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On Curvature Element-size Control in Metric Surface Mesh Generation

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Summary

A new procedure is suggested for controlling the element size distribution of surface meshes during automatic adaptive surface mesh generation. In order to ensure that the geometry of the surface can be accurately captured, the curvature properties of the surface are first analyzed. Based on the principal curvatures and principal directions of the surface, the curvature element-size requirement is defined in form of a metric tensor field. This element-size controlling metric tensor field, which can either be isotropic or anisotropic depends on the user requirement, is then employed to control the element size distribution during mesh generation. The suggested procedure is local, adaptive and can be easily used with many parametric surface mesh generators. As the proposed scheme defines the curvature element-size requirement in an implicit manner, it can be combined with any other user defined element size specification using the standard metric intersection procedure. This eventually leads to a simple implementation procedure and a high computational efficiency. Numerical examples indicate that the new procedure can effectively control the element size of surface meshes in the cost of very little additional computational effort.

Keywords: Automatic surface mesh generation, Metric triangulation, Principal curvatures and directions, Curvature element-size control, Parametric surfaces

1. Introduction

Surface mesh generation is one of the most important and yet difficult prerequisites for shell analysis [1-4], metal surface forming studies [5] as well as mesh generation in three dimensions [6-8]. Recently, with the rapid increase in computational speed of small to medium size computers and many advancements in adaptive finite element analysis [9-12], quite a number of new automatic surface mesh generation schemes [6,13-19] based on different approaches (generalized mapping, advancing front technique (AFT) and Delaunay triangulation method (DTM)) had been suggested. When deriving an automatic mesh generation scheme for 3D curved surfaces, the two central issues of interests are

- (1) To generate well-graded, good quality finite element meshes with element size distribution compatible with the user specification.
- (2) To ensure that the finite element meshes generated are good adaptive approximations of the target surfaces.

In the area of controlling the grading and the element size distribution of the mesh, the metric specification approach [19,20-23] was proved to be simple to implement and easy to use in practice. It can be conveniently used with both the AFT and the DTM. Furthermore, by carefully defining the metric specification over the problem domains [24], both isotopic and anisotropic finite element meshes with good grading and stretching characteristics satisfying the user requirements can be obtained. In addition, the metric specification is a very general approach and it can be applied to 2D, 3D surface as well as 3D mesh generation problems [25].

In the area of obtaining a good approximation of the original surface, the most straightforward solution is to limit the size of elements (or their edge lengths) in the mesh. Obviously, a global and uniform bound on the maximum allowable element or edge size is not an ideal and practical solution since the target surface to be discretized may consist of areas with very different curvatures. A more practical approach is to adaptively estimate the maximum allowable element size based on the local curvatures of the surface. Lohner [15] discussed the surface gridding techniques which can be operated on discretely defined surfaces and highlighted some essential procedures for post-generation and recovery when the AFT is used. A simple interpolatory subdivision procedure was suggested by Kobbelt [16] for the discretization of surfaces into uniform or adaptive structural meshes. It was shown in reference [16] that with a suitable choice of weighting parameters, the algorithm can achieve a

quadric convergent rate for C^1 surfaces. For parametric surfaces that can be represented by a smooth mapping procedure, based on the local bounds of the second derivatives of the surface, Sheng and Hirsch [26] and Piegl and Richard [27] suggested methods to adaptively estimate the maximum triangle-size bound of the elements in the mesh. However, their methods are isotopic in the sense that they did not make use of the directional nature of local surface curvatures for element size estimation. Hence, in some situations their methods may produce an unnecessary large number of elements at where the difference between the principal curvatures of the surface is great. To overcome this disadvantage, Anglada et al [28] extended and improved the element-size controlling procedure by using more precise bounds and took into consideration the directional behaviour of local surface curvatures and successfully reduced the number of triangles in the resulting triangulation.

The objective of this paper is to introduce a new approach for the proper control of element size during automatic adaptive surface mesh generation. The controlling scheme is anisotropic and will take into account the local and directional behaviour of the surface. Unlike many other controlling schemes suggested previously, the element-size controlling procedure will not be embedded *explicitly* into the automatic mesh generator. Instead, the curvature element-size requirement will be defined *implicitly* through the metric specification approach. One advantage of this approach is that it allows the user to employ the controlling scheme in conjunction with many metric surface mesh generators. Furthermore, the curvature element-size controlling metric can be regarded as the input of the mesh generator and therefore no modification of the surface mesh generator will be required. By combining the curvature element-size specification metric with the user defined metric, the surface mesh generator can then discretize the target surface into well-shaped elements with element size distribution compatible with the user requirement while the target surface is accurately approximated.

In the next section, a brief summary of the properties of parametric surfaces, which are essential ingredients to control the element size during mesh generation, will first be given. It will then be followed by the description of the element-size controlling requirement used in this study. Based on the results obtained, the construction of the curvature element-size controlling metric and its practical implementation will be given. Finally, several numerical examples will be provided to demonstrate the performance, effectiveness and reliability of the suggested scheme.

2. Basic differential geometry of parametric surfaces

First and second fundamental forms of parametric surfaces

In this section, a brief summary of some basic geometrical properties of parametric surfaces will be given in order to facilitate the description of the construction procedure of the curvature element-size controlling metric. More details related to the applications of differential geometry to surface design can be found in reference [29].

