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A Tighter Correlation Lower Bound for Quasi-Complementary Sequence Sets

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Abstract—Levenshtein improved the famous Welch bound on aperiodic correlation for binary sequences by utilizing some properties of the weighted mean square aperiodic correlation. Following Levenshtein's idea, a new correlation lower bound for quasi-complementary sequence sets (QCSSs) over the complex roots of unity is proposed in this paper. The derived lower bound is shown to be tighter than the Welch bound for QCSSs when the set size is greater than some value. The conditions for meeting the new bound with equality are also investigated.

Index Terms—Welch Bound, Levenshtein Bound, quasi-complementary sequence set (QCSS), mutually orthogonal complementary sequence set (MOCSS), Golay complementary pair.

I. INTRODUCTION

In the late 1950's, Golay [1] introduced the concept of “complementary pair”, which is defined as a pair of sequences whose aperiodic auto-correlations sum to zero for all non-zero time-shifts. In [2], Tseng and Liu extended the notion of the Golay complementary pair to a set of 2 or more sequences. The resultant sequence set has zero non-trivial aperiodic auto-correlation and cross-correlation sums, and are called the mutually orthogonal complementary sequence set (MOCSS) in this paper. Later, Suehiro and Hatori [3] devised a construction of the so-called complete complementary sequences, whose set size meets the upper bound of MOCSS. MOCSS have found many potential applications due to their perfect aperiodic correlation properties. Particularly, the application of MOCSS to asynchronous multi-carrier code-division multiple-access (MC-CDMA) communications was proposed by Chen, Yeh, and Suehiro [4] targeting at achieving zero multipath interference and zero multiuser interference, and it has been introduced as a promising technology for the next generation CDMA [5]. In [6], Davis and Jedwab pioneered the explicit construction of the Golay complementary pairs by use of the cosets of the first-order Reed-Muller code within the second-order Reed-Muller code, and showed its application in balancing the peak-to-average power ratio (PAPR), the code rate, and the error-correction capability in code-keying

orthogonal frequency-division multiplexing (OFDM) systems. This construction was further generalized to that for complete complementary sequence sets by associating each second-order coset of the first-order Reed-Muller code with a graph [7], [8]. Other practical applications of MOCSS include inter-symbol interference (ISI) channel estimation [9], [10] and radar waveform design [11]. Also, some zero correlation zone sequences [12]–[14] are constructed based on MOCSS.

It is noted that the set size of MOCSS is upper bounded by the number of channels¹, which limits the application of MOCSS in MC-CDMA communications when a large number of users are to be supported. To overcome this limitation on the set size, we can allow the aperiodic auto- or cross-correlation sums of a complementary sequence set to take uniformly low (but non-zero) values. Intuitively, this relaxed version of complementary sequence set is expected to have a larger set size. The resultant sequence set is called a quasi-complementary sequence set (QCSS).

In [15], Welch obtained a collection of correlation lower bounds by evaluating the mini-max value of the inner products of a vector set, including the aperiodic correlation lower bound for QCSSs, i.e.,

$$\delta_{\max}^2 \geq M^2 N^2 \frac{\frac{K}{M} - 1}{K(2N - 1) - 1}, \quad (1)$$

where $\delta_{\max}$ denotes the maximum aperiodic correlation magnitude, K the set size, M the number of channels (which is equal to or greater than 2), and N the elementary sequence length. The aforementioned set size upper bound for MOCSS, namely, $K \leq M$, can also be obtained from (1) by setting $\delta_{\max} = 0$. On the other hand, when $\delta_{\max}$ is a small positive value, it is easy to obtain $K > M$. Also, setting $M = 1$, (1) is reduced to the lower bound for the conventional (single-channel) sequence set [15]

$$\delta_{\max}^2 \geq N^2 \frac{K - 1}{K(2N - 1) - 1}. \quad (2)$$

In [16], Levenshtein showed that in the case of binary sequence sets, the Welch bound in (2) can be strengthened for set size $K \geq 4$ and sequence length $N \geq 2$. The basic idea behind the Levenshtein bound is that the weighted mean square aperiodic correlation of any sequence subset over the complex roots of unity should be equal to or greater than that of the whole set which includes all possible complex roots-of-unity sequences. The strengthening of the Welch bound

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¹In practice, each elementary sequence of a complementary sequence should be sent out in a separate channel. For instance, in a multicarrier CDMA system based on MOCSS, a distinct channel means a distinct sub-carrier.

was extended to complex roots-of-unity sequence sets by Boztaş [17]. Later, Peng and Fan derived the lower bound on aperiodic correlation for low correlation zone (LCZ) sequence sets by an approach similar to Levenshtein's [18]. Recently, correlation lower bounds for LCZ complementary sequences are given in [19]. In addition, optimal and near-optimal QCSSs (with respect to a periodic correlation lower bound), each of which is constructed by modulating a Singer difference set with an optimal quadriphase sequence set, are presented in [20]. To date however, there has been no similar effort on tightening the aperiodic correlation lower bound of QCSSs given in (1). Since a QCSS is capable of supporting more MC-CDMA users than a MOCSS, a tighter correlation lower bound will give us a closer insight on the tradeoffs between these sequence parameters, i.e., K, M, N and $\delta_{\max}$, which is useful for the finding of optimal QCSSs. Also, in CDMA study (including asynchronous MC-CDMA), a tight lower bound on the maximum aperiodic auto- and cross- correlations determines the maximum possible amount of multiple-access interference (MAI) and multi-path interference (MPI), which in turn determines the worst-case signal-to-noise ratio (SNR) or the maximum number of supportable CDMA users under a given bit-error-rate (BER) constraint [21].

Motivated by Levenshtein's idea, a new aperiodic correlation lower bound for QCSS over the complex roots of unity is derived in this paper. By properly choosing the weight vector in the bounding function, we show that the derived lower bound (which is a generalization of the Levenshtein bound) is tighter than the Welch bound for QCSS in (1) when the set size K is greater than a threshold value.

In the literature, another important development is to characterize the condition for sequence sets which can meet the Welch bound with equality. This work was initiated by Massey in 1991 [22], [23]. Later, Rupf and Massey discovered that the sum capacity of a synchronous CDMA system is maximized by the Welch bound equality (WBE) sequence sets [24]. For asynchronous CDMA systems, Mow found that the maximum worst-case signal-to-noise ratio (SNR) is achieved if and only if the WBE sequences form a complementary sequence [25], [21]. Since then, a lot of works have been concerned on the WBE sequences [26]-[30]. In [31], Liu and Guan have shown that the Levenshtein bound can be met with equality by the weighted-correlation complementary sequences. In this paper, we examine the conditions for meeting the proposed aperiodic correlation lower bound of QCSSs with equality. We will show that in conventional single-channel case (i.e., $M = 1$), they are reduced to the conditions for meeting the Levenshtein bound with equality [31].

