Existence and Uniqueness of Solution for Perturbed Nonautonomous Systems with Nonuniform Exponential Dichotomy

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1. Introduction

The notion of exponential dichotomy, introduced by Perron in [1], plays an important role in the theory of differential equations and dynamical systems (also see [2–5]). It is well known that if linear system \( \dot{x}(t) = A(t)x(t) \) admits an (uniform) exponential dichotomy, the nonlinear term \( f(t, x) \) is bounded and has a small Lipschitz constant, then the nonlinear system \( \dot{x}(t) = A(t)x(t) + f(t, x) \) has a unique bounded solution (see [6]). However, many scholars argued that (uniform) exponential dichotomy restricted the behavior of dynamical systems. For this reason, we need a more general concept of hyperbolicity. Recently, Barreira and Valls [7, 8] have introduced the notion of nonuniform exponential dichotomy. General nonuniform exponential dichotomy has also been proposed (see [9–11]). Many properties of nonuniform exponential dichotomy have been extensively studied. For example, the topological conjugacies between linear and nonlinear perturbations were explored and some new Grobman-Hartman type theorems for nonuniform exponential dichotomy were established (see [12, 13]). However, the essential condition of these results is based on the assumption that the nonlinear term satisfies \( |f(t, x)| \leq \mu e^{-\varepsilon t} \). Under the same condition, Zhang et al. studied nonlinear perturbations of nonuniform exponential dichotomy on measure chains (see [14]).

However, the condition \( |f(t, x)| \leq \mu e^{-\varepsilon t} \) is very restricted. There are few functions satisfying \( |f(t, x)| \leq \mu e^{-\varepsilon t} \). Thus, it is necessary to find a more conservative condition for the nonlinear term \( f(t, x) \). In this paper, our main objective is to explore the existence and uniqueness of solution to the perturbed nonautonomous system with a relatively conservative assumption \( |f(t, x)| \leq \mu \). Finally, we prove that if \( |f(t, x)| \leq \mu \), the perturbed nonautonomous system with nonuniform exponential dichotomy has a unique solution \( x(t) \) satisfying \( |x(t)| = O(e^{\varepsilon t}) \).
The outline of this paper is arranged as follows. Next section is to state our main results. In Section 3, we prove the main results.

2. Main Results

In this section, we will state our main theorems. First, we introduce the definition of nonuniform exponential dichotomy.

Consider systems
\[
\begin{align*}
\ddot{x}(t) &= A(t)x(t), \\
\dot{x}(t) &= A(t)x(t) + f(t,x),
\end{align*}
\]  
(1)  
(2)
where \( t \in \mathbb{R} \), \( x \in \mathbb{R}^n \), \( A(t) \) is a continuous matrix function, and \( f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is a continuous function.

Let \( T_A(t,s) \) be the evolution operator satisfying \( x(t) = T_A(t,s)x(s) \), \( t, s \in \mathbb{R} \), where \( x(t) \) is a solution of (1).

Definition 1. Linear system (1) is said to admit a nonuniform exponential dichotomy if there exists a projection \( P(t) \) (\( P^2 = P \)) and constants \( \alpha > 0 \), \( K > 0 \), \( \epsilon \geq 0 \), such that
\[
\begin{align*}
|T_A(t,s)P(s)| \leq Ke^{-\alpha|t-s|} \cdot e^{\epsilon|s|}, & \quad t \geq s, \\
|T_A(t,s)Q(s)| \leq Ke^{\alpha|t-s|} \cdot e^{-\epsilon|s|}, & \quad t \leq s,
\end{align*}
\]  
(3)
where \( P(t) + Q(t) = \text{Id} \) (identity), \( T_A(t,s)P(s) = P(t)T_A(t,s), t, s \in \mathbb{R} \).

Remark 2. When \( \epsilon \equiv 0 \), system (1) is said to have an exponential dichotomy; and when \( \epsilon \equiv 0, \alpha \equiv 0 \), system (1) is said to have a uniform dichotomy.

To present our main results, we give a theorem under the trivial condition \( |f(t,x)| \leq \mu e^{-\epsilon|t|} \).

Theorem 3. Suppose that linear system (1) admits a nonuniform exponential dichotomy. For \( t \in \mathbb{R}, x, x_1, x_2 \in \mathbb{R}^n \), if the nonlinear term \( f(t,x) \) satisfies
\[
\begin{align*}
(H_1) & \quad |f(t,x)| \leq \mu e^{-\epsilon|t|}, \\
(H_2) & \quad |f(t,x_1) - f(t,x_2)| \leq re^{-\epsilon|t|}|x_1 - x_2|, \\
(H_3) & \quad 4Kr < \alpha,
\end{align*}
\]  
(4)
where \( \mu, r, \epsilon, \alpha \) are all positive constants, then nonlinear system (2) has a unique bounded solution \( \bar{x}(t) \) satisfying
\[
\begin{align*}
\bar{x}(t) = & \int_t^\infty T_A(t,s)P(s)f(s,\bar{x}(s)) \, ds \\
& - \int_t^\infty T_A(t,s)Q(s)f(s,\bar{x}(s)) \, ds.
\end{align*}
\]  
(5)

Discussion. One of the essential conditions of Theorem 3 is \((H_1)\). However, this condition is very restricted. There are few functions satisfying \( |f(t,x)| \leq \mu e^{-\epsilon|t|} \). Thus, it is necessary to find a more conservative condition for the nonlinear term \( f(t,x) \). The main objective of this paper is to prove that the perturbed system has a unique solution under \( |f(t,x)| \leq \mu \). But Theorem 3 cannot be valid yet. For this case, we have the following.

