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**Heuristics with Guaranteed Performance Bounds for a Manufacturing System
with Product Recovery**

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Heuristics with Guaranteed Performance Bounds for a Manufacturing System with Product Recovery

Abstract

We consider a manufacturing system with product recovery. The system manufactures a new product as well as remanufactures the product from old, returned items. The items remanufactured with the returned products are as good as new and satisfy the same demand as the new item. The demand rate for the new item and the return rate for the old item are deterministic and constant. The relevant costs are the holding costs for the new item and the returned item, and the fixed setup costs for both manufacturing and remanufacturing. The objective is to determine the lot sizes and production schedule for manufacturing and remanufacturing so as to minimize the long-run average cost per unit time. We first develop a lower bound among all classes of policies for the problem. We then show that the optimal integer ratio policy for the problem obtains a solution whose cost is at most 1.5 percent more than the lower bound.

Keywords: *inventory, remanufacturing, product returns, lot sizing and scheduling, performance bounds*

1. Introduction

With increasing concerns over depletion of natural resources, as well as the impact of disposal of old products on the environment, reverse logistics or closed loop supply chain has generated greater interest among both academic researchers and practitioners. One aspect of closed loop supply chain is product recovery.

Product recovery which includes repairing and refurbishing of old products as well as remanufacturing of new items with old, returned products is being increasingly adopted because of both environmental as well as economic considerations. Metal scrap brokers, waste paper recycling, and deposit systems for soft drink bottles are all examples (Fleischmann et al. 1997). The electronics industry is one among the most important to develop closed loop supply chains for value added recovery. And large computer companies such as IBM have implemented process and technology changes to improve the

management of returns, including assisting clients with the return process (Fleischmann, Van Nunen and Grave 2003).

The management and planning of manufacturing and remanufacturing for production systems with product recovery has attracted the attention of many researchers. If there are returned products that can be remanufactured into new items and the demand can be met with newly manufactured or remanufactured products, it leads to a practical lot sizing and production scheduling problem involving manufacturing and remanufacturing.

In this paper, we consider a manufacturing system with product recovery. Demand for the product occurs at a constant rate, and must be met without any backlogging. A portion of the old units of the product are returned by the customers. The rate of product return per unit time is assumed to be constant. All the returned units of the product are recovered and remanufactured into new items. The costs incurred include the inventory holding costs for the new item and the returned item, and the fixed setup cost incurred for every batch of manufacturing and remanufacturing. The objective is to find production lot sizes for manufacturing and remanufacturing as well the production schedule for the same, so as to minimize the long-run average cost.

One of the first papers to consider lot sizing and scheduling in manufacturing systems with product recovery was Schrady (1967). The production for manufacturing and remanufacturing is assumed to occur at an infinite rate (or performed by an external supplier). For this model, Schrady proposed a $(1, R)$ policy where every manufacturing setup is followed by an integer number, R , of identical remanufacturing setups. The order quantity for manufacturing and remanufacturing was kept stationary.

Schrady's model was extended for finite production rate for remanufacturing by Nahmias and Rivera (1979). They proposed a class of heuristic policies that is similar to Schrady's. Koh et al. (2002) extended the work of Nahmias and Rivera by allowing the

remanufacturing rate to be either larger or smaller than the demand rate. Again as in Nahmias and Rivera (1979) and Schrady (1967), they consider only the class of $(1, R)$ policies.

Teunter (2001) extended the work of Schrady (1967) to consider two classes of control policies: (i) a $(1, R)$ policy where every manufacturing setup is followed by an integer number R of remanufacturing setups, and (ii) a $(P, 1)$ policy, where every remanufacturing setup is followed by an integer number P of manufacturing cycles. It would be appropriate to label the class of $(1, R)$ and $(P, 1)$ policies together as *integer-ratio policies*. An integer ratio policy in this problem context can be defined as a policy where the ratio of the average number of manufacturing setups to the average number of remanufacturing setups per unit time is either an integer or a reciprocal of an integer.

Teunter (2004) extended the work of Teunter (2001) to consider finite production rates for manufacturing and remanufacturing. Teunter (2001) only considered the relaxed version of the problem where P and R are allowed to be non-integers. Heuristic solutions were obtained for these classes of policies by Teunter (2004).

All the above papers assume deterministic demand and return rate for the product. Papers that deal with product returns in inventory systems with stochastic demand and return rates include Decroix (2006), Decroix, Song and Zipkin (2005), Heyman (1977), Fleishmann, Kuik and Dekker (2002), Muckstadt and Issac (1981) and van der Laan et al. (1996). All these papers either assume that returns can be directly used without any remanufacturing, or assume that there are no setup costs. Papers involving stochastic demand and return rates and setup costs for manufacturing and remanufacturing include van der Laan et al. (1999) and van der Laan and Teunter (2006).

As mentioned earlier, we focus on the problem with deterministic demand and return rates in this paper. The prior literature on the problem has focused only on heuristic policies.

Specifically, two classes of heuristic policies (belonging to the class of integer ratio policies) have been proposed, but none of the heuristics provide any a priori performance guarantee. Given that the inventory level of the returned item depends in a complex way on the sequence of manufacturing and remanufacturing setups, optimal policies are difficult to find. Even if optimal policies can be obtained, it may be hard to implement as a repeatable cycle as each cycle might contain complex sequences of the manufacturing and remanufacturing batches. Therefore in this paper, we attempt to find simple heuristics with a priori performance guarantees.

The contribution of this paper is as follows. We first develop a lower bound among all classes of policies for the problem. We then develop closed form expressions to determine the optimal integer ratio policy and prove that the cost of this policy is at most 1.5% more than the lower bound among all policies for the problem. We also show that the best of the two heuristic policies proposed by Teunter (2001, 2004) obtains a solution that is at most 6% more than the lower bound under certain conditions; however when the production rate for manufacturing and remanufacturing is close to the demand rate, we show that Teunter's heuristic can result in solutions with arbitrarily large costs.

The remainder of this paper is organized as follows. In section 2, we develop the model, and provide the notation and initial analysis. In section 3, we develop the lower bound among all classes of policies for the problem. In section 4, we develop performance bounds for integer ratio policies. We also provide a few numerical examples in this section to illustrate these performance bounds. Finally we end the paper with a few concluding remarks.