As in the previous studies [19], the geometry of target surface is defined by a bi-variate mapping of the form

$$(x, y, z)^T = \mathbf{r}(u, v) \quad (1)$$

and the relationship between, $\mathbf{dx}$, the elementary vector in the 3D space and $\mathbf{du}$, the corresponding vector in the parametric space is given by

$$\mathbf{dx} = (dx, dy, dz)^T = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = (\mathbf{r}_u \ \mathbf{r}_v) \mathbf{du} \quad (2)$$

where the vector $\mathbf{r}_u$ and $\mathbf{r}_v$ are tangents to the constant parameter lines on the surface. In general, they are not unit vectors nor are they orthogonal. From Eqn. 2, the well-known *first fundamental form* of the surface which governs the length scale transformation between the 3D and the parametric space is given by

$$\mathbf{dx}^T \mathbf{dx} = \mathbf{du}^T \begin{bmatrix} \mathbf{r}_u \cdot \mathbf{r}_u & \mathbf{r}_u \cdot \mathbf{r}_v \\ \mathbf{r}_u \cdot \mathbf{r}_v & \mathbf{r}_v \cdot \mathbf{r}_v \end{bmatrix} \mathbf{du} = \mathbf{du}^T \begin{bmatrix} E & F \\ F & G \end{bmatrix} \mathbf{du} \quad (3a)$$

such that

$$E = \mathbf{r}_u \cdot \mathbf{r}_u \quad F = \mathbf{r}_u \cdot \mathbf{r}_v \quad G = \mathbf{r}_v \cdot \mathbf{r}_v \quad (3b)$$

and

$$|\mathbf{r}_u \times \mathbf{r}_v| = EG - F^2 \geq 0 \quad (3c)$$

The unit normal vector $\hat{\mathbf{n}}$, of the tangential plane (Fig. 1) can be constructed by taking the cross product

$$\hat{\mathbf{n}} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \quad (4)$$

Similar to Eqn. 2, the rate of change of $\hat{\mathbf{n}}$ can be expressed as

$$d\hat{\mathbf{n}} = \hat{\mathbf{n}}_u du + \hat{\mathbf{n}}_v dv = (\hat{\mathbf{n}}_u \hat{\mathbf{n}}_v) d\mathbf{u} \quad (5)$$

and the *second fundamental form* of the surface (c.f. Eqn. 3a) is given by

$$d\hat{\mathbf{n}}^T d\mathbf{x} = d\mathbf{u}^T \begin{bmatrix} \hat{\mathbf{n}}_u \cdot \mathbf{r}_u & \hat{\mathbf{n}}_u \cdot \mathbf{r}_v \\ \hat{\mathbf{n}}_v \cdot \mathbf{r}_u & \hat{\mathbf{n}}_v \cdot \mathbf{r}_v \end{bmatrix} d\mathbf{u} = -d\mathbf{u}^T \begin{bmatrix} L & M \\ M & N \end{bmatrix} d\mathbf{u} \quad (6a)$$

$$\text{where} \quad L = -\hat{\mathbf{n}}_u \cdot \mathbf{r}_u = \hat{\mathbf{n}} \cdot \mathbf{r}_{uu} \quad M = -\hat{\mathbf{n}}_u \cdot \mathbf{r}_v = \hat{\mathbf{n}} \cdot \mathbf{r}_{uv} \quad N = -\hat{\mathbf{n}}_v \cdot \mathbf{r}_v = \hat{\mathbf{n}} \cdot \mathbf{r}_{vv} \quad (6b)$$

Note that Eqn. 6b follows from the fact that the vector $\hat{\mathbf{n}}$ is orthogonal to both $\mathbf{r}_u$ and $\mathbf{r}_v$ and therefore

$$\hat{\mathbf{n}} \cdot \mathbf{r}_u = 0 \quad \text{and} \quad \hat{\mathbf{n}} \cdot \mathbf{r}_v = 0 \quad (7a)$$

Differentiating Eqn. 7a with respect to u and v in turn will give

$$\begin{aligned} \hat{\mathbf{n}} \cdot \mathbf{r}_{uu} + \hat{\mathbf{n}}_u \cdot \mathbf{r}_u &= 0 \\ \hat{\mathbf{n}} \cdot \mathbf{r}_{uv} + \hat{\mathbf{n}}_u \cdot \mathbf{r}_v &= 0 \\ \hat{\mathbf{n}} \cdot \mathbf{r}_{vv} + \hat{\mathbf{n}}_v \cdot \mathbf{r}_v &= 0 \end{aligned} \quad (7b)$$

and finally leads to Eqn. 6b.

Principal curvatures and principal directions

At any point $\mathbf{P}$ on a parametric surface, one can always construct the tangential plane ($\mathbf{r}_u, \mathbf{r}_v$) and the unit norm vector $\hat{\mathbf{n}}$ using Eqns. 2 and 4. As any direction in the parametric space can always be represented by two infinitesimal quantities du and dv . It is more convenient to use their ratio, c , to represent the direction in the parametric space. That is

$$c = \frac{dv}{du} \quad (8)$$

will represent a given direction in the parametric space.

In general, the curve of intersection of a plane containing $\hat{\mathbf{n}}$ and the surface has a curvature κ (Fig. 2). It is well known that as the plane is rotated about the normal, the curvatures changes and it can be shown that unique direction for which the normal curvature is a minimum and a

maximum exists. The curvatures in these *principal directions* are called the *principal curvatures* and are denoted as $\kappa_{\max}$ and $\kappa_{\min}$. In the 3D space, the principal directions are represented by the vectors $\mathbf{d}_{\max}$ and $\mathbf{d}_{\min}$ respectively (Fig. 3) and they are orthogonal [29]. With the above notations, it can be shown that [29] the corresponding directions of $\mathbf{d}_{\max}$ and $\mathbf{d}_{\min}$ in the parametric space, $\mathbf{c}_{\max}$ and $\mathbf{c}_{\min}$, are the solutions of the following quadratic equation

$$(\text{FN}-\text{GM})c^2+(\text{EN}-\text{GL})c+(\text{EM}-\text{FL})=Ac^2+Bc+D=0 \quad (9a)$$

where

$$A=\text{FN}-\text{GM}, \quad B=\text{EN}-\text{GL}, \quad D=\text{EM}-\text{FL} \quad (9b)$$

In general, one or more of the coefficients A, B and D may equal to zero and according to their values, the point $\mathbf{P}$ can be classified as either an *ordinary point* or an *umbilical point*.

