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Robust H_∞ receding horizon control for a class of coordinated control problems involving dynamically decoupled subsystems

Ajay Gautam, Yun-Chung Chu, and Yeng Chai Soh*

Abstract—This paper presents a robust receding-horizon-control (RHC)-based scheme for a class of coordinated control problems involving dynamically decoupled subsystems which are required to reach a consensus condition in some optimal way. A general case of constrained subsystems having possibly uncertain time-varying dynamics in addition to external disturbances is considered and a suitable H_∞ -based near-consensus condition is defined as the target condition to be achieved. The proposed scheme employs computationally efficient subsystem-level RHC policies in the H_∞ -based minmax-cost framework together with a distributed subgradient-based method to optimize and update the consensus signal and the subsystem control inputs at regular intervals. Furthermore, the proposed framework allows the incorporation of computational delays in the RHC policy formulation so that the desired control performance is always guaranteed. The performance of the proposed control scheme is illustrated with some simulation examples.

Index Terms—Coordinated control, consensus control, synchronization, receding horizon control (RHC), robust RHC, MPC, uncertain systems

I. INTRODUCTION

In recent years, there have been considerable research interests in the area of cooperative behavior in networked systems. One important problem in this area is that of the coordinated control of the subsystems, called agents, of a large multi-agent system in such a way that the subsystems reach an agreement or consensus in a certain quantity of interest in order to achieve a shared goal [1]. Such a problem arises in applications including those related to multi-vehicle systems, such as formation flight [2], rendezvous in space [3], cooperative air traffic control [4], vehicle platooning in automated highways [5] and so on. This paper deals with consensus-related coordinated control problems in which the subsystems have decoupled but time-varying and possibly uncertain dynamics affected by external bounded disturbances, and are constrained in their states and inputs.

There are extensive existing works on consensus-related coordinated control problems (see, e.g., [1], [6] for detailed surveys). Many of the recent papers employ algorithms that are inspired by natural models of consensus such as those found in schools of fish and flocks of birds (e.g., [7], [8]). In their basic forms, these algorithms are simple, decentralized interaction

rules for agents with one-dimensional, first-order dynamics, interacting with neighboring members. Research results on the area have focussed on analyzing the convergence characteristics of the consensus rules under various inter-subsystem communications conditions (see, e.g., [3], [9]–[11]), or on synthesizing appropriate rules for specific conditions (see, e.g., [12], [13]). Investigations have also been made on algorithms that deal with subsystems with higher order dynamics such as double integrator dynamics (see, e.g., [14], [6]) or more general multi-input multi-output dynamics (e.g., [15]–[21]). In [15] and [16], the stability of multi-vehicle formations is analyzed using tools from graph/matrix and control theories. [17] has considered a convex synthesis for consensus in a more general setting in which the subsystems may be dynamically coupled. In [18], the authors have considered uncertain subsystem dynamics and explored conditions for consensus with a desired H_∞ performance under switching communications and time delay. Papers [19]–[24] explore results on convergence analysis and/or control synthesis for problems dealing with a time-varying consensus trajectory, sometimes referred to as synchronization problems, under various conditions of state measurement and information exchange.

On-line-optimization-based control approaches, popularly known as receding horizon control (RHC) or model predictive control (MPC) techniques [25], provide an alternative to biologically inspired algorithms for consensus-related coordinated control problems. These approaches allow the optimization of the subsystem control inputs for a given objective while systematically incorporating the state and input constraints in the optimization process - something that traditional consensus algorithms do not readily offer. This can be particularly important if the subsystems need to satisfy hard constraints on their states and inputs and an optimality in terms of the overall cost to reach the consensus condition is desired. MPC-based approaches have been employed in the context of distributed control of systems with dynamically decoupled subsystems in several papers such as [26]–[28]. In [26], subsystems are coupled in the cost function alone and the stability is achieved with parallel local RHC computations by adding a move suppression term to each local cost function. The authors in [28] have considered subsystems with coupling constraints and used a scheme that solves local robust RHC optimization subproblems sequentially, ensuring the future feasibility of the overall control problem. Subsystems with coupled constraints and cost functions have been considered in [27] in which local cost minimizations are carried out in parallel while ensuring

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that the local cost functions behave as Lyapunov functions. Consensus-related coordinated control problems have also been explored in the RHC framework in works such as [29]–[33]. In [29], a predictive mechanism with unconstrained inputs has been used for improved consensus in having a collective behavior among agents. The authors in [30] have used the geometric properties of the optimal path to prove the asymptotic convergence of an RHC-based consensus algorithm for systems with single or double integrator dynamics. In [31], a sequential optimization approach has been proposed to solve consensus and synchronization problems in a distributed RHC framework for linear and nonlinear systems. In [32] and [33], the authors have presented a negotiation algorithm to directly optimize the consensus signal in an MPC setting employing a distributed subgradient approach. The latter paper has also extended the results to an RHC implementation and analyzed the stability properties under such an implementation. In these papers, the authors have considered solutions based on the standard finite-horizon RHC optimization for subsystems with time-invariant dynamics and without uncertainties and disturbances. In practical situations, uncertainties and disturbances are often present and a solution based on the standard RHC optimization may not be reliable and robust.

The purpose of this paper is to present a robust framework for an efficient RHC-based solution of consensus/synchronization problems involving subsystems that have multi-dimensional, and possibly uncertain, time-varying dynamics including unknown additive disturbances. For this class of subsystem dynamics, the presence of possibly non-vanishing disturbances makes an exact consensus condition difficult to achieve, and the presence of hard constraints on the subsystems' states and inputs brings additional challenges that cannot be efficiently handled with the traditional finite-horizon RHC laws. We, therefore, propose a suitably defined H_∞ -based near-consensus condition as the target condition, and use a recently developed, uncertainty-based, closed-loop RHC formulation [34] in the H_∞ -based worst-case cost framework to achieve such a condition. With this formulation, which uses off-line-optimized dynamic control policies for the subsystems, the otherwise complex overall minmax optimization problem of minimizing, on-line, the worst-case cost to reach the near-consensus condition can be decomposed into simpler convex optimization subproblems with the consensus signal as the coupling variable, and can be solved in a decentralized way using the subgradient method [35] such as in [32]. We propose to employ an approximate subgradient method based on dual decomposition to optimize the consensus trajectory and present an overall control scheme which allows us to update the consensus trajectory in real-time, albeit possibly at a frequency different from that of the computation and application of the control inputs in the subsystems. The overall scheme ensures that the on-line-optimized consensus signal eventually converges to either a constant value or a periodic trajectory, and hence, the subsystems reach a near-consensus condition described by an H_∞ -like performance relating the offset of the subsystem states from an ideal consensus condition to the disturbances affecting the subsystems.

This paper also explores a way to appropriately handle

computational delays in the distributed optimization of the consensus signal in real time. Essentially, depending on the expected computational delay, the dynamics of the local control policies are structured in such a way that the predicted admissible control actions can be implemented when the consensus signal is being determined. Since the local dynamic control policies are optimized off-line such that they guarantee the desired individual control performance, the overall control scheme ensures that the desired H_∞ -based near-consensus condition is achieved in an expected way.

The rest of the paper is organized as follows. We give a description of the control problem in Section II. In Section III, we discuss the uncertainty-based dynamic control policies which we employ as the local control policies in the subsystems. We also present the results on the subgradient-based distributed optimization of the consensus signal and the details of the algorithm for overall control in this section. In Section IV, we discuss the handling of computational delays in the distributed computation of the consensus signal. Finally, we end the paper with some concluding remarks in Section V.

Notations: The signs $\succ, \preceq$ etc. denote positive/negative (semi-) definiteness of matrices. I (I_n) denotes an identity matrix (of size $n \times n$) and $\mathbf{1}$ denotes a vector of all ones. For a signal $x(t)$, $x(t+i|t)$ represents the value of $x(t+i)$ predicted at time t . For a vector x , $x_{[i]}$ denotes its i th component. For a matrix X , $X_{[i,:]}$ represents its i th row and $X_{[i,j]}$ denotes the element on its i th row and j th column. $\|X\|$ represents the spectral norm of X . $X \otimes Y$ represents the Kronecker product of X and Y , and $\text{diag}(X, Y)$ denotes a block diagonal matrix with blocks X and Y . In a symmetric matrix, $*$ represents the transpose of the diagonally opposite element. $\mathbb{Z}_+$ and $\mathbb{R}_+$ represent the sets of non-negative integers and real numbers respectively. $\mathbb{N}_n$ represents the set $\{1, 2, \dots, n\}$. $\oplus$ denotes set addition and $\ominus$ denotes set difference. Co denotes convex hull.

II. PROBLEM DESCRIPTION

We consider a system comprising N dynamically decoupled subsystems or agents, labeled $j = 1, 2, \dots, N$, described by

$$x_j(t+1) = A_j(t)x_j(t) + B_j(t)u_j(t) + D_j(t)w_j(t), \quad (1)$$

where $x_j(t) \in \mathbb{R}^{n_{xj}}$ is the state of subsystem j , $u_j(t) \in \mathbb{R}^{n_{uj}}$ is the control input applied to it and $w_j(t) \in \mathbb{R}^{n_{wj}}$ is the external disturbance affecting it. Each subsystem $j \in \mathbb{N}_N$ is supposed to satisfy the following assumptions:

A1: The disturbance $w_j(t)$ and the time-varying matrices $\varphi_j(t) = [A_j(t) \ B_j(t) \ D_j(t)]$ lie in polytopic sets $\mathbb{W}_j$ and Ω_j respectively, i.e., $\forall t \in \mathbb{Z}_+$,

$$w_j(t) \in \mathbb{W}_j = \text{Co}\{\varphi_j^{(1)}, \dots, \varphi_j^{(\nu_j)}\}, \quad (2)$$

$$\varphi_j(t) = \sum_{r=1}^{h_j} \theta_{j[r]}(t) \varphi_j^{(r)} \in \Omega_j = \text{Co}\{\varphi_j^{(1)}, \dots, \varphi_j^{(h_j)}\}, \quad (3)$$

where $\varphi_j^{(r)} = [A_j^{(r)} \ B_j^{(r)} \ D_j^{(r)}]$, $r = 1, \dots, h_j$ and $\theta_j(t) \in \mathbb{R}_+^{h_j}$ is such that $\theta_j(t)^T \mathbf{1} = 1$.

A2: The state and the input are constrained to lie in bounded polyhedra containing the origin in their interior, i.e.,

$$\begin{aligned} x_j(t) &\in \mathbb{X}_j = \{x \mid M_{x_j} x \leq \mathbf{1}\} \\ u_j(t) &\in \mathbb{U}_j = \{u \mid M_{u_j} u \leq \mathbf{1}\} \end{aligned} \quad (4)$$

where $M_{x_j} \in \mathbb{R}^{m_{x_j} \times n_{x_j}}$ and $M_{u_j} \in \mathbb{R}^{m_{u_j} \times n_{u_j}}$.

A3: The time-varying subsystem (1) is stabilizable via linear state feedback.

We are interested in the control of the overall system in such a way that the subsystems ultimately reach a ‘consensus condition’ in an optimal way. The consensus condition is expressed in terms of the ‘consensus components’ of subsystem states, namely, outputs $s_j(t) = C_j x_j(t) \in \mathbb{R}^{n_\sigma}$, $j = 1, \dots, N$, where C_j , $j = 1, \dots, N$ are suitably-defined full-row-rank matrices.

Definition 1 (Ideal asymptotic consensus). *The subsystems in (1) are said to asymptotically reach consensus as time $t \rightarrow \infty$ if their outputs satisfy*

$$\lim_{t \rightarrow \infty} (s_j(t) - s_k(t)) = 0, \quad \forall j, k \in \mathbb{N}_N. \quad (5)$$

The common signal $\sigma(t) \in \mathbb{R}^{n_\sigma}$ satisfying $\lim_{t \rightarrow \infty} (s_j(t) - \sigma(t)) = 0, \forall j \in \mathbb{N}_N$ is then referred to as the consensus signal.

It is generally not possible for the subsystems to achieve the ideal condition (5) in the presence of possibly unknown disturbances (and possibly uncertain subsystem dynamics). We therefore look for a situation where the condition is achieved by some appropriate nominal dynamics of the subsystems. We make the following assumptions in this regard.

A4: There exist matrices $S_j \in \mathbb{R}^{n_{x_j} \times n_\sigma}$, $U_j \in \mathbb{R}^{n_{u_j} \times n_\sigma}$, $j = 1, \dots, N$, and matrices $A_\zeta(t) \in \mathbb{R}^{n_\sigma \times n_\sigma}$ and $C_\zeta \in \mathbb{R}^{n_\sigma \times n_\sigma}$ such that, for each $j \in \mathbb{N}_N$,

$$\bar{A}_j(t)S_j + \bar{B}_j(t)U_j = S_j A_\zeta(t), \quad C_j S_j = C_\zeta \quad (6)$$

where matrices $\bar{A}_j(t)$ and $\bar{B}_j(t)$ define an appropriate nominal or desired dynamics model of subsystem j and satisfy $[\bar{A}_j(t) \quad \bar{B}_j(t)] = \sum_{r=1}^{h_j} \theta_{j[r]}(t) [\bar{A}_j^{(r)} \quad \bar{B}_j^{(r)}]$ for some $[\bar{A}_j^{(r)} \quad \bar{B}_j^{(r)}]$, $r = 1, \dots, h_j$.

A5: (a) Matrix $A_\zeta(t)$ is either periodic with a period $T > 1$ or constant (i.e., of period $T = 1$). All eigenvalues of the monodromy matrix $\bar{A}_\zeta = A_\zeta(t+T-1)A_\zeta(t+T-2)\dots A_\zeta(t)$ lie on the unit circle and are simple.

(b) The eigenvalues of $\bar{A}_\zeta$ are of the form $e^{\sqrt{-1} \frac{p_i}{q_i} 2\pi}$, $i = 1, \dots, n_\sigma$, where p_i, q_i are integers.

