

A New Weight Vector for a Tighter Levenshtein Bound on Aperiodic Correlation

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Abstract—The Levenshtein bound on aperiodic correlation, which is a function of the weight vector, is tighter than the Welch bound for sequence sets over the complex roots of unity when $M \geq 4$ and $n \geq 2$, where M denotes the set size and n the sequence length. Although it is known that the tightest Levenshtein bound is equal to the Welch bound for $M \in \{1, 2\}$, it is unknown whether the Levenshtein bound can be tightened for $M = 3$, and Levenshtein, in his 1999 paper, postulated that the answer may be negative. A new weight vector is proposed in this paper which leads to a tighter Levenshtein bound for $M = 3, n \geq 3$ and $M \geq 4, n \geq 2$. In addition, the explicit form of the weight vector (which is derived by relating the quadratic minimization to the Chebyshev polynomials of the second kind) in Levenshtein’s paper is given. Interestingly, this weight vector also yields a tighter Levenshtein bound for $M = 3, n \geq 3$ and $M \geq 4, n \geq \sqrt{M}$, a fact not noticed by Levenshtein.

Index Terms—Welch Bound, Levenshtein Bound, Aperiodic Correlation Lower Bound.

I. INTRODUCTION

Aperiodic correlation plays an important role in determining the system performance in asynchronous communications. For an asynchronous direct-sequence code-division multiple-access (DS-CDMA) system, Pursley showed that the worst-case probability of detection errors and the average signal-to-noise ratio are determined by the maximum non-trivial aperiodic correlation of the spreading sequences [1].

For any equi-energy sequence set¹, Welch proposed the following aperiodic correlation lower bound in [2], i.e.,

$$\delta_{\max}^2 \geq n^2 \frac{M-1}{M(2n-1)-1}, \quad (1)$$

where δ_{\max} denotes the maximum aperiodic correlation magnitude over all non-trivial auto-correlations and cross-correlations, M denotes the set size and n the sequence length. There was little progress in the research aiming at tightening the Welch bound on aperiodic correlation² for many years until

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¹i.e., all sequences in such a sequence set have identical energy.

²From now on, unless otherwise specified, “the Welch bound” refers to the lower bound on aperiodic correlation.

Levenshtein’s publication in 1999 [3].

The Levenshtein bound has been shown to be tighter than the Welch bound for a binary sequence set with the set size $M \geq 4$ and sequence length $n \geq 2$ [3]. This bound also holds for any sequence set over the complex roots of unity, as shown by Boztaş [4]. The Levenshtein bound is derived based on the fact that the weighted mean square aperiodic correlation of any sequence subset over the complex roots of unity is equal to or greater than that of the whole set which includes all sequences over the complex roots of unity. This idea was adopted by Peng and Fan to derive the aperiodic correlation lower bound of low correlation zone sequence sets [5]. Recently, Liu, Guan and Mow have derived the generalized Levenshtein bound for quasi-complementary sequence sets [6] and shown that it is tighter than the corresponding Welch bound [2]. In addition, Liu and Guan have shown that the Levenshtein bound is met with equality by the weighted-correlation complementary sequences [7]. The readers are referred to [8]–[12] for more information on perfect-/quasi- complementary sequences.

In Levenshtein’s technique for aperiodic correlation bounds, a weight vector with $2n-1$ non-negative elements which sum to 1 is used. The Levenshtein bound is a fractional quadratic function whose tightness depends on the weight vector. It is noted that when $M = 3$, Levenshtein was unable to find a weight vector which leads to a tighter bound and he mentioned in [3] that: “Thus for $M = 1$ and $M = 2$, the Welch bound cannot be improved by the method under consideration. This is probably also the case for $M = 3$. ”

In this paper, we present a new weight vector which takes the shape of “positive-cycle-of-a-sine-wave” and show that it leads to a tighter Levenshtein bound for $M = 3, n \geq 3$ and $M \geq 4, n \geq 2$. This settles the open problem in [3] on tightening the Levenshtein bound for $M = 3$.

In [3, *Theorem 2*], Levenshtein presented an improved lower bound resulting from the quadratic minimization using Lagrange multipliers and the Chebyshev polynomials of the second kind. We derive the explicit form of the weight vector and show that it also takes a similar shape of “positive-cycle-of-a-sine-wave” (yet different from our proposed one). Interestingly, this weight vector also gives rise to a tighter Levenshtein bound for $M = 3, n \geq 3$ and $M \geq 4, n \geq \sqrt{M}$. This fact, however, was not observed by Levenshtein in [3]. When $M \geq 4$, the condition $n \geq \sqrt{M}$ comes specifically from the constraints of the lower bound in [3, *Theorem 2*]. In contrast, our proposed weight vector is less restrictive and applicable for a tighter Levenshtein bound for $M \geq 4, n \geq 2$.

This paper is organized as follows. In Section II, notations

and a review of the Levenshtein bound are given. In Section III, the proposed weight vector and the resultant Levenshtein bound are presented, followed by the newly derived explicit form of the weight vector in [3, *Theorem 2*] and a comparison of existing known weight vectors. Finally, this paper is concluded in Section IV.

II. PRELIMINARIES

A. Notations

Let \mathbf{A} be a sequence set with M length- n sequences.

$$\begin{aligned} \mathbf{A} &= \{\mathbf{a}^0, \mathbf{a}^1, \dots, \mathbf{a}^u, \dots, \mathbf{a}^{M-1}\}, \quad 0 \leq u \leq M-1 \\ \mathbf{a}^u &= (a_0^u, a_1^u, \dots, a_t^u, \dots, a_{n-1}^u), \quad 0 \leq t \leq n-1. \end{aligned}$$

\mathbf{A} is called a sequence set over the complex roots of unity if every entry belongs to $E = \{1, \xi^1, \dots, \xi^{H-1}\}$, where $\xi = \exp\left(\frac{2\pi\sqrt{-1}}{H}\right)$ and H is a positive integer greater than 1. In particular, when $H = 2$, \mathbf{A} is a binary sequence set. In this paper, we focus on the aperiodic correlation lower bounds for sequence sets over the complex roots of unity.

The aperiodic correlation function $\rho_{\mathbf{a}^u, \mathbf{a}^v}(\tau)$ of \mathbf{A} is defined as

$$\rho_{\mathbf{a}^u, \mathbf{a}^v}(\tau) = \begin{cases} \sum_{t=0}^{n-1-\tau} a_t^u (a_{t+\tau}^v)^*, & 0 \leq \tau \leq (n-1); \\ \sum_{t=0}^{n-1+\tau} a_{t-\tau}^u (a_t^v)^*, & -(n-1) \leq \tau \leq -1; \\ 0, & |\tau| \geq n. \end{cases} \quad (2)$$

When $u \neq v$, $\rho_{\mathbf{a}^u, \mathbf{a}^v}(\tau)$ is called the aperiodic cross-correlation function; otherwise, it is called the aperiodic auto-correlation function and will be written as $\rho_{\mathbf{a}^u}(\tau)$ for simplicity.

For any $\mathbf{A} \subseteq E^n$, we define the following quantities:

$$\begin{aligned} \delta_a &= \max\{|\rho_{\mathbf{a}^u}(\tau)| : 0 \leq u \leq M-1, 0 < \tau \leq n-1\}, \\ \delta_c &= \max\{|\rho_{\mathbf{a}^u, \mathbf{a}^v}(\tau)| : u \neq v, 0 \leq u, v \leq M-1, \\ &\quad 0 \leq \tau \leq n-1\}, \end{aligned}$$

Furthermore, we define δ_{\max} , the maximum aperiodic correlation magnitude of \mathbf{A} , as $\delta_{\max} = \max\{\delta_a, \delta_c\}$.

The i -cyclic shift of any column vector $\mathbf{x} = (x_0, x_1, \dots, x_{n-1})^T$ is denoted by

$$T^i \mathbf{x} = (x_{n-i}, x_{n-i+1}, \dots, x_{n-1}, x_0, x_1, \dots, x_{n-i-1})^T,$$

where $0 \leq i \leq n-1$ and $(\cdot)^T$ denotes the transpose of (\cdot) .