Theorem 4. Suppose that linear system (1) admits a nonuniform exponential dichotomy with the estimates (3). For \( t \in \mathbb{R}, x, x_1, x_2 \in \mathbb{R}^n \), if \( f(t,x) \) satisfies
\[
\begin{align*}
(H_1) & \quad |f(t,x)| \leq \mu, \\
(H_2) & \quad |f(t,x_1) - f(t,x_2)| \leq re^{-\epsilon|t|}|x_1 - x_2|, \\
(H_3) & \quad 4Kr < \alpha - \epsilon,
\end{align*}
\]  
(6)
where \( \alpha - \epsilon \) is a positive constant, then system (2) has a unique solution \( x(t) \) satisfying
\[
|x(t)| = O(e^{\epsilon|t|}).
\]  
(7)

Remark 5. The method used to prove Theorem 3 cannot be applied to this case. To see how to overcome the difficulty, one can refer to the main proof of Theorem 4. The technique employed to prove Theorem 4 is very skillful and interesting, which is the highlight of this paper.

3. Proofs of Main Results

In what follows, to prove Theorem 3, a preliminary lemma is needed.

Lemma 6 (see [15], Lemma 4). If system (1) admits a nonuniform exponential dichotomy, then system (1) has no nontrivial bounded solutions; that is, \( x(t) = 0 \) is the unique bounded solution of (1).

3.1. Proof of Theorem 3. Let \( B = \{ \varphi(t) \mid \varphi(t) \text{ is continuous and } |\varphi(t)| \leq 2K \mu \alpha^{-1} \} \), for \( \forall \varphi \in B \), define a mapping \( \mathcal{T}_1 \):
\[
\mathcal{T}_1 \varphi(t) = \int_t^\infty T_A(t,s)P(s)f(s,\varphi(s)) \, ds \\
- \int_t^\infty T_A(t,s)Q(s)f(s,\varphi(s)) \, ds.
\]  
(8)
From (3) and \((\overline{H}_1)\) and \((\overline{H}_2)\), we have
\[
|\mathcal{T}_1 \varphi(t)| \leq \int_t^\infty Ke^{-\alpha|t-s|}e^{\epsilon|s|} \cdot \mu e^{-\epsilon|t|} \, ds \\
+ \int_t^{+\infty} Ke^{\alpha|t-s|}e^{-\epsilon|s|} \cdot \mu e^{-\epsilon|t|} \, ds
\]  
(9)
\[
= K \mu \alpha^{-1} + K \mu \alpha^{-1}
\]  
(10)
\[
= 2K \mu \alpha^{-1}.
\]  
(11)
Therefore, $\mathcal{T}_1 \varphi(t) \in B$, which implies $\mathcal{T}_1$ maps $B$ onto itself. On the other hand,

$$
\left| \mathcal{T}_1 \varphi_1(t) - \mathcal{T}_1 \varphi_2(t) \right|
\leq \int_{-\infty}^{t} T_A(t, s) P(s) re^{-\epsilon|s|} |\varphi_1(s) - \varphi_2(s)| \, ds 
+ \int_{t}^{+\infty} T_A(t, s) Q(s) re^{-\epsilon|s|} |\varphi_1(s) - \varphi_2(s)| \, ds
\leq \int_{-\infty}^{t} K re^{-\alpha(t-s)} |\varphi_1(s) - \varphi_2(s)| \, ds 
+ \int_{t}^{+\infty} K re^{\alpha(t-s)} |\varphi_1(s) - \varphi_2(s)| \, ds
\leq 2Kr\alpha^{-1} \sup_{s \geq 0} |\varphi_1(s) - \varphi_2(s)|
\leq \frac{1}{2} \sup_{s \geq 0} |\varphi_1(s) - \varphi_2(s)|.
$$

(8)

Then $\mathcal{T}_1$ is a contraction mapping. Therefore, in $B$, there exists a unique fixed point $\varphi_0(t)$, such that

$$
\varphi_0(t) = T_A(t, 0) \varphi_0(0) 
- \int_{-\infty}^{t} T_A(t, s) P(s) f(s, \varphi_*(s)) \, ds 
+ \int_{t}^{+\infty} T_A(t, s) Q(s) f(s, \varphi_*(s)) \, ds.
$$

(9)

Differentiating the above equality, we see that $\varphi_0(t)$ satisfies (2). Now we are going to show that the solution of system (2) satisfying $(\mathcal{H}_1), (\mathcal{H}_2)$, and $(\mathcal{H}_3)$ is unique. Assume that system (2) has another bounded solution $\varphi^*(t)$ satisfying $(\mathcal{H}_1), (\mathcal{H}_2)$, and $(\mathcal{H}_3)$; we have