Here, assumption **A4**, together with **A3**, ensures that the output $C_j \bar{x}_j$ of the nominal dynamics model $\bar{x}_j(t+1) = \bar{A}_j(t)\bar{x}_j(t) + \bar{B}_j(t)\bar{u}_j(t)$ of each subsystem $j \in \mathbb{N}_N$ can be made, through some local linear state feedback control, to asymptotically track the output $C_\zeta \bar{\zeta}(t)$ of a common exogenous system, which we call the consensus exosystem, defined by the dynamics

$$\bar{\zeta}(t+1) = A_\zeta(t)\bar{\zeta}(t), \quad (7)$$

where $\bar{\zeta}(t) \in \mathbb{R}^{n_\sigma}$. Conditions in (6) are in the form of regulator equations [36], and when the matrices $A_\zeta(t)$, $\bar{A}_j(t)$ and $\bar{B}_j(t)$ are all time-invariant, they resemble the standard regulator equations that are necessary and sufficient for the solvability of the linear output regulation problem in which the output of a linear system is required to track an output of an exosystem. The solvability of an output consensus/synchronization problem involving dynamically decoupled subsystems, under necessary assumptions on inter-subsystem communications, closely follows from the ability of each subsystem to track the output of a common exosystem, and, it has been shown

that these equations are indeed necessary and sufficient for the solvability of such a problem with linear feedback control inputs in the case of subsystems with time-invariant dynamics [24]. The sufficiency of the conditions in (6) in the case of time-varying nominal subsystem dynamics follows in a straightforward way. We also assume that the exosystem defined by the pair $(A_\zeta(t), C_\zeta)$ in (6) is observable, with its dimension typically satisfying $n_\sigma \leq n_\zeta \leq \min(n_{x_1}, n_{x_2}, n_{x_N})$. Next, assumption **A5(a)** implies that all modes of the exosystem dynamics are marginally stable. This is justified since, on the one hand, the fact that subsystem states and inputs are assumed to be bounded necessitates that the consensus signal remains bounded, and, on the other, as we envisage an asymptotic consensus with a non-trivial (non-zero) $\sigma(t)$, it makes no sense to have some components of the exosystem state tending to zero as $t \rightarrow \infty$. Part (b) of **A5** is motivated by certain convenient features it allows, such as the existence of a polyhedral invariant set for system $\phi(t+1) = \bar{A}_\zeta \phi(t)$ [37, Chap. 4].

Assumptions **A4-A5** are general in the sense that they can be satisfied for a broad range of consensus-related problems dealing with dynamically decoupled linear subsystems. Apart from the typical consensus problems involving subsystems with single or double integrator dynamics and the common synchronization problems involving linear oscillators, several related problems such as those arising in satellite formation flying (see, examples at the end of Section III) readily satisfy the assumptions with $\bar{A}_j(t) = A_j(t)$ and $U_j = 0$. When the time-varying description of subsystem dynamics in (1) is intended to represent model uncertainties, suitable constant nominal matrices (such as those lying at the center of Ω_j) can be chosen to satisfy **A4** with a constant $A_\zeta(t) = A_\zeta$. When the time-varying dynamics are the same for all subsystems, we may choose a similarly time-varying $A_\zeta(t)$ to represent the dynamics of the consensus exosystem.

Now, before proceeding to state the overall control problem, we note that the satisfaction of equations in (6) implies that, for each subsystem $j \in \mathbb{N}_N$, there exists some linear state feedback controller which ensures that, in the ideal situation where $A_j(t) = \bar{A}_j(t)$, $B_j(t) = \bar{B}_j(t)$ and $w_j(t) = 0, \forall t \in \mathbb{Z}_+$, we have $\lim_{t \rightarrow \infty} (x_j(t) - S_j \bar{\zeta}(t)) \rightarrow 0$ and $\lim_{t \rightarrow \infty} (u_j(t) - U_j \bar{\zeta}(t)) \rightarrow 0$ for some $\bar{\zeta}(t)$ satisfying (7) [24], [38]. The overall control objective in an ideal situation would be to asymptotically drive the state of each subsystem $j \in \mathbb{N}_N$ to some $S_j \varsigma(t)$, and hence its output to $\sigma(t) = C_\zeta \varsigma(t)$, where $\varsigma(t)$ eventually follows the dynamics in (7) but is not known *a-priori*, in some optimal way, typically with inputs obtained by minimizing the overall predicted quadratic cost of the form

$$\sum_{j=1}^N \sum_{i=0}^{\infty} \left\{ \|Q_j^{\frac{1}{2}} (x_j(t+i|t) - S_j \varsigma(t+i|t))\|^2 + \|R_j^{\frac{1}{2}} (u_j(t+i|t) - U_j \varsigma(t+i|t))\|^2 \right\} \quad (8)$$

with $Q_j \succ 0$ and $R_j \succ 0$, $j = 1, \dots, N$. Since we also wish to ensure that the consensus condition is maintained in the steady state with as small control effort as possible, we assume that matrices S_j and U_j satisfying (6), if they are non-unique, are determined such that the matrix $U_j^T R U_j$

is minimized in some sense (e.g., by minimizing its trace or determinant). Now, in an actual situation, we need to take into account the presence of disturbances and non-ideal subsystem dynamics while specifying the control objective and the cost function. To elaborate, for each subsystem $j \in \mathbb{N}_N$, let $\tilde{x}_j(t) = x_j(t) - S_j \varsigma(t)$ and $\tilde{u}_j(t) = u_j(t) - U_j \varsigma(t)$, and let $\tilde{D}_j^\Delta(t)$ denote the quantity $(A_j(t) - \bar{A}_j(t))S_j + (B_j(t) - \bar{B}_j(t))U_j$. Then, defining $\Delta\varsigma(t) = \varsigma(t+1) - A_\varsigma(t)\varsigma(t)$, the dynamics of the error-state $\tilde{x}_j(t)$, that directly follow from (1) and (6), can be expressed as

$$\tilde{x}_j(t+1) = A_j(t)\tilde{x}_j(t) + B_j(t)\tilde{u}_j(t) + \tilde{D}_j(t)\tilde{w}_j(t) - S_j\Delta\varsigma(t) \quad (9)$$

where $\tilde{D}_j(t)$ and $\tilde{w}_j(t)$ are the same as $D_j(t)$ and $w_j(t)$ respectively if $D_j^\Delta(t) = 0$, $\forall t \in \mathbb{Z}_+$ but are otherwise defined as $\tilde{D}_j(t) = [D_j(t) \quad D_j^\Delta(t)\Upsilon_j^\dagger]$ and $\tilde{w}(t) = [w_j(t)^\top \quad (\Upsilon_j^\dagger \varsigma(t))^\top]^\top$ with some appropriate scaling matrices $\Upsilon_j^\dagger$ and Υ_j such that $D_j^\Delta(t)\Upsilon_j^\dagger\Upsilon_j\varsigma(t) = D_j^\Delta(t)\varsigma(t)$ for all $t \in \mathbb{Z}_+$. Clearly, a reasonable objective in a non-ideal situation will be to drive the error-state $\tilde{x}_j(t)$ of each subsystem $j \in \mathbb{N}_N$ close to the origin in some sense while ensuring that $\Delta\varsigma(t)$ eventually tends to zero (and hence $\varsigma(t)$ follows the dynamics in (7)) as $t \rightarrow \infty$. We next define a relevant near-consensus condition based on an H_∞ -like performance criterion.

Definition 2 (H_∞ -based near-consensus). *The subsystems in (1) are said to achieve a near-consensus condition specified by a given parameter γ as time $t \rightarrow \infty$ if the subsystems satisfy the condition*

$$\sup_{\kappa \in \mathbb{Z}_+} \sum_{t=0}^{\kappa} \left\{ \|\tilde{x}(t)\|^2 - \gamma^2 \|\tilde{w}(t)\|^2 \right\} \leq \Theta_0 \quad (10)$$

with $\tilde{x}(t) = [\tilde{x}_1(t)^\top \dots \tilde{x}_N(t)^\top]^\top$ and $\tilde{w}(t) = [\tilde{w}_1(t)^\top \dots \tilde{w}_N(t)^\top]^\top$ for some $\varsigma(t)$ eventually following the dynamics in (7) and some Θ_0 depending the subsystem initial states $x_j(0)$, $j = 1, \dots, N$. Condition (10) implies that the subsystem outputs satisfy $\sum_{t=0}^{\kappa} \sum_{j=1}^N \|s_j(t) - \sigma(t)\|^2 \leq \beta\gamma^2 \sum_{t=0}^{\kappa} \sum_{j=1}^N \|\tilde{w}_j(t)\|^2 + \beta\Theta_0$, $\forall \kappa \in \mathbb{Z}_+$ for a signal $\sigma(t) = C_\varsigma\varsigma(t)$, with $\beta = \lambda_{\max}(\mathbf{C}^\top \mathbf{C})$ where $\mathbf{C} = \text{diag}(C_1, C_2, \dots, C_N)$.

We wish to achieve the H_∞ -based near-consensus condition of Definition 2 by employing a scheme that involves the real-time minimization of an H_∞ -like performance index defined with suitable parameters γ_j , $j = 1, \dots, N$. Noting that the subsystem matrices $\tilde{\rho}_j(t) = [A_j(t) \quad B_j(t) \quad \tilde{D}_j(t)]$ of (9) lie in an appropriately defined polytope $\tilde{\Omega}_j$ and assuming that $\varsigma(t)$ is constrained to lie in a compact set $\mathcal{X}_\varsigma$, (and hence $\sigma(t)$ is constrained in the set $C_\varsigma\mathcal{X}_\varsigma$), we wish to minimize, with suitable allowable values of the predicted subsystem control inputs and a suitable sequence of $\varsigma(t+i|t)$, the index

$$J(t) = \sum_{j=1}^N J_j(t),$$

$$J_j(t) = \sup_{\kappa \in \mathbb{Z}_+} \sum_{i=0}^{\kappa} \left\{ \|Q_j^{\frac{1}{2}} \tilde{x}_j(t+i|t)\|^2 + \|R_j^{\frac{1}{2}} \tilde{u}_j(t+i|t)\|^2 - \gamma_j^2 \|\tilde{w}_j(t+i|t)\|^2 \right\}, \quad (11)$$

evaluated for a worst-case performance over $\tilde{\rho}_j(t+i|t) \in \tilde{\Omega}_j$, $\tilde{w}_j(t+i|t) \in \tilde{\mathbb{W}}_j$, $i \in \mathbb{Z}_+$, $j = 1, \dots, N$, where $\tilde{\mathbb{W}}_j$ is the same as $\mathbb{W}_j$ if $D_j^\Delta(t) = 0$, $\forall t \in \mathbb{Z}_+$ and equal to $\mathbb{W}_j \times \Upsilon_j \mathcal{X}_\varsigma$ otherwise.

Minimizing the performance index in (11) requires the solution of a large distributed optimization problem under the given inter-subsystem communications conditions (the assumptions on which will be discussed later). Such a problem is computationally challenging if the computations are to be carried out in a receding horizon fashion for a traditional MPC optimization for each subsystem such that the system constraints are guaranteed to be satisfied at all times. In the following, we discuss appropriate parameterizations of the variables in the optimization problem minimizing the cost in (11) and present the RHC scheme designed to efficiently solve the overall problem of achieving the near-consensus condition.

III. RHC SCHEME FOR H_∞ -BASED COORDINATED CONTROL

In the minimization of the index in (11), the quantities to be decided include the predicted subsystem inputs $u_j(t+i|t)$, $i \in \mathbb{Z}_+$, $j = 1, \dots, N$, and the predicted sequence $\varsigma(t+i|t)$, $i \in \mathbb{Z}_+$, which is common to all subsystems and is required to eventually follow the dynamics in (7). If the current value of $\varsigma(t)$ is assumed to be known, we can instead consider $\tilde{u}_j(t+i|t)$, $i \in \mathbb{Z}_+$, $j = 1, \dots, N$ and $\Delta\varsigma(t+i|t) = \varsigma(t+i+1|t) - A_\varsigma(t+i)\varsigma(t+i|t)$, $i \in \mathbb{Z}_+$ as the decision variables. Clearly, it is not possible to determine these quantities for all $i \in \mathbb{Z}_+$. Furthermore, the presence of constraints and uncertainties in subsystem dynamics necessitates a closed-loop approach in the prediction of future control inputs. Otherwise, the resulting control will be unnecessarily conservative. A tractable and computationally efficient way to deal with the problem for the class of polytopic subsystem dynamics is to employ the uncertainty-based dynamic policies explored in [34]. In the following, we discuss the dynamic policies that we consider for the problem and then present the details of the overall control scheme.

A. Parameterization of control decision variables

We parameterize the predicted future decision variables $\Delta\varsigma(t+i|t)$, $i \in \mathbb{Z}_+$, and $\tilde{u}_j(t+i|t)$, $i \in \mathbb{Z}_+$, $j = 1, \dots, N$ as the future outputs of stable dynamical systems. In particular, we express $\Delta\varsigma(t+i|t)$ as

$$\Delta\varsigma(t+i|t) = C_\vartheta \vartheta(t+i|t), \quad i \in \mathbb{Z}_+ \quad (12a)$$

where $\vartheta(t+i|t) \in \mathbb{R}^{n_\vartheta}$, $i \in \mathbb{Z}_+$ are the predicted states of a Schur stable autonomous system

$$\vartheta(t+1) = A_\vartheta \vartheta(t) \quad (12b)$$

starting from $\vartheta(t|t) = \vartheta(t)$. On the other hand, we parameterize $\tilde{u}_j(t+i|t)$ for each $j \in \mathbb{N}_N$ as

$$\tilde{u}_j(t+i|t) = K_j \tilde{x}_j(t+i|t) + H_j \xi_j(t+i|t), \quad i \in \mathbb{Z}_+ \quad (13a)$$

where K_j is a constant static feedback gain for subsystem (1), which satisfies some optimality condition, typically the unconstrained LQ or H_∞ optimality for a nominal model. The quantities $\xi_j(t+i|t) \in \mathbb{R}^{n_{\xi_j}}$, $i \in \mathbb{Z}_+$, that define the perturbations $H_j \xi_j(t+i|t)$, $i \in \mathbb{Z}_+$ to the standard optimal static feedback input, are parameterized as the predicted states,

starting from an initial state $\xi_j(t|t) = \xi_j(t)$, of a time-varying controller system described by the dynamics

$$\xi_j(t+1) = G_j(t)\xi_j(t) + E_j(t)\vartheta(t) + F_j(t)\tilde{w}_j(t) \quad (13b)$$

where $G_j(t)$, $E_j(t)$ and $F_j(t)$ follow the same kind of time-variation as the subsystem matrices, i.e.,

$$\begin{aligned} [G_j(t) \quad E_j(t) \quad F_j(t)] &= \sum_{r=1}^{h_j} \theta_{j[r]}(t) [G_j^{(r)} \quad E_j^{(r)} \quad F_j^{(r)}] \\ &\in \text{Co} \left\{ [G_j^{(r)} \quad E_j^{(r)} \quad F_j^{(r)}], \quad r = 1, \dots, h_j \right\}. \end{aligned} \quad (13c)$$

The motivation behind the use of the particular forms of parameterizations in (12) and (13) is to avoid conservativeness as much as possible while making sure that the resulting control optimization problem remains tractable and decomposable. The time-invariant, autonomous dynamics in (12) do not involve any subsystem-specific uncertainties, and result in a deterministic parameterization of $\Delta\varsigma(t+i|t)$ ensuring $\Delta\varsigma(t+i|t) \rightarrow 0$ as $t \rightarrow \infty$. The subsystem controller dynamics in (13b), on the other hand, incorporate possibly uncertain or unknown quantities $\theta_j(t)$ and $w_j(t)$ associated with the subsystem, and allow the predicted future input perturbations to assume different values for different predicted future realizations of $\theta_j(t)$ and $w_j(t)$. This reduces the conservativeness that would result from a deterministic prediction of perturbations (e.g., [34], [39]–[41]). Note that the policy in (13a-b) would be more general with arbitrary controller matrices $G_j(t)$, $E_j(t)$ and $F_j(t)$ but, in order to have a tractable control problem, they are assumed to lie in a polytope as stated in (13c) [34].

