let $1 \leq r \leq k - 2$. Then there exists a monic irreducible polynomial $p(X) \in \mathbb{F}_q[X]$ of degree $k - r$ [9, Cor. 2.11]. By the choice of $p(X)$, observe that $p(\alpha_i) \neq 0$ for all i . Hence, the matrix

$$G_C = \begin{pmatrix} v_1 p(\alpha_1) & \dots & v_q p(\alpha_q) & 0 \\ v_1 \alpha_1 p(\alpha_1) & \dots & v_q \alpha_q p(\alpha_q) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ v_1 \alpha_1^{r-2} p(\alpha_1) & \dots & v_q \alpha_q^{r-2} p(\alpha_q) & 0 \\ v_1 \alpha_1^{r-1} p(\alpha_1) & \dots & v_q \alpha_q^{r-1} p(\alpha_q) & v_{q+1} \end{pmatrix} \quad (4.1)$$

is a generator matrix of a $[q + 1, r, q - r + 2]_q$ -MDS code C .

Observe that, for all $g(X) \in \mathbb{F}_q[X]_r$, $p(X)g(X)$ is also a polynomial in $\mathbb{F}_q[X]_k$. Moreover, the coefficient of X^{k-1} in $p(X)g(X)$ is given by the coefficient of X^{r-1} in $g(X)$. Thus, $C \subset E$, leading to the following construction of AQMDS code of length $q + 1$.

Theorem 4. Let $q \geq 3$. Let j, k be positive integers such that $3 \leq k \leq q$ and $2 \leq j \leq k - 1$. Then there exists an AQMDS code with parameters $[[q + 1, j, d_z/d_x]]_q$ where $\{d_z, d_x\} = \{q - k + 2, k - j + 1\}$.

Proof. Let $r = k - j$. Apply Theorem 2 with $C_1 = C^\perp$ and $C_2 = E$. $\square$

Note that Theorem 4 gives AQMDs codes with parameters $[[q + 1, j, d_z/d_x]]_q$ with $j \geq 2$. The next proposition gives the necessary and sufficient conditions for the existence of pure CSS AQMDs codes with $j = 1$.

Proposition 4. *Let n, k be positive integers such that $k \leq n - 1$. There exists a pair of nested MDS codes $C \subset C'$ with parameters $[n, k, n - k + 1]_q$ and $[n, k + 1, n - k]_q$, respectively, if and only if there exists an MDS code with parameters $[n + 1, k + 1, n - k + 1]_q$.*

Equivalently, there exists a pure CSS AQMDs code with parameters $[[n, 1, d_z/d_x]]_q$ where $\{d_z, d_x\} = \{n - k, k + 1\}$ if and only if there exists an MDS code with parameters $[n + 1, k + 1, n - k + 1]_q$.

Proof. Let G be a generator matrix of C . Pick $\mathbf{w} \in C' \setminus C$ and observe that $\begin{pmatrix} G \\ \mathbf{w} \end{pmatrix}$ is a generator matrix for C' . It can be verified that

$$\begin{pmatrix} \mathbf{0} & G \\ 1 & \mathbf{w} \end{pmatrix}$$

is a generator matrix of an $[n + 1, k + 1, n - k + 1]_q$ -MDS code.

Conversely, let D be an $[n + 1, k + 1, n - k + 1]_q$ -MDS code with $k \leq n - 1$. Shortening the code D at the last coordinate yields an $[n, k, n - k + 1]_q$ -MDS code C . Puncturing the code D at the last coordinate gives an $[n, k + 1, n - k]_q$ -MDS code C' . A quick observation confirms that $C \subset C'$. $\square$

This proposition leads to the following characterization.

Corollary 1. *Assuming the validity of the MDS conjecture, there exists a pure CSS AQMDs code with parameters $[[q + 1, 1, d_z/d_x]]_q$ if and only if q is even and $\{d_z, d_x\} = \{3, q - 1\}$.*

Proof. There exists a $[2^m + 2, 3, 2^m]_{2^m}$ -MDS code (see [10, Ch. 11, Th. 10]). By Proposition 4, an AQMDs code with the indicated parameters exists.