$$
\varphi^*(t) = T_A(t, 0) \varphi^*(0) 
+ \int_{0}^{t} T_A(t, s) T^{-1}(s, s) f(s, \varphi^*(s)) \, ds
= T_A(t, 0) \varphi^*(0) 
+ \int_{0}^{t} T_A(t, s) P(s) f(s, \varphi^*(s)) \, ds 
+ \int_{0}^{t} T_A(t, s) Q(s) f(s, \varphi^*(s)) \, ds
= T_A(t, 0) \varphi^*(0) 
+ \int_{-\infty}^{t} T_A(t, s) P(s) f(s, \varphi^*(s)) \, ds 
- \int_{-\infty}^{t} T_A(t, s) Q(s) f(s, \varphi^*(s)) \, ds
- \int_{0}^{+\infty} T_A(t, s) P(s) f(s, \varphi^*(s)) \, ds 
+ \int_{0}^{+\infty} T_A(t, s) Q(s) f(s, \varphi^*(s)) \, ds
= T_A(t, 0) \left[ \varphi^*(0) 
- \left( \int_{-\infty}^{0} T_A(0, s) P(s) f(s, \varphi^*(s)) \, ds 
- \int_{0}^{+\infty} T_A(0, s) Q(s) f(s, \varphi^*(s)) \, ds \right) \right]
+ \left( \int_{-\infty}^{t} T_A(t, s) P(s) f(s, \varphi^*(s)) \, ds 
- \int_{t}^{+\infty} T_A(t, s) Q(s) f(s, \varphi^*(s)) \, ds \right)
\leq 2KR\alpha^{-1} \sup_{s \geq 0} |\varphi_1(s) - \varphi_2(s)|
\leq \frac{1}{2} \sup_{s \geq 0} |\varphi_1(s) - \varphi_2(s)|.
$$

(10)

By calculating, we get

$$
\int_{-\infty}^{t} T_A(t, s) P(s) f(s, \varphi^*(s)) \, ds 
- \int_{-\infty}^{t} T_A(t, s) Q(s) f(s, \varphi^*(s)) \, ds 
\leq 2KR\alpha^{-1} \sup_{s \geq 0} |\varphi_1(s) - \varphi_2(s)|
\leq \frac{1}{2} \sup_{s \geq 0} |\varphi_1(s) - \varphi_2(s)|.
$$

(11)

As $\varphi^*(t)$ is bounded, we obtain

$$
T_A(t, 0) \left[ \varphi^*(0) 
- \left( \int_{-\infty}^{0} T_A(0, s) P(s) f(s, \varphi^*(s)) \, ds 
- \int_{0}^{+\infty} T_A(0, s) Q(s) f(s, \varphi^*(s)) \, ds \right) \right]
+ \left( \int_{-\infty}^{t} T_A(t, s) P(s) f(s, \varphi^*(s)) \, ds 
- \int_{t}^{+\infty} T_A(t, s) Q(s) f(s, \varphi^*(s)) \, ds \right)
\leq 2KR\alpha^{-1} \sup_{s \geq 0} |\varphi_1(s) - \varphi_2(s)|
\leq \frac{1}{2} \sup_{s \geq 0} |\varphi_1(s) - \varphi_2(s)|.
$$

(12)

is bounded. In addition, the formula above is the solution of system (1), so it is a bounded solution. From Lemma 6, we have

$$
T_A(t, 0) \left[ \varphi^*(0) 
- \left( \int_{-\infty}^{0} T_A(0, s) P(s) f(s, \varphi^*(s)) \, ds 
- \int_{0}^{+\infty} T_A(0, s) Q(s) f(s, \varphi^*(s)) \, ds \right) \right]
= 0.
$$

(13)

Therefore,

$$
\varphi^*(t) = \int_{-\infty}^{t} T_A(t, s) P(s) f(s, \varphi^*(s)) \, ds 
- \int_{-\infty}^{t} T_A(t, s) Q(s) f(s, \varphi^*(s)) \, ds.
$$

(14)
From (3), (H₂), and (H₃), we have
\[ |\varphi_0(t) - \varphi^*(t)| \leq \int_{-\infty}^{t} K e^{-\alpha(t-s)} e^{\varepsilon|s|} \cdot r e^{-\varepsilon|s|} |\varphi_0(s) - \varphi^*(s)| ds + \int_{t}^{\infty} K e^{\alpha(t-s)} e^{\varepsilon|s|} \cdot r e^{-\varepsilon|s|} |\varphi_0(s) - \varphi^*(s)| ds \]
\[ = 2Kr e^{-\alpha t} |\varphi_0(t) - \varphi^*(t)| \]
\[ = \frac{1}{2} \sup_{t \in \mathbb{R}} |\varphi_0(t) - \varphi^*(t)|. \]

That is, \( \sup_{t \in \mathbb{R}} |\varphi_0(t) - \varphi^*(t)| \leq (1/2) \sup_{t \in \mathbb{R}} |\varphi_0(t) - \varphi^*(t)| \), which implies \( \varphi_0(t) = \varphi^*(t) \). Then the uniqueness is proved. The proof of Theorem 3 is complete.

3.2. Proof of Theorem 4. To prove Theorem 4, a standard method is to employ a linear transformation \( x = e^{\varepsilon t} y \). However, \( x = e^{\varepsilon t} y \) is not differentiable at \( t = 0 \). Thus, we cannot use such transformation directly. We have to discuss by dividing into two pieces \( t \geq 0 \) and \( t \leq 0 \).

Consider system
\[ \dot{x}(t) = B(t)v(t) + F(t,v), \]
where \( u \in \mathbb{R}^n \), \( B(t) \) is a continuous matrix function, and \( F : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous function.