2. Notation and Analysis

The notation used throughout the paper is given in Table 1. The item under consideration need not be an end product. It could be a component that goes into a finished

product or a remanufactured finished product. The demand rate for the item is D . The return rate for old units of the item is fD , where $0 < f < 1$. Deterministic demand and return rates are reasonable assumptions when the mean demand and return rates are large and the variability in the rates are relatively very small in relation to the mean rate. This would be the case, when the installed base of the item (or the products in which the item is a component) is sufficiently large. Moreover, the insights and solutions derived from a deterministic model would be good starting points for developing heuristic policies for the problem with stochastic demand and return rates¹. The production rate for manufacturing is $p \geq D$, and the production rate for remanufacturing is $r \geq D$. As in many of the earlier papers, we assume that all returned items are remanufactured (or alternatively fD is the rate of return of old units that are good enough to be remanufactured). Besides Teunter (2001) had shown that it is optimal to either remanufacture all the returned items or not remanufacture at all.

Table 1: Notation

D	Demand rate per year
fD	Return rate per year, where $0 < f < 1$.
p	Rate of production for manufacturing (of new items), $\infty \geq p \geq D$
r	Rate of production for remanufacturing, $\infty \geq r \geq D$
K_p	Setup cost per production lot for manufacturing
K_r	Setup cost per production lot for remanufacturing
h_r	Holding cost per unit of the returned item per year
h_s	Holding cost per unit of the new item per year, $h_s > h_r > 0$.
$h_s - h_r$	Echelon holding cost for the new item created from returned items
Q_p	Production lot-size for manufacturing in a stationary policy
Q_r	Production lot-size for remanufacturing in a stationary policy
$\hat{p}$	Average number of manufacturing setups per year
$\hat{R}$	Average number of remanufacturing setups per year

¹ For example, one plausible heuristic policy for the problem with stochastic demands might be to use a reorder level, r , for the new (manufactured or remanufactured) items that ensures a certain service level, and use the order quantities derived from the deterministic model as the order quantities for the stochastic model. When the inventory position of the new items reaches the reorder level, a lot is remanufactured if there is sufficient inventory of returned items, otherwise a lot is manufactured.

The relevant costs are the setup cost for manufacturing (remanufacturing), K_p (K_r), and the holding cost for the returned item and the new item which are incurred at rates h_r and h_s respectively. Note that in some situations, the holding cost rate, h_s^r , for new items that are remanufactured might be lower than the holding cost rate, h_s^p , for those that are manufactured from new inputs (Teunter, 2001). Even though, we assume that $h_s = h_s^r = h_s^p$, the results we develop in this paper would be applicable even when $h_s^r \neq h_s^p$ (so long as $h_s^r, h_s^p > h_r$). The objective is to find the lot-sizing and scheduling policy that minimizes the long-run average total cost per unit time.

Since all the returned items are remanufactured, the average demand met from remanufacturing is fD and the average demand met through manufacturing of new items is $(1-f)D$.

Lemma 1: In the optimal policy for the problem, the inventory level of the returned item, I_r , will reach the value, zero, in a recurrent manner.

Proof: Assume the contrary and let the lowest level of I_r reached in the steady state in the optimal policy be δ . Over a long time horizon, $\tilde{T}$, the holding cost incurred for this level of returned item inventory is $\tilde{T}\delta h_r$. As $\tilde{T} \rightarrow \infty$, this cost would increase linearly and become very large. Therefore, it would make sense to increase the batch sizes of (at least) a few remanufacturing lots to get rid of this inventory. The additional holding cost incurred on the new item would certainly be lower [If one batch of remanufacturing lot was increased from q to $(q + \delta)$, the maximum increase in the holding cost for the new item would be $\frac{(q + \delta)}{D}\delta h_s < \tilde{T}\delta h_r$, as $\tilde{T} \rightarrow \infty$. Hence I_r will recurrently reach the value, zero, in an optimal policy. $\square$

Therefore, without loss of generality, we can assume that the initial value of $I_r = 0$.

Clearly, if there was a large amount of inventory of returned items, then we would schedule several batches of remanufacturing lots at the beginning (as the implicit assumption is that the cost per unit of remanufactured item is lower than that for a newly manufactured item). As we are minimizing the steady state long run average cost per unit time, across the infinite horizon, this initial period until I_r becomes zero can be ignored.

Let the average number of manufacturing and remanufacturing setups per year be $\hat{P}$ and $\hat{R}$ respectively. Even assuming a stationary policy, [where the lot size for every manufacturing (remanufacturing) setup is identical and equal to Q_p (Q_r)], the optimal production schedule can be difficult to obtain. For a given $\hat{P}$, $\hat{R}$, let $\hat{P}/\hat{R} = P/R$, (where P , R are integers without common factors other than 1). In this case, for every P manufacturing setups, there will be R remanufacturing setups. If $h_r = 0$, then the problem of scheduling the manufacturing and remanufacturing lots become straight forward as the only relevant costs are the setup cost for manufacturing/remanufacturing (which is fixed for a given $\hat{P}$, $\hat{R}$), and the holding cost for the new item. The inventory holding cost for the returned item (which is zero with $h_r = 0$) is not affected by the sequence and schedule of the manufacturing and remanufacturing setups. Therefore one could just have P manufacturing setups followed by R remanufacturing setups, with the only requirement that the ratio of the total cycle time for the P manufacturing setups to the total cycle time for R remanufacturing setups is $(1-f)/f$. In this case for the time duration $t_p = (P/\hat{P})(1-f)$, there are P manufacturing setups and one needs to minimize the inventory holding cost for the new item for this duration. Clearly it is optimal in this case to have identical lot size, $Q_p = (1-f)D/\hat{P}$, for each manufacturing setup. Similarly, it is optimal to have identical lot size, $Q_r = fD/\hat{R}$, for each remanufacturing setup. Note that

this result is valid even when $h_s^r \neq h_s^p$. Also a manufacturing or remanufacturing lot would be setup only at the point when the inventory level of the new item is zero.

However, if $h_r > 0$, then the cost of holding inventory of the returned items is influenced by the manufacturing and remanufacturing lot sizes as well as their production sequences. Therefore, the optimal lot sizes for manufacturing and remanufacturing in general need not be stationary.

