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**An automatic adaptive refinement procedure for
the reproducing kernel particle method
Part I: Stress recovery and *a posteriori* error estimation**

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Abstract

In this study, an adaptive refinement procedure using the reproducing kernel particle method (RKPM) for the solution of 2D elastostatic problems is suggested. This adaptive refinement procedure is based on the Zienkiewicz and Zhu (Z-Z) error estimator for the *a posteriori* error estimation and an adaptive finite point mesh generator for new point mesh generation. The presentation of the work is divided into two parts. In Part I, concentration will be paid on the stress recovery and the *a posteriori* error estimation processes for the RKPM. The proposed error estimator is different from most recovery type error estimators suggested previously in such a way that, rather than using the least squares fitting approach, the recovery stress field is constructed by an extraction function approach. Numerical studies using 2D benchmark boundary value problems indicated that the recovered stress field obtained is more accurate and converges at a higher rate than the RKPM stress field. In Part II of the study, concentration will be shifted to the development of an adaptive refinement algorithm for the RKPM.

KEYWORDS: *Reproducing Kernel Particle Method; Stress Recovery Procedure; A posteriori error estimation; Adaptive refinement.*

1. Introduction

In the finite element method (FEM), two most popular types of procedures for the *a posteriori* error estimation are, namely, the residual based [1-3] and the post-processing (recovery) based [4-7] approaches. The residual type error estimator, which appears to be first proposed by Babuska and Rheinboldt [1], is based on the evaluation the equilibrium residuals, the inter-element traction jumps and the surface traction equilibrium to assess the error of the FE solution. The recovery based error estimator, which is often referred as the Zienkiewicz-Zhu (ZZ) error estimator [4], is based on the recovery of a continuous “smooth” stress field by applying some post-processing techniques to the FE solution. Since the unprocessed stress field obtained from the FE analysis is discontinuous along the inter-element boundaries, the continuous recovered stress field could often lead to accurate error estimation. As a result, the ZZ error estimator is one of the most popular types of *a posteriori* error estimators for the FEM.

Unlike the FEM, the shape functions employed in many meshless methods are often C^1 functions and thus the raw stress field is continuous within the whole problem domain. As a result, the procedure to derive an *a posteriori* error estimator for the meshless methods could be formulated in different ways when compared with those for the FEM. Chung and Belytschko [8] developed local and global error estimators for the element-free Galerkin method (EFGM) [9, 10]. Their error estimators are recovery type estimators which estimate the discretization error of the EFGM solution by comparing the differences between the EFGM stress field and a post-processed stress field [8]. Liu *et. al.* [11] proposed an approximate measure to properly detect the high-scale response using the built-in feature of the reproducing kernel particle methods (RKPM) [12]. Duarte and Oden [13] derived their *a posteriori* error estimator for their *h-p* cloud method by calculating the interior and boundary residuals. Other approaches for the error estimation of meshless method which are based on local Taylor series expansion were suggested by Haussler-Combe and Korn [14] and Gavete *et. al.* [15, 16]. Recently, Lee and Zhou [17, 18] carried out a comprehensive study on the theories and natures of some commonly used stress recovery schemes for the EFGM and concluded that these schemes are, in fact, particular cases of a general recovery scheme.

The main objective of this research is to develop an automatic adaptive RKPM refinement procedure for the solution of elliptic boundary value problems (BVP). In Part I of this study, concentration will be paid on the development of a new *a posteriori* error estimator based on the extraction function technique. Note that the *a posteriori* error estimator discussed in this study is different from the least-square fitting type stress recovery schemes and the error estimators discussed in references [17] and [18] but is more closely related to the extraction scheme proposed by Babuska and Miller [6, 7]. The basic idea of the proposed error estimator is to make use of the superconvergent property of some extraction functions which can be obtained in a way similar to the formulation of the boundary element method. Furthermore, besides numerical verifications, some theoretical discussions on the convergence characteristics of the recovery stress field and the effectivity index of the error estimator will be given. In Part II of the study, attention will be shifted to the development of an adaptive refinement procedure for the RKPM based on the error estimation scheme discussed in this paper. The efficiency of the adaptive scheme developed will be demonstrated by applying the procedure to some standard 2D elastostatic benchmark problems.

2. The model problem and the RKPM

2.1 The modelled BVP

In this study, the model problem employed is the standard elliptic BVP of linear 2D elastostatic formulation over the problem domain Ω with boundary $\Gamma = \Gamma_u \cup \Gamma_t$ (Fig. 1).

The BVP is defined by the strong form

$$\mathbf{L}^T \boldsymbol{\sigma} + \mathbf{b} = 0 \quad (1a)$$

such that $\Gamma_u \cap \Gamma_t = \emptyset$ and

$$\begin{aligned} \mathbf{u} &= \bar{\mathbf{u}}, \text{ on the essential boundary } \Gamma_u \\ \mathbf{Q}\boldsymbol{\sigma} &= \mathbf{QDLu} = \bar{\mathbf{t}}, \text{ on the natural boundary } \Gamma_t \end{aligned} \quad (1b)$$

In Eqn. 1a, $\boldsymbol{\sigma} = [\sigma_{xx}, \sigma_{yy}, \tau_{xy}]^T = [\sigma_{11}, \sigma_{22}, \sigma_{12}]^T$ and $\mathbf{b} = [b_x, b_y]^T = [b_1, b_2]^T$ are the stress tensor and the body force vector respectively. $\bar{\mathbf{u}}$ and $\bar{\mathbf{t}}$ are the prescribed

displacements and tractions along Γ_u and Γ_t respectively. In Eqn. 1a, the linear differential operator $\mathbf{L}$ is defined as

$$\mathbf{L}^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \quad (2)$$

In Eqn. 1b, $\mathbf{D}$ is the material matrix which is a function of the Young's modulus E and the Poisson ratio ν . $\mathbf{Q}$ is a linear operator which relates the boundary stresses and the boundary tractions and is given by

$$\mathbf{Q} = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix} \quad (3)$$

The relationships among the displacement vector $\mathbf{u}$, the strain tensor, $\boldsymbol{\varepsilon}$ and the stress tensor $\boldsymbol{\sigma}(\mathbf{u})$ are

$$\boldsymbol{\varepsilon} = \mathbf{L}\mathbf{u}, \boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{L}\mathbf{u} \quad (4)$$

2.2 The RKPM formulation employed

The modelled BVP described in Section 2.1 can be solved by the RKPM in which an approximation $\hat{\mathbf{u}}$ to $\mathbf{u}$ is constructed by employing a *corrected kernel* in the form

$$\hat{\mathbf{u}} = \int_{\Omega} \bar{w}_d(\mathbf{x} - \mathbf{s}, \mathbf{y} - \mathbf{t}) \mathbf{u}(\mathbf{s}, \mathbf{t}) d\Omega = \int_{\Omega} C(\mathbf{x}, \mathbf{y}, \mathbf{s}, \mathbf{t}) w_d(\mathbf{x} - \mathbf{s}, \mathbf{y} - \mathbf{t}) \mathbf{u}(\mathbf{s}, \mathbf{t}) d\Omega \quad (5)$$

where $w_d(\mathbf{x} - \mathbf{s}, \mathbf{y} - \mathbf{t})$ is the *kernel function*, $C(\mathbf{x}, \mathbf{y}, \mathbf{s}, \mathbf{t})$ is the *correction function* and $\bar{w}_d(\mathbf{x} - \mathbf{s}, \mathbf{y} - \mathbf{t})$ is the *corrected kernel function*. In numerical computation, the discrete form of the continuous approximation function (Eqn. 5) is used [11, 12]

$$\hat{\mathbf{u}} = \sum_{i=1}^{NP} \bar{w}_d(\mathbf{x} - \mathbf{x}_i, \mathbf{y} - \mathbf{y}_i) \mathbf{u}(\mathbf{x}_i, \mathbf{y}_i) \Delta A_i \quad (6)$$

where $(\mathbf{x}_i, \mathbf{y}_i)$ are the coordinates of node i . NP is the number of nodes in the domain. $\mathbf{u}(\mathbf{x}_i, \mathbf{y}_i)$ is the nodal value of node i and ΔA_i is the area of the domain of influence of node i with support size h_i . In matrix form the approximation of displacement can be written as

$$\hat{\mathbf{u}}(x, y) = \sum_{i=1}^{NP} (C(x, y, x_i, y_i) w_d(x - x_i, y - y_i) \Delta V_i) \hat{\mathbf{u}}_i = \sum_{i=1}^{NP} N_i(x, y) \hat{\mathbf{u}}_i = \mathbf{N}^T \hat{\mathbf{u}} \quad (7)$$

In Eqn. 7, $\mathbf{N}$ is the matrix of the RKPM shape functions and $\hat{\mathbf{u}}$ is the vector of nodal displacement values. N_i and $\hat{\mathbf{u}}_i$ are the shape function and the nodal value of node i respectively.

