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2014

Chen, W., Wen, C., Hua, S., & Sun, C. (2014). Distributed Cooperative Adaptive Identification and Control for a Group of Continuous-Time Systems With a Cooperative PE Condition via Consensus. *IEEE Transactions on Automatic Control*, 59(1), 91-106.

<https://hdl.handle.net/10356/103818>

<https://doi.org/10.1109/TAC.2013.2278135>

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Distributed Cooperative Adaptive Identification and Control for a Group of Continuous-Time Systems with a Cooperatively PE Condition via Consensus

Weisheng Chen, *Member, IEEE*, Changyun Wen, *Fellow, IEEE*, Shaoyong Hua, Changyin Sun

Abstract—In the paper, we first address the uniformly exponential stability (UES) problem of a group of distributed cooperative adaptive systems in a general framework. Inspired by consensus theory, distributed cooperative adaptive laws are proposed to estimate the unknown parameters of these systems. It is shown that not only the whole closed-loop system is stable, but also both the identification/tracking error and the parameter estimation error converge to zero uniformly exponentially under a cooperatively persistent excitation (PE) condition of a regressor matrix in each system which is weaker than the traditionally defined PE condition. The effects of network topology on UES of the closed-loop system are also explored. If the topology is time-invariant, it needs to be undirected and connected. However, when the topology is time-varying, it is just required that the integration of the topology over an interval with fixed length is undirected and connected. The established results are then employed to identify and control several classes of linearly parameterized systems. Simulation examples are also provided to demonstrate the effectiveness and applications of the proposed distributed cooperative adaptive laws.

Index Terms—Linearly parameterized system, uniformly exponential stability (UES), persistent excitation (PE), distributed cooperative adaptive law, system identification and control, network topology, consensus.

I. INTRODUCTION

THE goal of this paper is to propose originally a distributed cooperative adaptive strategy for a group of uncertain systems, and to analyze its advantages over the conventional decentralized adaptive methods. The main motivation arises from the recent development of consensus theory [1-12] and adaptive control theory [12-32]. As background of this work, the consensus theory and the adaptive control theory will be firstly reviewed in the following two subsections, respectively.

A. Consensus Theory and Cooperative Strategy

A cooperative strategy proposed for a group of agents is to achieve a common objective in a cooperative way among individual agents in the group. It has attracted much attention

in the field of optimization and computation [1] since 1990s. As a fundamental cooperative strategy, consensus requires that the states of each agent reach a common value via local communication. However, it has received attention only recently. In [2], Jadbabaie et al. analyzed the consensus of the Vicsek model [3] theoretically. Since then, the study on consensus of multi-agent systems has been extensively active in the area of control theory. Until now, many interesting results have been established, see for examples the survey paper [4] and the book [5]. When consensus theory becomes more and more mature, many researchers begin to focus on its applications in other areas such as wireless sensor network [6], flocking problems [7] and so on. In this paper we try to explore a new application of the consensus theory in the fields of identification and adaptive control. Convergence of parameter estimates and uniformly exponential stability (UES) of adaptive systems are established without the requirement that the signals in each agent satisfy the traditionally defined PE condition. Instead they only need to be cooperatively persistent exciting, a weaker condition newly defined in this paper.

B. Adaptive Control Theory and PE Condition

The investigation of uncertain systems has also attracted tremendous attention from many researchers in the past several decades. Linearly parameterized systems, as one of the most important uncertain systems, are usually controlled by adaptive method. Adaptive control has been a subject of active research for more than a half century. Many interesting works have been done and fruitful results have been reported, see for examples [13]-[17]. Adaptive control theory has also found its wide applications including the synchronization of complex dynamic systems [8]-[11] and [18]-[23]. Note that most of the closed-loop adaptive systems can be represented by the following state-space form

$$\dot{z} = A(t, \chi)z + B(t, \chi)^T(\theta - \hat{\theta}) \quad (1)$$

$$\dot{\hat{\theta}} = C(t, \chi)^T z \quad (2)$$

where $z \in R^n$ stands for an identification/tracking error being the difference between actual states and identifier/reference states; $\theta \in R^m$ represents an unknown constant vector and $\hat{\theta}$ denotes its estimate; $\chi = [z^T, \hat{\theta}^T]^T \in R^{n+m}$; t_0 denotes the initial time, $A : [t_0, +\infty) \times R^{n+m} \rightarrow R^{n \times n}$, $B : [t_0, +\infty) \times R^{n+m} \rightarrow R^{m \times n}$ and $C : [t_0, +\infty) \times R^{n+m} \rightarrow R^{n \times m}$ are time-varying matrices that are allowed to depend on external

This work was supported in part by National Natural Science Foundation of China under Grants 61174213 and 61125306, and Program for New Century Excellent Talents in University under Grant NCET-10-0665.

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signals and initial conditions of the system. Such a closed-loop adaptive system can be found in many existing literatures, e.g., system (4.116)-(4.117) in [15].

One of the most important issues in adaptive control is how to guarantee the convergence of parameter estimates and exponential stability of the closed-loop adaptive system, which can be achieved if the regressor matrix $B(t, \chi)$ satisfies the PE condition. The concept of PE was firstly introduced by Astrom and Bohlin [24] to express the idea that the input signal of a plant should be sufficiently rich such that all the modes of the plant are excited [25]. Since then, many works have been done along this direction [26]-[32]. In fact, the PE condition has been extensively used. In [27], the synchronization problem was studied for a linear time-vary system and a chaotic system under the PE conditions. Tyukin et al. [28] addressed the problem of parameter estimation in nonlinearly parameterized systems. Recently, Wang and Hill [29], [30] considered the PE property of radial basis functions (RBFs) and its applications to deterministic learning control.

It is worthy to point out that the PE condition has an important effect on the stability of closed-loop system consisting of (1) and (2), especially on its exponential stability. When $B(t, \chi)$ is free of χ , i.e., $B(t, \chi) = B(t)$, uniformly globally exponential stability (UGES) can be guaranteed by assuming that $B(t)$ is PE, bounded and globally Lipschitz in [33]. However, when $B(t, \chi)$ depends on χ , UGES of the system cannot be obtained since it is difficult to check that $B(t, \chi)$ satisfies the above conditions. In this case, it has been shown in [35] that the system is (locally) exponentially convergent trajectory by trajectory. Note that UES is more significant than exponential convergence for each trajectory. For example, UES ensures certain useful properties like robustness and total stability, whereas exponential convergence cannot, as shown by a counterexample in [37]. Motivated by this viewpoint, in [26] and [38] UES was established for linear time-varying systems with the PE condition.

In this paper, we will show that for a group of linearly parameterized closed-loop error systems, the PE condition for UES of closed-loop systems can be further relaxed if we introduce appropriately the network topologies based on the consensus theory.

C. Our Works in This Paper

Motivated by the aforementioned discussions, we will focus on the problem of parameter estimation for a group of linearly parameterized systems and its applications in adaptive control. Suppose that the i th closed-loop adaptive error system is given by

$$\dot{z}_i = A_i(t, \chi_i)z_i + B_i(t, \chi_i)^T(\theta - \hat{\theta}_i), i = 1, \dots, N \quad (3)$$

where $\chi_i, z_i, A_i(t, \chi_i), B_i(t, \chi_i), \hat{\theta}_i$ and θ are defined similarly to $\chi, z, A(t, \chi), B(t, \chi), \hat{\theta}$ and θ in (1) and (2).

The main property of system (3) is that although each system has a different time-varying structure (i.e., $A_i(t, \chi_i), B_i(t, \chi_i)$), the unknown parameter vector θ is the same. Note that there indeed exist many real-world linearly parameterized systems whose closed-loop adaptive error

systems can be represented by (3). For example, a group of systems with different nonlinear functions working together in the same environment may have the same unknown physical parameters such as temperature and gravity, and thus each system can be described by the form of (3). Moreover, for a group of systems with the same system functions, their closed-loop adaptive error systems can be also described by (3) when the systems have different input signals or initial conditions. A typical example is adaptive formation control of N identical mobile robots [43] where all robots have the same dynamic model but different dynamic behaviors. In fact, in most of the existing literatures on formation control or coordinated control such as [41], the mobile robots used in simulation examples or experiments are identical. These practical examples are one of our main motivations to address the system in (3).

Both centralized and decentralized schemes can be applied to address the adaptation problem of system (3), e.g., [13]-[17] and [39], [40]. The former [13]-[17] only involves one centralized adaptive law which is allowed to use the information of all systems, whereas the latter [39], [40] has one adaptive law for each system which only uses the local information of the corresponding system. Both schemes have their own advantages and drawbacks. For example, the implementation of a centralized scheme is very complicated and also expensive, especially for complex systems. On the other hand, a decentralized scheme is ease of design and implementation. However, simplicity of the design makes the analysis of the overall designed system quite difficult and limits its ability to manipulate the whole system and therefore system performances, due to the lack of necessary global information. Inspired by these observations, in this paper we will propose a new adaptive scheme by establishing a network topology among systems, namely distributed cooperative adaptive scheme, where an adaptive law is designed for each system and it uses the information from itself and the systems in its neighborhood instead of all systems or just itself alone. Clearly, a distributed cooperative adaptive law is a tradeoff between the centralized adaptive law and the decentralized adaptive law. It essentially combines their advantages, but removes their drawbacks. For example, if we employ a decentralized adaptive law in the form of (2) to estimate θ for each system in (3), i.e.,

$$\dot{\hat{\theta}}_i = C_i(t, \chi_i)^T z_i, \quad (4)$$

then the estimated parameter vector $\hat{\theta}_i$ can be guaranteed to converge to its true value θ only under the assumption that $B_i(t, \chi_i)$ satisfies the PE condition for every $i = 1, 2, \dots, N$. However, when using the distributed cooperative adaptive law to be proposed, the PE condition is relaxed.

The novelty and main contributions of this paper are summarized as follows.

- (i) Instead of the classical decentralized adaptive law, a distributed cooperative adaptive law is proposed where a network topology is established and the estimates of unknown parameters can be exchanged among systems in its neighborhood by considering them as state variables. To the best of our knowledge, there has been no reports on such a class of distributed cooperative adaptive problems so far.

- (ii) Motivated by the consensus theory of multi-agent systems, we establish the UES of closed-loop adaptive error systems by employing Lyapunov stability theory and the algebraic graph theory under a newly defined cooperatively PE condition rather than the conventional PE condition required in all the current relevant literatures such as [26], [38]. The cooperatively PE condition basically means that it is not necessary for the signals in every local system to be persistently exciting, instead it is sufficient for the systems in the neighborhood network to cooperatively provide PE signals to the adaptive laws. The effects of network topologies on the performance of closed-loop systems are also examined in two cases, i.e., the time-invariant topology and the time-varying topology.
- (iii) Furthermore, the proposed distributed cooperative adaptive law is applied to identify two groups of classic parameterized systems and design adaptive controllers for two classes of linearly parameterized systems. Simulation examples also illustrate the advantages of the distributed cooperative adaptive law over the traditional decentralized adaptive law.

The rest of this paper is organized as follows. After introducing some necessary preliminaries in Section II, we present the convergence results of the distributed cooperative adaptive law under a general framework in Section III. In Sections IV and V, we further address the applications of the distributed cooperative adaptive law in the identification and control of several groups of parameterized systems. The paper is concluded in Section VI.

Notation: R and R^+ denote the set of real numbers and the set of nonnegative real numbers, respectively; R^n denotes the set of $n \times 1$ real vectors, and $R^{m \times n}$ denotes the set of $m \times n$ matrices; I_m denotes the $m \times m$ identity matrix; $\mathbf{1}_N$ is a column vector with all N elements being 1; $e_i \in R^N$ represents a unit vector whose i th element is one; $\text{diag}(G_i)$ denotes a block diagonal matrix with diagonal block G_i ; $\|\cdot\|$ is the 2-norm of a vector or a matrix. For a measurable matrix-valued function $\phi : R^+ \rightarrow R^{n \times m}$, $\|\phi\|_2 = (\int_0^\infty \|\phi(t)\|^2 dt)^{1/2}$. We use B_r to note the open ball $B_r := \{x \in R^n : \|x\| < r\}$ with r being an arbitrary positive constant. Let $A \in R^{n \times n}$ and $B \in R^{n \times n}$ be two symmetric matrices, then $A \geq B$ means $A - B$ is a positive semi-definite matrix.

