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On Coupling of Reproducing Kernel Particle method and Boundary Element Method

*C. K. Lee, S. T. Lie and Y. Y. Shuai

*School of Civil and Environmental Engineering
Nanyang Technological University
Nanyang Avenue
Singapore 639798*

*email: ccklee@ntu.edu.sg

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Abstract

In this paper a new coupling method to merge the Reproducing Kernel Particle Method (RKPM) and the Boundary Element Method (BEM) is proposed. In this formulation, the whole problem domain will first be divided into two subdomains in which the RKPM and the BEM are applied respectively. A simple and direct coupling procedure is then applied to preserve the compatibility of the solution along the interface of the two subdomains. Unlike other coupling procedures suggested previously, the present coupling procedure neither requires modification of the RKPM and the BEM formulations nor the use of finite elements along the interface boundary between the two subdomains. The validity and efficiency of the coupling procedure are demonstrated by using it to solve five benchmark elastostatic stress analysis problems. In addition, a simple analysis of the *a priori* convergence rate of the coupling procedure is given. Preliminary assessment of the convergence characteristics of the coupling procedure is done by carrying out uniform refinements on the benchmark problems.

KEYWORDS: Meshless method; Reproducing Kernel Particle Method; Boundary Element Method; A priori error estimation.

1. Introduction

Recently, considerable research efforts have been devoted to the development of the meshless methods [1-13] which do not require a mesh to discretize the problem domain and the approximate functions can be constructed entirely in terms of a set of scattered nodes. Such characteristics of the meshless method make it an alternative of the finite element method (FEM) for the analysis of problems that involve model refinement, crack propagations and large deformation. Among the many meshless methods suggested, the *element free Galerkin method* (EFGM) is one of most the popular approaches used by many researchers. The EFGM was proposed by Belytschko et al. [3-5] to improve the accuracy of the diffuse element method suggested by Nayroles et al. [6]. Besides the EFGM, another class of important meshless method is the *reproducing kernel particle method* (RKPM) which was proposed by Liu et al. [7-11]. The RKPM was originally introduced to improve the accuracy of the *smoothed particle hydrodynamics method* (SPH) [1,2]. In the RKPM, the kernel function is modified by introducing a correction function to satisfy the reproducing conditions.

Even though it is accepted that the meshless methods could offer potential solutions for many problems in which traditional FEM deems to be inconvenient to use, there are still technical difficulties yet to be overcome. The two most important difficulties include:

- (1) The shape functions of most meshless methods lack the Kronecker delta function property. This makes it difficult to impose the essential boundary conditions of the problem. Towards this end, many different approaches had been suggested to alleviate this problem [12,14-16]. Note that many of the suggested methods require either modifications of the meshless formulation the use of finite elements.
- (2) When comparing with the traditional FEM or the boundary element method (BEM), meshless methods require higher computational cost for the construction of shape functions and also for the formation of global equations system. This makes the meshless method too expensive for problems requiring a dense distribution of points over a large domain area or for problems with infinite or semi-infinite domains. Toward this end, many strategies had been suggested to couple the meshless methods with either FEM or BEM [17-20] so that the meshless formulations can be restricted to regions in which their unique element free advantages are beneficial. Note that many of the suggested methods so far either require the introduction of a layer of

finite elements along the interface boundary of the meshless and the FEM/BEM domains or one must modify the formulations of the FEM/BEM to enforce the interface displacement compatibility condition.

The main objective of this paper is to suggest a new procedure for the direct coupling of the RKPM and the BEM. A unit displacement method is used here to enforce the displacement compatibility condition along the interface boundary of the RKPM and the BEM domains. In the presented method, it is neither necessary to introduce any interface elements nor to modify the formulations of the RKPM and the BEM. In the next section, brief descriptions of the RKPM and the BEM formulations used in this study are first given. It is then followed by a detailed description of the coupling technique. A simple analysis on the *a priori* convergence rate of the coupling procedure developed is presented in Section 4. In Section 5, five selected benchmark examples are employed to demonstrate the efficiency of the proposed coupled BE-RKPM. Uniform refinements are carried out to show the accuracy and convergent characteristics of the coupled BE-RKPM. Finally, conclusions and discussions on potential areas of future developments are given.

2. The modelled problem, the RKPM and the BEM formulations

2.1 The modelled problem

Consider the standard 2D linear elastostatic stress analysis problem in the domain Ω with boundary $\Gamma = \Gamma_u \cup \Gamma_t$ defined by

$$\nabla \boldsymbol{\sigma} + \mathbf{b} = 0 \quad (1a)$$

$$\text{and} \quad \mathbf{u} = \bar{\mathbf{u}}, \text{ on essential boundary } \Gamma_u \text{ and } \boldsymbol{\sigma} \mathbf{n} = \bar{\mathbf{t}} \text{ on natural boundary } \Gamma_t \quad (1b)$$

In Eqn. 1, $\boldsymbol{\sigma}$ and $\mathbf{b}$ are the stress tensor and the body force vector respectively. $\bar{\mathbf{u}}$ and $\bar{\mathbf{t}}$ are, respectively, the prescribed displacements and tractions along Γ_u and Γ_t . $\mathbf{n}$ is the unit outward normal along Γ and $\Gamma_u \cap \Gamma_t = \emptyset$. The relationship between $\mathbf{u}$ and $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} = \mathbf{D} \mathbf{L} \mathbf{u} \quad (2)$$

In Eqn. 2, $\mathbf{D}$ is the material matrix and the operator $\mathbf{L}$ is defined as

$$\mathbf{L}^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \quad (3)$$

2.2 The RKPM and the BEM formulations used

In this section, concise summaries of the RKPM and the BEM formulations used are given in order to make clear the description of the coupling procedure. Details of the formulations can be found in references [16] and [21].

2.2.1 The RKPM formulation

The key idea of the RKPM is to construct an approximation $\mathbf{u}^R$ to $\mathbf{u}$ by employing a *corrected kernel*. In 2D problems, a corrected kernel can be written as

$$\mathbf{u}^R = \int_{\Omega} \bar{w}_d(\mathbf{x} - \mathbf{s}, \mathbf{y} - \mathbf{t}) \mathbf{u}(\mathbf{s}, \mathbf{t}) d\Omega = \int_{\Omega} C(\mathbf{x}, \mathbf{y}, \mathbf{s}, \mathbf{t}) w_d(\mathbf{x} - \mathbf{s}, \mathbf{y} - \mathbf{t}) \mathbf{u}(\mathbf{s}, \mathbf{t}) d\Omega \quad (4)$$

where $w_d(\mathbf{x} - \mathbf{s}, \mathbf{y} - \mathbf{t})$ is the *kernel function*, $C(\mathbf{x}, \mathbf{y}, \mathbf{s}, \mathbf{t})$ is the *correction function* and $\bar{w}_d(\mathbf{x} - \mathbf{s}, \mathbf{y} - \mathbf{t})$ is the *corrected kernel function*. The kernel function is required to satisfy five conditions: non-negativity, local support property, normality, monotonicity and approaching to Dirac delta function.

In practical computation, the corresponding discrete form of the continuous approximation function (Eqn. 4) can be written as [8,16]

$$\mathbf{u}^R = \sum_{i=1}^{NP} \bar{w}_d(\mathbf{x} - \mathbf{x}_i, \mathbf{y} - \mathbf{y}_i) \mathbf{u}(\mathbf{x}_i, \mathbf{y}_i) \Delta V_i \quad (5)$$

where $(\mathbf{x}_i, \mathbf{y}_i)$ are the coordinates of node i . NP is the number of nodes in the domain. $\mathbf{u}(\mathbf{x}_i, \mathbf{y}_i)$ and ΔV_i are, respectively, the nodal value and the domain of influence of node i . In matrix form the approximation of displacement function can be written as

$$\mathbf{u}^R(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{NP} N_i^R(\mathbf{x}, \mathbf{y}) \hat{\mathbf{u}}_i^R = (\mathbf{N}^R)^T \hat{\mathbf{u}}^R \quad (6)$$

In Eqn. 6, $\mathbf{N}^R$ is the matrix of the RKPM shape functions and $\hat{\mathbf{u}}^R$ is the vector of nodal displacement values. N_i^R and $\hat{\mathbf{u}}_i^R$ are the shape function and the nodal value of node i respectively. N_i^R can be expressed as

$$N_i^R(\mathbf{x}, \mathbf{y}) = C(\mathbf{x}, \mathbf{y}, \mathbf{x}_i, \mathbf{y}_i) w_d(\mathbf{x} - \mathbf{x}_i, \mathbf{y} - \mathbf{y}_i) \Delta V_i \quad (7)$$

In the present RKPM formulation, the essential boundary condition (Eqn. 1b) is enforced by employing the penalty multiplier method and the amended variational principles [16]. The final form of the functional to be minimized can be expressed as

$$\Pi = \frac{1}{2} \int_{\Omega} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\Omega - \int_{\Omega} \mathbf{u}^T \mathbf{b} d\Omega - \int_{\Gamma_t} \mathbf{u}^T \bar{\mathbf{t}} d\Gamma + \frac{\alpha}{2} \int_{\Gamma_u} (\mathbf{u} - \bar{\mathbf{u}})^T (\mathbf{u} - \bar{\mathbf{u}}) d\Gamma \quad (8)$$