B. Review of the Levenshtein bound

A real-valued vector $\mathbf{w} = (w_0, w_1, \dots, w_{2n-2})^T$ is called a weight vector if

$$w_i \geq 0, \quad \sum_{i=0}^{2n-2} w_i = 1. \quad (3)$$

For $0 \leq s, t \leq 2n-2$, define

$$l_{s,t,n} := \min\{|t-s|, 2n-1-|t-s|\}. \quad (4)$$

For $\mathbf{q}_a = (a, 1, 2, \dots, n-1, n-1, \dots, 2, 1)^T$, let

$$\mathcal{Q}_a = [\mathbf{q}_a, T^1 \mathbf{q}_a, T^2 \mathbf{q}_a, \dots, T^{2n-2} \mathbf{q}_a]$$

and

$$\begin{aligned} Q_{2n-1}(\mathbf{w}, a) &= a \sum_{i=0}^{2n-2} w_i^2 + \sum_{s,t=0}^{2n-2} l_{s,t,n} w_s w_t \\ &= \mathbf{w}^T \mathcal{Q}_a \mathbf{w}. \end{aligned} \quad (5)$$

Lemma 1: (The Levenshtein bound [3, *Theorem 1*]) For any sequence set $\mathbf{A} \subseteq E^n$ of size M and with any weight vector \mathbf{w} ,

$$\delta_{\max}^2 \geq n - \frac{Q_{2n-1}\left(\mathbf{w}, \frac{n(n-1)}{M}\right)}{1 - \frac{1}{M} \sum_{i=0}^{2n-2} w_i^2}. \quad (6)$$

A weaker simplified version of (6) is given below,

$$\delta_{\max}^2 \geq n - Q_{2n-1}\left(\mathbf{w}, \frac{n^2}{M}\right). \quad (7)$$

Remark 1: Applying the following weight vector

$$w_i = \begin{cases} \frac{1}{m}, & i \in \{0, 1, \dots, m-1\}; \\ 0, & i \in \{m, m+1, \dots, 2n-2\} \end{cases} \quad (8)$$

where $1 \leq m \leq n$, the lower bound in (6) reduces to [3, *Corollary 2*], i.e.,

$$\delta_{\max}^2 \geq \frac{nMm - n^2 - \frac{M(m^2-1)}{3}}{mM-1}, \quad 1 \leq m \leq n. \quad (9)$$

Moreover, by properly choosing the parameter m in (9), we have [3, *Corollary 3*], i.e.,

$$\delta_{\max}^2 \geq n - \frac{2n}{\sqrt{3M}}, \quad M \geq 3, n \geq 2. \quad (10)$$

In order to perform the convex optimization for the Levenshtein bound in (6), the following two results by Berlekamp [13] are recalled, i.e.,

- 1) The principal eigenvalue, and the secondary eigenvalues of the circulant matrix \mathcal{Q}_a in (5) are

$$\lambda_0 = a + (n-1)n,$$

and

$$\lambda_k = a - \frac{1 - (-1)^k \cos \frac{\pi k}{2n-1}}{2 \sin^2 \frac{\pi k}{2n-1}}, \quad k = 1, \dots, 2n-2$$

respectively.

- 2) The quadratic function $Q_{2n-1}(\mathbf{w}, a)$ is convex if all secondary eigenvalues $\lambda_k \geq 0$, and the minimum of which is achieved by $\mathbf{w} = \frac{1}{2n-1}(1, 1, \dots, 1)^T$.

Now, let $a = \frac{n(n-1)}{M}$ and substitute it into the above two results, one can show that all secondary eigenvalues of \mathcal{Q}_a are non-negative for $M = 1$ or 2 only. Therefore, we have the following remark.

Remark 2: When $M = 1$ or 2 , the tightest Levenshtein bound is equal to the Welch bound and is achieved by $\mathbf{w} = \frac{1}{2n-1}(1, 1, \dots, 1)^T$; When $M \geq 3$, however, not all secondary eigenvalues of \mathcal{Q}_a are non-negative, and thus the quadratic term $Q_{2n-1}\left(\mathbf{w}, \frac{n(n-1)}{M}\right)$ in (6) will become non-convex for $M \geq 3$. In the latter case, the Levenshtein bound in (9) is only known to be tighter for $M \geq 4, n \geq 2$ by setting $m = n$.

When $M = 3$ and $n = 2$, for any weight vector \mathbf{w} , we have

$$Q_{2n-1}\left(\mathbf{w}, \frac{n(n-1)}{M}\right) = 1 - \frac{1}{3} \sum_{i=0}^2 w_i^2.$$

Hence the Levenshtein bound in (6) in this case can be reduced to

$$\delta_{\max}^2 \geq n - \frac{Q_{2n-1}\left(\mathbf{w}, \frac{n(n-1)}{M}\right)}{1 - \frac{1}{M} \sum_{i=0}^{2n-2} w_i^2} = 1.$$

By computing the corresponding Welch bound in (1) yields the following observation.

Remark 3: When $M = 3$ and $n = 2$, the tightest Levenshtein bound is also equal to the Welch bound and is achieved by any weight vector.

We emphasize that the weight vector in (8) gives rise to a tighter Levenshtein bound (compared with the Welch bound) for $M \geq 4, n \geq 2$ only. Thus, it is of interest to find a new weight vector which leads to a tighter Levenshtein bound for $M = 3, n \geq 3$, and $M \geq 4, n \geq 2$. We will present such a new weight vector in the next section.

III. PROPOSED WEIGHT VECTOR FOR A TIGHTER LEVENSCHTEIN BOUND

A. Proposed Weight Vector and the Resultant Levenshtein Bound

Consider the following ‘‘positive-cycle-of-a-sine-wave’’ weight vector \mathbf{w} ,

$$w_i = \begin{cases} \tan \frac{\pi}{2m} \sin \frac{\pi i}{m}, & i \in \{0, 1, \dots, m-1\}; \\ 0, & i \in \{m, m+1, \dots, 2n-2\}, \end{cases} \quad (11)$$

where $2 \leq m \leq 2n-1$. It can be easily verified that

$$\sum_{i=0}^{2n-2} w_i = 1 \quad \text{and} \quad \sum_{i=0}^{2n-2} w_i^2 = \frac{m}{2} \tan^2 \frac{\pi}{2m}. \quad (12)$$

We present below the main result of this paper by substituting the proposed \mathbf{w} to (6).

Proposition 1:

$$\delta_{\max}^2 \geq n - \frac{n(n-1)m \tan^2 \frac{\pi}{2m} + 2MQ_{2n-1}(\mathbf{w}, 0)}{2M - m \tan^2 \frac{\pi}{2m}}, \quad (13)$$

where $Q_{2n-1}(\mathbf{w}, 0)$ is defined by (14) or (15) depending on the range of m .

1) for $2 \leq m \leq n$,

$$Q_{2n-1}(\mathbf{w}, 0) = \frac{m}{4} \left(1 - \tan^2 \frac{\pi}{2m}\right). \quad (14)$$

2) for $n < m \leq 2n-1$,

$$\begin{aligned} Q_{2n-1}(\mathbf{w}, 0) &= -\frac{3m-4n+2}{4} - \frac{m}{4} \tan^2 \frac{\pi}{2m} + \frac{m-n-1}{2} \cos \frac{n\pi}{m} \\ &+ \left(\frac{2m-2n+1}{4} \tan \frac{\pi}{2m} + \frac{3}{4 \tan \frac{\pi}{2m}} \right) \sin \frac{n\pi}{m}. \end{aligned} \quad (15)$$

Proof: see Appendix A. ■

Remark 4: For $m = n+1$, we note that

$$Q_{2n-1}(\mathbf{w}, 0) = \frac{m}{4} \left(1 - \tan^2 \frac{\pi}{2m}\right). \quad (16)$$

In this case, $Q_{2n-1}(\mathbf{w}, 0)$ takes the same form as that in (14). For $m = 2n-1$,

$$Q_{2n-1}(\mathbf{w}, 0) = \frac{3}{4 \sin \frac{\pi}{2m}} - \frac{m}{4 \cos^2 \frac{\pi}{2m}}. \quad (17)$$

Remark 5: Setting $m = n+1$ in *Proposition 1*, we assert that the resultant lower bound is tighter than the Welch bound in (1) for $M = 3, n \geq 3$ and $M \geq 4, n \geq 2$. In contrast, the Welch bound cannot be tightened for $M \in \{1, 2\}$ and $M = 3, n = 2$ (shown in *Remark 3*).

