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On error estimation and adaptive refinement for element free Galerkin method

Part I: Stress recovery and *a posteriori* error estimation

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Summary

In this study, an adaptive refinement procedure using the element free Galerkin method (EFGM) for the solution of 2D elastostatic problem is suggested. This refinement procedure is based on the well-known Zienkiewicz and Zhu (Z-Z) error estimator for the *a posteriori* error estimation and a simple point refinement scheme for new point mesh generation. The presentation of the work is divided into two parts. In Part I, concentration will be paid on the stress recovery and the *a posteriori* error estimation processes for the EFGM. A comprehensive study on the theories and natures of some commonly used recovery schemes for the EFGM is given. The essential features and relationships among the studied recovery schemes are highlighted. It will be demonstrated that all these recovery schemes are particular cases of a general recovery scheme called the *weighed continuous moving-least-square over stationary least square (MLS-SLS) recovery scheme*. A numerical experience has also been carried out to study the performances of a number of different MLS-SLS recovery procedures generated by applying different fitting parameters to the continuous MLS-SLS recovery scheme. In Part II of the study, attention will be focused on the mesh refinement and the development of the adaptive algorithms for the proposed automatic adaptive EFGM procedure.

KEYWORDS: *Element free Galerkin method, stress recovery, moving-least-square fitting, a posteriori error estimation, adaptive refinement*

1. Introduction

The objective of meshless methods is to alleviate the constrained structure of the mesh and to construct the approximation entirely in terms of nodes. These methods could be used to solve a large number of problems that cannot be easily treated by traditional finite element method (FEM). Among many meshless methods developed [1-9], the *Element Free Galerkin Method* (EFGM) is one of the most promising and well-developed methods for engineering applications. The EFGM differs from the FEM by using the *moving-least-square* (MLS) interpolation. In the EFGM, no element is needed and consequently the configuration of nodes is quite flexible. The global stiffness equations are formed by the Galerkin procedure and a background cell structure is often used for numerical integration. Essential boundary conditions can be treated by either modifying the functional used [10] or by coupling with finite elements [11].

The main purpose of this research is to develop an adaptive EFGM refinement procedure for the solution of elastostatic problems. In Part I of the study, based on the well-known Zienkiewicz-Zhu (Z-Z) error estimator [12], an approach for the *a posteriori* error estimation for the EFGM is investigated. As the essence of the Z-Z error estimator is to replace the exact stress by a recovery stress field for error estimation, a number of different recovery procedures for the construction of accurate smoothed stress field is proposed. Essential features and relationships among these stress recovery procedures will be discussed. The performances of these procedures will be assessed through some benchmark problems and the *a posteriori* error estimation scheme for the EFGM will be proposed.

In Part II of the study, based on the error estimation scheme developed, an adaptive EFGM refinement procedure will be implemented. Analyses of some 2D elastostatic benchmark problems are carried out to assess the efficiency of the suggested procedure.

2. Solution of linear elasticity by the EFGM

2.1 The modelled problem and MLS formulation

The problem to be solved is the typical linear elasticity problem expressed in the form

$$\mathbf{L}^T \mathbf{D} \mathbf{L} \mathbf{u} - \mathbf{f} = \mathbf{L}^T \boldsymbol{\sigma} - \mathbf{f} = 0 \quad (1a)$$

in a given 2D domain Ω with boundary Γ and body forces $\mathbf{f}$. The boundary conditions are

$$\mathbf{u} = \hat{\mathbf{u}} \text{ on the essential boundary } \Gamma_u \quad \mathbf{Q} \mathbf{D} \mathbf{L} \mathbf{u} = \hat{\mathbf{t}} \text{ on the traction boundary } \Gamma_t \quad (1b)$$

such that $\Gamma = \Gamma_u \cup \Gamma_t$ and $\Gamma_u \cap \Gamma_t = \emptyset$. The matrices $\mathbf{Q}$ and $\mathbf{L}$ are given by

$$\mathbf{Q} = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix} \text{ and } \mathbf{L}^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \quad (2)$$

and $\mathbf{n}=(n_x, n_y)$ is the outward normal vector along Γ . The stress tensor, $\boldsymbol{\sigma}$ can be computed from the displacement vector $\mathbf{u}$ by differentiation

$$\boldsymbol{\sigma} = \mathbf{D}\mathbf{L}\mathbf{u} \quad (3)$$

where $\mathbf{D}$ is the usual material matrix.

When the EFGM is used for the solution of Eqn. 1, the global approximation solution $\mathbf{u}^h$ is constructed by forming a local MLS approximation over Ω_x with radius h , the domain of influence of $\mathbf{x}$. The local MLS approximation $\mathbf{u}_L$ inside Ω_x is expressed as

$$\mathbf{u}_L(\mathbf{x}) = \sum_{j=1}^m p_j(\mathbf{x}) a_j(\mathbf{x}) = \mathbf{p}^T(\mathbf{x}) \mathbf{a}(\mathbf{x}) \quad \mathbf{x} \in \Omega_x \quad (4)$$

where $\mathbf{p}(\mathbf{x})$ is a complete basis of size m and order p , whose components $p_j(\mathbf{x})$ are polynomial basis functions of $\mathbf{x}$. In 2D problems, the commonly used bases are the linear basis, $\mathbf{p}^T=(1, x, y)$, and the quadratic basis, $\mathbf{p}^T=(1, x, y, x^2, xy, y^2)$.

The coefficients $a_j(\mathbf{x})$ are obtained by performing a weighted MLS fitting for $\mathbf{u}_L$ using the functional J defined as

$$J = \sum_{i=1}^N w_i(\mathbf{x}) [\hat{\mathbf{u}}_i - \mathbf{u}_L(\mathbf{x}_i)]^2 = \sum_{i=1}^N w_i(\mathbf{x}) [\hat{\mathbf{u}}_i - \mathbf{p}^T(\mathbf{x}_i) \mathbf{a}(\mathbf{x}_i)]^2 \quad (5)$$

where $w_i(\mathbf{x})$ are weight functions with compact supports and N is the number of nodal points in Ω_x where $w_i \neq 0$. The term $\hat{\mathbf{u}}_i$ is the parameter associate with the i th node. In the current study, the weight functions are the tensor product cubic spline functions of the form

$$w_i(\mathbf{x}) = w(\mathbf{x} - \mathbf{x}_i) = w_x w_y = w(r_x) w(r_y) \quad (6)$$

$$\text{with } w(\bar{s}) = \begin{cases} \frac{2}{3} - 4\bar{s} + 4\bar{s}^3 & \text{for } \bar{s} \leq \frac{1}{2} \\ \frac{4}{3} - 4\bar{s} + 4\bar{s}^2 - \frac{4}{3}\bar{s}^3 & \text{for } \frac{1}{2} < \bar{s} \leq 1 \\ 0 & \text{for } \bar{s} > 1 \end{cases} \quad \bar{s} = \frac{|\mathbf{x} - \mathbf{x}_i|}{h} \quad (7)$$

$$\text{where } r_x = \frac{\|\mathbf{x} - \mathbf{x}_i\|}{d_{mx}} \quad r_y = \frac{\|y - y_i\|}{d_{my}} \quad (8)$$

$$\text{and } d_{mx} = d_m c_{xi} \quad d_{my} = d_m c_{yi} \quad (9)$$

In Eqn. 9, c_{xi} and c_{yi} are the local characteristic dimension of nodal spacing. They are computed as the distances to the third closest node from the current node $\mathbf{x}_i$ from $\mathbf{x}$. d_m is the support scaling parameters which is normally dependent on the problem [3,10].