Ordinary Point

For an ordinary point, not all the coefficients A, B and D are equal to zero. In case that A is not equal to zero, the principal directions $\mathbf{c}_{\max}$ and $\mathbf{c}_{\min}$ can be obtained by solving Eqn. 9a. It should be noted that the two roots of Eqn. 9a are real since the term

$$B^2-4AD=(\text{EN}-\text{GL})^2-4(\text{FN}-\text{GM})(\text{EM}-\text{FL}) \quad (10a)$$

can be expressed as

$$\{(\text{EN}-\text{GL})-[2\text{F}(\text{EM}-\text{FL})/\text{E}]\}^2+4(\text{EG}-\text{F}^2)(\text{EM}-\text{FL})^2/\text{E}^2 \geq 0 \quad (10b)$$

since from Eqn. 3c, $\text{EG}-\text{F}^2 > 0$.

In the special case that $A=0$, $B \neq 0$ and $D \neq 0$, it appears that only one solution exists. However, it must be remembered that $c = \infty$, i.e. $du=0$, is also a possible solution. In this case, one can replace c by $f=1/c$ and Eqn. 9a can be written as

$$Bf+Df^2=0 \quad (11a)$$

which gives

$$f=0 \text{ and } f=-B/D \quad (11b)$$

as the principal directions.

In case that $A=D=0$ and $B \neq 0$, Eqn. 9a will reduce to the special case that

$$cB=0 \text{ or } fB=0 \quad (12a)$$

and obviously, the principal directions will be given by

$$c=0 \text{ and } f=0 \quad (12b)$$

That is, the direction of the parametric lines $u=\text{constant}$ and $v=\text{constant}$ are the principal directions in the parametric space.

Once the two roots of Eqn. 9a, c_1 and c_2 , are found, the principal directions in the parametric space can be represented by the vectors, $\mathbf{v}_i$, $i=1,2$ such that

$$\mathbf{v}_i = \left(\frac{1}{\sqrt{1+c_i^2}} \quad \frac{c_i}{\sqrt{1+c_i^2}} \right)^T \quad \text{if } c_i \neq \infty \quad (13a)$$

otherwise, one may use

$$\mathbf{v}_i = (0 \quad 1)^T \quad \text{if } c_i = \infty \text{ or } f_i = 0 \quad (13b)$$

and the corresponding unit vector in the 3D space can be obtained by using Eqn. 2 such that (Fig. 4)

$$\mathbf{d}_i = \frac{(\mathbf{r}_u \mathbf{r}_v) \mathbf{v}_i}{|(\mathbf{r}_u \mathbf{r}_v) \mathbf{v}_i|}, \quad i=1,2 \quad (14)$$

The values of the principal curvatures, κ_i , $i=1,2$, corresponding to c_i can be obtained by using the following equation [29]

$$\kappa_i = \begin{cases} \frac{L + Mc_i}{E + Fc_i} = \frac{M + Nc_i}{F + Gc_i} & \text{if } c_i \neq \infty \\ \frac{M}{F} = \frac{N}{G} & \text{otherwise} \end{cases} \quad (15a)$$

and

$$\kappa_{\max} = \max(\kappa_1, \kappa_2), \quad \kappa_{\min} = \min(\kappa_1, \kappa_2) \quad (15b)$$

Umbilical Point

In the special case that all the coefficients are equal to zero ($A=B=D=0$), the point $\mathbf{P}$ is an umbilical point and Eqn. 9a gives no information about the principal directions. In order to obtain the principal directions at $\mathbf{P}$, one has to include more terms in Eqn. 2 and Eqn. 9a will become a cubic equation. This implies one or three principal directions which are not

orthogonal. In the worse case that if all the coefficients in the cubic equation are again equal to zero, one have to include more terms in Eqn. 2 and this leads to more and more principal directions. Eventually, one may come at last to the limited case of a spherical point for which all directions are principal directions.

Despite the fact that principal directions are not known unless a higher order analysis is carried out, it can be shown that at an umbilical point, the surface normal curvature in any direction is given by [29]

$$\kappa = \frac{L}{E} = \frac{M}{F} = \frac{N}{G} \quad (16)$$

Hence, for the purpose of controlling element size according to the curvature of the surface, it is not necessary to carry out a higher order analysis to obtain the exact principal directions for an umbilical point. Therefore, in this study whenever it is found that $A=B=D=0$, it will be assumed that the principal directions are at the same directions as the parametric lines by setting $\mathbf{v}_1=(0,1)^T$ and $\mathbf{v}_2=(1,0)^T$.

Surface normal curvature in a general direction

If $\kappa_{\min}$ and $\kappa_{\max}$ are the principal curvatures of the surface in the unit principal directions $\mathbf{d}_{\min}$ and $\mathbf{d}_{\max}$ respectively and $\mathbf{q}$ is a unit vector on the tangential plane inclined at an angle θ with $\mathbf{d}_{\min}$ (Fig. 5), then the surface normal curvature along the $\mathbf{q}$ direction, $\kappa(\theta)$, is given by [29]

$$\kappa(\theta) = \kappa_{\min} \cos^2 \theta + \kappa_{\max} \sin^2 \theta \quad (17)$$

It should be noted that in Eqn. 17, the angle θ is measured on the tangential plane rather than in the parametric space. In addition, one should be reminded that the values of $\kappa_{\min}$ and $\kappa_{\max}$ are signed (i.e. they can either be less than, equal to or greater than zero) and Eqn. 17 means that for a certain value of θ , the normal surface curvature at the point may equal to zero even if both $\kappa_{\min}$ and $\kappa_{\max}$ are not equal to zero. For example, suppose that $\kappa_{\max} = -\kappa_{\min} = a > 0$, then the value of $\kappa(\theta)$ will equal to zero when $\theta = \pi/4$.