3. Lower Bound

We first develop a lower bound $Z_L(\hat{P}, \hat{R})$ on the long run average cost per year for any class of policies for a given $\hat{P}, \hat{R}$. Let

$$Z_L(\hat{P}, \hat{R}) = Z_L^S(\hat{P}, \hat{R}) + Z_L^R(\hat{P}, \hat{R}) \quad (1)$$

where $Z_L^S(\hat{P}, \hat{R})$ is a lower bound on the sum of the average setup cost per year for manufacturing and remanufacturing, and the average holding cost per year for the new item, and $Z_L^R(\hat{P}, \hat{R})$ is a lower bound on the average holding cost per year for the returned item.

Ignore the cost of holding inventory of the returned item for the moment. Then, as discussed earlier, for a given $(\hat{P}, \hat{R})$, the cost of holding the inventory of the new item is minimized by having identical lot sizes $Q_p = (1-f)D/\hat{P}$ and $Q_r = fD/\hat{R}$ in each setup for manufacturing and remanufacturing. Since the production rate for manufacturing is p , the average inventory of the new item held during a cycle involving a single manufacturing setup is $(1/2)Q_p(1-D/p)$. Similarly, the average inventory of the new item held during a cycle involving a single remanufacturing setup is $(1/2)Q_r(1-D/r)$.

Since the demand satisfied by manufacturing and remanufacturing are $(1-f)D$ and fD per year respectively, the total average duration of all manufacturing and remanufacturing cycles in a year is $(1-f)$ and f respectively. Therefore,

$$Z_L^S(\hat{P}, \hat{R}) = K_p \hat{P} + K_r \hat{R} + (1/2)h_s Q_p (1 - D/p)(1 - f) + (1/2)h_s Q_r (1 - D/r)f \quad (2)$$

Note that when $h_s^r \neq h_s^p$, (2) would be replaced by

$$Z_L^S(\hat{P}, \hat{R}) = K_p \hat{P} + K_r \hat{R} + (1/2)h_s^p Q_p (1 - D/p)(1 - f) + (1/2)h_s^r Q_r (1 - D/r)f$$

We now develop a lower bound on the average holding cost per year for the inventory of the returned item. First, consider a general sequence of several cycles of manufacturing setups followed by several cycles of remanufacturing setups. The inventory of the returned item will build up at the rate fD during the cycles of manufacturing setups. And during the cycles of remanufacturing setups, the inventory of the returned item will deplete at rate $r - fD$ during the production and build up at rate fD when there is no production (see Figure 1).

Figure 1: Inventory plot of the returned and new item

Let I_r be the inventory level of the returned items and let I_p be the inventory level of the new items remanufactured from the returned items. As the inventory of the returned items is converted into new items and then sold to customers, it is good to use the echelon inventory concept to study the returned products inventory. The returned products inventory at the returned item echelon incur a holding cost of h_r . When it is converted to a new item after remanufacturing, it moves to the new item echelon and incur an additional echelon holding cost of $h_s - h_r$. (Note: when $h_s^r \neq h_s^p$, the echelon holding cost would be $h_s^r - h_r$). The echelon inventory graph for the returned item and the new item generated from the returned item is as shown in Figure 2. The echelon inventory level of the returned items, $\tilde{I}_r = I_r + I_p$.

The echelon inventory level, $\tilde{I}_r$, of the returned item (during the series of remanufacturing cycles) is not affected by the lot size for remanufacturing. Therefore, the lot size decisions for remanufacturing depend only on the echelon inventory of the new item generated from remanufacturing. For a fixed number of remanufacturing setups during the time duration of the sequence of remanufacturing cycles, the echelon holding cost for the inventory of the new item is therefore minimized by having identical lot sizes. Therefore, the presence of a positive holding cost rate for the returned item will not influence the stationarity of the lot sizes for remanufacturing (or manufacturing) in the lower bound.

Figure 2: Echelon inventory graph for the returned item and new item remanufactured from the returned item

Now consider the inventory graph (Figure 3) for the echelon inventory of the returned item (which includes inventory of the returned item as well as inventory of the new item generated from the returned item through remanufacturing). During the cycles of manufacturing setups, the echelon inventory level will increase at rate fD and during cycles of remanufacturing setups, the echelon inventory level will deplete at rate $D - fD$.

From Lemma 1, we know that the returned item inventory level, I_r , will hit zero recurrently in optimal policy. Consider a time horizon, T , where $I_r = 0$, at the beginning as well as at the end of the horizon. As the returns are accumulated at a steady rate, fD , I_r can reach the value zero only at the completion of production of a remanufacturing lot. Therefore, it implies that the production of a remanufacturing lot was completed just before

the beginning and at the end of the time horizon, T . Since the lot sizes for remanufacturing are stationary and identical in the lower bound, this implies that the inventory, I_p , of new items at the beginning and the end of the time horizon is identical. Since the value of both $I_r = 0$, and I_p , are identical at the beginning and the end of the time horizon, T , it implies that all returns, fDT , during the period are remanufactured in the period and the total quantity of new items manufactured is $(1 - f)DT$ (as the demand during the period is met with the total quantity of manufactured and remanufactured items).

Figure 3: Echelon inventory graph for the returned item

Let x be the number of different sequences of manufacturing and remanufacturing setups in the time horizon, T . We prove in Theorem A.1 in the appendix that to minimize the holding cost of the echelon inventory of returned items, all the x sequences should be identical and should have identical total lot sizes for the remanufacturing cycles (and correspondingly for the manufacturing cycles) in the sequence. In other words, a lower bound on the long run average total cost per year can be calculated by assuming that the lot sizes for manufacturing, Q_p , and remanufacturing, Q_r , are stationary. Let $\tilde{P} = \hat{P}/\hat{R}$, and $\tilde{R} = \hat{R}/\hat{P}$.