Minimization of J with respect to $\mathbf{a}(\mathbf{x})$ yields the following system of linear equations

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) - \mathbf{C}(\mathbf{x})\hat{\mathbf{u}} = 0 \quad \text{or} \quad \mathbf{a}(\mathbf{x}) - \mathbf{A}(\mathbf{x})^{-1}\mathbf{C}(\mathbf{x})\hat{\mathbf{u}} = 0 \quad (10)$$

where

$$\mathbf{A}(\mathbf{x}) = \sum_{i=1}^N w_i(\mathbf{x}) \mathbf{p}^T(\mathbf{x}_i) \mathbf{p}(\mathbf{x}_i) \quad (11)$$

and

$$\mathbf{C}(\mathbf{x}) = [w_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1), w_2(\mathbf{x})\mathbf{p}(\mathbf{x}_2), \dots, w_N(\mathbf{x})\mathbf{p}(\mathbf{x}_N)] \quad (12)$$

The approximation $\mathbf{u}_L(\mathbf{x})$ can be then written as

$$\mathbf{u}_L(\mathbf{x}) = \sum_{i=1}^N \Phi_i(\mathbf{x}) \hat{\mathbf{u}}_i = \mathbf{\Phi}(\mathbf{x}) \hat{\mathbf{u}} \quad \text{for } \mathbf{x} \in \Omega_x \quad (13)$$

where

$$\mathbf{\Phi} = [\Phi_1(\mathbf{x}), \dots, \Phi_N(\mathbf{x})] = \mathbf{p}^T(\mathbf{x}) \mathbf{A}^{-1}(\mathbf{x}) \mathbf{B}(\mathbf{x}) \quad (14)$$

Based on local approximation, the approximation $\mathbf{u}^h$ over the whole domain can be formed by combining all the local approximations

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i=1}^N \Phi_i(\mathbf{x}) \hat{\mathbf{u}}_i \quad \text{for } \mathbf{x} \in \Omega \quad (15)$$

In Eqn. 15, $\Phi_i(\mathbf{x})$ are the MLS shape functions of the EFGM. The global stiffness equation systems can be obtained substituting the MLS shape functions into the weak form formulation and is given by

$$\mathbf{K} \hat{\mathbf{u}} = \mathbf{F} \quad (16)$$

In Eqn. 16, the entries of stiffness matrix $\mathbf{K}$ and load vector $\mathbf{F}$ are defined as

$$\mathbf{K}_{ij} = \int_{\Omega} \mathbf{B}_i^T \mathbf{D} \mathbf{B}_j d\Omega \quad (17a)$$

and

$$\mathbf{F}_i = \int_{\Omega} \mathbf{\Phi}_i^T \mathbf{f} d\Omega + \int_{\Gamma_t} \mathbf{\Phi}_i \hat{\mathbf{t}} d\Gamma \quad (17b)$$

In Eqn. 17, and the matrix of shape functions and their derivatives are expressed in the forms

$$\mathbf{\Phi}_i = \begin{bmatrix} \Phi_i & 0 \\ 0 & \Phi_i \end{bmatrix} \quad \mathbf{B}_i = \mathbf{L}^T \mathbf{\Phi}_i = \begin{bmatrix} \Phi_{i,x} & 0 \\ 0 & \Phi_{i,y} \\ \Phi_{i,y} & \Phi_{i,x} \end{bmatrix} \quad (18)$$

2.2 *Background cells for numerical integration*

Unlike the finite element method in which $\mathbf{K}$ and $\mathbf{F}$ can be formed by summing up in an element by element manner, in the EFGM, *background cells* [3] are used for the numerical integration of $\mathbf{K}$ and $\mathbf{F}$. Gaussian quadrature is performed over all the background cells and traction boundary segments. Each quadrature point contributes nonzero entries only to the equations corresponding to those nodes whose domains of influence cover that quadrature point. Thus, in the final equations, no coupling will occur between nodes that are not within a specified distance (which relates to the support size) and the corresponding entries in $\mathbf{K}$ vanish. If the nodes are numbered by sorting in the x direction, $\mathbf{K}$ will be banded.

In the current implementation, in order to integrate the weak form equation accurately to ensure the convergence of the EFGM [13], 4×4 Gaussian quadrature points within each cell are used while 4-points integration is used along the traction boundary (Fig. 1).

For problems with a rectangular geometry, rectangular integration cells that cover the problem domain exactly are preferred and they should exactly cover the problem domain. However, for curved geometries, if a rectangular integration cell structure is used, integration cells which is partially belongs to the problem domain will exist (e.g. cell ABCD in Fig. 1) and special formulations are required for these cells. To avoid such difficulties, quadrilateral background cells with sizes equal to the average spacing of the nodes and conformal with the curve boundary are used.

For non-convex domains or domains with crack edges, the visibility criterion [14] is used to decide whether a node should be included in the MLS procedure. When evaluating the MLS approximation at a given point, a node will be included only if the point is visible from that node.

2.3 *Treatment for essential boundary conditions*

In EFGM, the fact that the MLS approximation does not satisfy the Kronecker delta conditions complicates the imposition of essential boundary condition. In the current implementation, the coupled finite element-element free Galerkin method (FE-EFGM) [11] is adopted for the imposition of the essential boundary conditions. The coupling is accomplished by introducing a layer of finite elements along the essential boundary (Fig. 2). In these interface elements, a hybrid displacement approximation that satisfies the displacement continuity condition is defined.

3. Error estimation and stress recovery procedure for EFGM

3.1 Definition of error and the Zienkiewicz-Zhu error estimator

The error of the EFGM solution can be defined as the difference between the exact and the EFGM solution. i.e.

$$\mathbf{e}_u = \mathbf{u} - \mathbf{u}^h \quad \mathbf{e}_\sigma = \sigma - \sigma^h \quad (19)$$

In Eqn. 19, σ^h is the EFGM stress obtained by differentiating the EFGM displacements

$$\sigma^h(\mathbf{x}) = \mathbf{D} \mathbf{L} \mathbf{u}^h(\mathbf{x}) = \mathbf{D} \sum_{i=1}^N \mathbf{B}_i(\mathbf{x}) \hat{\mathbf{u}}_i \quad (20)$$

The global error in energy norm can be calculated as

$$\|\mathbf{e}_u\| = \left\{ \int_{\Omega} (\mathbf{L} \mathbf{e}_u)^T \mathbf{D}^{-1} (\mathbf{L} \mathbf{e}_u) d\Omega \right\}^{1/2} = \left\{ \int_{\Omega} \mathbf{e}_\sigma^T \mathbf{D}^{-1} \mathbf{e}_\sigma d\Omega \right\}^{1/2} \quad (21)$$

The relative error in the energy norm, η , is defined as

$$\eta = \|\mathbf{e}_u\| / \|\mathbf{u}\| \quad \|\mathbf{u}\| = \left\{ \int_{\Omega} \sigma^T \mathbf{D}^{-1} \sigma d\Omega \right\}^{1/2} \quad (22)$$

In this study, the well known Zienkiewicz-Zhu (Z-Z) error estimator [12] will be used for error estimation. The essence of the Z-Z error estimator is to first obtain a smoothed stress, σ^p , by means of a smoothing process. The difference between σ^p and σ^h will then be used to calculate the estimate error norm, $\overline{\|\mathbf{e}_u\|}$

$$\overline{\|\mathbf{e}_u\|} = \left\{ \int_{\Omega} (\sigma^p - \sigma^h)^T \mathbf{D}^{-1} (\sigma^p - \sigma^h) d\Omega \right\}^{1/2} \quad (23a)$$

To study the performance of the error estimator, the *recovery error norm*

$$\|\mathbf{e}_u^p\| = \left\{ \int_{\Omega} (\sigma - \sigma^p)^T \mathbf{D}^{-1} (\sigma - \sigma^p) d\Omega \right\}^{1/2} \quad (23b)$$

will also be computed. In addition, in order to assess the efficiency of the error estimator, the commonly used *effectivity index* θ [12].

$$\theta = \frac{\overline{\|\mathbf{e}_u\|}}{\|\mathbf{e}_u\|} \quad (24)$$

will be calculated. Note that an error estimator is *asymptotically exact* if it converges to unity as the number of degrees of freedom of the model is increased.

3.2 Stress recovery procedures for the EFGM

As the central idea of the Z-Z error estimator builds around the construction of a smoothed or projected stress field as an approximation of the exact stress, one can expect that the accuracy of the recovered stress will affect the efficiency of the error

estimator. In this study, a number of different recovery schemes are proposed and their performances will be tested. In the following sections, a systematic description of the recovery schemes examined will be presented.