3. Curvature element-size control

In surface mesh generation, the objective of curvature element-size control is to guarantee that the geometry of the target surface is accurately captured up to a prescribed tolerance. As shown in Fig. 6, if l is the chord (or edge) length of an element in the mesh approximating the

arc with length equal to $s \geq l$ and δ is the maximum distance between the edge and the arc, then there are usually two ways [14,26-28] to prescribe the curvature element-size requirement for a surface mesh.

- (1) To prescribe the maximum value of δ such that it is less than a given tolerance τ for all the elements in the mesh. That is

$$\delta \leq \tau \quad (18)$$

- (2) To prescribe the maximum ratio between $s-l$ and s such that it is less than a given tolerance ε for all the edges in the mesh. That is

$$\frac{s-l}{s} \leq \varepsilon \quad (19a)$$

With some simple calculations, it can be proved that these two requirements are actually equivalent in the sense that setting $\varepsilon \rightarrow 0$ implies $\tau \rightarrow 0$ (Details of the proof are given in Appendix A). In the current study, Eqn. 19a will be used for curvature element-size control as it will simplify the construction procedure of the curvature element-size controlling metric. When estimating the element-size requirement, it is always desired to consider the limiting case of the tolerance condition such that Eqn. 19a is rewritten as

$$\frac{s-l}{s} = \varepsilon \quad \text{or} \quad l = (1.0 - \varepsilon)s \quad (19b)$$

With this requirement, it can be proved that [14] the relationship between s , ε and κ can be written as

$$s = \frac{1}{\kappa} \sqrt{40(1 - (1 - 1.2\varepsilon)^{0.5})} \quad \text{or} \quad l = \frac{(1 - \varepsilon)}{\kappa} \sqrt{40(1 - (1 - 1.2\varepsilon)^{0.5})} = \frac{g(\varepsilon)}{\kappa} \quad (20)$$

where $g(\varepsilon) = (1 - \varepsilon) \sqrt{40(1 - (1 - 1.2\varepsilon)^{0.5})}$ is a function of the prescribed tolerance ε only. If the value of ε is constant in the problem domain, it will only need to pre-compute $g(\varepsilon)$ once before the mesh generation process is started. From Eqn. 20, it can be seen that the desired edge length is inversely proportional to the curvature κ and will depend on the angle between the edge to be generated and the principal directions. From the results obtained from Eqns. 17 and 20, the desired element-size, $h(\theta)$, for a given edge making an angle θ with the principal direction $\mathbf{d}_{\min}$ on the tangential plane should equal to

$$h(\theta) = \frac{g(\varepsilon)}{\kappa(\theta)} = \frac{g(\varepsilon)}{\kappa_{\min} \cos^2 \theta + \kappa_{\max} \sin^2 \theta} \quad (21)$$

However, as mentioned in the last section, the value of $\kappa(\theta)$ may equal to zero. Thus, it is necessary to modify Eqn. 21 to the form

$$h(\theta) = \frac{g(\varepsilon)}{|\kappa_{\min}| \cos^2 \theta + |\kappa_{\max}| \sin^2 \theta} \leq \frac{g(\varepsilon)}{\kappa_{\min} \cos^2 \theta + \kappa_{\max} \sin^2 \theta} \quad (22)$$

Note that in practice $|\kappa_{\min}|$ and $|\kappa_{\max}|$ may equal to zero (e.g. for a cylinder, one may has $|\kappa_{\min}|=0$ and $|\kappa_{\max}|>0$ and for a flat surface $|\kappa_{\min}|=|\kappa_{\max}|=0$). It is often required to prescribe a global maximum element-size limit, $h_{\max}>0$, and a global minimum element-size limit, $h_{\min}>0$, during mesh generation. From these two limits, the equivalent global minimum curvature, $\kappa_{\min}$, and the global maximum curvature, $\kappa_{\max}$, can be obtained as

$$\kappa_{\max} = \frac{g(\varepsilon)}{h_{\min}} > 0 \quad \text{and} \quad \kappa_{\min} = \frac{g(\varepsilon)}{h_{\max}} > 0 \quad (23)$$

The next step is to rectify the principal curvature values in such a way that they will within the global range $[\kappa_{\min}, \kappa_{\max}]$. Hence, the rectified principal curvature values, $\bar{\kappa}_{\max}$ and $\bar{\kappa}_{\min}$, will be defined as

$$\begin{aligned} \bar{\kappa}_{\max} &= \min[\kappa_{\max}, \max(|\kappa_{\max}|, \kappa_{\min})] \\ \bar{\kappa}_{\min} &= \min[\kappa_{\max}, \max(|\kappa_{\min}|, \kappa_{\min})] \end{aligned} \quad (24)$$

and the following edge-size function will be used to control element size during mesh generation

$$h(\theta) = \frac{g(\varepsilon)}{\bar{\kappa}_{\min} \cos^2 \theta + \bar{\kappa}_{\max} \sin^2 \theta} \quad (25)$$

Note that the edge-size function defined in Eqn. 25, is anisotropic as it depends on the principal directions at the point. Furthermore, it also depends on, θ , the direction of the edge to be generated.

4. Curvature element-size controlling metric

In 3D surface mesh generation, it is more convenient to express the element size requirement in form of a 3D metric field and subsequently using the definition of the surface to transform it to the parametric space for mesh generation. Suppose that it is now required to generate an edge at the point $\mathbf{P}$ in the $\mathbf{q}$ direction such that the angle between the unit vector $\mathbf{q}$ and the principal direction $\mathbf{d}_{\min}$ is equal to θ (Fig. 5), then the length of the edge (or element size) should equal to $h(\theta)$ prescribed by Eqn. 25. Hence, if $\mathbf{M}$ is the desired 3D curvature element-size controlling metric, it should satisfy the following unit length condition [19-22]

$$\sqrt{(h(\theta)\mathbf{q})^T \mathbf{M} (h(\theta)\mathbf{q})} = 1 \quad (26)$$