Proof: See Appendix B. ■

B. A Comparison of Existing Known Weight Vectors

In this subsection, we shall first explicitly reveal the weight vector in [3, *Theorem 2*]. We shall show that the weight vector therein is also capable of tightening the Levenshtein bound for $M = 3$. A summary of existing known weight vectors and their asymptotic bounds will be given.

Lemma 2: [3, *Theorem 2*] Let $M \leq n^2$ and $\cos \varphi = 1 - M/n^2$. Also, let r be a positive integer, with $r\varphi < \pi/2 + \varphi/2$ and $2 \leq 2r \leq n$. For $m = 2r$ and $\varphi_0 = \frac{\pi - m\varphi + \varphi}{2}$, define the following weight vector.

$$w_i = \begin{cases} \frac{\sin \frac{\varphi}{2}}{\sin \frac{m\varphi}{2}} \sin(\varphi_0 + i\varphi), & i \in \{0, 1, \dots, m-1\}; \\ 0, & i \in \{m, m+1, \dots, 2n-2\}. \end{cases} \quad (18)$$

Substituting the weight vector in (18) to the weaker Levenshtein bound in (7), we have

$$\delta_{\max}^2 \geq n - f(r), \quad (19)$$

where

$$\begin{aligned} f(r) &= r + \frac{\sin r\varphi - \sin(r\varphi - \varphi)}{2(1 - \cos \varphi) \sin r\varphi} - \frac{1}{2} \\ &= r + \frac{1}{2 \tan \frac{\varphi}{2} \tan r\varphi}, \end{aligned} \quad (20)$$

is monotonically decreasing over $1 \leq 2r \leq n$. Moreover, substituting the weight vector in (18) to the Levenshtein bound in (6), we have

$$\delta_{\max}^2 \geq \frac{n - f(r)}{1 - \frac{1}{M} \sum_{i=0}^{2n-2} w_i^2}, \quad (21)$$

where

$$\sum_{i=0}^{2n-2} w_i^2 = \frac{\sin^2 \frac{\varphi}{2}}{2 \sin^2 \frac{m\varphi}{2}} \left[m - \cos(\varphi(m-1) + 2\varphi_0) \frac{\sin \varphi m}{\sin \varphi} \right]. \quad (22)$$

Proof: Note that the proof of (19) can be found in [3, *Theorem 2*]. In what follows, we first prove the weight vector in (18). To this end, we closely follow the notations in [3,

Theorem 2] for ease of presentation. By relating [3, (31)] to the Chebyshev polynomials of the second kind, we have

$$\begin{aligned} y_i &= \frac{\sin(i\varphi + \varphi) - \sin i\varphi}{\sin r\varphi} \\ &= \frac{\cos i\varphi \sin \varphi + \sin i\varphi (\cos \varphi - 1)}{\sin r\varphi} \\ &= \frac{2 \sin \frac{\varphi}{2} \cos \left(i\varphi + \frac{\varphi}{2} \right)}{\sin r\varphi}, \quad \text{for } k = 0, 1, \dots, r-1. \end{aligned} \quad (23)$$

The weight vector in (18) follows by recalling [3, (27)], i.e.,

$$y_k = 2w_{r-k-1} = 2w_{r+k}, \quad \text{for } k = 0, 1, \dots, r-1. \quad (24)$$

Next, we prove the lower bound in (21). By more or less the same approach in the Proof of *Lemma 4*, the weight square sum in (22) can be proved. By

$$\begin{aligned} f(r) &= Q_{2n-1} \left(\mathbf{w}, \frac{n^2}{M} \right) \\ &= Q_{2n-1}(\mathbf{w}, 0) + \frac{n^2}{M} \sum_{i=0}^{2n-2} w_i^2, \end{aligned} \quad (25)$$

we have

$$\begin{aligned} Q_{2n-1} \left(\mathbf{w}, \frac{n(n-1)}{M} \right) &= Q_{2n-1}(\mathbf{w}, 0) + \frac{n(n-1)}{M} \sum_{i=0}^{2n-2} w_i^2 \\ &= f(r) - \frac{n}{M} \sum_{i=0}^{2n-2} w_i^2. \end{aligned} \quad (26)$$

The lower bound in (21) follows by (6) and (26). ■

Remark 6: Note that our proposed weight vector in (11) is not a special case of the weight vector in (18). Although both weight vectors take the shape of “positive-cycle-of-a-sine-wave”, our proposed weight vector has a phase increment of π/m , whereas that in (18) has a phase increment of $\cos^{-1}(1 - M/n^2)$ and is more restrictive.

By setting $r = \left\lfloor \frac{\pi}{2\varphi} + \frac{1}{2} \right\rfloor$, Levenshtein showed that the optimized lower bound in (19) (over different r) can be reduced as follows.

Lemma 3: [3, *Corollary 4*]

$$\delta_{\max}^2 \geq n - \left\lfloor \frac{\pi n}{\sqrt{8M}} \right\rfloor, \quad \text{for } 5 \leq M \leq n^2. \quad (27)$$

In contrast to *Remark 5*, we note that

Remark 7: The lower bound in (21) is tighter than the Welch bound in (1) for the following cases:

- 1) $M = 3, n \geq 3$;
- 2) $M \geq 4, n \geq \sqrt{M}$.

Proof: See Appendix C. ■

Recently, we have proposed another weight vector, called “quadratic weight vector” [15]. It may be viewed as the quadratic approximation of our proposed weight vector in this paper and gives rise to a tighter Levenshtein bound for $M = 3, n \geq 4$ and $M \geq 4, n \geq 2$. We summarize existing known weight vectors in Table I.

For every fixed n , optimizing the lower bound in *Proposition 1* over $2 \leq m \leq 2n - 2$ is challenging. For ease of comparison, we consider sufficiently large n and optimize the lower bounds rising from those weight vectors³ in Table I. We present below the following proposition.

Proposition 2: For $M = 3$ and $M = 4$, the respective asymptotic lower bound (optimized over different m) resulted from our proposed weight vector is tighter than that from other weight vectors, as shown in Table II.

Proof: See Appendix D. ■

IV. CONCLUSIONS

In this paper, we have proposed a new weight vector in (11) which takes the shape of “positive-cycle-of-a-sine-wave” for the Levenshtein bound. We have shown that the resultant Levenshtein bound presented in *Proposition 1* is tighter than the Welch bound for $M = 3, n \geq 3$ and $M \geq 4, n \geq 2$, where M and n denote the set size and the sequence length, respectively. This settles the open problem left in [3] on tightening the Levenshtein bound for $M = 3$. Since the Welch bound cannot be tightened for $M \in \{1, 2\}$ and $M = 3, n = 2$ (as shown in *Remark 3*), the proposed weight vector leads to a tighter Levenshtein bound (over the Welch bound) for all (non-trivial) possible (M, n) cases.

Secondly, we have derived in (18) the explicit expression of Levenshtein’s weight vector (which is obtained by relating the quadratic minimization solution to the Chebyshev polynomials of the second kind) in [3, *Theorem 2*]. An interesting observation, as shown in *Remark 7*, is that the weight vector in (18) also gives rise to a tighter Levenshtein bound for $M = 3, n \geq 3$ and $M \geq 4, n \geq \sqrt{M}$. This fact, however, was not recognized by Levenshtein in [3].

Although the weight vector in (18) also takes the shape of “positive-cycle-of-a-sine-wave”, it has a phase increment of $\cos^{-1}(1 - M/n^2)$, whereas our proposed one has a phase increment of π/m (where m is the number of nonzero elements in the weight vector). Note that for $M \geq 4$, the weight vector in (18) is valid for a tighter Levenshtein bound for $n \geq \sqrt{M}$ only (due to the inherent constraints in [3, *Theorem 2*]), whereas our proposed weight vector is valid for any $n \geq 2$. In conclusion, we think our proposed weight vector is interesting as it is simpler, less restrictive, and leads to a slightly tighter asymptotic bound for $M \in \{3, 4\}$ (as shown in Table II).