3.2.1 The T-Belytschko's Stress Recovery Scheme (TB Scheme)

This is a popular recovery stress scheme first suggested by Tabbara et al [15] and then subsequently used by Belytschko [16] for error estimation in the EFGM. The recovery stress field can be written in a simple form:

$$\boldsymbol{\sigma}^{\text{TB}}(\mathbf{x}) = \sum_{i=1}^{n_{\text{TB}}} \Phi_i^{\text{TB}}(\mathbf{x}) \boldsymbol{\sigma}^h(\mathbf{x}_i) \quad (25a)$$

In Eqn. 25a, $\boldsymbol{\sigma}^h(\mathbf{x}_i)$ is the EFGM stress evaluated at node i , Φ_i^{TB} is a set of recovery MLS shape functions and n_{TB} is the number of nodal points whose support includes the point $\mathbf{x}$. By using Eqn. 20, the recovery stress, $\boldsymbol{\sigma}^{\text{TB}}(\mathbf{x})$ can be rewritten as

$$\boldsymbol{\sigma}^{\text{TB}}(\mathbf{x}) = \sum_{i=1}^{n_{\text{TB}}} \Phi_i^{\text{TB}}(\mathbf{x}) \left(\mathbf{D} \sum_{j=1}^{N_i} \mathbf{B}_j(\mathbf{x}_i) \hat{\mathbf{u}}_j \right) \quad (25b)$$

where N_i is the number of nodal points associate with the support of the i th node. The recovery MLS shape functions, Φ_i^{TB} , can be constructed through a similar procedure as in the case of MLS shape functions construction in the EFGM analysis. In practice, in order to obtain an accurate recovered stress field, Φ_i^{TB} are constructed by using reduced supports [16] on the same nodal points of the original (Fig. 3).

By comparing Eqn. 25a with Eqn. 20, one can see that the EFGM stress is expressed in terms of the first derivatives of the MLS shape functions. However, the recovered stress is expressed in terms of another set of MLS shape functions. This means that with an appropriate choice of the reduced supports, $\boldsymbol{\sigma}^{\text{TB}}(\mathbf{x})$ could be more accurate than $\boldsymbol{\sigma}^h(\mathbf{x})$.

From Eqn. 25, one can also interpret that $\boldsymbol{\sigma}^{\text{TB}}(\mathbf{x})$ is computed by taking the product of MLS shape functions and the nodal EFGM stress. Thus, it follows that the TB scheme is in fact nothing more than performing another MLS fitting over the nodal EFGM stress. An example for the 1D case is shown in Fig. 4.

3.2.2 The Discrete MLS over Stationary least-square (MLS-SLS) fitting Scheme

From the experience obtained in the FE error estimation, it is found that in many cases, *stationary least-square* (SLS) fitting can improve the accuracy of the FE stress. A least-square fitting scheme is said to be stationary if the coefficients used are *independent* of $\mathbf{x}$. Consider a typical node $\mathbf{x}_i$ and its SLS fitting patch Ω_i (Fig. 5), the assumed stress field inside Ω_i , $\sigma_L^{ZZ}(\bar{\mathbf{x}})$, is defined as

$$\sigma_L^{ZZ}(\bar{\mathbf{x}}) = \sum_{j=1}^{m_q} q_j(\bar{\mathbf{x}}) \mathbf{c}_j = \mathbf{q}^T(\bar{\mathbf{x}}) \mathbf{c} \quad (26)$$

where $\bar{\mathbf{x}}$ is the coordinates of a local coordinate system with origin at $\mathbf{x}_i$. $\mathbf{c}$ is a set of coefficients which are *independent* of $\bar{\mathbf{x}}$. $\mathbf{q}$ is a complete polynomial basis of order q with m_q terms. The number of nodal points used, n_i^{ZZ} must satisfies the condition $n_i^{ZZ} \geq m_q$. The coefficients $\mathbf{c}$ are obtained by minimizing the functional J_{ZZ} with respect to $\mathbf{c}$.

$$J_{ZZ} = \sum_{j=1}^{n_i^{ZZ}} w_j^{ZZ} (\sigma_L^{ZZ}(\bar{\mathbf{x}}_j) - \sigma^h(\bar{\mathbf{x}}_j))^2 = \sum_{j=1}^{n_i^{ZZ}} w_j^{ZZ} (\mathbf{q}^T(\bar{\mathbf{x}}_j) \mathbf{c} - \sigma^h(\bar{\mathbf{x}}_j))^2 \quad \mathbf{x}_j \in \Omega_i \quad (27)$$

In Eqn. 27, w_j^{ZZ} is a set of nodal weights so that $\mathbf{c}$ can be solved as:

$$\mathbf{G}\mathbf{c}=\mathbf{d} \quad \text{or} \quad \mathbf{c}=\mathbf{G}^{-1}\mathbf{d}, \quad \mathbf{G} = \sum_{j=1}^{n_i^{ZZ}} w_j^{ZZ} \mathbf{q}^T(\bar{\mathbf{x}}_j) \mathbf{q}(\bar{\mathbf{x}}_j), \quad \mathbf{d} = \sum_{j=1}^{n_i^{ZZ}} w_j^{ZZ} \mathbf{q}^T(\bar{\mathbf{x}}_j) \sigma^h(\bar{\mathbf{x}}_j) \quad (28a)$$

and

$$\sigma_L^{ZZ}(\mathbf{x}) = \mathbf{q}^T(\bar{\mathbf{x}}) \mathbf{G}^{-1} \mathbf{d} \quad (28b)$$

After repeating the local SLS fittings over all the nodes, one can construct a global stress field, $\sigma^{ZZ}(\mathbf{x})$, by using the MLS fitting procedure over all the $\sigma_L^{ZZ}(\mathbf{x}_i)$ obtained.

That is

$$\sigma^{ZZ}(\mathbf{x}) = \sum_{i=1}^{N_{ZZ}} \Phi_i^{ZZ}(\mathbf{x}) \sigma_L^{ZZ}(\mathbf{x}_i) \quad (29)$$

In Eqn. 29, $\Phi_i^{ZZ}(\mathbf{x})$ is a set of MLS shape functions (which could equal to $\Phi_i^{TB}(\mathbf{x})$). In general, $\sigma^{ZZ}(\mathbf{x})$ is the stress obtained from MLS fitting over the local SLS fittings of the stress computed from EFGM. An example for the 1D case is shown in Fig. 6. Assume that the first nodal point $\mathbf{x}_j$, $j=1$, inside Ω_i is the node $\mathbf{x}_i$ itself, i.e. $\mathbf{x}_1=\mathbf{x}_i$. Consider the special case in which the weight functions for SLS fitting are defined as

$$w_j^{ZZ} = \begin{cases} 1 & i = 1 \\ 0 & i = 2, \dots, n_i^{ZZ} \end{cases} \quad (30)$$

The equation $\mathbf{G}\mathbf{c}=\mathbf{d}$ (Eqn. 28a) will then be reduced to

$$\mathbf{q}^T(\bar{\mathbf{x}}_i)\mathbf{q}(\bar{\mathbf{x}}_i)\mathbf{c} = \mathbf{q}^T(\bar{\mathbf{x}}_i)\boldsymbol{\sigma}^h(\bar{\mathbf{x}}_i) \quad (31)$$

Since the local coordinates of $\mathbf{x}_i$ is (0,0), Eqn. 28a can be written as

$$\begin{bmatrix} 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ \cdot \\ \cdot \\ c_{mq} \end{bmatrix} = \begin{bmatrix} \sigma^h(\mathbf{x}_i) \\ 0 \\ \cdot \\ 0 \end{bmatrix} \quad (32)$$

It implies that $c_1 = \sigma^h(\mathbf{x}_i)$ with $c_2, \dots, c_{mq}$ undetermined. The expression of the local stress $\sigma_L^{ZZ}(\mathbf{x}_i)$ (Eqn. 26) will be reduced to

$$\sigma_L^{ZZ}(\mathbf{x}_i) = c_1 = \sigma^h(\mathbf{x}_i) \quad (33)$$

Finally, in Eqn. 29, if one selects $\Phi_i^{ZZ}(\mathbf{x}) = \Phi_i^{TB}(\mathbf{x})$, then $\sigma^{ZZ}(\mathbf{x}) = \sigma^{TB}(\mathbf{x})$. That is, the TB scheme can be considered as a special case of the discrete MLS-SLS scheme with the weight coefficients defined by Eqn. 30.