In Eqn. 8, the factor $\alpha \gg 1$ is a penalty parameter used to enforce the essential boundary condition [16]. By using Eqn. 6, the global equations system can be obtained as

$$\mathbf{K}^R \hat{\mathbf{u}}^R = \mathbf{f}^R \quad (9)$$

such that
$$\mathbf{K}_{ij}^R = \int_{\Omega} (\mathbf{B}_i^R)^T \mathbf{D} \mathbf{B}_j^R d\Omega + \alpha \int_{\Gamma_u} \mathbf{N}_i^R \mathbf{S} \mathbf{N}_j^R d\Gamma \quad (10a)$$

$$\mathbf{f}_i^R = \int_{\Omega} \mathbf{N}_i^R \mathbf{b} d\Omega + \int_{\Gamma_t} \mathbf{N}_i^R \bar{\mathbf{t}} d\Gamma + \alpha \int_{\Gamma_u} \mathbf{N}_i^R \mathbf{S} \bar{\mathbf{u}} d\Gamma \quad (10b)$$

$$\mathbf{B}_i^R = \begin{bmatrix} \mathbf{N}_{i,x}^R & 0 \\ 0 & \mathbf{N}_{i,y}^R \\ \mathbf{N}_{i,y}^R & \mathbf{N}_{i,x}^R \end{bmatrix}, \mathbf{S} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \text{ and } S_i = \begin{cases} 1 & \text{if } \hat{\mathbf{u}}_i^R \text{ is prescribed on } \Gamma_u \\ 0 & \text{if } \hat{\mathbf{u}}_i^R \text{ is not prescribed on } \Gamma_u \end{cases} \quad (11)$$

2.2.2 The BEM formulation

For 2D problems with governing equation and boundary equations defined by Eqn. 1, the principle of virtual displacements can be written as

$$\int_{\Omega} (\nabla \boldsymbol{\sigma} + \mathbf{b}) \mathbf{u}^* d\Omega = \int_{\Gamma_u} (\mathbf{u} - \bar{\mathbf{u}}) \mathbf{t}^* d\Gamma - \int_{\Gamma_t} (\mathbf{t} - \bar{\mathbf{t}}) \mathbf{u}^* d\Gamma \quad (12)$$

In Eqn. 12, $\mathbf{u}^*$ and $\mathbf{t}^*$ are the virtual displacement and the virtual surface tractions respectively. By using integrated by parts on the first term of Eqn. 12, one may obtain

$$\int_{\Omega} \mathbf{b} \mathbf{u}^* d\Omega + \int_{\Omega} (\nabla \boldsymbol{\sigma}^*) \mathbf{u} d\Omega = \int_{\Gamma_u} (\mathbf{u}^* \mathbf{t} - \bar{\mathbf{u}} \mathbf{t}^*) d\Gamma - \int_{\Gamma_t} (\mathbf{u} \mathbf{t}^* - \mathbf{u}^* \bar{\mathbf{t}}) d\Gamma \quad (13)$$

By introducing the fundamental solution of the form

$$\nabla \boldsymbol{\sigma}^* + \boldsymbol{\delta}^i = 0 \quad (14)$$

where, $\boldsymbol{\delta}^i$ is the Dirac delta function, the domain integral equation can be reduced to the boundary integral equation (BIE) of the form [21]

$$\mathbf{c}^i \mathbf{u}^i + \int_{\Gamma} \mathbf{u} \mathbf{t}^* d\Gamma = \int_{\Gamma} \mathbf{t} \mathbf{u}^* d\Gamma + \int_{\Omega} \mathbf{b} \mathbf{u}^* d\Omega \quad (15)$$

Furthermore, it is assumed that the $\mathbf{u}$ and $\mathbf{t}$ are interpolated along the boundaries as

$$\mathbf{u} \approx \mathbf{u}^B = (\mathbf{N}_u^B)^T \hat{\mathbf{u}}^B \text{ and } \mathbf{t} \approx \mathbf{t}^B = (\mathbf{N}_t^B)^T \hat{\mathbf{t}}^B \quad (16)$$

In Eqn. 16, $\mathbf{N}_u^B$ and $\mathbf{N}_t^B$ are the interpolation functions of $\mathbf{u}$ and $\mathbf{t}$ along Γ_t and Γ_u respectively. $\hat{\mathbf{u}}^B$ and $\hat{\mathbf{t}}^B$ are respectively the vectors of nodal values of $\mathbf{u}$ and $\mathbf{t}$ along Γ_t and Γ_u . Substituting the interpolation functions into Eqn. 15, the final boundary integration equation will become

$$\mathbf{H} \hat{\mathbf{u}}^B = \mathbf{G} \hat{\mathbf{t}}^B + \mathbf{Q} \quad (17)$$

$$\text{where } \mathbf{H}_i = \mathbf{c}^i + \int_{\Gamma} (\mathbf{N}_u^B)^T \mathbf{t}^* d\Gamma, \quad \mathbf{G}_i = \int_{\Gamma} (\mathbf{N}_t^B)^T \mathbf{u}^* d\Gamma, \quad \mathbf{Q}_i = \int_{\Omega} \mathbf{b} \mathbf{u}^* d\Omega \quad (18)$$

3 Formulation of the coupled BE-RKPM

3.1 Partition of problem domain

In the coupled formulation, the problem domain Ω is partitioned into two disjoint subdomains Ω^R and Ω^B such that $\Omega = \Omega^R \cup \Omega^B$ (Fig. 1). The RKPM and the BEM are employed in Ω^R and Ω^B respectively. The boundaries of Ω^R and Ω^B are denoted as Γ^R and Γ^B respectively. It is assumed that *both* essential and natural boundary conditions are present in Ω^R and Ω^B . The boundary conditions are given by

For Ω^R , $\Gamma^R = \Gamma_u^R \cup \Gamma_t^R \cup \Gamma^{RB}$:

$$\text{Essential boundary} \quad \mathbf{u} = \bar{\mathbf{u}}^R \text{ on } \Gamma_u^R \quad (19a)$$

$$\text{Natural boundary} \quad \mathbf{t} = \bar{\mathbf{t}}^R \text{ on } \Gamma_t^R \quad (19b)$$

For Ω^B , $\Gamma^B = \Gamma_u^B \cup \Gamma_t^B \cup \Gamma^{BR}$:

$$\text{Essential boundary} \quad \mathbf{u} = \bar{\mathbf{u}}^B \text{ on } \Gamma_u^B \quad (20a)$$

$$\text{Natural boundary} \quad \mathbf{t} = \bar{\mathbf{t}}^B \text{ on } \Gamma_t^B \quad (20b)$$

Since the outward norm vectors of the boundaries Γ^{RB} and Γ^{BR} are opposite to each another, along the interface boundary

$$\Gamma^{RB} = -\Gamma^{BR} \text{ and } \mathbf{t}^{RB} = -\mathbf{t}^{BR} \text{ and } \mathbf{u}^{RB} = \mathbf{u}^{BR} \quad (20c)$$

In Eqn. 20c, $\mathbf{t}^{RB}$ ($\mathbf{u}^{RB}$) and $\mathbf{t}^{BR}$ ($\mathbf{u}^{BR}$) are the tractions (displacements) along the interface boundary with respect to Ω^R and Ω^B respectively.

3.2 Solution by the RKPM in Ω^R

For the solution by the RKPM in Ω^R , it is assumed that the interface boundary between Ω^R and Ω^B , Γ^{RB} , is an essential boundary such that

$$\mathbf{u} = \mathbf{u}^{RB} \text{ on } \Gamma^{RB} \quad (21)$$

It will be shown in Section 3.6 that the use of Eqn. 21 is equivalent to enforce the displacement compatibility condition along the interface boundary in a weak sense and will greatly simplify the formulation of the coupling procedure.

Following the formulation in Section 2.2.1, the approximation function $\mathbf{u}^R$ (c.f. Eqn. 6) inside Ω^R can be introduced as

$$\mathbf{u}^R(x, y) = \sum_{i=1}^{NP} N_i^R(x, y) \hat{\mathbf{u}}_i^R = (\mathbf{N}^R)^T \hat{\mathbf{u}}^R \text{ for } (x, y) \in \Omega^R \quad (22a)$$

By using Eqn. 22a, the displacements along Γ^{RB} , $\mathbf{u}^{RB}$, can be expressed in the general form as

$$\mathbf{u}^{RB} \approx (\mathbf{N}^{RB})^T \hat{\mathbf{u}}^{RB} + (\mathbf{N}^C)^T \hat{\mathbf{u}}^C \quad (22b)$$

In Eqn. 22b, $\mathbf{N}^{RB}$ and $\hat{\mathbf{u}}^{RB}$ are, respectively, the RKPM shape functions and the nodal parameters along the interface boundary. $\mathbf{N}^C$ and $\hat{\mathbf{u}}^C$ are, respectively, sets of RKPM shape functions and nodal values associate with some nodes near the interface boundary. Note that the second term in Eqn. 22b exists because the RPKM shape functions lack the Kronecker delta function property.