That is, a vector with length $h(\theta)$ in the $\mathbf{q}$ direction will appear as a unit vector in the normalized space. Now consider the 3D curvature element-size controlling metric, $\mathbf{M}_{c3D}$, of the form

$$\mathbf{M}_{c3D} = [\mathbf{d}_{\min}, \mathbf{d}_{\max}, \hat{\mathbf{n}}] \begin{bmatrix} \left(\frac{\bar{\kappa}_{\min}}{g(\varepsilon)} \cos\theta \right)^2 & \frac{\bar{\kappa}_{\min} \bar{\kappa}_{\max}}{g(\varepsilon)^2} \cos\theta \sin\theta & 0 \\ \frac{\bar{\kappa}_{\min} \bar{\kappa}_{\max}}{g(\varepsilon)^2} \cos\theta \sin\theta & \left(\frac{\bar{\kappa}_{\max}}{g(\varepsilon)} \sin\theta \right)^2 & 0 \\ 0 & 0 & k \end{bmatrix} [\mathbf{d}_{\min}, \mathbf{d}_{\max}, \hat{\mathbf{n}}]^T \quad (27)$$

where k is any real number greater than or equal to zero.

Since

$$\mathbf{q} \cdot \mathbf{d}_{\min} = \cos\theta, \mathbf{q} \cdot \mathbf{d}_{\max} = \sin\theta \quad \text{and} \quad \mathbf{q} \cdot \hat{\mathbf{n}} = 0 \quad (28)$$

through simple evaluation, it can be proved that $\mathbf{M}_{c3D}$ will satisfy the condition stated in Eqn. 26. If the metric field defined by $\mathbf{M}_{c3D}$ is employed to control the element size distribution during mesh generation, then the output finite element mesh generated should able to satisfy the element size requirement stated in Eqn. 25. However, since the metric $\mathbf{M}_{c3D}$ depends on both the principal curvatures and the angle θ , if $\mathbf{M}_{c3D}$ is employed to control the element size, one must embed the metric specification explicitly in the mesh generator. That is, during each element formation step, before a new segment is created one has to first compute the terms $\mathbf{d}_{\min}$, $\mathbf{d}_{\max}$, $\bar{\kappa}_{\max}$, $\bar{\kappa}_{\min}$ and then check explicitly whether Eqn. 25 is satisfied or not according

the value of θ . This approach will inevitably require modifications of the existing mesh generation scheme and requires more computational cost during mesh generation.

An alternative way to enforce the curvature element-size requirement is to approximate the metric $\mathbf{M}_{c3D}$ by another metric which is independent of θ so that it can be applied implicitly during mesh generation. Consider the following 3D pseudo-curvature element-size controlling metric, $\mathbf{M}_{pc3D}$, of the form

$$\mathbf{M}_{pc3D} = [\mathbf{d}_{\min}, \mathbf{d}_{\max}, \hat{\mathbf{n}}] \begin{bmatrix} \left(\frac{\bar{\kappa}_{\min}}{g(\varepsilon)}\right)^2 & 0 & 0 \\ 0 & \left(\frac{\bar{\kappa}_{\max}}{g(\varepsilon)}\right)^2 & 0 \\ 0 & 0 & k \end{bmatrix} [\mathbf{d}_{\min}, \mathbf{d}_{\max}, \hat{\mathbf{n}}]^T \quad (29)$$

which is independent of θ . With some straightforward calculations, it can be shown that it will satisfy the following unit length conditions

$$\begin{aligned} \sqrt{\left(\frac{g(\varepsilon)}{\bar{\kappa}_{\min}} \mathbf{d}_{\min}\right)^T \mathbf{M}_{pc3D} \left(\frac{g(\varepsilon)}{\bar{\kappa}_{\min}} \mathbf{d}_{\min}\right)} &= 1 \\ \sqrt{\left(\frac{g(\varepsilon)}{\bar{\kappa}_{\max}} \mathbf{d}_{\max}\right)^T \mathbf{M}_{pc3D} \left(\frac{g(\varepsilon)}{\bar{\kappa}_{\max}} \mathbf{d}_{\max}\right)} &= 1 \end{aligned} \quad (30)$$

That is, in the two principal directions, the metric $\mathbf{M}_{pc3D}$ will satisfy the unit length condition exactly. By using Eqn. 28, in the general direction $\mathbf{q}$, a vector with length $h(\theta)$ will have a length equal to $h_p(\theta)$ in the normalized space such that

$$h_p(\theta) = \sqrt{(\mathbf{h}(\theta)\mathbf{q})^T \mathbf{M}_{pc3D} (\mathbf{h}(\theta)\mathbf{q})} = \frac{\sqrt{\bar{\kappa}_{\min}^2 \cos^2 \theta + \bar{\kappa}_{\max}^2 \sin^2 \theta}}{\bar{\kappa}_{\min} \cos^2 \theta + \bar{\kappa}_{\max} \sin^2 \theta} \quad (31)$$

Note that in the particular case that $\bar{\kappa}_{\min} = \bar{\kappa}_{\max}$, Eqn. 31 will be reduced to

$$h_p(\theta) \equiv 1 \quad (32)$$

and the unit length condition will be satisfied for all the values of θ .