3.2.3 The continuous MLS-SLS Scheme

Consider the more general continuous form of the functional J_{ZZ} in the discrete MLS-SLS scheme (Eqn. 27)

$$J_{ZZ} = \int_{\Omega_j} (\sigma_L^{ZZ}(\bar{\mathbf{x}}) - \sigma^h(\bar{\mathbf{x}}))^T (\sigma_L^{ZZ}(\bar{\mathbf{x}}) - \sigma^h(\bar{\mathbf{x}})) d\Omega \quad (34)$$

If numerical integration is used to evaluate Eqn. 34 with integration points set as the nodes inside the fitting patch and weight equal to w_j^{ZZ} , Eqn. 34 can be expressed as

$$J_{ZZ} = \int_{\Omega_j} (\sigma_L^{ZZ}(\bar{\mathbf{x}}) - \sigma^h(\bar{\mathbf{x}}))^T (\sigma_L^{ZZ}(\bar{\mathbf{x}}) - \sigma^h(\bar{\mathbf{x}})) d\Omega \approx \sum_{j=1}^{n_i^{ZZ}} w_j^{ZZ} (\sigma_L^{ZZ}(\bar{\mathbf{x}}_j) - \sigma^h(\bar{\mathbf{x}}_j))^2 \quad (35)$$

Therefore, the discrete MLS-SLS can in turn be considered as a special case of the continuous MLS-SLS scheme defined by Eqn. 34. Furthermore, if a matrix $\mathbf{H}$ (e.g. the inverse of the material matrix, $\mathbf{D}^{-1}$) is inserted into Eqn. 35, J_{ZZ} will become

$$J_{ZZ} = \int_{\Omega_j} (\sigma_L^{ZZ}(\bar{\mathbf{x}}) - \sigma^h(\bar{\mathbf{x}}))^T \mathbf{H} (\sigma_L^{ZZ}(\bar{\mathbf{x}}) - \sigma^h(\bar{\mathbf{x}})) d\Omega \quad (36)$$

In Eqn. 36, the components of the stress are coupled and one can see that Eqn. 35 is a particular case of Eqn. 36 where $\mathbf{H}$ is a 3 by 3 identity matrix, $\mathbf{I}_3$.

3.2.4 The MLS-MLS scheme and relationships among different recovery schemes

An even more general stress recovery scheme can be constructed if one uses the MLS fitting instead of the SLS fitting in Eqn. 36 for the local stress construction. That is

$$\boldsymbol{\sigma}_L^{ZZ}(\mathbf{x}) = \sum_{j=1}^{m_q} \mathbf{q}_j(\bar{\mathbf{x}}) \mathbf{c}_j(\bar{\mathbf{x}}) = \mathbf{q}^T(\bar{\mathbf{x}}) \mathbf{c}(\bar{\mathbf{x}}) \quad (37)$$

where $\mathbf{c}(\bar{\mathbf{x}})$ is now *dependent* on $\bar{\mathbf{x}}$. In this case, the recover scheme is called a MLS-MLS scheme and J_{ZZ} can also be expressed in either discrete or continuous forms.

From the above descriptions of different recovery schemes, one can classify them into two main categories, namely, MLS-MLS and MLS-SLS schemes. In all these schemes, MLS fitting (Eqn. 29) is used to construct the global stress field while different SLS fitting schemes are employed to generate the local stress field $\boldsymbol{\sigma}_L^{ZZ}(\mathbf{x})$. A schematic diagram is given in Fig. 7 to illustrate the relationships among these schemes.

4. Testing of different recovery schemes

4.1 Scope of testing and parameters used

In this section, numerical studies will be carried out to investigate the accuracies of different recovery schemes and their performances in the *a posteriori* error estimation. However, due to limitations of the length of this paper, only the discrete MLS-SLS scheme (Section 3.2.2) and the TB scheme (3.2.1) will be tested. Furthermore, the TB scheme is considered as a particular case of the discrete MLS-SLS scheme. Since any MLS-SLS scheme can be divided into two stages, namely, (i) SLS fitting over the EFGM stress field and (ii) global MLS fitting, numerical studies will be carried out to investigate the influences of various parameters used in these two stages on the accuracy of the recovery stress.

4.1.1 Parameters tested in the SLS fitting

(1) Order of complete polynomial basis $\mathbf{q}(\mathbf{x})$, q

In the numerical study, both linear and quadratic polynomial expansions will be used in the displacement approximation (Eqn. 4). Thus, the order of SLS fitting, q should be at least equal to the EFGM displacement approximation, p . In this study, two different orders of SLS fitting with $q=p$ and $q=p+1$ will be tested. (Table 1).

(2) *Number of nodes inside Ω_i , n_i^{ZZ}*

Once the value of q is fixed, the minimum value of n_i^{ZZ} will be defined such that $n_i^{ZZ} \geq m_q$ where m_q is the number of used terms in the fitting (e.g. $q=1$, $m_q=3$ and $q=2$, $m_q=6$ and so on). In general, the size of the patch should be as small as possible to maintain the local character of the SLS fitting. However, it is sometime necessary to include more nodes to maintain the stability of the fitting procedure [17]. As a result, in the numerical study, for a given value of q , three different values of n_i^{ZZ} will be tested as shown in Table 1.

(3) *Nodal integration weights used, w_j^{ZZ}*

In the current numerical experiment, the following three different schemes will be used to define the integration weights for the SLS fitting (Eqn. 35).

(i) *The TB scheme*

In this scheme, Eqn. 30 will be used so that the MLS-SLS scheme will be reduced to the TB scheme as shown in the second row of Table 2.

(ii) *The uniform scheme*

In this scheme, all w_j^{ZZ} are set equal to $1/n_i^{ZZ}$ as shown in the third row of Table 2.

(iii) *The Distance scheme*

In this scheme, the weights will be defined based on the distance of a node from $\mathbf{x}_i$ by the following steps:

- (a) The patch node $\mathbf{x}_i$ will be assigned as the first node. Other nodes in the patch are labelled in an ascending order according to their distances from $\mathbf{x}_i$. An example is shown in Fig. 8 with $n_i^{ZZ}=6$. If the distances of two nodes are equal their orders will be determined based on their global node numbers.
- (b) The weight for the patch node $\mathbf{x}_i$ is set equal to w , i.e. $w_1^{ZZ}=w$. In this study, three different values of $w=1/2, 2/3$ and $3/4$ will be tested.
- (c) The weights for other nodes w_j^{ZZ} , $j=2, \dots, n_i^{ZZ}$, will be computed according to the distances from $\mathbf{x}_i$ as shown in the last row of Table 2.

4.1.2 Implementation of the MLS fitting

As pointed out in Section 3.2.1, a reduced support size is needed during the MLS fitting in order to obtain an accurate recovered stress field. In the current study, the support

size of the MLS shape functions $\Phi_i^{ZZ}(\mathbf{x})$ used is defined as (c.f. Eqn. 9):

$$d_{mx}^p = d_m^p c_{xi} \quad \text{and} \quad d_{my}^p = d_m^p c_{yi} \quad (38)$$

In Eqn. 38, c_{xi} and c_{yi} are local characteristic dimension of nodal spacing defined in Eqn. 9. d_m^p is the support scaling parameters for the recovery procedure. Similar to d_m , its value is again dependent on both the problem and the order of polynomial used [16]. The actual values used in numerical examples will be given in the next section.

5 Numerical Examples

In this section, four benchmark problems will be employed to assess the performance of the recovery procedures when different parameters are used. Uniform refinements will be carried out for both linear and quadratic EFGM analyses. Cubic spline function (Eqn. 7) and tensor product weight (Eqn. 8) are employed to define the weight function. The support size parameters used for the MLS fitting in the EFG analysis (Eqn. 9) and the stress recovery procedure (Eqn. 38) are listed in Table 3. During the recovery procedures, it is found that the accuracy of the recovery stress will not be affected if the order of numerical integration used for the formation of recovery matrix $\mathbf{G}$ (Eqn. 28a) is not less than that used in the EFGM (4×4 Gauss points per cell). However, a higher order (8×8) numerical integration is used for the evaluation of the error norms (Eqn. 23) to ensure the precision of error computations. The accuracies of the different recovery procedures are assessed by evaluating (i) the effectivity index, θ (Eqn. 24) and (ii) the relative error in energy norm of the recovered stress field, η^p , defined as

$$\eta^p = \frac{\|\mathbf{e}_u^p\|}{\|\mathbf{u}\|} \quad (39)$$

5.1 The tested problems and point meshes used

Problem 1: Timoshenko Beam (non-singular problem, convex domain)

In the first example, a beam subjected to a parabolic traction at the free end as shown in Fig. 9 is considered. Plane stress conditions are assumed and the smooth exact solution is given by Timoshenko and Goodier [18]

$$\sigma_x(x, y) = -\frac{Py}{I}(L - x), \quad \sigma_y(x, y) = 0.0, \quad \tau_{xy}(x, y) = -\frac{P}{2I}(D^2/4 - y^2) \quad (40)$$

The four point meshes used are shown in Fig. 10. Note that in these meshes, a layer of 4-node finite elements is inserted along the fixed end for the imposition of the essential boundary conditions. Results obtained from the EFGM solution and the performance of different stress recovery schemes are summarized in Tables 4 and 5 for linear and quadratic analyses respectively. Note that in Table 5, due to the regular nodal distribution, the recovery matrix will be singular in the case of cubic SLS fitting for mesh 1 and therefore no result is available.