**Lemma 7.** Suppose that system \( v(t) = B(t)v(t) \) admits a nonuniform exponential dichotomy; that is, its evolution operator \( T_B(t,s) \) satisfies
\[ |T_B(t,s)P(s)| \leq Ke^{-\lambda(t-s)} e^{\varepsilon|s|}, \quad t \geq s, \]
\[ |T_B(t,s)Q(s)| \leq Ke^{-\lambda(t-s)} e^{\varepsilon|s|}, \quad t \leq s, \]
where \( \lambda \) is a positive constant. In addition,
\[ |F(t,v_1) - F(t,v_2)| \leq \varepsilon e^{-\varepsilon|t|} |v_1 - v_2|, \]
\[ 4Kr < \lambda. \]

If \( |F(t,v)| \leq \mu e^{-\varepsilon|t|} \) for \( t \geq 0 \), then for any \( a \in \mathbb{R}^n \), system (16) has a unique solution \( v^*(t) \) satisfying the following:
(i) \( |v^*(t)| < +\infty \), for \( t \leq 0 \);
(ii) \( Q(0)v^*(0) = Q(0)a \);
(iii) in \( \mathbb{R}^+ \), \( v^*(t) \) satisfies integral equation
\[ v^*(t) = T_B(t,0)P(0)a + \int_{0}^{t} T_B(t,s)P(s)F(s,v^*(s))ds \]
\[ - \int_{t}^{+\infty} T_B(t,s)Q(s)F(s,v^*(s))ds. \]

If \( |F(t,v)| \leq \mu e^{-\varepsilon|t|} \) for \( t \leq 0 \), then for any \( a \in \mathbb{R}^n \), system (16) has a unique solution \( v^*(t) \) satisfying the following:
(i) \( |v^*(t)| < +\infty \), for \( t \leq 0 \);
(ii) \( Q(0)v^*(0) = Q(0)a \);
(iii) in \( \mathbb{R}^+ \), \( v^*(t) \) satisfies integral equation
\[ v^*(t) = T_B(t,0)P(0)a + \int_{0}^{t} T_B(t,s)P(s)F(s,v^*(s))ds \]
\[ - \int_{t}^{+\infty} T_B(t,s)Q(s)F(s,v^*(s))ds. \]
Differentiating the above equality, we see that \( v^+(t) \) satisfies the system (16).

From (22) and (24), we know that \( |v^+(t)| < +\infty \) for \( t \geq 0 \), and it is easy to demonstrate that \( P(0)v^+(0) = P(0)a \). Now we are going to show the uniqueness of \( v^+(t) \). If there is another bounded function \( \bar{v}^+ \) satisfying (i), (ii), and (iii) on \( \mathbb{R}^+ \), in view of (iii) and (18) we have

\[
|v^+(t) - \bar{v}^+(t)| \\
\leq \int_{0}^{t} Ke^{-\lambda(t-s)} e^{\epsilon t} \cdot re^{-\epsilon s} |\bar{v}^+(s) - v^+(s)| ds \\
+ \int_{t}^{\infty} Ke^{\lambda(t-s)} e^{\epsilon s} \cdot re^{-\epsilon s} |\bar{v}^+(s) - v^+(s)| ds \\
\leq 2KrA^{-1} \|\bar{v}^+ - v^+\|. 
\]

which implies \( \|\bar{v}^+ - v^+\| \leq (1/2)\|\bar{v}^+ - v^+\| \). Hence, \( v^+(t) \equiv \bar{v}^+(t) \). The proof of the existence and uniqueness of \( v^-(t) \) is similar to that of \( v^+(t) \). The proof of Lemma 7 is complete. \( \square \)

**Lemma 8.** Suppose that \( \alpha > 0, \delta > 0, C, L, \) and \( M \) are nonnegative constants and that \( v(t) \) is a nonnegative bounded continuous function which satisfies two of the following inequalities:

\[
v(t) \leq Ce^{-\alpha t} + L \int_{0}^{t} e^{-\alpha(t-s)} v(s) ds \\
+ M \int_{t}^{\infty} e^{\delta(t-s)} v(s) ds, \quad (t \geq 0), \\
v(t) \leq Ce^{\alpha t} + L \int_{0}^{t} e^{\alpha(t-s)} v(s) ds \\
+ M \int_{t}^{\infty} e^{-\delta(t-s)} v(s) ds, \quad (t \leq 0). 
\]

In addition, if \( \gamma = L/\alpha + M/\delta < 1 \), then for \( t \geq 0 \) or \( t \leq 0 \), one has

\[
v(t) \leq (1 - \gamma)^{-1} Ce^{-(\alpha / (\alpha - \gamma) - 1)L t}. 
\]

Proof. The proof is straightforward by Lemma 6.2 of Chapter 3 in [6]. \( \square \)

**Lemma 9.** For any \( a \in \mathbb{R}^n \), system (2) has a unique solution \( x^+(t) \) with the following properties:

(i) \( |x^+(t)e^{\epsilon t}| < +\infty \), for \( t \geq 0 \);
(ii) \( P(0)x^+(0) = P(0)a \);
(iii) \( x^+(t) \) on \( \mathbb{R}^+ \) satisfies integral equation

\[
x^+(t) = T_A(t,0)P(0)a + \int_{0}^{t} T_A(t,s)P(s)f(s,x^+(s)) ds \\
- \int_{t}^{\infty} T_A(t,s)Q(s)f(s,x^+(s)) ds. 
\]

Similarly, for any \( a \in \mathbb{R}^n \), system (2) also has a unique solution \( x^-(t) \) with the following properties:

(i) \( |x^-(t)e^{-\epsilon t}| < +\infty \), for \( t \leq 0 \);
(ii) \( Q(0)x^-(-0) = Q(0)a \);
(iii) \( x^-(-t) \) on \( \mathbb{R}^- \) satisfies integral equation

\[
x^-(-t) = T_A(-t,0)Q(0)a + \int_{0}^{-t} T_A(-t,s)Q(s)f(s,x^-(s)) ds \\
- \int_{-t}^{\infty} T_A(-t,s)Q(s)f(s,x^-(s)) ds. 
\]