Case 1: $\hat{P}/\hat{R} = \tilde{P} \geq 1$

In this case, the average number of manufacturing setups for every remanufacturing setup is $\tilde{P}$, which need not be an integer. Hence to achieve the lowest holding cost for the echelon inventory of the returned item, we might need complex sequences of manufacturing and remanufacturing setups, where the number of manufacturing cycles in a sequence need not be the same for every sequence. As proved in theorem A.1, identical total lot size for the remanufacturing and manufacturing cycles achieve the lowest echelon inventory of the returned item. Therefore a lower bound on the average inventory of the returned item can be achieved by assuming an average cycle of $\tilde{P} = \hat{P}/\hat{R}$ manufacturing setups followed by one remanufacturing setup (even though this might not be a feasible schedule due to $\tilde{P}$ not being an integer). As the average number of remanufacturing setups per year is $\hat{R}$, the remanufacturing lot size $Q_r = \frac{Df}{\hat{R}}$. In order to remanufacture this lot size Q_r , the lowest possible inventory of the returned item required at the beginning of a remanufacturing setup is $Q_r - \frac{Q_r}{r} Df$. This inventory goes down to zero and then increases to $Q_r - \frac{Q_r}{r} Df$ at the end of a series of manufacturing setups (or the beginning of the next remanufacturing setup). Therefore, a lower bound on the inventory holding cost for the returned item when $\hat{P} \geq \hat{R}$ is

$$Z_L^P(\hat{P}, \hat{R}) = \frac{1}{2} h_r Q_r \left(1 - \frac{D}{r} f \right), \quad \text{for } \hat{P} \geq \hat{R} \quad (3)$$

Case 2: $\hat{R}/\hat{P} = \tilde{R} \geq 1$

In this case, the average number of remanufacturing setups for every manufacturing setup is $\tilde{R}$. Therefore a lower bound on the average echelon inventory of the returned item can be achieved by assuming an average cycle of $\tilde{R}$ remanufacturing setups followed by

one manufacturing setup. Since the amount of returns generated ($T D f$) during the period T is equal to the quantity remanufactured, the inventory of the returned item would be completely used up (and equal to zero) at the end of production of the last remanufacturing cycle before a manufacturing setup. Thereafter the inventory of the returned item would increase.

As the remanufacturing lot size is Q_r , the time required to deplete the inventory of the remanufactured items after completion of production is $\frac{Q_r}{D} \left(1 - \frac{D}{r}\right)$, and during this time inventory of the returned item would build up at rate $D f$. Therefore, at the commencement of a manufacturing setup, the inventory level of the returned item is $Q_r \left(1 - \frac{D}{r}\right) f$. As the duration of the manufacturing cycle is $\frac{Q_p}{D}$, the total inventory of the returned item at the end of the manufacturing cycle is $Q_r \left(1 - \frac{D}{r}\right) f + Q_p f$. Therefore, the average inventory of the returned item from the end of production of the last remanufacturing cycle (before the manufacturing setup) to the beginning of the next set of $\tilde{R}$ remanufacturing cycles is $\frac{1}{2} \left(Q_r \left(1 - \frac{D}{r}\right) f + Q_p f \right)$.

To calculate the average inventory of the returned item from the beginning of the first remanufacturing setup till the end of production of the $\tilde{R}th$ remanufacturing setup, we first calculate the average echelon inventory of the returned item.

At the beginning of the first remanufacturing setup, there is no inventory of the new item. The echelon inventory of the returned item depletes at the rate $D - f$. At the end of production of $\tilde{R}th$ remanufacturing cycle, the inventory of the returned item is fully exhausted, however the inventory of the new item (remanufactured from returned items) is

$Q_r \left(1 - \frac{D}{r}\right) f$. Therefore, the average echelon inventory of the returned item during the $\tilde{R}$ remanufacturing cycles is $Q_r \left(1 - \frac{D}{r}\right) f + \frac{1}{2} Q_p f$.

As the remanufacturing lot size is Q_r , the average inventory of the new item during this period is $\frac{1}{2} Q_r \left(1 - \frac{D}{r}\right) f$. Hence average inventory of the returned item during the $\tilde{R}$ remanufacturing cycles is the difference between the average echelon inventory of the returned item and the average inventory of the new item which is $Q_r \left(1 - \frac{D}{r}\right) f + \frac{1}{2} Q_p f - \frac{1}{2} Q_r \left(1 - \frac{D}{r}\right) f = \frac{1}{2} Q_r \left(1 - \frac{D}{r}\right) f + \frac{1}{2} Q_p f$. Therefore, the average inventory of the returned item during both the manufacturing and remanufacturing cycles is the same. Therefore, the lower bound on the inventory holding cost for the returned item in case 2 is

$$Z_L^R(\hat{P}, \hat{R}) = \frac{1}{2} h_r \left(Q_r \left(1 - \frac{D}{r}\right) f + Q_p f \right), \quad \text{for } \hat{R} \geq \hat{P} \quad (4)$$

Note that since $Q_p = \frac{D(1-f)}{\hat{P}}$, and $Q_r = \frac{Df}{\hat{R}}$, in the derivation of the lower bound, $\hat{P} \geq \hat{R}$

is equivalent to $Q_p \leq Q_r \frac{1-f}{f}$, and $\hat{R} \geq \hat{P}$ is equivalent to $Q_p \geq Q_r \frac{1-f}{f}$. Therefore, the

lower bound on the total cost for a given $\hat{P}$, $\hat{R}$ (or correspondingly Q_p , Q_r) is:

$$Z_L^1(Q_p, Q_r) = K_p \frac{D(1-f)}{Q_p} + K_r \frac{Df}{Q_r} + \frac{1}{2} h_s Q_p \left(1 - \frac{D}{p}\right) (1-f) + \frac{1}{2} h_s Q_r \left(1 - \frac{D}{r}\right) f + \frac{1}{2} h_r Q_r \left(1 - \frac{D}{r}\right) f, \quad \text{for } Q_p \leq Q_r \frac{1-f}{f} \quad (5)$$

$$Z_L^2(Q_p, Q_r) = K_p \frac{D(1-f)}{Q_p} + K_r \frac{Df}{Q_r} + \frac{1}{2} h_s Q_p \left(1 - \frac{D}{p}\right) (1-f) + \frac{1}{2} h_s Q_r \left(1 - \frac{D}{r}\right) f$$

$$+ \frac{1}{2} h_r \left(Q_r \left(1 - \frac{D}{r}\right) f + Q_p f \right), \quad \text{for } Q_p \geq Q_r \frac{1-f}{f} \quad (6)$$

Note that when $h_s^r \neq h_s^p$, h_s in the 3rd term of (5) and (6) would be replaced with h_s^p and h_s in the 4th term of (5) and (6) would be replaced with h_s^r . The rest of the results in the paper will then still carry through (after appropriate redefinition of y_p^1 , y_r^1 , y_p^2 , and y_r^2 , below).