In the RKPM formulation, the essential boundary condition (Eqn. 1b) is enforced by the penalty multiplier method and the amended variational principles [19]. The final form of the functional to be minimized can be expressed as

$$\Pi = \frac{1}{2} \int_{\Omega} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\Omega - \int_{\Omega} \mathbf{u}^T \mathbf{b} d\Omega - \int_{\Gamma_t} \mathbf{u}^T \bar{\mathbf{t}} d\Gamma + \frac{\alpha}{2} \int_{\Gamma_u} (\mathbf{u} - \bar{\mathbf{u}})^T (\mathbf{u} - \bar{\mathbf{u}}) d\Gamma \quad (8)$$

In Eqn. 8, the factor $\alpha=5 \times 10^6 E$ is the penalty parameter employed to enforce the essential boundary condition [19, 28]. It shall be noted that the value of α selected here is somehow arbitrary. However, numerical results obtained from this (see Section 4.2) and previous studies [19, 28] shown that so long as $\alpha \gg E$, the boundary conditions will be properly enforced for almost all practical problems. By using Eqns. 7 and 8 and the standard Galerkin procedure, the global equations system can be obtained as

$$\mathbf{K} \hat{\mathbf{u}} = \mathbf{f} \quad (9a)$$

such that
$$\mathbf{K}_{ij} = \int_{\Omega} (\mathbf{B}_i)^T \mathbf{D} \mathbf{B}_j d\Omega + \alpha \int_{\Gamma_u} N_i \mathbf{S} N_j d\Gamma \quad (9b)$$

$$\mathbf{f}_i = \int_{\Omega} N_i \mathbf{b} d\Omega + \int_{\Gamma_t} N_i \bar{\mathbf{t}} d\Gamma + \alpha \int_{\Gamma_u} N_i \mathbf{S} \bar{\mathbf{u}} d\Gamma \quad (9c)$$

$$\mathbf{B}_i = \begin{bmatrix} N_{i,x} & 0 \\ 0 & N_{i,y} \\ N_{i,y} & N_{i,x} \end{bmatrix}, \mathbf{S} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \text{ and } S_i = \begin{cases} 1 & \text{if } \hat{\mathbf{u}}_i \text{ is prescribed on } \Gamma_u \\ 0 & \text{if } \hat{\mathbf{u}}_i \text{ is not prescribed on } \Gamma_u \end{cases} \quad (9d)$$

After the vector $\hat{\mathbf{u}}$ is solved, the RKPM solution $\hat{\mathbf{u}}$ could be computed by using Eqn. 7. The RPKM stress field $\hat{\boldsymbol{\sigma}}(\hat{\mathbf{u}})$ and boundary traction $\hat{\mathbf{t}}(\hat{\mathbf{u}})$, which are continuous over Ω , can be computed as

$$\hat{\boldsymbol{\sigma}} = \mathbf{D} \mathbf{L} \hat{\mathbf{u}} \quad \hat{\mathbf{t}} = \mathbf{Q} \hat{\boldsymbol{\sigma}} = \mathbf{Q} \mathbf{D} \mathbf{L} \hat{\mathbf{u}} \quad (10)$$

The pointwise error of the RKPM stress field is defined as the difference between the exact solution and the RKPM solution

$$\mathbf{e}_\sigma = \sigma - \hat{\sigma} \quad (11a)$$

and the error in energy norm is calculated as

$$\|\mathbf{e}\| = \left\{ \int_{\Omega} (\mathbf{L}(\mathbf{u} - \hat{\mathbf{u}}))^T \mathbf{D}(\mathbf{L}(\mathbf{u} - \hat{\mathbf{u}})) d\Omega \right\}^{1/2} = \left\{ \int_{\Omega} (\mathbf{e}_\sigma)^T \mathbf{D}^{-1}(\mathbf{e}_\sigma) d\Omega \right\}^{1/2} \quad (11b)$$

3. Extraction equation for stress recovery and *a posteriori* error estimation

3.1 The extraction equation

The model problem define in Eqn. 1 can also be expressed abstractly in an extraction operator form as [6]

$$\sigma = \Psi\sigma(\mathbf{u}) + \Phi \quad (12a)$$

where Ψ is an extraction operator such that Ψ^{-1} exists to assure that Eqn. 12a has a unique solution. Φ is the forcing term [6] which may be computed using only the problem's input data (the body force $\mathbf{b}$ in Eqn. 1a). Since the BVP has a unique solution $\mathbf{u}$ (and hence, σ), when the exact solution is replaced by the approximated RKPM solution $\hat{\sigma} = \mathbf{D}\mathbf{L}\hat{\mathbf{u}}$, an extracted (or recovered) stress field $\tilde{\sigma} \approx \sigma$ can be obtained as

$$\tilde{\sigma} = \Psi\hat{\sigma} + \Phi \quad (12b)$$

The pointwise error $\tilde{\mathbf{e}}_\sigma$ and the error norm of the recovered stress field $\|\tilde{\mathbf{e}}\|$ can then be defined as (c.f. Eqn. 11)

$$\tilde{\mathbf{e}}_\sigma = \sigma - \tilde{\sigma} \quad (13a)$$

$$\|\tilde{\mathbf{e}}\| = \left\{ \int_{\Omega} (\tilde{\mathbf{e}}_\sigma)^T \mathbf{D}^{-1}(\tilde{\mathbf{e}}_\sigma) d\Omega \right\}^{1/2} \quad (13b)$$

Obviously, if the recovered stress is more accurate than the RKPM stress, it could be used to estimate the error of the RKPM solution and the estimated error norm $\|\tilde{\mathbf{e}}\|$ [4] can be written as

$$\|\bar{\mathbf{e}}\| = \left\{ \int_{\Omega} (\bar{\mathbf{e}}_{\sigma})^T \mathbf{D}^{-1}(\bar{\mathbf{e}}_{\sigma}) d\Omega \right\}^{1/2}, \quad \bar{\mathbf{e}}_{\sigma} = \tilde{\sigma} - \hat{\sigma} \quad (14)$$

From Eqns. 11, 13 and 14, the three pointwise errors, $\mathbf{e}_{\sigma}$, $\tilde{\mathbf{e}}_{\sigma}$ and $\bar{\mathbf{e}}_{\sigma}$ are related by

$$\mathbf{e}_{\sigma} = \tilde{\mathbf{e}}_{\sigma} + \bar{\mathbf{e}}_{\sigma} \quad (15)$$

By applying the triangle inequality to the norms of the terms on both sides of Eqn. 15, it can be shown that

$$\left| 1 - \frac{\|\tilde{\mathbf{e}}\|}{\|\mathbf{e}\|} \right| \leq \frac{\|\bar{\mathbf{e}}\|}{\|\mathbf{e}\|} \leq \left| 1 + \frac{\|\tilde{\mathbf{e}}\|}{\|\mathbf{e}\|} \right| \quad (16a)$$