The necessary condition follows from combining the MDS conjecture and Proposition 4. Assume that there exists a $[[q + 1, 1, d_z/d_x]]_q$ -AQMDs code Q with $d_x \geq 2$. If q is odd, the existence of Q would imply the existence of a nontrivial MDS code of length $q + 2$, contradicting the MDS conjecture. For even q , suppose $\{d_z, d_x\} \neq \{q - 1, 3\}$. Without loss of generality, assume $d_z \geq d_x \neq 3$. Then there exists a nested pair $[q + 1, q + 1 - d_x, d_x + 1]_q \subset [q + 1, q + 2 - d_x, d_x]_q$. By Proposition 4, there exists a $[q + 2, q + 2 - d_x, d_x + 1]_q$ -MDS code. If $d_x = 2$, then $q + 2 - d_x = q \notin \{3, q - 1\}$, contradicting the MDS conjecture. If $d_x > 3$, then $d_z < q - 1$ and $3 < q + 2 - d_z \leq q + 2 - d_x < q - 1$, a contradiction to the MDS conjecture. $\square$

5. AQMDS Codes of Length $n = 2^m + 2 \geq 6$ with $d_z = d_x = 4$

MDS codes of length $q+2$ are known to exist for $q = 2^m$, and $k \in \{3, 2^m - 1\}$ [10, Ch. 11, Th. 10]. Let $v_1, v_2, \dots, v_{q+2}$ be nonzero elements in $\mathbb{F}_q$ and fix $\alpha_q = 0$ in the notations of Section 3.

For $m \geq 2$, a generator matrix for $k = 3$ or a parity check matrix for $k = 2^m - 1$ is given by

$$H = \begin{pmatrix} v_1 & \cdots & v_{q-1} & v_q & 0 & 0 \\ v_1\alpha_1 & \cdots & v_{q-1}\alpha_{q-1} & 0 & v_{q+1} & 0 \\ v_1\alpha_1^2 & \cdots & v_{q-1}\alpha_{q-1}^2 & 0 & 0 & v_{q+2} \end{pmatrix}. \quad (5.1)$$

Let C be a $[2^m + 2, 2^m - 1, 4]_{2^m}$ -code with parity check matrix H given in (5.1). Let D be the $[2^m + 2, 3, 2^m]_{2^m}$ -code whose generator matrix G is given by

$$G = \begin{pmatrix} v_1^{-1} & \cdots & v_{q-1}^{-1} & v_q^{-1} & 0 & 0 \\ v_1^{-1}\alpha_1^{-1} & \cdots & v_{q-1}^{-1}\alpha_{q-1}^{-1} & 0 & v_{q+1}^{-1} & 0 \\ v_1^{-1}\alpha_1^{-2} & \cdots & v_{q-1}^{-1}\alpha_{q-1}^{-2} & 0 & 0 & v_{q+2}^{-1} \end{pmatrix}. \quad (5.2)$$

The following theorem gives a construction of an AQMDS code of length $q + 2$.

Theorem 5. Let $q = 2^m \geq 4$. Then there exists an AQMDS code with parameters $[[2^m + 2, 2^m - 4, 4/4]]_{2^m}$.

Proof. First we prove that $D \subset C$ by showing that $M = (m_{i,j}) := GH^T = \mathbf{0}$. Note that

$$m_{i,j} = \sum_{l=1}^{q+2} g_{i,l} \cdot h_{j,l}$$

for $1 \leq i, j \leq 3$. If $i = j$, then $m_{i,j} = q = 0$. If $i \neq j$, the desired conclusion follows since

$$\sum_{i=1}^{q-1} \alpha_i = \sum_{i=1}^{q-1} \alpha_i^{-1} = 0 \text{ and } \sum_{i=1}^{q-1} \alpha_i^{-2} = \sum_{i=1}^{q-1} \alpha_i^2 = \left(\sum_{i=1}^{q-1} \alpha_i \right)^2 = 0.$$

Applying Theorem 2 with $C_1 = D^\perp$ and $C_2 = C$ completes the proof. $\square$

6. AQMDS Codes with $d_z \geq d_x = 2$, an Alternative Look

In the previous sections, suitable pairs of GRS or extended GRS codes were chosen for the CSS construction. This section singles out the case of $d_x = 2$ where the particular type of the MDS code chosen is inessential. The following theorem gives a construction on an AQC with $d_x = 2$.

Theorem 6 ([6, Th. 7]). Let C be a linear (not necessarily MDS) $[n, k, d]_q$ -code with $k \geq 2$. If C has a codeword $\mathbf{u}$ such that $\text{wt}(\mathbf{u}) = n$, then there exists an $[[n, k - 1, d/2]]_q$ -AQC.