In Section II, necessary notations and a review of the Levenshtein bound are given. In Section III, the proposed lower bound for QCSSs will be present, followed by an analysis of its tightness and the conditions for meeting it with equality in Section IV. Finally, this work is concluded in Section V.

II. PRELIMINARIES

A. Notations

A sequence set $\mathbf{A}$ contains K length- N sequences $\mathbf{a}^u$, each of which has all of its entries taking values from a set, called the alphabet. In symbols,

$$\mathbf{A} \triangleq \{\mathbf{a}^0, \mathbf{a}^1, \dots, \mathbf{a}^u, \dots, \mathbf{a}^{K-1}\}, \quad 0 \leq u \leq K-1;$$

$$\mathbf{a}^u \triangleq (a_{0,u}^u, a_{1,u}^u, \dots, a_{t,u}^u, \dots, a_{N-1,u}^u), \quad 0 \leq t \leq N-1.$$

In this paper, the alphabet of interest is the set of complex roots of unity, i.e., $E \triangleq \{1, \xi^1, \dots, \xi^{H-1}\}$ with $\xi \triangleq \exp \frac{2\pi\sqrt{-1}}{H}$, where the alphabet size H is an integer greater than 1. Let E^N denote the set of all possible length- N vectors over E . We say the sequence set $\mathbf{A}$ is defined over E and we have $\mathbf{A} \subseteq E^N$.² The aperiodic correlation function $\rho_{\mathbf{a}^u, \mathbf{a}^v}(\tau)$ of two sequences $\mathbf{a}^u, \mathbf{a}^v \in \mathbf{A}$ is defined as

$$\rho_{\mathbf{a}^u, \mathbf{a}^v}(\tau) \triangleq \begin{cases} \sum_{t=0}^{N-1-\tau} a_t^u (a_{t+\tau}^v)^*, & 0 \leq \tau \leq N-1; \\ \sum_{t=0}^{N-1+\tau} a_{t-\tau}^u (a_t^v)^*, & -(N-1) \leq \tau \leq -1; \\ 0, & |\tau| \geq N, \end{cases} \quad (3)$$

where $(\cdot)^*$ denotes the complex conjugate of $(\cdot)$. When $u = v$, it is called the aperiodic auto-correlation function; otherwise, it is called the aperiodic cross-correlation function.

A complementary sequence set $\mathcal{C}$ contains K complementary sequences. Each complementary sequence consists of M (≥ 2) length- N elementary sequences defined over the alphabet E . In symbols,

$$\mathcal{C} \triangleq \{\mathbf{C}^0, \mathbf{C}^1, \dots, \mathbf{C}^u, \dots, \mathbf{C}^{K-1}\}, \quad 0 \leq u \leq K-1;$$

$$\mathbf{C}^u \triangleq \{\mathbf{c}_{0,u}^u, \mathbf{c}_{1,u}^u, \dots, \mathbf{c}_{p,u}^u, \dots, \mathbf{c}_{M-1,u}^u\}, \quad 0 \leq p \leq M-1;$$

$$\mathbf{c}_p^u \triangleq (c_{p,0}^u, c_{p,1}^u, \dots, c_{p,t}^u, \dots, c_{p,N-1}^u), \quad 0 \leq t \leq N-1.$$

The cardinality (i.e., the set size) of $\mathcal{C}$ (or $\mathbf{A}$) is denoted by $|\mathcal{C}|$ (respectively, $|\mathbf{A}|$). In a multicarrier CDMA transmission of a complementary sequence, all elementary sequences are sent and received in separate non-interfering channels. Therefore, it is helpful to write each complementary sequence as a two-dimensional matrix by vertically stacking all of its indexed elementary sequences row by row, e.g., for $0 \leq u \leq K-1$,

$$\mathbf{C}^u = \begin{bmatrix} \mathbf{c}_{0,u}^u \\ \mathbf{c}_{1,u}^u \\ \vdots \\ \mathbf{c}_{M-1,u}^u \end{bmatrix} = \begin{bmatrix} c_{0,0}^u & c_{0,1}^u & \cdots & c_{0,N-1}^u \\ c_{1,0}^u & c_{1,1}^u & \cdots & c_{1,N-1}^u \\ \vdots & \vdots & \ddots & \vdots \\ c_{M-1,0}^u & c_{M-1,1}^u & \cdots & c_{M-1,N-1}^u \end{bmatrix}. \quad (4)$$

In this paper, $M = |\mathbf{C}^u|$ is also referred to as the number of channels. Let $E^{M \times N}$ denote the set of all possible M -channel length- N complementary sequences. Then an M -channel length- N complementary sequence set $\mathcal{C}$ can be viewed as a set of K matrices (each of order $M \times N$) which satisfies $\mathcal{C} \subseteq E^{M \times N}$.

The aperiodic correlation function $\rho_{\mathbf{C}^u, \mathbf{C}^v}(\tau)$ of two complementary sequences $\mathbf{C}^u, \mathbf{C}^v \in \mathcal{C}$ is defined as the aperiodic

²In this paper, it is helpful to think of a sequence "set" as an *ordered* set of sequences indexed by some subscript(s) or superscript(s), and a length- N sequence can also be viewed as an N -dimensional *row* vector.

correlation sum, i.e.,

$$\rho_{\mathbf{C}^u, \mathbf{C}^v}(\tau) \triangleq \sum_{p=0}^{M-1} \rho_{\mathbf{c}_p^u, \mathbf{c}_p^v}(\tau), \quad 0 \leq u, v \leq K-1. \quad (5)$$

Next, the aperiodic auto-correlation tolerance δ_a and the aperiodic cross-correlation tolerance δ_c of $\mathcal{C}$ are respectively defined as

$$\begin{aligned} \delta_a &\triangleq \max \left\{ |\rho_{\mathbf{C}^u, \mathbf{C}^u}(\tau)| : 0 \leq u \leq K-1, 0 < \tau \leq N-1 \right\}; \\ \delta_c &\triangleq \max \left\{ |\rho_{\mathbf{C}^u, \mathbf{C}^v}(\tau)| : u \neq v, 0 \leq u, v \leq K-1, \right. \\ &\quad \left. 0 \leq \tau \leq N-1 \right\}. \end{aligned}$$

Moreover, the aperiodic tolerance (also called the maximum aperiodic correlation magnitude) $\delta_{\max}$ is defined to be $\delta_{\max} \triangleq \max\{\delta_a, \delta_c\}$. When $0 < \delta_{\max} \ll MN$, $\mathcal{C}$ is called a quasi-complementary sequence set (QCSS); when $\delta_{\max} = 0$, it is called a mutually orthogonal complementary sequence set (MOCSS). In particular, when $M = 1$, $\delta_{\max}$ is reduced to the aperiodic tolerance for conventional (single-channel) sequence sets.