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³except for the optimization of the lower bound from the quadratic weight vector in [15] which is very close to our proposed weight vector in this paper.

TABLE I: Existing Known Weight Vectors of the Levenshtein Bound

from	weight vector \mathbf{w}	constraints	tighter (M, n) zone
[3]	$\frac{1}{2n-1}(1, 1, \dots, 1)^T$	n.a.	n.a.
[3]	$\begin{cases} \frac{1}{m}, & 0 \leq i \leq m-1; \\ 0, & m \leq i \leq 2n-2. \end{cases}$	$1 \leq m \leq n$	$M \geq 4, n \geq 2$
[3]	$\begin{cases} \frac{\sin \frac{\varphi}{2}}{\sin \frac{m\varphi}{2}} \sin(\varphi_0 + i\varphi), & 0 \leq i \leq m-1; \\ 0, & m \leq i \leq 2n-2. \end{cases}$ where $\varphi_0 = \frac{\pi - m\varphi + \varphi}{2}$.	$\begin{cases} \cos \varphi = 1 - M/n^2, \\ M \leq n^2, m = 2r, \\ r\varphi < \pi/2 + \varphi/2, \\ 2 \leq 2r \leq n. \end{cases}$	$\begin{cases} 1 : M = 3, n \geq 3; \\ 2 : M \geq 4, n \geq \sqrt{M}. \end{cases}$
[15]	$\begin{cases} \frac{6i(m-i)}{m(m^2-1)}, & 0 \leq i \leq m-1; \\ 0, & m \leq i \leq 2n-2. \end{cases}$	$2 \leq m \leq n+1$	$\begin{cases} 1 : M = 3, n \geq 4; \\ 2 : M \geq 4, n \geq 2. \end{cases}$
Proposed	$\begin{cases} \tan \frac{\pi}{2m} \sin \frac{\pi i}{m}, & 0 \leq i \leq m-1; \\ 0, & m \leq i \leq 2n-2. \end{cases}$	$2 \leq m \leq 2n-2$	$\begin{cases} 1 : M = 3, n \geq 3; \\ 2 : M \geq 4, n \geq 2. \end{cases}$

TABLE II: Asymptotic Lower Bound Comparison

correlation lower bound	$M = 3$	$M = 4$	$M \geq 5$
The Welch bound ¹	$\delta_{\max}^2 \gtrsim 0.3333n$	$\delta_{\max}^2 \gtrsim 0.3750n$	$\delta_{\max}^2 \gtrsim \left[\frac{1}{2} - \frac{1}{2M}\right] n$
[3, Corollary 2] ²	$\delta_{\max}^2 \geq 0.3333n$	$\delta_{\max}^2 \geq 0.4227n$	$\delta_{\max}^2 \geq \max \left\{ 1 - \frac{2}{\sqrt{3M}}, n, \frac{nM-n^2}{M-1} \right\}$
[3, Theorem 2] ³	$\delta_{\max}^2 \gtrsim 0.3528n$	$\delta_{\max}^2 \gtrsim 0.4442n$	$\delta_{\max}^2 \gtrsim \left 1 - \frac{\pi}{\sqrt{8M}} \right n$
Proposition 1 ⁴	$\delta_{\max}^2 \gtrsim 0.3594n$	$\delta_{\max}^2 \gtrsim 0.4446n$	$\delta_{\max}^2 \gtrsim \max \left\{ \left 1 - \frac{\pi}{\sqrt{8M}} \right n, \frac{nM-n^2}{M-1} \right\}$

¹ shown in (1), from the weight vector in the 1st row of Table I;

² shown in (9), from the weight vector in the 2nd row of Table I;

³ shown in (19), from the weight vector in the 3rd row of Table I. Note that $5 \leq M \leq n^2$;

⁴ from the proposed weight vector in (11).

APPENDIX A PROOF OF Proposition 1

To prove Proposition 1, we first introduce the following lemma.

Lemma 4: For $x \neq 0$, let $\mathcal{C}(n, x) = \sum_{\tau=1}^{n-1} \cos \tau x$, $\mathcal{S}(n, x) = \sum_{\tau=1}^{n-1} \sin \tau x$, then

$$\begin{aligned} \mathcal{C}(n, x) &= \frac{\sin \frac{nx}{2} \cos \frac{n-1}{2} x}{\sin \frac{x}{2}} - 1, \\ \mathcal{S}(n, x) &= \frac{\sin \frac{nx}{2} \sin \frac{n-1}{2} x}{\sin \frac{x}{2}}. \end{aligned} \quad (28)$$

Proof: Let $\mathcal{E}(n, x) = \sum_{\tau=0}^{n-1} \exp \sqrt{-1} \tau x$. Then, $\mathcal{C}(n, x) = \text{Re} \{ \mathcal{E}(n, x) \} - 1$, and $\mathcal{S}(n, x) = \text{Im} \{ \mathcal{E}(n, x) \}$, where $\text{Re} \{ x \}$ and $\text{Im} \{ x \}$ denote the real part and the imaginary part of complex-valued data x , respectively. By using the identity

$$\begin{aligned} \mathcal{E}(n, x) &= \sum_{\tau=0}^{n-1} \exp \sqrt{-1} x \tau = \frac{1 - \exp \sqrt{-1} n x}{1 - \exp \sqrt{-1} x} \\ &= \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}} \left(\cos \frac{n-1}{2} x + \sqrt{-1} \sin \frac{n-1}{2} x \right), \end{aligned}$$

the proof of this lemma follows. ■

For the proposed weight vector \mathbf{w} in (11), we have

$$\begin{aligned} Q_{2n-1} \left(\mathbf{w}, \frac{n(n-1)}{M} \right) \\ &= Q_{2n-1}(\mathbf{w}, 0) + \frac{n(n-1)}{M} \sum_{i=0}^{2n-2} w_i^2 \\ &= Q_{2n-1}(\mathbf{w}, 0) + \frac{n(n-1)m}{2M} \tan^2 \frac{\pi}{2m}. \end{aligned} \quad (29)$$

Thus, our main task is to derive $Q_{2n-1}(\mathbf{w}, 0) = \mathbf{w}^T \mathcal{Q}_0 \mathbf{w}$, where

$$\mathcal{Q}_0 = [\mathbf{q}_0, T^1 \mathbf{q}_0, T^2 \mathbf{q}_0, \dots, T^{2N-2} \mathbf{q}_0]$$

and

$$\mathbf{q}_0 = (0, 1, 2, \dots, n-1, n-1, \dots, 2, 1)^T.$$

We denote by $\langle \mathbf{a}, \mathbf{b} \rangle$ the inner product of vectors \mathbf{a} and \mathbf{b} . By the inner product properties, we have

$$\mathbf{w}^T (T^s \mathbf{q}_0) = \langle \mathbf{w}, T^s \mathbf{q}_0 \rangle = \langle \mathbf{q}_0, T^{-s} \mathbf{w} \rangle = \mathbf{q}_0^T (T^{-s} \mathbf{w}),$$

where $0 \leq s \leq 2n-2$. Hence, we have

$$Q_{2n-1}(\mathbf{w}, 0) = \mathbf{w}^T \mathcal{Q}_0 \mathbf{w} = \mathbf{q}_0^T \Omega \mathbf{w}, \quad (30)$$

with

$$\begin{aligned} \Omega &= [\mathbf{w}, T^{-1} \mathbf{w}, T^{-2} \mathbf{w}, \dots, T^{-(2n-2)} \mathbf{w}] \\ &= \begin{bmatrix} \mathbf{w}^T \\ (T^{-1} \mathbf{w})^T \\ (T^{-2} \mathbf{w})^T \\ \vdots \\ (T^{-(2n-2)} \mathbf{w})^T \end{bmatrix}. \end{aligned} \quad (31)$$