The linear equations system for Ω^R can be obtained as (c.f. Eqn. 9)

$$\mathbf{K}^R \hat{\mathbf{u}}^R = \mathbf{f}^R + \mathbf{f}^{RB} \quad (23)$$

where $\mathbf{K}_{ij}^R = \int_{\Omega^R} \mathbf{B}_i^T \mathbf{D} \mathbf{B}_j d\Omega + \alpha \int_{\Gamma_u^R + \Gamma^{RB}} N_i^R \mathbf{S} N_j^R d\Gamma$ (c.f. Eqn. 10a) (24a)

$$\mathbf{f}_i^R = \int_{\Omega^R} N_i^R \mathbf{b} d\Omega + \int_{\Gamma_t^R} N_i^R \bar{\mathbf{t}}^R d\Gamma + \alpha \int_{\Gamma_u^R} N_i^R \mathbf{S} \bar{\mathbf{u}}^R d\Gamma$$
 (c.f. Eqn. 10b) (24b)

and $\mathbf{f}_i^{RB} = \alpha \int_{\Gamma^{RB}} N_i^R \mathbf{S} (\mathbf{N}^{RB})^T \hat{\mathbf{u}}^{RB} d\Gamma$ (24c)

Note that Eqn. 23 cannot be solved alone since the vector $\hat{\mathbf{u}}^{RB}$ is not known.

3.3 Solution by the BEM in Ω^B

For Ω^B , the following boundary conditions are used

$$\mathbf{u} = \bar{\mathbf{u}}^B \text{ on } \Gamma_u^B \text{ and } \mathbf{t} = \bar{\mathbf{t}}^B \text{ on } \Gamma_t^B \quad (25a)$$

$$\mathbf{u} = \mathbf{u}^{\text{RB}} \quad \text{and} \quad \mathbf{t} = -\mathbf{t}^{\text{RB}} \quad \text{on} \quad \Gamma^{\text{BR}} \quad (25\text{b})$$

By using the BIE formulation described in Section 2.2.2, the following BIE can be obtained for Ω^{B} (c.f. Eqn. 15)

$$\mathbf{c}^{\text{i}} \mathbf{u}^{\text{i}} + \int_{\Gamma^{\text{B}} + \Gamma^{\text{BR}}} \mathbf{u} \mathbf{t}^* d\Gamma = \int_{\Gamma^{\text{B}} + \Gamma^{\text{BR}}} \mathbf{t} \mathbf{u}^* d\Gamma + \int_{\Omega^{\text{B}}} \mathbf{b} \mathbf{u}^* d\Omega \quad (26)$$

If the unknown displacement on the tractions boundary $\Gamma_{\mathbf{t}}^{\text{B}}$ is denoted as $\mathbf{u}_{\Gamma_{\mathbf{t}}}^{\text{B}}$ and the unknown tractions on the boundary $\Gamma_{\mathbf{u}}^{\text{B}}$ is denoted as $\mathbf{t}_{\Gamma_{\mathbf{u}}}^{\text{B}}$, then the following equation can be arrived by substituting the boundary conditions in Eqn. 25 into Eqn. 26.

$$\begin{aligned} & \mathbf{c}^{\text{i}} \mathbf{u}^{\text{i}} + \int_{\Gamma_{\mathbf{u}}^{\text{B}}} \bar{\mathbf{u}}^{\text{B}} \mathbf{t}^* d\Gamma + \int_{\Gamma_{\mathbf{t}}^{\text{B}}} \mathbf{u}_{\Gamma_{\mathbf{t}}}^{\text{B}} \mathbf{t}^* d\Gamma + \int_{\Gamma^{\text{BR}}} \mathbf{u}^{\text{RB}} \mathbf{t}^* d\Gamma \\ &= \int_{\Gamma_{\mathbf{u}}^{\text{B}}} \mathbf{t}_{\Gamma_{\mathbf{u}}}^{\text{B}} \mathbf{u}^* d\Gamma + \int_{\Gamma_{\mathbf{t}}^{\text{B}}} \bar{\mathbf{t}}^{\text{B}} \mathbf{u}^* d\Gamma + \int_{\Gamma^{\text{BR}}} (-\mathbf{t}^{\text{RB}}) \mathbf{u}^* d\Gamma + \int_{\Omega^{\text{B}}} \mathbf{b} \mathbf{u}^* d\Gamma \end{aligned} \quad (27)$$

Furthermore, it is assumed that *inside* Ω^{B} , the known terms $\bar{\mathbf{u}}^{\text{B}}$ and $\bar{\mathbf{t}}^{\text{B}}$ and the unknown terms $\mathbf{u}_{\Gamma_{\mathbf{t}}}^{\text{B}}$, $\mathbf{t}_{\Gamma_{\mathbf{u}}}^{\text{B}}$, $\mathbf{t}^{\text{RB}}$ and $\mathbf{u}^{\text{RB}}$ are interpolated in the following manner (c.f. Eqn. 16)

$$\bar{\mathbf{u}}^{\text{B}} = (\mathbf{N}_{\mathbf{u}}^{\text{B}})^{\text{T}} \hat{\mathbf{u}}^{\text{B}}, \quad \bar{\mathbf{t}}^{\text{B}} = (\mathbf{N}_{\mathbf{t}}^{\text{B}})^{\text{T}} \hat{\mathbf{t}}^{\text{B}} \quad (28\text{a})$$

$$\mathbf{u}_{\Gamma_{\mathbf{t}}}^{\text{B}} = (\mathbf{N}_{\mathbf{u}}^{\text{B}})^{\text{T}} \hat{\mathbf{u}}_{\Gamma_{\mathbf{t}}}^{\text{B}}, \quad \mathbf{t}_{\Gamma_{\mathbf{u}}}^{\text{B}} = (\mathbf{N}_{\mathbf{t}}^{\text{B}})^{\text{T}} \hat{\mathbf{t}}_{\Gamma_{\mathbf{u}}}^{\text{B}} \quad (28\text{b})$$

$$\mathbf{t}^{\text{RB}} = (\mathbf{N}_{\mathbf{t}}^{\text{B}})^{\text{T}} \hat{\mathbf{t}}^{\text{RB}}, \quad \mathbf{u}^{\text{RB}} = (\mathbf{N}_{\mathbf{u}}^{\text{B}})^{\text{T}} \hat{\mathbf{u}}^{\text{RB}} \quad (28\text{c})$$

In Eqns. 28, $\hat{\mathbf{u}}^{\text{B}}$, $\hat{\mathbf{t}}^{\text{B}}$, $\hat{\mathbf{u}}_{\Gamma_{\mathbf{t}}}^{\text{B}}$ and $\hat{\mathbf{t}}_{\Gamma_{\mathbf{u}}}^{\text{B}}$ are the vectors of boundary nodal values corresponding to the terms $\bar{\mathbf{u}}^{\text{B}}$, $\bar{\mathbf{t}}^{\text{B}}$, $\mathbf{u}_{\Gamma_{\mathbf{t}}}^{\text{B}}$ and $\mathbf{t}_{\Gamma_{\mathbf{u}}}^{\text{B}}$ respectively. $\hat{\mathbf{t}}^{\text{RB}}$ and $\hat{\mathbf{u}}^{\text{RB}}$ are respectively the nodal values of $\mathbf{t}^{\text{RB}}$ and $\mathbf{u}^{\text{RB}}$ of the interface boundary nodes. By using Eqns. 17 and 18, the linear equations system for Ω^{R} can be obtained as

$$\begin{bmatrix} \mathbf{H}^{\text{RB}} & \mathbf{H}_{\Gamma_{\mathbf{u}}}^{\text{B}} & \mathbf{H}_{\Gamma_{\mathbf{t}}}^{\text{B}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}^{\text{RB}} \\ \hat{\mathbf{u}}_{\Gamma_{\mathbf{u}}}^{\text{B}} \\ \hat{\mathbf{u}}_{\Gamma_{\mathbf{t}}}^{\text{B}} \end{bmatrix} = \begin{bmatrix} -\mathbf{G}^{\text{RB}} & \mathbf{G}_{\Gamma_{\mathbf{u}}}^{\text{B}} & \mathbf{G}_{\Gamma_{\mathbf{t}}}^{\text{B}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{t}}^{\text{RB}} \\ \hat{\mathbf{t}}_{\Gamma_{\mathbf{u}}}^{\text{B}} \\ \hat{\mathbf{t}}_{\Gamma_{\mathbf{t}}}^{\text{B}} \end{bmatrix} + \mathbf{Q} \quad (29\text{a})$$

where

$$\mathbf{H}^{\text{RB}} = \mathbf{c}^{\text{i}} + \int_{\Gamma^{\text{RB}}} (\mathbf{N}_{\mathbf{u}}^{\text{B}})^{\text{T}} \mathbf{t}^* d\Gamma, \quad \mathbf{H}_{\Gamma_{\mathbf{u}}}^{\text{B}} = \mathbf{c}^{\text{i}} + \int_{\Gamma_{\mathbf{u}}^{\text{B}}} (\mathbf{N}_{\mathbf{u}}^{\text{B}})^{\text{T}} \mathbf{t}^* d\Gamma, \quad \mathbf{H}_{\Gamma_{\mathbf{t}}}^{\text{B}} = \mathbf{c}^{\text{i}} + \int_{\Gamma_{\mathbf{t}}^{\text{B}}} (\mathbf{N}_{\mathbf{u}}^{\text{B}})^{\text{T}} \mathbf{t}^* d\Gamma \quad (29\text{b})$$

$$\mathbf{G}^{\text{RB}} = \int_{\Gamma^{\text{RB}}} (\mathbf{N}_t^{\text{B}})^T \mathbf{u}^* d\Gamma, \quad \mathbf{G}_{\Gamma_u}^{\text{B}} = \int_{\Gamma_u^{\text{B}}} (\mathbf{N}_t^{\text{B}})^T \mathbf{u}^* d\Gamma, \quad \mathbf{G}_{\Gamma_t}^{\text{B}} = \int_{\Gamma_t^{\text{B}}} (\mathbf{N}_t^{\text{B}})^T \mathbf{u}^* d\Gamma, \quad \mathbf{Q} = \int_{\Omega^{\text{B}}} \mathbf{b} \mathbf{u}^* d\Omega \quad (29c)$$