Let C be an $[n, k, n - k + 1]_q$ -MDS code. Ezerman *et al.* [5] showed that C has a codeword $\mathbf{u}$ with $\text{wt}(\mathbf{u}) = n$, except when either C is the dual of the binary repetition code of odd length $n \geq 3$, or C is a simplex code with parameters $[q + 1, 2, q]_q$. Hence, the following corollary can be derived.

Corollary 2. *The following statements hold:*

- (1) *For even integers n , there exists an $[[n, n - 2, 2/2]]_2$ -AQMDS code.*
- (2) *For positive integers $n, q \geq 3$, there exists an $[[n, n - 2, 2/2]]_q$ -AQMDS code.*
- (3) *Given positive integers $q \geq n \geq 4$, there exists an AQMDS code for $2 \leq k \leq n - 2$ with parameters $[[n, k - 1, d_z/2]]_q$ with $d_z = n - k + 1$.*
- (4) *Given $q \geq 4$, there exists an AQMDS code for $3 \leq k \leq q - 1$ with parameters $[[q + 1, k - 1, d_z/2]]_q$ with $d_z = q - k + 2$.*
- (5) *Given positive integer $m \geq 2$ and $q = 2^m$, there exists an AQMDS code with parameters $[[2^m + 2, 2, 2^m/2]]_{2^m}$ and an AQMDS code with parameters $[[2^m + 2, 2^m - 2, 4/2]]_{2^m}$.*

Wang *et al.* [14, Cor. 3.4] gave a different proof of the existence of $[[n, n - 2, 2/2]]_q$ -AQMDS codes Q for $n, q \geq 3$.

In this section, it is shown for $d_x = 2$ that the specific construction of the classical MDS codes used in the CSS construction is inconsequential. This is useful as there are many classical MDS codes which are not equivalent to the GRS codes (see [11], for instance).

7. Summary

While the ingredients to construct a pure AQC under the CSS construction, namely a pair of nested codes, the knowledge on the codimension and the dual distances of the codes, are all classical, computing the exact set of parameters and establishing the optimality of the resulting AQC are by no means trivial.

This work shows how to utilize the wealth of knowledge available regarding classical MDS codes to completely classify under which conditions there exists a particular pure CSS AQMDS code and how to construct such a code explicitly. Outside the MDS framework, more work needs to be done in determining the exact values of d_x and d_z and in establishing optimality.

We summarize the results of the paper in the following theorem.

Theorem 7. Let q be a prime power, n, k be positive integers and j be a nonnegative integer. Assuming the validity of the MDS conjecture, there exists a pure CSS AQMDS code with parameters $[[n, j, d_z/d_x]]_q$, where $\{d_z, d_x\} = \{n - k - j + 1, k + 1\}$ if and only if one of the following holds:

- (1) *[Prop. 2, Prop. 3] q is arbitrary, $n \geq 2$, $k \in \{1, n - 1\}$, and $j \in \{0, n - k\}$.*
- (2) *[Cor. 2] $q = 2$, n is even, $k = 1$, and $j = n - 2$.*
- (3) *[Cor. 2] $q \geq 3$, $n \geq 2$, $k = 1$, and $j = n - 2$.*

- (4) [Prop. 2, Prop. 3, Th. 3] $q \geq 3$, $2 \leq n \leq q$, $k \leq n-1$, and $0 \leq j \leq n-k$.
- (5) [Prop. 2, Prop. 3, Th. 4] $q \geq 3$, $n = q+1$, $k \leq n-1$, and $j \in \{0, 2, \dots, n-k\}$.
- (6) [Cor. 1] $q = 2^m$, $n = q+1$, $j = 1$, and $k \in \{2, 2^m-2\}$.
- (7) [Prop. 2, Prop. 3, Th. 5, Cor. 2] $q = 2^m$ where $m \geq 2$, $n = q+2$,

$$\begin{cases} k = 1, \text{ and } j \in \{2, 2^m-2\}, \\ k = 3, \text{ and } j \in \{0, 2^m-4, 2^m-1\}, \text{ or,} \\ k = 2^m-1, \text{ and } j \in \{0, 3\}. \end{cases}$$

As a concluding remark, note that all AQMDS codes constructed here are pure CSS codes. The existence of a degenerate CSS AQMDS code or an AQMDS code derived from non-CSS method with parameters different from those in Theorem 7 remains an open question.

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