Let \( T_C(t,s) \) be the evolution operator of the linear system \( v(t) = (A(t) - \epsilon t)v(t) \). Since \( x(t) = T_A(t,0)x(0) \), \( x = e^{-\epsilon t}x \), we have \( T_C(t,s) = e^{-(t-s)}T_A(t,s) \). Hence, from (3), we obtain

\[
|T_C(t,s)P(s)| \leq Ke^{-(\alpha / (\alpha - \gamma) - 1)L s}, \quad \text{for } t \geq s, \\
|T_C(t,s)Q(s)| \leq Ke^{-(\alpha / (\alpha - \gamma) - 1)L s}, \quad \text{for } t \leq s. 
\]

Since \( 4Kr < \alpha - \epsilon \), system (30) satisfies all conditions of Lemma 7. Therefore, for any \( a \in \mathbb{R}^n \), system (30) has a unique solution \( v^+(t) \) with the following properties:

(i) \( |v^+(t)| < +\infty \), for \( t \geq 0 \);
(ii) \( P(0)v^+(0) = P(0)a \);
(iii) \( v^+(t) \) on \( \mathbb{R}^+ \) satisfies integral equation

\[
v^+(t) = T_C(t,0)P(0)a + \int_{0}^{t} T_C(t,s)P(s)f(s,v^+(s)) ds \\
- \int_{t}^{\infty} T_C(t,s)Q(s)f(s,v^+(s)) ds 
\]

\[
= e^{\epsilon t}T_A(t,0)P(0)a + \int_{0}^{t} e^{-\epsilon(t-s)}T_A(t,s)P(s)e^{-\epsilon s}f(s,v^+(s)e^{\epsilon s}) ds \\
- \int_{t}^{\infty} e^{-\epsilon(t-s)}T_A(t,s)Q(s)e^{-\epsilon s}f(s,v^+(s)e^{\epsilon s}) ds. 
\]
Hence,
\[ v^+(t) e^{\varepsilon t} = T_A(t,0) P(0) a + \int_0^t T_A(t,s) P(s) f(s, v^+(s) e^{\varepsilon s}) ds - \int_t^{+\infty} T_A(t,s) Q(s) f(s, v^+(s) e^{\varepsilon s}) ds. \] (34)

Let \( x^+(t) = v^+(t) e^{\varepsilon t} \), we have
\[ x^+(t) = T_A(t,0) P(0) a + \int_0^t T_A(t,s) P(s) f(s, x^+(s)) ds - \int_t^{+\infty} T_A(t,s) Q(s) f(s, x^+(s)) ds. \] (35)

Then \( x^+(t) \) is the solution of system (2) and it satisfies all conditions of Lemma 9.

The proof for the existence and uniqueness of \( x^+(t) \) is similar to that of \( x^+(t) \), so we omit it. This completes the proof of Lemma 9.

**Lemma 10.** If \( |T_A(t,0) P(0) a| \leq Me^{\varepsilon |t|} \), then \( a = 0 \).

**Proof.** If \( a \neq 0 \), then \( P(0) a \neq 0 \) or \( Q(0) a \neq 0 \). Without loss of generality, we assume \( P(0) a \neq 0 \), then we have
\[ |T_A(t,0) P(0) a| = |T_A(t,0) P(0) T_A^{-1}(s,0) T_A(s,0) P(0) a| \]
\[ = |T_A(t,0) T_A(0,s) P(s) T_A(s,0) P(0) a| \]
\[ = |T_A(t,s) P(s) T_A(s,0) P(0) a| \]
\[ \leq |T_A(t,s) P(s)| \cdot |T_A(s,0) P(0) a|. \] (36)

Since \( |T_A(t,s) P(s)| \leq Ke^{-\alpha(t-s)} e^{\varepsilon |t|} \) for \( t \geq s \),
\[ |T_A(t,0) P(0) a| \leq Ke^{-\alpha(t-s)} e^{\varepsilon |t|} \cdot |T_A(s,0) P(0) a|, \] (37)

which implies
\[ |T_A(s,0) P(0) a| \geq \frac{|T_A(t,0) P(0) a|}{Ke^{-\alpha(t-s)} e^{\varepsilon |t|}}. \] (38)

Taking \( t = 0 \), we obtain
\[ |T_A(s,0) P(0) a| \geq \frac{|P(0) a|}{Ke^{-\alpha s} e^{\varepsilon s}} \]
\[ = K^{-1} e^{-(\alpha-\varepsilon s)} |P(0) a| \quad (s \leq 0). \] (39)

Therefore, when \( s \leq 0 \),
\[ |T_A(s,0) P(0) a| e^{\varepsilon |t|} \geq K^{-1} e^{-\alpha s} |P(0) a| \rightarrow +\infty \quad \text{as} \quad s \rightarrow -\infty. \] (40)

On the other hand, when \( s \leq 0 \), from (3), we have
\[ |T_A(s,0) Q(0) a| \leq Ke^{\alpha s} = Ke^{\alpha s}, \] (41)

hence,
\[ \left| \frac{T_A(s,0) Q(0) a}{e^{\varepsilon |t|}} \right| \leq Ke^{(\alpha+\varepsilon)s}. \] (42)

It follows from (40) and (42) that
\[ \left| \frac{T_A(s,0) a}{e^{\varepsilon |t|}} \right| = \left| \frac{T_A(s,0) (P(s) + Q(s)) a}{e^{\varepsilon |t|}} \right| \]
\[ \geq \frac{|T_A(s,0) P(s) a|}{e^{\varepsilon |t|}} - \frac{|T_A(s,0) Q(s) a|}{e^{\varepsilon |t|}} \]
\[ \geq \frac{|T_A(s,0) P(s) a|}{e^{\varepsilon |t|}} - Ke^{(\alpha+\varepsilon)s}. \] (43)

From the above inequality, we know that \( |T_A(s,0) a|/e^{\varepsilon |t|} \rightarrow +\infty \) as \( s \rightarrow -\infty \), which contradicts the original condition \( |T_A(s,0) a|/e^{\varepsilon |t|} \leq M \) and it implies \( a = 0 \). This ends the proof of Lemma 10.