3.4 Relationship between the vectors $\hat{\mathbf{t}}^{\text{RB}}$ and $\hat{\mathbf{u}}^{\text{RB}}$

Since in the BEM formulation both $\hat{\mathbf{t}}^{\text{RB}}$ and $\hat{\mathbf{u}}^{\text{RB}}$ on Γ^{RB} are unknowns, Eqn. 27 could not be solved independently. In fact, if one would like to solve Eqn. 27, it is first required to obtain the relationship between $\mathbf{u}^{\text{RB}}$ and $\mathbf{t}^{\text{RB}}$. It means that one has to express the nodal traction vector $\hat{\mathbf{t}}^{\text{RB}}$ in terms of the nodal displacement vector $\hat{\mathbf{u}}^{\text{RB}}$ and this could be resolved from the RKPM formulation in Ω^{R} (Eqn. 22) by using the *unit displacement method* described below.

As shown in Fig. 2, assume that there are m points on Γ^{RB} . For the i th node along Γ^{RB} , the unit base vectors, $\mathbf{e}_{2i-1}$ and $\mathbf{e}_{2i}$, in the x and y directions respectively are defined as

$$\mathbf{e}_{2i-1} = (\underbrace{0, 0, \dots, 1}_{2i-1 \text{ terms}}, 0, \dots, 0)^T \text{ and } \mathbf{e}_{2i} = (\underbrace{0, 0, \dots, 1}_{2i \text{ terms}}, 0, \dots, 0)^T \quad (30a)$$

The nodal displacement vector $\hat{\mathbf{u}}^{\text{RB}}$ can then be written in component form as

$$\hat{\mathbf{u}}^{\text{RB}} = (u_1^{\text{RB}}, u_2^{\text{RB}}, \dots, u_{2m}^{\text{RB}})^T \text{ or } \hat{\mathbf{u}}^{\text{RB}} = [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{2m}] \begin{bmatrix} u_1^{\text{RB}} \\ u_2^{\text{RB}} \\ \vdots \\ u_{2m}^{\text{RB}} \end{bmatrix} \quad (30b)$$

In Eqn. 30b, the terms u_{2i-1}^{RB} and u_{2i}^{RB} are the x and y directions displacement components at node i respectively. Now reconsider Eqn. 23 again and decompose the displacement vector $\hat{\mathbf{u}}^{\text{R}}$ into two parts, $\hat{\mathbf{u}}_1^{\text{R}}$ and $\hat{\mathbf{u}}_2^{\text{R}}$ such that

$$\hat{\mathbf{u}}^{\text{R}} = \hat{\mathbf{u}}_1^{\text{R}} + \hat{\mathbf{u}}_2^{\text{R}} \quad \text{and} \quad \mathbf{K}^{\text{R}} \hat{\mathbf{u}}_1^{\text{R}} = \mathbf{f}^{\text{R}}, \quad \mathbf{K}^{\text{R}} \hat{\mathbf{u}}_2^{\text{R}} = \mathbf{f}^{\text{RB}} \quad (31)$$

In Eqn. 31, the first component $\hat{\mathbf{u}}_1^{\text{R}}$ is the nodal values induced by internal body forces and non-interface boundary conditions. It can be solved as $\hat{\mathbf{u}}_1^{\text{R}} = (\mathbf{K}^{\text{R}})^{-1} \mathbf{f}^{\text{R}}$ without the consideration of Ω^{B} . The second component $\hat{\mathbf{u}}_2^{\text{R}}$ is the nodal values induced by the interface boundary conditions. From Eqn. 24c, $\mathbf{f}^{\text{RB}}$ can be expressed as

$$\begin{aligned}
\mathbf{f}^{\text{RB}} &= \alpha \int_{\Gamma^{\text{RB}}} \mathbf{N}^{\text{R}} \mathbf{S} (\mathbf{N}^{\text{RB}})^{\text{T}} \hat{\mathbf{u}}^{\text{RB}} d\Gamma \\
&= \left(\alpha \int_{\Gamma^{\text{RB}}} \mathbf{N}^{\text{R}} \mathbf{S} (\mathbf{N}^{\text{RB}})^{\text{T}} [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{2m}] d\Gamma \right) \begin{bmatrix} \mathbf{u}_1^{\text{RB}} \\ \mathbf{u}_2^{\text{RB}} \\ \vdots \\ \mathbf{u}_{2m}^{\text{RB}} \end{bmatrix} \quad (\text{from Eqn. 30b}) \quad (32a) \\
&= \begin{bmatrix} \mathbf{f}_{\mathbf{e}_1}^{\text{RB}} & \mathbf{f}_{\mathbf{e}_2}^{\text{RB}} & \dots & \mathbf{f}_{\mathbf{e}_{2m}}^{\text{RB}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^{\text{RB}} \\ \mathbf{u}_2^{\text{RB}} \\ \vdots \\ \mathbf{u}_{2m}^{\text{RB}} \end{bmatrix} = \mathbf{F}^{\text{RB}} \hat{\mathbf{u}}^{\text{RB}}
\end{aligned}$$

$$\text{where } \mathbf{f}_{\mathbf{e}_i}^{\text{RB}} = \alpha \int_{\Gamma^{\text{RB}}} \mathbf{N}^{\text{R}} \mathbf{S} (\mathbf{N}^{\text{RB}})^{\text{T}} \mathbf{e}_i d\Gamma \quad \text{and} \quad \mathbf{F}^{\text{RB}} = \begin{bmatrix} \mathbf{f}_{\mathbf{e}_1}^{\text{RB}} & \mathbf{f}_{\mathbf{e}_2}^{\text{RB}} & \dots & \mathbf{f}_{\mathbf{e}_{2m}}^{\text{RB}} \end{bmatrix} \quad (32b)$$

Let $\hat{\mathbf{u}}_{\mathbf{e}_i}^{\text{R}}$ be the nodal value induced by the unit base vector $\mathbf{e}_i$ such that

$$\mathbf{K}^{\text{R}} \hat{\mathbf{u}}_{\mathbf{e}_i}^{\text{R}} = \mathbf{f}_{\mathbf{e}_i}^{\text{RB}} \quad \text{or} \quad \hat{\mathbf{u}}_{\mathbf{e}_i}^{\text{R}} = (\mathbf{K}^{\text{R}})^{-1} \mathbf{f}_{\mathbf{e}_i}^{\text{RB}} \quad (33)$$

then by Eqns. 31 and 32, the vector $\hat{\mathbf{u}}_2^{\text{R}}$ can be expressed as

$$\hat{\mathbf{u}}_2^{\text{R}} = \sum_{i=1}^{2m} \hat{\mathbf{u}}_{\mathbf{e}_i}^{\text{R}} u_i^{\text{RB}} = \sum_{i=1}^{2m} (\mathbf{K}^{\text{R}})^{-1} \mathbf{f}_{\mathbf{e}_i}^{\text{RB}} u_i^{\text{RB}} = (\mathbf{K}^{\text{R}})^{-1} \sum_{i=1}^{2m} \mathbf{f}_{\mathbf{e}_i}^{\text{RB}} u_i^{\text{RB}} = (\mathbf{K}^{\text{R}})^{-1} \mathbf{F}^{\text{RB}} \hat{\mathbf{u}}^{\text{RB}} \quad (34)$$

Hence, by combining Eqns. 22a, 31, 32 and 34 the displacement $\mathbf{u}^{\text{R}}$ inside Ω^{R} and along Γ^{RB} can be expressed as

$$\begin{aligned}
\mathbf{u}^{\text{R}} &= (\mathbf{N}^{\text{R}})^{\text{T}} \hat{\mathbf{u}}^{\text{R}} \\
&= (\mathbf{N}^{\text{R}})^{\text{T}} (\hat{\mathbf{u}}_1^{\text{R}} + \hat{\mathbf{u}}_2^{\text{R}}) \\
&= (\mathbf{N}^{\text{R}})^{\text{T}} \left((\mathbf{K}^{\text{R}})^{-1} \mathbf{f}^{\text{R}} + (\mathbf{K}^{\text{R}})^{-1} \mathbf{F}^{\text{RB}} \hat{\mathbf{u}}^{\text{RB}} \right) \\
&= (\mathbf{N}^{\text{R}})^{\text{T}} (\mathbf{K}^{\text{R}})^{-1} (\mathbf{f}^{\text{R}} + \mathbf{F}^{\text{RB}} \hat{\mathbf{u}}^{\text{RB}})
\end{aligned} \quad (35)$$