The right cyclic shift operation is denoted by T so that for any L -dimensional vector $\mathbf{a} = (a_0, a_1, \dots, a_{L-1})$,

$$T^i \mathbf{a} = (a_{L-i}, a_{L-i+1}, \dots, a_{L-1}, a_0, a_1, \dots, a_{L-i-1})$$

represents the vector resulting from cyclically shifting $\mathbf{a}$ to the right by i positions. Clearly, we have $T^L \mathbf{a} = \mathbf{a}$.

For two complex-valued L -dimensional vectors $\mathbf{a}$ and $\mathbf{b}$, denote by $\langle \mathbf{a}, \mathbf{b} \rangle$ their inner product, i.e., $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=0}^{L-1} a_i b_i^*$.

For $0 \leq i \leq p-1$ and $0 \leq j \leq q-1$, the i th row, j th column, and (i, j) th element of any $p \times q$ matrix $\mathbf{H}$ are denoted by $\mathbf{H}(i, :)$, $\mathbf{H}(:, j)$, and $\mathbf{H}(i, j)$, respectively.

In the sequel, to avoid the possible confusion with a complementary sequence or a complementary sequence set, in single-channel case (i.e., $M = 1$), we shall also refer to a sequence as a ‘‘conventional sequence’’ and a sequence set as a ‘‘conventional sequence set’’.

B. Review of the Levenshtein Bound

For the ease of reference and comparison, our notations here closely follow those in [16]. Without specified otherwise, a conventional sequence is also viewed as a *row* vector. For two conventional sequence sets $\mathbf{A}, \mathbf{B} \subseteq E^N$, define

$$\begin{aligned} F(\mathbf{A}, \mathbf{B}) &\triangleq \frac{1}{|\mathbf{A}||\mathbf{B}|} \sum_{\mathbf{x} \in \mathbf{A}} \sum_{\mathbf{y} \in \mathbf{B}} \sum_{s=0}^{2N-2} \sum_{t=0}^{2N-2} \\ &\quad \left| \langle T^s(\mathbf{x}, 0^{N-1}), T^t(\mathbf{y}, 0^{N-1}) \rangle \right|^2 w_s w_t, \end{aligned} \quad (6)$$

where 0^{N-1} means a vector of $N-1$ zeros, $(\mathbf{x}, 0^{N-1})$ denotes the vector concatenation of vectors $\mathbf{x}$ and 0^{N-1} , and the weight vector $\mathbf{w} = (w_0, w_1, \dots, w_{2N-2})$ is constrained by

$$w_i \geq 0, \quad i = 0, 1, \dots, 2N-2, \quad \text{and} \quad \sum_{i=0}^{2N-2} w_i = 1. \quad (7)$$

Denote the quadratic form

$$\begin{aligned} Q_{2N-1}(\mathbf{w}, a) &\triangleq \mathbf{w} \mathbf{Q}_{2N-1} \mathbf{w}^T \\ &= a \sum_{i=0}^{2N-2} w_i^2 + \sum_{s,t=0}^{2N-2} \tau_{s,t,N} w_s w_t, \end{aligned} \quad (8)$$

where $\mathbf{Q}_{2N-1}$ is a $(2N-1) \times (2N-1)$ matrix with all of its diagonal entries equal to a , and for $s \neq t$, its off-diagonal entries $\mathbf{Q}_{2N-1}(s, t) = \tau_{s,t,N}$ with

$$0 \leq \tau_{s,t,N} \triangleq \min \{|t-s|, 2N-1-|t-s|\} \leq N-1. \quad (9)$$

We have the following remark.

Remark 1:

$$\begin{aligned} &\langle (\mathbf{x}, 0^{N-1}), T^\tau(\mathbf{y}, 0^{N-1}) \rangle \\ &= \begin{cases} [\rho_{\mathbf{y}, \mathbf{x}}(\tau)]^*, & \text{for } 0 \leq \tau \leq N-1; \\ \rho_{\mathbf{x}, \mathbf{y}}(-\tau), & \text{for } -(N-1) \leq \tau \leq -1, \end{cases} \\ &\langle T^\tau(\mathbf{x}, 0^{N-1}), (\mathbf{y}, 0^{N-1}) \rangle \\ &= \begin{cases} \rho_{\mathbf{x}, \mathbf{y}}(\tau), & \text{for } 0 \leq \tau \leq N-1; \\ [\rho_{\mathbf{y}, \mathbf{x}}(-\tau)]^*, & \text{for } -(N-1) \leq \tau \leq -1. \end{cases} \end{aligned} \quad (10)$$

By (10), we have

$$\begin{aligned} &\left| \langle T^s(\mathbf{x}, 0^{N-1}), T^t(\mathbf{y}, 0^{N-1}) \rangle \right|^2 \\ &\quad + \left| \langle T^t(\mathbf{x}, 0^{N-1}), T^s(\mathbf{y}, 0^{N-1}) \rangle \right|^2 \\ &= |\rho_{\mathbf{x}, \mathbf{y}}(\tau_{s,t,N})|^2 + |\rho_{\mathbf{y}, \mathbf{x}}(\tau_{s,t,N})|^2. \end{aligned} \quad (11)$$

Although the key results from [16] were derived for binary sequences, their extensions from the alphabet $\{1, -1\}$ to E are straightforward and are summarized in the following lemma.

