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On Adaptive Refinement Analysis for the Coupled Boundary Element Method – Reproducing Kernel Particle Method

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Abstract

An adaptive refinement procedure for the coupled boundary element-reproducing kernel particle method (coupled BE-RKPM) is suggested. The procedure is developed by combining a h -adaptive BEM refinement scheme based on the residual approach and an adaptive RKPM refinement scheme based on an extraction type *a posteriori* error estimator. During the adaptive refinement process for the coupled analysis, the displacement compatibility condition along the interface boundary of the BEM and the RKPM domains is maintained by using a penalty approach and a simple node spacing adjustment procedure. Thus, no specially formulated finite element is required along the interface boundary between the RKPM and the BEM domains. Numerical examples show that the suggested procedure significantly reduces the effect of singularity and nearly optimal convergence rate is obtained.

KEYWORDS: *Coupled boundary element-reproducing Kernel Particle Method; A priori and posteriori error estimations; Adaptive refinement.*

1. Introduction

In recent years, due to the rapid development of the meshless methods such as the reproducing kernel particle method (RKPM) [1], much work has been done in the coupling of the FEM and the meshless methods [2-4] and also the BEM with the meshless methods [5-7]. The main advantage of these coupling procedures is that the more expensive FEM or meshless methods can be applied to only a small but critical sub-domain, such as region near a crack tip, while the BEM is used in the remaining part of the infinite or semi infinite domain. Since in many cases the coupled methods are employed for solving of problems with high accuracy requirement, adaptive refinement is also an important topic of research for both the meshless methods and the coupled method. Early work on the research of the adaptive meshless methods can be found in the publications by Liu et al. [8, 9] while efforts to develop adaptive coupling procedures were reported in references [10] and [11]. The main objective of this paper is to suggest an adaptive refinement procedure for the coupled BE-RKPM developed by the authors [7]. In the suggested procedure, adaptive refinement in the RKPM domain will be carried out using the procedure developed in references [12] and [13] while the bisection procedure developed by Chen et al. [14] will be employed for the refinement in the BEM domain. In the next section, a brief description on the modelled problem and the coupled BE-RKPM used will be given. It is then followed by the reviews on the *a priori* and *a posteriori* error estimations for the coupled BE-RKPM. In Section 4, the adaptive refinement procedure for the coupled BE-RKPM will be described. Numerical examples are employed in Section 5 to demonstrate the efficiency of the proposed adaptive procedure and finally conclusions of the study will be given.

2. Formulations for the BEM, the RKPM and the coupled BE-RKPM

2.1 The modelled problem

The model problem considered is the 2D elastostatic problem over the domain Ω with boundary $\Gamma = \Gamma_u \cup \Gamma_t$ defined by the equation

$$\mathbf{L}^T \boldsymbol{\sigma} + \mathbf{b} = 0 \quad (1a)$$

such that $\Gamma_u \cap \Gamma_t = \emptyset$ and

$$\begin{aligned} \mathbf{u} &= \bar{\mathbf{u}}, \text{ on the essential boundary } \Gamma_u \\ \mathbf{T}\boldsymbol{\sigma} &= \mathbf{T}\mathbf{D}\mathbf{L}\mathbf{u} = \bar{\mathbf{t}}, \text{ on the natural boundary } \Gamma_t \end{aligned} \quad (1b)$$

In Eq. (1), $\boldsymbol{\sigma}$ and $\mathbf{b}$ are the stress tensor and the body force vector respectively. $\bar{\mathbf{u}}$ and $\bar{\mathbf{t}}$ are the prescribed displacements and tractions. $\mathbf{L}$ and $\mathbf{T}$ is a linear operator and a directional matrix respectively and they are defined as

$$\mathbf{L}^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \text{ and } \mathbf{T} = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix} \quad (2)$$

$\mathbf{D}$ is the material matrix depends on E , the Young modulus and ν , the Poisson's ratio.

2.2 The RKPM formulation employed

In the RKPM an approximation $\mathbf{u}^R$ to $\mathbf{u}$ is constructed by employing a *discrete corrected kernel* in the form

$$\mathbf{u}^R(x, y) = \sum_{i=1}^{NP} N_i^R(x, y, \Delta A_i) \hat{\mathbf{u}}_i^R = (\mathbf{N}^R)^T \hat{\mathbf{u}}^R \quad (3)$$

where NP is the number of nodes in the mesh. N_i^R and $\hat{\mathbf{u}}_i^R$ are respectively the shape function and the nodal parameter of node i . ΔA_i is the area of Ω_i , the domain of influence of node i with support size h_i . $\mathbf{N}^R$ is the matrix of the RKPM shape functions and $\hat{\mathbf{u}}^R$ is the

vector of nodal parameters. In the RKPM, the essential boundary condition is enforced by the penalty method [7]. The final system of equations for the RKPM can be expressed as

$$\mathbf{K}^R \hat{\mathbf{u}}^R = \mathbf{f}^R \quad (4a)$$

$$\mathbf{K}_{ij}^R = \int_{\Omega} (\mathbf{B}_i^R)^T \mathbf{D} \mathbf{B}_j^R d\Omega + \alpha \int_{\Gamma_u} N_i^R \mathbf{S} N_j^R d\Gamma \text{ and } \mathbf{f}_i^R = \int_{\Omega} N_i^R \mathbf{b} d\Omega + \int_{\Gamma_t} N_i^R \bar{\mathbf{t}} d\Gamma + \alpha \int_{\Gamma_u} N_i^R \mathbf{S} \bar{\mathbf{u}} d\Gamma \quad (4b)$$

$$\mathbf{B}_i^R = \begin{bmatrix} N_{i,x}^R & 0 \\ 0 & N_{i,y}^R \\ N_{i,y}^R & N_{i,x}^R \end{bmatrix}, \mathbf{S} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \text{ and } S_i = \begin{cases} 1 & \text{if } \hat{\mathbf{u}}_i^R \text{ is prescribed on } \Gamma_u \\ 0 & \text{if } \hat{\mathbf{u}}_i^R \text{ is not prescribed on } \Gamma_u. \end{cases} \quad (4c)$$