Thus, from Eqn. 2, the stress inside Ω^{R} and along Γ^{RB} , $\boldsymbol{\sigma}^{\text{R}}$, is defined by

$$\boldsymbol{\sigma}^{\text{R}} = \mathbf{D} \mathbf{L} (\mathbf{N}^{\text{R}})^{\text{T}} (\mathbf{K}^{\text{R}})^{-1} (\mathbf{f}^{\text{R}} + \mathbf{F}^{\text{RB}} \hat{\mathbf{u}}^{\text{RB}}) \quad (36)$$

Along the boundary Γ^{RB} , the relationship between the stress, $\boldsymbol{\sigma}^{\text{RB}}$ and traction, $\mathbf{t}^{\text{RB}}$ can be expressed as (Fig. 3),

$$\mathbf{t}^{\text{RB}} = \mathbf{T} \boldsymbol{\sigma}^{\text{RB}}, \quad \mathbf{T} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad (37)$$

If the vector $\hat{\mathbf{t}}^{\text{RB}}$ is written in component form as

$$\hat{\mathbf{t}}^{\text{RB}} = (\mathbf{t}_1^{\text{RB}}, \mathbf{t}_2^{\text{RB}}, \dots, \mathbf{t}_{2m}^{\text{RB}})^{\text{T}} \quad (38)$$

then using Eqns. 36 and 37, at the i th node of the interface boundary, one can write

$$\begin{aligned} \begin{pmatrix} \mathbf{t}_{2i-1}^{\text{RB}} \\ \mathbf{t}_{2i}^{\text{RB}} \end{pmatrix} &= \left(\mathbf{TDL}(\mathbf{N}^{\text{R}})^{\text{T}} \right) \Big|_{\mathbf{p}_i} (\mathbf{K}^{\text{R}})^{-1} \left(\mathbf{f}^{\text{R}} + \mathbf{F}^{\text{RB}} \Big|_{\mathbf{p}_i} \hat{\mathbf{u}}^{\text{RB}} \right) \\ &= \begin{pmatrix} \mathbf{t}_{2i-1}^0 \\ \mathbf{t}_{2i}^0 \end{pmatrix} + \begin{bmatrix} \mathbf{W}_{2i-1}^{\text{RB}} \\ \mathbf{W}_{2i}^{\text{RB}} \end{bmatrix} \hat{\mathbf{u}}^{\text{RB}} \end{aligned} \quad (39a)$$

In Eqn. 39a, $\mathbf{p}_i = (x_i, y_i)$ are the coordinates of the i th nodes. The terms $(\mathbf{t}_{2i-1}^0, \mathbf{t}_{2i}^0)^{\text{T}}$ and

the $2 \times (2m)$ matrix $\begin{bmatrix} \mathbf{W}_{2i-1}^{\text{RB}} \\ \mathbf{W}_{2i}^{\text{RB}} \end{bmatrix}$ are given by

$$\begin{pmatrix} \mathbf{t}_{2i-1}^0 \\ \mathbf{t}_{2i}^0 \end{pmatrix} = \left(\mathbf{TDL}(\mathbf{N}^{\text{R}})^{\text{T}} \right) \Big|_{\mathbf{p}_i} (\mathbf{K}^{\text{R}})^{-1} \mathbf{f}^{\text{R}} \quad \text{and} \quad \begin{pmatrix} \mathbf{W}_{2i-1}^{\text{RB}} \\ \mathbf{W}_{2i}^{\text{RB}} \end{pmatrix} = \left(\mathbf{TDL}(\mathbf{N}^{\text{R}})^{\text{T}} \right) \Big|_{\mathbf{p}_i} (\mathbf{K}^{\text{R}})^{-1} \mathbf{F}^{\text{RB}} \quad (39b)$$

Thus, by repeating Eqn. 39b for all m nodes along the interface boundary, the final relationship between $\hat{\mathbf{t}}^{\text{RB}}$ and $\hat{\mathbf{u}}^{\text{RB}}$ can be written as

$$\hat{\mathbf{t}}^{\text{RB}} = \mathbf{W}^{\text{RB}} \hat{\mathbf{u}}^{\text{RB}} + \hat{\mathbf{t}}^0 \quad (40a)$$

such that

$$\mathbf{W}^{\text{RB}} = \begin{bmatrix} \mathbf{W}_1^{\text{RB}} \\ \mathbf{W}_2^{\text{RB}} \\ \vdots \\ \mathbf{W}_{2m}^{\text{RB}} \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{t}}^0 = \begin{pmatrix} \mathbf{t}_1^0 \\ \mathbf{t}_2^0 \\ \vdots \\ \mathbf{t}_{2m}^0 \end{pmatrix} \quad (40b)$$

3.5 Coupling of the BEM and the RKPM

Once the relationship between $\hat{\mathbf{t}}^{\text{RB}}$ and $\hat{\mathbf{u}}^{\text{RB}}$ is known, it is now possible to solve the coupled equations system. By using Eqns. 23 and 32a for Ω^{R} , Eqn. 29a for Ω^{B} and Eqn. 40a for Γ^{RB} , the final equation for the whole domain can be summarized as

Inside Ω^{R} : $\mathbf{K}^{\text{R}} \hat{\mathbf{u}}^{\text{R}} = \mathbf{f}^{\text{R}} + \mathbf{F}^{\text{RB}} \hat{\mathbf{u}}^{\text{RB}} \quad (41)$

Inside Ω^{B} : $\begin{bmatrix} \mathbf{H}^{\text{RB}} & \mathbf{H}_{\Gamma_u}^{\text{B}} & \mathbf{H}_{\Gamma_t}^{\text{B}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}^{\text{RB}} \\ \hat{\mathbf{u}}_{\Gamma_u}^{\text{B}} \\ \hat{\mathbf{u}}_{\Gamma_t}^{\text{B}} \end{bmatrix} = \begin{bmatrix} -\mathbf{G}^{\text{RB}} & \mathbf{G}_{\Gamma_u}^{\text{B}} & \mathbf{G}_{\Gamma_t}^{\text{B}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{t}}^{\text{RB}} \\ \hat{\mathbf{t}}_{\Gamma_u}^{\text{B}} \\ \hat{\mathbf{t}}_{\Gamma_t}^{\text{B}} \end{bmatrix} + \mathbf{Q} \quad (29a)$

Along Γ^{RB} : $\hat{\mathbf{t}}^{\text{RB}} = \mathbf{W}^{\text{RB}} \hat{\mathbf{u}}^{\text{RB}} + \hat{\mathbf{t}}^0 \quad (40a)$

To solve for the unknown terms $\hat{\mathbf{u}}^R$, $\hat{\mathbf{u}}^{RB}$, $\hat{\mathbf{u}}_{\Gamma_t}^B$, $\hat{\mathbf{t}}^{RB}$ and $\hat{\mathbf{t}}_{\Gamma_u}^B$, one only need to use Eqn. 40a to eliminate $\hat{\mathbf{t}}^{RB}$ from Eqn. 29a so that

$$\left[\left(\mathbf{H}^{RB} + \mathbf{G}^{RB} \mathbf{W}^{RB} \right) \quad -\mathbf{G}_{\Gamma_u}^B \quad \mathbf{H}_{\Gamma_t}^B \right] \begin{bmatrix} \hat{\mathbf{u}}^{RB} \\ \hat{\mathbf{t}}_{\Gamma_u}^B \\ \hat{\mathbf{u}}_{\Gamma_t}^B \end{bmatrix} = \left[-\mathbf{G}^{RB} \quad -\mathbf{H}_{\Gamma_u}^B \quad \mathbf{G}_{\Gamma_t}^B \right] \begin{bmatrix} \hat{\mathbf{t}}^0 \\ \hat{\mathbf{u}}_{\Gamma_u}^B \\ \hat{\mathbf{t}}_{\Gamma_t}^B \end{bmatrix} + \mathbf{Q} \quad (42)$$

and solve for the unknowns $\hat{\mathbf{u}}^{RB}$, $\hat{\mathbf{t}}_{\Gamma_u}^B$ and $\hat{\mathbf{u}}_{\Gamma_t}^B$. After $\hat{\mathbf{u}}^{RB}$ is computed, it can be substituted back into Eqn. 42 for the solution of $\hat{\mathbf{u}}^R$.