The cost functions for the $(P, 1)$ and $(1, R)$ policy derived by Teunter (2001, 2004) are the same as (5) and (6) respectively. Note however that the above lower bounds are new.

Let,

$$x_p^1 = K_p D(1-f), \quad y_p^1 = \frac{1}{2} h_s \left(1 - \frac{D}{p}\right) (1-f)$$

$$x_r^1 = K_r D(1-f), \quad y_r^1 = \left(\frac{1}{2} h_s \left(1 - \frac{D}{r}\right) f + \frac{1}{2} h_r \left(1 - \frac{D}{r}\right) f \right) \frac{f}{(1-f)}$$

$$x_p^2 = K_p Df, \quad y_p^2 = \frac{1}{2} h_s \left(1 - \frac{D}{p}\right) \frac{(1-f)^2}{f} + \frac{1}{2} h_r (1-f)$$

$$x_r^2 = K_r Df, \quad y_r^2 = \frac{1}{2} h_s \left(1 - \frac{D}{r}\right) f + \frac{1}{2} h_r \left(1 - \frac{D}{r}\right) f$$

$$P = \frac{Q_r (1-f)}{Q_p f}, \quad R = \frac{Q_p f}{Q_r (1-f)} = \frac{1}{P}$$

Then, (5) and (6) can be rewritten as

$$Z_L^1(P, Q_p) = \frac{x_p^1}{Q_p} + y_p^1 Q_p + \frac{x_r^1}{P Q_p} + y_r^1 P Q_p, \quad \text{for } P \geq 1 \quad (7)$$

$$Z_L^2(R, Q_r) = \frac{x_p^2}{R Q_r} + y_p^2 R Q_r + \frac{x_r^2}{Q_r} + y_r^2 Q_r, \quad \text{for } R \geq 1 \quad (8)$$

It can be seen that when $P = R = 1$, $Z_L^1 = Z_L^2$.

A lower bound on the total cost can therefore be obtained by optimizing

$Z_L^1(P, Q_p)$ and $Z_L^2(R, Q_r)$ and choosing the lower of the two bounds. Note that $Z_L^1(Z_L^2)$ is convex in $Q_p(Q_r)$ for a given $P(R)$. Therefore, optimizing with respect to $Q_p(Q_r)$, we get

$$Z_L^1(P) = 2\sqrt{\left(x_p^1 + \frac{x_r^1}{P}\right)(y_p^1 + y_r^1 P)} = 2\sqrt{x_p^1 y_p^1 + x_r^1 y_r^1 + \frac{x_r^1 y_p^1}{P} + x_p^1 y_r^1 P}, \quad P \geq 1 \quad (9)$$

and similarly,

$$Z_L^2(R) = 2\sqrt{\left(\frac{x_p^2}{R} + x_r^2\right)(y_p^2 R + y_r^2)} = 2\sqrt{x_p^2 y_p^2 + x_r^2 y_r^2 + \frac{x_p^2 y_r^2}{R} + x_r^2 y_p^2 R}, \quad R \geq 1 \quad (10)$$

The terms in the square root in the above expression are convex in $P(R)$. However, $P(R) \geq 1$. Therefore, the value of $P^*(R^*)$ that optimizes $Z_L^1(Z_L^2)$ are:

$$P^* = \begin{cases} \sqrt{\frac{x_r^1 y_p^1}{x_p^1 y_r^1}}, & \text{if } \sqrt{\frac{x_r^1 y_p^1}{x_p^1 y_r^1}} > 1 \\ 1 & \text{otherwise} \end{cases} \quad (11)$$

$$R^* = \begin{cases} \sqrt{\frac{x_p^2 y_r^2}{x_r^2 y_p^2}}, & \text{if } \sqrt{\frac{x_p^2 y_r^2}{x_r^2 y_p^2}} > 1 \\ 1 & \text{otherwise} \end{cases} \quad (12)$$

The corresponding value of Q_p^* and Q_r^* that optimize Z_L^1 and Z_L^2 respectively are

$$Q_p^* = \sqrt{\frac{x_p^1 + \frac{x_r^1}{P^*}}{y_p^1 + y_r^1 P^*}} \quad \text{and} \quad Q_r^* = \sqrt{\frac{x_r^2 + \frac{x_p^2}{R^*}}{y_r^2 + y_p^2 R^*}} \quad (13)$$

The value of the lower bounds can be obtained by substituting (11) into (9) and (12) into (10). The best among the two lower bounds Z_L^1 and Z_L^2 can be chosen and therefore, the lower bound on the long run average cost for the problem is

$$Z_L = \begin{cases} Z_L^1, & \text{if } Z_L^1 \leq Z_L^2 \\ Z_L^2 & \text{otherwise} \end{cases} \quad (14)$$

4. Performance Bounds of Integer-Ratio Policies

In this section, we obtain the optimal integer ratio policy (i.e., the optimal among all $(P, 1)$ and $(1, R)$ policies) and prove that its cost is at most 1.5% more than the lower bound among all policies for the problem.

As shown in Teunter (2004) for the $(P, 1)$ policy, the long run average cost function is given by the expression (5) with the additional restriction that $P = \frac{Q_r(1-f)}{Q_p f}$ is an integer.

And for the $(1, R)$ policy, the long run average cost function is given by expression (6) with the additional restriction that $R = \frac{Q_p f}{Q_r(1-f)}$ is an integer.