Problem 2: Infinite Plate with a Hole (non-singular problem, concave domain)

The second example is a portion of an infinite plate with a central circular hole subjected to a unit tensile load as shown in Fig. 11. Due to symmetry, only the upper right quadrant of the plate is modeled. Essential boundary conditions are imposed on the left and bottom edges by inserting 4-node finite elements there. Traction boundary conditions given by the exact solution are imposed on the right and top edges. The exact solution is given by [18]:

$$\begin{aligned}\sigma_x(x, y) &= 1 - \frac{a^2}{r^2} \left(\frac{3}{2} \cos 2\beta + \cos 4\beta \right) + \frac{3a^4}{2r^4} \cos 4\beta \\ \sigma_y(x, y) &= -\frac{a^2}{r^2} \left(\frac{1}{2} \cos 2\beta - \cos 4\beta \right) - \frac{3a^4}{2r^4} \cos 4\beta \\ \tau_{xy}(x, y) &= -\frac{a^2}{r^2} \left(\frac{1}{2} \sin 2\beta + \sin 4\beta \right) + \frac{3a^4}{2r^4} \sin 4\beta\end{aligned}\tag{41}$$

The three point meshes used are shown in Fig. 12 (only the distributions of points are shown). The results obtained are listed in Tables 6 and 7 respectively.

Problem 3: Region near the Tip of a Crack (singular problem, convex domain)

The third example is a region near a crack tip (Fig. 13). Due to symmetry, only half of the domain is modeled in the analysis. The plate is loaded by tractions corresponding to the first symmetric mode of the stress intensity factor solution given by [19]

$$\begin{aligned}\sigma_x(x, y) &= \frac{K_I}{\sqrt{2\pi}} \cos \frac{\beta}{2} \left(1 - \sin \frac{\beta}{2} \sin \frac{3\beta}{2} \right) \\ \sigma_y(x, y) &= \frac{K_I}{\sqrt{2\pi}} \cos \frac{\beta}{2} \left(1 + \sin \frac{\beta}{2} \sin \frac{3\beta}{2} \right) \\ \tau_{xy}(x, y) &= \frac{K_I}{\sqrt{2\pi}} \left(\sin \frac{\beta}{2} \cos \frac{\beta}{2} \cos \frac{3\beta}{2} \right)\end{aligned}\tag{42}$$

In Eqn. 42, K_I is set equal to $\sqrt{2\pi}$. For this problem, a singular point exists at the crack tip (0,0) with strength $\lambda=0.5$. The three point meshes used in which a layer of finite elements is inserted along the edge AB is shown in Fig. 14 (only the positions of nodes are shown). The results obtained are listed in Tables 8 and 9 respectively.

Example 4: L-Shape Domain (singular problem, concave domain)

In the final problem, the L-shape domain shown in Fig. 15 is loaded by tractions which are corresponding to the following stress field [19]

$$\begin{aligned}\sigma_x &= \lambda r^{(\lambda-1)} \{ (2 - Q(\lambda + 1)) \cos((\lambda - 1)\beta) - (\lambda - 1) \cos((\lambda - 3)\beta) \} \\ \tau_{xy} &= \lambda r^{(\lambda-1)} \{ (\lambda - 1) \sin((\lambda - 3)\beta) + Q(\lambda + 1) \sin((\lambda - 1)\beta) \} \\ \sigma_y &= \lambda r^{(\lambda-1)} \{ (2 + Q(\lambda + 1)) \cos((\lambda - 1)\beta) + (\lambda - 1) \cos((\lambda - 3)\beta) \}\end{aligned}\quad (43)$$

In Eqn. 43, $Q=0.543075579$ and $\lambda=0.544483737$. For this problem, a singular point exists at the re-entrant corner point A. The three point meshes are shown in Fig. 16. Note that for these meshes, three and one finite elements are placed around the points A and B respectively to enforce the essential boundary conditions. The results obtained for this example are listed in Table 10 and 11 respectively.

5.2 Conclusions and Comments on the numerical results

5.2.1 Accuracy of recovery stress when compared with EFGM stress

From previous study [16], it was found that due to high frequency content of derivatives of the MLS shape functions, the EFGM stresses may produce spurious oscillations at high gradient stress regions. Detailed inspection of the recovery stresses indicated that in many cases, such oscillations are reduced significantly by the recovery procedures. For Problems 1 and 2 in which the exact solutions are smooth, the accuracies of recovery stresses are better than the EFGM stresses (e.g. in Table 5, mesh 2, $\eta \approx 5\%$ while all recovery schemes yielded $\eta^p < 1.5\%$). For Problems 3 and 4 in which singular point exists, the accuracies of the recovery stresses are not always better than the EFGM stress. However, a slightly faster convergence rate for the recovered stress can be observed. (e.g. Table 10, $\eta < \eta^p$ for mesh 1 while $\eta^p < \eta$ for almost all recovery schemes for the final mesh.)

5.2.2 Effectivity Indices

From Tables 4 to 7, one can see that for smooth problems, the values of θ for all the recovery schemes are very close to the ideal value of 1.0. Furthermore, θ converges to unity as the meshes are refined uniformly. However, for Problems 3 and 4 in which a singular point exists, even though the values of θ are all reasonably close to unity, no such convergence trend can be identified as the meshes are refined.

5.2.3 Performance of different recovery schemes

(1) Effect of the order of complete polynomial bases used in the SLS fitting, q

As pointed out in Section 3.2.1, the order of complete polynomial q used in the SLS fitting should be at least equal to the order of the EFGM analysis, p . Thus, two values of $q=p$ and $q=p+1$ are tested. Numerical results indicates that for Problems 1 and 2 in which the exact solutions are smooth, for both $p=1$ and $p=2$, a value of $q=p+1$ led to improvements for most schemes. For example, in Table 5, mesh 3 with $w_1^{ZZ}=1/n_j^{ZZ}$, the results obtained from $q=p+1=3$ are more accurate than $q=2$. It seems that for smooth problems, the accuracy of recovered stress could be improved by increasing value of q . For Problems 3 and 4 in which singular point exists, it is interestingly found that the optimal value of q is dependent on p . For the case of $p=1$, using $q=2$ yields slightly better results than when $q=p$. (e.g. Table 8, mesh 2 with $w_1^{ZZ}=2/3$). However, the combination of $p=2$, $q=3$ often led to worse recovered stress fields (e.g. Table 9, mesh 2 with, $w_1^{ZZ}=2/3$).

(2) Effect of the number of nodes inside the recovery patch in the SLS fitting, n_i^{ZZ}

As pointed out in Section 4.1.1, n_i^{ZZ} should be kept to a minimum but large enough to ensure the stability of the recovery procedure. The results obtained indicate that the optimal number of nodes needed dependent on both p and q .

For Problem 1, when $p=1$, the minimum required number (i.e. $n_i^{ZZ}=3$ for $q=1$ and $n_i^{ZZ}=6$ for $q=2$) is the best in almost all cases. However, when $p=2$, one and a half time of minimum required number (i.e. $n_j^{ZZ}=9$ for $q=2$ and $n_i^{ZZ}=15$ for $q=3$) seems to be the best choice.

For Problem 2, it is found that in most cases the minimum number of nodes led to the best results. In addition, the accuracy of the recovery stress dropped as n_i^{ZZ}

increased. However, there are exceptions for both $p=1$ and $p=2$ that the minimum number of nodes used does not lead to the best result. (e.g. Table 6, mesh 3, $q=2$, a value of $n_i^{ZZ}=6$ gives the worst result for all values of w_1^{ZZ} .)