Therefore,

$$\begin{aligned} & \Omega \mathbf{w} \\ &= \left[\langle \mathbf{w}, \mathbf{w} \rangle, \langle T^{-1} \mathbf{w}, \mathbf{w} \rangle, \dots, \langle T^{-(2n-2)} \mathbf{w}, \mathbf{w} \rangle \right]^T \\ &= \left[\langle \mathbf{w}, \mathbf{w} \rangle, \langle T^{-1} \mathbf{w}, \mathbf{w} \rangle, \langle T^{-2} \mathbf{w}, \mathbf{w} \rangle, \dots, \langle T^{-(n-1)} \mathbf{w}, \mathbf{w} \rangle, \right. \\ & \quad \left. \langle T^{n-1} \mathbf{w}, \mathbf{w} \rangle, \langle T^{n-2} \mathbf{w}, \mathbf{w} \rangle, \dots, \langle T^1 \mathbf{w}, \mathbf{w} \rangle \right]^T, \end{aligned} \quad (32)$$

and

$$\begin{aligned} Q_{2n-1}(\mathbf{w}, 0) &= \left[\sum_{\tau=1}^{n-1} \tau \langle T^\tau \mathbf{w}, \mathbf{w} \rangle + \sum_{\tau=1}^{n-1} \tau \langle T^{-\tau} \mathbf{w}, \mathbf{w} \rangle \right] \\ &= 2 \sum_{\tau=1}^{n-1} \tau \langle T^\tau \mathbf{w}, \mathbf{w} \rangle. \end{aligned} \quad (33)$$

We remark that

1) For $2 \leq m \leq n$,

$$\langle T^\tau \mathbf{w}, \mathbf{w} \rangle = \begin{cases} \rho_{\mathbf{w}}(\tau), & 0 \leq \tau \leq m-1; \\ 0, & m-1 < \tau \leq n-1; \end{cases} \quad (34)$$

2) For $n < m \leq 2n-1$,

$$\langle T^\tau \mathbf{w}, \mathbf{w} \rangle = \begin{cases} \rho_{\mathbf{w}}(\tau), & \text{for } 0 \leq \tau \leq 2n-1-m; \\ \rho_{\mathbf{w}}(\tau) + \rho_{\mathbf{w}}(2n-1-\tau), & \text{for } 2n-1-m < \tau \leq n-1. \end{cases} \quad (35)$$

For the proposed weight vector \mathbf{w} in (11), when $0 \leq \tau \leq m-1$, we have

$$\begin{aligned} \rho_{\mathbf{w}}(\tau) &= \tan^2 \frac{\pi}{2m} \sum_{i=0}^{m-1-\tau} \sin \frac{\pi i}{m} \sin \frac{\pi(i+\tau)}{m} \\ &= \frac{\tan^2 \frac{\pi}{2m}}{2} \sum_{i=0}^{m-1-\tau} \left(\cos \frac{\pi \tau}{m} - \cos \frac{\pi(2i+\tau)}{m} \right) \\ &= \frac{\tan^2 \frac{\pi}{2m}}{2} \left((m-\tau) \cos \frac{\pi \tau}{m} + \frac{\sin \frac{\pi \tau}{m}}{\tan \frac{\pi}{m}} \right). \end{aligned} \quad (36)$$

Taking (32)-(36) into account, we have

1) For $2 \leq m \leq n$,

$$\begin{aligned} Q_{2n-1}(\mathbf{w}, 0) &= 2 \sum_{\tau=1}^{m-1} \rho_{\mathbf{w}}(\tau) \tau \\ &= \left(mA - B + \frac{C}{\tan \frac{\pi}{m}} \right) \tan^2 \frac{\pi}{2m}, \end{aligned} \quad (37)$$

where

$$\begin{aligned} A &= \sum_{\tau=1}^{m-1} \tau \cos \frac{\pi \tau}{m}, \\ B &= \sum_{\tau=1}^{m-1} \tau^2 \cos \frac{\pi \tau}{m}, \\ C &= \sum_{\tau=1}^{m-1} \tau \sin \frac{\pi \tau}{m}. \end{aligned} \quad (38)$$

2) For $n < m \leq 2n-1$,

$$\begin{aligned} & Q_{2n-1}(\mathbf{w}, 0) \\ &= 2 \left(\sum_{\tau=1}^{n-1} \rho_{\mathbf{w}}(\tau) \tau + \sum_{\tau=n}^{m-1} \rho_{\mathbf{w}}(\tau) (2n-1-\tau) \right) \\ &= \tan^2 \frac{\pi}{2m} \left(m\bar{A} - \bar{B} + \frac{\bar{C}}{\tan \frac{\pi}{m}} + (2n-1)mD \right. \\ & \quad \left. - (2n-1+m)E + F + \frac{2n-1}{\tan \frac{\pi}{m}} G - \frac{H}{\tan \frac{\pi}{m}} \right), \end{aligned} \quad (39)$$

with

$$\begin{aligned} \bar{A} &= \sum_{\tau=1}^{n-1} \tau \cos \frac{\pi \tau}{m}, \quad \bar{B} = \sum_{\tau=1}^{n-1} \tau^2 \cos \frac{\pi \tau}{m}, \\ \bar{C} &= \sum_{\tau=1}^{n-1} \tau \sin \frac{\pi \tau}{m}, \quad D = \sum_{\tau=n}^{m-1} \cos \frac{\pi \tau}{m}, \\ E &= \sum_{\tau=n}^{m-1} \tau \cos \frac{\pi \tau}{m}, \quad F = \sum_{\tau=n}^{m-1} \tau^2 \cos \frac{\pi \tau}{m}, \\ G &= \sum_{\tau=n}^{m-1} \sin \frac{\pi \tau}{m}, \quad H = \sum_{\tau=n}^{m-1} \tau \sin \frac{\pi \tau}{m}. \end{aligned} \quad (40)$$

Next, we derive the trigonometric sums of multiple angles in (38) and (40). For ease of presentation, let $x = \pi/m$.

By (28), we have

$$\begin{aligned} A &= \frac{\partial \mathcal{S}(m, x)}{\partial x} = \frac{m}{2} - \frac{1}{2 \sin^2 \frac{x}{2}}, \\ \bar{A} &= \frac{\partial \mathcal{S}(n, x)}{\partial x} = \frac{-1 + \cos nx + 2n \sin \frac{x}{2} \sin \frac{2n-1}{2} x}{4 \sin^2 \frac{x}{2}}, \end{aligned} \quad (41)$$

$$E = A - \bar{A},$$

$$D = \mathcal{C}(m, x) - \mathcal{C}(n, x) = \frac{1}{2} \left(1 - \frac{\sin \frac{2n-1}{2} x}{\sin \frac{x}{2}} \right).$$

Meanwhile,

$$\begin{aligned} C &= -\frac{\partial \mathcal{C}(m, x)}{\partial x} = \frac{m}{2 \tan \frac{x}{2}}, \\ \bar{C} &= -\frac{\partial \mathcal{C}(n, x)}{\partial x} = \frac{\sin nx}{4 \sin^2 \frac{x}{2}} - \frac{n \cos \frac{2n-1}{2} x}{2 \sin \frac{x}{2}}, \end{aligned} \quad (42)$$

$$H = C - \bar{C},$$

$$G = \mathcal{S}(m, x) - \mathcal{S}(n, x) = \frac{1}{2} \left(\frac{1}{\tan \frac{x}{2}} + \frac{\cos \frac{2n-1}{2} x}{\sin \frac{x}{2}} \right).$$

In addition,

$$\begin{aligned} B &= \frac{\partial C}{\partial x} = \frac{m}{4} \left(2m-1 - \frac{1}{\sin^2 \frac{x}{2}} - \frac{1}{\tan^2 \frac{x}{2}} \right), \\ \bar{B} &= \frac{\partial \bar{C}}{\partial x} = \frac{(2n-1) \cos nx}{4 \sin^2 \frac{x}{2}} - \frac{\sin \frac{2n-1}{2} x}{4 \sin^3 \frac{x}{2}} + \frac{n^2 \sin \frac{2n-1}{2} x}{2 \sin \frac{x}{2}}, \\ F &= \sum_{\tau=n}^{m-1} \tau^2 \cos \tau x = B - \bar{B}. \end{aligned} \quad (43)$$

Substituting A, B, C into (37), and letting $x = \pi/m$, it is easy to get (14).