Hence, the estimated error norm will be asymptotically exact if the convergent rate of the recovered stress field $\tilde{\sigma}$ is higher than that of $\hat{\sigma}$ as N_{DOF} , the number of degrees of freedom of the problem, approaches infinity. In this case, the effectivity index of the error estimator

$\theta = \frac{\|\bar{\mathbf{e}}\|}{\|\mathbf{e}\|}$ will converge to unity [20]. That is,

$$\lim_{N_{\text{DOF}} \rightarrow \infty} \frac{\|\tilde{\mathbf{e}}\|}{\|\mathbf{e}\|} = 0 \Rightarrow \lim_{N_{\text{DOF}} \rightarrow \infty} \frac{\|\bar{\mathbf{e}}\|}{\|\mathbf{e}\|} = \lim_{N_{\text{DOF}} \rightarrow \infty} \theta = 1 \quad (16b)$$

3.2 Convergence rate of the recovered stress field

From Eqn. 16, it can be seen that if the extraction equation (Eqn. 12b) is employed to generate a recovered stress field for the *a posteriori* error estimation of the RKPM solution, the convergence rate of the recovered solution will affect the performance and reliability of the *a posteriori* error estimator. The recovered stress field should, ideally, converge at a higher rate than the RKPM stress field as this will ensure the asymptotically exactness of the effectivity index. Toward this end, it could be proved that if the extraction operator Ψ (Eqn. 12) is compact and $\hat{\mathbf{u}}$ is the RKPM solution, then the recovered stress field $\tilde{\sigma}$ generated will converge at a higher rate than σ . Details of the convergence properties of the extracted stress field could be found in references [21] to [25] and a brief outline of the proof will be given here.

From Eqns. 12 to 14, the relationship between $\tilde{\mathbf{e}}_{\sigma}$ and $\mathbf{e}_{\sigma}$ can also be written as

$$\tilde{\mathbf{e}}_{\sigma} = \Psi \mathbf{e}_{\sigma} \quad (17a)$$

It can be shown that there exists a self-adjoint orthogonal projection operator $\hat{\mathbf{P}}$ such that [21]

$$\hat{\mathbf{P}}(\mathbf{e}_{\sigma}) = \mathbf{0} \quad (17b)$$

Hence, combining Eqns. 17a with 17b, one can write

$$\tilde{\mathbf{e}}_{\sigma} = \Psi(\mathbf{I} - \hat{\mathbf{P}})\mathbf{e}_{\sigma} \quad (17c)$$

Furthermore, the projection operator is strong convergent but not uniformly convergent to the identity operator itself. That is

$$\lim_{N_{\text{DOF}} \rightarrow \infty} \|\Psi(\mathbf{I} - \hat{\mathbf{P}})\sigma\| = 0 \quad \forall \sigma \in \mathbf{H}^0(\Omega) \quad (18)$$

but $\lim_{N_{\text{DOF}} \rightarrow \infty} \|\Psi(\mathbf{I} - \hat{\mathbf{P}})\| = 0$ is not assured. In Eqn. 18, $\mathbf{H}^0(\Omega)$ is the zero order Sobolev space defined over Ω . Hence, in order to establish the convergence property of the term $\frac{\|\tilde{\mathbf{e}}\|}{\|\mathbf{e}\|}$, further analysis on the convergence of the operator $\Psi(\mathbf{I} - \hat{\mathbf{P}})$ is needed [23].

Now consider the definition of the term $\|\Psi(\mathbf{I} - \hat{\mathbf{P}})\|$

$$\|\Psi(\mathbf{I} - \hat{\mathbf{P}})\| = \|\Psi^* - \hat{\mathbf{P}}\Psi^*\| = \sup_{\mathbf{z} \in \mathbf{H}^0(\Omega), \|\mathbf{z}\|=1} \|(\Psi^* \mathbf{z} - \hat{\mathbf{P}}\Psi^* \mathbf{z})\| \quad (19a)$$

where Ψ^* is the adjoint of Ψ . Assuming that $\lim_{N_{\text{DOF}} \rightarrow \infty} \|\Psi(\mathbf{I} - \hat{\mathbf{P}})\| \neq 0$, then there exists an $\varepsilon > 0$ and a sequence $\{\mathbf{z}_{N_{\text{DOF}}}\} \subset \mathbf{H}^0(\Omega)$ such that

$$\|(\Psi^* \mathbf{z}_{N_{\text{DOF}}} - \hat{\mathbf{P}}\Psi^* \mathbf{z}_{N_{\text{DOF}}})\| > \varepsilon, \quad \|\mathbf{z}_{N_{\text{DOF}}}\| = 1, \quad \forall \mathbf{z}_{N_{\text{DOF}}} \in \mathbf{H}^0(\Omega) \quad (19b)$$

Since Ψ is compact, Ψ^* will also be compact. Hence, the sequence $\{\mathbf{z}_{N_{\text{DOF}}}\}$ can always be selected in such a way that the expression $\Psi^* \mathbf{z}_{N_{\text{DOF}}}$ will converge to an element $\mathbf{z} \in \mathbf{H}^0(\Omega)$ as the RKPM model is refined. That is

$$\lim_{N_{\text{DOF}} \rightarrow \infty} \Psi^* \mathbf{z}_{N_{\text{DOF}}} = \mathbf{z} \quad (19c)$$

Now by using the identity

$$\begin{aligned}
(\Psi^* - \hat{\mathbf{P}}\Psi^*)\mathbf{z}_{N_{\text{DOF}}} &\equiv \Psi^*\mathbf{z}_{N_{\text{DOF}}} - \hat{\mathbf{P}}\Psi^*\mathbf{z}_{N_{\text{DOF}}} - \mathbf{z} + \hat{\mathbf{P}}\mathbf{z}_{N_{\text{DOF}}} + \mathbf{z} - \hat{\mathbf{P}}\mathbf{z}_{N_{\text{DOF}}} \\
&\equiv (\mathbf{I} - \hat{\mathbf{P}})(\Psi^*\mathbf{z}_{N_{\text{DOF}}} - \mathbf{z}) + (\mathbf{z} - \hat{\mathbf{P}}\mathbf{z}_{N_{\text{DOF}}})
\end{aligned} \tag{20a}$$

one can write

$$\begin{aligned}
\|\Psi^*\mathbf{z}_{N_{\text{DOF}}} - \hat{\mathbf{P}}\Psi^*\mathbf{z}_{N_{\text{DOF}}}\| &= \|(\mathbf{I} - \hat{\mathbf{P}})(\Psi^*\mathbf{z}_{N_{\text{DOF}}} - \mathbf{z}) + (\mathbf{z} - \hat{\mathbf{P}}\mathbf{z}_{N_{\text{DOF}}})\| \\
&\leq \|(\mathbf{I} - \hat{\mathbf{P}})(\Psi^*\mathbf{z}_{N_{\text{DOF}}} - \mathbf{z})\| + \|\mathbf{z} - \hat{\mathbf{P}}\mathbf{z}_{N_{\text{DOF}}}\|
\end{aligned} \tag{20b}$$

However, as $N_{\text{DOF}} \rightarrow \infty$, $\Psi^*\mathbf{z}_{N_{\text{DOF}}} - \mathbf{z} \rightarrow \mathbf{0}$ and $\hat{\mathbf{P}}\mathbf{z}_{N_{\text{DOF}}} \rightarrow \mathbf{z}$, the two RHS terms of Eqn. 20b will converge to zero which results in a contradiction of the assumption of Eqn. 19b. Hence, it can be conclude that

$$\lim_{N_{\text{DOF}} \rightarrow \infty} \|\Psi(\mathbf{I} - \hat{\mathbf{P}})\| = 0 \tag{21}$$

From Eqn. 17c, it can be seen that

$$\begin{aligned}
\|\tilde{\mathbf{e}}\| &= \|\Psi(\mathbf{I} - \hat{\mathbf{P}})\mathbf{e}_\sigma\| \\
&\leq \|\Psi(\mathbf{I} - \hat{\mathbf{P}})\| \cdot \|\mathbf{e}\|
\end{aligned} \tag{22a}$$