In case that $\bar{\kappa}_{\min} \neq \bar{\kappa}_{\max}$, let η be the ratio $\bar{\kappa}_{\min}/\bar{\kappa}_{\max}$ such that $\kappa_{gmin}/\kappa_{gmax} \leq \eta \leq 1$, then Eqn. 31 can be rewritten as

$$h_p(\theta) = \frac{\sqrt{\eta^2 \cos^2 \theta + \sin^2 \theta}}{\eta \cos^2 \theta + \sin^2 \theta} \quad (33)$$

A graph showing the variation of $h_p(\theta)$ with different values of η is plotted in Fig. 7. From Fig. 7, it can be seen that the value of $h_p(\theta)$ is always greater than or equal to 1. Furthermore, the smaller the value of η , the greater the value of $h_p(\theta)$. This implies that when using $\mathbf{M}_{pc3D}$ to control the edge length during mesh generation, one can expect that the size of the elements generated will be in general greater than the desired value. However, from Fig. 7 it should be pointed out that the value of $h_p(\theta)$ is not significantly larger than unity unless (i) the value of η is very small (e.g. if $\eta=0.1$, $h_p(\theta) < 2$, $\eta=0.05$, $h_p(\theta) < 2.5$) and (ii) the value of θ is close to zero (i.e. when the direction of the vector $\mathbf{q}$ is close to the minimum principal direction $\mathbf{d}_{\min}$ where large edge length is required). In fact, it is found that when $\eta=0.01$, the $h_p(\theta) > 3$ only when $\theta \leq 17^\circ$. In addition, it should be mentioned that when deriving Eqn. 25, some safety factors had already been included by applying some conservative assumptions in Eqns. 19b and 22. Hence, in practice, it is found that $\mathbf{M}_{pc3D}$ will lead to good edge size distribution if the an addition safety factor, ω , defined as

$$\omega = 0.25\eta + 0.75 \quad (34)$$

is applied to reduce the edge size when η is close to zero. Consequently, in this study, the element-size controlling metric used in the mesh generation procedure will be given by

$$\mathbf{M}_{pc3D} = [\mathbf{d}_{\min}, \mathbf{d}_{\max}, \hat{\mathbf{n}}] \begin{bmatrix} \left(\frac{\bar{\kappa}_{\min}}{\omega \cdot g(\epsilon)} \right)^2 & 0 & 0 \\ 0 & \left(\frac{\bar{\kappa}_{\max}}{\omega \cdot g(\epsilon)} \right)^2 & 0 \\ 0 & 0 & k \end{bmatrix} [\mathbf{d}_{\min}, \mathbf{d}_{\max}, \hat{\mathbf{n}}]^T \quad (35)$$

Element-size controlling metric in the parametric space and implementation

The curvature element-size controlling metric given in Eqn. 35 defines the element size requirement in the 3D space. In parametric surface mesh generation, the mesh generation process will be carried out in the parametric space. Thus, the $\mathbf{M}_{pc3D}$ metric must be first transformed to the parametric space for mesh generation. This can be done by defining the parametric curvature element-size controlling metric, $\mathbf{M}_{pcuv}$ as [22]

$$\mathbf{M}_{\text{pcuv}} = (\mathbf{r}_u \ \mathbf{r}_v)^\top \mathbf{M}_{\text{pc3D}} (\mathbf{r}_u \ \mathbf{r}_v) \quad (36)$$

In addition to curvature element-size control, the users would normally like to define their own element size requirement by another user defined metric, $\mathbf{M}_{\text{u3D}}$, in the 3D space. Again, its counterpart, $\mathbf{M}_{\text{uuv}}$, in the parametric space can be obtained by the transformation

$$\mathbf{M}_{\text{uuv}} = (\mathbf{r}_u \ \mathbf{r}_v)^\top \mathbf{M}_{\text{u3D}} (\mathbf{r}_u \ \mathbf{r}_v) \quad (37)$$

Eventually, the combined effect of the two metric tensors in Eqns. 36 and 37 can be obtained by the standard metric intersection procedure described in references [19] and [22]. As a result, the final metric tensor used for parametric mesh generation is

$$\mathbf{M}_{\text{pcuv}} \cap \mathbf{M}_{\text{uuv}} \quad (38)$$

By using the metric tensor defined in Eqn. 38 to control the element size, both the curvature element-size and the user defined element size requirements can be enforced implicitly using the background mesh technique which is commonly used in adaptive mesh generation [9,11,12]. Firstly, both $\mathbf{M}_{\text{pc3D}}$ and $\mathbf{M}_{\text{u3D}}$ will be computed at all the nodal points of the background parametric mesh. Secondly, Eqns. 36 to 38 will be employed to obtain the final metric tensor which will be used by the mesh generator for mesh generation. Finally, the element-size controlling metric will be interpreted as input data by the surface mesh generator and hence no modification of the mesh generator is needed. The only additional computational effort needed is to implement a small preprocessor to compute the principal curvatures information and the $\mathbf{M}_{\text{pcuv}}$ metric at the nodal points of the background mesh and to carry out the metric intersection procedure (Eqn. 38). The speed of the mesh generation scheme will not be impaired.

It should be noted that the metric field shown in Eqn. 35 is anisotropic and hence the final 3D surface mesh generated will be an anisotropic mesh. Should the user prefer to obtain an isotopic mesh, the metric tensor in Eqn. 35 can be modified to the from

$$\mathbf{M}_{\text{pci3D}} = [\mathbf{d}_{\min}, \mathbf{d}_{\max}, \hat{\mathbf{n}}] \begin{bmatrix} \left(\frac{\bar{\kappa}_{\max}}{g(\varepsilon)} \right)^2 & 0 & 0 \\ 0 & \left(\frac{\bar{\kappa}_{\max}}{g(\varepsilon)} \right)^2 & 0 \\ 0 & 0 & k \end{bmatrix} [\mathbf{d}_{\min}, \mathbf{d}_{\max}, \hat{\mathbf{n}}]^\top \quad (39)$$

and the isotopic controlling metric, $\mathbf{M}_{\text{pci3D}}$ will lead to an isotopic 3D surface mesh, of course, provide that the user defined metric $\mathbf{M}_{\text{u3D}}$ (Eqn. 37) is also isotopic.

One final remark that worth to be mentioned here is the value of the constant k used in Eqns. 35 and 39. As the actual mesh generation processes are carried out in the parametric space, it is equivalent to generate edges and elements at the tangential plane of the point under consideration. Hence, in principle, the value of k will not affect the result of the generation process.

5. Numerical examples

In this section, three mesh generation examples will be presented to demonstrate the effectiveness of the proposed curvature element-size controlling scheme. In order to test the efficiency of the proposed scheme, for each example, meshes will be generated with and without the curvature element-size control.