In Eq. (4b), the factor $\alpha=5 \times 10^6 E$ is the penalty term to enforce the essential boundary condition. After $\hat{\mathbf{u}}^R$ is solved, the RPKM stress $\boldsymbol{\sigma}^R$ is computed as

$$\boldsymbol{\sigma}^R = \mathbf{D} \mathbf{L} \hat{\mathbf{u}}^R \quad (5)$$

2.3 The BEM formulation employed

In the BEM formulation, Eq. (1) is rewritten in terms of the virtual displacement $\mathbf{u}^*$, the virtual surface traction $\mathbf{t}^*$ and the fundamental solution $\nabla \boldsymbol{\sigma}^* + \boldsymbol{\delta}^i = 0$ where $\boldsymbol{\delta}^i$ is the Dirac delta function as

$$c^i \mathbf{u}^i + \int_{\Gamma} \mathbf{u} \mathbf{t}^* d\Gamma = \int_{\Gamma} \mathbf{t} \mathbf{u}^* d\Gamma + \int_{\Omega} \mathbf{b} \mathbf{u}^* d\Omega \quad (6)$$

$\mathbf{u}$ and $\mathbf{t}$ are approximated as

$$\mathbf{u} \approx \mathbf{u}^B = (\mathbf{N}_u^B)^T \hat{\mathbf{u}}^B \text{ and } \mathbf{t} \approx \mathbf{t}^B = (\mathbf{N}_t^B)^T \hat{\mathbf{t}}^B \quad (7)$$

where $\mathbf{N}_u^B$ and $\mathbf{N}_t^B$ are the BE shape functions of $\mathbf{u}$ and $\mathbf{t}$ along Γ_t and Γ_u respectively. $\hat{\mathbf{u}}^B$ and $\hat{\mathbf{t}}^B$ are respectively the nodal values of $\mathbf{u}^B$ and $\mathbf{t}^B$. Substituting Eq. (7) into Eq. (6), the final system of equations for the BEM can be expressed as

$$\mathbf{H} \hat{\mathbf{u}}^B = \mathbf{G} \hat{\mathbf{t}}^B + \mathbf{Q} \quad (8a)$$

$$\mathbf{H} = \mathbf{c} + \int_{\Gamma} (\mathbf{N}_u^B)^T \mathbf{t}^* d\Gamma, \quad \mathbf{G} = \int_{\Gamma} (\mathbf{N}_t^B)^T \mathbf{u}^* d\Gamma, \quad \mathbf{Q} = \int_{\Omega} \mathbf{b} \mathbf{u}^* d\Omega \quad (8b)$$

After $\hat{\mathbf{u}}^B$ and $\hat{\mathbf{t}}^B$ are solved, the BEM stress $\boldsymbol{\sigma}^B$ can be computed as

$$\boldsymbol{\sigma}^B = \boldsymbol{\Psi}^B(\hat{\mathbf{u}}^B, \hat{\mathbf{t}}^B) + \boldsymbol{\Phi}^B \quad (9)$$

where $\boldsymbol{\Psi}^B$ and $\boldsymbol{\Phi}^B$ are boundary integral operators depend on the fundamental solution and the body force term [12].

2.4 The couple BE-RKPM

In the directly coupling BE-RKPM formulation [7], the problem domain Ω is partitioned into two disjoint subdomains Ω^R and Ω^B (Fig. 1) where the RKPM and the BEM are applied respectively. The boundaries of Ω^R and Ω^B are denoted as Γ^R and Γ^B respectively with the following boundary conditions:

$$\text{For } \Omega^R: \quad \Gamma^R = \Gamma_u^R \cup \Gamma_t^R \cup \Gamma^{RB} \text{ such that } \mathbf{u} = \bar{\mathbf{u}}^R \text{ on } \Gamma_u^R \text{ and } \mathbf{t} = \bar{\mathbf{t}}^R \text{ on } \Gamma_t^R \quad (10a)$$

$$\text{For } \Omega^B: \quad \Gamma^B = \Gamma_u^B \cup \Gamma_t^B \cup \Gamma^{BR} \text{ such that } \mathbf{u} = \bar{\mathbf{u}}^B \text{ on } \Gamma_u^B \text{ and } \mathbf{t} = \bar{\mathbf{t}}^B \text{ on } \Gamma_t^B \quad (10b)$$

The displacements and tractions compatibility conditions along Γ^{RB} can be expressed as

$$\mathbf{t}^{RB} = -\mathbf{t}^{BR} \quad \text{and} \quad \mathbf{u}^{RB} = \mathbf{u}^{BR} \quad (11)$$

where $\mathbf{t}^{RB}$ ($\mathbf{u}^{RB}$) and $\mathbf{t}^{BR}$ ($\mathbf{u}^{BR}$) are the tractions (displacements) along Γ^{RB} with respect to Ω^R and Ω^B respectively. For Ω^R , Γ^{RB} is treated as an essential boundary such that $\mathbf{u} = \mathbf{u}^{RB}$ on Γ^{RB} and $\mathbf{u}^{RB}$, the displacements along Γ^{RB} can be expressed as

$$\mathbf{u}^{RB} \approx (\mathbf{N}^{RB})^T \hat{\mathbf{u}}^{RB} + (\mathbf{N}^C)^T \hat{\mathbf{u}}^C \quad (12)$$