Eqn. 22a, implies that

$$\frac{\|\tilde{\mathbf{e}}\|}{\|\mathbf{e}\|} \leq \|\Psi(\mathbf{I} - \hat{\mathbf{P}})\| \Rightarrow \lim_{N_{\text{DOF}} \rightarrow 0} \frac{\|\tilde{\mathbf{e}}\|}{\|\mathbf{e}\|} \leq \lim_{N_{\text{DOF}} \rightarrow 0} \|\Psi(\mathbf{I} - \hat{\mathbf{P}})\| = 0 \tag{22b}$$

Hence, from Eqn. 16b, one can conclude that if the recovered stress field is constructed by Eqn. 12b, the *a posteriori* estimator will be asymptotically exact in the sense that

$$\lim_{N_{\text{DOF}} \rightarrow \infty} \frac{\|\tilde{\mathbf{e}}\|}{\|\mathbf{e}\|} = \lim_{N_{\text{DOF}} \rightarrow \infty} \theta = 1 \tag{23}$$

However, it should be note that Eqn. 23 only indicates that the convergence rate of the recovered stress is higher than, but is not always *one* order higher than (i.e. superconvergence) the RKPM stress. Numerical examples given in Section 4 and Part II of this study [26] will show that the difference between the convergence rates of the two stress fields are depended on the smoothness of the solution and the point mesh pattern used.

3.3 The extraction procedure

In the present study, the extraction operator Ψ used is a boundary integral operator while the load term Φ is a domain integral operator and Eqn. 12b can be written in the form

$$[\tilde{\sigma}(\mathbf{p})] = \int_{\Gamma_R} \hat{\Psi}(\hat{\sigma}(\hat{\mathbf{u}})) d\Gamma + \int_{\Omega_R} \bar{\Phi} d\Omega \quad (24a)$$

where $[\tilde{\sigma}(\mathbf{p})]$ is the recovered stress at a given point $\mathbf{p} = (x_1^p, x_2^p)$ inside the problem domain (Fig. 1) with the stress components arranged in a matrix form such that

$$[\tilde{\sigma}(\mathbf{p})] = \begin{bmatrix} \tilde{\sigma}_{xx} & \tilde{\tau}_{xy} \\ \tilde{\tau}_{xy} & \tilde{\sigma}_{yy} \end{bmatrix} = \begin{bmatrix} \tilde{\sigma}_{11} & \tilde{\sigma}_{12} \\ \tilde{\sigma}_{12} & \tilde{\sigma}_{22} \end{bmatrix} \quad (24b)$$

The terms $\hat{\Psi}(\hat{\sigma}(\hat{\mathbf{u}}))$ and $\bar{\Phi}$ are 2 by 2 matrices and are functions of the RKPM solution $\hat{\sigma}(\hat{\mathbf{u}})$ (Eqn. 6) and the body force vector, $\mathbf{b}$ (Eqn. 1a) such that

$$\hat{\Psi}(\hat{\sigma}(\hat{\mathbf{u}})) = \begin{bmatrix} \hat{\Psi}_{11}(\hat{\sigma}(\hat{\mathbf{u}})) & \hat{\Psi}_{12}(\hat{\sigma}(\hat{\mathbf{u}})) \\ \hat{\Psi}_{12}(\hat{\sigma}(\hat{\mathbf{u}})) & \hat{\Psi}_{22}(\hat{\sigma}(\hat{\mathbf{u}})) \end{bmatrix} \text{ and } \bar{\Phi} = \begin{bmatrix} \bar{\Phi}_{11}(\mathbf{b}) & \bar{\Phi}_{12}(\mathbf{b}) \\ \bar{\Phi}_{12}(\mathbf{b}) & \bar{\Phi}_{22}(\mathbf{b}) \end{bmatrix} \quad (24c)$$

In Eqn. 24a, Ω_R is a local subdomain encloses the point $\mathbf{p}$ with boundary Γ_R and outward normal vector $\mathbf{n}_R$ (Fig. 1.)

In order to obtain the expressions for the operators $\hat{\Psi}$ and $\bar{\Phi}$, one can consider the weak form of the governing equation which can be written as

$$\int_{\Omega} \sigma^* (\mathbf{L}^T \sigma + \mathbf{b}) d\Omega = 0 \quad (25)$$

In Eqn. 25, σ^* is the test function used. Suppose that σ^* is selected to satisfy the fundamental equation such that

$$\mathbf{L}^T \sigma^* + \delta^i = 0 \quad (26)$$

where δ^i is the Dirac delta function, then after integration by part for Eqn. 25 twice, the following integral equation can be obtained for the displacement at point $\mathbf{p}$

$$c_{lk}^p u_k(\mathbf{p}) + \int_{\Gamma} u_k t_{lk}^* d\Gamma = \int_{\Gamma} t_k u_{lk}^* d\Gamma + \int_{\Omega} b_k u_{lk}^* d\Omega, \text{ sum over } k, l, k = 1, 2 \quad (27a)$$

In Eqn. 27a, u_{lk}^* and t_{lk}^* are, respectively, the 2D fundamental displacement and surface tractions solution to Eqn. 26. For any point $\mathbf{q} = (x_1^q, x_2^q)$ located either inside the domain Ω or along the domain boundary Γ , u_{lk}^* and t_{lk}^* are given by [27]

$$u_{lk}^*(\mathbf{q}) = \frac{1}{8\pi G(1-\nu)} \left(\left((3-4\nu) \ln\left(\frac{1}{r}\right) \right) \delta_{lk} + r_l r_k \right) \quad (27b)$$

$$t_{lk}^*(\mathbf{q}) = \frac{-1}{4\pi(1-\nu)r} \left(\frac{\partial r}{\partial \mathbf{n}} \left((1-2\nu) \delta_{lk} + 2r_l r_k \right) - (1-2\nu) (r_l n_k - r_k n_l) \right) \quad (27c)$$

where

$$G = \frac{E}{2(1-\nu)}, \quad r = \sqrt{(x_1^q - x_1^p)^2 + (x_2^q - x_2^p)^2} \quad \text{and} \quad r_\alpha = \frac{\partial r}{\partial x_\alpha}, \quad \text{for } \alpha = l, k \quad (27d)$$

n_l and n_k are the direction cosines of the outward normal $\mathbf{n}$ at the boundary point $\mathbf{q}$ with respect to the x_l and the x_k axes respectively. The term c_{lk}^p in Eqn. 27a is defined as

$$c_{lk}^p = \begin{cases} \delta_{lk} & \mathbf{p} \in \Omega \\ \frac{1}{2} \delta_{lk} & \mathbf{p} \in \Gamma \text{ (smooth boundary point)} \\ \delta_{lk} + \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} t_{lk}^* d\Gamma & \mathbf{p} \in \Gamma \text{ (non-smooth boundary point)} \end{cases} \quad (27e)$$

By using the constitutive equation (Eqn. 4) and Eqn. 27, the stress at the point $\mathbf{p}$ can be written as [27]

$$\sigma_{ij}(\mathbf{p}) = \int_{\Gamma} (-u_k s_{kij}^* + t_k d_{kij}^*) d\Gamma + \int_{\Omega} b_k d_{kij}^* d\Omega \quad (28a)$$

where the terms s_{kij}^* and d_{kij}^* at a point $\mathbf{q}$ are given by

$$d_{kij}^* = \frac{1}{4\pi(1-\nu)r} \left((1-2\nu) (r_i \delta_{jk} + r_j \delta_{ik} - r_k \delta_{ij}) + 2r_i r_j r_k \right) \quad (28b)$$

$$s_{kij}^* = \frac{G}{2\pi(1-\nu)r^2} \left\{ 2r_m n_m \left((1-2\nu) r_k \delta_{ij} + \nu(r_j \delta_{ik} + r_i \delta_{jk}) - 4r_i r_j r_k \right) + 2\nu(n_i r_j + n_j r_i) r_k \right. \\ \left. + (1-2\nu) (2n_k r_i r_j + n_j \delta_{ik} + n_i \delta_{jk} - (1-4\nu) n_k \delta_{ij}) \right\} \quad (28c)$$

with sum over m and $i, j, k, l, m = 1, 2$.