As expressions (5) and (6) are equivalent to expressions (7) and (8), and therefore equivalent to expressions (9) and (10), the cost for $(P, 1)$ and $(1, R)$ policies can be written as

$$TC^{(P, 1)} = 2\sqrt{x_p^1 y_p^1 + x_r^1 y_r^1 + \frac{x_r^1 y_p^1}{P} + x_p^1 y_r^1 P}, \quad P \geq 1, \text{ and integer} \quad (15)$$

$$TC^{(1, R)} = 2\sqrt{x_p^2 y_p^2 + x_r^2 y_r^2 + \frac{x_p^2 y_r^2}{R} + x_r^2 y_p^2 R}, \quad R \geq 1, \text{ and integer} \quad (16)$$

Theorem 1

(i) Let $TC^{*(P, 1)}$ be the cost corresponding to the optimal $(P, 1)$ policy. Then

$$TC^{*(P, 1)} / Z_L^1 \leq 1.015.$$

(ii) Let $TC^{*(1, R)}$ be the cost corresponding to the optimal $(1, R)$ policy. Then

$$TC^{*(1, R)} / Z_L^2 \leq 1.015$$

Proof:

We prove the first result below. The proof for the second result is similar. As

$(TC^{*(P, 1)})^2$ is convex, and P is required to be an integer, clearly the optimal integer value

of P is either $\lceil P^* \rceil$ or $\lfloor P^* \rfloor$ whichever gives the lower cost.

When the value of P^* in the lower bound Z_L^1 is binding and equal to 1 (or any other integer value), then clearly that itself is an integer solution and therefore, $TC^{*(P, 1)}/Z_L^1 = 1$.

We now prove the result for the case when P^* is non-binding (i.e. > 1) and non-integer. In this case, it can be shown that integer value of P that minimize $TC^{(P, 1)}$ satisfies

$$P(P-1) \leq (P^*)^2 = \frac{x_r^1 y_p^1}{x_p^1 y_r^1} \leq P(P+1) \quad (17)$$

The left inequality of (17) can be obtained from the fact that $TC^{(P, 1)} \leq TC^{(P-1, 1)}$ and the right inequality of (17) can be obtained from $TC^{(P, 1)} \leq TC^{(P+1, 1)}$. Note that as P^* is non-binding and greater than 1, both the inequalities will hold. From (17), it can be seen that,

$$\frac{1}{\sqrt{2}} \leq \frac{P}{P^*} \leq \sqrt{2} \quad (18)$$

where the optimal P^* in the lower bound Z_L^1 is non-integer. From (15) and (9),

$$\left(\frac{TC^{*(P, 1)}}{Z_L^1} \right)^2 = \frac{x_p^1 y_p^1 + x_r^1 y_r^1 + \frac{x_r^1 y_p^1}{P} + x_p^1 y_r^1 P}{x_p^1 y_p^1 + x_r^1 y_r^1 + \frac{x_r^1 y_p^1}{P^*} + x_p^1 y_r^1 P^*} \quad (19)$$

Let

$$A = x_p^1 y_p^1 + x_r^1 y_r^1 \quad (20)$$

$$B = \frac{x_r^1 y_p^1}{P^*} + x_p^1 y_r^1 P^* \quad (21)$$

$$C = \frac{x_r^1 y_p^1}{P} + x_p^1 y_r^1 P \quad (22)$$

Substituting $P^* = \sqrt{\frac{x_r^1 y_p^1}{x_p^1 y_r^1}}$ into expression (21), we get $B = 2\sqrt{x_p^1 x_r^1 y_p^1 y_r^1}$. From (11) and the

fact that P^* is non-binding, we have

$$\frac{x_r^1 y_p^1}{P^*} = x_p^1 y_r^1 P^* \quad (23)$$

Therefore, from (23), (21) and (22), we get

$$\frac{C}{B} = \frac{x_r^1 y_p^1 / P}{2 x_r^1 y_p^1 / P^*} + \frac{x_p^1 y_r^1 P}{2 x_p^1 y_r^1 P^*} = \frac{1}{2} \left(\frac{P}{P^*} + \frac{P^*}{P} \right) = \frac{1}{2} \left(\varepsilon + \frac{1}{\varepsilon} \right)$$

where $\varepsilon = \frac{P}{P^*}$ and from (18), $\frac{1}{\sqrt{2}} \leq \varepsilon \leq \sqrt{2}$. The expression $\left(\varepsilon + \frac{1}{\varepsilon} \right)$ is convex and

therefore its maximum value in the range $\frac{1}{\sqrt{2}} \leq \varepsilon \leq \sqrt{2}$ is at $\varepsilon = \sqrt{2}$ or $\frac{1}{\sqrt{2}}$. Therefore,

$$\frac{C}{B} \leq \frac{1}{2} \left(\sqrt{2} + \frac{1}{\sqrt{2}} \right) = \frac{3}{2\sqrt{2}} = 1.06 \quad (24)$$

Now from (19), (20), (21), (22) and (24),

$$\left(\frac{TC^{*(P, 1)}}{Z_L^1} \right)^2 = \frac{A+C}{A+B} \leq \frac{A+1.06B}{A+B} \quad (25)$$

From Lemma A.1 (proved in the appendix), we know that $B \leq A$. Let $B = \delta A$ where

$$0 < \delta \leq 1. \text{ Then, } \left(\frac{TC^{*(P, 1)}}{Z_L^1} \right)^2 \leq \frac{A+1.06\delta A}{A+\delta A} = \frac{1+1.06\delta}{1+\delta} \quad (26)$$

It can be shown (by taking the first derivative with respect to δ), that the right hand side of

(26) is increasing in δ . As $0 < \delta \leq 1$, $\left(\frac{TC^{*(P, 1)}}{Z_L^1} \right)^2 \leq \frac{1+1.06\delta}{1+\delta} \leq \frac{1+1.06}{1+1} = 1.03$. Therefore,

$$\frac{TC^{*(P, 1)}}{Z_L^1} \leq \sqrt{1.03} = 1.015. \quad \square$$

Theorem 2

Let $TC_{IRP}^* = \min \{ TC^{*(P, 1)}, TC^{*(1, R)} \}$, which is the cost of best among the optimal $(P, 1)$

and $(1, R)$ policies. $TC_{IRP}^* / Z_L \leq 1.015$.