3.6 Some remarks about the coupling procedure

3.6.1 Compatibility of displacement along the interface boundary Γ^{RB}

Recalling Eqns. 22b and 28c for the approximation of displacement $\mathbf{u}^{RB}$, along the interface boundary, two interpolation functions ($\mathbf{N}^{RB}$ and $\mathbf{N}_u^B$) are employed.

$$\text{For } \Omega^R: \quad \mathbf{u}^{RB} \approx \left(\mathbf{N}^{RB} \right)^T \hat{\mathbf{u}}^{RB} + \left(\mathbf{N}^C \right)^T \hat{\mathbf{u}}^C \quad (22b)$$

$$\text{For } \Omega^B: \quad \mathbf{t}^{RB} = \left(\mathbf{N}_t^B \right)^T \hat{\mathbf{t}}^{RB}, \quad \mathbf{u}^{RB} = \left(\mathbf{N}_u^B \right)^T \hat{\mathbf{u}}^{RB} \quad (28c)$$

Since in general, $\mathbf{N}^{RB} \neq \mathbf{N}_u^B$ and the vector $\hat{\mathbf{u}}^C \neq 0$, even though the same nodal vector $\hat{\mathbf{u}}^{RB}$ is used in Eqns. 22b and 28c, the displacement along the interface boundary is not fully compatible. However, since in the RKPM formulation, the interface boundary displacement is taken as an essential boundary condition (Eqn. 21) and the interface nodal vector $\hat{\mathbf{u}}^{RB}$ is obtained by solving the BIE in Ω^B , the interface essential boundary condition can be interpreted as

$$\mathbf{u} = \left(\mathbf{N}_u^B \right)^T \hat{\mathbf{u}}^{RB} \text{ on } \Gamma^{RB} \quad (43)$$

As the penalty formulation (Eqn. 8) is employed to enforce the essential boundary condition, one can expect that as the computational model is refined, the integral

$$\int_{\Gamma^{RB}} \left(\mathbf{u} - \left(\mathbf{N}_u^B \right)^T \hat{\mathbf{u}}^{RB} \right)^T \left(\mathbf{u} - \left(\mathbf{N}_u^B \right)^T \hat{\mathbf{u}}^{RB} \right) d\Gamma \quad (44)$$

will converge to zero. Thus, in the present coupling procedure, the compatibility condition along the interface boundary is imposed in a weak sense similar to the case of the essential boundary condition along Γ_u^R (Eqn. 19a). This implies that the

convergence rate of the coupled formulation should not be lower than that of the RKPM.

3.6.2 Computational cost of the coupling procedure

From Eqn. 39a, one can see that in order to obtain the matrix $\mathbf{W}^{\text{RB}}$ and the vector $\hat{\mathbf{t}}^0$, one has to compute the products $\hat{\mathbf{u}}_1^{\text{R}} = (\mathbf{K}^{\text{R}})^{-1} \mathbf{f}^{\text{R}}$ (Eqn. 31) and $(\mathbf{K}^{\text{R}})^{-1} \mathbf{F}^{\text{RB}}$ (Eqn. 34) which involve the term $(\mathbf{K}^{\text{R}})^{-1}$. It should be noted that in actual computation, the term $(\mathbf{K}^{\text{R}})^{-1}$ will never be computed explicitly. To obtain $\hat{\mathbf{u}}_1^{\text{R}}$ and $(\mathbf{K}^{\text{R}})^{-1} \mathbf{F}^{\text{RB}}$, it is only required to assembly and factorize $\mathbf{K}^{\text{R}}$ once and then back-substitute $2m+1$ times for the vector $\mathbf{f}^{\text{R}}$ and the $2\text{NP} \times 2m$ matrix $\mathbf{F}^{\text{RB}}$. After the vector $\hat{\mathbf{u}}^{\text{RB}}$ is found, from Eqns. 31 and 34, the displacement vector $\hat{\mathbf{u}}^{\text{R}}$ can be obtained as

$$\hat{\mathbf{u}}^{\text{R}} = \hat{\mathbf{u}}_1^{\text{R}} + \hat{\mathbf{u}}_2^{\text{R}} = \hat{\mathbf{u}}_1^{\text{R}} + (\mathbf{K}^{\text{R}})^{-1} \mathbf{F}^{\text{RB}} \hat{\mathbf{u}}^{\text{RB}} \quad (45)$$

and no direct solution of Eqn. 41 is required. Since the number of interface nodes is normally much smaller than the number of nodes in Ω^{R} , the additional computational cost required for the coupling procedure is modest when compared with that required for the RKPM.

4. *A priori* error estimation for the Coupled BE-RKPM

4.1 Definition of error norm

The errors in stress $\mathbf{e}_{\sigma}$, and displacement $\mathbf{e}_{\mathbf{u}}$, are defined as the differences between the exact and the approximated solutions

$$\mathbf{e}_{\mathbf{u}} = \mathbf{u} - \mathbf{u}^{\text{h}} \text{ and } \mathbf{e}_{\sigma} = \sigma - \sigma^{\text{h}} \quad (46)$$

In Eqn. 46, $\mathbf{u}^{\text{h}}$ and σ^{h} are, respectively, the approximated displacement and stress obtained by the RKPM, the BEM or the coupled BE-RKPM. In general, such pointwise error measures are difficult to use in practice and the *energy norm of the error* is often used instead. For 2D elastostatic problems defined by Eqn. 1, the energy norm of the exact solution $\|\mathbf{u}\|_{\Omega}$, and the energy norm of the error $\|\mathbf{e}_{\mathbf{u}}\|_{\Omega}$, of the approximated solution are defined as [22]

$$\|\mathbf{u}\|_{\Omega}^2 = \int_{\Omega} (\mathbf{L}\mathbf{u})^{\text{T}} \mathbf{D}(\mathbf{L}\mathbf{u}) d\Omega = \int_{\Omega} \sigma^{\text{T}} \mathbf{D}^{-\text{T}} \sigma d\Omega \quad (47a)$$

$$\|\mathbf{e}_u\|_{\Omega}^2 = \int_{\Omega} (\mathbf{L}\mathbf{e}_u)^T \mathbf{D}(\mathbf{L}\mathbf{e}_u) d\Omega = \int_{\Omega} (\mathbf{e}_{\sigma})^T \mathbf{D}^{-T} \mathbf{e}_{\sigma} d\Omega \quad (47b)$$

The absolute value of the energy norm varies from problem to problem and is not particularly useful by itself. Hence, the relative error in energy norm, η , defined as

$$\eta = \frac{\|\mathbf{e}_u\|_{\Omega}}{\|\mathbf{u}\|_{\Omega}} \quad (48)$$

is usually adopted as a measure of the accuracy of the solution.

4.2 *A priori error estimation for the RKPM*

For the RKPM, if the model is under uniform h - refinement, the *a priori* error estimation can be written in the form [23]

$$\|\mathbf{e}_u\| \leq C^R (h^R)^{\min(p^R, \lambda)} \quad (49a)$$

In Eqn. 49a, h^R is the spacing of nodal points in the domain. The term p^R is the order of the RKPM approximation such that $p^R = q-1$ and $q \geq 2$ is the completed order of polynomial used in the correction function (Eqn. 4) of the RKPM formulation. C^R is a constant independent of both h^R and p^R . $\lambda < 1$ is the *strength of the singularity* of the problem [22]. For 2D problems, h^R is inversely proportional to $(\text{NDOF}^R)^2$ where NDOF^R is the number of degree of freedoms of the RKPM. Thus, in terms of NDOF^R , the *a priori* error estimation for the RKPM can be written as

$$\|\mathbf{e}_u\| \leq C^R (\text{NDOF}^R)^{-\min(p^R, \lambda)/2} \quad (49b)$$

4.3 *A priori error estimation for the BEM*

For the BEM, the *a priori* error estimation under uniform h - refinement can be written as [24]

$$\|\mathbf{e}_u\| \leq C^B (h^B)^{\min(p^B, \lambda)} \quad (50a)$$

where h^B is the size of boundary elements used, $p^B \geq 1$ is the order of the interpolation functions $\mathbf{N}_u^B$ and $\mathbf{N}_t^B$ (Eqn. 16) used in the BEM. C^B is a constant independent of both h^B and p^B . For the BEM, since only the boundary of the problem domain is discretized, h^B is only inversely proportional to NDOF^B , the number of degree of freedoms of the BEM. Thus, in terms of NDOF^B , Eqn. 50a can be written in the form

$$\|\mathbf{e}_u\| \leq C^B (\text{NDOF}^B)^{-\min(p^B, \lambda)} \quad (50b)$$

4.4 *A priori error estimation for the coupled BE-RKPM*

Since when the coupled BE-RKPM is applied, the problem domain Ω is divided into two disjoint subdomains, Ω^R and Ω^B , the following equations can be established

$$\|\mathbf{e}_u\|^2 = \|\mathbf{e}_u\|_R^2 + \|\mathbf{e}_u\|_B^2 \quad (51a)$$

$$\text{NDOF} = \text{NDOF}^R + \text{NDOF}^B \quad (51b)$$

In Eqn. 51, $\|\mathbf{e}_u\|_R$ and $\|\mathbf{e}_u\|_B$ are the energy norms of error in Ω^R and Ω^B respectively while NDOF and $\|\mathbf{e}_u\|$ are respectively the total degree of freedoms and error norm for the whole problem domain. Now assume that under uniform h - refinement, the *a priori* error estimation for the whole problem domain can be written as

$$\|\mathbf{e}_u\| = C (\text{NDOF})^{-\gamma} \quad (52)$$

where C is another constant independent of the overall convergence rate γ and NDOF. Now let $a = \min(p^R, \lambda)/2$ and $b = \min(p^B, \lambda)$ be the convergence rates of the RKPM and the BEM respectively. Upon substituting Eqns. 49b, 50b and 52 into Eqn. 51a, one can write

$$\begin{aligned} \frac{C^2}{\text{NDOF}^{2\gamma}} &= \frac{(C^R)^2}{(\text{NDOF}^R)^{2a}} + \frac{(C^B)^2}{(\text{NDOF}^B)^{2b}} \\ &\geq \frac{(C^R)^2}{\text{NDOF}^{2a}} + \frac{(C^B)^2}{\text{NDOF}^{2b}} \\ &\geq \frac{(C^R)^2}{\text{NDOF}^{2\max(a, b)}} + \frac{(C^B)^2}{\text{NDOF}^{2\max(a, b)}} \\ &= \frac{(C^R)^2 + (C^B)^2}{\text{NDOF}^{2\max(a, b)}} \end{aligned} \quad (53a)$$

As a result, as $\text{NDOF} \rightarrow \infty$, one can conclude that $\gamma \leq \max(a, b)$.