In practice, the exact displacement and tractions are not known. Hence, the recovered stress at point $\mathbf{p}$, $\tilde{\sigma}_{ij}(\mathbf{p})$ can be obtained by replacing the exact displacement term u_k and the

exact traction term t_k by their corresponding RKPM solutions. Furthermore, the domain of integration can be limited to a local subdomain Ω_R (with boundary Γ_R) which encloses $\mathbf{p}$. Thus, the extracted stress $\tilde{\sigma}_{ij}(\mathbf{p})$ could be obtained by using the expression (c.f. Eqn. 28a)

$$\tilde{\sigma}_{ij}(\mathbf{p}) = \int_{\Gamma_R} (-\hat{u}_k s_{kij}^* + \hat{t}_k d_{kij}^*) d\Gamma + \int_{\Omega_R} b_k d_{kij}^* d\Omega \quad (29a)$$

Eqn. 29a can be written in vector form as

$$\tilde{\sigma}_{ij}(\mathbf{p}) = \int_{\Gamma_R} (-\mathbf{s}_{ij}^*)^T \hat{\mathbf{u}} + (\mathbf{d}_{ij}^*)^T \hat{\mathbf{t}} d\Gamma + \int_{\Omega_R} (\mathbf{d}_{ij}^*)^T \mathbf{b} d\Omega \quad (29b)$$

where

$$\mathbf{s}_{ij}^* = [s_{1ij}^*, s_{2ij}^*]^T \quad \text{and} \quad \mathbf{d}_{ij}^* = [d_{1ij}^*, d_{2ij}^*]^T \quad (29c)$$

By using Eqn. 10 which relates $\hat{\mathbf{t}}$ and $\hat{\mathbf{u}}$, Eqn. 29b can be rewritten as

$$\begin{aligned} \tilde{\sigma}_{ij}(\mathbf{p}) &= \int_{\Gamma_R} (-\mathbf{s}_{ij}^*)^T \hat{\mathbf{u}} + (\mathbf{d}_{ij}^*)^T (\mathbf{QDL}\hat{\mathbf{u}}) d\Gamma + \int_{\Omega_R} (\mathbf{d}_{ij}^*)^T \mathbf{b} d\Omega \\ &= \int_{\Gamma_R} (\mathbf{d}_{ij}^*)^T (\mathbf{QDL}\hat{\mathbf{u}}) - (\mathbf{s}_{ij}^*)^T \hat{\mathbf{u}} d\Gamma + \int_{\Omega_R} (\mathbf{d}_{ij}^*)^T \mathbf{b} d\Omega \\ &= \int_{\Gamma_R} \left((\mathbf{d}_{ij}^*)^T (\mathbf{QDL}) - (\mathbf{s}_{ij}^*)^T \right) \hat{\mathbf{u}} d\Gamma + \int_{\Omega_R} (\mathbf{d}_{ij}^*)^T \mathbf{b} d\Omega \end{aligned} \quad (29d)$$

Hence, by comparing Eqn. 24c with Eqn. 29d, it can be concluded that

$$\hat{\Psi}_{ij}(\hat{\sigma}(\hat{\mathbf{u}})) = \left((\mathbf{d}_{ij}^*)^T (\mathbf{QDL}) - (\mathbf{s}_{ij}^*)^T \right) \hat{\mathbf{u}} \quad \text{and} \quad \bar{\Phi}_{ij}(\mathbf{b}) = (\mathbf{d}_{ij}^*)^T \mathbf{b} \quad (30)$$

Note that with the use of the subdomain Ω_R , the normal vector $\mathbf{n}$ and its components n_l and n_k in Eqn. 27c should now be replaced by their subdomain counterparts $\mathbf{n}_R$, n_{Rl} and n_{Rb} (Fig. 1) respectively.

It should be noted that Eqns. 27 to 30 are valid for the case of plane strain problems, for plane stress problems, one should replace the term ν by $(1 - \nu)/\nu$.

3.4 Computation of the recovered stress field $[\tilde{\sigma}(\mathbf{p})]$ and a posteriori error estimation

It should be noted that the recovery equation (Eqn. 29d) holds irrespective of the size and shape of the local recovery subdomain Ω_R . In this study, in order to reduce the

computational efforts needed for the recovery procedure, for any point $\mathbf{p}$ inside the problem domain under consideration, Ω_R is selected as a circular disc with radius equal to r_R (Fig. 2). Hence, if the minimum distance between $\mathbf{p}$ and the domain boundary is greater than or equal to r_R , the boundary of the recovery subdomain, Γ_R is simply a circle $\mathcal{C}(\mathbf{p}, r_R)$ with centre at $\mathbf{p}$ and radius equal to r_R (Fig. 2). In case that the minimum distance between $\mathbf{p}$ and the domain boundary is less than r_R , Γ_R will consist of two parts such that $\Gamma_R = \Gamma_{Ri} \cup \Gamma_{Rb}$ where Γ_{Rb} is the intersection of $\mathcal{C}(\mathbf{p}, r_R)$ with Γ (Fig. 3). In both cases, Eqn. 29d could be evaluated by applying a suitable numerical integration scheme along Γ_R . Details of the numerical integration scheme adopted in this study can be found in the Appendix of this paper.

By using Eqns. 28 to 30, the components of the extraction operators $\hat{\Psi}$ and $\bar{\Phi}$ can be obtained for any point inside Ω . However, when the point $\mathbf{p}$ is lying on the domain boundary, some terms defined by Eqns. 28b and 28c that contain expressions like $1/r$ or $1/r^2$ will be singular. In this case, in order to overcome such singular problem, a simple procedure involving the creation of two “offsetting” points and linear interpolation as described below is applied (Fig. 4). If the point $\mathbf{p}$ under consideration is lying on the domain boundary, two additional points $\mathbf{p}_1$ and $\mathbf{p}_2$ which are lying on the inward normal at $\mathbf{p}$ will be established. The distances between the points $\mathbf{p}$ and $\mathbf{p}_1$, $\mathbf{p}_1$ and $\mathbf{p}_2$ are set equal to a small distance Δp . The recovered stresses at these two points $\tilde{\sigma}(\mathbf{p}_1)$ and $\tilde{\sigma}(\mathbf{p}_2)$ are first computed using Eqn. 29d. Finally, the recovered stress the boundary point $\mathbf{p}$, $\tilde{\sigma}(\mathbf{p})$ will be interpolated as

$$\tilde{\sigma}(\mathbf{p}) = 2\tilde{\sigma}(\mathbf{p}_1) - \tilde{\sigma}(\mathbf{p}_2) \quad (31)$$

It should be mentioned that since linear relation is assumed in Eqn. 31, in order to obtain accurate recovered stress at $\mathbf{p}$, the offsetting distance Δp should be a small value when compared with h_p , the size of domain of influence associated with point $\mathbf{p}$ for the RKPM (Section 2.2). In the current implementation, Δp is set equal to $0.05h_p$.