Proof:

Case 1 $Z_L = Z_L^1$: In this case, we pick the best $(P, 1)$ policy. Clearly from Theorem 2,

$$\frac{TC_{IRP}^*}{Z_L} = \frac{TC_{IRP}^*}{Z_L^1} \leq \frac{TC^{*(P, 1)}}{Z_L^1} \leq 1.015.$$

Case 2 $Z_L = Z_L^2$: In this case, we would pick the best $(1, R)$ policy. The proof is then similar to case 1. $\square$

4.1. Analysis of Teunter's Heuristic

Teunter (2001, 2004) developed heuristics for determining $(P, 1)$ and $(1, R)$ policies and recommends choosing the policy with the lower cost. We analyze below the effectiveness of Teunter's heuristics. Instead of writing Z_L^1 and correspondingly $TC^{(P, 1)}$ as in (9), we could write

$$Z_L^1(Q_p, Q_r) = Z_{Lp}^1(Q_p) + Z_{Lr}^1(Q_r) \quad (27)$$

where, $Z_{Lp}^1(Q_p) = \frac{x_p^1}{Q_p} + y_p^1 Q_p$, $Z_{Lr}^1(Q_r') = \frac{x_r^1}{Q_r'} + y_r^1 Q_r'$, $Q_r' = Q_r(1-f)/f$, and $Q_r' \geq Q_p$.

Teunter (2004) treats the cost function as separable in Q_p and Q_r' with the constraint that $Q_r'/Q_p \geq 1$ and integer for the $(P, 1)$ policy. Teunter uses the following expressions to determine a heuristic $(P, 1)$ policy: Q_p^* and $Q_r'^*$ are determined as follows.

$$Q_p^* = \sqrt{x_p^1/y_p^1} \quad \text{and} \quad Q_r'^* = \begin{cases} \sqrt{x_r^1/y_r^1} & \text{if } \sqrt{x_r^1/y_r^1} \geq Q_p^* \\ \sqrt{x_p^1/y_p^1} & \text{otherwise} \end{cases} \quad (28)$$

Let $P^* = Q_r'^*/Q_p^*$. A heuristic $(P, 1)$ policy is then obtained by choosing $Q_p = Q_p^*$, and $Q_r' = P Q_p^*$ and where $P = \lfloor P^* \rfloor$ or $\lceil P^* \rceil$ whichever gives the lower cost. A heuristic for

the $(1, R)$ policy is also obtained in a similar fashion with the only difference that the value of Q_p instead of Q_r is modified in that case.

Assume that the optimal value $Q_p^* = \sqrt{x_p^1/y_p^1}$ and $Q_r^* = \sqrt{x_r^1/y_r^1}$ obtained by treating the cost function in (27) as separable functions satisfy the constraint $Q_r^* \geq Q_p^*$. Substituting these values into (27) will then give the value of the lower bound. Then, similar to (17) and (18), it is easy to show that $1/\sqrt{2} \leq P/P^* \leq \sqrt{2}$ and therefore similar to the derivation of (24) it can be shown that $Z_{Lr}^1(PQ_p^*) \leq 1.06Z_{Lr}^1(Q_r^*)$. Hence when the constraint $Q_r(1-f)/f \geq Q_p$ is not binding in the lower bound, Z_L^1 , Teunter's heuristic $(P, 1)$ policy obtains a solution whose cost is not more than $1.06Z_L^1$. Similarly when the constraint $Q_p f/(1-f) \geq Q_r$ is not binding in the lower bound, Z_L^2 , Teunter's heuristic $(1, R)$ policy obtains a solution whose cost is not more than $1.06Z_L^2$.

Therefore for many cases, Teunter's heuristic can guarantee a solution within six percent of the lower bound. However, when the aforementioned constraints are both binding, there can be instances when adjusted value of P (and similarly R) is not within the $\sqrt{2}$ interval. This can happen especially when the production rate for manufacturing, p and remanufacturing, r are close to the demand rate. Let $p = r = (d + \varepsilon)$, where $\varepsilon > 0$, is a infinitesimally small positive number. Let M be a very large positive number. Substituting $p = r = (d + \varepsilon)$ into (5) and (6), we get

$$Z_L^1(Q_p, Q_r) = K_p \frac{D(1-f)}{Q_p} + K_r \frac{Df}{Q_r} + Q_p \varepsilon + Q_r \varepsilon + \frac{1}{2} h_r Q_r (1-f), \quad \text{for } Q_p \leq Q_r \frac{1-f}{f} \quad (29)$$

$$Z_L^2(Q_p, Q_r) = K_p \frac{D(1-f)}{Q_p} + K_r \frac{Df}{Q_r} + Q_p \varepsilon + Q_r \varepsilon + \frac{1}{2} h_r Q_p f, \quad \text{for } Q_p \geq Q_r \frac{1-f}{f} \quad (30)$$

From (29), for Teunter's heuristic $(P, 1)$ policy, $Q_r = Q_p = M$, as Q_r is adjusted to a multiple of the Q_p^* that is obtained by optimizing the terms in Q_p separately. Clearly, in the

best lower bound as well at the optimal $(P, 1)$ policy, Q_p and Q_r should be jointly optimized subject to the restriction that $Q_p \leq Q_r (1 - f)/f$. Similarly, from (30), it is easy to show that Teunter's heuristic $(1, R)$ policy can result in values of $Q_r = Q_p = M$. Therefore, in cases where the production rate is close to the demand rate, cost of Teunter's heuristic policy can be very large compared to the lower bound.

4.2. Numerical Example

We illustrate the performance bounds obtained above using two sets of numerical examples. In the first set of numerical examples, the production rate is much larger than the demand rate. The parameters used for these set of examples are given in Table 2.

Table 2: Parameter Settings for the Example Set 1

D	p	r	K_p	K_r	h_s	h_r
1000	5000	3000	20	5	10	2

The return rate f used are (0.15, 0.25, 0.35, 0.45, 0.55, 0.65, 0.75, 0.85, and 0.95). The cost of the lower bound, the cost of the solution obtained by our proposed optimal integer ratio policy and that of Teunter's heuristic policy are provided in Table 3. From the results shown in Table 3, we can see that the performance of the optimal integer ratio policy is within 1.5% of the lower bound (worst case 0.8557%), and the performance of Teunter's heuristic policy is within 6% of the lower bound (worst case 3.165%).