For Problems 3 and 4, for both $p=1$ and $p=2$ and all values of q , the minimum value of n_i^{ZZ} lead to the most accurate recovered stress. (e.g. Table 10, mesh 1, $q=1$, $w_1^{ZZ}=3/4$, a value of $n_i^{ZZ}=3$ gives the best result.)

(3) *Effect of nodal integration weight, w_1^{ZZ} , used in the SLS fitting*

As described in Section 4.1.1, three different schemes (TB, uniform and distance) for assigning the values of w_1^{ZZ} are tested. The results obtained indicated that for Problems 2 and 4, in general, the larger the value of w_1^{ZZ} , the more accurate the recovery stress resulted. However, some cases exist that for some combinations of p , q and n_i^{ZZ} , the recovery stresses obtained are better when w_1^{ZZ} is reduced. (e.g. Table 8, mesh 2, when $q=2$ and $n_j^{ZZ}=6$, the results for $w_1^{ZZ}=1/n_i^{ZZ}=1/6$ are the best.) In general, by looking at all results obtained as a whole, the TB scheme ($w_1^{ZZ}=1$) outperform the other two schemes for these two problems.

For Problem 1, it is found that there often exist some cases in which a value of $w_1^{ZZ}<1$ leads to better results than the TB scheme. Such cases exist for both $p=1$ and $p=2$. For example, in Table 4, when $q=1$ and $n_i^{ZZ}=3$, the results for the uniform scheme are better than the TB scheme for all meshes. Another example can be found in Table 5, when $q=2$ and $n_i^{ZZ}=10$, the uniform scheme again outperformed the TB scheme. For Problem 3, it is found that the choice of $w_1^{ZZ}=0.5$ resulted in the best results in almost all cases while the TB scheme yielded results with slightly lower accuracy.

6. Conclusion

In this paper, a study on the theory and nature of different recovery schemes for the EFGM has been done. The essential features and relationships among these schemes are described and discussed. It is shown that many commonly used stress recovery schemes for the EFGM can be considered as a particular case of the weighed continuous MLS-SLS recovery scheme. Numerical experience has also been carried out to study the

performances of a number of stress recovery procedures based on different fitting parameters. The results obtained suggest that:

1. The TB scheme is the simplest to implement and requires a minimum amount of computational cost. It often leads to the most accurate recovery stress in all problems tested with good effectivity indices. However, in all the tested problems there exist some cases that, with an appropriate combination of different parameters, the discrete MLS-SLS scheme is able to outperform the TB scheme. Moreover, it is rather difficult to determine such a combination in advance. Furthermore, it is found that in all these cases, the differences between the best MLS-SLS scheme and the TB scheme are marginal. Therefore, in practice, it can be concluded that the TB scheme is preferable than the discrete MLS-SLS scheme.
2. The performances of all the discrete MLS-SLS recovery schemes (including the TB scheme) tested are largely affected by the smoothness of the exact solution. For smooth problems, the recovered stress fields are more accurate than the EFGM stress. The effectivity indices obtained are close to the ideal value of unity and asymptotic convergence is observed. For non-smooth problems, when a uniform refinement is carried out, no asymptotic convergence of the effectivity index is observed. In addition, the recovered stress fields sometimes may be even less accurate than the EFGM solutions but a slightly better convergence rate is still observed.

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Figure 1. A background cell structure for numerical integration in EFGM

Figure 2. A layer of elements along essential boundary

Figure 3. Stress recovery for TB scheme

Figure 4. MLS of $\sigma^h(x_i)$ in 1D case

Figure 5. Local SLS fitting over node x_i

Figure 6. Discrete MLS-SLS fitting in 1D

Figure 7. Relationship among different recovery schemes

Figure 8. An example for nodal numbering in the SLS fitting

Figure 9. Problem 1: A cantilever beam under end shear

Figure 10. Uniformly refined meshes for Problem 1

Figure 11. Problem 2: A plate with a central hole

Figure 12. Uniformly refined meshes for Problem 2

Figure 13. Problem 3: A region near a crack tip

Figure 14. Uniformly refined meshes for Problem 3

Figure 15. Problem 4: L-shaped domain

Figure 16. Uniformly refined meshes for Problem 4

Complete basis p used in EFGM approximation	Values of q	Number of terms in the SLS basis	Values of n_i^{ZZ}
linear, p=1	1	3	3 / 6 / 9
	2	6	6 / 9 / 12
quadratic, p=2	2	6	6 / 9 / 12
	3	10	10 / 15 / 20

Table 1. Values of q and n_i^{ZZ} used in the SLS fitting

Schemes	w_1^{ZZ}	Values of $w_j^{ZZ}, j=2, \dots, n_i^{ZZ}$
TB	1	0
Uniform	$1/n_i^{ZZ}$	$1/n_i^{ZZ}$
Distance	$w=1/2, 2/3, 3/4$	$\frac{1-w}{l_j \left(\frac{1}{l_2} + \dots + \frac{1}{l_i} + \frac{1}{l_{n_i^{ZZ}}} \right)}$

Table 2. Values of w_j^{ZZ} used in SLS fitting

Benchmark Problems	Linear EFGM analysis		Quadratic EFGM analysis	
	d_m	d_m^p	d_m	d_m^p
1	1.5	1.2	2.7	2.3
2	1.5	1.3	2.0	1.7
3	2.0	1.5	2.7	2.2
4	2.5	2.0	3.5	3.0

Table 3. Support size parameters used in numerical examples

Mesh	$\eta(\%)$	q	n_i^{ZZ}	$\eta^{p*}(\%)$					θ				
				w_1^{ZZ}					w_1^{ZZ}				
				$u^\#$	1/2	2/3	3/4	1^+	$u^\#$	1/2	2/3	3/4	1^+
1	21.872	1	3	18.550	18.586	18.632	18.642	18.698	1.045	1.046	1.047	1.047	1.048
			6	19.950	19.221	18.975	18.883	18.698	1.140	1.090	1.072	1.065	
			9	21.134	19.201	18.940	18.851	18.698	1.213	1.097	1.076	1.067	
		2	6	18.734	18.697	18.696	18.696	18.698	1.056	1.051	1.049	1.049	
			9	18.931	18.754	17.727	18.718	18.698	1.068	1.054	1.051	1.050	
			12	18.948	18.733	18.713	18.707	18.698	1.107	1.061	1.055	1.053	
2	10.015	1	3	7.267	7.283	7.302	7.313	7.348	1.042	1.041	1.041	1.041	1.041
			6	8.551	7.944	7.708	7.603	7.348	1.169	1.105	1.079	1.068	
			9	8.962	7.995	7.732	7.619	7.348	1.229	1.119	1.088	1.075	
		2	6	7.356	7.347	7.347	7.347	7.348	1.044	1.041	1.041	1.041	
			9	7.476	7.381	7.365	7.360	7.348	1.053	1.044	1.042	1.042	
			12	8.035	7.464	7.407	7.387	7.348	1.125	1.057	1.049	1.047	
3	6.351	1	3	4.040	4.049	4.059	4.064	4.080	1.034	1.033	1.033	1.033	1.034
			6	4.785	4.430	4.291	4.229	4.080	1.135	1.085	1.065	1.056	
			9	4.960	4.446	4.295	4.231	4.080	1.180	1.096	1.072	1.061	
		2	6	4.081	4.079	4.080	4.080	4.080	1.035	1.034	1.034	1.034	
			9	4.132	4.093	4.087	4.085	4.080	1.040	1.035	1.034	1.034	
			12	4.514	4.153	4.118	4.105	4.080	1.102	1.047	1.041	1.038	
4	4.625	1	3	2.648	2.653	2.659	2.661	2.671	1.028	1.028	1.028	1.028	1.028
			6	3.133	2.900	2.808	2.768	2.671	1.111	1.070	1.054	1.046	
			9	3.224	2.905	2.808	2.766	2.671	1.145	1.070	1.059	1.050	
		2	6	2.670	2.670	2.670	2.670	2.671	1.029	1.028	1.028	1.028	
			9	2.697	2.677	2.674	2.673	2.671	1.032	1.029	1.029	1.029	
			12	2.966	2.719	2.696	2.688	2.671	1.085	1.039	1.034	1.032	

Table 4. Results obtained for Problem 1, linear analysis

Legends for Tables 4 to 11.

+: TB scheme as a particular case of discrete MLS-SLS scheme when $w_1^{ZZ} = 1.0$.