Given the parameterizations in (12) and (13), the problem of minimizing the worst-case value of the index $J(t)$ in (11) over $\tilde{\varphi}_j(t+i|t) \in \tilde{\Omega}_j$, $\tilde{w}_j(t+i|t) \in \tilde{\mathbb{W}}_j$, $i \in \mathbb{Z}_+$, $j = 1, \dots, N$ involves, as decision variables, the matrices C_ϑ , A_ϑ and H_j , $G_j^{(r)}$, $E_j^{(r)}$, $F_j^{(r)}$, $r = 1, \dots, h_j$, $j = 1, \dots, N$, and the vectors $\vartheta(t)$ and $\xi_j(t)$, $j = 1, \dots, N$. However, the resulting problem is computationally intensive to be solved on-line. To simplify on-line computations, we determine all the matrix variables off-line and allow the vectors $\vartheta(t)$ and $\xi_j(t)$, $j = 1, \dots, N$ to be optimized on-line to minimize the cost index. Matrices C_ϑ and A_ϑ are chosen depending on how we envisage $\varsigma(t+i|t)$ to evolve. The trivial choice of $C_\vartheta = I$ and $A_\vartheta = 0$, for example, would imply that only one update of $\varsigma(t)$ (at time $t+1$) is foreseen and $\varsigma(t+i+2|t) = A_\varsigma(t+i+1)\varsigma(t+i+1|t)$, $\forall i \in \mathbb{Z}_+$. The controller matrices H_j , $G_j^{(r)}$, $E_j^{(r)}$ and $F_j^{(r)}$, $r = 1, \dots, h_j$ determine the feasibility domain as well as the predicted control performance of the local control policy of each subsystem $j \in \mathbb{N}_N$. So, we wish to compute these matrices such that they allow a feasible invariant set¹ of the largest possible size for the subsystem error-state. Clearly, the dimensions n_ϑ and n_{ξ_j} of $\vartheta(t)$ and $\xi_j(t)$, $j \in \mathbb{N}_N$ affect the features of the control scheme, including the subsystem feasibility domains, the control performance and the off-line and on-line computational complexities, and they are chosen for a proper trade-off among these features.

¹A feasible invariant set for a controlled system refers to a positively invariant set which is constraint-admissible in the sense that all system constraints are satisfied for any state lying within this set.

B. Computation of controller matrices and invariant sets

To discuss the determination of subsystem controller matrices, we consider the augmented subsystem error-state dynamics

$$\chi_j(t+1) = \Psi_j(t)\chi_j(t) + \mathcal{D}_j(t)\tilde{w}_j(t) \quad (14)$$

where $\chi_j(t) = [\tilde{x}_j(t)^\top \quad \vartheta(t)^\top \quad \xi_j(t)^\top]^\top$ is the augmented subsystem error-state. Defining $\mathcal{A}_j(t) = A_j(t) + B_j(t)K_j$, the augmented subsystem matrices can be written as

$$\Psi_j(t) = \begin{bmatrix} \mathcal{A}_j(t) & -S_j C_\vartheta & B_j(t)H_j \\ 0 & A_\vartheta & 0 \\ 0 & E_j(t) & G_j(t) \end{bmatrix}, \quad \mathcal{D}_j(t) = \begin{bmatrix} \tilde{D}_j(t) \\ 0 \\ F_j(t) \end{bmatrix}$$

and they lie in a polytope, i.e., $[\Psi_j(t) \quad \mathcal{D}_j(t)] \in \text{Co} \left\{ [\Psi_j^{(r)} \quad \mathcal{D}_j^{(r)}], \quad r = 1, \dots, h_j \right\}$ where

$$\Psi_j^{(r)} = \begin{bmatrix} \mathcal{A}_j^{(r)} & -S_j C_\vartheta & B_j^{(r)} H_j \\ 0 & A_\vartheta & 0 \\ 0 & E_j^{(r)} & G_j^{(r)} \end{bmatrix}, \quad \mathcal{D}_j^{(r)} = \begin{bmatrix} \tilde{D}_j^{(r)} \\ 0 \\ F_j^{(r)} \end{bmatrix},$$

and $\mathcal{A}_j^{(r)} = A_j^{(r)} + B_j^{(r)}K_j$. Clearly, a feasible invariant set $\mathcal{S}_{\tilde{x}_j}$ for the subsystem error-state is given by the projection, on the $\tilde{x}_j$ -subspace, of a positively invariant set $\mathcal{S}_{\chi_j}$ for the state $\chi_j(t)$ of (14) on which the constraints in (4) are satisfied. While determining the controller matrices, we consider an ellipsoidal invariant set for $\chi_j(t)$, defined as $\mathcal{E}_{\chi_j} = \{\chi \mid \chi^\top \mathcal{W}_j^{-1} \chi \leq 1\}$, $\mathcal{W} \succ 0$, and maximize the volume of its appropriate projection under necessary conditions. We also impose a constraint

$$\vartheta(t) \in \mathcal{X}_\vartheta = \{\vartheta \mid M_\vartheta \vartheta \leq 1\}, \quad M_\vartheta \in \mathbb{R}^{m_\vartheta \times n_\vartheta} \quad (15)$$

on $\vartheta(t)$ such that the set $C_\vartheta \mathcal{X}_\vartheta$, in which $\Delta\varsigma(t)$ must lie, has a desired size. We now present the conditions to be used in the determination of the controller matrices for each subsystem.

Lemma 1 (Invariance). *The ellipsoidal set $\mathcal{E}_{\chi_j}$ is robust positively invariant for system (14) if there exist a matrix $R_{w_j} \succ 0$ and scalars $\alpha_j^{(r)} > 0$, $r = 1, \dots, h_j$ satisfying*

$$\begin{bmatrix} \mathcal{W}_j & \Psi_j^{(r)} \mathcal{W}_j & \mathcal{D}_j^{(r)} \\ * & \alpha_j^{(r)} \mathcal{W}_j & 0 \\ * & * & (1 - \alpha_j^{(r)}) R_{w_j} \end{bmatrix} \succ 0, \quad r = 1, \dots, h_j, \quad (16)$$

$$1 - \tilde{w}_j^{(i)\top} R_{w_j} \tilde{w}_j^{(i)} \geq 0, \quad i = 1, \dots, \tilde{\nu}_j$$

where $\tilde{w}_j^{(i)}$, $i = 1, \dots, \tilde{\nu}_j$ are the vertices of the set $\tilde{\mathbb{W}}_j$.

Proof: The proof follows from the lines of reasoning used in the proof of Lemma 1 in [34] in which a slightly different description of the disturbance set is considered. ■

Lemma 2 (Feasibility). *Given any $\varsigma(t) \in \mathcal{X}_\varsigma$, the ellipsoidal set $\mathcal{E}_{\chi_j}$ will be feasible w.r.t. the constraints in (4) if*

$$\begin{bmatrix} Z_j & \begin{bmatrix} \tilde{M}_{x_j} & 0 & 0 \\ 0 & M_\vartheta & 0 \\ \tilde{M}_{u_j} K & 0 & \tilde{M}_{u_j} H_j \end{bmatrix} \mathcal{W}_j \\ * & \mathcal{W}_j \end{bmatrix} \succ 0, \quad (17)$$

$$Z_{j[r,r]} \leq 1, \quad r = 1, \dots, m_{u_j} + m_{x_j} + m_\vartheta$$

where $\tilde{M}_{x_j} = (\text{diag}(b_{[1]}, \dots, b_{[m_{x_j}]})^{-1} M_{x_j})$ with $b_{[l]} = \min_{\varsigma \in \mathcal{X}_\varsigma} (1 - M_{x_j} [l, :] S_j \varsigma)$ and $\tilde{M}_{u_j} = (\text{diag}(d_{[1]}, \dots, d_{[m_{u_j}]})^{-1} M_{u_j})$ with $d_{[l]} = \min_{\varsigma \in \mathcal{X}_\varsigma} (1 - M_{u_j} [l, :] U_j \varsigma)$.

Proof: Noting that $\tilde{x}_j(t) = x_j(t) - S_j \varsigma(t)$ lies in the set $\mathbb{X}_j \ominus S_j \mathcal{X}_\varsigma$, and $\tilde{u}_j(t) = u_j(t) - U_j \varsigma(t)$ lies in the set $\mathbb{U}_j \ominus U_j \mathcal{X}_\varsigma$, the proof follows from standard results (e.g., [42]). ■

Lemma 3 (Boundedness). *If there exists a matrix $\mathcal{P}_j \succeq 0$ that satisfies the LMIs*

$$\begin{bmatrix} \mathcal{P}_j & \Psi_j^{(r)\top} \mathcal{P}_j & C_{z_j}^\top & 0 \\ * & \mathcal{P}_j & 0 & \mathcal{P}_j \mathcal{D}_j^{(r)} \\ * & * & I & 0 \\ * & * & * & \gamma_j^2 I \end{bmatrix} \succeq 0, \quad r = 1, 2, \dots, h_j \quad (18)$$

for a positive scalar γ_j , then, for all $t \in \mathbb{Z}_+$,

$$\|\mathcal{P}_j^{\frac{1}{2}} \chi_j(t+1)\|^2 - \|\mathcal{P}_j^{\frac{1}{2}} \chi_j(t)\|^2 \leq -\|z_j(t)\|^2 + \gamma_j^2 \|\tilde{w}_j(t)\|^2 \quad (19)$$

and, hence, we have

$$\sup_{\kappa \in \mathbb{Z}_+} \sum_{t=0}^{\kappa} \{\|z_j(t)\|^2 - \gamma_j^2 \|\tilde{w}_j(t)\|^2\} \leq \|\mathcal{P}_j^{\frac{1}{2}} \chi_j(0)\|^2 \quad (20)$$

for the output signal $z_j(t) = C_{z_j} \chi_j(t)$.

Proof: See e.g., [43]. ■

Conditions (16) and (17) respectively ensure the invariance of the set $\mathcal{E}_{\chi_j}$ in the presence of model uncertainties and disturbances and the feasibility of the set w.r.t. the constraints for the j th subsystem. Condition (18) with C_{z_j} defined as $C_{z_j} = \begin{bmatrix} Q_j^{1/2} & 0 & 0 \\ R_j^{1/2} K_j & 0 & R_j^{1/2} H_j \end{bmatrix}$ so that $z_j(t)^\top z_j(t) = \|Q_j^{\frac{1}{2}} \tilde{x}_j(t)\|^2 + \|R_j^{\frac{1}{2}} \tilde{u}_j(t)\|^2$ guarantees that the predicted worst-case value of $J_j(t)$ in (11) is bounded by $\|\mathcal{P}_j^{\frac{1}{2}} \chi_j(0)\|^2$.

While determining the subsystem controller matrices, we wish to maximize the size of the projection $\mathcal{E}_{\tilde{x}_j}$ of the set $\mathcal{E}_{\chi_j}$ on the $\tilde{x}_j$ -subspace. However, since the subsystems will be required to negotiate for a common value of $\vartheta(t)$ in real-time, we must ensure that a reasonable range of values of $\vartheta(t)$ will be feasible for any value of $\tilde{x}_j(t)$ in the set $\mathcal{E}_{\tilde{x}_j}$. Therefore, we rather maximize the size of the projection of $\mathcal{E}_{\chi_j}$ on the $\tilde{x}_j$ - ϑ -space, which is given by $\mathcal{E}_{\tilde{x}_j-\vartheta} = \{\zeta \mid \zeta^\top (\Lambda_j \mathcal{W}_j \Lambda_j^\top)^{-1} \zeta \leq 1\}$ where $\Lambda_j = [I_{n_{x_j}+n_\vartheta} \quad 0]$. For this, we solve the problem

$$\max_{\mathcal{W}_j, \mathcal{P}_j, Z_j, R_{w_j}, H_j, G_j^{(r)}, E_j^{(r)}, F_j^{(r)}, \alpha_j^{(r)} \quad (r=1, \dots, h_j)} \log \det(\Lambda_j \mathcal{W}_j \Lambda_j^\top) \quad (21)$$

subject to conditions (16), (17) and (18).

The constraints in problem (21) are bilinear matrix inequalities (BMIs) and the problem can be solved by using an alternating semidefinite programming (SDP) optimization approach, making use of the special structure of the problem (assuming $E_j^{(r)} = 0$, $r = 1, \dots, h_j$) [34]. Note that we consider ellipsoidal feasible sets while computing the controller matrices since this leads to a computationally tractable problem. However, the sets $\mathcal{E}_{\chi_j}$ and $\mathcal{E}_{\tilde{x}_j-\vartheta}$ obtained from (21) are conservative in size. The corresponding largest feasible invariant sets for the resulting system (14) are (better approximated as) polyhedral sets $\mathcal{X}_{\chi_j}$ and $\mathcal{X}_{\tilde{x}_j-\vartheta}$ when $\mathbb{W}_j$ is a polytope, and, once the controller matrices are determined, such polyhedral sets can be easily computed (see, e.g., [37, Chap. 5], [44, App. A]) and used in the on-line computations.