II. PRELIMINARIES

A. Algebraic Graph Theory

In this paper, the network topology among N systems is used to describe their interconnections and it is modeled as a weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with the set of nodes $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$, the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = (a_{ij})_{N \times N}$ with nonnegative adjacency elements. Node v_i denotes the i th system. An edge in $\mathcal{G}$ is denoted by an unordered pair $e_{ij} = (i, j)$. $e_{ij} \in \mathcal{E}$ if and only if there is information exchange between system i and system j , and $e_{ij} \in \mathcal{E} \Leftrightarrow e_{ji} \in \mathcal{E}$. The adjacency element a_{ij} represents the communication quality between system i and system j . Note that $e_{ij} \in \mathcal{E} \Leftrightarrow a_{ij} > 0$. Assume

$a_{ij} = a_{ji}$, which means that A is symmetric. The Laplacian matrix $\mathcal{L}$ of the graph $\mathcal{G}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$ where $\mathcal{D} = \text{diag}(d_1, d_2, \dots, d_N)$ and $d_i = \sum_{j=1}^N a_{ij}$. For any two nodes $v_i, v_j \in \mathcal{V}$, if there is a path between them, then $\mathcal{G}$ is connected.

Consider a collection of simple graphs $\mathcal{G}_1, \dots, \mathcal{G}_m$, with the same vertex set $\mathcal{V}$ for some $m > 1$. Denote $\mathcal{G}_{1, \dots, m}$ as their union graph, which is also a simple graph. Its vertex set is still $\mathcal{V}$, its edge set equals the union of the edge sets of all the graphs in the collection, and the connection weight between the edge i and the edge j is the sum of nonzero a_{ij} of $\mathcal{G}_1, \dots, \mathcal{G}_m$. Moreover, this collection, $\mathcal{G}_1, \dots, \mathcal{G}_m$, is jointly connected if its union graph $\mathcal{G}_{1, \dots, m}$ is connected.

Lemma 1 [31]: Let $\mathcal{L}$ be the Laplacian matrix associated with the undirected graph $\mathcal{G}$ of N nodes. Then $\mathcal{L}$ has at least one zero eigenvalue and all its nonzero eigenvalues are positive. Furthermore, $\mathcal{L}$ has a simple zero eigenvalue and all other eigenvalues are positive if and only if the graph $\mathcal{G}$ is connected.

B. Kronecker Product

Definition 1 [32]: Let $A \in R^{m \times n}$ and $B \in R^{p \times q}$. The Kronecker Product (or tensor product) of A and B is defined as the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in R^{mp \times nq}.$$

Lemma 2 [32]: Let $A \in R^{n \times n}$ have eigenvalues $\lambda_i, i = 1, \dots, n$, and let $B \in R^{m \times m}$ have eigenvalues $\mu_j, j = 1, \dots, m$. Then the mn eigenvalues of $A \otimes B$ are $\lambda_1\mu_1, \dots, \lambda_1\mu_m, \lambda_2\mu_1, \dots, \lambda_2\mu_m, \dots, \lambda_n\mu_1, \dots, \lambda_n\mu_m$. Moreover, if $x_1, \dots, x_p$ are linearly independent right eigenvectors of A corresponding to $\lambda_1, \dots, \lambda_p (p \leq n)$, and $z_1, \dots, z_q$ are linearly independent right eigenvectors of B corresponding to $\mu_1, \dots, \mu_q (q \leq m)$, then $x_i \otimes z_j \in R^{mn}$ are linearly independent right eigenvectors of $A \otimes B$ corresponding to $\lambda_i\mu_j$.

Lemma 3 [32]: Let $A \in R^{m \times n}$, $B \in R^{r \times s}$, $C \in R^{n \times p}$, and $D \in R^{s \times t}$. Then, $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ and $(A \otimes B)^T = A^T \otimes B^T$.

C. λ -UES and λ -uPE Condition of Parameterized Time-Varying Systems

Consider a parameterized time-varying system of the form

$$\dot{x} = f(t, \lambda, x), \quad x(t_0) = x_0, t \geq t_0 \quad (5)$$

where $x \in R^n$ is the system state vector; $\lambda \in \Omega$ is a constant vector, with $\Omega \subset R^q$ being a closed set; $t_0 \in R^+$ denotes the initial time; $f : [t_0, \infty) \times \Omega \times R^n \rightarrow R^n$ is uniformly locally Lipschitz in t and λ , and $f(t, \lambda, 0) = 0$. The solution of system (5) starting from (t_0, x_0) is denoted as $x(t, \lambda, t_0, x_0)$ or simply, $x(t, \lambda)$.

Definition 2 (λ -ULES and λ -UGES) [26]: The origin of system (5) is said to be λ -uniformly locally exponentially

stable (λ -ULES) if there exist $r > 0$, $k_\lambda > 0$ and $\gamma_\lambda > 0$ such that, for all $t \geq t_0$ and $\lambda \in \Omega$,

$$\|x(t, \lambda, t_0, x_0)\| \leq k_\lambda \|x_0\| e^{-\gamma_\lambda(t-t_0)},$$

if $\|x_0\| < r$. Furthermore, the system is said to be λ -uniformly globally exponentially stable (λ -UGES) if the exponential bound holds for all $(t_0, x_0) \in R^+ \times R^n$. In the rest of this subsection, without loss of generality, assume $t_0 = 0$.

Definition 3 (λ -uPE) [26]: A continuous function $\phi(\cdot, \cdot) : R^+ \times \Omega \rightarrow R^{m \times n}$ is λ -uniformly persistently exciting (λ -uPE) if there exist two positive constants α and T_0 such that for each $\lambda \in \Omega$

$$\int_t^{t+T_0} \phi(\tau, \lambda) \phi(\tau, \lambda)^T d\tau \geq \alpha I_m, \quad \forall t \geq 0. \quad (6)$$

Lemma 4 [26]: For the system $\dot{x} = -\Phi(t, \lambda) \Phi(t, \lambda)^T x$ where $\lambda \in \Omega$ is an unknown constant vector, assume that $\Phi(\cdot, \cdot)$ is λ -uPE and there exists a constant $\phi_M > 0$ such that for all $t \geq 0$ and all $\lambda \in \Omega$, $\|\Phi(t, \lambda)\| \leq \phi_M$. Then, the system is λ -UGES with $k_\lambda = 1$, $\gamma_\lambda \geq \alpha/T_0(1 + \phi_M^2 T_0)^2$.

Now consider a parameterized linear time-varying (LTV) system of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A(t, \lambda) & B(t, \lambda)^T \\ -C(t, \lambda) & -D(t, \lambda) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (7)$$

where $x_1 \in R^n$ and $x_2 \in R^m$ are the states; $\lambda \in \Omega$ is an unknown constant vector; $A(t, \lambda) : R^+ \times \Omega \rightarrow R^{n \times n}$, $B(t, \lambda) : R^+ \times \Omega \rightarrow R^{m \times n}$, $C(t, \lambda) : R^+ \times \Omega \rightarrow R^{m \times n}$ and $D(t, \lambda) : R^+ \times \Omega \rightarrow R^{m \times m}$ are system matrices. Further assume $D(t, \lambda)$ is positive semi-definite. To analyze the exponential stability of system (7), we need the following assumptions.

Assumption 1: There exists $\phi_M > 0$ such that for all $t \geq 0$ and $\lambda \in \Omega$, $\max\{\|B(t, \lambda)\|, \|D(t, \lambda)\|, \|\frac{dB(t, \lambda)}{dt}\|\} \leq \phi_M$.

Assumption 2: There exist symmetric matrices $P(t, \lambda)$ and $Q(t, \lambda)$ such that $P(t, \lambda)B(t, \lambda)^T = C(t, \lambda)^T$ and $A(t, \lambda)^T P(t, \lambda) + P(t, \lambda)A(t, \lambda) + \dot{P}(t, \lambda) \leq -Q(t, \lambda)$. Furthermore there exist p_m, q_m, p_M and $q_M > 0$ such that, for all $(t, \lambda) \in R^+ \times \Omega$, $p_m I_n \leq P(t, \lambda) \leq p_M I_n$ and $q_m I_n \leq Q(t, \lambda) \leq q_M I_n$.

Remark 1: Assumptions 1 and 2 are similar to those in [26] with minor differences. Since λ is a constant vector, all matrices in Assumptions 1 and 2 are parameter-dependent. The equality and inequalities in Assumption 2 will be used to construct a parameter-dependent Lyapunov function in the proof of Lemma 5 below. Note that the idea of using parameter-dependent equalities or inequality to construct the parameter-dependent Lyapunov function has been widely used in existing works such as [26] and [42]. As shown in the proofs of Theorems 4, 5 and 6 later, the closed-loop systems of several typical linearly parameterized systems indeed satisfy Assumptions 1 and 2.

The following lemma gives the conditions for guaranteeing that system (7) is λ -UGES.

Lemma 5 : System (7) under Assumptions 1 and 2, is λ -UGES if there exist two positive constants T_0 and α such that for all $(t, \lambda) \in R^+ \times \Omega$,

$$\int_t^{t+T_0} [B(\tau, \lambda)B(\tau, \lambda)^T + D(\tau, \lambda)] d\tau \geq \alpha I_m, \quad \forall t \geq 0. \quad (8)$$

Proof: See the Appendix. ■

Remark 2: Obviously, Theorem 1 in [26] is a special case of Lemma 5 in which $D(t, \lambda) = 0$ and thus $B(t, \lambda)$ must satisfy λ -uPE condition. However, Lemma 5 does not require that $B(t, x)$ must be λ -uPE due to the presence of $D(t, \lambda)$. For the parameterized systems without λ -uPE condition, the closed-loop systems can be guaranteed to be λ -UGES by suitably introducing $D(t, \lambda)$ into the adaptive scheme. This is a key technique used in this paper.

D. UES and u-PE Condition of State-Dependent Time-Varying Systems

Consider the system

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, t \geq t_0 \quad (9)$$

where $f : [t_0, \infty) \times R^n \rightarrow R^n$ is piecewise continuous in t and locally Lipschitz in x on $[t_0, \infty) \times R^n$, and $f(t, 0) = 0$. The solution of system (9) starting from initial condition (t_0, x_0) is denoted as $x(t, t_0, x_0)$ or simply, $x(t)$.

Definition 2' (ULES and UGES) [38]: The origin $x = 0$ of system (9) is said to be uniformly locally exponentially stable (ULES) if there exist positive constants γ_1, γ_2 and r such that for any $(t_0, x_0) \in R^+ \times B_r$, all the corresponding solutions satisfy

$$\|x(t, t_0, x_0)\| \leq \gamma_1 \|x_0\| e^{-\gamma_2(t-t_0)}, \quad \forall t \geq t_0. \quad (10)$$

Furthermore, the origin $x = 0$ of the system is said to be uniformly globally exponentially stable (UGES) if there exist two positive constants γ_1 and γ_2 such that (10) holds for all $(t_0, x_0) \in R^+ \times R^n$.

Let $\phi : R^+ \times R^n \rightarrow R^{m \times n}$ be such that $\phi(\cdot, t_0, x_0)$ is locally integrable for each solution $x(\cdot, t_0, x_0)$.

Definition 3' (u-PE) [38]: The pair (ϕ, f) is called uniformly persistently exciting (u-PE) (or simply, ϕ is u-PE) if, for each $r > 0$, there exist positive constants α and T_0 such that for all $(t_0, x_0) \in R^+ \times B_r$, all the corresponding solutions satisfy

$$\int_t^{t+T_0} \phi(\tau, x(\tau, t_0, x_0)) \phi(\tau, x(\tau, t_0, x_0))^T d\tau \geq \alpha I_m, \quad \forall t \geq t_0. \quad (11)$$

Furthermore, if (11) holds for all $(t_0, x_0) \in R^+ \times R^n$, the pair (ϕ, f) is called uniformly globally persistently exciting (ug-PE) (or simply, ϕ is ug-PE).