For (15), we show the derivation below.

Since

$$\begin{aligned}
m\bar{A} - 2nE &= -mn + \frac{2n-m}{4\sin^2\frac{x}{2}} + \frac{(2n+m)\cos nx}{4\sin^2\frac{x}{2}} \\
&\quad + \frac{(2n^2+mn)\sin\frac{2n-1}{2}x}{2\sin\frac{x}{2}}, \\
F - \bar{B} &= \frac{m(2m-1)}{4} - \frac{m}{4\sin^2\frac{x}{2}} - \frac{m}{4\tan^2\frac{x}{2}} \\
&\quad - \frac{(2n-1)\cos nx}{2\sin^2\frac{x}{2}} + \frac{\sin\frac{2n-1}{2}x}{2\sin^3\frac{x}{2}} \\
&\quad - \frac{n^2\sin\frac{2n-1}{2}x}{\sin\frac{x}{2}},
\end{aligned} \tag{44}$$

therefore,

$$\begin{aligned}
m\bar{A} - 2nE + F - \bar{B} &= \frac{-4mn + 2m^2 - m}{4} + \frac{n-m}{2\sin^2\frac{x}{2}} - \frac{m}{4\tan^2\frac{x}{2}} \\
&\quad + \frac{(m-2n+2)\cos nx}{4\sin^2\frac{x}{2}} + \frac{\sin nx \cos\frac{x}{2}}{2\sin^3\frac{x}{2}} - \frac{\cos nx}{2\sin^2\frac{x}{2}} \\
&\quad + \frac{mn\cos(n-1)x}{2\sin^2\frac{x}{2}} - \frac{mn\cos nx}{2\sin^2\frac{x}{2}}.
\end{aligned} \tag{45}$$

Meanwhile

$$\begin{aligned}
\frac{\bar{C} - H}{\tan x} &= \frac{\sin nx}{2\tan x \sin^2\frac{x}{2}} - \frac{m}{2\tan x \tan\frac{x}{2}} - \frac{n\sin nx}{\tan x} - \frac{n\cos nx}{\tan x \tan\frac{x}{2}}, \\
(2n-1)\left(mD + \frac{G}{\tan x}\right) &= \frac{2n-1}{2m} + \frac{2n-1}{2\tan\frac{x}{2}\tan x} - \frac{(2n-1)\cos(n-1)x}{4m\sin^2\frac{x}{2}} \\
&\quad + \frac{(2n-1)\cos nx}{4m\sin^2\frac{x}{2}} + \frac{(2n-1)\sin nx}{2\tan x} + \frac{(2n-1)\cos nx}{2\tan x \tan\frac{x}{2}}, \\
(1-m)E &= \frac{(1-m)m}{2} + \frac{m-1}{4\sin^2\frac{x}{2}} - \frac{(mn-m-n+1)\cos nx}{4\sin^2\frac{x}{2}} \\
&\quad + \frac{(mn-n)\cos(n-1)x}{4\sin^2\frac{x}{2}}.
\end{aligned} \tag{46}$$

We now combine (45)-(46) as follows

$$\begin{aligned}
m\bar{A} - (2n+m-1)E + F - \bar{B} + \frac{\bar{C} - H}{\tan x} \\
+ (2n-1)\left(mD + \frac{G}{\tan x}\right) \\
= I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{47}$$

where

$$\begin{aligned}
I_1 &= \sin nx \left(\frac{\cos\frac{x}{2}}{\sin^3\frac{x}{2}} - \frac{1}{2\sin x \sin^2\frac{x}{2}} - \frac{1}{2\tan x} \right), \\
I_2 &= \frac{(m-n-1)\cos nx}{4\sin^2\frac{x}{2}} + \frac{(m-n)\cos(n-1)x}{4\sin^2\frac{x}{2}}, \\
I_3 &= \frac{2n-2m-1}{4\sin^2\frac{x}{2}}, \\
I_4 &= \frac{2n-m-1-\cos nx}{2\tan\frac{x}{2}\tan x}.
\end{aligned} \tag{48}$$

Note the four equations below.

$$\begin{aligned}
I_1 \tan^2\frac{x}{2} &= \sin nx \left(\frac{\cos\frac{x}{2}}{\sin^3\frac{x}{2}} - \frac{1}{2\sin x \sin^2\frac{x}{2}} - \frac{1}{2\tan x} \right) \tan^2\frac{x}{2} \\
&= \sin nx \left(\frac{\tan\frac{x}{2}}{4} + \frac{3}{4\tan\frac{x}{2}} \right),
\end{aligned} \tag{49}$$

$$\begin{aligned}
I_2 \tan^2\frac{x}{2} &= \left(\frac{(m-n-1)\cos nx}{4\sin^2\frac{x}{2}} + \frac{(m-n)\cos(n-1)x}{4\sin^2\frac{x}{2}} \right) \tan^2\frac{x}{2} \\
&= \frac{2m-2n-1-\tan^2\frac{x}{2}}{4} \cos nx + \frac{m-n}{2} \tan\frac{x}{2} \sin nx,
\end{aligned} \tag{50}$$

$$I_3 \tan^2\frac{x}{2} = \frac{2n-2m-1}{4} \left(1 + \tan^2\frac{x}{2} \right), \tag{51}$$

$$\begin{aligned}
I_4 \tan^2\frac{x}{2} &= \frac{2n-m-1-\cos nx}{2\tan\frac{x}{2}\tan x} \tan^2\frac{x}{2} \\
&= -\frac{1-\tan^2\frac{x}{2}}{4} \cos nx + \frac{2n-m-1}{4} \left(1 - \tan^2\frac{x}{2} \right).
\end{aligned} \tag{52}$$

Substituting $x = \pi/m$, we arrive at (15) by (39). Thus, we complete the proof for *Proposition 1*.

APPENDIX B PROOF OF Remark 5

For $m = n + 1$, denote by ϵ the bound improvement which is the resultant lower bound in *Proposition 1* subtracted by the Welch bound in (1). That is,

$$\begin{aligned}
\epsilon &= \frac{n(n-1)(M+1)}{(2n-1)M-1} \\
&\quad - \frac{n(n-1)(n+1)\tan^2\frac{\pi}{2n+2} + 2MQ_{2n-1}(\mathbf{w}, 0)}{2M - (n+1)\tan^2\frac{\pi}{2n+2}}.
\end{aligned}$$

We first prove the following two conclusions:

- (A) : $\min_{M \geq 3} \epsilon = \epsilon|_{M=3}$ for any fixed $n \geq 2$;
(B) : $\epsilon|_{M \geq 4, n=2} > 0$, $\epsilon|_{M=3, n=2} = 0$ and $\epsilon|_{M=3, n \geq 3} > 0$;
After some manipulations, ϵ can be written as,

$$\epsilon = A(Ba + C), \tag{53}$$

where $a = \frac{n(n-1)}{M}$, and

$$\begin{aligned} A &= \frac{n(n-1)}{\left((2n-1)(n-1)n-a\right)\left(\frac{2n(n-1)}{(n+1)\tan^2\frac{\pi}{2n+2}}-a\right)}, \\ B &= \frac{2n(n-1)}{(n+1)\tan^2\frac{\pi}{2n+2}} - 2n^2(n-1) + \frac{2Q_{2n-1}(\mathbf{w}, 0)}{(n+1)\tan^2\frac{\pi}{2n+2}}, \\ C &= \frac{2n^2(n-1)^2 - 2Q_{2n-1}(\mathbf{w}, 0)(2n-1)(n-1)n}{(n+1)\tan^2\frac{\pi}{2n+2}}. \end{aligned}$$

Note that by (16),

$$Q_{2n-1}(\mathbf{w}, 0) = \frac{(n+1)(1 - \tan^2\frac{\pi}{2n+2})}{4}.$$

Since

$$\tan^2\frac{\pi}{2n+2} > \left(\frac{\pi}{2n+2}\right)^2,$$

we have

$$B < \left(\frac{8}{\pi^2} - 2\right)n^3 + \left(2 + \frac{2}{\pi^2}\right)n^2 - \frac{4}{\pi^2}n + \left(\frac{2}{\pi^2} - \frac{1}{2}\right) < 0,$$

for any $n \geq 2$. Therefore, for a given n , the minimum of ϵ (with the change of M) is achieved by $M = 3$, and this proves (A).