Furthermore, from Eqn. 51b, one can always write

$$\text{NDOF}^B = \xi \text{NDOF} \text{ and } \text{NDOF}^R = (1 - \xi) \text{NDOF} \text{ for some } 0 < \xi < 1 \quad (53b)$$

Hence,

$$\begin{aligned}
\frac{C^2}{\text{NDOF}^{2\gamma}} &= \frac{(C^R)^2}{(\text{NDOF}^R)^{2a}} + \frac{(C^B)^2}{(\text{NDOF}^B)^{2b}} \\
&\leq \frac{(C^R)^2}{(\text{NDOF}^R)^{2\min(a,b)}} + \frac{(C^B)^2}{(\text{NDOF}^B)^{2\min(a,b)}} \\
&= \frac{(C^R)^2}{((1-\xi)\text{NDOF})^{2\min(a,b)}} + \frac{(C^B)^2}{(\xi\text{NDOF})^{2\min(a,b)}} \quad (53c) \\
&= \frac{(C^R)^2}{(1-\xi)^{2\min(a,b)}} \frac{1}{\text{NDOF}^{2\min(a,b)}} + \frac{(C^B)^2}{\xi^{2\min(a,b)}} \frac{1}{\text{NDOF}^{2\min(a,b)}} \\
&\leq \max\left(\frac{(C^R)^2}{\xi^{2\min(a,b)}}, \frac{(C^B)^2}{(1-\xi)^{2\min(a,b)}}\right) \frac{1}{\text{NDOF}^{2\min(a,b)}}
\end{aligned}$$

Thus, as $\text{NDOF} \rightarrow \infty$, $\gamma \geq \min(a,b)$. Finally, combining Eqns. 53a and 53c, one has

$$\min(a,b) \leq \gamma \leq \max(a,b) \quad (54)$$

Thus, the convergence rates of the coupled BE-RKPM will always be between the convergence rates of the RKPM and the BEM.

In case that no singularity is present and the orders of approximation of the RKPM (p^R) and the BEM (p^B) used are the same, then from the Eqn. 49b, 50b and 54

$$p^R/2 < \gamma < p^B \quad (55a)$$

In case that singularity is present, since $\min(p^R, p^B) > \lambda$, from Eqns. 49 and 50, γ will be in the range

$$\lambda/2 < \gamma < \lambda \quad (55b)$$

5. Numerical examples

In this section numerical studies on five benchmark problems are carried out to test the performance of the coupled BE-RKPM. In the first four examples, the exact solutions of the problems are known. Uniform refinements are carried out by using four different models with different nodes/boundary segments density and a total of 48 analyses (4 Examples $\times$ 4 Models $\times$ 3 Methods) are carried out. In the last example, a problem with semi-infinite domain is investigated and the results obtained by the suggested coupling procedure are compared with the results obtained by the coupled BE-FEM suggested by Brebbia and Georgion [25] and the coupled BE-EFGM suggested by Liu and Gu [20]. In all the analyses, the value of α (Eqn. 8) used

in the RKPM formulation is set equal to $5 \times 10^6 E$ where E is the Young's modulus of the problem while linear approximation ($p^R = p^B = 1$) is adopted. Hence, when no singularity is present in the problem domain, the optimal convergence rates for the RKPM and the BEM solutions should be equal to 0.5 and 1 respectively. In the first four examples, in order to compare the effectiveness of different methods, the relative error η (Eqn. 48), and the actual convergence rate $R(\eta)$, defined as

$$R(\eta) = \frac{\log\left(\frac{\eta_i}{\eta_{i+1}}\right)}{\log\left(\frac{\text{NDOF}_{i+1}}{\text{NDOF}_i}\right)} \quad (56)$$

obtained by the RKPM, the BEM and the coupled BE-RKPM are computed. In Eqn. 56, η_i and NDOF_i are, respectively, the relative error and the degree of freedoms of the i th refinement model.

5.1 Numerical examples with uniform refinement

Example 1: Timoshenko beam (convex domain, smooth exact solution)

The first example is a cantilever beam loaded under end shear force as shown in Fig. 4. The exact solution of this problem is given by Timoshenko and Goodier [26]

$$\sigma_x = -\frac{P y}{I}(L - x), \quad \sigma_y = 0.0, \quad \tau_{xy} = -\frac{P}{2I}(H^2/4 - y^2), \quad I = H^3/12 \quad (57)$$

The models used in the BEM, the RKPM and the Coupled BE-RKPM are shown in Figs. 5, 6 and 7 respectively. The results obtained are summarized in Table 1 while the rates of convergence are plotted in Fig. 8, respectively.

Example 2: Infinite plate with hole (concave domain, smooth exact solution)

In the second example, an infinite plate with a circular hole as shown Fig. 9 is considered. The exact solution of this problem is given by [26]

$$\begin{aligned} \sigma_x &= 1 - \frac{1}{r^2} \left(\frac{3}{2} \cos 2\theta + \cos 4\theta \right) + \frac{3}{2r^4} \cos 4\theta \\ \sigma_y &= -\frac{1}{r^2} \left(\frac{1}{2} \cos 2\theta - \cos 4\theta \right) - \frac{3}{2r^4} \cos 4\theta \\ \tau_{xy} &= -\frac{1}{r^2} \left(\frac{1}{2} \sin 2\theta + \sin 4\theta \right) + \frac{3}{2r^4} \sin 4\theta \end{aligned} \quad (58)$$

For this example, the models used in different analyses are shown in Figs. 10 to 12 while the results obtained are summarized in Table 2 and Fig. 13.

Example 3: Domain near a crack Tip (convex domain, singular exact solution)

In the third example, the problem domain is a region near the tip of a crack as shown in Fig. 14. It is loaded by tractions given by an exact solution corresponding to the first symmetric mode of the stress intensity factor solution. The stress field in the domain is given by [22]

$$\begin{aligned}\sigma_x &= \frac{1}{\sqrt{r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) \\ \sigma_y &= \frac{1}{\sqrt{r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) \\ \sigma_{xy} &= \frac{1}{\sqrt{r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}\end{aligned}\tag{59}$$

Exploiting symmetry, only half of the domain is modelled. In this example, a singularity exists at (0,0) with $\lambda=0.5$. For this problem, the models used in the BEM, the RKPM and the coupled BE-RKPM analyses are shown in Figs. 15, 16 and 17 respectively. The results obtained are summarized in Table 3 while the rates of convergence are plotted in Fig. 18.

Example 4: L-Shaped Domain (concave domain, singular exact solution)

In the last example, a L-shaped plane as shown in Fig. 19 is considered. It is loaded by tractions corresponding to the following exact stress solution [22]

$$\begin{aligned}\sigma_x &= \lambda r^{(\lambda-1)} \{ (2 - Q(\lambda+1)) \cos((\lambda-1)\theta) - (\lambda-1) \cos((\lambda-3)\theta) \} \\ \sigma_y &= \lambda r^{(\lambda-1)} \{ (2 + Q(\lambda+1)) \cos((\lambda-1)\theta) + (\lambda-1) \cos((\lambda-3)\theta) \} \\ \tau_{xy} &= \lambda r^{(\lambda-1)} \{ (\lambda-1) \sin((\lambda-3)\theta) + Q(\lambda+1) \sin((\lambda-1)\theta) \}\end{aligned}\tag{60}$$

In Eqn. 60, $Q=0.543075579$ and $\lambda=0.544483737$. The traction conditions satisfy equilibrium and hence only rigid-body constraints have to be applied at points C and F. In this example, similar to Example 3, a singularity exists at point C. The models used for the BEM, the RKPM and the coupled BE-RKPM are shown in Figs. 20, 21 and 22 respectively. The results obtained are summarized in Table 4 and the rates of convergence are plotted in Fig. 23.