where $\mathbf{N}^{RB}$ and $\hat{\mathbf{u}}^{RB}$ are the RKPM shape functions and the nodal parameters along Γ^{RB} respectively. $\mathbf{N}^C$ and $\hat{\mathbf{u}}^C$ are, respectively, RKPM shape functions and nodal parameters associates with some nodes near Γ^{RB} since the RPKM shape functions lack the Konecker delta function property. The equations system for Ω^R can be expressed as (c.f. Eq. (4))

$$\mathbf{K}^R \hat{\mathbf{u}}^R = \mathbf{f}^R + \mathbf{f}^{RB} \quad (13a)$$

$$\mathbf{K}_{ij}^R = \int_{\Omega^R} \mathbf{B}_i^T \mathbf{D} \mathbf{B}_j d\Omega + \alpha \int_{\Gamma_u^R + \Gamma^{RB}} N_i^R \mathbf{S} N_j^R d\Gamma \quad (13b)$$

$$\mathbf{f}_i^R = \int_{\Omega^R} N_i^R \mathbf{b} d\Omega + \int_{\Gamma_t^R} N_i^R \bar{\mathbf{t}}^R d\Gamma + \alpha \int_{\Gamma_u^R} N_i^R \mathbf{S} \bar{\mathbf{u}}^R d\Gamma \quad \text{and} \quad \mathbf{f}_i^{RB} = \alpha \int_{\Gamma^{RB}} N_i^R \mathbf{S} (\mathbf{N}^{RB})^T \hat{\mathbf{u}}^{RB} d\Gamma \quad (13c)$$

For Ω^B where BEM is applied, the following boundary conditions are used

$$\mathbf{u} = \bar{\mathbf{u}}^B \quad \text{on} \quad \Gamma_u^B \quad \text{and} \quad \mathbf{t} = \bar{\mathbf{t}}^B \quad \text{on} \quad \Gamma_t^B \quad (14a)$$

$$\mathbf{u} = \mathbf{u}^{RB} \quad \text{and} \quad \mathbf{t} = -\mathbf{t}^{RB} \quad \text{on} \quad \Gamma^{RB} \quad (14b)$$

If the known terms $\bar{\mathbf{u}}^B$ and $\bar{\mathbf{t}}^B$ and the unknowns $\mathbf{u}_{\Gamma_t}^B$, $\mathbf{t}_{\Gamma_u}^B$, $\mathbf{t}^{RB}$ and $\mathbf{u}^{RB}$ are interpolated as

$$\bar{\mathbf{u}}^B = (\mathbf{N}_u^B)^T \hat{\mathbf{u}}_{\Gamma_u}^B, \quad \bar{\mathbf{t}}^B = (\mathbf{N}_t^B)^T \hat{\mathbf{t}}_{\Gamma_t}^B \quad (15a)$$

$$\mathbf{u}_{\Gamma_t}^B = (\mathbf{N}_u^B)^T \hat{\mathbf{u}}_{\Gamma_t}^B, \quad \mathbf{t}_{\Gamma_u}^B = (\mathbf{N}_t^B)^T \hat{\mathbf{t}}_{\Gamma_u}^B \quad (15b)$$

$$\mathbf{t}^{RB} = (\mathbf{N}_t^B)^T \hat{\mathbf{t}}^{RB}, \quad \mathbf{u}^{RB} = (\mathbf{N}_u^B)^T \hat{\mathbf{u}}^{RB} \quad (15c)$$

where $\hat{\mathbf{u}}_{\Gamma_u}^B$, $\hat{\mathbf{t}}_{\Gamma_t}^B$, $\hat{\mathbf{u}}_{\Gamma_t}^B$, $\hat{\mathbf{t}}_{\Gamma_u}^B$, $\hat{\mathbf{t}}^{RB}$ and $\hat{\mathbf{u}}^{RB}$ are the nodal values corresponding to $\bar{\mathbf{u}}^B$, $\bar{\mathbf{t}}^B$, $\mathbf{u}_{\Gamma_t}^B$,

$\mathbf{t}_{\Gamma_u}^B$, $\mathbf{t}^{RB}$ and $\mathbf{u}^{RB}$ respectively, the equations system for Ω^B can be expressed as (c.f. Eq. (8))

$$\begin{bmatrix} \mathbf{H}^{RB} & \mathbf{H}_{\Gamma_u}^B & \mathbf{H}_{\Gamma_t}^B \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}^{RB} & \hat{\mathbf{u}}_{\Gamma_u}^B & \hat{\mathbf{u}}_{\Gamma_t}^B \end{bmatrix}^T = \begin{bmatrix} -\mathbf{G}^{RB} & \mathbf{G}_{\Gamma_u}^B & \mathbf{G}_{\Gamma_t}^B \end{bmatrix} \begin{bmatrix} \hat{\mathbf{t}}^{RB} & \hat{\mathbf{t}}_{\Gamma_u}^B & \hat{\mathbf{t}}_{\Gamma_t}^B \end{bmatrix}^T + \mathbf{Q} \quad (16a)$$

$$\mathbf{H}^{RB} = \mathbf{c} + \int_{\Gamma^{RB}} (\mathbf{N}_u^B)^T \mathbf{t}^* d\Gamma, \quad \mathbf{H}_{\Gamma_u}^B = \mathbf{c} + \int_{\Gamma_u^B} (\mathbf{N}_u^B)^T \mathbf{t}^* d\Gamma, \quad \mathbf{H}_{\Gamma_t}^B = \mathbf{c} + \int_{\Gamma_t^B} (\mathbf{N}_u^B)^T \mathbf{t}^* d\Gamma \quad (16b)$$

$$\mathbf{G}^{RB} = \int_{\Gamma^{RB}} (\mathbf{N}_t^B)^T \mathbf{u}^* d\Gamma, \quad \mathbf{G}_{\Gamma_u}^B = \int_{\Gamma_u^B} (\mathbf{N}_t^B)^T \mathbf{u}^* d\Gamma, \quad \mathbf{G}_{\Gamma_t}^B = \int_{\Gamma_t^B} (\mathbf{N}_t^B)^T \mathbf{u}^* d\Gamma, \quad \mathbf{Q} = \int_{\Omega^B} \mathbf{b} \mathbf{u}^* d\Omega \quad (16c)$$