Next, we deal with the proof of (B). When $M \geq 4$ and $n = 2$, we have

$$\epsilon|_{M \geq 4, n=2} = \frac{M(M-3)}{(3M-1)(2M-1)} > 0.$$

When $M = 3$, we have

$$\begin{aligned} \frac{Ba + C}{n(n-1)} &= \frac{5n^2 - 9n + 2 + (3n^2 + n - 2)\tan^2\frac{\pi}{2n+2}}{3(n+1)\tan^2\frac{\pi}{2n+2}} \\ &\quad - \frac{2n^2(n-1)}{3}. \end{aligned} \quad (54)$$

Note that when $n = 2$, $\epsilon|_{M=3} = 0$ because the left-hand term in (54) equals to 0.

On the other hand, by [14, page 42], we have,

$$\begin{aligned} \tan \theta &= \sum_{i=1}^{\infty} \frac{\mathcal{B}_{2i}(-4)^i(1-4^i)}{(2i)!} \theta^{2i-1} \\ &= \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \frac{17\theta^7}{315} + \dots \\ &\leq \theta \left[1 + \frac{\theta^2}{3} (1 + \theta^2 + \theta^4 + \dots) \right] \\ &= \theta \left[1 + \frac{\theta^2}{3(1-\theta^2)} \right], \end{aligned}$$

for small angle $0 \leq \theta < \frac{\pi}{4}$, where \mathcal{B}_s are the Bernoulli numbers. Thus, let

$$\eta = \frac{2\theta^2}{3 - 2\theta^2},$$

we have

$$\frac{1}{\tan^2 \theta} \geq \frac{1}{\theta^2} (1 - \eta).$$

Furthermore, let $\theta = \pi/(2n+2)$. For $n \geq 3$, (54) can be bounded as

$$\frac{Ba + C}{n(n-1)} \geq \frac{20(1-\eta) - 2\pi^2}{3\pi^2} n^3 + f, \quad (55)$$

where

$$\begin{aligned} f &= \frac{-16(1-\eta) + 2\pi^2}{3\pi^2} n^2 + \frac{-28(1-\eta) + 3\pi^2}{3\pi^2} n \\ &\quad + \frac{8(1-\eta) - 2\pi^2}{3\pi^2} \\ &\geq \frac{-16(1-\eta) + 2\pi^2}{3\pi^2} \times 9 + \frac{-28(1-\eta) + 3\pi^2}{3\pi^2} \times 3 \\ &\quad + \frac{8(1-\eta) - 2\pi^2}{3\pi^2} \\ &= \frac{25\pi^2 - 220}{3\pi^2} > 0. \end{aligned}$$

Note that when $n \geq 11$,

$$20(1-\eta) - 2\pi^2 = \frac{60 - 80\theta^2}{3 - 2\theta^2} - 2\pi^2 > 0,$$

implying that $\epsilon|_{M=3} > 0$ by (53) and (55). The proof of (B) will be complete by directly calculating the values of $\epsilon|_{M=3}$ for $2 \leq n \leq 12$ as shown in Table III (on the top of next page).

By (A) and (B), it follows that when $m = n+1$, the lower bound in *Proposition 1* is tighter than the Welch bound in (1) for $M = 3, n \geq 3$, and $M \geq 4, n \geq 2$.

APPENDIX C PROOF OF Remark 7

Denote by ε_0 the bound improvement which is the optimized lower bound in (19) (over different r) subtracted by the Welch bound in (1). Similarly, denote by ε_1 the optimized lower bound in (21) subtracted by the Welch bound in (1). That is,

$$\begin{aligned} \varepsilon_0 &= \frac{n(n-1)(M+1)}{(2n-1)M-1} - \min_r f(r), \\ \varepsilon_1 &= \frac{(M-1)n^2}{(2n-1)M-1} - \frac{n - \min_r f(r)}{1 - \frac{1}{M} \sum_{i=0}^{2n-2} w_i^2}, \end{aligned} \quad (56)$$

where $\sum_{i=0}^{2n-2} w_i^2$ is given in (22). Note that $\varepsilon_1 > \varepsilon_0$ because the lower bound in (19) is a weaker Levenshtein bound.

Our main task is to prove that $\varepsilon_1 > 0$ for $M = 3, n \geq 3$ and $M \geq 4, n \geq \sqrt{M}$. We note that (1), $f(r)$ in (26) is monotonically decreasing with r ; (2), $\varphi > \frac{\sqrt{2M}}{n}$ as $2 \sin^2 \frac{\varphi}{2} = \frac{M}{n^2}$.

1) For $M = 3$, it is readily to show that $\pi/\varphi + 1 > n$. Hence,

$$\min_{2 \leq 2r \leq n} f(r) = \frac{n}{2} + \frac{1}{2 \tan \frac{\varphi}{2} \tan \frac{n\varphi}{2}}. \quad (57)$$

When $n\varphi/2 < \pi/2$, we have

$$\frac{1}{\tan \frac{\varphi}{2} \tan \frac{n\varphi}{2}} \leq \frac{1}{\tan \frac{\sqrt{6}}{2n} \tan \frac{\sqrt{6}}{2}} \leq n \cdot \frac{2}{\sqrt{6} \tan \frac{\sqrt{6}}{2}}. \quad (58)$$

TABLE III: The values of $\epsilon|_{M=3}$ for $2 \leq n \leq 12$

n	2	3	4	5	6	7	8	9	10
$\epsilon _{M=3}$	0	0.0039	0.0165	0.0299	0.0425	0.0541	0.0648	0.0747	0.0841

TABLE IV: The values of $\epsilon_1|_{M=3}$ for $2 \leq n \leq 12$

n	2	3	4	5	6	7	8	9	10	11	12
$\epsilon_1 _{M=3}$	0	0.0066	0.0162	0.0292	0.0441	0.0603	0.0773	0.0948	0.1127	0.1308	0.1492

Then, by (57) and (58), we have

$$\begin{aligned} \epsilon_0|_{M=3} &\geq \left[\frac{4(n-1)}{3(2n-1)-1} - \frac{1}{2} - \frac{1}{\sqrt{6} \cdot \tan \frac{\sqrt{6}}{2}} \right] \cdot n \\ &\geq \left[\frac{2n-2}{3n-2} - \frac{1}{2} - \frac{1}{\sqrt{6} \cdot \tan \frac{\sqrt{6}}{2}} \right] \cdot n \\ &> 0, \quad \text{for } n \geq 13. \end{aligned} \quad (59)$$

On the other hand, for $2 \leq n \leq 12$, we directly calculate the values of $\epsilon_1|_{M=3}$ as shown in Table IV.

When $n\varphi/2 \geq \pi/2$, by dropping off $\frac{1}{2 \tan \frac{\varphi}{2} \tan \frac{n\varphi}{2}}$ in (57) and repeating more or less the same proof in the above completes the proof.

- 2) For $M = 4$, we also have $\pi/\varphi + 1 > n$ and thus (57). By a similar proof for $M = 3$, we prove that $\epsilon_1 > 0$ for $M = 4, n \geq \sqrt{M} = 2$.
- 3) For $M \geq 5$, we have $\pi/\varphi + 1 \leq n$. In this case, the maximized lower bound in (19) (over different r) in fact has been shown in [3, Corollary 4]. That is, for $5 \leq M \leq n^2$, we have

$$\delta_{\max}^2 \geq n - \left\lceil \frac{\pi n}{\sqrt{8M}} \right\rceil, \quad (60)$$

by setting

$$r = \left\lfloor \frac{\pi}{2\varphi} + \frac{1}{2} \right\rfloor.$$

Therefore,

$$\epsilon_0 = \frac{n(n-1)(M+1)}{(2n-1)M-1} - \left\lceil \frac{\pi n}{\sqrt{8M}} \right\rceil \geq g(n) \quad (61)$$

where

$$g(n) = n \cdot \left[\frac{(n-1)(M+1)}{(2n-1)M-1} - \frac{\pi}{\sqrt{8M}} \right] - 1. \quad (62)$$

For $M \geq 5$ and $n \geq \sqrt{M}$, it is readily to show that $g(n)$ is monotonically increasing. Hence,

$$\min_{n \geq \sqrt{M}} g(n) = g(\sqrt{M}) > 0, \quad \text{for } M \geq 21. \quad (63)$$

It is noted that

$$g(6\sqrt{M}) > 0, \quad \text{for } M \geq 5. \quad (64)$$

By (61), (63) and (64), we assert that $\epsilon_1 > \epsilon_0 > 0$ for (1): $5 \leq M \leq 20$ and $n \geq 6\sqrt{M}$ and (2): $M \geq 21$ and $n \geq \sqrt{M}$.