Remark 3: Note that if we parameterize (9) by denoting the initial condition (t_0, x_0) as λ , then Definitions 2' and 3' become Definitions 2 and 3, respectively. An important conclusion pointed out in [26, below Definition 1 of p.16], is that the state-dependent time-varying linear system $\dot{x} = \tilde{A}(t, x)x$ is ULES (or UGES) only if $\dot{\tilde{x}} = \tilde{A}(t, \lambda)\tilde{x}$ is λ -UGES with $\Omega = R^+ \times B_r$ (or $\Omega = R^+ \times R^n$), where $\tilde{A}(t, \lambda) := \tilde{A}(t, x(t, t_0, x_0))$. Based on this, we immediately obtain Lemmas 4' and 5' by slightly modifying Lemmas 4 and 5.

Lemma 4' [26]: For the system $\dot{x} = -\Phi(t, x)\Phi(t, x)^T x$, if $(\Phi, -\Phi\Phi^T)$ is u-PE (or ug-PE) and there exists a constant $\phi_m > 0$ such that for all $t \geq t_0$ and $(t_0, x_0) \in R^+ \times B_r$ (or

$(t_0, x_0) \in R^+ \times R^n$, $\|\Phi(t, x)\| \leq \phi_M$. Then the system is ULES (or UGES).

Now, consider the state-dependent LTV system of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A(t, x) & B(t, x)^T \\ -C(t, x) & -D(t, x) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (12)$$

$$:= F(t, x)x, \quad x(t_0) = x_0$$

where $x_1 \in R^n$ and $x_2 \in R^m$ are the states, and $x = [x_1^T, x_2^T]^T$; $A(t, x) : [t_0, \infty) \times R^{n+m} \rightarrow R^{n \times n}$, $B(t, x) : [t_0, \infty) \times R^{n+m} \rightarrow R^{n \times m}$, $C(t, x) : [t_0, \infty) \times R^{n+m} \rightarrow R^{m \times n}$ and $D(t, x) : [t_0, \infty) \times R^{n+m} \rightarrow R^{m \times m}$ are system matrices. To analyze the exponential stability of system (12), we need the following assumptions.

Assumption 1': For each fixed $r > 0$, there exists $\phi_M > 0$ such that, for all $t \geq t_0$ and $(t_0, x_0) \in R^+ \times B_r$, $\max\{\|B(t, x)\|, \|D(t, x)\|, \|\frac{dB(t, x(t))}{dt}\|\} \leq \phi_M$. Further assume that $D(t, x)$ is symmetric.

Assumption 2': For each fixed $r > 0$, there exist symmetric matrices $P(t, x)$ and $Q(t, x)$ such that, for all $t \geq t_0$ and $(t_0, x_0) \in R^+ \times B_r$, $A(t, x)^T P(t, x) + P(t, x)A(t, x) + \dot{P}(t, x) \leq -Q(t, x)$ and $P(t, x)B(t, x)^T = C(t, x)^T$. Furthermore, $\exists p_m, q_m, p_M$ and $q_M > 0$ such that $p_m I_n \leq P(t, x) \leq p_M I_n$ and $q_m I_n \leq Q(t, x) \leq q_M I_n$.

Lemma 5': Under Assumptions 1' and 2', the system (12), is ULES, where $r > 0$ is any fixed constant, if there exist two positive constants T_0 and α such that for all $(t_0, x_0) \in R^+ \times B_r$,

$$\begin{aligned} & \int_t^{t+T_0} [B(\tau, x(\tau, t_0, x_0))B(\tau, x(\tau, t_0, x_0))^T \\ & + D(\tau, x(\tau, t_0, x_0))] d\tau \\ & \geq \alpha I_m, \forall t \geq t_0. \end{aligned} \quad (13)$$

Furthermore, if Assumptions 1' and 2' and condition (13) hold for all $(t_0, x_0) \in R^+ \times R^{n+m}$, then system (12) is UGES.

E. Cooperatively PE Condition

One of our main contributions is to relax the PE condition imposed on every local system. The new condition only requires that persistently exciting signals are cooperatively generated by the systems in a neighborhood network. With respect to this, we now give a formal definition on cooperatively PE condition.

Definition 4: A group of matrix-valued functions $\phi_i : [t_0, +\infty) \rightarrow R^{m \times n}$, $i = 1, 2, \dots, N$ is said to satisfy a cooperatively PE condition if there exist two positive constants T_0 and α_0 such that

$$\int_t^{t+T_0} \left[\sum_{i=1}^N \phi_i(\tau)\phi_i(\tau)^T \right] d\tau \geq \alpha_0 I_m, \forall t \geq t_0. \quad (14)$$

Remark 4: Note that the cooperatively PE condition is much weaker than the normal PE condition. It is possible that each $\phi_i(t)$ may not be PE, but all the ϕ_i for $i = 1, 2, \dots, N$ cooperatively satisfy the PE condition in the sense that the matrix $[\phi_1(t), \phi_2(t), \dots, \phi_N(t)]$ formed by them is PE. For example, neither $\phi_1(t) = [\sin t, 0]^T$ nor $\phi_2(t) = [0, \cos t]^T$

satisfies the PE condition. However, the matrix $[\phi_1, \phi_2]$ is PE as

$$\phi_1(t)\phi_1^T(t) + \phi_2(t)\phi_2^T(t) = \begin{bmatrix} \sin^2(t) & 0 \\ 0 & \cos^2(t) \end{bmatrix}$$

indeed satisfies (14). That is, $\phi_1(t)$ and $\phi_2(t)$ satisfy the cooperatively PE condition.

Similarly, we define the cooperatively u-PE condition for a group of matrix-valued functions depending on dynamic systems. Consider N systems, where the i th system is described by

$$\dot{x}_i = f_i(t, x_i), \quad x_i(t_0) = x_{i0}, t \geq t_0 \quad (15)$$

where $f_i : [t_0, \infty) \times R^n \rightarrow R^n$ is piecewise continuous in t and locally Lipschitz in x_i on $[t_0, \infty) \times R^n$. The solution of system (15) starting from initial condition (t_0, x_{i0}) is denoted as $x_i(t, t_0, x_{i0})$.

Definition 4': A group of matrix-valued functions $\phi_i(t, x_i)$, $i = 1, 2, \dots, N$, is said to satisfy a cooperatively u-PE condition if for each $r > 0$, there exist two positive constants T_0 and α such that for all $(t_0, x_{i0}) \in R^+ \times B_r$, all the corresponding solutions satisfy

$$\begin{aligned} & \int_t^{t+T_0} \left[\sum_{i=1}^N \phi_i(\tau, x_i(\tau, t_0, x_{i0}))\phi_i(\tau, x_i(\tau, t_0, x_{i0}))^T \right] d\tau \\ & \geq \alpha I_m, \forall t \geq t_0. \end{aligned} \quad (16)$$

If (16) holds for all $(t_0, x_{i0}) \in R^+ \times R^n$, then $\phi_i(t, x_i)$, $i = 1, 2, \dots, N$, are said to satisfy a cooperatively ug-PE condition.

III. DISTRIBUTED COOPERATIVE ADAPTIVE SCHEME

A. Distributed Cooperative Adaptive Law Under Time-Invariant Topologies

Inspired by the consensus theory [5], instead of the decentralized adaptive laws (4), we propose the following distributed cooperative adaptive law for the i th local system (3)

$$\dot{\hat{\theta}}_i = C_i(t, \chi_i)^T z_i - \gamma \sum_{j \in \mathcal{N}_i} a_{i,j}(\hat{\theta}_i - \hat{\theta}_j) \quad (17)$$

where $\gamma > 0$ is a design parameter; $\mathcal{N}_i$ denotes the set of systems from which system i can receive information; $a_{i,j}$ are the (i, j) th entry of the adjacency matrix $\mathcal{A}$ of the graph $\mathcal{G}$ which represents the interconnection (also called the network topology) among all systems in (3); $a_{i,j} > 0$ if $a_{i,j} \in \mathcal{N}_i$, and $a_{i,j} = 0$, otherwise. In this subsection, we assume that $\mathcal{A}$ is time-invariant. As mentioned in the introduction, the distributed cooperative adaptive law (17) is a tradeoff between the centralized adaptive law and the decentralized adaptive law since each local system has a local adaptive law which uses the information from itself and the systems in its neighborhood.

By defining the parameter estimation error vector $\tilde{\theta}_i = \theta - \hat{\theta}_i$, we have

$$\dot{\tilde{\theta}}_i = -C_i(t, \chi_i)^T z_i - \gamma \sum_{j \in \mathcal{N}_i} a_{i,j}(\tilde{\theta}_i - \tilde{\theta}_j). \quad (18)$$

Define $z = [z_1^T, \dots, z_n^T]^T$, $\tilde{\theta} = [\tilde{\theta}_1^T, \dots, \tilde{\theta}_n^T]^T$ and $\chi = [\chi_1^T, \dots, \chi_n^T]^T$. Then the overall closed-loop system including (3) and (18) can be rewritten as

$$\begin{bmatrix} \dot{z} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} A(t, \chi) & B(t, \chi)^T \\ -C(t, \chi) & -\gamma \mathcal{L} \otimes I_m \end{bmatrix} \begin{bmatrix} z \\ \tilde{\theta} \end{bmatrix} \quad (19)$$

where $A(t, \chi) = \text{diag}\{A_i(t, \chi_1), \dots, A_N(t, \chi_N)\}$, $B(t, \chi) = \text{diag}\{B_1(t, \chi_1), \dots, B_N(t, \chi_N)\}$, $C(t, \chi) = \text{diag}\{C_1(t, \chi_1), \dots, C_N(t, \chi_N)\}$, and $\mathcal{L}$ is the Laplacian matrix of the graph $\mathcal{G}$.

We are now in a position to give our first main result.

Theorem 1: Consider the closed-loop adaptive system (19) consisting of (3) and (17). Suppose that Assumptions 1' and 2' hold for $A_i(t, \chi_i)$, $B_i(t, \chi_i)$ and $C_i(t, \chi_i)$, $i = 1, 2, \dots, N$. Then the closed-loop system (19) is ULES with any fixed constant $r > 0$, if the topology is undirected and connected, and $B_i(t, \chi_i)$, $i = 1, 2, \dots, N$, satisfy the cooperatively u-PE condition. In addition, if Assumptions 1' and 2' holds for all $(t_0, \chi_{i0}) \in R^+ \times R^{n+m}$ and $B_i(t, \chi_i)$, $1 \leq i \leq N$, satisfy the cooperatively ug-PE condition, then system (19) is UGES.

Remark 5: Note that for the decentralized adaptive law (4), each $B_i(t, \chi_i)$ must satisfy the u-PE condition in order to guarantee that the closed-loop system is ULES. This condition is relaxed here by the proposed distributed cooperative adaptive law (17), as it is just required that $B_i(t, \chi_i)$ satisfy the cooperatively u-PE condition.