Lemma 1: For any conventional sequence set $\mathbf{A} \subseteq E^N$, we have

1) (Lemma 1 of [16])

$$F(\mathbf{A}, \mathbf{A}) \leq \frac{1}{K} \left((N^2 - \delta_{\max}^2) \sum_{i=0}^{2N-2} w_i^2 + K \delta_{\max}^2 \right). \quad (12)$$

2) (Lemmas 2 and 3 of [16])

$$F(\mathbf{A}, \mathbf{A}) \geq F(E^N, E^N) = \sum_{s,t=0}^{2N-2} (N - \tau_{s,t,N}) w_s w_t. \quad (13)$$

3) (Theorem 1 of [16])

$$\delta_{\max}^2 \geq N - \frac{Q_{2N-1} \left(\mathbf{w}, \frac{N(N-1)}{K} \right)}{1 - \frac{1}{K} \sum_{i=0}^{2N-2} w_i^2}. \quad (14)$$

4) (Corollary 2 of [16])

$$\delta_{\max}^2 \geq \frac{3NKm - 3N^2 - K(m^2 - 1)}{3(mK - 1)}, \quad 1 \leq m \leq N. \quad (15)$$

5) (Corollary 3 of [16])

$$\delta_{\max}^2 \geq N - \frac{2N}{\sqrt{3K}}, \quad K \geq 3, N \geq 2. \quad (16)$$

III. PROPOSED LOWER BOUNDS OF QUASI-COMPLEMENTARY SEQUENCE SETS (QCSSs)

For two complementary sequence sets $\mathcal{C}, \mathcal{D} \subseteq E^{M \times N}$, given $\mathbf{w}$ as defined in (7), we define

$$F(\mathcal{C}, \mathcal{D}) \triangleq \frac{1}{|\mathcal{C}||\mathcal{D}|} \sum_{\mathbf{X} \in \mathcal{C}} \sum_{\mathbf{Y} \in \mathcal{D}} \sum_{s=0}^{2N-2} \sum_{t=0}^{2N-2} |\langle T^s(\mathbf{X}, 0^{N-1}), T^t(\mathbf{Y}, 0^{N-1}) \rangle|^2 w_s w_t, \quad (17)$$

where $\mathbf{X}$ and $\mathbf{Y}$ can be written as

$$\begin{aligned} \mathbf{X} &= \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m, \dots, \mathbf{x}_{M-1}\}, \\ \mathbf{x}_m &= (x_{m,0}, x_{m,1}, \dots, x_{m,N-1}) \\ \mathbf{Y} &= \{\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_m, \dots, \mathbf{y}_{M-1}\}, \\ \mathbf{y}_m &= (y_{m,0}, y_{m,1}, \dots, y_{m,N-1}) \end{aligned}$$

and

$$\begin{aligned} &|\langle T^s(\mathbf{X}, 0^{N-1}), T^t(\mathbf{Y}, 0^{N-1}) \rangle|^2 \\ &\triangleq \left| \sum_{m=0}^{M-1} \langle T^s(\mathbf{x}_m, 0^{N-1}), T^t(\mathbf{y}_m, 0^{N-1}) \rangle \right|^2. \end{aligned} \quad (18)$$

For any elementary sequence $\mathbf{x}_m \in \mathbf{X}$ ($0 \leq m \leq M-1$), construct a matrix $\mathbf{H}_{\mathbf{x}_m}$ of size $(2N-1) \times (2N-1)$, such that the s th ($0 \leq s \leq 2N-2$) row of $\mathbf{H}_{\mathbf{x}_m}$ is

$$\mathbf{H}_{\mathbf{x}_m}(s, :) = T^s(\mathbf{x}_m, 0^{N-1}). \quad (19)$$

Also define

$$f_{i,j,s}(\mathbf{x}_m, \mathbf{x}_n) \triangleq \mathbf{H}_{\mathbf{x}_m}(s, i) [\mathbf{H}_{\mathbf{x}_n}(s, j)]^* w_s. \quad (20)$$

Then, we have (21) and consequently (22) in the top of next page. Therefore, we have the following lemma.

Lemma 2: For any complementary sequence sets $\mathcal{C}, \mathcal{D} \subseteq E^{M \times N}$,

$$(F(\mathcal{C}, \mathcal{D}))^2 \leq F(\mathcal{C}, \mathcal{C})F(\mathcal{D}, \mathcal{D}). \quad (23)$$

In what follows, we will drop the subscripts of $\tau_{s,t,N}$ (defined in (9)) for ease of presentation, although τ may vary with different values of s, t , and N . Since

$$\begin{aligned} &\sum_{\mathbf{Y} \in E^{M \times N}} |\rho_{\mathbf{X}, \mathbf{Y}}(\tau)|^2 \\ &= \sum_{\mathbf{Y} \in E^{M \times N}} \left| \sum_{m=0}^{M-1} \rho_{\mathbf{x}_m, \mathbf{y}_m}(\tau) \right|^2 \\ &= \sum_{\mathbf{Y} \in E^{M \times N}} \left(\sum_{m,n=0}^{M-1} \sum_{i,j=0}^{N-1-\tau} \xi^{\bar{x}_{m,i} - \bar{y}_{m,i+\tau}} \xi^{-\bar{x}_{n,j} + \bar{y}_{n,j+\tau}} \right) \\ &= \sum_{m,n=0}^{M-1} \sum_{i,j=0}^{N-1-\tau} \xi^{\bar{x}_{m,i} - \bar{x}_{n,j}} \left(\sum_{\mathbf{Y} \in E^{M \times N}} \xi^{-\bar{y}_{m,i+\tau} + \bar{y}_{n,j+\tau}} \right) \\ &= S_1 + S_2, \end{aligned}$$

where $x_{m,i} = \xi^{\bar{x}_{m,i}}$ and $y_{m,i+\tau} = \xi^{\bar{y}_{m,i+\tau}}$ for $\bar{x}_{m,i}, \bar{y}_{m,i+\tau} \in \{0, 1, \dots, H-1\}$, S_1 and S_2 are the summation terms classified according to a) : $m = n$ and $i = j$; b) : $m = n$ and

$i \neq j$, or $m \neq n$ and $i = j$, or $m \neq n$ and $i \neq j$, respectively. For case a), we have

$$S_1 = \sum_{\mathbf{Y} \in E^{M \times N}} \sum_{m=0}^{M-1} \sum_{i=0}^{N-1-\tau} 1 = H^{MN} M(N-\tau). \quad (24)$$

For case b), when $\mathbf{Y}$ runs over $E^{M \times N}$, the quantity in the following equation, i.e.,

$$-\bar{y}_{m,i+\tau} + \bar{y}_{n,j+\tau} \quad (25)$$

is equi-distributed over the set $\{0, 1, \dots, H-1\}$. Hence,

$$\sum_{\mathbf{Y} \in E^{M \times N}} \xi^{-\bar{y}_{m,i+\tau} + \bar{y}_{n,j+\tau}} = 0, \quad (26)$$

implying that $S_2 = 0$. Therefore, we have

$$\sum_{\mathbf{Y} \in E^{M \times N}} |\rho_{\mathbf{X}, \mathbf{Y}}(\tau)|^2 = H^{MN} M(N-\tau). \quad (27)$$