**Proof of Theorem 4.** For any solution \( x(t) \) of system (2), it can be written as follows:
\[ x(t) = T_A(t,0) x(0) + \int_0^t T_A(t,s) T_A^{-1}(s,s) f(s, x(s)) ds + \int_0^t T_A(t,s) P(s) f(s, x(s)) ds - \int_0^t T_A(t,s) Q(s) f(s, x(s)) ds. \]
(44)
Let $\xi(t)$ be any $n$-variable continuous function defined on $\mathbb{R}$. From (3) and (H₁), we have

$$
\left| \int_{-\infty}^{t} T_A(t, s) P(s) f(s, \xi(s)) \, ds \right| \leq \int_{-\infty}^{t} K e^{\alpha(t-s)} e^{\mu |s|} \, ds = K \mu e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha s} e^{\mu |s|} \, ds.
$$

(45)

For $t \geq 0$,

$$
e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha s} e^{\mu |s|} \, ds = e^{-\alpha t} \left( \int_{-\infty}^{0} e^{\alpha s} e^{-\alpha s} \, ds + \int_{0}^{t} e^{\alpha s} e^{\mu |s|} \, ds \right)
= e^{-\alpha t} \left( \frac{1}{\alpha - \varepsilon} + \frac{1}{\alpha + \varepsilon} (e^{(\alpha+\varepsilon)t} - 1) \right)
\leq \frac{1}{\alpha - \varepsilon} e^{-\alpha t} + \frac{1}{\alpha + \varepsilon}.
$$

(46)

for $t \leq 0$,

$$
e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha s} e^{\mu |s|} \, ds = e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha s} e^{\mu |s|} \, ds
\leq \frac{1}{\alpha - \varepsilon} e^{-\alpha t}.
$$

(47)

Hence, for any $t \in \mathbb{R}$, we have

$$
e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha s} e^{\mu |s|} \, ds \leq \frac{1}{\alpha - \varepsilon} e^{-\alpha t} + \frac{1}{\alpha + \varepsilon}.
$$

(48)

Therefore, for any $t \in \mathbb{R}$, we have

$$
\left| \int_{-\infty}^{t} T_A(t, s) P(s) f(s, \xi(s)) \, ds \right| \leq \frac{K \mu}{\alpha - \varepsilon} e^{-\alpha t} + \frac{K \mu}{\alpha + \varepsilon}.
$$

(49)

By the same calculation, for any $t \in \mathbb{R}$, we have

$$
\left| \int_{t}^{+\infty} T_A(t, s) Q(s) f(s, \xi(s)) \, ds \right| \leq \frac{K \mu}{\alpha - \varepsilon} e^{-\alpha t} + \frac{K \mu}{\alpha + \varepsilon}.
$$

(50)

In Lemma 9, $x^+(t)$ and $x^-(t)$ are uniquely determined by $a$; we denote them by $x^+(t, a)$ and $x^-(t, a)$, respectively. Let $\rho = (2K\mu)/(\alpha - \varepsilon)$, denote by $G$ the closed sphere on $\mathbb{R}^n$ whose center is at the origin of the coordinate system and whose radius is $\rho$. For any $a \in G$, we define a mapping $T_2 : G \to \mathbb{R}^n$ as follows:

$$
T_2 a = \int_{0}^{t} T_A(0, s) P(s) f(s, x^-(s, a)) \, ds - \int_{0}^{+\infty} T_A(0, s) Q(s) f(s, x^+(s, a)) \, ds.
$$

(51)

It follows from (3) and (H₁) that

$$
|T_2 a| \leq \int_{-\infty}^{0} K e^{\alpha t} e^{\mu |s|} \, ds + \int_{0}^{+\infty} K e^{-\alpha t} e^{\mu |s|} \, ds
= \frac{K \mu}{\alpha - \varepsilon} + \frac{K \mu}{\alpha + \varepsilon}
= \frac{2K \mu}{\alpha - \varepsilon} = \rho,
$$

(52)

which implies that $T_2$ maps $G$ onto itself. Now we are going to show that $T_2$ is continuous. For any $a_1, a_2 \in G$, from (3) and (H₂), we have

$$
|T_2 a_1 - T_2 a_2| \leq \int_{-\infty}^{0} K e^{\alpha t} e^{\mu |s|} \cdot r e^{-\mu |s|} \left| x^-(s, a_1) - x^-(s, a_2) \right| \, ds
+ \int_{0}^{+\infty} K e^{-\alpha t} e^{\mu |s|} \cdot r e^{-\mu |s|} \left| x^+(s, a_1) - x^+(s, a_2) \right| \, ds
\leq \int_{-\infty}^{0} K r e^{\alpha t} \left| x^-(s, a_1) - x^-(s, a_2) \right| \, ds
+ \int_{0}^{+\infty} K r e^{-\alpha t} \left| x^+(s, a_1) - x^+(s, a_2) \right| \, ds.
$$