By using Eqns. 12b, 24, 29 and 31, it is possible to compute the recovered stress at any location inside the problem domain. In this study, the recovered stress tensor will be extracted at all the nodes of the point mesh and the recovered stress field $\tilde{\sigma}(x, y)$ will be constructed by re-using the RKPM shape functions such that

$$\tilde{\boldsymbol{\sigma}}(x, y) = \sum_{i=1}^{NP} N_i \tilde{\boldsymbol{\sigma}}_i \quad (32)$$

In Eqn. 32, $\tilde{\boldsymbol{\sigma}}_i$ is the recovered stress tensor at node i . Eqn. 14 will then be used to compute the estimated pointwise error ($\bar{\mathbf{e}}_{\boldsymbol{\sigma}}$) and the error norm ($\|\bar{\mathbf{e}}\|$) of the RKPM solution. Note that since the integral in Eqn. 14 is referred to the whole problem domain, the error norm computed is a global *a posteriori* error estimator. In practice, since the recovered stress is available at any location inside the problem domain, it is also possible to compute the estimated error norm for any subdomain Ω_s as (c.f. Eqn. 14)

$$\|\bar{\mathbf{e}}\|_{\Omega_s} = \left\{ \int_{\Omega_s} (\bar{\mathbf{e}}_{\boldsymbol{\sigma}})^T \mathbf{D}^{-1} (\bar{\mathbf{e}}_{\boldsymbol{\sigma}}) d\Omega \right\}^{1/2} \quad (33)$$

4. Numerical verification

4.1 Scope of study

In this section, numerical studies will be carried out to investigate the accuracy and the convergence rate of the recovered stress field. In order to thoroughly test the performance of the recovery procedure, numerical verifications will be carried out by applying the RKPM for the solution of four classic benchmark problems with different domain geometry (convex and concave) and with different smoothness of the exact solution (smooth and singular). For each benchmark problem, linear RKPM is applied and four uniform meshes will be employed to obtain the RKPM solutions. Since the exact solutions for these problems are known, it is possible to compute the relative error of the RKPM solution η_{RKPM} and the relative error of the recovered stress η_{Rv} such that

$$\eta_{\text{RKPM}} = \frac{\|\mathbf{e}\|}{\|\mathbf{u}\|} \quad \text{and} \quad \eta_{\text{Rv}} = \frac{\|\tilde{\mathbf{e}}\|}{\|\mathbf{u}\|} \quad (34a)$$

In Eqn. 34a, $\|\mathbf{u}\|$ is the exact energy norm of the solution

$$\|\mathbf{u}\| = \left\{ \int_{\Omega} \boldsymbol{\sigma}^T \mathbf{D}^{-1} \boldsymbol{\sigma} d\Omega \right\}^{1/2} \quad (34b)$$

Furthermore, since linear RKPM is used, when uniform refinement is carried out, it is expected that R_{RKPM} , the convergence rate of η_{RKPM} , defined as

$$R_{\text{RKPM}} = \frac{\log(\eta_{\text{RKPM},i}/\eta_{\text{RKPM},i+1})}{\log(N_{\text{DOF},i+1}/N_{\text{DOF},i})} \quad (35)$$

will be given by

$$R_{\text{RKPM}} = \min(p/2, \lambda/2) \quad (36)$$

In Eqn. 36, $p=1$ is the order of approximation of the RKPM and $\lambda < 1$ is the strength of singularity of the problem [28]. In Eqn. 35, $\eta_{\text{RKPM},i}$ and $N_{\text{DOF},i}$ are, respectively, the relative error and the degrees of freedom for the i th RKPM refinement analysis.

Similarly, the convergence rate of the recovered solution, R_{RV} can be computed as

$$R_{\text{RV}} = \frac{\log(\eta_{\text{RV},i}/\eta_{\text{RV},i+1})}{\log(N_{\text{DOF},i+1}/N_{\text{DOF},i})} \quad (37)$$

with $\eta_{\text{RV},i}$ is the relative error for the recovered solution obtained from the i th refinement analysis. From Eqn. 16b, it can be seen that the effectivity index of the *a posteriori* error estimator will be asymptotically exact if $R_{\text{RV}} > R_{\text{RKPM}}$.

In the numerical study, for each RKPM solution obtained, three different recovered stress fields will be computed by using three different values of r_{R} (Fig. 2) such that

$$r_{\text{R}} = kh \quad k=1, 2 \text{ and } 3 \quad (38)$$

In Eqn. 38, h is the size of the domain of influence used in the RKPM. Since uniform refinement is employed, h will be a constant for all the nodes in the mesh and reduced uniformly in each refinement step. Hence, the value r_{R} will also be reduced uniformly as the meshes are refined. However, for a given selected value of k , the ratio between r_{R} and h will remain as a constant in successive refinements. For the different values of r_{R} tested, values of η_{RV} , R_{RV} and the corresponding values of effectivity index θ are computed.

Note that the accuracy of the recovery stress could be improved [6] by adopting a bigger subdomain Ω_{R} in the recovered procedure. However, it should be mentioned that a larger Ω_{R} will lead to a higher computational cost for the integration of Eqn. 29 and thus in practice, an optimal value of r_{R} should be used.

4.2 The patch test

One necessary condition for the asymptotically exactness of the *a posteriori* error estimator is that the recovered stress field *must* be able to reproduce the exact stress field if the solution of the RKPM solution is exact. Since linear RKPM is employed for the solution in the numerical study, a simple patch test for the recovery procedure can be carried out by considering the model problem shown in Fig. 5. In Fig. 5, a rectangular plate under plane stress conditions is loaded by uniform traction force on one side. Due to the boundary conditions imposed, a constant stress state of $\sigma_{xx}=1.0$ will be achieved and can be solved exactly by the RKPM solution. The recovered stress $\tilde{\sigma}_{xx}$ along the line $x=1.0$ obtained by using different values of the penalty term α in the RKPM formulation (Eqn. 9b) are listed in Table 1. It could be seen from Table 1 that for all the penalty values used, the recovered stress obtained are all very close to the exact value of unity. Note that the small difference ($<0.5\%$) was due to the error introduced when Eqn. 29 was computed by numerical integration [26].

4.3 Numerical examples

Example 1: Timoshenko beam (convex domain, smooth exact solution)

The first example is a cantilever beam loaded under uniform end shear force (Fig. 6). The exact stress field of this problem is given by [29]

$$\sigma_x = -\frac{P y}{I}(L - x), \quad \sigma_y = 0.0, \quad \tau_{xy} = -\frac{P}{2I}(H^2/4 - y^2), \quad I = H^3/12 \quad (39)$$

The four uniform meshes used for the RKPM solutions are shown in Fig. 7. The numerical results obtained which including the relative errors, convergence rates and effectivity indices obtained by using the three different values of r_R are summarized in Table 2.

Example 2: Infinite plate with hole (concave domain, smooth exact solution)

In the second example, an infinite plate with a circular hole as shown Fig. 8 is considered. The exact solution of this problem is given by [29]

$$\begin{aligned}
\sigma_x &= 1 - \frac{1}{r^2} \left(\frac{3}{2} \cos 2\vartheta + \cos 4\vartheta \right) + \frac{3}{2r^4} \cos 4\vartheta \\
\sigma_y &= -\frac{1}{r^2} \left(\frac{1}{2} \cos 2\vartheta - \cos 4\vartheta \right) - \frac{3}{2r^4} \cos 4\vartheta \\
\tau_{xy} &= -\frac{1}{r^2} \left(\frac{1}{2} \sin 2\vartheta + \sin 4\vartheta \right) + \frac{3}{2r^4} \sin 4\vartheta
\end{aligned} \tag{40}$$

For this example, the meshes employed and the numerical results obtained are shown in Fig. 9 and Table 3 respectively.

Example 3: Domain near a crack Tip (convex domain, singular exact solution)

In the third example, the problem domain is a region near the tip of a crack (Fig. 10). The plate is loaded by tractions corresponding to the first symmetric mode of the stress intensity factor solution. The exact stress field in the domain is given by [29]

$$\begin{aligned}
\sigma_x &= \frac{1}{\sqrt{r}} \cos \frac{\vartheta}{2} \left(1 - \sin \frac{\vartheta}{2} \sin \frac{3\vartheta}{2} \right) \\
\sigma_y &= \frac{1}{\sqrt{r}} \cos \frac{\vartheta}{2} \left(1 + \sin \frac{\vartheta}{2} \sin \frac{3\vartheta}{2} \right) \\
\sigma_{xy} &= \frac{1}{\sqrt{r}} \sin \frac{\vartheta}{2} \cos \frac{\vartheta}{2} \cos \frac{3\vartheta}{2}
\end{aligned} \tag{41}$$

Exploiting symmetry, only half of the domain is modelled. Note that in this example, a singularity exists at (0,0) with $\lambda=0.5$ and it is expected that the convergence rate of the RKPM solution will be reduced from the usual value of $p/2=0.5$ to $\lambda/2=0.25$. For this problem, the meshes used in the uniform refinement are shown in Fig. 11 while the results obtained are summarized in Table 4.