Since both $\hat{\mathbf{t}}^{RB}$ and $\hat{\mathbf{u}}^{RB}$ on Γ^{RB} are unknowns, Eq. (16a) could not be solved directly and it

is required to relate $\mathbf{u}^{RB}$ with $\mathbf{t}^{RB}$. Towards this end, by using the unit displacement method

[7], the relationship between $\hat{\mathbf{t}}^{RB}$ and $\hat{\mathbf{u}}^{RB}$ can be written as

$$\hat{\mathbf{t}}^{RB} = \mathbf{W}^{RB} \hat{\mathbf{u}}^{RB} + \hat{\mathbf{t}}^0 \quad (17)$$

where the matrix $\mathbf{W}^{RB}$ and the vector $\hat{\mathbf{t}}^0$ can be computed from the known terms $\mathbf{N}^R$, $\mathbf{K}^R$, $\mathbf{f}^R$ and $\mathbf{N}^C$. $\hat{\mathbf{t}}^{RB}$ can then be eliminated from Eq. (16a) for solving $\hat{\mathbf{u}}^{RB}$, $\hat{\mathbf{t}}_{\Gamma_u}^B$ and $\hat{\mathbf{u}}_{\Gamma_t}^B$ and $\hat{\mathbf{u}}^R$.

3. *A priori and a posteriori error estimations*

3.1 *A priori error estimation for the coupled BE-RKPM*

The error in stress $\mathbf{e}_\sigma$ is defined as the difference between the exact and the computed stress

$$\mathbf{e}_\sigma = \boldsymbol{\sigma} - \boldsymbol{\sigma}^h \quad (18)$$

In Eq. (18), $\boldsymbol{\sigma}^h$ is the stress obtained by either the RKPM or the BEM. The energy norm of the exact solution $\|\mathbf{u}\|_\Omega$, and the energy norm of the error $\|\mathbf{e}_u\|_\Omega$, are defined as

$$\|\mathbf{u}\|_\Omega^2 = \int_\Omega (\mathbf{L}\mathbf{u})^T \mathbf{D}(\mathbf{L}\mathbf{u}) d\Omega = \int_\Omega \boldsymbol{\sigma}^T \mathbf{D}^{-T} \boldsymbol{\sigma} d\Omega \quad (19)$$

$$\|\mathbf{e}_u\|_\Omega^2 = \int_\Omega (\mathbf{L}\mathbf{e}_u)^T \mathbf{D}(\mathbf{L}\mathbf{e}_u) d\Omega = \int_\Omega (\mathbf{e}_\sigma)^T \mathbf{D}^{-T} \mathbf{e}_\sigma d\Omega \quad (20)$$

In practice, the relative error in energy norm, η , defined as

$$\eta = \|\mathbf{e}_u\|_\Omega / \|\mathbf{u}\|_\Omega \quad (21)$$

is adopted as a measure of the accuracy of the solution. With the definition of the error norm, the *a priori* error estimation for the coupled BE-RKPM can be summarized as

$$\|\mathbf{e}_u\|_\Omega = C (NDOF)^{-\gamma} \quad (22)$$

where C is constant independent of the overall convergence rate γ and $NDOF$, the total numbers of degree of freedom. It can be shown that for problems with singularity under uniform refinement, γ will be given by [7]

$$\lambda/2 < \gamma < \lambda \quad (23)$$

where $\lambda < 0.5$ is the strength of singularity of the exact solution. However, the value of γ can be improved by using an adaptive refinement to obtain a mesh with an *equilibration* of error

among all the degree of freedoms. In this case, if the singular point is located inside Ω^R , γ will be given by

$$\lambda/2 < p^R/2 \approx \gamma \quad (24)$$

where p^R is the order of approximation for the RKPM.

3.2 *A posteriori error estimation for the RKPM*

In this study, the Z-Z approach [13] is adopted for the *a posteriori* estimation for the RKPM solution. The basic idea of the Z-Z approach is to replace the exact stress field in Eq. (20) with a recovered stress field $\tilde{\sigma}^R$ such that the estimated error norm $\|\bar{e}_u^R\|$ is computed as

$$\bar{e}_\sigma^R = \tilde{\sigma}^R - \sigma^R, \quad \|\bar{e}_u^R\|_\Omega = \sqrt{\int_\Omega (\bar{e}_\sigma^R)^T \mathbf{D}^{-T} (\bar{e}_\sigma^R) d\Omega} = \sqrt{\int_\Omega (\tilde{\sigma}^R - \sigma^R)^T \mathbf{D}^{-T} (\tilde{\sigma}^R - \sigma^R) d\Omega} \quad (25)$$