When no curvature element-size control is imposed, an isotopic uniform metric of the form

$$\mathbf{M}_{\text{uni}} = \begin{bmatrix} \frac{1}{h_i^2} & 0 & 0 \\ 0 & \frac{1}{h_i^2} & 0 \\ 0 & 0 & \frac{1}{h_i^2} \end{bmatrix}, \quad h_i = \frac{h_0}{i+1}, i = 1, 2, 3, 4 \quad (40)$$

will be used to define a uniform element size field during refinement. In Eqn. 40, h_0 is the initialize element size which will be set equal to one third of length of the longest side of the target surface. h_i is the element size used during the i th refinement and a total for four refinements will be carried out.

When curvature element-size control is carried out, the uniform metric tensor defined in Eqn. 40 will be intersected with the element-size controlling metric given in Eqn. 35. Hence, the input metric for mesh generation will be of the form

$$\mathbf{M}_{\text{pcuv}} \cap (\mathbf{r}_u \ \mathbf{r}_v)^T \mathbf{M}_{\text{uni}} (\mathbf{r}_u \ \mathbf{r}_v) \quad (41)$$

A value of $\varepsilon=0.002$ (Eqn. 19a) such that $g(\varepsilon)=0.21915482$ (Eqn. 20) will be used in all the examples tested. The value of the constant k in $\mathbf{M}_{\text{pc3D}}$ will set equal to zero and again totally four refinements will be carried out. For both the cases of with and without element-size

control, the values of the global maximum and minimum element size, $h_{\max}$ and $h_{\min}$ (Eqn. 23) will be set equal to h_0 and $0.001h_0$ respectively.

In order to assess the effectiveness and performance of the curvature element-size controlling scheme, the edge lengths of all the mesh generated and the corresponding arc lengths will be computed. For a given line segment AB in the parametric space (Fig. 8), if (u_A, v_A) and (u_B, v_B) are the parametric coordinates of the end points A and B respectively, then the arc length $\overline{AB}$ can be computed as

$$\overline{AB} = \begin{cases} \left| \int_{u_1}^{u_2} \sqrt{E + 2Fc + Gc^2} du \right|, & \text{if } u_1 \neq u_2 \quad c = \frac{v_2 - v_1}{u_2 - u_1} \\ \left| \int_{v_1}^{v_2} \sqrt{G} dv \right|, & \text{otherwise} \end{cases} \quad (42)$$

Eqn. 42 will be evaluated by high-order (6-point) numerical integration rule. The edge size quality of the mesh will be measured by computing the arc length deviation factor, Δ , defined as

$$\Delta = \left(\frac{\overline{AB} - l_{AB}}{\overline{AB}} \right), \quad l_{AB} = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2} \quad (43)$$

for all the edges in the mesh. In Eqn. 43, (x_A, y_A, z_A) and (x_B, y_B, z_B) are the 3D coordinates of the points A and B respectively. Ideally, for a mesh with prefect curvature element-size distribution, the Δ values of all the edges should be less than or equal to ε . In practice, for a target surface with rapidly changing curvatures, the curvature element-size distribution of the mesh can be considered as good if there is less than 5% of edges with Δ outside the range $[0, 2\varepsilon]$.

In the three examples given in this section, the target surfaces selected are either cubic or quadric non-uniform rational B-spline (NURBS) surfaces [30] such that the principal curvatures of the surfaces vary continuously in the parametric space. However, in all examples, there exist some regions at where rapid changes in the principal curvatures and the principal directions occur. The exact analytical definitions of the surfaces will not be given here since they are lengthy but not very informative. A better and more concise picture of the geometries of the surfaces can be obtained by referring to the parametric nets of the surfaces as shown in Figs. 9a, 10a and 11a for Examples 1, 2 and 3 respectively. The initial meshes used in the numerical examples are shown in Figs. 9b, 10b and 11b and the generation

procedures are initialized by computing the metric tensors over the nodal points of these initial meshes. For both the cases of with and without curvature elements-size control, four refinements are carried out for all the examples problems. The final meshes generated without curvature element-size control are shown in Figs. 9c, 10c and 11c while Figs. 9d, 10d and 11d show the corresponding meshes obtained with element-size control. Furthermore, Table 1 summarize the characteristics of all the final meshes generated while the distributions of the arc length deviation factor Δ are shown in Figs. 9e, 10e and 11f for Examples 1, 2 and 3 respectively.

Examples	Curvature element-size control	NN	NE	NEDGE
1	Without	379	638	1016
	With	1989	3578	5566
2	Without	344	582	927
	With	2295	4456	6750
3	Without	300	533	832
	With	1181	2147	3327

Table 1. Characteristics of the final mesh generated

NN= Number of nodes in the mesh,

NE= Number of elements in the mesh

NEDGE = Number of edges in the mesh

A tolerance value of $\varepsilon=0.002$ was used in all the examples with element-size control.

Value of the constant k (Eqn. 35) was set equal to 0.0 for all the cases.

By comparing Fig. 9c with Fig. 9d, Fig. 10c with Fig. 10d and Fig. 11c with Fig. 11d, it can be seen that by controlling the element size distribution using $\mathbf{M}_{pc3D}$, the geometry of the target surfaces is more accurately captured, especially at where the changes in curvatures of the surfaces are large and rapid. The plots shown in Figs. 9e, 10e and 11f also confirm that whenever element-size control was employed, the percentages of edges with arc length deviation factor less than ε is greatly increased. In fact, in all the three examples more than 96% of all the edges in the final meshes generated with Δ inside the range $[0, 2\varepsilon]$ while when no control was employed only 66%-68% of edges with Δ inside the range $[0, 2\varepsilon]$.