Example 4: L-Shaped Domain (concave domain, singular exact solution)

In the last example, a L-shaped plane (Fig. 12) is considered. It is loaded by tractions corresponding to the following exact stress solution [30]

$$\begin{aligned}
\sigma_x &= \lambda r^{(\lambda-1)} \{ (2 - Q(\lambda+1)) \cos((\lambda-1)\vartheta) - (\lambda-1) \cos((\lambda-3)\vartheta) \} \\
\sigma_y &= \lambda r^{(\lambda-1)} \{ (2 + Q(\lambda+1)) \cos((\lambda-1)\vartheta) + (\lambda-1) \cos((\lambda-3)\vartheta) \} \\
\tau_{xy} &= \lambda r^{(\lambda-1)} \{ (\lambda-1) \sin((\lambda-3)\vartheta) + Q(\lambda+1) \sin((\lambda-1)\vartheta) \}
\end{aligned} \tag{42}$$

In Eqn. 42, $Q=0.543075579$ and $\lambda=0.544483737$. The traction conditions satisfy equilibrium and hence only rigid-body constraints have to be applied at points C and F. In

this example, a singularity exists at point C and hence the convergence rate under uniform refinement will be equal to $\lambda/2 \approx 0.27$. The meshes used for the RKPM solution are shown in Fig. 13 and the numerical results obtained are tabulated in Table 5.

4.4 Discussions on the numerical results obtained

From the numerical results listed in Tables 2 to 5, it can be observed that

- (1) Regarding the accuracy and convergence rates of the RKPM solutions, it can be seen that in Examples 1 and 2 where the exact solution is smooth, the convergence rate of the RKPM solutions is approximately equal to the theoretical rate of $p/2=0.5$. However, such normal convergence rate was reduced to $\lambda/2$ in Examples 3 and 4 when singularity exists in the problem domain.
- (2) Regarding the accuracy of the recovered stress field, it can be seen from Tables 2 to 5 that in all four benchmark problems and for all the meshes used, the recovered stress fields are more accurate than the corresponding RKPM solutions. Furthermore, as predicted and proved in [6], the larger the size of the recovery radius r_R used, the more accurate the recovered stress. Tables 2 to 4 also show that the relative errors of the recovered stress fields obtained in all the cases are roughly equal to one half of the corresponding RKPM solution and these agree with the prediction of Eqn. 22a.
- (3) Regarding the convergence rates of the recovered solutions, it can be seen that they are all greater than their corresponding RKPM counterparts. In all the examples, the convergence rates of the recovered solutions were improved by increasing the recovery radius from h to $2h$. However, no further improvement could be found by changing r_R from $2h$ to $3h$. Hence, it appears that a value of $r_R=2h$ could be used in practice to optimize the efficiency of the *a posteriori* error estimator. Note that in Examples 1 and 2, the recovered stresses are nearly superconvergent with convergence rates > 0.8 . However, in Examples 3 and 4, despite that the convergence rates of the recovered solutions are higher than the RKPM solutions, they were affected by the present of singularity and are still less than the normal convergence rate of the RKPM solution.
- (4) Regarding the effectivity indices of the *a posteriori* error estimator, it can be seen that a higher value of r_R leads to better values of θ . In particular, in Examples 1 and 2 when the recovered stress fields are almost superconvergent, the values of θ are converging

to unity even the final meshes used in these two examples are relatively coarse with $\text{DOF} < 600$. For Examples 3 and 4, since the convergence rates of the recovered stress fields were affected by the singularity, the values of θ obtained are not as good as the results obtained in Examples 1 and 2. However, even in these cases, it can still be seen from Tables 4 and 5 that values of θ were converging as the meshes are refined.

5. Conclusions

In this paper, a new method is proposed for the stress recovery procedure and the *a posteriori* error estimation for the RKPM. The basic principle of the stress recovery procedure is to employ some compact extraction functions which are similar to the boundary integral technique for the generation of the recovery stress field. The recovery stress field is then compared with the RKPM solution for the *a posteriori* error estimation.

In order to validate the accuracy and convergence rate of the recovered stress field obtained and the reliability of the *a posteriori* error estimator, four numerical examples involving benchmark 2D boundary value problems that with different domain geometry and smoothness of the exact solution were employed. Uniform refinements were carried out and recovery subdomains with different sizes were constructed to investigate the effect of subdomain size on the performance of the recovery procedure. The numerical results indicated that the recovered stress field is more accurate and converges at a higher rate than the original RKPM stress field. Furthermore, it was also shown that both the accuracy and convergence rate of the recovered stress field can be improved by increasing the size of the recovery subdomain. For the performance of the *a posteriori* error estimation scheme, since the recovery stress field converged at a higher rate than the RKPM solution, asymptotic convergence of the effectivity indices was observed. However, as the convergence rate of the recovered stress was affected by the present of singularity, the performance of the error estimator was affected for such problems and lower values of effectivity indices were obtained when compared with the cases that the exact solutions are smooth.

One obvious potential area of application of the present stress recovered procedure and the *a posteriori* error estimation scheme is to employ them for the development of an automatic adaptive refinement procedure for the RKPM and this will be reported in Part II of this study.

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Appendix: Integration schemes for Equation 37a

Recall that the recovered stress at a point $\mathbf{p}$, $\tilde{\sigma}_{ij}(\mathbf{p})$ is computed by the using Eqn. 29a

$$\tilde{\sigma}_{ij}(\mathbf{p}) = \int_{\Gamma_R} (-\hat{u}_k s_{kij}^* + \hat{t}_k d_{kij}^*) d\Gamma + \int_{\Omega_R} b_k d_{kij}^* d\Omega \quad (29a)$$

where

$$d_{kij}^* = \frac{1}{4\pi(1-\nu)r} \left((1-2\nu)(r_i \delta_{jk} + r_j \delta_{ik} - r_k \delta_{ij}) + 2r_i r_j r_k \right) \quad (28b)$$

$$s_{kij}^* = \frac{G}{2\pi(1-\nu)r^2} \left\{ \begin{aligned} &2r_m n_m \left((1-2\nu)r_k \delta_{ij} + \nu(r_j \delta_{ik} + r_i \delta_{jk}) - 4r_i r_j r_k \right) + 2\nu(n_i r_j + n_j r_i)r_k \\ &+ (1-2\nu)(2n_k r_i r_j + n_j \delta_{ik} + n_i \delta_{jk} - (1-4\nu)n_k \delta_{ij}) \end{aligned} \right\} \quad (28c)$$

For the first boundary integral on the RHS of Eqn. 29a, as mentioned in Section 3.4, two possible cases are required to be considered.

Case 1: The minimum distance between $\mathbf{p}$ and the domain boundary is greater than or equal to r_R

In this case, the boundary of the recovery subdomain, Γ_R is complete a circle $\mathcal{C}(\mathbf{p}, r_R)$ (Fig. A1). Since in this case the terms $1/r$ and $1/r^2$ in d_{kij}^* and s_{kij}^* are constant along $\mathcal{C}(\mathbf{p}, r_R)$, the first RHS integral of Eqn. 29a could be computed by first divides the circle $\mathcal{C}(\mathbf{p}, r_R)$ into a number of equal size arc segments that subtend an angle ω at $\mathbf{p}$. Gaussian integration scheme is then applied to each arc segment. It is found that the accuracy of the integration result is only sensitive to the subtended angle ω but not the order of Gaussian rule used. Further numerical investigations found that optimal results could be obtained by setting $\omega=4^\circ$ and using the single point rule for each arc segment.