In this study, an extraction approach [12] is employed to recover the stress at all RKPM nodes so that

$$\tilde{\sigma}_i^R(x_i, y_i) = \Psi^R(x_i, y_i, r_{R,i}) \sigma^B + \Phi^R(x_i, y_i, r_{R,i}) \quad \text{For } i=1, \dots, NP \quad (26)$$

where $\tilde{\sigma}_i^R(x_i, y_i)$ is the extracted stress at node i . $\Psi^R(x_i, y_i, r_{R,i})$ and $\Phi^R(x_i, y_i, r_{R,i})$ are two local extraction operators similar to the integral operators Ψ^B and Φ^B used in the stress evaluation of the BEM (Eq. (9)). $r_{R,i}$ is the size of the *recovery subdomain* for node i which is a function of the support size h_i . $\tilde{\sigma}^R(x, y)$, the recovered stress at a point (x, y) is constructed by re-using the shape functions of the RKPM such that

$$\tilde{\sigma}^R(x, y) = \sum_{i=1}^{NP} N_i^R \tilde{\sigma}_i^R \quad (27)$$

3.3 *A posteriori error estimation for the BEM*

In this study, the method suggested by Chen et al. [14] is adopted for the *a posteriori* estimation for the BEM solution. Firstly, the direct boundary stress $\hat{\sigma}^B(\mathbf{p}) = [\hat{\sigma}_x^B, \hat{\sigma}_y^B, \hat{\tau}_{xy}^B]^T$ at a given boundary point $\mathbf{p}$ (Fig. 2) is computed by the following steps.

- (i) The tractions $\mathbf{t}^B(\mathbf{p}) = [t_1^B, t_2^B]^T$ and the displacement $\mathbf{u}^B(\mathbf{p}) = [u_1^B, u_2^B]^T$ are computed.
- (ii) The direct boundary stress with respect to the local $\mathbf{n-t}$ coordinate system (Fig. 2),

$[\hat{\sigma}_n^B, \hat{\sigma}_t^B, \hat{\tau}_{nt}^B]^T$ are computed as [14]

$$\begin{aligned}\hat{\sigma}_n^B &= t_1^B \cos \theta + t_2^B \sin \theta, \quad \hat{\tau}_{nt}^B = -t_1^B \sin \theta + \hat{t}_2^B \cos \theta \\ \hat{\sigma}_t^B &= \left(2\bar{G} \left(-\frac{\partial u_1^B}{\partial \mathbf{t}} \sin \theta + \frac{\partial u_2^B}{\partial \mathbf{t}} \cos \theta \right) + \mu (t_1^B \cos \theta + t_2^B \sin \theta) \right) / (1 - \mu)\end{aligned}\quad (28)$$

where $\bar{\lambda} = \frac{E\mu}{(1+\mu)(1-2\mu)}$ and $\bar{G} = \frac{E}{2(1+\mu)}$ so that $\mu = \nu$ or $(1-\nu)/\nu$ for plane strain

and plane stress problems respectively.

- (iii) The direct boundary stress, $\hat{\sigma}^B(\mathbf{p})$ with respect to the global $\mathbf{x-y}$ coordinate system is computed by standard tensor transformation.

Secondly, the *residual* of boundary stress at point $\mathbf{p}$, $\mathbf{e}_R^B(\mathbf{p})$, is defined as the difference between the exact boundary stress $\boldsymbol{\sigma}(\mathbf{p})$ and $\hat{\sigma}^B(\mathbf{p})$

$$\mathbf{e}_R^B(\mathbf{p}) = \boldsymbol{\sigma}(\mathbf{p}) - \hat{\sigma}^B(\mathbf{p}) \quad (29)$$

and the *residual norm*, $\|\mathbf{e}_R^B\|_\Gamma$, over Γ is defined as [14] (c.f. Eq. (20))

$$\|\mathbf{e}_R^B\|_\Gamma^2 = \int_\Gamma (\mathbf{e}_R^B)^T \mathbf{D}^{-T} \mathbf{e}_R^B d\Gamma \quad (30)$$

At point $\mathbf{p}$, a more accurate recovery stress $\tilde{\sigma}^B(\mathbf{p})$ can be extracted by using Eq. (9) so that

$$\tilde{\sigma}^B(\mathbf{p}) = \boldsymbol{\Psi}^B(\hat{\mathbf{u}}^B, \hat{\mathbf{t}}^B) + \boldsymbol{\Phi}^B \quad (31)$$

Finally, since $\tilde{\sigma}^B(\mathbf{p})$ is more accurate than $\hat{\sigma}^B(\mathbf{p})$, it can replace the exact stress in Eqs.

(29) and (30) for error estimation. Thus, $\bar{\mathbf{e}}_R^B(\mathbf{p})$, the *estimated residual* at point $\mathbf{p}$, $\|\bar{\mathbf{e}}_R^B\|_\Gamma$ the

estimated residual norm, and $\|\bar{\mathbf{e}}_R^B\|_{\Gamma_i}$, the *estimated residual norm* for the i th element can be

defined as

$$\bar{\mathbf{e}}_R^B(\mathbf{p}) = \tilde{\boldsymbol{\sigma}}^B(\mathbf{p}) - \hat{\boldsymbol{\sigma}}^B(\mathbf{p}) \quad \|\bar{\mathbf{e}}_R^B\|_{\Gamma} = \sqrt{\int_{\Gamma} (\bar{\mathbf{e}}_R^B)^T \mathbf{D}^{-T} \bar{\mathbf{e}}_R^B d\Gamma} \quad \|\bar{\mathbf{e}}_R^B\|_{\Gamma_i} = \sqrt{\int_{\Gamma_i} (\bar{\mathbf{e}}_R^B)^T \mathbf{D}^{-T} \bar{\mathbf{e}}_R^B d\Gamma} \quad (32)$$

4. Adaptive refinement procedures

4.1 Adaptive refinement for the RKPM region

In this study, the procedure described in [13] is employed for the refinement of the RKPM mesh for Ω_R . During the adaptive refinement analysis, the exact energy norm $\|\mathbf{u}\|_{\Omega_R}$ will be approximated by $\|\mathbf{u}^R\|_{\Omega_R}$, the energy norm of the recovered stress (c.f. Eq. (19))

$$\|\mathbf{u}\|_{\Omega_R}^2 \approx \|\mathbf{u}^R\|_{\Omega_R}^2 = \int_{\Omega_R} (\boldsymbol{\sigma}^R)^T \mathbf{D}^{-T} \boldsymbol{\sigma}^R d\Omega \quad (33)$$

and the *estimated relative error* $\bar{\eta}_{RKPM}$ will be computed as

$$\bar{\eta}_{RKPM} = \chi \|\bar{\mathbf{e}}_u^R\|_{\Omega_R} / \|\mathbf{u}^R\|_{\Omega_R} \quad (34)$$