Similar to Remark 1, we have

Remark 2:

$$\begin{aligned} &\langle (\mathbf{X}, 0^{N-1}), T^\tau(\mathbf{Y}, 0^{N-1}) \rangle \\ &= \begin{cases} [\rho_{\mathbf{Y}, \mathbf{X}}(\tau)]^*, & \text{for } 0 \leq \tau \leq N-1; \\ \rho_{\mathbf{X}, \mathbf{Y}}(-\tau), & \text{for } -(N-1) \leq \tau \leq -1, \end{cases} \\ &\langle T^\tau(\mathbf{X}, 0^{N-1}), (\mathbf{Y}, 0^{N-1}) \rangle \\ &= \begin{cases} \rho_{\mathbf{X}, \mathbf{Y}}(\tau), & \text{for } 0 \leq \tau \leq N-1; \\ [\rho_{\mathbf{Y}, \mathbf{X}}(-\tau)]^*, & \text{for } -(N-1) \leq \tau \leq -1. \end{cases} \end{aligned} \quad (28)$$

By (28), we have the following equation.

$$\begin{aligned} &|\langle T^s(\mathbf{X}, 0^{N-1}), T^t(\mathbf{Y}, 0^{N-1}) \rangle|^2 \\ &\quad + |\langle T^s(\mathbf{Y}, 0^{N-1}), T^t(\mathbf{X}, 0^{N-1}) \rangle|^2 \\ &= |\rho_{\mathbf{X}, \mathbf{Y}}(\tau)|^2 + |\rho_{\mathbf{Y}, \mathbf{X}}(\tau)|^2. \end{aligned} \quad (29)$$

Hence, we have

$$\begin{aligned} &F(\{\mathbf{X}\}, E^{M \times N}) = F(\mathcal{C}, E^{M \times N}) = F(E^{M \times N}, E^{M \times N}) \\ &= \frac{1}{2 \cdot H^{MN}} \sum_{s,t=0}^{2N-2} \sum_{\mathbf{Y} \in E^{M \times N}} \left[|\rho_{\mathbf{X}, \mathbf{Y}}(\tau)|^2 + |\rho_{\mathbf{Y}, \mathbf{X}}(\tau)|^2 \right] w_s w_t \\ &= \sum_{s,t=0}^{2N-2} M(N-\tau) w_s w_t. \end{aligned} \quad (30)$$

Substituting $\mathcal{D} = E^{M \times N}$ into (23), and based on (30), we have

Lemma 3:

$$F(\mathcal{C}, \mathcal{C}) \geq F(E^{M \times N}, E^{M \times N}) = \sum_{s,t=0}^{2N-2} M(N-\tau) w_s w_t. \quad (31)$$

Expanding (17) accordingly, we obtain

Lemma 4: For any $\mathcal{C} \subseteq E^{M \times N}$ of set size K ,

$$F(\mathcal{C}, \mathcal{C}) \leq \frac{M^2 N^2}{K} \sum_{i=0}^{2N-2} w_i^2 + \left(1 - \frac{1}{K} \sum_{i=0}^{2N-2} w_i^2 \right) \delta_{\max}^2. \quad (32)$$

$$\begin{aligned}
F(\mathcal{C}, \mathcal{D}) &= \frac{1}{|\mathcal{C}||\mathcal{D}|} \sum_{\mathbf{X} \in \mathcal{C}} \sum_{\mathbf{Y} \in \mathcal{D}} \sum_{s=0}^{2N-2} \sum_{t=0}^{2N-2} |\langle T^s(\mathbf{X}, 0^{N-1}), T^t(\mathbf{Y}, 0^{N-1}) \rangle|^2 w_s w_t \\
&= \frac{1}{|\mathcal{C}||\mathcal{D}|} \sum_{\mathbf{X} \in \mathcal{C}} \sum_{\mathbf{Y} \in \mathcal{D}} \sum_{s=0}^{2N-2} \sum_{t=0}^{2N-2} \left| \sum_{m=0}^{M-1} \sum_{i=0}^{2N-2} \mathbf{H}_{\mathbf{x}_m}(s, i) [\mathbf{H}_{\mathbf{y}_m}(t, i)]^* \right|^2 w_s w_t \\
&= \frac{1}{|\mathcal{C}||\mathcal{D}|} \sum_{\mathbf{X} \in \mathcal{C}} \sum_{s=0}^{2N-2} \sum_{\mathbf{Y} \in \mathcal{D}} \sum_{t=0}^{2N-2} \left[\sum_{i,j=0}^{2N-2} \sum_{m,n=0}^{M-1} f_{i,j,s}(\mathbf{x}_m, \mathbf{x}_n) [f_{i,j,t}(\mathbf{y}_m, \mathbf{y}_n)]^* \right] \\
&= \frac{1}{|\mathcal{C}||\mathcal{D}|} \sum_{i,j=0}^{2N-2} \sum_{m,n=0}^{M-1} \underbrace{\sum_{\mathbf{X} \in \mathcal{C}} \sum_{s=0}^{2N-2} f_{i,j,s}(\mathbf{x}_m, \mathbf{x}_n)}_{|\mathcal{C}|^2 F(\mathcal{C}, \mathcal{C})} \underbrace{\sum_{\mathbf{Y} \in \mathcal{D}} \sum_{t=0}^{2N-2} [f_{i,j,t}(\mathbf{y}_m, \mathbf{y}_n)]^*}_{|\mathcal{D}|^2 F(\mathcal{D}, \mathcal{D})}
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\left| \sum_{i,j=0}^{2N-2} \sum_{m,n=0}^{M-1} \underbrace{\sum_{\mathbf{X} \in \mathcal{C}} \sum_{s=0}^{2N-2} f_{i,j,s}(\mathbf{x}_m, \mathbf{x}_n)}_{|\mathcal{C}|^2 F(\mathcal{C}, \mathcal{C})} \underbrace{\sum_{\mathbf{Y} \in \mathcal{D}} \sum_{t=0}^{2N-2} [f_{i,j,t}(\mathbf{y}_m, \mathbf{y}_n)]^*}_{|\mathcal{D}|^2 F(\mathcal{D}, \mathcal{D})} \right|^2 \\
&\leq \sum_{i,j=0}^{2N-2} \sum_{m,n=0}^{M-1} \left| \sum_{\mathbf{X} \in \mathcal{C}} \sum_{s=0}^{2N-2} f_{i,j,s}(\mathbf{x}_m, \mathbf{x}_n) \right|^2 \cdot \sum_{i,j=0}^{2N-2} \sum_{m,n=0}^{M-1} \left| \sum_{\mathbf{Y} \in \mathcal{D}} \sum_{t=0}^{2N-2} [f_{i,j,t}(\mathbf{y}_m, \mathbf{y}_n)]^* \right|^2 \\
&= \underbrace{\sum_{i,j=0}^{2N-2} \sum_{m,n=0}^{M-1} \sum_{\mathbf{X} \in \mathcal{C}} \sum_{s=0}^{2N-2} f_{i,j,s}(\mathbf{x}_m, \mathbf{x}_n) \sum_{\mathbf{Y} \in \mathcal{C}} \sum_{t=0}^{2N-2} [f_{i,j,t}(\mathbf{y}_m, \mathbf{y}_n)]^*}_{|\mathcal{C}|^2 F(\mathcal{C}, \mathcal{C})} \\
&\quad \cdot \underbrace{\sum_{i,j=0}^{2N-2} \sum_{m,n=0}^{M-1} \sum_{\mathbf{X} \in \mathcal{D}} \sum_{s=0}^{2N-2} f_{i,j,s}(\mathbf{x}_m, \mathbf{x}_n) \sum_{\mathbf{Y} \in \mathcal{D}} \sum_{t=0}^{2N-2} [f_{i,j,t}(\mathbf{y}_m, \mathbf{y}_n)]^*}_{|\mathcal{D}|^2 F(\mathcal{D}, \mathcal{D})}.
\end{aligned} \tag{22}$$