(53)

From (3) and the condition (iii) of Lemma 9, for $t \geq 0$, we have

$$
\left| x^+(t, a_1) - x^+(t, a_2) \right|
\leq T_A(t, 0) P(0) \left| a_1 - a_2 \right|
+ \int_{0}^{t} K e^{-\alpha(t-s)} e^{\mu |s|} \cdot r e^{-\mu |s|} \left| x^+(s, a_1) - x^+(s, a_2) \right| \, ds
+ \int_{t}^{+\infty} K e^{\alpha(t-s)} e^{\mu |s|} \cdot r e^{-\mu |s|} \left| x^+(s, a_1) - x^+(s, a_2) \right| \, ds
\leq K e^{-\alpha t} \left| a_1 - a_2 \right|
+ \int_{0}^{t} K r e^{-\alpha(t-s)} \left| x^+(s, a_1) - x^+(s, a_2) \right| \, ds
+ \int_{t}^{+\infty} K r e^{\alpha(t-s)} \left| x^+(s, a_1) - x^+(s, a_2) \right| \, ds.
$$

(54)

Multiplying by $e^{-\alpha t}$ on both sides of the above inequality, for $t \geq 0$, we get

$$
e^{-\alpha t} \left| x^+(t, a_1) - x^+(t, a_2) \right|
\leq K e^{-\alpha t(t-s)} \left| a_1 - a_2 \right|
+ \int_{0}^{t} K r e^{-\alpha(t-s)} e^{-\mu |s|} \left| x^+(s, a_1) - x^+(s, a_2) \right| \, ds
+ \int_{t}^{+\infty} K r e^{\alpha(t-s)} e^{\mu |s|} \left| x^+(s, a_1) - x^+(s, a_2) \right| \, ds.
$$

(55)
By Lemma 9, for \( t \geq 0 \), \( e^{-\alpha t} |\chi^+(t, a_1) - \chi^+(t, a_2)| \) is a bounded function. And it follows from Lemma 8 that
\[
e^{-\alpha t} |\chi^+(t, a_1) - \chi^+(t, a_2)| \leq K |a_1 - a_2| e^{-[(\alpha + \varepsilon) - 1] \gamma t},
\]
where \( \gamma = K r/(\alpha + \varepsilon) + K r/(\alpha - \varepsilon) \). From (H3) and \( (\alpha + \varepsilon)^{-1} < (\alpha - \varepsilon)^{-1} \), we get
\[
\gamma \leq 2 K r (\alpha - \varepsilon)^{-1} < \frac{1}{2}.
\]
Therefore, for \( \alpha - 2 Kr > 0 \), we have
\[
e^{-\alpha t} |\chi^+(t, a_1) - \chi^+(t, a_2)| \leq 2 K |a_1 - a_2| e^{-[(\alpha - 2 K r) t] t} (t \geq 0).
\]
Hence,
\[
|\chi^+(t, a_1) - \chi^+(t, a_2)| \leq 2 K |a_1 - a_2| e^{-[(\alpha - 2 K r) t]} (t \geq 0).
\]
Similarly,
\[
|\chi^-(t, a_1) - \chi^-(t, a_2)| \leq 2 K |a_1 - a_2| e^{[(\alpha - 2 K r) t]} (t \leq 0).
\]
So from (53), it follows that
\[
|\mathcal{T}_2 a_1 - \mathcal{T}_2 a_2| \leq \int_{-\infty}^{0} K r e^{\alpha s} \cdot 2 K |a_1 - a_2| e^{-[(\alpha - 2 K r) s]} ds
\]
\[+ \int_{0}^{\infty} K r e^{-\alpha s} \cdot 2 K |a_1 - a_2| e^{[(\alpha - 2 K r) s]} ds \leq \int_{-\infty}^{0} 2 K^2 r |a_1 - a_2| e^{2[(\alpha - K r) s]} ds
\]
\[+ \int_{0}^{\infty} 2 K^2 r |a_1 - a_2| e^{-2[(\alpha - K r) s]} ds = \frac{2 K^2 r}{\alpha - K r} |a_1 - a_2|,
\]
which show that \( \mathcal{T}_2 \) is a continuous mapping. By fixed point theorem, \( \mathcal{T}_2 \) has at least one fixed point on \( G \). We denote this fixed point by \( a_0 \), then
\[
a_0 = \mathcal{T}_2 a_0 = \int_{-\infty}^{0} T_A(0, s) P(s) f(s, \chi^-(s, a_0)) ds - \int_{0}^{+\infty} T_A(0, s) Q(s) f(s, \chi^+(s, a_0)) ds.
\]
As \( P^2(s) = P(s), P(t)T_A(0, t) = T_A(t, s) P(s), P(s) + Q(s) = 1d \), we obtain
\[
P(0) a_0 = \int_{-\infty}^{0} T_A(0, s) P(s) f(s, \chi^-(s, a_0)) ds,
\]
\[
Q(0) a_0 = -\int_{0}^{+\infty} T_A(0, s) Q(s) f(s, \chi^+(s, a_0)) ds.
\]
From Lemma 9, we have
\[
\chi^+(0, a_0) = P(0) a_0 - \int_{0}^{+\infty} T_A(0, s) Q(s) f(s, \chi^+(s, a_0)) ds,
\]
\[
\chi^-(0, a_0) = Q(0) a_0 + \int_{-\infty}^{0} T_A(0, s) P(s) f(s, \chi^-(s, a_0)) ds.
\]
Hence,
\[
\chi^+(0, a_0) = \chi^-(0, a_0) = a_0.
\]
By the existence and uniqueness of the initial value problem, we conclude that \( \chi^+(t, a_0) = \chi^-(t, a_0) \). We can denote it by \( x_0(t) \). Hence,
\[
x_0(0) = a_0
\]
\[= \int_{-\infty}^{0} T_A(0, s) P(s) f(s, x(s, a_0)) ds - \int_{0}^{+\infty} T_A(0, s) Q(s) f(s, x(s, a_0)) ds.
\]
From the above equation, it follows from (44) that
\[
x_0(t) = \int_{-\infty}^{t} T_A(t, s) P(s) f(s, x(s)) ds - \int_{t}^{+\infty} T_A(t, s) Q(s) f(s, x(s)) ds.
\]
From (49) and (50), we have
\[
|x_0(t)| \leq \frac{2 K \mu}{\alpha - \varepsilon} e^{\varepsilon |t|} + \frac{2 K \mu}{\alpha - \varepsilon}, (-\infty < t < +\infty),
\]
which implies that \( x_0(t) \) satisfies (5); that is, \( x_0(t) = O(e^{\varepsilon |t|}) \).