In Eq. (34), $\chi=1.2$ is a compensational factor employed to improve the efficiency of the refinement procedure [13]. The objective of an adaptive refinement is to achieve an RKPM solution with relative error less than a user prescribed target η_{target}^R such that

$$\bar{\eta}_{RKPM} \leq \eta_{target}^R \quad \text{or} \quad \chi \|\bar{\mathbf{e}}_u^R\|_{\Omega_R} \leq \eta_{target}^R \|\mathbf{u}^R\|_{\Omega_R} \quad (35)$$

From Eq. (35), the *global refinement indicator*, ξ_g^R can be defined as

$$\xi_g^R = \frac{\chi \|\bar{\mathbf{e}}_u^R\|_{\Omega_R}}{\eta_{target}^R \|\mathbf{u}^R\|_{\Omega_R}} \quad (36)$$

A value of $\xi_g^R \leq 1$ implies that the target accuracy is satisfied. Otherwise, further refinement is needed. In case that refinement is needed, a *local refinement indicator* is computed by first using the partition-of-unity property of the RKPM shape functions to define the *estimated error norm contribution* [13] associates with the i th node $\|\bar{\mathbf{e}}^R\|_i$

$$\|\bar{\mathbf{e}}^R\|_i = \chi \sqrt{\int_{\Omega_i} N_i^R(\bar{\mathbf{e}}_\sigma^R)^T \mathbf{D}^{-T} N_i^R(\bar{\mathbf{e}}_\sigma^R) d\Omega} \quad (37)$$

Then by imposing the equilibration condition of $\|\bar{\mathbf{e}}^R\|_i$ among all the nodes for an optimal mesh,

$$\|\bar{\mathbf{e}}^R\|_i = \|\bar{\mathbf{e}}^R\|_j = \|\bar{\mathbf{e}}^R\|_{\Omega_R} / \sqrt{NP} \quad \text{For } 1 \leq i, j \leq NP \quad (38)$$

the local refinement indicator for node i , ξ_i^R , can be defined as

$$\xi_i^R = \frac{\|\bar{\mathbf{e}}^R\|_i}{\left(\|\bar{\mathbf{e}}^R\|_{\Omega_R} / \sqrt{NP}\right)} = \frac{\sqrt{NP} \|\bar{\mathbf{e}}^R\|_i}{\eta_{target}^R \|\mathbf{u}^R\|} \quad (39)$$

A value of $\xi_i^R > 1$ means that the mesh requires local refinement at node i .

With the definition of the global (ξ_g^R) and the local (ξ_i^R) refinement indicators, the new node spacing can be obtained by the following steps:

- (i) Retrieve h_i^{old} , the node spacing of the existing mesh.
- (ii) Compute the global refinement indicator, ξ_g^R , and the local refinement indicators, ξ_i^R , $i=1, \dots, NP$.
- (iii) The adaptive refinement procedure will be stopped if $\xi_g^R \leq 1$. Otherwise, the new node spacing h_i^{new} will be computed as [13]

$$h_i^{new} = \frac{h_i^{old}}{(\xi_i^R)^{-1/\beta_i}} \quad \text{where} \quad \beta_i = \begin{cases} p^R, & \text{if } \Omega_i \text{ contains no singular point} \\ \lambda, & \text{if } \Omega_i \text{ contains a singular point with strength} = \lambda \end{cases} \quad (40)$$

- (iv) Values of h_i^{new} will be fed into a point mesh generator [15] for new point mesh generation.

4.2 Adaptive refinement for the BEM region

In this study, the procedure described in [14] is employed for the refinement of the BE mesh for Ω_B . In the BEM, the *total boundary energy norm* $\|\mathbf{u}\|_\Gamma$ is defined as (c.f. Eq. (19))

$$\|\mathbf{u}\|_\Gamma^2 = \int_\Gamma \boldsymbol{\sigma}^T \mathbf{D}^{-T} \boldsymbol{\sigma} d\Gamma \quad (41)$$

During the refinement, the exact total boundary energy norm will be approximated by the total boundary energy norm of the recovery stress $\tilde{\boldsymbol{\sigma}}^B$

$$\|\mathbf{u}\|_\Gamma^2 \approx \|\tilde{\mathbf{u}}^B\|_\Gamma^2 = \int_\Gamma (\tilde{\boldsymbol{\sigma}}^B)^T \mathbf{D}^{-T} \tilde{\boldsymbol{\sigma}}^B d\Gamma \quad (42)$$

The *estimated relative residual error* of the BEM solution $\bar{\eta}_{BEM}$ is defined as

$$\bar{\eta}_{BEM} = \|\bar{\mathbf{e}}_R^B\|_\Gamma / \|\tilde{\mathbf{u}}^B\|_\Gamma \quad (43)$$

where $\|\bar{\mathbf{e}}_R^B\|_\Gamma$ is the estimated residual norm (Eq. (32)). The objective of the adaptive refinement is to achieve a BEM solution with $\bar{\eta}_{BEM}$ less than a user prescribed target η_{target}^B such that

$$\bar{\eta}_{BEM} \leq \eta_{target}^B \quad \text{or} \quad \|\bar{\mathbf{e}}_R^B\|_\Gamma \leq \eta_{target}^B \|\tilde{\mathbf{u}}^B\|_\Gamma \quad (44)$$