5.2 *Discussions on the numerical results with uniform refinement*

From the numerical results obtained, it can be concluded that

- (1) Regarding the accuracy of the solutions obtained, from Tables 1 to 4, it can be seen that in all the examples, the BEM solutions are always the most accurate solutions while the coupled BE-RKPM solutions are more accurate than the RKPM solutions. This can be explained by the fact that the BEM is a semi-analytical numerical method which makes use of the fundamental solution in its formulation and thus leads to more accurate solutions. Hence, in practical applications of the coupled BE-RKPM, it should be preferred to limit the RKPM subdomain only to where rapid refinement/remodelling/change in domain geometry occurs.
- (2) Regarding the convergence rates of the solutions, from Figs. 8 and 13, when no singularity is present in the problem domain, for both the RKPM and the BEM, the theoretical convergence rates of 0.5 and 1.0 are achieved. Furthermore, as predicted by Eqn. 55a, the rate of convergence of the coupled BE-RKPM solution is in the predicted range of (0.5,1.0). When singularity is present in the problem domain, as seen from Figs. 18 and 23, the convergence rates of all methods are affected as predicted by Eqn. 55b. Furthermore, the rate of convergence of the coupled BE-RKPM lies between the range $(\lambda/2, \lambda)$. Thus, in all the numerical results, the actual convergence rate obtained from the coupled BE-RKPM is always higher than the convergence rate obtained from the RKPM. Hence, one can conclude that the suggested coupling procedure does not impair the convergence rate of the solution despite that the compatibility conditions along the interface boundary is only imposed in a weak sense (Eqn. 44).
- (3) Regarding the computational cost of the solutions, detailed timings indicated that for the same point/mesh density, the BEM is approximately ten times faster than the RKPM. Furthermore, additional CPU time needed for the coupling solution is negligible when comparing with the CPU time used in the RKPM. Hence, by coupling the BEM with the RKPM, a significant saving of computational cost can be achieved, especially for problems involving infinite/semi-infinite domain.

5.3 *Example 5: A structure on a semi-infinite soil foundation*

In this example, the suggested coupled BE-RKPM is applied for the solution of a soil-structure interaction problem as shown in Fig. 24, which has been solved by Brebbia and Georgion [25] using the BEM and the coupled BE-FEM and by Liu and Gu [20]

using the coupled BE-EFGM. As shown in Fig. 24, the infinite soil foundation is treated by truncating the soil region at a finite distance from the structure [20,25]. Note that such truncation treatment is used here as it is convenient to compare the suggested coupled BE-RKPM with other researchers' solutions. As shown in Figs. 25 and 26, the RKPM and the BEM are used in regions 1 and 2 respectively. Two loading cases are applied (Fig. 25): Case1 considers five concentrated vertical loads along the top and Case 2 considers an additional load acting at a corner.

The results obtained by using the BEM [25], the BE-FEM [25], the coupling BE-EFGM [20] and the suggested coupled BE-RKPM are shown in Table 5. From the results, it can be seen that the coupled BE-RKPM is in excellent agreement with the results obtained by other methods. It should be noted that only in the coupled BE-RKPM, only 40 BE elements and 35 RKPM nodes are used.

6. Conclusions and future extensions

In this paper, a simple but direct procedure for the coupling of the reproducing kernel particle method (RKPM) and the boundary element method (BEM) is suggested. The suggested method requires neither modification of the BEM/RKPM nor the use of any interface element. In addition, the computational cost incurring in the coupling procedure is modest. Numerical results obtained by uniform refinement indicate that when comparing with the RKPM, the suggested coupled procedure can effectively reduce the computational cost needed for the solution while no adverse effect on the accuracy and convergence rate of the solution is observed. In fact, by merging the RKPM with the BEM, improvements in both the accuracy and the convergence rate are observed in all the examples tested.

Since in this study only uniform refinement is carried out, one immediate extension of the present research is to devise a reliable *a posteriori* error estimator for the coupled BE-RKPM. The *a posteriori* error estimator developed can be used for the development of an adaptive refinement procedure for the coupled method to further enhance the accuracy of the solution and to reduce computational cost needed.

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Figure 1: The coupling domain of RKPM and BEM

Figure 2: Unit displacement along the interface boundary

Figure 3: Transformation between nodal traction and nodal stress

Figure 4: Example 1: Cantilever beam under end shear force

Figure 5: BEM models used for Example 1

Figure 6: RKPM models used for Example 1

Figure 7: Coupling BE-RKPM models used for Example 1

Figure 8: Rates of convergence in energy norm for Example 1

Figure 9: Example 2: A plate with a central hole

Figure 10: BEM models used for Example 2

Figure 11: RKPM models used for Example 2

Figure 12: Coupling BE-RKPM models used for Example 2

Figure 13: Rates of convergence in energy norm for Example 2

Figure 14: Example 3: Region near a crack tip

(a) Model 1: 12 elements

(c) Model 3: 30 elements

(b) Model 2: 24 elements

(d) Model 4: 48 elements

Figure 15: BEM models used for Example 3

(a) Model 1: 15 points

(c) Model 3: 66 points

(b) Model 2: 45 points

(d) Model 4: 153 points

Figure 16: RKPM models used for Example 3

(a) Model 1: 18 elements, 9 points

(c) Model 3: 43 elements, 36 points

(b) Model 2: 34 elements, 25 points

(d) Model 4: 68 elements, 81 points

Figure 17: Coupling BE-RKPM models used for Example 3

Figure 18: Rates of convergence in energy norm for Example 3

Figure 19: Example 4: A L-shape domain

Figure 20: BEM models used for Example 4

Figure 21: RKPM models used for Example 4

Figure 22: Coupling BE-RKPM models used for Example 4

Figure 23: Rates of convergence in energy norm for Example 4

Figure 24: A structure standing on a semi-infinite soil foundation

Fig. 25: RKPM mesh of structure (region 1) and load cases

Fig. 26: Boundary Element mesh of soil (Region 2)

Table 1: Results of Example 1

Mesh	BEM			RKPM			Coupling BE-RKPM		
	NDOF	$\eta(\%)$	$R(\eta)$	NDOF	$\eta(\%)$	$R(\eta)$	NDOF	$\eta(\%)$	$R(\eta)$
1	48	16.2	$\diagdown$	54	24.3	$\diagdown$	54	18.6	$\diagdown$
2	72	10.4	1.09	170	12.1	0.61	136	9.3	0.75
3	104	7.1	1.03	252	9.3	0.66	184	7.3	0.801
4	144	4.5	1.40	594	6.1	0.49	402	4.3	0.677

Table 2: Results of Example 2

Mesh	BEM			RKPM			Coupling BE-RKPM		
	NDOF	$\eta(\%)$	$R(\eta)$	NDOF	$\eta(\%)$	$R(\eta)$	NDOF	$\eta(\%)$	$R(\eta)$
1	40	7.6	$\diagdown$	72	16.4	$\diagdown$	50	14.7	$\diagdown$
2	72	4.1	1.14	162	9.9	0.67	88	9.8	0.72
3	116	2.4	1.12	240	7.8	0.61	108	8.7	0.58
4	144	1.8	1.33	564	4.8	0.57	248	4.7	0.74

Table 3: Results of Example 3

Mesh	BEM			RKPM			Coupling BE-RKPM		
	NDOF	$\eta(\%)$	$R(\eta)$	NDOF	$\eta(\%)$	$R(\eta)$	NDOF	$\eta(\%)$	$R(\eta)$
1	48	24.4	$\diagdown$	30	30.3	$\diagdown$	66	23.5	$\diagdown$
2	96	18.4	0.41	90	21.4	0.32	138	18.7	0.31
3	120	16.8	0.41	132	19.3	0.27	184	16.8	0.29
4	192	14.2	0.36	306	15.8	0.24	338	14.3	0.27

Table 4: Results of Example 4

Mesh	BEM			RKPM			Coupling BE-RKPM		
	NDOF	$\eta(\%)$	$R(\eta)$	NDOF	$\eta(\%)$	$R(\eta)$	NDOF	$\eta(\%)$	$R(\eta)$
1	64	20.5	$\diagdown$	66	24.1	$\diagdown$	78	21.1	$\diagdown$
2	96	16.6	0.52	134	18.8	0.35	146	16.2	0.42
3	128	14.8	0.40	226	16.2	0.28	338	11.6	0.40
4	160	13.5	0.41	342	14.3	0.30	494	10.4	0.29

Table 5: Results of Example 5, vertical displacements along top of structure

Nodes	Displacements ($\times 10^{-4}$)			
	FE*	BE-FEM*	BE-EFGM#	BE-RKPM
Load Case 1				
1	1.41	1.40	1.42	1.42
2	1.34	1.33	1.33	1.35
3	1.32	1.32	1.32	1.33
4	1.34	1.33	1.33	1.35
5	1.41	1.40	1.42	1.42
Load Case 2				
1	-3.39	-3.55	-3.58	-3.59
2	-0.97	-1.05	-1.04	-1.06
3	1.35	1.35	1.34	1.32
4	3.61	3.70	3.68	3.72
5	6.00	6.17	6.13	6.18

*Results obtained from Reference [25]

#Results obtained from Reference [20]