Proof: We just prove the case of ULES as the case of UGES is directly obtained by replacing B_r with R^{n+m} . From Assumptions 1', 2' and Lemma 5', we only need to show that under the conditions of Theorem 1, there exists a positive constant α such that for all $(t_0, \chi_{i0}) \in R^+ \times B_r$, $i = 1, \dots, N$,

$$\Delta(t, t_0, \chi_0) := \int_t^{t+T_0} [B(\tau, \chi(\tau))B(\tau, \chi(\tau))^T + \gamma \mathcal{L} \otimes I_m] d\tau \geq \alpha I_{Nm}. \quad (20)$$

For notational simplicity, denote

$$H(t, t_0, \chi_0) = \text{diag}\{H_1(t, t_0, \chi_{i0}), \dots, H_N(t, t_0, \chi_{i0})\} \quad (21)$$

where $H_i(t, t_0, \chi_{i0}) = \int_t^{t+T_0} B_i(\tau, \chi_i(\tau))B_i(\tau, \chi_i(\tau))^T d\tau$. Then (20) can be rewritten as

$$\Delta(t, t_0, \chi_0) = H(t, t_0, \chi_0) + T_0 \gamma \mathcal{L} \otimes I_m \geq \alpha I_{Nm}. \quad (22)$$

Since the topology is undirected and connected, according to Lemma 1, $\mathcal{L}$ has only one zero eigenvalue whose unit eigenvector is $\frac{1}{\sqrt{N}}\mathbf{1}$, and correspondingly $\mathcal{L} \otimes I_m$ has m zero eigenvalues whose orthogonal unit eigenvectors are

$$\nu_1 = \frac{1}{\sqrt{N}}\mathbf{1} \otimes e_1, \dots, \nu_m = \frac{1}{\sqrt{N}}\mathbf{1} \otimes e_m \quad (23)$$

where $e_i \in R^N$. The other eigenvalues of $\mathcal{L} \otimes I_m$ are positive and denoted as $0 < \lambda_{m+1} \leq \dots \leq \lambda_{Nm}$, whose orthogonal unit eigenvectors are denoted correspondingly as $\nu_{m+1}, \dots, \nu_{Nm}$. For an arbitrary nonzero vector $\xi \in R^{Nm}$, it can be always expressed as

$$\xi = \sum_{i=1}^m c_i \nu_i + \sum_{i=m+1}^{Nm} c_i \nu_i. \quad (24)$$

On one hand, when $\sum_{i=m+1}^{Nm} c_i^2 \neq 0$, we have

$$\begin{aligned} \xi^T \Delta(t, t_0, \chi_0) \xi &= \xi^T H(t, t_0, \chi_0) \xi + \sum_{i=m+1}^{Nm} T_0 \gamma \lambda_i c_i^2 \\ &\geq T_0 \gamma \lambda_2 \sum_{i=m+1}^{Nm} c_i^2 > 0. \end{aligned} \quad (25)$$

On the other hand, when $\sum_{i=m+1}^{Nm} c_i^2 = 0$ which means $\sum_{i=1}^m c_i^2 \neq 0$ and $\xi = \sum_{i=1}^m c_i \nu_i$, we have

$$\begin{aligned} \xi^T \Delta(t, t_0, \chi_0) \xi &= \left(\sum_{i=1}^m c_i \nu_i \right)^T H(t, t_0, \chi_0) \left(\sum_{i=1}^m c_i \nu_i \right) \\ &= \mathcal{C}^T \mathcal{V}^T H(t, t_0, \chi_0) \mathcal{V} \mathcal{C} \end{aligned} \quad (26)$$

where $\mathcal{C} = [c_1, \dots, c_m]^T$ and $\mathcal{V} = [\nu_1, \dots, \nu_m]$. Since $B_i(t, \chi_i)$, $1 \leq i \leq N$, satisfy the cooperatively u-PE condition, it is easily verified from (21) and (23) that for all $(t_0, \chi_{i0}) \in R^+ \times B_r$,

$$\mathcal{V}^T H(t, t_0, \chi_0) \mathcal{V} = \sum_{i=1}^N H_i(t, t_0, \chi_{i0}) \geq \alpha_0 I_m. \quad (27)$$

Substituting (27) into (26) yields

$$\xi^T [H(t, t_0, \chi_0) + T_0 \gamma \mathcal{L} \otimes I_m] \xi \geq \alpha_0 \sum_{i=1}^m c_i^2 > 0. \quad (28)$$

Thus, we have proven that $\Delta(t, t_0, \chi_0)$ is a positive-definite matrix for all $t \geq t_0$ and all $(t_0, \chi_{i0}) \in R^+ \times B_r$. So, its eigenvalues are all positive.

We next show that there exists a positive constant α such that $\Delta(t, t_0, \chi_0) \geq \alpha I_{Nm}$ for all $t \geq t_0$ and all $(t_0, \chi_{i0}) \in R^+ \times B_r$. Equivalently, we prove that all the eigenvalues of the time-varying positive definite matrix $\Delta(t, t_0, \chi_0)$ must have a lower bound $\alpha > 0$. This is done through contradiction by assuming that there exists an eigenvalue $\underline{\lambda}(t, t_0, \chi_0)$ and three sequences $\{t^k\}_{k=1}^\infty$, $\{t_0^k\}_{k=1}^\infty$ and $\{\chi_0^k\}_{k=1}^\infty$ such that $\lim_{k \rightarrow \infty} \underline{\lambda}(t^k, t_0^k, \chi_0^k) = 0$. Denote the unit eigenvector of $\underline{\lambda}(t^k, t_0^k, \chi_0^k)$ as $\eta(t^k, t_0^k, \chi_0^k)$, i.e., $\|\eta(t^k, t_0^k, \chi_0^k)\| = 1$. Then we have

$$\begin{aligned} &\lim_{k \rightarrow \infty} \eta(t^k, t_0^k, \chi_0^k)^T \Delta(t^k, t_0^k, \chi_0^k) \eta(t^k, t_0^k, \chi_0^k) \\ &= \lim_{k \rightarrow \infty} \eta(t^k, t_0^k, \chi_0^k)^T \underline{\lambda}(t^k, t_0^k, \chi_0^k) \eta(t^k, t_0^k, \chi_0^k) \\ &= 0. \end{aligned} \quad (29)$$

However, $\eta(t^k, t_0^k, \chi_0^k)$ can be also rewritten as $\eta(t^k, t_0^k, \chi_0^k) = \sum_{i=1}^{Nm} c_i(t^k, t_0^k, \chi_0^k) \nu_i$ with $\sum_{i=1}^{Nm} c_i^2(t^k, t_0^k, \chi_0^k) = 1$. Similarly to the derivations of (25)-(28), on one hand, if $\sum_{i=m+1}^{Nm} c_i^2(t^k, t_0^k, \chi_0^k)$ has a positive lower bound $\underline{c}$, then

$$\begin{aligned} &\lim_{k \rightarrow \infty} \eta(t^k, t_0^k, \chi_0^k)^T \Delta(t^k, t_0^k, \chi_0^k) \eta(t^k, t_0^k, \chi_0^k) \\ &\geq \lim_{k \rightarrow \infty} T_0 \gamma \lambda_2 \sum_{i=m+1}^{Nm} c_i^2(t^k, t_0^k, \chi_0^k) \\ &\geq T_0 \gamma \lambda_2 \underline{c} > 0. \end{aligned} \quad (30)$$

On the other hand, if $\sum_{i=m+1}^{Nm} c_i^2(t^k, t_0^k, \chi_0^k)$ does not have a positive lower bound, then there must exist

three subsequences: $\{t^{k_j}\}_{j=1}^\infty$, $\{t_0^{k_j}\}_{j=1}^\infty$ and $\{\chi_0^{k_j}\}_{j=1}^\infty$ corresponding to $\{t^k\}_{k=1}^\infty$, $\{t_0^k\}_{k=1}^\infty$ and $\{\chi_0^k\}_{k=1}^\infty$ such that $\lim_{j \rightarrow \infty} \sum_{i=m+1}^{Nm+1} c_i^2(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) = 0$, which means $\lim_{j \rightarrow \infty} \sum_{i=1}^m c_i^2(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) = 1$. Denote $\eta(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) = \eta_1(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) + \eta_2(t^{k_j}, t_0^{k_j}, \chi_0^{k_j})$ where $\eta_1(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) = \sum_{i=1}^m c_i^2(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) \nu_i$ and $\eta_2(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) = \sum_{i=m+1}^{Nm+1} c_i^2(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) \nu_i$. Obviously, $\lim_{j \rightarrow \infty} \eta_2(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) = 0$, and then

$$\begin{aligned} & \lim_{j \rightarrow \infty} \eta(t^{k_j}, t_0^{k_j}, \chi_0^{k_j})^T \Delta(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) \eta(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) \\ &= \lim_{j \rightarrow \infty} \eta_1(t^{k_j}, t_0^{k_j}, \chi_0^{k_j})^T H(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) \\ & \quad + \lim_{j \rightarrow \infty} \eta_2(t^{k_j}, t_0^{k_j}, \chi_0^{k_j})^T \Delta(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) \eta(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) \\ & \quad + \lim_{j \rightarrow \infty} \eta_1(t^{k_j}, t_0^{k_j}, \chi_0^{k_j})^T \Delta(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) \eta_2(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) \\ &\geq \lim_{j \rightarrow \infty} \alpha_0 \sum_{i=1}^m c_i^2(t^{k_j}, t_0^{k_j}, \chi_0^{k_j}) \\ &= \alpha_0 > 0. \end{aligned} \quad (31)$$

Thus, both (30) and (31) contradict to (29). Therefore, we can conclude that there exists a positive constant α such that $\Delta(t, t_0, \chi_0) \geq \alpha I_{Nm}$ holds for all $t \geq t_0$ and all $(t_0, \chi_{i0}) \in R^+ \times B_r$. ■

Remark 6: Theorem 1 requires that the network topology must be undirected as it ensures that the adjacency matrix $\mathcal{A}$ is symmetric and positive semi-definite. This is because with this property, Lemma 5 or Lemma 5' is successfully used to obtain the exponential stability of system (19) as shown in the proof of Theorem 1. However, for the case where the network topology is directed, there is still no suitable method to establish the conclusion of Theorem 1. The main obstacle comes from the requirement that the matrix $D(\tau, \lambda)$ in Lemma 5 must be symmetric. How to relax this requirement is a research topic worthy for further investigation.

B. Distributed Cooperative Adaptive Law with a Time-Varying Topology

In practice, the network topology may be time-varying, and even unconnected at certain time. In this subsection, we further investigate the conditions under which the time-varying network topology can guarantee UES of a closed-loop system.

Firstly, we denote the Laplacian matrix of a time-varying topology $\mathcal{G}(t, \chi(t))$ as $\mathcal{L}(t, \chi(t))$. Assume that for an arbitrary fixed $r > 0$, $\mathcal{L}(t, \chi(t))$ is uniformly bounded for all $t \geq t_0$ and $(t_0, \chi_{i0}) \in R^+ \times B_r, i = 1, \dots, N$. It is easy to show that for a given constant $T_0 > 0$, $\int_t^{t+T_0} \mathcal{L}(\tau, \chi(\tau)) d\tau$ is still a Laplacian matrix corresponding to a time-varying graph. Now we give the main results for this case as follows.

Theorem 2: Consider the closed-loop adaptive system consisting of (3) and (17). Suppose that Assumptions 1' and 2' hold for $A_i(t, \chi_i), B_i(t, \chi_i)$ and $C_i(t, \chi_i), i = 1, \dots, N$, and further the network topology is undirected and time-varying. Then the closed-loop system (19) is ULES with any fixed constant $r > 0$, if for each fixed $r > 0$ and all $(t_0, \chi_{i0}) \in R^+ \times B_r, i = 1, \dots, N$, the following two conditions hold: (i) there exist three positive constants T_0, δ

and α_0 such that the matrix $\int_t^{t+T_0} \mathcal{L}(\tau, \chi(\tau)) d\tau$ has only one zero eigenvalue denoted as λ_1 and all its remaining nonzero eigenvalues satisfy $\delta \leq \lambda_2(t, t_0, \chi_0) \leq \dots \leq \lambda_N(t, t_0, \chi_0)$ for any $t \geq t_0$; (ii) $B_i(t, t_0, \chi_{i0}), 1 \leq i \leq N$, satisfy the cooperatively u-PE condition. Furthermore, if Assumptions 1'-2' and the conditions (i)-(ii) hold for all $(t_0, \chi_{i0}) \in R^+ \times R^{n+m}$, then the system (19) is UGES.

Proof: Similarly to the proof of Theorem 1, we just prove the case of ULES by showing that there exists a positive constant α such that for all $t \geq t_0$ and all $(t_0, \chi_{i0}) \in R^+ \times B_r, i = 1, \dots, N$,

$$\begin{aligned} \Delta(t, t_0, \chi_0) &:= \int_t^{t+T_0} [B(\tau, \chi(\tau)) B(\tau, \chi(\tau))^T \\ & \quad + \gamma \mathcal{L}(\tau, \chi(\tau)) \otimes I_m] d\tau \\ &\geq \alpha I_{Nm}. \end{aligned} \quad (32)$$

Denote

$$\Upsilon(t, t_0, \chi_0) = \int_t^{t+T_0} [\mathcal{L}(\tau, \chi(\tau)) \otimes I_m] d\tau. \quad (33)$$

Then, (32) can be rewritten equivalently as

$$\begin{aligned} \Delta(t, t_0, \chi_0) &= H(t, t_0, \chi_0) + \gamma \Upsilon(t, t_0, \chi_0) \\ &\geq \alpha I_{Nm}. \end{aligned} \quad (34)$$

Based on the conditions of Theorem 2 and according to Lemma 1, $\Upsilon(t, t_0, \chi_0)$ still has m zero eigenvalues whose orthogonal unit eigenvectors given by (23) are time-invariant. The other eigenvalues of $\Upsilon(t, t_0, \chi_0)$ are positive and time-varying, denoted as $0 < \lambda_{m+1}(t, t_0, \chi_0) \leq \dots \leq \lambda_{Nm}(t, t_0, \chi_0)$. Their corresponding orthogonal unit eigenvectors, $\nu_{m+1}(t, t_0, \chi_0), \dots, \nu_{Nm}(t, t_0, \chi_0)$, are also time-varying.