#: Uniform scheme in which $w_j^{ZZ} = 1/n_i^{ZZ}$.

*: The accuracy of recovery stress field in the highlighted cell is the best one in that row.

q, n_i^{ZZ} , w_1^{ZZ} : the order of polynomial basis used, the number of nodes inside the fitting patch and the weights used in the SLS fitting.

η : The relative global error (%) in energy norm of the FE-EFGM solution (Eqn. 22).

η^p : The relative global error (%) in energy norm of the recovered stress field (Eqn. 39).

θ : The effectivity index of the error estimator.

Mesh	$\eta(\%)$	q	n_i^{ZZ}	$\eta^{p*}(\%)$					θ				
				w_1^{ZZ}					w_1^{ZZ}				
				$u^\#$	1/2	2/3	3/4	1^+	$u^\#$	1/2	2/3	3/4	1^+
1	15.646	2	6	9.196	9.056	9.037	9.030	9.017	0.872	0.867	0.866	0.866	0.865
			9	9.040	9.027	9.022	9.021	9.017	0.869	0.866	0.866	0.866	
			12	9.222	9.045	9.031	9.027	9.017	0.880	0.868	0.867	0.866	
		3	10	/	/	/	/	9.017	/	/	/	/	
			15	/	/	/	/	9.017	/	/	/	/	
			20	/	/	/	/	9.017	/	/	/	/	
2	5.200	2	6	1.414	1.359	1.351	1.349	1.343	0.991	0.992	0.992	0.992	0.993
			9	1.353	1.345	1.344	1.344	1.343	0.990	0.992	0.992	0.992	
			12	1.663	1.337	1.333	1.334	1.343	1.028	0.997	0.995	0.994	
		3	10	1.326	1.337	1.340	1.341	1.343	0.992	0.992	0.993	0.993	
			15	1.346	1.301	1.317	1.325	1.343	1.001	0.993	0.993	0.993	
			20	1.405	1.308	1.320	1.327	1.343	1.009	0.994	0.993	0.993	
3	2.841	2	6	0.556	0.554	0.542	0.542	0.541	1.003	1.005	1.005	1.005	1.005
			9	0.531	0.537	0.539	0.540	0.541	1.003	1.004	1.005	1.005	
			12	0.578	0.530	0.534	0.536	0.541	1.010	1.005	1.005	1.005	
		3	10	0.523	0.538	0.539	0.540	0.541	1.006	1.006	1.005	1.005	
			15	0.524	0.528	0.533	0.536	0.541	1.011	1.006	1.006	1.005	
			20	0.539	0.529	0.534	0.536	0.541	1.012	1.006	1.006	1.006	
4	1.853	2	6	0.316	0.312	0.312	0.312	0.312	1.007	1.008	1.008	1.008	1.009
			9	0.303	0.309	0.310	0.311	0.312	1.007	1.008	1.008	1.008	
			12	0.308	0.303	0.303	0.308	0.312	1.007	1.008	1.008	1.008	
		3	10	0.303	0.310	0.311	0.311	0.312	1.009	1.009	1.009	1.009	
			15	0.304	0.306	0.309	0.309	0.312	1.012	1.009	1.009	1.009	
			20	0.308	0.306	0.308	0.309	0.312	1.013	1.009	1.009	1.009	

Table 5. Results obtained for Problem 1, quadratic analysis

Mesh	$\eta(\%)$	q	n_i^{ZZ}	$\eta^{p*}(\%)$					θ				
				w_1^{ZZ}					w_1^{ZZ}				
				$u^\#$	1/2	2/3	3/4	1^+	$u^\#$	1/2	2/3	3/4	1^+
1	10.231	1	3	11.905	11.835	11.800	11.786	11.756	0.786	0.773	0.766	0.763	0.756
			6	13.867	12.439	12.133	12.014	11.756	1.042	0.856	0.812	0.795	
			9	16.204	12.713	12.288	12.122	11.756	1.314	0.894	0.833	0.810	
		2	6	11.991	11.796	11.775	11.769	11.756	0.799	0.764	0.760	0.759	
			9	12.177	11.693	11.712	11.723	11.756	0.883	0.765	0.759	0.758	
			12	14.176	11.864	11.800	11.782	11.756	1.140	0.796	0.776	0.769	
2	5.741	1	3	6.672	6.561	6.498	6.471	6.404	0.939	0.921	0.911	0.907	0.898
			6	7.577	6.818	6.663	6.591	6.404	1.157	0.982	0.946	0.932	
			9	8.127	6.914	6.718	6.631	6.404	1.298	1.009	0.961	0.943	
		2	6	6.407	6.400	6.402	6.402	6.404	0.908	0.890	0.899	0.899	
			9	6.471	6.408	6.405	6.404	6.404	0.939	0.905	0.902	0.901	
			12	6.736	6.454	6.430	6.421	6.404	0.996	0.913	0.906	0.903	
3	3.449	1	3	3.525	3.477	3.448	3.435	3.404	1.071	1.057	1.048	1.045	1.037
			6	3.731	3.519	3.473	3.452	3.404	1.143	1.073	1.057	1.051	
			9	3.898	3.549	3.489	3.462	3.404	1.197	1.083	1.063	1.055	
		2	6	3.405	3.404	3.404	3.404	3.404	1.039	1.038	1.037	1.037	
			9	3.363	3.389	3.395	3.398	3.404	1.040	1.037	1.037	1.037	
			12	3.376	3.394	3.398	3.400	3.404	1.052	1.039	1.038	1.038	

Table 6. Results obtained for Problem 2, linear analysis

Mesh	$\eta(\%)$	q	n_i^{ZZ}	$\eta^{p*}(\%)$					θ				
				w_1^{ZZ}					w_1^{ZZ}				
				$u^\#$	1/2	2/3	3/4	1^+	$u^\#$	1/2	2/3	3/4	1^+
1	12.024	2	6	13.619	13.487	13.472	13.468	13.459	1.088	1.085	1.086	1.087	1.088
			9	13.722	13.382	13.407	13.421	13.459	1.142	1.087	1.087	1.087	
			12	14.520	13.397	13.413	13.424	13.459	1.223	1.091	1.088	1.088	
		3	10	13.459	13.459	13.456	13.457	13.459	1.086	1.086	1.087	1.087	
			15	13.470	13.468	13.466	13.463	13.459	1.089	1.088	1.088	1.088	
			20	14.010	13.499	13.497	13.483	13.459	1.095	1.093	1.092	1.091	
2	4.781	2	6	5.566	5.433	5.415	5.409	5.396	1.117	1.084	1.079	1.077	1.074
			9	5.552	5.428	5.413	5.407	5.396	1.120	1.084	1.079	1.077	
			12	5.875	5.489	5.447	5.431	5.396	1.199	1.099	1.087	1.083	
		3	10	5.575	5.523	5.512	5.483	5.396	1.184	1.168	1.105	1.091	
			15	5.433	5.420	5.418	5.411	5.396	1.084	1.081	1.080	1.078	
			20	5.484	5.470	5.466	5.443	5.396	1.134	1.100	1.091	1.086	
3	2.213	2	6	1.694	1.685	1.685	1.687	1.684	0.909	0.903	0.902	0.902	1.008
			9	1.649	1.668	1.675	1.678	1.684	0.941	0.908	0.905	0.904	
			12	1.769	1.682	1.682	1.683	1.684	1.014	0.918	0.910	0.907	
		3	10	1.691	1.770	1.683	1.683	1.684	0.920	0.909	0.906	0.907	
			15	1.743	1.700	1.694	1.687	1.684	1.015	1.014	1.010	1.009	
			20	1.932	1.847	1.722	1.686	1.684	1.017	1.015	1.014	1.008	