Finally, once the controller matrices are computed for each subsystem, an expression upper bounding the subsystem cost

function in (11) can be directly obtained from the result of Lemma 3. Given the subsystem initial state $x_j(t)$, and the values of $\varsigma(t)$ and $\vartheta(t)$, an upper bound of the worst-case value of $J_j(t)$ in (11) evaluated over $\tilde{\varphi}_j(t+i|t) \in \tilde{\Omega}_j$, $\tilde{w}_j(t+i|t) \in \tilde{\mathbb{W}}_j$, $i \in \mathbb{Z}_+$ is given by $\hat{J}_j(t) = \|\mathcal{P}_j^{\frac{1}{2}} [(x_j(t) - S_j \varsigma(t))^\top \quad \vartheta(t)^\top \quad \xi_j(t)^\top]^\top\|^2$, where $\mathcal{P}_j \succeq 0$ is a minimal matrix (e.g., in the minimum-trace sense) satisfying (18) and $\xi_j(t)$ is the controller initial state which is admissible in the sense that all system constraints are satisfied.

C. On-line control optimization problem and its distributed solution

1) On-line control optimization problem: It follows from the above discussion that, given the current value of $\varsigma(t)$, an upper bound of the worst-case value of $J(t)$ in (11) is given by $\sum_{j=1}^N \hat{J}_j(t) = \sum_{j=1}^N \|\mathcal{P}_j^{\frac{1}{2}} [(x_j(t) - S_j \varsigma(t))^\top \quad \vartheta(t)^\top \quad \xi_j(t)^\top]^\top\|^2$. Here, $\vartheta(t)$ and $\xi_j(t)$, $j = 1, \dots, N$ are variables whose values should be admissible in the sense that the constraint $\varsigma(t) \in \mathcal{X}_\varsigma$ and the constraints in (4) are satisfied at all times $t \in \mathbb{Z}_+$.

For each subsystem $j \in \mathbb{N}_N$, the constraints in (4) are guaranteed to be satisfied for any value of $\varsigma(t) \in \mathcal{X}_\varsigma$ if $\chi_j(t)$ lies in the set $\mathcal{X}_{\chi_j}$. However, since the set $\mathcal{X}_{\chi_j}$ is obtained with the assumption that $\tilde{x}_j(t)$ and $\tilde{u}_j(t)$ should lie in the sets $\mathbb{X}_j \ominus S_j \mathcal{X}_\varsigma$ and $\mathbb{U}_j \ominus U_j \mathcal{X}_\varsigma$ respectively, the constraint $\chi_j(t) \in \mathcal{X}_{\chi_j}$ may lead to conservative results when $\varsigma(t)$ is small and not close to the boundary of $\mathcal{X}_\varsigma$. The following result explores a way to use expanded feasible invariant sets on-line in order to reduce this conservativeness to some extent.

Lemma 4. *For a subsystem $j \in \mathbb{N}_N$, let $\bar{c}_j$ be defined as $\bar{c}_j = \sup\{c \geq 0 \mid (1+c)(\mathbb{X}_j \ominus S_j \mathcal{X}_\varsigma) \subseteq \mathbb{X}_j, (1+c)(\mathbb{U}_j \ominus U_j \mathcal{X}_\varsigma) \subseteq \mathbb{U}_j\}$. Then, $\vartheta(t)$ and $\xi_j(t)$ are constraint-admissible if they satisfy, for some c_j , $0 \leq c_j \leq \bar{c}_j$, the conditions*

$$[(x_j(t) - S_j \varsigma(t))^\top \quad \vartheta(t)^\top \quad \xi_j(t)^\top]^\top \in (1+c_j) \mathcal{X}_{\chi_j}, \quad (22a)$$

$$\varsigma(t+i|t) \in (1-\mu_j) \mathcal{X}_\varsigma, \forall i \in \mathbb{Z}_+ \quad (22b)$$

where $\mu_j = c_j/\bar{c}_j$ if $\bar{c}_j \neq 0$ and $\mu_j = 0$ if $\bar{c}_j = 0$.

Proof: See Appendix A. ■

Note that the first constraint in (22) can be easily expressed as a linear constraint with c_j as an additional variable. The constraint (22b) can also be expressed as a linear constraint when the set $\mathcal{X}_\varsigma$ is a polyhedral set. If $A_\varsigma(t) = A_\varsigma$ is constant, constraint (22b) can be replaced by $[\varsigma(t)^\top \quad \vartheta(t)^\top]^\top \in (1-\mu_j) \mathcal{X}_{\varsigma\vartheta}$, where $\mathcal{X}_{\varsigma\vartheta}$ is the largest set inside $\mathcal{X}_\varsigma \times \mathcal{X}_\vartheta$, which is invariant for the dynamics

$$\begin{bmatrix} \varsigma(t+1) \\ \vartheta(t+1) \end{bmatrix} = \begin{bmatrix} A_\varsigma & C_\vartheta \\ 0 & A_\vartheta \end{bmatrix} \begin{bmatrix} \varsigma(t) \\ \vartheta(t) \end{bmatrix}. \quad (23)$$

The same approach can be followed when $A_\varsigma(t)$ is time-varying by considering either T such largest sets, each lying in $\mathcal{X}_\varsigma \times \mathcal{X}_\vartheta$ and invariant to the dynamics specified by the system matrix $\begin{bmatrix} \bar{A}_\varsigma & \bar{C}_\vartheta \\ 0 & \bar{A}_\vartheta \end{bmatrix}$ where $\bar{A}_\vartheta = A_\vartheta^p$ with $p = T$, and $\bar{C}_\vartheta^\tau = \sum_{i=1}^T A_\varsigma(\tau+T-1) \cdot A_\varsigma(\tau+T-2) \dots A_\varsigma(\tau+i) C_\vartheta A_\vartheta^{i-1}$ for $\tau = 0, 1, \dots, T-1$, or a single largest set in $\mathcal{X}_\varsigma \times \mathcal{X}_\vartheta$, which is invariant to the dynamics specified by each of such matrices $\begin{bmatrix} \bar{A}_\varsigma & \bar{C}_\vartheta^\tau \\ 0 & \bar{A}_\vartheta \end{bmatrix}$, $\tau = 0, 1, \dots, T-1$.

Following the result of Lemma 4, the on-line control optimization problem can be written as

$$\min_{\vartheta(t), \xi_j(t), c_j, j=1, \dots, N} \sum_{j=1}^N \hat{J}_j(t) \quad (24)$$

$$\text{with } \hat{J}_j(t) = \left\| \mathcal{P}_j^{\frac{1}{2}} \left[(x_j(t) - S_j \varsigma(t))^T \quad \vartheta(t)^T \quad \xi_j(t)^T \right]^T \right\|^2$$

subject to conditions (22a,b) for $j = 1, \dots, N$.

We refer to problem (24) as the global control problem. It requires the determination of the global variable $\vartheta(t)$ along with the local-level control decision variables. Before we proceed to discuss the distributed solution of (24), we note that when $\vartheta(t)$ is known, problem (24) reduces into N uncoupled optimization problems which can be separately solved by the subsystems. The j th such problem can be written as

$$\min_{\xi_j(t), c_j} \hat{J}_j(t) \quad (25)$$

$$\text{with } \hat{J}_j(t) = \left\| \mathcal{P}_j^{\frac{1}{2}} \left[(x_j(t) - S_j \varsigma(t))^T \quad \vartheta(t)^T \quad \xi_j(t)^T \right]^T \right\|^2$$

subject to conditions (22a,b). We refer to (25) as the local control problem for the j th subsystem.

2) Distributed solution of the global control problem:

Problem (24) is a convex problem involving a quadratic objective function and convex constraints, and it can be efficiently solved if the subsystem state information is fully available and the computational resources are not limited. However, in a practical situation possibly with large number of geographically separated subsystems, a centralized solution at one of the subsystems or some other central location may not be appealing. If the subsystems are all equipped with some computational resources, it is natural to let the subsystems share the computational burden. It is obvious that problem (24) is of the form

$$\min_{(\varphi_j, \vartheta) \in \mathcal{S}_{\varphi_j}, j=1, \dots, N} \sum_{j=1}^N f_j(\varphi_j, \vartheta) \quad (26)$$

where $\mathcal{S}_{\varphi_j}$, $j = 1, \dots, N$ are convex sets. Also, for each $j \in \mathbb{N}_N$, φ_j represents the variables associated with subsystem j only, whereas ϑ is a coupling variable common to all. For a distributed solution of the optimization problem of the form (26), techniques based on primal or dual decompositions [45, Chap. 6] are commonly used. In this work, we consider a subgradient-based solution employing the decomposition of the dual function. Using a local instance of the common variable ϑ in each subproblem, problem (26) can be written as

$$\min_{\vartheta, (\varphi_j, v_j) \in \mathcal{S}_{\varphi_j}, j=1, \dots, N} \sum_{j=1}^N f_j(\varphi_j, v_j) \quad (27)$$

subject to the condition $v_1 = v_2 = \dots = v_N = \vartheta$. The dual function for (27) can be written as [45]

$$q(\lambda) = \min_{\vartheta, (\varphi_j, v_j) \in \mathcal{S}_{\varphi_j}, j=1, \dots, N} \left\{ \sum_{j=1}^N f_j(\varphi_j, v_j) + \lambda^T (v - (\mathbf{1} \otimes I_{n_\vartheta}) \vartheta) \right\} \quad (28)$$

where $v = [v_1^T \ v_2^T \ \dots \ v_N^T]^T$ and $\lambda = [\lambda_1^T \ \lambda_2^T \ \dots \ \lambda_N^T]^T$. The minimum of (28) w.r.t. ϑ occurs when $(\mathbf{1} \otimes I_{n_\vartheta})^T \lambda = 0$.

Let a set $\mathcal{S}_\lambda$ be defined as $\mathcal{S}_\lambda = \{\lambda \mid (\mathbf{1} \otimes I_{n_\vartheta})^T \lambda = 0\}$. Then, the dual problem can be written as

$$\max_{\lambda \in \mathcal{S}_\lambda} h(\lambda) \quad \text{with } h(\lambda) = \sum_{j=1}^N h_j(\lambda_j) \quad (29a)$$

where $h_j(\lambda_j)$ is defined as the optimal value of the problem

$$\min_{(\varphi_j, v_j) \in \mathcal{S}_{\varphi_j}} f_j(\varphi_j, v_j) + \lambda_j^T v_j. \quad (29b)$$

Here, (29a) is the dual master-problem and (29b) is the subproblem associated with the j th subsystem. We solve the master-problem using the projected subgradient method. The use of subgradient methods in solving various optimization problems has been extensively explored in the literature and their suitability for solving certain decomposable large-scale problems with distributed/decentralized computations has been well recognized (see e.g., [46], [35], [45], [47]). In (29), clearly, a subgradient of the function $-h_j(\lambda_j)$ at λ_j is given by $-v_j^*(\lambda_j)$, where $v_j^*(\lambda_j)$ is an optimal value of v_j in the subproblem (29b) [45, Chap. 6]. Since the constraint set $\mathcal{S}_\lambda$ is affine, the dual variable λ can be updated using the projection of the overall subgradient $-v^*(\lambda) = -[v_1^*(\lambda_1)^T \ \dots \ v_N^*(\lambda_N)^T]^T$ on the set $\mathcal{S}_\lambda$. The projection of $v^*(\lambda)$ on $\mathcal{S}_\lambda$ is given by $g_p(\lambda) = v^*(\lambda) - \bar{v}(\lambda)$ with $\bar{v}(\lambda) = (\mathbf{1} \otimes I_{n_\vartheta}) \hat{\vartheta}(\lambda)$ where $\hat{\vartheta}(\lambda) = \frac{1}{N} (\mathbf{1} \otimes I_{n_\vartheta})^T v^*(\lambda)$ represents the component-wise average of the optimal values $v_j^*(\lambda_j)$, $j = 1, \dots, N$ of the local instances of the coupling variable. We present the outline of the algorithm for the solution of the dual problem (29) in the following.

Algorithm 1: Distributed computation of ϑ

- i. Initialization: Set $k = 0$. Choose $\lambda^{(0)} \in \mathcal{S}_\lambda$.
- ii. Repeat for $k = 0, 1, 2, \dots$
 - a) For $j = 1, \dots, N$, solve (29b) to obtain $(\varphi_j^*(\lambda_j^{(k)}), v_j^*(\lambda_j^{(k)})) = \arg \min_{(\varphi_j, v_j) \in \mathcal{S}_{\varphi_j}} f_j(\varphi_j, v_j) + \lambda_j^{(k)T} v_j$
 - b) Compute $\bar{v}(\lambda^{(k)})$ and $g_p(\lambda^{(k)})$
 - i) $\bar{v}(\lambda^{(k)}) = \frac{1}{N} (\mathbf{1} \otimes I_{n_\vartheta}) (\mathbf{1} \otimes I_{n_\vartheta})^T v^*(\lambda^{(k)})$
where $v^*(\lambda^{(k)}) = [v_1^*(\lambda_1^{(k)})^T \ \dots \ v_N^*(\lambda_N^{(k)})^T]^T$
 - ii) $g_p(\lambda^{(k)}) = v^*(\lambda^{(k)}) - \bar{v}(\lambda^{(k)})$
 - c) Update λ
 $\lambda^{(k+1)} = \lambda^{(k)} + a_k g_p(\lambda^{(k)})$

Here, step 2a of Algorithm 1 involves the solution of subproblems which are convex problems. The dual variable is updated in step 2c where the quantities $a_k, k \in \mathbb{Z}_+$ are positive step-sizes that are suitably chosen *a-priori* and satisfy $\sum_{k=0}^\infty a_k \rightarrow \infty$. Clearly, step 2a and step 2c of Algorithm 1 can be carried out in the subsystems in a decentralized way. Step 2b, however, involves averaging of the optimal values v_j^* , $j = 1, \dots, N$ of the local instances of the coupling variable in order to compute the projected subgradient at each iteration instant k . This requires an exchange of information among the subsystems. Depending on the nature of the inter-subsystem communications architecture, the averaging step can be carried out suitably either in a decentralized manner or sequentially with the other two steps using the incremental variant of the subgradient method [45, Chap. 6] as in [32].