Now, we further show that there exists a positive constant α such that $\Delta(t, t_0, \chi_0) \geq \alpha I_{Nm}$ for all $t \geq t_0$ and all $(t_0, \chi_{i0}) \in R^+ \times B_r, i = 1, \dots, N$, namely it is uniformly positive definite. This is equivalent to prove that all its eigenvalues must have a common constant lower bound $\alpha > 0$. This is also achieved through contradiction by assuming that there exists an eigenvalue $\underline{\lambda}(t, t_0, \chi_0)$ and three sequences $\{t^k\}_{k=1}^\infty, \{t_0^k\}_{k=1}^\infty$ and $\{\chi_0^k\}_{k=1}^\infty$ such that $\lim_{k \rightarrow \infty} \underline{\lambda}(t^k, t_0^k, \chi_0^k) = 0$. The rest of the proof can be easily obtained by following similar steps to the proof of Theorem 1 and is omitted here. ■

Remark 7: Note that in the context of time-varying network topologies, Theorem 2 just requires that the integration of topologies over an interval with fixed length is connected. This is less demanding than the conditions in Theorem 1, where the topology is assumed to be connected all the time. The requirement on the network topologies in Theorem 2 includes some conditions in the existing works as its special cases such as the jointly-connected topologies in [36].

In reality, a network topology usually switches due to link failure or creation. To describe such varying topologies, we define a piecewise constant switching function $\sigma(t) : [0, \infty) \mapsto \mathcal{P} = \{1, 2, \dots, l\}$, where l denotes the total number of all possible undirected graphs. The graph at time t is denoted by $\mathcal{G}_{\sigma(t)}$, with the corresponding Laplacian

matrix $\mathcal{L}_{\sigma(t)}$. Now consider an infinite sequence of nonempty, bounded and contiguous time-intervals $[t_r, t_{r+1})$, $r = 0, 1, \dots$, with $t_0 = 0$ and $t_{r+1} - t_r \leq T_1$ for a constant $T_1 > 0$. In each interval $[t_r, t_{r+1})$, there is a sequence of subintervals $[t_{r_0}, t_{r_1}), [t_{r_1}, t_{r_2}), \dots, [t_{r_{m_r-1}}, t_{r_{m_r}})$, where $t_{r_0} = t_r$, $t_{r_{m_r}} = t_{r+1}$, and m_r is a positive integer, such that the network topology $\mathcal{G}_{\sigma(t)}$ switches at t_{r_j} and it does not change during the subinterval $[t_{r_j}, t_{r_{j+1}})$. Define the minimum dwell time as $T_2 := \min_{r \geq 0, 0 \leq j \leq m_r-1} \{t_{r_{j+1}} - t_{r_j}\}$, and suppose $T_2 > 0$. Evidently, there are at most $m_* = \lfloor T_1/T_2 \rfloor$ subintervals in each interval $[t_r, t_{r+1})$, where $\lfloor T_1/T_2 \rfloor$ denotes the maximum integer not larger than T_1/T_2 .

Corollary 1: Consider the closed-loop adaptive system consisting of (3) and (17). Suppose Assumptions 1' and 2' hold for $A_i(t, \chi_i)$, $B_i(t, \chi_i)$ and $C_i(t, \chi_i)$, $i = 1, 2, \dots, N$. Then the closed-loop system (19) is ULES with any fixed constant $r > 0$, if the following two conditions hold: (i) the network topology is undirected and switched, but the collection of graphs in each interval $[t_r, t_{r+1})$ is jointly-connected and the minimum dwell time $T_2 > 0$; (ii) $B_i(t, \chi_i)$, $1 \leq i \leq N$, satisfy the cooperatively u-PE condition. In addition, if Assumptions 1'-2' and the conditions (i)-(ii) hold for all $(t_0, \chi_{i0}) \in R^+ \times R^{n+m}$, then the system (19) is UGES.

Proof: We still only consider the case of ULES. Let $T = \max\{2T_1, T_0\}$, where T_0 is defined in the cooperatively u-PE condition. Consider the interval $\mathcal{I}(t) = [t, t+T)$. Obviously, on one hand, the cooperatively u-PE condition still holds if replacing T_0 by T , and on the other hand, the interval $\mathcal{I}(t)$ contains at least one $[t_r, t_{r+1})$ as its subinterval. Thus,

$$\begin{aligned} \int_t^{t+T} \mathcal{L}(\tau) d\tau &\geq \int_{t_r}^{t_r+t_{r+1}} \mathcal{L}(\tau) d\tau \\ &= \sum_{j=1}^{m_r} (t_{r_j} - t_{r_{j-1}}) \mathcal{L}_{r_j} \\ &\geq T_2 \sum_{j=1}^{m_r} \mathcal{L}_{r_j}. \end{aligned} \quad (35)$$

Since the collection of graphs in each interval $[t_r, t_{r+1})$ is jointly-connected, the constant matrix $\mathcal{L}_r := \sum_{j=1}^{m_r} \mathcal{L}_{r_j}$ is a Laplacian matrix of some connected graphs and the number of $\mathcal{L}_r$ is finite. Thus there exists a constant $\delta > 0$ such that $\lambda_1(\mathcal{L}_r) = 0$ and $\delta \leq \lambda_2(\mathcal{L}_r) \leq \dots \leq \lambda_N(\mathcal{L}_r)$. Together with (35), this implies that $\int_t^{t+T} \mathcal{L}(\tau) d\tau$ satisfies the conditions of Theorem 2. The proof is completed. ■

IV. APPLICATIONS TO DISTRIBUTED COOPERATIVE ADAPTIVE IDENTIFICATION OF UNCERTAIN SYSTEMS

In this section, we apply the above distributed cooperative adaptive scheme to the identification of two classes of parameterized systems described by, respectively, the linear “static” parametric model (SPM) and the linear “dynamic” parametric model (DPM). More details on such two models can be found in [14].

A. Distributed Cooperative Adaptive Identification of SPM

Consider a team of systems that can be described by the SPM

$$y_i = \phi_i(t)^T \theta, \quad i = 1, 2, \dots, N \quad (36)$$

where $y_i \in R^n$ is the system output, $\theta \in R^m$ is an unknown constant vector, and $\phi_i : [0, +\infty) \rightarrow R^{m \times n}$ is a known, continuous and uniformly bounded matrix-valued function.

The following identification model is widely used in existing literatures

$$\hat{y}_i = \phi_i(t)^T \hat{\theta}_i, \quad i = 1, 2, \dots, N \quad (37)$$

where $\hat{y}_i$ is the output of identification model, and $\hat{\theta}_i$ denotes the estimate of θ . Inspired by the idea of distributed cooperative adaptation and based on the results of the previous section, we propose the following distributed cooperative adaptive law

$$\dot{\hat{\theta}}_i = \rho \phi_i(t) (y_i - \hat{y}_i) - \gamma \sum_{j \in \mathcal{N}_i} a_{i,j}(t) (\hat{\theta}_i - \hat{\theta}_j) \quad (38)$$

where $\rho > 0$ is the adaptation gain, γ and $a_{i,j}$ are defined in (17), but it is time-varying here.

Define the model output errors and the parameter identification errors as $z_i = y_i - \hat{y}_i$ and $\tilde{\theta}_i = \theta - \hat{\theta}_i$, respectively. Further denote $z = [z_1^T, \dots, z_N^T]^T$, $\tilde{\theta} = [\tilde{\theta}_1^T, \dots, \tilde{\theta}_N^T]^T$ and $\Phi(t) = \text{diag}\{\phi_1(t), \dots, \phi_N(t)\}$. Then we have

$$z = \Phi(t)^T \tilde{\theta} \quad (39)$$

and

$$\dot{\tilde{\theta}} = -\rho \Phi(t) z - (\gamma \mathcal{L}(t) \otimes I_m) \tilde{\theta}. \quad (40)$$

Substituting (39) into (40) yields

$$\dot{\tilde{\theta}} = -[\rho \Phi(t) \Phi(t)^T + (\gamma \mathcal{L}(t) \otimes I_m)] \tilde{\theta}. \quad (41)$$

Now we have the following theorem.

Theorem 3: Consider the adaptive system consisting of (36)-(38). Suppose that the network topology is undirected. Then the closed-loop system (41) is UGES if there exist positive constants T_0, δ and α_0 such that the following two conditions hold: (i) the matrix $\int_t^{t+T_0} \mathcal{L}(\tau) d\tau$ has only one zero eigenvalue λ_1 and all its rest nonzero eigenvalues satisfy $\delta \leq \lambda_2(t) \leq \dots \leq \lambda_N(t)$ for any $t \geq t_0$; (ii) $\phi_i(t)$, $1 \leq i \leq N$, satisfy the cooperatively PE condition.

Proof: Since the network topology is undirected, so the matrix $\rho \Phi(t) \Phi(t)^T + (\gamma \mathcal{L}(t) \otimes I_m)$ is symmetric and positive semi-definite, which means that there exists a matrix $\Psi(t)$ such that

$$\rho \Phi(t) \Phi(t)^T + (\gamma \mathcal{L}(t) \otimes I_m) = \Psi(t) \Psi(t)^T.$$

Based on Lemma 4', $\Psi(t)$ is uniformly bounded from the uniform boundedness of $\phi_i(t)$. Now we need to verify that $\Psi(t)$ is u-PE, i.e., there exists $\alpha > 0$ such that

$$\int_t^{t+T_0} [\rho \Phi(\tau) \Phi(\tau)^T + \gamma \mathcal{L}(\tau) \otimes I_m] d\tau \geq \alpha I_m. \quad (42)$$

Clearly this has been shown in the proof of Theorem 2 under the conditions of this theorem. ■

B. Distributed Cooperative Adaptive Identification of DPM

Consider a group of systems given by the DPM

$$\dot{y}_i = \phi_i(t, y_i)^T \theta, \quad i = 1, 2, \dots, N \quad (43)$$

where $y_i, \phi_i(t, y_i)$ and θ are defined in (36), but $\phi_i(t, y_i)$ is allowed to depend on y_i .

The identification model for (43) is designed as

$$\dot{\hat{y}}_i = a_i(\hat{y}_i - y_i) + \phi_i(t, y_i)^T \hat{\theta}_i, \quad i = 1, \dots, N \quad (44)$$

where $a_i > 0$ is a design parameter; $\hat{\theta}_i$ and $\hat{y}_i$ are defined in (37). The distributed cooperative adaptive law is

$$\dot{\hat{\theta}}_i = \rho \phi_i(t, y_i)(y_i - \hat{y}_i) - \gamma \sum_{j \in \mathcal{N}_i} a_{i,j}(t)(\hat{\theta}_i - \hat{\theta}_j) \quad (45)$$

where ρ, γ and $a_{i,j}(t)$ are defined in (38). Define the model output error z and the parameter estimation error $\tilde{\theta}$ as that in (39). Then their dynamics are presented as follows

$$\begin{bmatrix} \dot{z} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} A & \Phi(t, y)^T \\ -\rho \Phi(t, y) & -\gamma \mathcal{L}(t) \otimes I_m \end{bmatrix} \begin{bmatrix} z \\ \tilde{\theta} \end{bmatrix} \quad (46)$$

where $y = [y_1, \dots, y_N]^T$, $A = \text{diag}\{a_1 I_n, \dots, a_N I_n\}$, and $\Phi(t, y)$ is similar to $\Phi(t)$ in (39).

Theorem 4: Consider the adaptive system consisting of (43)-(45). Suppose that the network topology is undirected. Then the closed-loop system (46) is ULES with any fixed constant $r > 0$, if there exist positive constants T_0, δ and α_0 such that the following two conditions hold: (i) the matrix $\int_t^{t+T_0} \mathcal{L}(\tau) d\tau$ has only one zero eigenvalue λ_1 and all its nonzero eigenvalues satisfy $\delta \leq \lambda_2(t) \leq \dots \leq \lambda_N(t)$ for any $t \geq t_0$; (ii) $\phi_i(t, y), 1 \leq i \leq N$, satisfy the cooperatively u-PE condition.