Eq. (44) leads to the following *global boundary refinement indicator*

$$\xi_g^B = \frac{\bar{\eta}_{BEM}}{\eta_{target}^B} = \frac{\|\bar{\mathbf{e}}_R^B\|_\Gamma}{\eta_{target}^B \|\tilde{\mathbf{u}}^B\|_\Gamma} \quad (45)$$

A value of $\xi_g^B > 1$ implies that further refinement is needed. In case that refinement is needed, a *local boundary refinement indicator* is computed by first defining the boundary energy norm $\|\tilde{\mathbf{u}}^B\|_{\Gamma_i}$ and the estimated relative residual error for the i th boundary element, $\bar{\eta}_{BEM,i}$ such that

$$\|\tilde{\mathbf{u}}^B\|_{\Gamma_i} = \sqrt{\int_{\Gamma_i} (\tilde{\boldsymbol{\sigma}}^B)^T \mathbf{D}^{-T} \tilde{\boldsymbol{\sigma}}^B d\Gamma} \quad \text{and} \quad \bar{\eta}_{BEM,i} = \|\bar{\mathbf{e}}_R^B\|_{\Gamma_i} / \|\tilde{\mathbf{u}}^B\|_{\Gamma_i} \quad (46)$$

From $\bar{\eta}_{BEM,i}$, the local boundary refinement indicator for the i th element ξ_i^B is computed as

$$\xi_i^B = \frac{\bar{\eta}_{BEM,i}}{\eta_{target}^B} = \frac{\|\bar{\mathbf{e}}_R^B\|_{\Gamma_i}}{\eta_{target}^B \|\tilde{\mathbf{u}}^B\|_{\Gamma_i}} \quad (47)$$

A value of $\xi_i^B > 1$ implies that further refinement of the i th boundary element is needed. With the definition of the global (ξ_g^B) and the local (ξ_i^B) boundary refinement indicators, the adaptive refinement procedure for the BEM can be summarized as the following steps:

- (i) Retrieve L_i , the boundary element size from the existing boundary mesh.
- (ii) Compute the global refinement indicators ξ_g^B and the local refinement indicators ξ_i^B , for $i=1, \dots, NBE$. NBE is the number of BE in Ω^B .
- (iii) The adaptive refinement procedure will be stopped if $\xi_g^B \leq 1$ is satisfied. Otherwise, the BE mesh will be refined according to steps (iv) and (v).
- (iv) Identify elements with $\xi_i^B > 1$. Subdivide them into two equal size elements.
- (v) Check all the boundary elements, subdivide the longer element of two neighbours elements if their length ratio is outside the range $[0.25, 4]$.

4.3 Nodes size adjustment along interface boundary

From Section 2.4, in the coupled BE-RKPM, the displacement compatibility condition along the interface boundary Γ^{RB} is enforced in a weak sense. Thus, it is required that the size of BE along Γ^{BR} and the RKPM node spacing near Γ^{RB} should be of the same order. Hence, if h_i and L_j are the RKPM node spacing near Γ^{RB} and the size of BE along Γ^{BR} respectively (Fig. 3). It is expected that during any refinement step

$$h_i \approx L_j \quad (48)$$

Eq. 48 can be enforced by making the end points of all BEs coincide with the RKPM nodes on Γ^{RB} . However, such requirement is not flexible in practice since different algorithms are often employed for the discretizations of Ω_B and Ω_R . In this study, a simple procedure summarized below is adopted to adjust the RKPM node spacing near Γ^{RB} before the new point mesh is generated.

- (i) Compute h_i^{new} and L_j by the procedures described in Sections 4.1 and 4.2 respectively.
- (ii) All the BE and their end points e_j that are lying along Γ^{RB} are identified.
- (iii) For a given RKPM node p_i , locate the corresponding nearest e_j on Γ^{RB} . If e_j is inside the domain of influence of p_i , adjust the value of h_i^{new} such that

$$h_i^{new} = \frac{1}{2}(L_j + L_{j+1}) \quad (49)$$

where L_j and L_{j+1} are the size of boundary element adjacent to e_j (Fig. 3).

4.4 Adaptive refinement for the coupled BE-RKPM

Based on the adaptive refinement strategies for the RKPM (Section 4.1) and the BEM (Section 4.2), together with the interface node spacing adjustment procedure (Section 4.3), the suggested refinement strategy for the coupled BE-RKPM can be summarised as follow:

- (i) Initialize the adaptive refinement procedure by setting the current number of refinement, $NCR=0$.
- (ii) Retrieve the existing RKPM and BEM meshes. Solve the coupled problem (Section 2.4).
- (iii) Perform the *a posteriori* error estimations for the RKPM (Section 3.2) and the BEM (Section 3.3) solutions. Compute the global and local refinement indicators for the RKPM (ξ_g^R and ξ_i^R , Section 4.1) and the BEM (ξ_g^B and ξ_i^B , Section 4.2) solutions.
- (iv) Stop the adaptive refinement procedure if any of the following conditions are satisfied:
 - (a) $\xi_g^R \leq 1$ and $\xi_g^B \leq 1$.

(b) $NCR = NMAXR$ where $NMAXR$ is the maximum number of refinement allowed.

Otherwise, set $NCR = NCR + 1$ and refine the mesh by steps (v) and (vi)

(v) Determine h_i^{new} for Ω_R (Section 4.1) and L_i for Ω_B (Section 4.2). Adjust the value of

h_i^{new} for nodes near Γ^{RB} using the procedure described in Section 4.3.

(vi) Generate the new point and boundary meshes and for the next cycle of analysis.

5. Numerical validations

In this section, the following two classic benchmark problems with singularity are employed to validate the performance of the adaptive refinement procedure for the coupled BE-RKPM.