Based on *Lemma 3* and *Lemma 4*, we present below the first theorem of this paper.

Theorem 1:

$$\delta_{\max}^2 \geq M \left(N - \frac{Q_{2N-1} \left(\mathbf{w}, \frac{N(MN-1)}{K} \right)}{1 - \frac{1}{K} \sum_{i=0}^{2N-2} w_i^2} \right). \tag{33}$$

Remark 3: When $M = 1$, *Theorem 1* is reduced to the Levenshtein bound in (14).

Remark 4: When $\mathbf{w} = \frac{1}{2N-1}(1, 1, \dots, 1)$ is applied to the quadratic function $Q_{2N-1} \left(\mathbf{w}, \frac{N(MN-1)}{K} \right)$, the proposed lower bound in *Theorem 1* is reduced to the Welch bound for QCSSs in (1).

IV. DISCUSSIONS ON THE PROPOSED LOWER BOUND FOR QCSSs

In this section, we discuss the tightness of the proposed lower bound in *Theorem 1*. We show that it is tighter than the Welch bound for QCSSs in (1) when a weight vector is properly chosen. Conditions of meeting the proposed lower bound with equality is also investigated.

A. Simplified Forms of the Proposed Lower Bound and Analysis of its Tightness

If the weight vector $\mathbf{w}$ in [16] with

$$w_i = \begin{cases} \frac{1}{m}, & i \in \{0, 1, \dots, m-1\}; \\ 0, & i \in \{m, m+1, \dots, 2N-2\} \end{cases} \tag{34}$$

for $1 \leq m \leq N$ is applied to *Theorem 1*, a new improved QCSS lower bound is obtained as follows.

Corollary 1: For $1 \leq m \leq N$,

$$\delta_{\max}^2 \geq \frac{3MNKm - 3M^2N^2 - MK(m^2 - 1)}{3(mK - 1)}. \tag{35}$$

The proposed lower bound in (35) is stronger than the Welch bound for QCSSs in (1) at least for $m = N - 1$ and $m = N$ if one of the following cases is fulfilled:

1) $3M + 1 \leq K \leq 4M - 1$, $M \geq 2$ and

$$N > \left\lfloor \frac{K - 1 + \sqrt{-3K^2 + (12M - 6)K + 12M + 1}}{2(K - 3M)} \right\rfloor; \tag{36}$$

2) $K \geq 4M$, $M \geq 2$ and $N \geq 2$.

Proof: See Appendix A. ■

By properly choosing m around $\sqrt{\frac{3M}{K}}N$ in (35), which is similar to the derivation for (16), we have the following simplified lower bound.

Corollary 2: For $K \geq 3M, M \geq 2, N \geq 2$,

$$\delta_{\max}^2 \geq MN \left(1 - 2\sqrt{\frac{M}{3K}} \right). \quad (37)$$

Remark 5: For sufficiently large K , *Corollary 2* shows that $\delta_{\max}^2$ tends to MN . In contrast, the Welch bound in (1) shows that $\delta_{\max}^2$ tends to $\frac{MN}{2}$.

By performing a convex analysis on the fractional quadratic term in (33), we present the second theorem of this paper as follows.

Theorem 2: The Welch bound for QCSSs in (1) cannot be improved by the proposed lower bound in *Theorem 1* when

$$K \leq \bar{K} \triangleq \left\lfloor 4(MN - 1)N \sin^2 \frac{\pi}{2(2N - 1)} \right\rfloor, \quad (38)$$

where $\lfloor x \rfloor$ denotes the biggest integer which is not greater than x . Note that $\bar{K}$ will decrease and converge to $\left\lfloor \frac{\pi^2 M}{4} \right\rfloor$ as N approaches positive infinity. Therefore, if $K \leq \left\lfloor \frac{\pi^2 M}{4} \right\rfloor$, the Welch bound for QCSSs cannot be improved for any $M \geq 2$ and $N \geq 2$.

Proof: See Appendix B. ■

B. Meeting the Proposed Lower Bound with Equality

Definition 1: For the weight vector $\mathbf{w}$ defined in (7), the *weighted aperiodic cross-correlation function* of length- N sequences $\mathbf{a}$ and $\mathbf{b}$ is defined as

$$\rho_{\mathbf{a}, \mathbf{b}; \mathbf{w}, \lambda}(\tau) = \begin{cases} (2N - 1) \sum_{t=0}^{N-1-\tau} a_t(b_{t+\tau})^* w_{t+\lambda}, & 0 \leq \tau \leq (N - 1); \\ (2N - 1) \sum_{t=0}^{N-1+\tau} a_{t-\tau}(b_t)^* w_{t+\lambda}, & -(N - 1) \leq \tau \leq -1; \\ 0, & |\tau| \geq N. \end{cases} \quad (39)$$

where $0 \leq \lambda \leq 2N - 2$.