Proof: According to Theorem 2, we just need to verify Assumptions 1' and 2'. Consider the Lyapunov function

$$V = \frac{1}{2} z^T z + \frac{1}{\rho} \tilde{\theta}^T \tilde{\theta}.$$

Its derivative is given by

$$\dot{V} = - \sum_{i=1}^N a_i z_i^T z_i - \gamma \tilde{\theta}^T \mathcal{L}(t) \otimes I_m \tilde{\theta}$$

which means V is bounded uniformly for all $(t_0, (z(t_0), \tilde{\theta}(t_0))) \in R^+ \times B_r$. Then Assumption 1' is easily verified based on the bound of V , and Assumption 2' also holds by letting $P = \rho I_{nN}$ and $Q = -2\rho A$. ■

C. Illustrative Examples

Example 1: Consider six identical mass-spring-dashpot systems of the form

$$M \ddot{x}_i = u_i - kx_i - f \dot{x}_i, \quad i = 1, 2, 3, 4, 5, 6$$

where k is the spring constant, f is the viscous-friction or damping coefficient, M is the mass of the system, u_i is the forcing input, and x_i is the displacement of the mass M .

The above systems can be described by the following SPM [14]

$$y_i = \phi_i(t, x_i)^T \theta, \quad i = 1, 2, 3, 4, 5, 6 \quad (47)$$

where $\phi_i(t, x_i)^T = \left[\frac{1}{\Lambda(s)} u_i, -\frac{s}{\Lambda(s)} x_i, -\frac{1}{\Lambda(s)} x_i \right]^T$, $\theta = \left[\frac{1}{M}, \frac{f}{M}, \frac{k}{M} \right]^T$, and $\frac{1}{\Lambda(s)}$ is a chosen stable filter with $\Lambda(s) = (s + \lambda)^2$ and $\lambda > 0$. The unknown parameter vector θ needs to be identified.

In simulation, let $M = 2, f = 2, k = 4$ and $\lambda = 1$. Six inputs are chosen as $u_1 = \sin(t), u_2 = 2 \cos(0.5t), u_3 = 3 \sin(2t), u_4 = 3 \cos(2t), u_5 = \sin(t) + 0.5 \cos(t)$ and $u_6 = 2 \sin(3t) + \cos(0.4t)$. The parameter estimates are initialized as $\hat{\theta}_1(0) = -1.5, \hat{\theta}_2(0) = -1, \hat{\theta}_3(0) = -0.5, \hat{\theta}_4(0) = 0, \hat{\theta}_5(0) = 0.5, \hat{\theta}_6(0) = 1$, and the other initial conditions are zero. Firstly, we employ the conventional decentralized adaptive law obtained by letting $a_{i,j}(t) = 0$ in (38). The simulation results with the adaptive gain $\rho = 30$ are shown in Fig. 1. We can see that each $\hat{\theta}_i(t)$ cannot converge to its true value. This is because each $\phi_i(t)$ does not satisfy the traditionally defined PE condition. However, it is easily verified that the conditions in Theorem 3 holds. Thus we use the distributed cooperative adaptive law (38) with a fixed connected topology graph given in Fig. 2 and $\rho = \gamma = 30$. The obtained simulation results are shown in Fig. 3. It can be seen that each $\hat{\theta}_i(t)$ indeed converges to its true value.

Example 2: Consider three systems described by the following DPM

$$\dot{y}_i = \phi_i(t, y_i)^T \theta, \quad i = 1, 2, 3 \quad (48)$$

where $\phi_1(t, y_i)^T = [\sin t, y_i e^{-t}, y_i e^{-2t}]$, $\phi_2(t, y_i)^T = [y_i e^{-3t}, \cos t, y_i e^{-4t}]$, $\phi_3(t, y_i)^T = [y_i e^{-2.5t}, y_i e^{-1.5t}, \frac{t}{t+1}]$, and $\theta = [3.12, -2.45, 0.67]^T$.

In our simulation studies, all initial conditions are set to be zero. Firstly, we also apply the conventional decentralized adaptive law obtained by letting $a_{i,j}(t) = 0$ in (45). The simulation results with $a_i = -1$ and $\rho = 1$ are shown in Fig. 4, where the estimated parameters cannot converge to their true values. Secondly, we use the distributed cooperative scheme (45). To illustrate the effectiveness of the cooperative adaptive scheme with the time-varying topology, we assume the network topology is switched periodically with a period π between two unconnected topologies $\mathcal{G}_1$ and $\mathcal{G}_2$ shown in Fig. 5. The corresponding adjacency matrices are given by

$$\mathcal{A}_1(t) = \begin{bmatrix} 0 & |\sin(t)| & 0 \\ |\sin(t)| & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{A}_2(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & |\cos(t)| \\ 0 & |\cos(t)| & 0 \end{bmatrix}.$$

The simulation results with $a_i = -1$ and $\rho = \gamma = 1$ are shown in Fig. 6. It can be observed that all $\hat{\theta}_i(t)$ indeed converge to their true values, which accords with Theorem 4.

Fig. 1: Decentralized adaptation: $\hat{\theta}_i$, $1 \leq i \leq 6$. Fig. 2: Fixed network topology: $\mathcal{G}$ Fig. 3: Cooperative adaptation: $\hat{\theta}_i$, $1 \leq i \leq 6$ (fixed topology).

V. APPLICATIONS TO DISTRIBUTED COOPERATIVE ADAPTIVE CONTROL

A. Distributed Cooperative Adaptive Control for a Class of Linear Systems

Consider a team of linear systems with the following form [34]

$$\dot{x}_i = Ax_i + bu_i, \quad i = 1, \dots, N \quad (49)$$

where $x_i \in R^n$ and $u_i \in R$ are, respectively, the state and the control input of the i th system; $A \in R^{n \times n}$ and $b \in R^n$ are defined in the companion form as

$$A = \begin{bmatrix} 0_{(n-1) \times 1} & I_{n-1} \\ a_1 & a_2 \cdots a_n \end{bmatrix}, \quad b = \begin{bmatrix} 0_{(n-1) \times 1} \\ 1 \end{bmatrix}$$

with $a = [a_1, a_2, \dots, a_n]^T$ being an unknown parameter vector.

Fig. 4: Decentralized adaptation: $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$. Fig. 5: Switched network topology: $\mathcal{G}_1$, $\mathcal{G}_2$ Fig. 6: Cooperative adaptation: $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$ (switched topology).

The reference model corresponding to the i th systems is chosen as

$$\dot{x}_{mi} = A_m x_{mi} + b r_i(t) \quad (50)$$

where $x_{mi} \in R^n$ and $r_i \in R$ are the state and the input of the reference model, respectively. The matrix $A_m \in R^{n \times n}$ is also in the companion form and is stable. Thus, for a given matrix $Q > 0$, there exists $P > 0$ such that

$$A_m P + P A_m^T = -Q. \quad (51)$$

The last row of A_m is denoted as $a_m = [a_{m1}, a_{m2}, \dots, a_{mn}]^T$. The objective is to determine the input u_i for the i th system such that $\lim_{t \rightarrow \infty} [x_i(t) - x_{mi}(t)] = 0$.

Instead of using decentralized adaptive control, a direct distributed cooperative adaptive control law is designed as

$$u_i = x_i^T \hat{\theta}_i + r_i(t) \quad (52)$$

with the distributed cooperative adaptive law

$$\dot{\hat{\theta}}_i = \rho x_i b^T P (x_i - x_{mi}) - \gamma \sum_{j \in \mathcal{N}_i} a_{i,j}(t) (\hat{\theta}_i - \hat{\theta}_j) \quad (53)$$

where ρ and γ are positive design parameters, and $\hat{\theta}_i$ denotes the estimation of the unknown parameter vector $\theta = a_m - a$.

Define the model tracking error and the parameter estimation error as $z_i = x_i - x_{mi}$ and $\tilde{\theta}_i = \theta - \hat{\theta}_i$, respectively. It follows from (49)-(53) that

$$\begin{bmatrix} \dot{z} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} A_z & \Phi(t, z)^T \\ -\rho\Phi(t, z) & -\gamma\mathcal{L}(t) \otimes I_m \end{bmatrix} \begin{bmatrix} z \\ \tilde{\theta} \end{bmatrix} \quad (54)$$

where $z = [z_1^T, \dots, z_N^T]^T$, $\tilde{\theta} = [\tilde{\theta}_1^T, \dots, \tilde{\theta}_N^T]^T$, $A_z = \text{diag}\{A_m, \dots, A_m\}$, and $\Phi(t, z)^T = \text{diag}\{bx_1(t)^T, \dots, bx_N(t)^T\}$. Note that $x_i(t) = x_{mi}(t) + z_i(t)$. Based on Theorem 2, we obtain the following result.

Theorem 5: Consider the closed-loop adaptive system consisting of (49)-(50) and (52)-(53). Suppose that the network topology is undirected. Then the closed-loop system (54) is ULES with any fixed constant $r > 0$, if there exist positive constants T_0, δ and α_0 such that the following two conditions hold: (i) the matrix $\int_t^{t+T_0} \mathcal{L}(\tau) d\tau$ has only one zero eigenvalue λ_1 and all its nonzero eigenvalues satisfy $\delta \leq \lambda_2(t) \leq \dots \leq \lambda_N(t)$ for any $t \geq t_0$; (ii) $x_{im}(t), 1 \leq i \leq N$, satisfy the cooperatively PE condition.

Proof: To employ Theorem 2, we firstly need to prove the uniform boundedness of $\Phi(t, z)$ and $\frac{d\Phi(t, z(t))}{dt}$. Consider the Lyapunov function

$$V(z, \tilde{\theta}) = \sum_{i=1}^N z_i^T P z_i + \frac{1}{2\rho} \sum_{i=1}^N \tilde{\theta}_i^T \tilde{\theta}_i. \quad (55)$$

It follows from (51) and (54) that

$$\dot{V}(t) \leq -\sum_{i=1}^N z_i^T Q z_i \leq -\lambda_{\min}(Q) z^T z, \quad (56)$$

which shows that $\lim_{t \rightarrow \infty} z_i(t) = 0$ and for each fixed $r > 0$, $\tilde{\theta}_i(t)$ are uniformly bounded for $(t_0, (z_i(t_0), \tilde{\theta}_i(t_0))) \in R^+ \times B_r$. Then it is easy to prove the uniform boundedness of $x_i(t)$ and $u_i(t)$, which implies the uniform boundedness of $\Phi(t, z)$ and $\frac{d\Phi(t, z(t))}{dt}$. Thus, it is verified that Assumptions 1' and 2' hold.

Moreover,

$$\Phi(t, z) = \Phi(x_m(t)) + bZ(t)$$

where $\Phi(x_m(t)) = \text{diag}\{bx_{m1}(t)^T, \dots, bx_{mN}(t)^T\}$ and $Z(t) = \text{diag}\{bz_1(t)^T, \dots, bz_N(t)^T\}$. Notice that (56) also implies that $z(t)$ is square integrable uniformly in $(t_0, (z_i(t_0), \tilde{\theta}_i(t_0))) \in R^+ \times B_r$. Thus, $bZ(t)$ is also square integrable uniformly. Using the same proof as in [38, Lemma 1-P1)], we conclude that “ $x_{im}(t), 1 \leq i \leq N$, satisfy the cooperatively PE condition” is equivalent to “ $x_i(t), 1 \leq i \leq N$, satisfy the cooperatively u-PE condition”.