Table 7. Results obtained for Problem 2, quadratic analysis

Mesh	$\eta(\%)$	q	n_i^{ZZ}	$\eta^{p*}(\%)$					θ				
				w_1^{ZZ}					w_1^{ZZ}				
				$u^\#$	1/2	2/3	3/4	1^+	$u^\#$	1/2	2/3	3/4	1^+
1	24.447	1	3	28.842	28.838	28.839	28.840	28.846	1.062	1.060	1.060	1.059	1.059
			6	34.055	30.501	29.647	29.355	28.846	1.572	1.286	1.191	1.150	
			9	38.201	31.385	30.080	29.623	28.846	1.829	1.350	1.229	1.197	
		2	6	27.794	28.035	28.365	28.507	28.846	1.147	1.065	1.060	1.059	
			9	30.354	28.682	28.704	28.736	28.846	1.256	1.093	1.075	1.069	
			12	30.909	28.645	28.677	28.714	28.846	1.341	1.107	1.082	1.074	
2	17.241	1	3	20.374	20.374	20.374	20.374	20.374	1.029	1.029	1.029	1.029	1.029
			6	24.112	21.580	20.961	20.749	20.374	1.480	1.226	1.143	1.108	
			9	26.997	22.210	21.275	20.945	20.374	1.680	1.277	1.173	1.129	
		2	6	19.640	19.808	20.038	20.137	20.374	1.105	1.032	1.028	1.028	
			9	21.441	20.261	20.275	20.297	20.374	1.208	1.058	1.042	1.037	
			12	21.861	20.241	20.259	20.284	20.374	1.290	1.071	1.048	1.042	
3	12.187	1	3	14.406	14.406	14.406	14.406	14.406	1.012	1.012	1.012	1.012	1.027
			6	17.052	15.261	14.823	14.672	14.406	1.434	1.196	1.117	1.085	
			9	19.091	15.707	15.045	14.811	14.406	1.615	1.240	1.145	1.104	
		2	6	13.888	14.007	14.169	14.238	14.406	1.084	1.039	1.011	1.011	
			9	15.160	14.326	14.336	14.352	14.406	1.185	1.039	1.025	1.020	
			12	15.458	14.312	14.325	14.342	14.406	1.263	1.051	1.031	1.024	

Table 8. Results obtained for Problem 3, linear analysis

Mesh	$\eta(\%)$	q	n_i^{ZZ}	$\eta^{p*}(\%)$					θ				
				w_1^{ZZ}					w_1^{ZZ}				
				$u^\#$	1/2	2/3	3/4	1^+	$u^\#$	1/2	2/3	3/4	1^+
1	24.010	2	6	27.371	26.477	27.092	27.357	27.962	0.950	0.864	0.858	0.858	0.864
			9	29.194	27.291	27.529	27.650	27.962	1.147	0.900	0.878	0.872	
			12	30.369	27.159	27.428	27.576	27.962	1.277	0.910	0.882	0.874	
		3	10	30.343	27.649	27.758	27.811	27.962	1.187	0.894	0.877	0.872	
			15	31.258	27.429	27.619	27.714	27.962	1.313	0.903	0.880	0.873	
			20	33.262	27.324	27.524	27.641	27.962	1.467	0.913	0.883	0.875	
2	16.872	2	6	19.280	18.599	19.012	19.190	19.596	0.968	0.889	0.886	0.888	0.895
			9	20.487	19.155	19.309	19.388	19.596	1.144	0.923	0.905	0.901	
			12	21.346	19.068	19.240	19.338	19.596	1.265	0.931	0.908	0.902	
		3	10	21.228	19.371	19.447	19.486	19.596	1.191	0.921	0.906	0.901	
			15	21.845	19.228	19.356	19.422	19.596	1.298	0.928	0.908	0.902	
			20	23.497	19.187	19.306	19.382	19.596	1.426	0.935	0.910	0.903	
3	11.928	2	6	13.685	13.154	13.447	13.573	13.861	0.975	0.888	0.886	0.888	0.896
			9	14.489	13.548	13.657	13.713	13.861	1.138	0.921	0.905	0.901	
			12	15.095	13.485	13.608	13.677	13.861	1.254	0.929	0.907	0.902	
		3	10	15.010	13.700	13.754	13.782	13.861	1.184	0.920	0.906	0.902	
			15	15.445	13.598	13.690	13.737	13.861	1.286	0.926	0.907	0.902	
			20	16.614	13.568	13.654	13.708	13.861	1.409	0.932	0.909	0.903	

Table 9. Results obtained for Problem 3, quadratic analysis

Mesh	$\eta(\%)$	q	n_i^{ZZ}	$\eta^{p*}(\%)$					θ				
				w_1^{ZZ}					w_1^{ZZ}				
				$u^\#$	1/2	2/3	3/4	1^+	$u^\#$	1/2	2/3	3/4	1^+
1	22.815	1	3	24.194	23.654	23.412	23.320	23.129	0.854	0.823	0.809	0.804	0.794
			6	32.071	27.356	25.592	24.847	23.129	1.436	1.111	0.984	0.928	
			9	34.580	27.676	25.760	24.955	23.129	1.575	1.131	0.974	0.935	
		2	6	23.352	23.109	23.110	23.114	23.129	0.818	0.795	0.794	0.794	
			9	27.265	23.956	23.547	23.407	23.129	1.077	0.850	0.822	0.812	
			12	29.718	24.109	23.723	23.524	23.129	1.236	0.795	0.839	0.824	
2	16.823	1	3	19.876	18.467	17.752	17.444	16.707	0.870	0.713	0.759	0.745	0.717
			6	20.867	18.099	17.268	17.008	16.707	1.140	0.881	0.794	0.762	
			9	24.138	18.762	17.573	17.188	16.707	1.341	0.918	0.809	0.769	
		2	6	16.712	16.671	16.684	16.690	16.707	0.726	0.717	0.717	0.717	
			9	19.238	17.187	16.949	16.867	16.707	0.916	0.745	0.730	0.725	
			12	20.210	17.671	16.815	16.771	16.707	1.037	0.717	0.730	0.724	
3	11.715	1	3	11.502	11.492	11.487	11.485	11.481	0.725	0.723	0.723	0.723	0.722
			6	14.367	12.407	11.838	11.657	11.481	1.129	0.879	0.794	0.764	
			9	16.523	12.807	12.010	11.756	11.481	1.316	0.911	0.806	0.769	
		2	6	11.595	11.476	11.477	11.478	11.481	0.738	0.722	0.722	0.722	
			9	13.183	11.806	11.641	11.586	11.481	0.911	0.747	0.733	0.729	
			12	14.040	12.476	11.599	11.549	11.481	1.016	0.722	0.732	0.728	

Table 10. Results obtained for Problem 4, linear analysis

Mesh	$\eta(\%)$	q	n_i^{ZZ}	$\eta^{p*}(\%)$					θ				
				w_1^{ZZ}					w_1^{ZZ}				
				$u^\#$	1/2	2/3	3/4	1^+	$u^\#$	1/2	2/3	3/4	1^+
1	19.904	2	6	24.979	24.613	24.587	24.581	24.573	0.963	0.927	0.923	0.922	0.920
			9	27.642	25.373	25.000	24.864	24.573	1.138	0.972	0.948	0.939	
			12	29.152	25.524	25.072	24.909	24.573	1.225	0.980	0.955	0.944	
		3	10	27.497	25.240	24.687	24.672	24.573	1.074	0.974	0.945	0.938	
			15	28.669	25.241	24.906	24.791	24.573	1.223	0.975	0.948	0.939	
			20	20.196	25.330	24.953	24.810	24.573	1.302	0.978	0.950	0.939	
2	13.973	2	6	16.189	15.939	15.952	15.962	15.992	0.901	0.862	0.860	0.860	0.861
			9	18.004	16.334	16.158	16.100	15.992	1.126	0.909	0.884	0.876	
			12	19.396	16.206	16.039	16.005	15.992	1.268	0.917	0.885	0.876	
		3	10	19.041	16.112	16.009	16.001	15.992	1.134	0.908	0.868	0.864	
			15	19.732	16.149	16.007	15.982	15.992	1.290	0.910	0.882	0.873	
			20	20.550	16.200	16.032	15.997	15.992	1.359	0.913	0.884	0.874	
3	9.696	2	6	11.384	11.127	11.116	11.114	11.113	0.813	0.769	0.766	0.765	0.764
			9	12.441	11.272	11.178	11.152	11.113	1.044	0.805	0.782	0.776	
			12	13.492	11.214	11.105	11.090	11.113	1.189	0.813	0.782	0.774	
		3	10	12.998	11.114	12.825	12.332	11.113	1.232	0.800	1.025	0.936	
			15	13.657	11.085	11.025	11.034	11.113	1.232	0.800	0.774	0.769	
			20	14.228	11.115	11.045	11.048	11.113	1.301	0.802	0.776	0.770	

Table 11. Results obtained for Problem 4, quadratic analysis