The rest of the work is to show that $\epsilon_1 > 0$ also holds for $5 \leq M \leq 20$ and $\sqrt{M} \leq n < 6\sqrt{M}$, which can be ensured by directly calculating the values of ϵ_1 in this region.

APPENDIX D

PROOF OF THE ASYMPTOTIC BOUNDS IN TABLE II

- 1) For the Welch bound in (1), the proof for the asymptotic lower bounds in Table II is straightforward and thus omitted.
- 2) By (10) (which is [3, Corollary 3]), we have

$$\delta_{\max}^2 \geq \begin{cases} 0.3333n, & \text{for } M = 3; \\ 0.4227n, & \text{for } M = 4. \end{cases} \quad (65)$$

In addition, setting $m = 1$ in (9) (which is [3, Corollary 2]), we have

$$\delta_{\max}^2 \geq \frac{nM - n^2}{M - 1}. \quad (66)$$

Therefore, for $M \geq 5$, we have the optimized lower bound as

$$\delta_{\max}^2 \geq \max \left\{ \left[1 - \frac{2}{\sqrt{3M}} \right] n, \frac{nM - n^2}{M - 1} \right\}.$$

- 3) For the Levenshtein bound in (19), we first consider $M \in \{3, 4\}$. Recall (57) in Appendix B, i.e.,

$$\min_{2 \leq 2r \leq n} f(r) = \frac{n}{2} + \frac{1}{2 \tan \frac{\varphi}{2} \tan \frac{n\varphi}{2}}, \quad \text{for } M \in \{3, 4\}. \quad (67)$$

Since $2 \sin^2 \frac{\varphi}{2} = \frac{M}{n^2}$, we have

$$\lim_{n \rightarrow +\infty} \tan \frac{\varphi}{2} = \frac{\sqrt{2M}}{2n}.$$

Therefore,

$$\lim_{n \rightarrow +\infty} \frac{1}{2 \tan \frac{\varphi}{2} \tan \frac{n\varphi}{2}} = \frac{n}{\sqrt{2M} \tan \frac{\sqrt{2M}}{2}}, \quad (68)$$

for $M \in \{3, 4\}$. By (19) and (68), we obtain

$$\delta_{\max}^2 \gtrsim \begin{cases} 0.3528n, & \text{for } M = 3; \\ 0.4442n, & \text{for } M = 4. \end{cases} \quad (69)$$

Recalling the lower bound (27) for sufficiently large n completes the proof.

- 4) Define $\gamma = \lim_{n \rightarrow +\infty} m/n$. For the Levenshtein bound in Proposition 1, note that

$$\begin{cases} \frac{\pi}{2m} < \tan \frac{\pi}{2m} < 2 \cdot \frac{\pi}{2m} = \frac{\pi}{m}, & \text{for } 2 \leq m \leq 2n-2; \\ \lim_{\substack{n \rightarrow +\infty \\ 1 < \gamma < 2}} \tan \frac{\pi}{2m} = \frac{\pi}{2m}, & \text{for } n < m \leq 2n-2. \end{cases} \quad (70)$$

By

$$0 \leq \lim_{n \rightarrow +\infty} \frac{m \tan^2 \frac{\pi}{2m}}{n} \leq \pi^2 \cdot \lim_{n \rightarrow +\infty} \frac{1}{mn} = 0, \quad (71)$$

we have

$$\lim_{n \rightarrow +\infty} \frac{m \tan^2 \frac{\pi}{2m}}{n} = 0. \quad (72)$$

Also,

$$\begin{aligned} \lim_{\substack{n \rightarrow +\infty \\ 1 < \gamma < 2}} \frac{\frac{2m-2n+1}{4} \tan \frac{\pi}{2m}}{n} &= \frac{\gamma-1}{2} \cdot \lim_{\substack{n \rightarrow +\infty \\ 1 < \gamma < 2}} \frac{\pi}{2m} = 0, \\ \lim_{\substack{n \rightarrow +\infty \\ 1 < \gamma < 2}} \frac{\frac{3}{4 \tan \frac{\pi}{2m}}}{n} &= \frac{3}{2\pi} \cdot \lim_{\substack{n \rightarrow +\infty \\ 1 < \gamma < 2}} \frac{m}{n} = \frac{3\gamma}{2\pi}. \end{aligned} \quad (73)$$

By (72) and (73), we have the following equation.

$$\begin{aligned} &\lim_{n \rightarrow +\infty} \frac{Q_{2n-1}(\mathbf{w}, 0)}{n} \\ &= \begin{cases} \frac{4-3\gamma}{4} + \frac{(\gamma-1) \cos \frac{\pi}{\gamma}}{2} + \frac{3\gamma \sin \frac{\pi}{\gamma}}{2\pi}, & \text{for } 1 < \gamma < 2; \\ \frac{\gamma}{4}, & \text{for } 0 < \gamma \leq 1. \end{cases} \end{aligned} \quad (74)$$

It is also noted that

$$\lim_{n \rightarrow +\infty} (n-1)m \tan^2 \frac{\pi}{2m} = \frac{\pi^2}{4\gamma}. \quad (75)$$

By (74) and (75), the asymptotic lower bound in *Proposition 1* can be simplified to

$$\delta_{\max}^2 \gtrsim \begin{cases} \left[\frac{3\gamma}{4} - \frac{(\gamma-1) \cos \frac{\pi}{\gamma}}{2} - \frac{3\gamma \sin \frac{\pi}{\gamma}}{2\pi} - \frac{\pi^2}{8\gamma M} \right] n, & \text{for } 1 < \gamma < 2; \\ \left[1 - \frac{\gamma}{4} - \frac{\pi^2}{8\gamma M} \right] n, & \text{for } 0 < \gamma \leq 1. \end{cases} \quad (76)$$

When $M \geq 5$, setting $\gamma = \frac{\pi}{\sqrt{2M}} < 1$, we by (76) have

$$\delta_{\max}^2 \gtrsim \left[1 - \frac{\pi}{\sqrt{8M}} \right] \cdot n. \quad (77)$$

Obviously, when M becomes sufficiently large (then $\gamma = \frac{\pi}{\sqrt{2M}}$ tends to 0), the optimized lower bound in *Proposition 1* will be achieved at $m = 2$, which is (66). Therefore, for $M \geq 5$, we have the optimized lower bound as follows.

$$\delta_{\max}^2 \gtrsim \max \left\{ \left[1 - \frac{\pi}{\sqrt{8M}} \right] n, \frac{nM - n^2}{M - 1} \right\}.$$

When $M \in \{3, 4\}$, we have $\frac{\pi}{\sqrt{2M}} > 1$ and thus the optimized lower bound in (76) will be achieved over $1 < \gamma < 2$. Since the analytical maximization of

$$\frac{3\gamma}{4} - \frac{(\gamma-1) \cos \frac{\pi}{\gamma}}{2} - \frac{3\gamma \sin \frac{\pi}{\gamma}}{2\pi} - \frac{\pi^2}{8\gamma M}, \quad (78)$$

is intractable for $1 < \gamma < 2$, we carry out the optimization by pinpointing the numerical maximization point of (78), i.e.,

$$\delta_{\max}^2 \gtrsim \begin{cases} 0.3594n, & \text{for } M = 3 \text{ and } \gamma = 1.3085; \\ 0.4446n, & \text{for } M = 4 \text{ and } \gamma = 1.1116. \end{cases} \quad (79)$$

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