We consider distributed averaging with linear iterations (see [12], [13]) for step 2b(i) in Algorithm 1. Let us consider that

the inter-subsystem communication architecture is represented by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ comprising of a set of nodes $\mathcal{N} = \mathbb{N}_N$ and a set of edges $\mathcal{E}$. This means that each node $j \in \mathcal{N}$ can directly communicate only with the set of its neighbours denoted by $\mathcal{N}_j = \{l \mid \{j, l\} \in \mathcal{E}\}$. Since the averaging of the quantities $v_j^*, j = 1, \dots, N$ has to be carried out component-wise, we consider, for each component, iterations of the form

$$\zeta(i+1) = W \zeta(i) \quad (30)$$

where $W \in \mathbb{R}^{N \times N}$ is a sparse weight matrix satisfying

$$W_{[i,j]} = 0, \quad \forall \{i, j\} \notin \left\{ \mathcal{E}, \{1, 1\}, \{2, 2\}, \dots, \{N, N\} \right\}. \quad (31)$$

We require that the iterates in (30) converge to the average of the initial values in ζ , i.e.,

$$\lim_{i \rightarrow \infty} \zeta(i) = \lim_{i \rightarrow \infty} W^i \zeta(0) = \frac{1}{N} \mathbf{1}^T \zeta(0) = \bar{\zeta} \quad (32)$$

Clearly, W defines the convergence properties of the averaging iterations (30). In particular, a quantity of interest is the per-step convergence factor [12] defined as

$$c = \sup_{\zeta(i) \neq \bar{\zeta}} \frac{\|\zeta(i+1) - \bar{\zeta}\|}{\|\zeta(i) - \bar{\zeta}\|} \quad (33)$$

We next note the following result from [12] regarding the requirements on the weight matrix W for convergent averaging.

Lemma 5. *Convergence condition (32) holds for any initial vector $\zeta(0)$ if and only if W has $\mathbf{1}$ as one of its left as well as one of its right eigenvectors, and the spectral radius of $W - \frac{1}{N} \mathbf{1} \mathbf{1}^T$ is less than unity. Moreover, the per-step convergence factor c is given by the spectral norm of $W - \frac{1}{N} \mathbf{1} \mathbf{1}^T$, i.e., $c = \|W - \frac{1}{N} \mathbf{1} \mathbf{1}^T\|$.*

A weight matrix W satisfying the condition in Lemma 5 exists if the graph $\mathcal{G}$ is connected, and one that leads to the fastest convergence can be obtained by minimizing the spectral norm of $W - \frac{1}{N} \mathbf{1} \mathbf{1}^T$ subject to the sparsity constraint (31) specified by the graph $\mathcal{G}$. This minimization problem can be expressed as an SDP problem and can be easily solved [12]. With W determined *a-priori*, we compute $\bar{v}(\lambda^{(k)})$ in step 2b(i) of Algorithm 1 using the iterations

$$\bar{v}(\lambda^{(k)}, l+1) = (W \otimes I_{n_{\bar{v}}}) \bar{v}(\lambda^{(k)}, l) \quad (34)$$

starting from $\bar{v}(\lambda^{(k)}, 0) = v^*(\lambda^{(k)})$. These iterations are completely decentralized and involve trivial computations in the subsystems. However, since the convergence to the average is only asymptotic, the iterations in (34) should be carried out to an accuracy that is sufficient to ensure the desired convergence of the subgradient algorithm. We now present the following result on the convergence of Algorithm 1 with the averaging in step 2b(i) approximated by decentralized averaging (34).

Proposition 6. *Let λ^* be an optimal point that maximizes $h(\lambda)$ in (29a) and h^* be the optimal value of h . Consider Algorithm 1 with $g_p(\lambda^{(k)})$ in step 2 approximated by $\hat{g}_p(\lambda^{(k)}) = v^*(\lambda^{(k)}) - \bar{v}(\lambda^{(k)}, \eta_k)$ where $\bar{v}^*(\lambda^{(k)}, \eta_k)$ is obtained from (34) with $\eta_k, k \geq 0$ satisfying*

$$\eta_k \geq \max \left(1, \frac{\ln(\epsilon_k^{-1} \|g_p(\lambda^{(k)})\| \|\lambda^* - \lambda^{(k)}\|)}{\ln c^{-1}} \right) \quad (35)$$

for a sequence of given small positive quantities $\{\epsilon_k\}, k \geq 0$. Let $\{\lambda^{(k)}\}$ be the sequence of λ generated by the algorithm. Then, we have

$$\liminf_{k \rightarrow \infty} (h^* - h(\lambda^{(k)})) \leq \delta \quad (36)$$

where $\delta = \limsup_{k \rightarrow \infty} \left(\frac{1}{2} a_k (1 + c^{2\eta_k}) \|g(\lambda^{(k)})\|^2 + \epsilon_k \right)$. Moreover, if $a_k \rightarrow 0$ and $\epsilon_k \rightarrow \epsilon$, we have

$$\liminf_{k \rightarrow \infty} (h^* - h(\lambda^{(k)})) \leq \epsilon. \quad (37)$$

Proof: See Appendix B. ■

Remark 1. *The use of an approximate projection of the subgradient $-v^*(\lambda^{(k)})$ of $-h(\lambda^{(k)})$ on the set S_λ in step 2b of Algorithm 1 can be considered as using the actual projection on S_λ of an approximate subgradient $\hat{g}(\lambda^{(k)}) = -\left(v^*(\lambda^{(k)}) - (\bar{v}(\lambda^{(k)}, \eta_k) - \bar{v}(\lambda^{(k)}))\right)$. It follows from condition (35) that $\hat{g}(\lambda^{(k)})$ belongs to the ϵ_k -subdifferential of $-h(\lambda^{(k)})$ at $\lambda^{(k)}$. The result of Proposition 6 can, therefore, be seen as a case of the convergence of an approximate subgradient projection method of optimization [48].*

Remark 2. *Condition (35) on η_k can be satisfied for a given sequence $\{\epsilon_k\}$ with a sufficiently large η_k at each k . Since $\mathcal{X}_\zeta$ is bounded, an upper bound of $\|g_p(\lambda^{(k)})\|$ can be easily computed. Similarly, since the subsystems' feasible invariant sets are bounded, an upper bound of $\|\lambda^* - \lambda^{(k)}\|$ can be estimated during the off-line design phase. These upper bounds give an upper bound of η_k for all k . However, since both $\|g_p(\lambda^{(k)})\|$ and $\|\lambda^* - \lambda^{(k)}\|$ decrease with the converging values of the quantities $v_j^*, j = 1, \dots, N$, a small η_k may usually be sufficient to ensure that a suitable sequence $\{\epsilon_k\}$ exists so that the algorithm converges with reasonable accuracy. Indeed, if $v_j^*(\lambda_j^{(k)}), j = 1, \dots, N$ obtained from the minimizations in (29b) are unique, $\eta_k = 1$ would ensure convergence since, in such a case, $v^*(\lambda^{(k)})$ actually coincides with the gradient and the norm of the gradient error $((W - \mathbf{1} \mathbf{1}^T / N) \otimes I_{n_{\bar{v}}}) v^*(\lambda^{(k)})$ is smaller than the gradient norm so that the approximate gradient is still a direction of ascent. In any case, since smaller η_k means poorer ascent in each iteration and larger η_k implies more inter-subsystem communications and more delay between iterations of the main algorithm, a suitable choice of η_k will involve a tradeoff between these factors.*

D. Overall RHC scheme for achieving near-consensus

The overall RHC scheme for attaining the near-consensus condition requires a suitable real-time solution of the global control problem (24). In an RHC framework, in order to utilize the full potential of the RHC paradigm, we would wish to compute all the decision variables at each time instant t . However, even though we envisage real-time updates in the trajectory of $\varsigma(t)$ because of factors such as the time-variability and uncertainty in subsystem dynamics and the specific form of the off-line optimized control policies considered in the subsystems, the determination of the optimal value of $\vartheta(t)$ is computationally intensive and requires extensive inter-subsystem communications. Therefore, it may be reasonable to optimize $\vartheta(t)$ at a different frequency, say at every T_{cc} time

Fig. 1: Overall control scheme for achieving the near-consensus condition.

steps, rather than optimizing it together with the local variables $\xi_j(t)$, $j = 1, \dots, N$ at each time step.

In the overall RHC scheme, we want $\varsigma(t)$ to eventually follow the dynamics in (7). However, in the presence of non-vanishing disturbances in the subsystems, this will not necessarily happen if we continue to optimize $\vartheta(t)$. For this reason, we apply appropriate checks before proceeding to optimize and update $\vartheta(t)$. The details of the approach are outlined in the scheme presented in Fig. 1. Essentially, the prior condition in step B3(ii) has been introduced only to avoid unnecessary computations since it ensures that the relative intensive optimizations in Step B3(iii) are carried out at each T_{cc} steps only if the overall cost with an updated value of $\vartheta(t)$ is certain to be lower than the overall cost at the time of its previous update. The posterior check of step B3(iv), where $P_\theta \succ 0$ is a suitably chosen matrix, is applied before accepting the new value $\vartheta^+(t)$ obtained from step B3(iii) and it is solely meant to ensure that $\Delta \varsigma(t)$ eventually converges to zero and that $\varsigma(t)$ follows the dynamics in (7). It is not employed for a posterior optimality check. Note that the aggregate costs $\hat{J}(t)$ and $\hat{J}^+(t)$ in step B3 can be estimated by employing an averaging of the local costs in a distributed way as discussed in the last section for step 2(b) of Algorithm 1. We now present the following result on the performance of the basic overall control scheme of Fig. 1.

Proposition 7. *Given a constraint-admissible value of $\varsigma(0)$, if problem (24) is feasible at $t = 0$, then the RHC scheme of Fig. 1 guarantees that (a) the control problem remains feasible*

at all times $t \in \mathbb{Z}_+$, (b) the cost monotonicity condition

$$\sum_{j=1}^N \hat{J}_j(t+1) - \sum_{j=1}^N \hat{J}_j(t) \leq - \sum_{j=1}^N \left\{ \|Q_j^{\frac{1}{2}} \tilde{x}_j(t)\|^2 + \|R_j^{\frac{1}{2}} \tilde{u}_j(t)\|^2 - \gamma_j^2 \|\tilde{w}_j(t)\|^2 \right\} \quad (38)$$

holds at all times $t \in \mathbb{Z}_+$, (c) the trajectory of $\varsigma(t)$ eventually follows the dynamics in (7), i.e., the trajectory $\{\varsigma(t+i)\}_{i=0}^\infty$ aligns with $\left\{ \left(\prod_{r=1}^i A_\varsigma(t+i-r) \right) \varsigma(t) \right\}_{i=0}^\infty$ as $t \rightarrow \infty$, and (d) the subsystems achieve the near-consensus condition of Definition 2 with $\gamma = \max(\gamma_1, \gamma_2, \dots, \gamma_N) / \sqrt{\lambda_{\min}(\mathbf{Q})}$, $\Theta_0 = \hat{J}(0) / \lambda_{\min}(\mathbf{Q})$, where $\mathbf{Q} = \text{diag}(Q_1, Q_2, \dots, Q_N)$.

Proof: Given the feasibility of problem (24) at $t = 0$, part (a) follows from Lemma 4. Next, note that when $\vartheta(t)$ is not optimized in real time, condition (38) immediately follows from (19), which is satisfied by the predicted closed-loop subsystems. Since any real-time optimization and update of the value of $\vartheta(t)$ in the scheme of Fig. 1 at some time t will only further reduce the predicted aggregate cost $\hat{J}(t) = \sum_{j=1}^N \hat{J}_j(t)$, part (b) of the statement follows.

To prove part (c), we first note that, without real-time updates, the signal $\varsigma(t)$, under the dynamics (7), is either constant or follows a periodic trajectory with a period $T_\varsigma = \bar{q}T$, where $\bar{q}$ is the lowest common multiple of the integers $q_1, q_2, \dots, q_{n_\varsigma-1}$ and q_{n_ς} mentioned in assumption A5. The case of a constant $\varsigma(t)$ can be considered as the one with $T_\varsigma = 1$. Next, for each $t \in \{0, 1, \dots, T_\varsigma - 1\}$, let $\{\varsigma_t(j)\}_{j=0,1,\dots}$ be a sequence with $\varsigma_t(j)$ defined as $\varsigma_t(j) = \varsigma(t + jT_\varsigma)$. We now show that, for each $t \in \{0, 1, \dots, T_\varsigma - 1\}$, $\varsigma_t(j)$ converges to some constant $\bar{\varsigma}_t$ as $j \rightarrow \infty$ and that $\bar{\varsigma}_{t+1} = A_\varsigma(t) \bar{\varsigma}_t$ with $\bar{\varsigma}_{T_\varsigma} = \bar{\varsigma}_0$.