Now we are going to prove that the solution of (2) which satisfies (5) is unique. We assume that system (2) has another solution \( x_0^*(t) \) satisfying (5). From (44), \( x_0^*(t) \) can be written as
\[
x_0^*(t) = T_A(t, 0) \left[ x_0^*(0) - \left( \int_{-\infty}^{0} T_A(0, s) P(s) f(s, x_0^*(s)) ds - \int_{0}^{+\infty} T_A(0, s) Q(s) f(s, x_0^*(s)) ds \right) \right]
\]
\[+ \left( \int_{-\infty}^{t} T_A(t, s) P(s) f(s, x_0^*(s)) ds - \int_{t}^{+\infty} T_A(t, s) Q(s) f(s, x_0^*(s)) ds \right).
\]
From (49) and (50), we get
\[
\left\| \int_{-\infty}^{t} T_A(t,s) P(s) f(s,x^*_0(s)) ds \right\| \leq \frac{K\mu}{\alpha - \epsilon} e^{\epsilon |s|} + \frac{K\mu}{\alpha - \epsilon},
\]
\[
\left\| \int_{t}^{+\infty} T_A(t,s) Q(s) f(s,x^*_0(s)) ds \right\| \leq \frac{K\mu}{\alpha - \epsilon} e^{\epsilon |s|} + \frac{K\mu}{\alpha - \epsilon}.
\]
(70)

It follows from \(|x^*_0(t)| = O(e^{\epsilon |t|})\) and Lemma 10 that
\[
x^*_0 - \left( \int_{-\infty}^{0} T_A(0,s) P(s) f(s,x^*_0(s)) ds - \int_{0}^{+\infty} T_A(0,s) Q(s) f(s,x^*_0(s)) ds \right) = 0.
\]
(71)

Therefore,
\[
x^*_0(t) = \int_{-\infty}^{t} T_A(t,s) P(s) f(s,x^*_0(s)) ds - \int_{t}^{+\infty} T_A(t,s) Q(s) f(s,x^*_0(s)) ds.
\]
(72)

From (67), (72), and (H2), we have
\[
| x_0(t) - x^*_0(t) | \leq \int_{-\infty}^{t} Ke^{-(s-t)} e^{\epsilon |s|} | x_0(s) - x^*_0(s) | ds
+ \int_{t}^{+\infty} Ke^{(s-t)} e^{\epsilon |s|} | x_0(s) - x^*_0(s) | ds
\]
(73)
\[
= \int_{-\infty}^{t} K e^{-(s-t)} | x_0(s) - x^*_0(s) | ds
+ \int_{t}^{+\infty} K e^{(s-t)} | x_0(s) - x^*_0(s) | ds.
\]

Let \( L = \sup_{t \in \mathbb{R}} e^{-\epsilon |t|} | x_0(t) - x^*_0(t) | \). Since \( x_0(t) \) and \( x^*_0(t) \) satisfy (5), \( e^{-\epsilon |t|} | x_0(s) - x^*_0(s) | \) is bounded. Thus, for \( t \geq 0 \), we have
\[
e^{-\epsilon |t|} | x_0(t) - x^*_0(t) |
\]
\[
e^{-\epsilon t} | x_0(t) - x^*_0(t) |
\]
\[
\leq \int_{-\infty}^{t} Ke^{-(s-t)} e^{\epsilon |s|} | x_0(s) - x^*_0(s) | ds
+ \int_{t}^{+\infty} Ke^{(s-t)} e^{\epsilon |s|} | x_0(s) - x^*_0(s) | ds
\]
\[
= \int_{-\infty}^{t} Ke^{-(s-t)} | x_0(s) - x^*_0(s) | ds
+ \int_{t}^{+\infty} Ke^{(s-t)} | x_0(s) - x^*_0(s) | ds.
\]
(74)

Similarly, we can prove that
\[
e^{-\epsilon |t|} | x_0(t) - x^*_0(t) | \leq \frac{3}{4} L, \quad \text{when } t \leq 0.
\]
(75)

Therefore,
\[
L = \sup_{t \in \mathbb{R}} e^{-\epsilon |t|} | x_0(t) - x^*_0(t) | \leq \frac{3}{4} L, \quad (-\infty < t + \infty).
\]
(76)

That is, \( L \leq (3/4)L \), which implies \( L = 0 \). Consequently, \( x_0(t) = x^*_0(t) \). This completes the proof of Theorem 4. \( \Box \)

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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