Problem 1: Domain near a crack tip (convex domain)

In the first problem, the domain is a region near the tip of a crack (Fig. 4) loaded by tractions corresponding to the first symmetric mode of the stress intensity factor solution [16]:

$$\begin{aligned}\sigma_x &= \frac{I}{\sqrt{r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \\ \sigma_y &= \frac{I}{\sqrt{r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \\ \sigma_{xy} &= \frac{I}{\sqrt{r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}\end{aligned}\tag{50}$$

Due to symmetry, only half of the domain is modelled and the exact solution of this problem contains a singularity at (0,0) with $\lambda=0.5$.

Problem 2: L-Shaped Domain (concave domain)

In the second problem, a L-shaped plane (Fig. 5) loaded by tractions which satisfy equilibrium corresponding to the following exact stress solution [17]

$$\begin{aligned}\sigma_x &= \lambda r^{(\lambda-1)} \{ (2 - Q(\lambda+1)) \cos((\lambda-1)\theta) - (\lambda-1) \cos((\lambda-3)\theta) \} \\ \sigma_y &= \lambda r^{(\lambda-1)} \{ (2 + Q(\lambda+1)) \cos((\lambda-1)\theta) + (\lambda-1) \cos((\lambda-3)\theta) \} \\ \tau_{xy} &= \lambda r^{(\lambda-1)} \{ (\lambda-1) \sin((\lambda-3)\theta) + Q(\lambda+1) \sin((\lambda-1)\theta) \}\end{aligned}\tag{51}$$

is considered. In Eq. 51, $Q=0.543075579$ and $\lambda=0.544483737$ is the strength of singularity at point C. Note that rigid-body constraints are applied at points C and F.

For both problems, linear RKPM with $p^R=1$ and linear BEM are applied. Both uniform and adaptive refinements are carried out. The initial meshes used and the partitions of the problem domain are shown in Figs. 6a and 8a for Problems 1 and 2 respectively. Note that the partitions of the problem domains are *unchanged* during all refinements and the singularity of the problem is located inside Ω^R . For uniform refinement, successive meshes are obtained by reducing the elements size and node spacing uniformly. For adaptive refinement, adaptive meshes are generated by applying the procedure described in Section 4. Values for the target relative error for the RKPM (η_{target}^R) and target relative residual error for the BEM (η_{target}^B) are set equal to 5% while the maximum number of refinement allow ($NMAXR$) is set equal to 3.

In order to assess the performance of the adaptive refinement procedure, the exact relative error of the coupled BE-RKPM solution for the *whole* problem domain η (Eq. (21)) is computed. In addition, the overall convergence rate R , defined as

$$R = \frac{\log(\eta_i / \eta_{i+1})}{\log(NDOF_{i+1} / NDOF_i)} \quad (52)$$

is also computed. Since linear formulations are used in both the BEM and the RKPM, for uniform refinement, the convergence rate will be given by

$$\lambda/2 \leq R \leq \lambda \quad (53)$$

If the adaptive refinement can eliminate the effect of singularity, it is expected the convergence rate will be improved to

$$\lambda/2 \leq 0.5 \approx R \quad (54)$$

The adaptive meshes generated and the convergence history are shown in Figs. (6)-(9) while the properties of the uniform and adaptive meshes generated are summarized in Tables (1) and (2). From the numerical results, it can be seen for uniform refinement, the convergence rate is significantly reduced by the singularity. For adaptive refinement, the convergence rate is much improved and a nearly optimal value of $R \approx 0.4$ is obtained. Figs. (6) and (8) also indicate that both the BEM and RKPM meshes are adaptively refined at region near the singularity. As a result, it can be concluded that the adaptive refinement procedure can largely reduce the effect of singularity.

6. Conclusions

In this paper, an adaptive refinement procedure for the coupled boundary element-reproducing kernel particle method (BE-RKPM) is suggested. A h -adaptive scheme based on the residual method is adopted for the refinement of the BEM while an adaptive scheme based on an extraction type *a posteriori* error estimator is used for the refinement of the RKPM. The displacement compatibility condition along the interface boundary is enforced in a weak sense by using a penalty approach and a simple node spacing adjustment procedure. As a result, no special interface finite element is needed along the interface boundary between the RKPM and the BEM domains. Numerical examples given demonstrated that the suggested procedure can largely reduce the effect of singularity and nearly optimal convergence rate is restored in examples contain singular point.

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Figure. 1: Partition of the problem main Ω for the coupled BE-RKPM

Figure 2: Computation of direct boundary stress components at the boundary point p

Figure 3: BE and RKPM nodes and adjustment of node spacing along Γ^{RB}

Figure 4: Problem 1: Region near a crack tip

Figure 5: Problem 2: A L-shape domain

Figure 6: Adaptive coupled BE-RKPM meshes used for Problem 1

Figure 7: Rate of convergence for Problem 1

Figure 9: Rate of convergence for Problem 2

Mesh	NBE	NP	$NDOF$	η	R
Uniform refinement					
1	18	9	66	23.2	---
2	32	16	124	19.05	0.31
3	36	25	146	17.91	0.38
4	72	36	300	14.59	0.25
Adaptive refinement					
1	18	9	66	23.2	---
2	32	63	206	15.1	0.38
3	50	235	574	9.9	0.41
4	78	391	902	8.3	0.39

Table 1 Results of Problem 1

Mesh	NBE	NP	$NDOF$	η	R
Uniform refinement					
1	40	33	178	16.46	---
2	56	142	436	11.885	0.36
3	80	178	548	10.9	0.38
4	100	224	688	10.0	0.38
Adaptive refinement					
1	40	33	178	16.46	---
2	65	58	280	13.3	0.47
3	100	392	992	7.5	0.45
4	162	513	1290	6.7	0.43

Table 2 Results of Problem 2