Let $\{t_s\}_{s=0,1,\dots}$ be the subsequence of time instants at which $\vartheta(t)$ is updated in step B3(v). Then, let us consider a sequence $\{\vartheta^{\Delta e}(t)\}_{t=0}^\infty$ with $\vartheta^{\Delta e}(t)$ defined as

$$\vartheta^{\Delta e}(t) = \begin{cases} \vartheta^\Delta(t) & \text{if } t \in \{t_0, t_1, \dots\} \\ 0 & \text{otherwise,} \end{cases}$$

where $\vartheta^\Delta(t)$, $t = t_0, t_1, \dots$ are defined in step B3(iv). Further, for a given $t \in \{0, \dots, T_\varsigma - 1\}$, let $V_t^{\Delta e}(j)$ represent the stacked vector $[\vartheta^{\Delta e}(t + jT_\varsigma)^\top \ \vartheta^{\Delta e}(t + jT_\varsigma + 1)^\top \ \dots \ \vartheta^{\Delta e}(t + jT_\varsigma + T_\varsigma - 1)^\top]^\top$. Then, letting $\vartheta_t(j) = \vartheta(t + jT_\varsigma)$, and noting the fact that $\prod_{i=1}^{T_\varsigma} A_\varsigma(t + T_\varsigma - i) = \bar{A}_\varsigma^q = I$, we have

$$\begin{bmatrix} \varsigma_t(j+1) \\ \vartheta_t(j+1) \end{bmatrix} = \begin{bmatrix} I & Y_t^1 \\ 0 & A_\theta^{T_\varsigma} \end{bmatrix} \begin{bmatrix} \varsigma_t(j) \\ \vartheta_t(j) \end{bmatrix} + \begin{bmatrix} Y_t^1 & Y_t^2 & \dots & Y_t^{T_\varsigma} \\ A_\theta^{T_\varsigma} & A_\theta^{T_\varsigma-1} & \dots & A_\theta \end{bmatrix} \times V_t^{\Delta e}(j) \quad (39)$$

where $Y_t^j = \sum_{r=0}^{T_\varsigma-j} \left(\prod_{i=1}^{T_\varsigma-j-r} A_\varsigma(t + T_\varsigma - i) \right) C_\theta A_\theta^r$, $j = 1, \dots, T_\varsigma$. Here, in the expression for Y_t^j , any null product $\prod_{i=1}^0 (\cdot)$ is replaced by I . Now, since all eigenvalues of A_θ lie inside the unit circle, the powers $\begin{bmatrix} I & Y_t^1 \\ 0 & A_\theta^{T_\varsigma} \end{bmatrix}^j$ of the matrix $\begin{bmatrix} I & Y_t^1 \\ 0 & A_\theta^{T_\varsigma} \end{bmatrix}$ converge to a constant matrix as $j \rightarrow \infty$. So, it follows from (39) that $\varsigma_t(j)$ converges as $j \rightarrow \infty$ if the series $\sum_{j=0}^\infty V_t^{\Delta e}(j)$ converges and in that case $\lim_{j \rightarrow \infty} \varsigma_t(j) = \bar{\varsigma}_t = \varsigma_t(0) + L_t^1 \vartheta_t(0) + [L_t^1 \ L_t^2 \ \dots \ L_t^{T_\varsigma}] \sum_{j=0}^\infty V_t^{\Delta e}(j)$, where $L_t^j = Y_t^j + Y_t^1 A_\theta^{T_\varsigma-j+1} (I - A_\theta^{T_\varsigma})^{-1} = \bar{Y}_t^j (I - A_\theta^{T_\varsigma})^{-1}$ with

$\bar{Y}_t^j = \sum_{r=0}^{T_\zeta-1} \left(\prod_{i=1}^{T_\zeta-1-r} A_\zeta(t+T_\zeta-i) \right) C_\vartheta A_\vartheta^{(T_\zeta-j+1+r) \bmod T_\zeta}$. The convergence of the series $\sum_{j=0}^\infty V_t^{\Delta e}(j)$ is obvious when the subsequence $\{t_s\}_{s=0,1,\dots}$ is finite. However, even when this subsequence is infinite, the checking condition in step B3(iv) guarantees the absolute convergence of the series since $\hat{J}(t_s) \leq \hat{J}(t_{s-1}) - \|P_\vartheta^{\frac{1}{2}} \vartheta^\Delta(t_s)\|$ implies (summing from $s=1$ to ∞), $\sum_{s=1}^\infty \|P_\vartheta^{\frac{1}{2}} \vartheta^\Delta(t_s)\| \leq \hat{J}(0)$. Hence, the convergence of $\zeta_t(j)$ follows. Next, following the expression for $\bar{\zeta}_t$, we can write $\bar{\zeta}_{t+1}$ as

$$\bar{\zeta}_{t+1} = \zeta(t+1) + L_{t+1}^1 \vartheta(t+1) + [L_{t+1}^1 \quad L_{t+1}^2 \quad \dots \quad L_{t+1}^{T_\zeta}] \times \left[\begin{array}{c} 0 \quad I \quad 0 \quad \dots \quad 0 \\ 0 \quad 0 \quad I \quad \dots \quad 0 \\ \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\ 0 \quad 0 \quad 0 \quad \dots \quad I \\ I \quad 0 \quad 0 \quad \dots \quad 0 \end{array} \right] \left\{ \sum_{j=0}^\infty V_t^{\Delta e}(j) - \begin{bmatrix} \vartheta^{\Delta e}(t) \\ 0 \end{bmatrix} \right\}.$$

Since $A_\zeta(t)L_t^j = A_\zeta(t+T_\zeta)L_t^j = L_{t+1}^{j-1}$ with $L_{t+1}^0 = L_{t+1}^{T_\zeta}$, and $C_\vartheta + L_{t+1}^1 A_\vartheta = L_{t+1}^{T_\zeta} = A_\zeta(t)L_t^1$, we have,

$$\begin{aligned} \bar{\zeta}_{t+1} &= A_\zeta(t)\zeta(t) + C_\vartheta(\vartheta(t) + \vartheta^{\Delta e}(t)) + L_{t+1}^1 A_\vartheta(\vartheta(t) + \vartheta^{\Delta e}(t)) \\ &\quad - L_{t+1}^{T_\zeta} \vartheta^{\Delta e}(t) + A_\zeta(t) [L_t^1 \quad L_t^2 \quad \dots \quad L_t^{T_\zeta}] \sum_{j=0}^\infty V_t^{\Delta e}(j) \\ &= A_\zeta(t) \left(\zeta(t) + L_t^1 \vartheta(t) + [L_t^1 \quad L_t^2 \quad \dots \quad L_t^{T_\zeta}] \sum_{j=0}^\infty V_t^{\Delta e}(j) \right) \\ &= A_\zeta(t) \bar{\zeta}_t. \end{aligned}$$

Hence, we have part (c) of the statement. Finally, since

$$\begin{aligned} &\sup_{\kappa \in \mathbb{Z}_+} \sum_{t=0}^\kappa \left\{ \lambda_{\min}(\mathbf{Q}) \|\tilde{x}(t)\|^2 - \left(\max_{j \in \mathbb{N}_N} \gamma_j^2 \right) \|\tilde{w}(t)\|^2 \right\} \\ &\leq \sup_{\kappa \in \mathbb{Z}_+} \sum_{t=0}^\kappa \left\{ \sum_{j=1}^N \|Q_j^{\frac{1}{2}} \tilde{x}_j(t)\|^2 - \sum_{j=1}^N \gamma_j^2 \|\tilde{w}_j(t)\|^2 \right\} \leq \hat{J}(0), \end{aligned}$$

where the last inequality is implied by the monotonicity relation (38), condition (10) will be satisfied with $\gamma = \max(\gamma_1, \gamma_2, \dots, \gamma_N) / \sqrt{\lambda_{\min}(\mathbf{Q})}$, $\Theta_0 = \hat{J}(0) / \lambda_{\min}(\mathbf{Q})$ and with $\zeta(t)$ eventually following the dynamics in (7). Hence, we have part (d) of the statement. ■

Remark 3. Proposition 7 expresses the parameter γ of condition (10) in terms of γ_j , $j = 1, \dots, N$ considered in the subsystem performance functions. These parameters are chosen a-priori such that they ensure desired control performances in the subsystems. However, they also affect the size of the feasible invariant sets $\mathcal{X}_{\tilde{x}_j}$, $j = 1, \dots, N$ for the controlled subsystem error-states. A larger γ_j usually allows a larger size of the set $\mathcal{X}_{\tilde{x}_j}$. Therefore, when an allowable overall value of γ is specified a-priori for a desired overall control performance, it may be reasonable to choose $\gamma_j = \sqrt{\lambda_{\min}(\mathbf{Q})} \gamma$ in order to allow, for each subsystem $j \in \mathbb{N}_N$, as large $\mathcal{X}_{\tilde{x}_j}$ as possible.

Example 1. For an illustration, we consider a 5-member system with LTV subsystem dynamics described by matrices

$$A_j(t) = \begin{pmatrix} a_j^{11} & a_j^{12} \\ 0 & a_j^{22} + \rho_j(t) \end{pmatrix}; \quad B_j(t) = \begin{pmatrix} 0 \\ 1 + 2\rho_j(t) \end{pmatrix} \quad (40)$$

where $a_j^{11} = a_j^{12} = 1$ and $|\rho_j(t)| \leq 0.05$, $j = 1, \dots, N$, and $a_j^{22} = 1.1$ for $j = 1, 2, 3$ and $a_j^{22} = 1.2$ for $j = 4, 5$. We assume $\|w_j(t)\|_\infty \leq 1$ with $D_j(t) = \text{diag}(0.4, 0.1)$, and consider constraints $|x_{j[1]}(t)| \leq 250$, $|x_{j[2]}(t)| \leq 10$ and $|u_j(t)| \leq 1$,

Fig. 2: Inter-subsystem communications structure for examples 1 and 3 (left) and that for example 2 (right).

$j = 1, \dots, N$. Next, for $C_j = [1 \quad 0]$, $j = 1, \dots, 5$, we choose $A_\zeta(t) = C_\zeta = 1$, $\bar{A}_j(t) = A_j(t)$, $\bar{B}_j(t) = B_j(t)$, $S_j = [1 \quad 0]^T$ and $U_j = 0$, $j = 1, \dots, 5$. In the cost expression, $Q_j = 2I$ and $R_j = 2$ are considered for $j = 1, 2, 3$, and $Q_j = I$ and $R_j = 1$ for $j = 4, 5$. Further, we select an LQ optimal K_j (for the average model with $\rho_j(t)=0$), and choose $\gamma_j = 2$ and $n_{\xi_j} = 5$ for each subsystem. We choose $A_\vartheta = 0$ and $C_\vartheta = 1$ in (12), consider $\mathcal{X}_\zeta = \{\zeta \mid |\zeta| \leq 50\}$ and $\mathcal{X}_\vartheta = 0.5\mathcal{X}_\zeta$, and determine the subsystem controller matrices (assuming $E_j^{(r)} = 0$, $r = 1, \dots, h_j$) and feasible sets as mentioned in Section III-B.

We consider the scheme of Fig. 1 with $T_{cc} = 3$ for the subsystem initial states shown in the first row of Table I. The assumed inter-subsystem communications structure is shown in Fig. 2 (left). For the computation of $\vartheta(t)$ with Algorithm 1, we choose a step-size rule of the form $a_k = \bar{a}/(\bar{b} + k)$, with suitably chosen $\bar{a} > 0$ and $\bar{b} \geq 1$. Fig. 3(a) compares the convergence of the local instances v_j , $j = 1, \dots, 5$ of $\vartheta(t)$ at $t = 0$ obtained with $a_k = 100/(100 + k)$ and with different values of $\eta_k = \eta$ in the averaging step (step 2b) of the algorithm. The rate of convergence with $\eta_k = 5$ is found to be almost the same as that would be obtained with the use of the actual average in step 2b. We consider $\eta_k = 5$ for all k .

In the simulations, $\rho_j(t)$ and $w_j(t)$ are chosen from uniform random distributions. With the scheme of Fig. 1, the subsystems finally reach the near-consensus condition with $\bar{\zeta} = 18.01$. Fig. 3(b) shows the convergence of $s_j(t) = x_{j[1]}(t)$, $j = 1, \dots, 5$ to the final value $\bar{\zeta}$. We compare the total regulation cost in achieving the near-consensus condition using the scheme of Fig 1 with the corresponding costs when the subsystems are driven to the neighbourhood of the origin under the same sets of realizations of $\rho_j(t)$ and $w_j(t)$. It is found that the cost is about 33.4% lower in the former case. When an update of the consensus signal is made only at time at $t = 0$, the cost is found to be about 31% lower.

Next, we consider slightly different subsystem dynamics given by $A_j(t)$ as in (40) with $a_j^{12} = \rho_j(t) \in [-0.02, 0.02]$ for $j = 1, \dots, 5$ and $a_j^{11} = a_j^{22} = 1$ for $j = 1, \dots, 3$, and $a_j^{11} = a_j^{22} = 1.02$ for $j = 4, 5$. Further, $B_j(t) = I$ and $D_j(t) = \text{diag}(0.2, 0.1)$ are given for all subsystems. We consider $C_j = S_j = A_\zeta = I$, $n_\zeta = 2$ and $\mathcal{X}_\zeta = \{\zeta \mid \|\zeta\|_\infty \leq 5\}$, and $A_\vartheta = 0$, $C_\vartheta = I$ and $\mathcal{X}_\vartheta = 0.5\mathcal{X}_\zeta$. So, we have $\bar{A}_j(t) = I$ for all subsystems, and $U_j = 0$ for $j = 1, 2, 3$ and $U_j = \text{diag}(-0.02, -0.02)$ for $j = 4, 5$. Further, we have $\tilde{w}_j(t) = w_j(t)$ and $\bar{D}_j(t) = D_j(t)$ for $j = 1, 2, 3$, and define $\bar{D}(t) = [D_j(t) \quad (A_j(t) - I) \begin{bmatrix} 9 \\ 5 \end{bmatrix}]$ and $\tilde{w}_j(t) = [w_j(t)^T \quad [0 \quad \frac{1}{5}]\zeta(t)]^T$ for $j = 4, 5$. Using all other details as in the earlier case, we apply the scheme of Fig. 1 for the initial subsystem states which are marked with asterisks in Fig. 3(c) which also shows the resulting trajectories of

TABLE I: Initial subsystem states for examples 1 and 3

	Member 1	Member 2	Member 3	Member 4	Member 5
Example 1	$[144.0 \ 1.2]^T$	$[-40.0 \ 2.2]^T$	$[136.0 \ -2.2]^T$	$[118.4 \ -0.9]^T$	$[-90.4 \ -2.0]^T$
Example 3	$[81.0 \ 0.7]^T$	$[-22.5 \ 1.3]^T$	$[76.5 \ -1.2]^T$	$[57.6 \ -0.5]^T$	$[-37.4 \ 1.1]^T$

Fig. 3: (a) Convergence of instances $v_j, j = 1, \dots, 5$ of the coupling variable in the first case of Example 1 to $v(0)$ with Algorithm 1 at time $t = 0$, (b) Actual convergence of the consensus components of subsystem states in the first case of Example 1 under the control scheme of Fig. 1, (c) Actual convergence of the subsystem states in the second case of Example 1.

the subsystem states, when random realizations of $\rho_j(t)$ and $w_j(t)$ are considered as before. The aggregate cost under the proposed scheme is found to be about 18% lower than the corresponding cost in a case where all subsystem states are driven to the origin.