Fig. 11e shows the final mesh obtained for Example 3 by setting $k=10000$ in Eqn. 35 during each refinement. It can be seen that the mesh generated is very similar to the one shown in Fig. 11d which is obtained by setting $k=0$. In addition, Fig. 11f indicates that the Δ distributions of the two meshes are in fact very similar while the information shown in Table 2 also demonstrates that the convergence of the procedure was not significantly affected by

the value of k used. This result reconfirms that as the element formation step is carried out in the parametric space, it is equivalent to generate the elements on the tangential plane span by the vectors $\mathbf{d}_{\min}$ and $\mathbf{d}_{\max}$. Hence, the value of k , which is effectively the element size in the normal direction $\hat{\mathbf{n}}$, has little effect on the mesh generation result.

Mesh	NN		NE		NEDGE	
	k=0	k=10000	k=0	k=10000	k=0	k=10000
1	362	365	652	631	986	995
2	579	576	1013	1007	1591	1582
3	942	920	1675	1628	2613	2547
4	1181	1174	2147	2133	3327	3306

Table 2. Characteristics of meshes generated by using different values of k , Example 3

The convergence history for Example 3 is shown in Fig. 12 (results from Examples 1 and 2 are very similar and hence are omitted here). From Fig. 12, it can be seen that the rate of convergence of the current controlling scheme is reasonably good and with only one or two iterations the percentage of edges with $\Delta > 5\epsilon$ is already reduced to less than 5%.

Finally, regarding the speed of the controlling procedure, as the curvature element-size requirements are defined implicitly as the input data of the mesh, it is found that the speed of the mesh generation procedure is virtually unaffected. Regarding the computational cost required for the calculation of the curvature element-size controlling metric, detailed timings indicate that the CPU time needed for the calculation was small and never accounted for more than 2.5% of the total mesh generation time used.

6. Conclusions

In this paper, a novel approach has been suggested to control the element size distribution during automatic surface mesh generation for parametric surfaces. The new scheme makes use of the principal curvature properties of the parametric surface and is local and adaptive. The curvature element-size controlling requirement is expressed implicitly as an input metric tensor field defining in the background mesh. The procedure is highly flexible in the sense that, depends on the user's requirement, the curvature element-size controlling metric tensor field can either be isotopic or anisotropic. Furthermore, it can be combined with the user's own node spacing specification through the simple procedure of metric intersection. As a result, the whole element-size controlling procedure can be implemented as a simple preprocessor

for the mesh generator. It can be conveniently used in conjunction with virtually all metric surface mesh generation schemes and no modification of the meshing scheme will be needed. Numerical examples given in the last section demonstrated that it can effectively improve the quality of the resulting meshes and the geometries of the target surfaces are accurately captured. The suggested element-size controlling scheme is highly efficient such that it will not affect the speed of the mesh generator. Detailed timings indicate that the CPU time needed to compute the element-size requirement is only a small fraction of the total CPU time needed for mesh generation.

Appendix A

Consider Fig. A1 where $AB=DC=\delta$, $AD=BC=l$ and the points E and F are the midpoints of the lines AD and BC respectively. As O is the centre of the arc $\overline{AD}$, $OA=OE=OD=r$. As the length of the arc $\overline{AE} = \frac{s}{2}$ is greater than the length of the line AE, one can write

$$\delta^2 + \left(\frac{l}{2}\right)^2 \leq \left(\frac{s}{2}\right)^2 \quad (A1)$$

which after simplification becomes

$$\begin{aligned} \delta &\leq \frac{1}{2} \sqrt{s^2 - l^2} \\ &\leq \frac{1}{2} \sqrt{(s-l)(s+l)} \\ &\leq \frac{1}{2} \sqrt{(\epsilon s)(2s)} \quad (\text{since } s \geq l \text{ and } \epsilon s \geq s-l) \\ &\leq \frac{1}{\sqrt{2}} s \sqrt{\epsilon} \end{aligned} \quad (A2)$$

Since the sum of the area of the triangle OAD and the rectangle ABCD is greater than the area of the circular sector OAC, one can write

$$\begin{aligned} \frac{1}{2} rs &\leq \frac{l(r-\delta)}{2} + l\delta \\ rs &\leq lr + l\delta \\ \frac{r(s-l)}{l} &\leq \delta \\ \frac{rs}{l} \epsilon &\leq \delta \end{aligned} \quad (A3)$$

Combining Eqn. A2 with Eqn. A3 gives

$$\frac{rs}{l} \epsilon \leq \delta \leq \frac{1}{\sqrt{2}} s \sqrt{\epsilon} \quad (A4)$$

which indicates that the two measures are equivalent in the sense that as $\epsilon \rightarrow 0$, it is also true that $\delta \rightarrow 0$.

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Figure 1. The unit normal vector $\hat{n}$

Figure 2. Curvature at the point **P**

Figure 3. Principal curvatures and directions at point **P**

3-D space

Parametric space

Figure 4. Principal directions in the parametric and the 3D spaces

3-D space

Figure 5. Curvature of the surface in the q direction

Figure 6. Curvature element-size deviation measurement

Figure 7. Variation of $h_p(\theta)$ with η

Figure 8. Computation of segment and arc lengths

Figure 9a. Geometry of Example 1

Figure 9b. Initial mesh for Example 1

Figure 9c. Final mesh for Example 1: Without element size control

Figure 9d. Final mesh for Example 1: With element size control

Figure 9e. Arc length deviation factor distributions for Example 1

Figure 10a. Geometry of Example 2

Figure 10b. Initial mesh for Example 2

Figure 10c. Final mesh for Example 2: Without element size control

Figure 10d. Final mesh for Example 2: With element size control

Figure 10e. Arc length deviation factor distributions for Example 2

Figure 11a. Geometry of Example 3

Figure 11b. Initial mesh for Example 3

Figure 11c. Final mesh for Example 3: Without element size control

Figure 11d. Final mesh for Example 3: With element size control ($k=0.0$)

Figure 11e. Final mesh for Example 3: With element size control
($k=10000$)

Figure 11f. Arc length deviation factor distributions for Example 3

Figure 12. Convergence history for Example 3: With element size control (k=0.0)

Figure A1. Relationship between δ and ε