Table 3: Cost of Lower Bound, Optimal Integer Ratio Policy and Teunter's Heuristic for Example Set 1

f	Lower bound (A)	Cost of the optimal integer policy (B)	Percent difference ((B-A)/B)	Cost of Teunter's policy (C)	Percent difference ((C-A)/A)
0.15	546.7871	547.0375	0.0458%	547.0719	0.0521%
0.25	517.8055	518.411	0.1169%	518.545	0.1428%
0.35	490.6581	490.6628	0.0010%	490.6816	0.0048%
0.45	468.7466	472.7579	0.8557%	483.5825	3.1650%
0.55	446.4243	446.6542	0.0515%	447.0843	0.1479%
0.65	423.4307	423.458	0.0064%	423.4935	0.0148%
0.75	399.2149	399.6874	0.1184%	400.1046	0.2229%
0.85	372.3254	372.3957	0.0189%	372.4343	0.0293%
0.95	336.5239	336.5264	0.0007%	336.527	0.0009%

In the second set of numerical examples, the production rate is close to the demand rate. The parameters used for these set of examples are provided in Table 4.

Table 4: Parameter Settings for the Example Set 2

D	$p = r$	K_p	K_r	h_s	h_r	f
1000	1000+x	20	20	10	5	0.5

The numerical results for these set of examples are provided in Table 5. As can be seen from these results, the optimal integer ratio policy obtains solutions that are the same as the lower bound in these cases, but Teunter's heuristic obtains solutions that are much higher than the lower bound.

Table 5: Cost of Lower Bound, Optimal Integer Ratio Policy and Teunter's Heuristic for Example set 2

$p = r$	Lower bound (A)	Cost of the optimal integer policy (B)	Percent difference ((B-A)/B)	Cost of Teunter's policy (C)	Percent difference ((C-A)/A)
1000.000001	316.228	316.228	0.00%	20411241.50	6454500.61%
1000.001	316.229	316.229	0.00%	64550.46	20312.60%
1001	317.017	317.017	0.00%	2064.70	551.29%
1010	323.961	323.961	0.00%	719.37	122.05%
1050	351.866	351.866	0.00%	450.75	28.10%
1100	318.385	318.385	0.00%	428.17	34.48%
1200	428.174	428.174	0.00%	447.99	4.63%
1300	464.096	464.096	0.00%	475.47	2.45%
1500	516.398	516.398	0.00%	521.75	1.04%

5. Conclusion

In this paper, we consider the problem of lot sizing and scheduling in manufacturing systems with product recovery. All the previous papers on the problem have focused on heuristic policies or specific classes of policies without any guaranteed performance bounds. In this paper, we develop simple heuristics with apriori performance guarantees.

We first develop a lower bound among all classes of policies for the problem, and then prove that the cost of optimal integer ratio policy (best of $(P, 1)$ or $(1, R)$ policy) obtains a solution whose cost is at most 1.5 percent more than the lower bound. We also show that the best of the two heuristic policies proposed by Teunter (2001, 2004) obtains a solution that is at most 6% more than the lower bound under certain conditions; however when the production rate for manufacturing and remanufacturing is close to the demand rate, we show that Teunter's heuristic can result in solutions with arbitrarily large costs.

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Appendix

Theorem A.1

In the finite horizon, T (where $I_r = 0$, at the beginning and the end of the horizon), let there be x sequences of a number of manufacturing cycles, followed by a number of remanufacturing cycles, with the manufacturing (remanufacturing) lot size in each cycle equal to Q_p (Q_r). The holding cost of the echelon inventory of the returned item for the period T is minimized by having all the x sequences identical with total manufacturing quantity (sum of the lot sizes in all the manufacturing setups) in each sequence, $\bar{Q} = TD(1-f)/x$ and correspondingly the total remanufacturing quantity in each sequence $= \bar{Q}f/(1-f) = TDf/x$.

Proof:

As already discussed, the production lot sizes do not impact the echelon inventory of the returned item. The other inventory costs in the lower bound for the problem are minimized by having stationary lot sizes Q_p and Q_r for manufacturing and remanufacturing. Given that the lot size for remanufacturing is Q_r , at the end of the previous remanufacturing sequence $i-1$, the echelon inventory of the returned item should at least be $Q_r(1-D/r)f$. Let the total lot size for manufacturing in the i^{th} sequence be q_i^P . Then the echelon inventory of the returned item a_i , at the end of the manufacturing cycles in the i^{th} sequence is at least $a_i = q_i^P f + Q_r(1-D/r)f$. Clearly to minimize the echelon inventory of returned items, the remanufacturing cycles should try to use up as much of the returned items as possible. Since the remanufacturing lot size is Q_r , the lowest possible value of echelon inventory b_i at the end of i^{th} sequence of manufacturing and remanufacturing is $Q_r(1-D/r)f$. Therefore, the average echelon inventory of the returned item in the i^{th}

sequence is $Q_r(1-D/r)f + (1/2)q_i^P f$.

To minimize the echelon inventory of the returned item, the length of the i^{th} sequence of remanufacturing cycles should be large enough to use up all the returned products generated. Hence the duration of the i^{th} sequence for manufacturing and remanufacturing is $t_i = (q_i^P/D) + (q_i^P f / (D(1-f))) = q_i^P / (D(1-f))$. The lower bound on the average echelon inventory of the returned item for the horizon T over x sequences is therefore

$$\sum_{i=1}^x t_i (Q_r(1-D/r)f + (1/2)q_i^P f) = TQ_r(1-D/r)f + \sum_{i=1}^x (1/2D)(q_i^P)^2 f / (1-f)$$

where $T = \sum_{i=1}^x q_i^P / (D(1-f))$, and $q_i^P \geq 0$. Using the Kuhn-Tucker conditions for the minimizing the average echelon inventory, it is easily shown that $q_i^P = \bar{Q} = TD(1-f)/x$, for each of the x sequences and correspondingly, the total remanufacturing quantity in each sequence is $\bar{Q}f / (1-f) = TDf / x$. This proves the result. $\square$

Lemma A.1: Let A and B be defined according to (20) and (21) respectively, then $A \geq B$.

Proof: Let $\alpha = \sqrt{x_p^1 y_p^1}$ and $\beta = \sqrt{x_r^1 y_r^1}$, Then $A = \alpha^2 + \beta^2$, and $B = 2\alpha\beta$. Therefore,

$$A - B = \alpha^2 + \beta^2 - 2\alpha\beta = (\alpha - \beta)^2 \geq 0. \quad \square$$