Example 2. This example deals with a consensus problem arising in the reconfiguration of a satellite formation. We consider a simple cross-track pendulum formation in which the motion of the satellites (relative to some reference point moving along an orbit called the reference orbit) is limited to only one direction — the direction normal to the plane of the reference orbit. The (approximate) discrete-time dynamics of this motion are described by the system matrices $A_j(t) = \Phi(\tau, t)_{\tau=T_s}$, $B_j(t) = \left(\int_0^{\varrho T_s} \Phi(T_s - \tau, t) d\tau \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with

$$\Phi(\tau, t) = \begin{bmatrix} \cos(\psi(t)\tau) & \frac{1}{\psi(t)} \sin(\psi(t)\tau) \\ -\psi(t) \sin(\psi(t)\tau) & \cos(\psi(t)\tau) \end{bmatrix} \quad (41)$$

where T_s is the sampling period, ϱ is the fraction of T_s during which the control input is applied and $\psi(t) = \bar{\omega}^{1/4} \omega(t)^{3/4}$ where $\omega(t)$ is the angular velocity of the reference point and $\bar{\omega}$ is its mean value [49]. Here, the state $x_j(t) \in \mathbb{R}^2$ represents the displacement-velocity pair and the input $u_j(t) \in \mathbb{R}$ represents the control acceleration. And, with a suitably chosen matrix $D_j(t) = D$, the disturbance $w_j(t) \in \mathbb{R}^2$ is assumed to satisfy $\|w_j(t)\|_\infty \leq 1$. The dynamics of this example result in cross-track oscillations about the reference point. We consider a problem of configuring a set of $N = 6$ satellites, initially in oscillations with various amplitudes and phases, in such a way that they move with the same amplitude and with the same or uniformly spaced phases. The communications structure used in this example is shown in Fig. 2 (right).

First we assume $\psi(t) = \omega(t) = \bar{\omega}$ and wish to configure the satellites such that the phase separation between neighboring members is 60° . We consider the following data: $\bar{\omega} = 0.001$, $T_s = 30s$, $\varrho = 1/3$, $D = \text{diag}(0.15, 0.002)$, and the following constraints: $|u_k| \leq 3 \text{ mm/s}^2$, $|x_{[1]}(t)| \leq 3\text{km}$, $|x_{[2]}(t)| \leq 3\text{m/s}$.

We let $Q_j = Q = \text{diag}(10^{-4}, 1)$ and $R_j = R = 10^2$, use the LQ optimal gain as K_j and choose $\gamma_j = 0.01$ for all subsystems. Next, we assume $A_\zeta(t) = A_\zeta = \begin{bmatrix} \cos \bar{\omega} T_s & \sin \bar{\omega} T_s \\ -\sin \bar{\omega} T_s & \cos \bar{\omega} T_s \end{bmatrix}$, $C_\zeta = I$ and choose $C_j = \begin{bmatrix} \cos(2\pi j/N) & \sin(2\pi j/N) \\ -\sin(2\pi j/N) & \cos(2\pi j/N) \end{bmatrix} \text{diag}(1, 1/\bar{\omega})$, $S_j = C_j^{-1}$, $\bar{A}_j(t) = A_j(t)$ and $U_j = 0$, $j = 1, \dots, N$. We obtain control policies considering $A_\vartheta = 0$, $C_\vartheta = I$, $\mathcal{X}_\zeta = \{\zeta \mid |\zeta_{[1]}| \leq 0.8\text{km}, |\zeta_{[2]}| \leq 0.8\text{m/s}\}$ and $\mathcal{X}_\vartheta = 0.5\mathcal{X}_\zeta$.

The scheme of Fig. 1 is applied with $T_{cc} = 3$, $\eta_k = 5$ and the other details as in the previous example. The initial satellite states are chosen from oscillations with arbitrary amplitudes (of up to 1.5km) and phases, and $w_j(t)$ for each member is chosen from a uniform random distribution. The components of the resulting $\zeta(t)$ starting from $\zeta(0) = 0$ are shown in Fig. 4(a) where the time instants at which the signal is updated are marked with asterisks. The corresponding subsystem state trajectories are shown in Fig. 4(b) where the initial and the final states are marked with asterisks and circles respectively. When we compare the costs of achieving the desired configuration, it is found that the total cost and the control input cost are about 26% and 12% lower for an on-line-determined configuration with the proposed scheme than for a pre-determined configuration specified by $\sigma(0) = \zeta(0) = (500\text{m}, 0)$.

Next, we consider the case of a time varying $\omega(t)$ resulting from a reference orbit of eccentricity 0.1 so that $\omega(t) \in [0.00082, 0.00122]$ and $\bar{\omega} = 0.001$. We wish to align the satellites in oscillations such that their amplitudes and phases are synchronized. With all other details remaining the same as in the previous case, we choose $A_\zeta(t) = A_j(t)$ and $C_j = S_j = C_\zeta = I$. We approximate the elements of the system matrices in (41) by their first order Taylor approximations in terms of $\omega(t)$ about $\bar{\omega}$ so that we have polytopic subsystem matrices defined with 2 vertices. We determine the subsystem controller matrices and apply the scheme of Fig. 1 for an arbitrarily chosen sets of subsystem initial states. The resulting synchronization of the position components of the member

Fig. 4: (a) Time variation of the components of $\varsigma(t)$ and (b) member state trajectories under the scheme of Fig. 1 in the first case of Example 2. (c) Synchronization of the position components of the member states in the second case of Example 2.

states are shown in Fig. 4(c).

IV. HANDLING COMPUTATIONAL DELAYS IN RHC-BASED COORDINATED CONTROL

In the basic RHC-based scheme of Fig. 1, we have assumed that the computations can be performed accurately in a very small time so that computational delays and their effects on the control performance are negligible. Practically, however, the distributed optimization of $\vartheta(t)$ requires extensive iterative computations involving inter-subsystem exchange of information. If the delays are significant, it may not be possible to guarantee the desired performance. One of the straightforward ways to take computational delays into account is to first project the subsystem state information forward into an appropriate future time instant and then to optimize the future value of $\vartheta(t)$. However, projecting the state information of a subsystem forward to a future time instant using the actual uncertain dynamics model will result in a set of possible values, rather than an exact value, of the future state, and solving the control problem for sets of possible subsystem states at the chosen future time instant will likely result in conservativeness or even infeasibility. Alternatively, we may project the subsystem state using a nominal dynamics model but, since this projecting mechanism is not perfect, the optimization result may require a further check for the relative optimality of the computed value of the global variable.

An alternative to the above mentioned ways of handling computational delays is to compute the global variable based on the current subsystem state information and then use a mechanism to project the optimization result forward into the future. In this section, we discuss an approach in which we use the current subsystem state information and a suitable mechanism to optimize the predicted values of $\vartheta(t)$ from a chosen future time instant. In particular, the proposed approach employs time-delayed version of augmented subsystem error-state dynamics for each subsystem and, with some assumptions, it allows us to optimize the predicted future $\vartheta(t)$ such that the desired control performance can be guaranteed. In the following, we discuss the details of the approach.

A. Computation of time-delayed version of local controller dynamics

Let us consider that we wish to allow T_c time steps for the computation of $\vartheta(t)$ using Algorithm 1. Then, we let $\vartheta(t)$, A_ϑ and C_ϑ have the structure $\vartheta(t) = [\vartheta^{[0]T} \ \vartheta^{[1]T} \ \dots \ \vartheta^{[p]T}]^T$,

$$A_\vartheta = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_\vartheta^{[q]} \\ 0 & 0 & 0 & \dots & A_\vartheta^{[p]} \end{bmatrix}, \quad C_\vartheta = [I \ 0 \ 0 \ \dots \ 0] \quad (42)$$

where $p = T_c$ and $q = T_c - 1$. Here, the components $\vartheta^{[0]}, \dots, \vartheta^{[p-1]}$ of $\vartheta(t)$ give the predicted $\Delta\varsigma(t+i|t)$, $i = 0, \dots, p-1$. Predicted $\Delta\varsigma(t+i|t)$, $i \geq p$ will depend on the matrices $A_\vartheta^{[q]}$ and $A_\vartheta^{[p]}$ which should be appropriately chosen. They may have the same form as A_ϑ and C_ϑ of (42), in which case $\vartheta^{[p]}$ will represent several further values of $\Delta\varsigma(t+i|t)$.

In a similar way, we may consider $\xi_j(t)$ to have the form $\xi_j(t) = [\xi_j^{[0]}(t)^T \ \xi_j^{[1]}(t)^T \ \dots \ \xi_j^{[p]}(t)^T]^T$, and let the matrices $G_j^{(r)}$, $F_j^{(r)}$ and H_j have the structure

$$G_j = \begin{bmatrix} G_j^{[00]}(r) & G_j^{[01]}(r) & 0 & \dots & 0 \\ G_j^{[10]}(r) & G_j^{[11]}(r) & G_j^{[12]}(r) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_j^{[q0]}(r) & G_j^{[q1]}(r) & G_j^{[q2]}(r) & \dots & G_j^{[qp]}(r) \\ G_j^{[p0]}(r) & G_j^{[p1]}(r) & G_j^{[p2]}(r) & \dots & G_j^{[pp]}(r) \end{bmatrix}, \quad (43)$$

$$F_j = \begin{bmatrix} F_j^{[0]}(r) \\ F_j^{[1]}(r) \\ \vdots \\ F_j^{[q]}(r) \\ F_j^{[p]}(r) \end{bmatrix}, \quad H_j = [H_j^{[0]} \ 0 \ \dots \ 0].$$

Here the dimensions of the components of $\xi_j(t)$ are appropriately chosen. Furthermore, we let the matrix $E_j^{(r)}$ have the same structure as that of $G_j^{(r)}$.

It is obvious from the structure of the controller matrices in (42) and (43) that the component $\vartheta^{[p]}$ of $\vartheta(t)$ and $\xi_j^{[p]}$ of ξ_j will not be used in the update of the subsystem error-state until T_c steps later. Hence they can be freely optimized employing

the distributed optimization algorithm within T_c time steps in order to determine the future updates of the consensus signal.

Employing the special structure of the controller matrices enables us to consider optimization of the future updates of the consensus signal. However depending upon the assumed computation time T_c , this may significantly increase the dimension n_θ and n_{ξ_j} , $j \in \mathbb{N}_N$ of the decision variables and hence the size of the optimization problems. Nevertheless, since the determination of the controller matrices and the invariant sets are carried out off-line, the increase in the on-line computational burden will likely remain modest. For the off-line computation of the controller matrices, we can use the alternating SDP optimization approach mentioned in the last section. The initial feasible solution can be obtained by solving an appropriate BMI feasibility problem using available methods such as cone-complementarity linearization [50].

B. Overall RHC scheme for achieving near-consensus with delayed computations

Fig. 5 outlines details of the modified version of the RHC-based overall control scheme that employs control computations based on the controller structures discussed above.

The modified RHC scheme of Fig. 5 basically follows the same set of steps as in the basic scheme of Fig. 1. The steps in B3 of the modified scheme parallel those in B3 of the basic scheme of Fig. 1 but are split into two groups which are now executed at separate time instants. Moreover, in the modified scheme, we choose to re-optimize the local variables once the solution of the global problem becomes available at times $t = nT_{cc} + T_c$ in order to update them as per the current subsystem state information. This re-optimization was not employed in the basic scheme of Fig. 1 since there was no computational delay. Another difference is that in the modified scheme, we do not optimize the local control variables for $T_c - 2$ time steps once the distributed optimization of the future consensus signal update is initiated at some time $t = nT_{cc}$. This means that the control actions predicted at time $t = nT_{cc}$ will have to be used for the actual implementation at times $t = nT_{cc} + 1$ to $t = nT_{cc} + T_c - 1$. So, we ensure that these predicted control actions, which depend on the disturbances and the time-variation of the subsystem dynamics, are actually implementable. We consider the following assumptions in this context.

A6: For each $j \in \mathbb{N}_N$, the value of the time-varying parameter $\theta_j(t)$ defining the subsystem matrices is known at the time instant $t + 1$.

A7: For each $j \in \mathbb{N}_N$, $D_j(t)$ has full column rank at all $t \in \mathbb{Z}_+$.

We have the following result on the performance of the overall RHC scheme of Fig. 5.

Proposition 8. *Given that $T_c = 1$ or that assumptions A6-A7 hold, if constraint-admissible values of $\varsigma(0)$ and $\vartheta(0)$ are available for the given initial conditions at time $t = 0$, then the RHC scheme of Fig. 5 guarantees that (a) the control problem remains feasible at all times $t \in \mathbb{Z}_+$, (b) the control actions to be implemented are computable for each subsystem at all times $t \in \mathbb{Z}_+$, (c) the cost monotonicity condition (38) holds at all times $t \in \mathbb{Z}_+$, (d) the trajectory of $\varsigma(t)$ eventually follows the dynamics in (7), and (e) the subsystems achieve*

A. Off-line (Before time $t = 0$) :

1. Select T_c and n_θ . Choose appropriate matrices C_θ and A_θ in the form of (42).
2. For each subsystem $j \in \mathbb{N}_N$, select K_j and γ_j . Select n_{ξ_j} and determine $H_j, G_j^{(r)}, E_j^{(r)}, F_j^{(r)}$ ($r = 1, \dots, h_j$) of the form in (43) using (21). Also, determine $\mathcal{X}_{\chi_j}$ and P_j .
3. Choose $T_{cc} > T_c$ based on the nature of subsystem dynamics.