Definition 2: The *aperiodic weighted correlation function* $\rho_{\mathcal{C}^u, \mathcal{C}^v; \mathbf{w}, \lambda}(\tau)$ of $\mathcal{C}$ is defined as

$$\rho_{\mathcal{C}^u, \mathcal{C}^v; \mathbf{w}, \lambda}(\tau) = \sum_{m=0}^{M-1} \rho_{\mathbf{c}_m^u, \mathbf{c}_m^v; \mathbf{w}, \lambda}(\tau), \quad (40)$$

where $0 \leq u, v \leq K - 1$. $\mathcal{C}$ is called a *weighted-correlation mutually orthogonal complementary sequence set* (WC-MOCSS) characterized by $\mathbf{w}$, if for any $0 \leq \lambda \leq 2N - 2$, it satisfies

$$\rho_{\mathcal{C}^u, \mathcal{C}^v; \mathbf{w}, \lambda}(\tau) = 0, \quad \text{where } u \neq v \text{ or } u = v, \tau \neq 0. \quad (41)$$

Next, we shall show how to meet the proposed lower bound in *Theorem 1*, which is obtained by *Lemma 3* and *Lemma 4*, with equality. To meet the inequality (32) in *Lemma 4* with equality, it is clear that all non-trivial aperiodic correlations of a QCSS should take identical amplitude. Hence, our main task is to work out the conditions to meet the inequality (31)

in *Lemma 3* with equality. Tracing back the proof in Section III, one just needs to work out the conditions for (22) with equality.

Note that the equality of (22) is met if and only if

$$\sum_{\mathbf{X} \in \mathcal{C}} \sum_{s=0}^{2N-2} f_{i,j,s}(\mathbf{x}_m, \mathbf{x}_n) = c \sum_{\mathbf{Y} \in \mathcal{D}} \sum_{t=0}^{2N-2} f_{i,j,t}(\mathbf{y}_m, \mathbf{y}_n), \quad (42)$$

for any constant c .

For the left-hand term in (42), we have

$$\begin{aligned} & \sum_{\mathbf{X} \in \mathcal{C}} \sum_{s=0}^{2N-2} f_{i,j,s}(\mathbf{x}_m, \mathbf{x}_n) \\ &= \sum_{\mathbf{X} \in \mathcal{C}} \sum_{s=0}^{2N-2} \mathbf{H}_{\mathbf{x}_m}(s, i) [\mathbf{H}_{\mathbf{x}_n}(s, j)]^* w_s \\ &= \sum_{\mathbf{X} \in \mathcal{C}} \langle \mathbf{H}_{\mathbf{x}_m}(:, i), \mathbf{w} \cdot \mathbf{H}_{\mathbf{x}_n}(:, j) \rangle, \\ &= \sum_{\mathbf{X} \in \mathcal{C}} \sum_{t=0}^{N-\tau-1} x_{m,t} [x_{n,t+\tau}]^* w_{\lambda-t \bmod 2N-1}, \\ &= \sum_{\mathbf{X} \in \mathcal{C}} \sum_{t=0}^{N-\tau-1} x_{m,t} [x_{n,t+\tau}]^* \tilde{w}_{t-\lambda \bmod 2N-1} \\ &= \frac{1}{2N-1} \sum_{\mathbf{X} \in \mathcal{C}} \rho_{\mathbf{x}_m, \mathbf{x}_n; \tilde{\mathbf{w}}, -\lambda}(\tau) \end{aligned} \quad (43)$$

where $\tilde{\mathbf{w}} = \{\tilde{w}_i = w_{2N-1-i}, 0 \leq i \leq 2N - 2\}$ is called the reversal of $\mathbf{w}$, and

- 1) $\lambda = \min\{i, j\}$ if $\tau = |i - j|$;
- 2) $\lambda = \max\{i, j\}$ if $\tau = 2N - 1 - |i - j|$.

For the right-hand term in (42), setting $\mathcal{D} = E^{M \times N}$, we have

$$\begin{aligned} & \sum_{\mathbf{Y} \in \mathcal{D}} \sum_{s=0}^{2N-2} f_{i,j,s}(\mathbf{y}_m, \mathbf{y}_n) \\ &= \sum_{t=0}^{N-\tau-1} \tilde{w}_{t-\lambda \bmod 2N-1} \cdot \sum_{\mathbf{Y} \in E^{M \times N}} y_{m,t} [y_{n,t+\tau}]^* \\ &= H^{MN-2} \sum_{t=0}^{N-\tau-1} \tilde{w}_{t-\lambda \bmod 2N-1} \\ & \quad \cdot \left[\sum_{y_{m,t}=\xi^0}^{\xi^{H-1}} y_{m,t} \right] \cdot \left[\sum_{y_{n,t+\tau}=\xi^0}^{\xi^{H-1}} y_{n,t+\tau} \right]^* \\ &= 0, \end{aligned} \quad (44)$$

if $m \neq n$ or $m = n, \tau \neq 0$.

Note that when i and j range over $\{0, 1, \dots, 2N - 2\}$, $-\lambda$ will range over $\{0, 1, \dots, 2N - 2\}$ as well. Therefore, by (42)-(44), for any $0 \leq -\lambda \leq 2N - 2$,

$$\sum_{\mathbf{X} \in \mathcal{C}} \rho_{\mathbf{x}_m, \mathbf{x}_n; \tilde{\mathbf{w}}, -\lambda}(\tau) = 0, \quad \text{where } m \neq n \text{ or } m = n, \tau \neq 0. \quad (45)$$

For the complementary sequence set $\mathcal{C}$ with set size K and number of channels M in the above discussion, re-write it into

the ordered matrix form on the top of (46), where each column corresponds to a complementary matrix of $\mathcal{C}$. Now consider another complementary sequence set $\bar{\mathcal{C}}$ with set size M , and number of channels K , which is shown on the bottom of (46).

$$\mathcal{C} = \begin{bmatrix} \mathbf{c}_0^0 & \mathbf{c}_1^0 & \cdots & \mathbf{c}_{M-1}^{K-1} \\ \mathbf{c}_0^1 & \mathbf{c}_1^1 & \cdots & \mathbf{c}_{M-1}^{K-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{c}_{M-1}^0 & \mathbf{c}_{M-1}^1 & \cdots & \mathbf{c}_{M-1}^{K-1} \end{bmatrix}_{M \times K} \quad (46)$$

$$\bar{\mathcal{C}} = \begin{bmatrix} \mathbf{c}_0^0 & \mathbf{c}_1^0 & \cdots & \mathbf{c}_{M-1}^0 \\ \mathbf{c}_0^1 & \mathbf{c}_1^1 & \cdots & \mathbf{c}_{M-1}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{c}_0^{K-1} & \mathbf{c}_1^{K-1} & \cdots & \mathbf{c}_{M-1}^{K-1} \end{bmatrix}_{K \times M}$$

By (46), it is clear that $\bar{\mathcal{C}}$ is obtained from $\mathcal{C}$ by the transpose operation. For ease of presentation, from now on, we shall call $\bar{\mathcal{C}}$ “the transpose of $\mathcal{C}$ ”.