Case 2: The minimum distance between $\mathbf{p}$ and the domain boundary is less than r_R

In this case, Γ_R will consist of two parts such that $\Gamma_R = \Gamma_{Ri} \cup \Gamma_{Rb}$ where Γ_{Rb} is the intersection of $\mathcal{C}(\mathbf{p}, r_R)$ with Γ (Fig. A2). Obviously, contribution from the first part Γ_{Ri} can be computed in the same way as in the first case. For the second part, since the terms $1/r$ and $1/r^2$ in d_{kij}^* and s_{kij}^* are no longer constant along the line Γ_{Rb} with length equal to L (Fig.

A2), a slightly more complicated integration scheme is needed. Firstly, Γ_{Rb} will be divided into a number of segments with length ΔL such that

$$\Delta L = \frac{L}{N_{\text{bseg}}} \leq \frac{D}{6} \quad N_{\text{bseg}} = \text{Int}\left(\frac{6L}{D}\right) + 1 \quad (\text{A1})$$

In Eqn. A1, D is the shortest distance between $\mathbf{p}$ and Γ and $\text{Int}(R)$ is the integer part for the real number R . Secondly, the contribution from Γ_{Rb} is then computed by applying 6-point Gaussian rule for each segment.

For the second domain integral on the RHS of Eqn. 29a, if the body force term $\mathbf{b}$ is zero or constant, this integral can be computed analytically. In case that $\mathbf{b}$ is not constant within the subdomain, the special Gaussian integration rules described in reference [31] could be used.

Fig. A1: Integration of Eqn. 37a when $\Gamma_R = C(\mathbf{p}, r_R)$

Fig. A2: Integration of Eqn. 37a when $\Gamma_R = \Gamma_{Ri} \cup \Gamma_{Rb}$

Fig. 1: The problem main Ω

Fig. 2: Local recovery subdomain Ω_R with circular boundary for an interior point

Fig. 3: Local recovery subdomain for a point close to the domain boundary

Fig. 4: Recovery of stress for boundary point p

Fig. 5: The patch test for the recovery procedure

Fig. 6: Example 1: Cantilever beam under end shear force

Fig. 7: Uniform refinement for Example 1

Fig. 8: Example 2: A plate with a central hole

Fig. 9: Uniform refinement for Example 2

Figure 12: Example 4: A L-shape domain

Fig. 13: Uniform refinement for Example 4

Node coordinates		(1.0, 0)	(1.0, 0.25)	(1.0, 0.5)	(1.0, 0.75)	(1.0, 1.0)
Penalty coefficient α (Eqn. 9b)	$10^1 E$	1.0027	0.99978	0.99937	0.99994	1.0031
	$10^3 E$	1.0027	0.99979	0.99937	0.99995	1.0031
	$10^6 E$	1.0027	0.99979	0.99937	0.99995	1.0031
	$10^9 E$	1.0027	0.99982	0.99938	0.99993	1.0031

Table 1: Patch test for the recovered stress value, $\tilde{\sigma}_x$ along the line $x=1$

Mesh	1	2	3	4
DOFs	54	170	252	594
h	5	2.5	2.0	1.25
η_{RKPM} (%)	14.53	6.58	5.10	2.89
η_{Rv} ($r_R=h$) (%)	7.73	2.92	2.1	1.02
η_{Rv} ($r_R=2h$) (%)	7.03	2.62	1.87	0.91
η_{Rv} ($r_R=3h$) (%)	6.73	2.51	1.8	0.88
R_{RKPM}		0.69	0.65	0.66
R_{Rv} ($r_R=h$)		0.85	0.84	0.83
R_{Rv} ($r_R=2h$)		0.86	0.85	0.84
R_{Rv} ($r_R=3h$)		0.86	0.85	0.84
θ ($r_R=h$)	0.59	0.67	0.73	0.82
θ ($r_R=2h$)	0.63	0.71	0.78	0.83
θ ($r_R=3h$)	0.67	0.73	0.8	0.85

Table 2: Results for Example 1

Legends for Tables 2 to 5

DOFs = Degrees of freedom of the RKPM mesh.

h = Uniform size of domain of influence for the RKPM.

r_R = Radius of the recovery subdomain, Ω_R (Fig. 2).

η_{RKPM} = Relative error of the RKPM solution (Eqn. 34).

η_{Rv} = Relative error of the recovered stress field (Eqn. 34).

R_{RKPM} = Convergence rate of the RKPM solution (Eqn. 35).

R_{Rv} = Convergence rate of the recovered solution (Eqn. 37)

θ = Effectivity index (Eqn. 16b).

Mesh	1	2	3	4
DOFs	72	164	240	564
h	1.0	0.625	0.5	0.3125
η_{RKPM} (%)	7.78	4.48	3.52	2.13
η_{Rv} ($r_{\text{R}}=h$) (%)	4.63	2.26	1.63	0.81
η_{Rv} ($r_{\text{R}}=2h$) (%)	4.25	2.06	1.49	0.73
η_{Rv} ($r_{\text{R}}=3h$) (%)	4.09	1.98	1.42	0.70
R_{RKPM}		0.67	0.63	0.59
R_{R} ($r_{\text{R}}=h$)		0.87	0.85	0.82
R_{R} ($r_{\text{R}}=2h$)		0.88	0.85	0.83
R_{R} ($r_{\text{R}}=3h$)		0.88	0.87	0.83
θ ($r_{\text{R}}=h$)	0.62	0.66	0.72	0.82
θ ($r_{\text{R}}=2h$)	0.65	0.70	0.75	0.84
θ ($r_{\text{R}}=3h$)	0.68	0.73	0.78	0.86

Table 3: Results for Example 2

Mesh	1	2	3	4
DOFs	30	90	132	306
h	0.5	0.25	0.2	0.125
η_{RKPM} (%)	29.7	20.9	18.8	15.2
η_{Rv} ($r_{\text{R}}=h$) (%)	19.12	12.45	10.93	8.42
η_{Rv} ($r_{\text{R}}=2h$) (%)	17.97	11.71	10.24	7.82
η_{Rv} ($r_{\text{R}}=3h$) (%)	17.21	11.09	9.7	7.41
R_{RKPM}		0.32	0.28	0.25
R_{R} ($r_{\text{R}}=h$)		0.39	0.34	0.31
R_{R} ($r_{\text{R}}=2h$)		0.39	0.35	0.32
R_{R} ($r_{\text{R}}=3h$)		0.40	0.35	0.32
θ ($r_{\text{R}}=h$)	0.4	0.48	0.5	0.53
θ ($r_{\text{R}}=2h$)	0.46	0.54	0.57	0.61
θ ($r_{\text{R}}=3h$)	0.49	0.56	0.6	0.63

Table 4: Results for Example 3

Mesh	1	2	3	4
DOFs	66	134	226	342
h	0.3535	0.2357	0.17675	0.1414
η_{RKPM} (%)	23.5	18.5	15.9	14.3
$\eta_{\text{Rv}} (r_{\text{R}}=h)$ (%)	15.02	11.4	9.49	8.31
$\eta_{\text{Rv}} (r_{\text{R}}=2h)$ (%)	14.12	10.71	8.87	7.74
$\eta_{\text{Rv}} (r_{\text{R}}=3h)$ (%)	13.52	10.18	8.43	7.35
R_{RKPM}		0.34	0.29	0.26
$R_{\text{R}} (r_{\text{R}}=h)$		0.39	0.35	0.32
$R_{\text{R}} (r_{\text{R}}=2h)$		0.39	0.36	0.33
$R_{\text{R}} (r_{\text{R}}=3h)$		0.4	0.36	0.33
$\theta (r_{\text{R}}=h)$	0.42	0.47	0.54	0.58
$\theta (r_{\text{R}}=2h)$	0.46	0.55	0.59	0.63
$\theta (r_{\text{R}}=3h)$	0.5	0.58	0.63	0.66

Table 5: Results for Example 4