B. On-line (Repeat at each time instant $t = 0, 1, 2, \dots$) :

1. For each $j \in \mathbb{N}_N$, obtain the state measurement $x_j(t)$.
2. If $t = 0$, choose feasible values of $\varsigma(0)$ and $\vartheta(0)$. Set $t_0 = 0$, $s = 1$.
3. If $t = nT_{cc} + i$, where $n \in \{0, 1, 2, \dots\}$ and $i = 0$ or T_c , then,
 - i) If $t = nT_{cc}$, where $n \in \{0, 1, 2, \dots\}$, then
 - Set $updateFlag = 1$.
 - Solve problem (25) for each $j \in \mathbb{N}_N$ with the existing value of $\vartheta(t)$ to obtain $\xi_j(t)$, and compute $\hat{J}(t) = \sum_{j=1}^N \hat{J}_j(t)$. If $t > 0$ and $\hat{J}(t) \geq \hat{J}(t_{s-1})$, set $updateFlag = 0$.
 - If $updateFlag = 1$, start a fresh run of Algorithm 1 to solve (24) with the current states $x_j(t)$, $j = 1, \dots, N$ and the current values of $\vartheta^{[p]}(t)$ and $\xi_j^{[p]}(t)$, $p = 0, \dots, T_c - 1$, $j = 1, \dots, N$ to obtain $\vartheta^{[T_c]}(t)$. (The computations started here can last till $t = nT_{cc} + T_c$.)
 - ii) If $t = nT_{cc} + T_c$, where $n \in \{0, 1, 2, \dots\}$ and if $updateFlag = 1$, then
 - Obtain the value of $\vartheta(t - T_c)$ from the latest run of Algorithm 1 started at $t - T_c$. Denote it by $\vartheta^+(t - T_c)$ and set $\vartheta^+(t) = A_\theta^{T_c} \vartheta^+(t - T_c)$.
 - For each $j \in \mathbb{N}_N$, solve (25) with $\vartheta^+(t)$ and denote the computed local variable by $\xi_j^+(t)$. Compute the total cost $\sum_{j=1}^N \hat{J}_j(t)$ and denote it by $\hat{J}^+(t)$. Let $\vartheta^-(t) = A_\theta \vartheta(t - 1)$ and $\vartheta^\Delta(t) = \vartheta^+(t) - \vartheta^-(t)$. If $\hat{J}^+(t) > \hat{J}(t_{s-1}) - \|P_\theta^{\frac{1}{2}} \vartheta^\Delta(t)\|$, set $updateFlag = 0$.
 - If $updateFlag = 1$, set $\vartheta(t) = \vartheta^+(t)$, $\xi_j(t) = \xi_j^+(t)$, $j = 1, \dots, N$, $\hat{J}(t) = \hat{J}^+(t)$, $t_s = t$, $s = s + 1$.
4. If $t = nT_{cc} + i$, $i \in \{1, \dots, T_c\}$ and $updateFlag = 0$, or if $t = nT_{cc} + i$, $i \in \{T_c + 1, \dots, T_{cc} - 1\}$, solve (25) with the existing value of $\vartheta(t)$ to obtain $\xi_j(t)$ for each $j \in \mathbb{N}_N$. Else, if $t = nT_{cc} + i$, $i \in \{1, \dots, T_c - 1\}$ and $updateFlag = 1$, for each $j \in \mathbb{N}_N$, obtain $\xi_j(t) = G_j(t - 1)\xi_j(t - 1) + E_j(t - 1)\vartheta(t - 1) + F_j(t - 1)\tilde{w}_j(t - 1)$.
5. For each subsystem, $j \in \mathbb{N}_N$, apply the control input $u_j(t) = U_j \varsigma(t) + K_j(x_j(t) - S_j \varsigma(t)) + H_j \xi_j(t)$.
6. Set $\varsigma(t + 1) = A_\varsigma \varsigma(t) + C_\theta \vartheta(t)$, $\vartheta(t + 1) = A_\theta \vartheta(t)$.

Fig. 5: Overall control scheme for achieving the near-consensus condition with delayed computations.

the near-consensus condition of Definition 2 with γ and Θ_0 as mentioned in Proposition 7.

Proof: Part (a) of the statement follows from the argument used for proving part (a) of Proposition 7. To see part (b), note that if $T_c = 1$, the subsystem control actions $\xi_j(t)$, $j \in \mathbb{N}_N$ will always be obtained through the solution of the local control problems based on the existing value of $\vartheta(t)$. So, implementable control actions will always be computable. On the other hand, if assumptions A6 and A7 hold, controller matrices $G_j(t - 1)$, $E_j(t - 1)$ and $F_j(t - 1)$ will be known at time t since $\theta_j(t - 1)$ becomes known, and $w_j(t - 1)$ can be computed from the value of $\tilde{D}_j(t - 1)\tilde{w}_j(t - 1) = \tilde{x}_j(t) - A_j(t - 1)\tilde{x}_j(t - 1) - B_j(t - 1)\tilde{u}_j(t - 1) + S_j \Delta \varsigma(t - 1)$, where $x_j(t)$ is the subsystem state measured at time t . The predicted control actions are, therefore, computable at times $t = nT_{cc} + i$, $n \in \{0, 1, 2, \dots\}$, $i \in \{1, 2, T_c - 1\}$ in step B4 of the control scheme. Next, since the controller matrices considered for each subsystem satisfy the boundedness condition (18),

Fig. 6: Consensus trajectories for $T_c = 2$ and $T_c = 0$ in Example 3.

the corresponding cost monotonicity conditions hold and part (c) of the statement follows. Finally, the proofs of parts (d) and (e) of the statement follow from the lines of reasoning used in proving parts (c) and (d) of Proposition 7. ■

Note that assumption **A6** means that the subsystem matrices can no longer be always unknown or indeterminable. They may be unknown *a priori* but they need to be determinable *a posteriori*. Assumption **A7**, on the other hand, is just a convenient assumption to guarantee that $w_j(t-1)$ can be determined uniquely at time t if required for the computation of the control action $u_j(t)$. Such uniqueness, however, is not really necessary for the results of this paper.

Example 3. We again consider the first case of Example 1 but assuming $\rho_j(t) = \sin \frac{2\pi t}{T_\rho}$ with $T_\rho = 50$. We choose $T_c = 2$ and, accordingly, consider the controller matrices of the form (43) with $n_{\xi_j} = 5$ for all subsystems. We also choose $n_\vartheta = 3$ and select $A_\zeta^{[2]} = 0$ and $A_\vartheta^{[1]} = 1$ in (42). The components $\xi_j^{[0]}$, $\xi_j^{[1]}$ and $\xi_j^{[2]}$ are chosen to have dimensions 1, 1 and 3 respectively. The controller matrices and the feasible invariant sets are computed as mentioned in Section III-B.

We consider the control scheme of Fig. 5 with $T_{cc} = 3$ for a set of subsystem initial states shown in the third row of Table I. For a random realization of $w_j(t)$, $j = 1, \dots, 5$, the resulting consensus signal trajectory is shown in Fig. 6 (solid line). The figure also shows the consensus signal trajectory that would result with the same set of initial conditions and with the same set of random realizations of the disturbances if there were no computational delays and $\vartheta(t)$ could be optimized and implemented at each time $t = nT_{cc}$. When the total regulation cost is compared with the corresponding cost of regulation to the origin, it is found that the cost in this example with $T_c = 2$ is about 34% lower when compared with the regulation to the origin. Without the computational delay (i.e. if $T_c = 0$), however, the reduction would be about 41%.

V. CONCLUSION

We have presented an RHC-based scheme for a class of consensus-related coordinated control problems that involve autonomously actuated subsystems with time-varying dynamics and additive disturbances. The proposed scheme employs a computationally efficient RHC algorithm for an H_∞ -like performance in conjunction with an approximate subgradient method suitable for distributed implementation in order to regularly optimize the consensus trajectory and the control

inputs for the subsystems. We have also explored a way to handle computational delays in the distributed computation of the consensus signal updates in order to ensure the desired overall control performance. Simulation results illustrate the effective performance of the proposed control scheme.

APPENDIX

A. Proof of Lemma 4

If $\bar{c}_j = 0$, the result is obvious. So, we assume $\bar{c}_j > 0$. Let $\tilde{\mathbb{U}}_j = \mathbb{U}_j \cup U_j \mathcal{X}_\zeta$. Then, since $\bar{c}_j \tilde{\mathbb{U}}_j \subseteq U_j \mathcal{X}_\zeta$ from the definition of $\bar{c}_j$, it follows that $u_j(t) \in (1 - c_j/\bar{c}_j)U_j \mathcal{X}_\zeta \oplus (1 + c_j)\tilde{\mathbb{U}}_j = \mathbb{U}_j \oplus (c_j/\bar{c}_j)(U_j \mathcal{X}_\zeta \ominus \bar{c}_j \tilde{\mathbb{U}}_j) \subseteq \mathbb{U}_j$. In the similar way, we also have $x_j(t) \in \mathbb{X}_j$. Next, since $\mathcal{X}_{\chi_j}$ is invariant for (14), any set $(1 + c_j)\mathcal{X}_{\chi_j}$, $c_j \geq 0$ is also invariant for (14). This can be easily seen by noting that $\Psi_j^{(r)} \mathcal{X}_{\chi_j} + \mathcal{D}_j^{(r)} \tilde{\mathbb{W}}_j \subseteq \mathcal{X}_{\chi_j}$ implies $\Psi_j^{(r)}(1 + c_j)\mathcal{X}_{\chi_j} + \mathcal{D}_j^{(r)} \tilde{\mathbb{W}}_j \subseteq (1 + c_j)\mathcal{X}_{\chi_j}$ for any $c_j > 0$. So, we have $\chi_j(t + i|t) \in (1 + c_j)\mathcal{X}_{\chi_j}$, $\forall i \in \mathbb{Z}_+$. This, together with condition (22b), ensures that $u_j(t + i|t) \in \mathbb{U}_j$, $x_j(t + i|t) \in \mathbb{X}_j$, $\forall i \in \mathbb{Z}_+$. Hence, the result follows. ■

B. Proof of Proposition 6

For any k , we have

$$\begin{aligned} & \|\lambda^* - \lambda^{(k+1)}\|^2 \\ &= \|\lambda^* - \lambda^{(k)} - a_k \hat{g}_p(\lambda^{(k)})\|^2 \\ &= \|\lambda^* - \lambda^{(k)}\|^2 + a_k^2 \|\hat{g}_p(\lambda^{(k)})\|^2 - 2a_k \hat{g}_p(\lambda^{(k)})^\top (\lambda^* - \lambda^{(k)}) \\ &= \|\lambda^* - \lambda^{(k)}\|^2 + a_k^2 \|\hat{g}_p(\lambda^{(k)})\|^2 - 2a_k v^*(\lambda^{(k)})^\top (\lambda^* - \lambda^{(k)}) \\ & \quad + 2a_k (\tilde{v}(\lambda^{(k)}, \eta_k) - \bar{v}(\lambda^{(k)}))^\top (\lambda^* - \lambda^{(k)}) \end{aligned}$$

where we have used the fact $\bar{v}(\lambda^{(k)})^\top (\lambda^* - \lambda^{(k)}) = 0$ to obtain the third equality above. Next, since

$$\begin{aligned} & \tilde{v}(\lambda^{(k)}, \eta_k) - \bar{v}(\lambda^{(k)}) \\ &= ((W \otimes I_{n_\vartheta})^{\eta_k} - \mathbf{1}^\top / N \otimes I_{n_\vartheta}) v^*(\lambda^{(k)}) \\ &= ((W \otimes I_{n_\vartheta})^{\eta_k} - \mathbf{1}^\top / N \otimes I_{n_\vartheta}) (v^*(\lambda^{(k)}) - \bar{v}(\lambda^{(k)})) \\ &= ((W - \mathbf{1}^\top / N)^{\eta_k} \otimes I_{n_\vartheta}) g_p(\lambda^{(k)}), \end{aligned}$$

$\|\tilde{v}(\lambda^{(k)}, \eta_k) - \bar{v}(\lambda^{(k)})\| \leq c^{\eta_k} \|g_p(\lambda^{(k)})\|$ and $\|\hat{g}_p(\lambda^{(k)})\| = \|g_p(\lambda^{(k)}) + \bar{v}(\lambda^{(k)}) - \tilde{v}(\lambda^{(k)}, \eta_k)\| \leq (1 + c^{\eta_k}) \|g_p(\lambda^{(k)})\|$. Moreover, since $-v^*(\lambda^{(k)})$ is a subgradient of $-h(\lambda^{(k)})$, $-(h^* - h(\lambda^{(k)})) \geq -v^*(\lambda^{(k)})^\top (\lambda^* - \lambda^{(k)})$. So, we have,

$$\begin{aligned} & \|\lambda^* - \lambda^{(k+1)}\|^2 \\ & \leq \|\lambda^* - \lambda^{(k)}\|^2 + a_k^2 (1 + c^{\eta_k})^2 \|g_p(\lambda^{(k)})\|^2 - 2a_k (h^* - h(\lambda^{(k)})) \\ & \quad + 2a_k c^{\eta_k} \|g_p(\lambda^{(k)})\| \|\lambda^* - \lambda^{(k)}\| \\ & \leq \|\lambda^* - \lambda^{(k)}\|^2 + a_k^2 (1 + c^{\eta_k})^2 \|g_p(\lambda^{(k)})\|^2 - 2a_k (h^* - h(\lambda^{(k)})) \\ & \quad + 2a_k \epsilon_k \end{aligned}$$

where the last inequality follows from (35). Applying this inequality successively for $k = k-1$ until $k = 0$, we get

$$\begin{aligned} \|\lambda^* - \lambda^{(k+1)}\|^2 & \leq \|\lambda^* - \lambda^{(0)}\|^2 + \sum_{i=0}^k a_i^2 (1 + c^{\eta_i})^2 \|g_p(\lambda^{(i)})\|^2 \\ & \quad - 2 \sum_{i=0}^k a_i (h^* - h(\lambda^{(i)})) + 2 \sum_{i=0}^k a_i \epsilon_i. \end{aligned}$$

Since $\|\lambda^* - \lambda^{(k+1)}\|^2 \geq 0$, we can write this as

$$\begin{aligned} & \frac{\sum_{i=0}^k a_k \left(h^* - h(\lambda^{(k)}) \right)}{\sum_{i=0}^k a_k} \\ & \leq \frac{\frac{1}{2} \|\lambda^* - \lambda^{(0)}\|^2 + \sum_{i=0}^k a_k \left(\epsilon_k + \frac{1}{2} a_k (1 + c^{\eta_k})^2 \|g_p(\lambda^{(k)})\|^2 \right)}{\sum_{i=0}^k a_k}. \end{aligned} \quad (\text{A.1})$$

Since $\sum_{i=0}^k a_k \rightarrow \infty$ as $k \rightarrow \infty$, using the properties of the Cesàro averages of sequences [48] and the facts that $h^* - h(\lambda^{(k)})$ is convex and $\|g_p(\lambda^{(k)})\|$ is bounded, conclusions (36) and (37) directly follow from (A.1). ■

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