Then the result is immediately obtained based on Theorem 2. ■

B. Distributed Cooperative Adaptive Control for a Class of Nonlinear Systems

Consider a group of well-known parametric strict-feedback systems, where the i th ($i = 1, \dots, N$) system is described by

the following form [15]

$$\Sigma_i : \begin{cases} \dot{x}_{i,j} &= x_{i,j+1} + \varphi_{i,j}(x_{i,j})^T \theta, \quad j = 1, \dots, n-1 \\ x_{i,n} &= \beta_i(x_i) u_i + \varphi_{i,n}(x_i)^T \theta \\ y_i &= x_{i,1} \end{cases} \quad (57)$$

where $\bar{x}_{i,j} = [x_{i,1}, x_{i,2}, \dots, x_{i,j}]^T \in R^n$, $j = 1, \dots, n-1$, $x_i = [x_{i,1}, x_{i,2}, \dots, x_{i,n}]^T \in R^n$, $y_i \in R$ and $u_i \in R$ are the state, the output and the control input of the i th system, respectively; $\varphi_{i,j} : R^j \rightarrow R^m$ and $\beta_i : R^n \rightarrow (0, +\infty)$ are known smooth functions, and $\theta \in R^m$ is an unknown parameter vector.

For a given reference signal $y_{r,i}(t)$ for the output y_i to track, the following decentralized adaptive backstepping control can be designed based on the approach in [15]:

Control law

$$u_i = \frac{1}{\beta_i(x_i)} [\alpha_{i,n}(x_i, \hat{\theta}_i, \bar{y}_{r,i}^{(n-1)}) + y_{r,i}^{(n)}]. \quad (58)$$

Adaptive law

$$\dot{\hat{\theta}}_i = \rho \left[\tau_{i,n}(x_i, \hat{\theta}_i, \bar{y}_{r,i}^{(n-1)}) \right] \quad (59)$$

where $\rho > 0$ is the adaptation gain; $\alpha_{n,i}(\cdot, \cdot, \cdot)$ and $\tau_{n,i}(\cdot, \cdot, \cdot)$ are the same as those in [15, Table 4.1]; $\hat{\theta}_i$ denotes the estimate of θ . Then the closed-loop system is given by (see eqs. (4.194) and (4.195) in [15])

$$\begin{cases} \dot{z}_i &= A_{z_i}(z_i, \hat{\theta}_i) z_i + W_i(z_i, \hat{\theta}_i)^T \tilde{\theta}_i \\ \dot{\tilde{\theta}}_i &= -\rho W_i(z_i, \hat{\theta}_i) z_i \end{cases} \quad (60)$$

where $\tilde{\theta}_i = \theta - \hat{\theta}_i$. The other notations $z_i, A_{z_i}(\cdot, \cdot)$ and $W_i(\cdot, \cdot)$ are defined similarly to that in [15].

However, the decentralized control law (58) and adaptive law (59) cannot guarantee that each estimated parameter $\hat{\theta}_i$ converges to its true value if each $W_i(z_i, \hat{\theta}_i)$ cannot satisfy the PE condition, as illustrated by Example 4.13 in [15]. Based on Theorem 2, we propose the following distributed cooperative adaptive law

$$\dot{\hat{\theta}}_i = \rho \tau_{i,n}(x_i, \hat{\theta}_i, \bar{y}_{r,i}^{(n-1)}) - \gamma \sum_{j \in \mathcal{N}_i} a_{i,j}(t) (\hat{\theta}_i - \hat{\theta}_j) \quad (61)$$

with $\gamma > 0$.

Remark 8: The main difference of the distributed cooperative adaptive law (61) from the conventional decentralized adaptive law (59) is the use of the network topology term $\sum_{j \in \mathcal{N}_i} a_{i,j}(t) \times (\hat{\theta}_i - \hat{\theta}_j)$. Information can be exchanged locally among systems. It is natural to expect that the adaptive law (61) has more advantages over the adaptive law (59). For example, as shown in Theorem 6 below, if using (61), each $\hat{\theta}_i$ can tend to its true values even when $W_i(z_i, \hat{\theta}_i), 1 \leq i \leq N$ just satisfy the cooperatively u-PE condition, instead of the traditionally defined PE condition.

By replacing (59) with (61), the closed-loop system (60) can be rewritten as

$$\begin{bmatrix} \dot{z} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} A_z(t, z, \hat{\theta}) & \Phi(t, z, \tilde{\theta})^T \\ -\rho\Phi(t, z, \hat{\theta}) & -\gamma\mathcal{L}(t) \otimes I_m \end{bmatrix} \begin{bmatrix} z \\ \tilde{\theta} \end{bmatrix} \quad (62)$$

where $z = [z_1^T, \dots, z_N^T]^T$, $\tilde{\theta} = [\tilde{\theta}_1^T, \dots, \tilde{\theta}_N^T]^T$, $A_z(t, z, \tilde{\theta}) = \text{diag}\{A_{z_1}(z_1, \tilde{\theta}_1), \dots, A_{z_N}(z_N, \tilde{\theta}_N)\}$, and $\Phi(t, z, \tilde{\theta})^T = \text{diag}\{W_1(z_1, \tilde{\theta}_1)^T, \dots, W_N(z_N, \tilde{\theta}_N)^T\}$. Based on Theorem 2, we obtain the following result.

Theorem 6: Consider a group of systems (57) with the control law (58) and the distributed cooperative adaptive law (61). Suppose the network topology is undirected. Then the closed-loop system (62) is ULES with any fixed constant $r > 0$, if there exist three positive constants T_0, δ and α_0 such that the following two conditions hold: (i) the matrix $\int_t^{t+T_0} \mathcal{L}(\tau) d\tau$ has only one zero eigenvalue λ_1 and all its nonzero eigenvalues satisfy $\delta \leq \lambda_2(t) \leq \dots \leq \lambda_N(t)$ for any t ; (ii) $\varphi_{i,j}(\bar{x}_{i,j}(t))$, $1 \leq i \leq N$, $1 \leq j \leq n$, satisfy the cooperatively u-PE condition, where $\bar{x}_{i,j} := [x_{i,1}, \dots, x_{i,j}]^T$.

Proof: Firstly, consider the Lyapunov function $V = \frac{1}{2} \sum_{i=1}^N z_i^T z_i + \frac{1}{2\rho} \sum_{i=1}^N \tilde{\theta}_i^T \tilde{\theta}_i$ whose derivative, based on (62), is given by

$$\begin{aligned} \dot{V} &= - \sum_{i=1}^N c_i z_i^T z_i - \frac{\gamma}{\rho} \tilde{\theta}^T [\mathcal{L}(t) \otimes I_m] \tilde{\theta} \\ &\leq - \sum_{i=1}^N c_i z_i^T z_i. \end{aligned} \quad (63)$$

Using the same derivations as in [15], we have that $\lim_{t \rightarrow \infty} z_i(t) = 0$ and for each fixed $r > 0$, $\hat{\theta}_i$ and x_i are bounded uniformly in $(t_0, (z_i(t_0), \hat{\theta}_i(t_0)) \in R_{\geq 0} \times B_r$, which implies the uniform boundedness of $\Phi(t, z, \tilde{\theta})$ and $\frac{d\Phi(t, z, \tilde{\theta})}{dt}$ and A_z . Thus, Assumptions 1' and 2' hold. Based on Theorem 2, system (62) is ULES if we can prove that $W_i(z_i, \hat{\theta}_i)$, $1 \leq i \leq N$, satisfy the cooperatively u-PE condition.

In fact, $W_i(z_i, \hat{\theta}_i)^T$ can be rewritten as $W_i(z_i, \hat{\theta}_i)^T = M_i(z_i, \hat{\theta}_i) F_i(\bar{x}_{i,n})^T$, where $F_i(\bar{x}_{i,n})^T = [\varphi_{i,1}(x_{i,1})^T, \dots, \varphi_{i,n}(x_{i,n})^T]^T$ and

$$M_i(z_i, \hat{\theta}_i)^T = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -\frac{\partial \alpha_{i,1}}{\partial x_{i,1}} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -\frac{\partial \alpha_{i,n-1}}{\partial x_{i,1}} & \dots & -\frac{\partial \alpha_{i,n-1}}{\partial x_{i,n-1}} & 1 \end{bmatrix}.$$

Obviously, $M_i(z_i, \hat{\theta}_i)^T$ is uniformly nonsingular since z_i and $\hat{\theta}_i$ are uniformly bounded. Thus, we just need to show that $F_i(\bar{x}_{i,n})$, $1 \leq i \leq N$, satisfy the cooperatively u-PE condition, which is equivalent to that $\varphi_{i,j}(\bar{x}_{i,j}(t))$, $1 \leq i \leq N$, $1 \leq j \leq n$, satisfy the cooperatively u-PE condition. The proof is completed. ■

Remark 9: Although there exist some works on the stability of closed-loop system consisting of (57)-(59), see for example [15, Theorem 4.12], only globally asymptotic stability is established. To the best of our knowledge, there have been no reports on the UES of system (62) so far. In Theorem 6, we just obtain a sufficient condition for the ULES because the results depends on r . Note that it is very close to the UGES property since r is fixed but arbitrary, which actually means the uniformly globally asymptotical stability (UGAS). Moreover, a sufficient condition for $\varphi_{i,j}(\bar{x}_{i,j}(t))$, $1 \leq i \leq N$, $1 \leq j \leq n$, to meet the cooperatively u-PE condition is that $\varphi_{i,1}(y_{r,i}(t))$, $1 \leq i \leq N$, satisfy the cooperatively PE

condition, which can be proved by following the same proof as in [38, Lemma 1-P1]. The latter can be checked more easily since it does not depend on the system states.

C. Simulation Examples

Example 3: Consider the following four linear systems described by (49) with $a = [1, 3, 5, 2]^T$. Their reference models are given by (50), where $a_m = [2, -6, -7, -4]^T$ and the input signals are chosen as $r_1(r) = \sin(t)$, $r_2(t) = \cos(0.5t)$, $r_3(t) = \sin(2t)$ and $r_4(t) = \sin(\cos(t))$, respectively. From Subsection V-A, $\theta = a_m - a = [-3, -9, -12, -6]^T$. By solving (51) with $Q = I_4$,

$$P = \begin{bmatrix} 2.6500 & 2.9500 & 1.5500 & 0.2500 \\ 2.9500 & 6.0750 & 3.7000 & 0.5750 \\ 1.5500 & 3.7000 & 3.7500 & 0.6000 \\ 0.2500 & 0.5750 & 0.6000 & 0.2750 \end{bmatrix}.$$

Fig. 7: Decentralized adaptation: $x_i - x_{mi}$, $i = 1, 2, 3, 4$.

Fig. 8: Decentralized adaptation: $\hat{\theta}_i$, $i = 1, 2, 3, 4$.

Fig. 9: Switched network topology: $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$.

Firstly, we employ the conventional decentralized adaptive control law obtained by setting $\gamma = 0$ in (52) and (53). Simulation results with the adaptive gain $\rho = 1$ are shown in Figs. 7 and 8. We can see that all tracking errors converge to zero, but all $\hat{\theta}_i(t)$ cannot converge to their true values. The reason is that $x_{m1}(t), x_{m2}(t), x_{m3}(t)$ and $x_{m4}(t)$ do not satisfy the PE condition. However, it is easy to verify that the conditions shown in Theorem 5 hold. We further use the distributed cooperative adaptive law (53) to conduct simulation studies. Assume the network topology is switched between three unconnected topologies $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ shown in Fig. 9 with the following switching law

$$\sigma(t) = \begin{cases} 1, & t \in [2k\pi, 2k\pi + \frac{1}{2}\pi) \\ 2, & t \in [2k\pi + \frac{1}{2}\pi, 2k\pi + \frac{3}{2}\pi) \\ 3, & t \in [2k\pi + \frac{3}{2}\pi, 4k\pi) \end{cases}$$

Their time-varying adjacency matrices are given by

$$\begin{aligned} \mathcal{A}_1(t) &= \begin{bmatrix} 0 & |\sin(t)| & 0 & 0 \\ |\sin(t)| & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \mathcal{A}_2(t) &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ \mathcal{A}_3(t) &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & |\cos(t)| & 0 \\ 0 & |\cos(t)| & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The simulation results with $\rho = \gamma = 1$ are shown in Figs. 10 and 11, from which it can be seen that all tracking errors converge to zeroes, and each $\hat{\theta}_i(t)$ converges to its true value.

Example 4: Consider the adaptive formation control problem of three identical two-wheel mobile robots. The dynamic of the i th ($i = 1, 2, 3$) robot is described as [43]

$$\dot{x}_i \sin(\theta_i) - \dot{y}_i \cos(\theta_i) + d\dot{\theta}_i = 0 \quad (64)$$

$$M_0 T(\theta_i) \ddot{p}_i + M_0 \dot{T}(\theta_i) \dot{p}_i = S^T(\theta_i) B(\theta_i) \tau_i \quad (65)$$

where $p_i = [x_i, y_i, \theta_i]^T$ denotes the general coordinates of the i th robot; $M_0, T(\theta_i), S(\theta_i)$ and $B(\theta_i)$ are given as

$$\begin{aligned} M_0 &= \begin{bmatrix} m & 0 \\ 0 & \frac{I}{d} \end{bmatrix}, & T(\theta_i) &= \begin{bmatrix} \cos \theta_i & \sin \theta_i \\ -\sin \theta_i & \cos \theta_i \end{bmatrix}, \\ S(\theta_i) &= \begin{bmatrix} \cos \theta_i & -d \sin \theta_i \\ \sin \theta_i & d \cos \theta_i \\ 0 & 1 \end{bmatrix}, & B(\theta_i) &= \begin{bmatrix} \cos \theta_i & \cos \theta_i \\ \sin \theta_i & \sin \theta_i \\ L & -L \end{bmatrix}. \end{aligned}$$

where physical parameters m, d, I and L are defined in [43]. The formation reference of the i th robot is expressed as $p_i^d(t) = \sum_{j \in \Omega_i} \bar{a}_{ij}(p_j(t) + D_{ij}^d(t))$ in which Ω_i is defined as the set of robot i 's neighbors, and $\bar{a}_{ij}$ is a projection vector that describes how robot i 's neighbors affect its reference trajectory with $\sum_{j \in \Omega_i} \bar{a}_{ij} = 1$. $D_{ij}^d(t)$ represents the ideal reference distance that robot i should keep from robot j .

By defining $e_i = p_i - p_i^d, z_i = \dot{e}_i + e_i, \dot{p}_i^r = \dot{p}_i^d - e_i, z_i = \dot{p}_i - \dot{p}_i^r$ and $\tilde{z}_i = T(\theta_i) z_i$, the system (65) is transformed to

$$M_0 \dot{\tilde{z}}_i = \Phi(\theta_i, \dot{p}_i, \ddot{p}_i, \dot{p}_i^r) \tilde{z}_i + S^T(\theta_i) B(\theta_i) \tau_i. \quad (66)$$

Note that in (66) we use the fact that $-M_0 T(\theta_i) \ddot{p}_i^r + M_0 \dot{T}(\theta_i) \dot{p}_i^r = \Phi(\theta_i, \dot{p}_i, \ddot{p}_i, \dot{p}_i^r) \vartheta$, where $\vartheta = [m, \frac{m}{d}, \frac{I}{d^2}, \frac{I}{d}]^T$

Fig. 10: Cooperative adaptation: $x_i - x_{mi}$ (switched topology).

Fig. 11: Cooperative adaptation: $\hat{\theta}_i$ (switched topology).

and $\Phi = [\phi_i^{11}, \phi_i^{12}, 0, 0; 0, 0, \phi_i^{21}, \phi_i^{22}]$ with $\phi_i^{11} = \cos \theta_i \dot{x}_i^r + \sin \theta_i \dot{y}_i^r$, $\phi_i^{12} = (\sin \theta_i \dot{x}_i - \cos \theta_i \dot{y}_i)(\sin \theta_i \dot{x}_i^r - \cos \theta_i \dot{y}_i^r)$, $\phi_i^{21} = -\sin \theta_i \dot{x}_i^r + \cos \theta_i \dot{y}_i^r$, $\phi_i^{22} = (\cos \theta_i \dot{x}_i + \sin \theta_i \dot{y}_i)(\sin \theta_i \dot{x}_i^r - \cos \theta_i \dot{y}_i^r)$. Based on (66), we design the control law

$$\tau_i = (S^T(\theta_i) B(\theta_i))^{-1} [-c \tilde{z}_i - \Phi(\theta_i, \dot{p}_i, \ddot{p}_i^r, \dot{p}_i^r) \hat{\vartheta}_i] \quad (67)$$

with the cooperative adaptive law

$$\dot{\hat{\vartheta}}_i = \rho \Phi(\theta_i, \dot{p}_i, \ddot{p}_i^r, \dot{p}_i^r) \tilde{z}_i - \gamma \sum_{j \in \mathcal{N}_i} a_{i,j} (\hat{\vartheta}_i - \hat{\vartheta}_j). \quad (68)$$

Assume the network topology is fixed and shown in Fig. 2 with three nodes. Let Robot 1 be the leader. Its reference point is $p_1^d(t) = [3.3 \sin(\frac{\pi}{2}t + 2), 3 \cos(\frac{\pi}{2}t + 2)]^T$, and the reference points of other robots are $p_2^d(t) = p_1^d(t) + D_{21}(t)$ and $p_3^d(t) = p_2^d(t) + D_{32}(t)$ with $D_{21}(t) = -\frac{1}{6}p_1^d(t)$ and $D_{32}(t) = -\frac{1}{5}p_2^d(t)$. For simulation, we take $m = 1, d = 2$ and $I = 3$, which means $\vartheta = [1, 0.5, 0.75, 1.5]$. The initial conditions are set to be $[x_1, y_1]^T = [-5, 5]^T, [x_2, y_2]^T = [-5, -5]^T, [x_3, y_3]^T = [5, 5]^T$ and the remaining initial conditions are set to be zero. We first employ the decentralized adaptive control law obtained by letting $\gamma = 0$ in (68). The simulation results with $c = 2$ and $\rho = 1$ are shown in Figs. 12 and 13. Fig. 12 shows the trajectories of three robots, from which we can see that all robots form a regular order.

Fig. 12: The traces of mobile robots.

Fig. 13: Decentralized adaptation: $\hat{\vartheta}_1$, $\hat{\vartheta}_2$, $\hat{\vartheta}_3$.

The estimated parameters are shown in Fig. 13, where they do not converge to their true values. Then, we further use the distributed cooperative adaptive law (68). The simulation results with $c = 2$, $\rho = 1$ and $\gamma = 1$ are shown in Figs. 14 and 15, where we can see that the three robots still form a regular order, and all of their estimated parameters converge to their true values.

VI. CONCLUSIONS

In this paper, we have proposed a distributed cooperative adaptive scheme. Its main advantage is that the UES of closed-loop system is ensured only under a weaker cooperatively PE condition due to the introduction of a network topology, instead of the conventional PE condition. Furthermore, the proposed distributed cooperative adaptive scheme has been successfully applied to the identification and control of several linearly parameterized systems, which sufficiently shows the significance of the proposed adaptive scheme.

It is worthy to point out that this paper aims to make the first step for studying the distributed cooperative adaptive scheme. Owing to certain technical obstacles, there are still a number of open problems for future research, for example, the following topics:

- (i) Presently, the network topologies are undirected. An interesting problem is to address the case of directed network topologies.
- (ii) It is worthy to extend the distributed cooperative adaptive scheme to other type of systems such as discrete-

Fig. 14: The traces of mobile robots.

Fig. 15: Cooperative adaptation: $\hat{\vartheta}_1$, $\hat{\vartheta}_2$, $\hat{\vartheta}_3$.

time systems.

- (iii) Another important issue is to explore further potential applications of the distributed cooperative adaptive scheme. For example, it is interesting to study the distributed cooperative adaptive formation control of vessels and satellites since they can be modeled as linearly parameterized systems.
- (iv) Some other interesting problems include investigating the effects of certain network factors such as sampling, quantization, time delay and so on. Furthermore, the robustness of the cooperative adaptive scheme with respect to uncertainties like unmodelled dynamics and external disturbances is also an important issue to be addressed.

APPENDIX
PROOF OF LEMMA 5

The main line is along with Theorem 1 in [26]. Define a coordinate transformation of (7)

$$\xi := \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ -B(t, \lambda) & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (69)$$

Under Assumption 1 there exists $t_M > 0$ such that $\|\xi\| \leq t_M \|x\|$ or $\|x\| \leq t_M^{-1} \|\xi\|, \forall (t, \lambda) \in R^+ \times \Omega$, where $x = [x_1^T, x_2^T]^T$. Thus, the system (7) is λ -UGES if and only if the following system is λ -UGES:

$$\begin{aligned} \dot{\xi} = & \underbrace{\begin{bmatrix} A(t, \lambda) & B(t, \lambda)^T \\ -R_1(t, \lambda) & -B(t, \lambda)B(t, \lambda)^T - D(t, \lambda) \end{bmatrix}}_{\bar{A}(t, \lambda)} \xi \\ & + \underbrace{\begin{bmatrix} B(t, \lambda)^T B(t, \lambda) \\ R_1(t, \lambda) - R_2(t, \lambda) \end{bmatrix}}_{\mathcal{K}(t, \lambda)} \xi_1 \end{aligned} \quad (70)$$

where $R_1(t, \lambda) := P_B^{-1}(t, x)B(t, \lambda)P(t, \lambda)$ and $R_2(t, \lambda) = D(t, \lambda)B(t, \lambda) + \dot{B}(t, \lambda) + B(t, \lambda) \times [P(t, \lambda) + A(t, \lambda) + B(t, \lambda)^T B(t, \lambda)]$ and $P_B(t, \lambda)$ is a positive definite matrix to be defined.

We now follow the same procedure as in Theorem 1 in [26]. Suppose $\dot{\zeta} = \bar{A}(t, \lambda)\zeta$ is λ -UGES. Then, consider the Lyapunov function

$$V_1(t, \lambda, x) = \frac{1}{2} x^T R(t, \lambda) x$$

where $R(t, \lambda) = \text{dig}\{P(t, \lambda), I\}$. Its time derivative along (7) yields

$$\dot{V}_1(t, \lambda, x) \leq q_m \|x_1\|^2.$$

Hence

$$\|\xi_1\|_2 = \|x_1\|_2 \leq c_* \|x_0\| \leq c_* t_M^{-1} \|\xi_0\|$$

where $c_* := \sqrt{(p_M + 1)/q_m}$. Under Assumptions 1 and 2, there exists k_M such that $\|\mathcal{K}(t, \lambda)\| \leq k_M$ for all $\lambda \in \Omega$ and $t \geq 0$. Then, according to [26, Lemma 4], we conclude that the system (70) is λ -UGES. Now, it is only left to prove that $\dot{\zeta} = \bar{A}(t, \lambda)\zeta$ is λ -UGES.

Since $D(t, \lambda)$ is positive semi-definite, there exists a bounded matrix $H(t, \lambda)$ such that $D(t, \lambda) = H(t, \lambda)H(t, \lambda)^T$. Thus,

$$\begin{aligned} & B(t, \lambda)B(t, \lambda)^T + D(t, \lambda) \\ &= [B(t, \lambda), H(t, \lambda)][B(t, \lambda), H(t, \lambda)]^T \\ &= \Psi(t, \lambda) \end{aligned}$$

which, together with the condition (8), implies that $[B(t, \lambda), H(t, \lambda)]$ satisfies the λ -uPE condition. It follows from Lemma 4 that the system $\dot{\bar{\theta}} = -\Psi(t, \lambda)\bar{\theta}$ is λ -UGES. Therefore, from [26, Lemma 1], there exists a positive definite matrix $P_B(t, \lambda)$ such that for all $(t, \lambda) \in R^+ \times \Omega$,

$$c_1 I \leq P_B(t, \lambda) \leq c_2 I \quad (71)$$

$$\dot{P}_B(t, \lambda) - P_B(t, \lambda)\Psi(t, \lambda) - \Psi(t, \lambda)P_B(t, \lambda) = -I. \quad (72)$$

where c_1 and c_2 are positive constants. Consider the Lyapunov function

$$V(t, \lambda, \zeta) := \zeta_1^T P(t, \lambda) \zeta_1 + \zeta_2^T P_B(t, \lambda) \zeta_2.$$

Based on Assumption 2 and (72), its time derivative along $\dot{\zeta} = \bar{A}(t, \lambda)\zeta$, is

$$\dot{V}(t, \lambda, \zeta) = -\zeta_1^T Q(t, \lambda) \zeta_1 - \zeta_2^T \zeta_2 \leq -\epsilon V(t, \lambda, \zeta).$$

where $\epsilon := \min\{q_m, 1\}$, which implies that $\dot{\zeta} = \bar{A}(t, \lambda)\zeta$ is λ -UGES. This completes the proof.

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