Note that (45) can be written explicitly as follows.

$$\sum_{p=0}^{K-1} \rho_{\mathbf{c}_m^p, \mathbf{c}_n^p; \tilde{\mathbf{w}}, -\lambda}(\tau) = 0, \text{ for any } m \neq n \text{ or } m = n, \tau \neq 0. \quad (47)$$

Recalling *Definition 2*, we obtain the following theorem.

Theorem 3: The proposed lower bound in *Theorem 1*, which is characterized by $\mathbf{w}$, is met with equality if and only if all non-trivial aperiodic correlations of QCSS $\mathcal{C}$ have identical amplitude, and the transpose of $\mathcal{C}$ is a weighted-correlation mutually orthogonal complementary sequence set (WC-MOCSS) characterized by the reversal of $\mathbf{w}$.

V. CONCLUSIONS

A new aperiodic correlation lower bound for quasi-complementary sequence sets (QCSSs) over complex roots of unity has been proposed in *Theorem 1* of this paper. As a generalization of the Levenshtein bound [16], the proposed lower bound is a function of the set size K , the channel number M , the elementary sequence length N , and the weight vector $\mathbf{w}$. It is reduced to the Welch bound for QCSSs when $\mathbf{w} = \frac{1}{2N-1}(1, 1, \dots, 1)$ is applied to the bounding function. The key to obtain the lower bound in *Theorem 1* is (31) in *Lemma 3*. The basic idea of (31) is that the “weighted mean square aperiodic correlation” (i.e., $F(\mathcal{C}, \mathcal{C})$) of any sequence subset over the complex roots of unity should be equal to or greater than that of the whole set which includes all possible complex roots-of-unity sequences. While Levenshtein shows that (31) holds for $M = 1$ (i.e., conventional single-channel sequence sets), we have made a contribution to show that (31) also holds for any integer $M \geq 2$ (i.e., QCSSs), where M denotes the number of channels.

In *Corollary 1*, we have shown that, by applying the weight vector in (34), the proposed lower bound is tighter than the Welch bound for QCSSs in one of the following sets of conditions:

- 1) $3M + 1 \leq K \leq 4M - 1$, $M \geq 2$ and

$$N > \frac{K - 1 + \sqrt{-3K^2 + (12M - 6)K + 12M + 1}}{2(K - 3M)};$$

- 2) $K \geq 4M$, $M \geq 2$ and $N \geq 2$.

Furthermore, for sufficiently large K , by *Remark 5*, the lower bound in *Corollary 1* approaches $\sqrt{MN}$, whereas the Welch bound for QCSSs approaches $\sqrt{\frac{MN}{2}}$. Interestingly, the same was obtained by Levenshtein for conventional sequence sets by setting $M = 1$ [16].

Finally, we have shown in *Theorem 3* that the derived lower bound can be met with equality if and only if all non-trivial aperiodic correlations of a QCSS take identical amplitude, and the transpose of the QCSS is a weighted-correlation mutually orthogonal complementary sequence set (WC-MOCSS). Interestingly, to meet the Levenshtein bound (i.e., $M = 1$) with equality, the latter condition can be simplified to that in [31]: The conventional sequence set should form a weighted-correlation complementary sequence which has zero aperiodic weighted-correlation auto-correlation sums for all non-zero time-shifts.

APPENDIX A PROOF OF COROLLARY 1

Proof: For $1 \leq m \leq N$, Levenshtein showed that [16]

$$Q_{2N-1} \left(\mathbf{w}, \frac{N(MN - 1)}{K} \right) = \frac{N(MN - 1)}{Km} + \frac{m^2 - 1}{3m}. \quad (48)$$

The proof of (35) follows by substituting (48) into (33). Next, let us analyze the tightness of (35).

Let $\epsilon|_{K,m}$ denote the subtraction of the lower bound in (35) by the Welch bound for QCSSs, i.e.,

$$\epsilon|_{K,m} = \frac{MNKm - M^2N^2 - \frac{MK(m^2-1)}{3}}{Km - 1} - M^2N^2 \frac{\frac{K}{M} - 1}{K(2N - 1) - 1}. \quad (49)$$

In addition, define

$$\alpha \triangleq \frac{KM^2N^3}{(Km - 1)(K(2N - 1) - 1)}, \quad (50)$$

and

$$\beta \triangleq \frac{K}{M} \left(\frac{1}{3} - \frac{N - 1}{3N^2} \right) + \frac{N + 1}{3MN^2} - 1. \quad (51)$$

Note that

$$\begin{aligned} \epsilon|_{m=N-1} &= \alpha|_{m=N-1} \cdot \beta, \\ \epsilon|_{m=N} &= (1 - 1/N) \cdot \alpha|_{m=N} \cdot \beta. \end{aligned} \quad (52)$$

For $3M + 1 \leq K \leq 4M - 1$, by (52), the lower bound in (35) is tighter than the Welch bound for QCSSs at least for $m = N - 1$ and $m = N$ if $\beta > 0$, i.e.,

$$N > \frac{K - 1 + \sqrt{-3K^2 + (12M - 6)K + 12M + 1}}{2(K - 3M)}.$$

For $K \geq 4M$, we have

$$\begin{aligned}
& \epsilon|_{K \geq 4M, m=N} \\
&= (1 - 1/N) \cdot \alpha|_{m=N} \cdot \beta \\
&\geq (1 - 1/N) \cdot \alpha|_{m=N} \cdot \left[4 \left(\frac{1}{3} - \frac{N-1}{3N^2} \right) + \frac{N+1}{3MN^2} - 1 \right] \\
&> (1 - 1/N) \cdot \alpha|_{m=N} \cdot \left[\frac{1}{3} - \frac{4N-4}{3N^2} \right] \\
&> \frac{KM^2(N-1)(N-2)^2}{3(KN-1)(K(2N-1)-1)} \\
&\geq 0.
\end{aligned} \tag{53}$$

By (53), the lower bound in (35) is tighter than the Welch bound for QCSSs for $K \geq 4M, M \geq 2, N \geq 2, m = N$. Similarly, we can by (52) prove the case for $K \geq 4M, M \geq 2, N \geq 2, m = N-1$. Thus, we complete the proof. ■

APPENDIX B PROOF OF THEOREM 2

Proof: Two conclusions made by Berlekamp [32] are recalled first:

- 1) The principal eigenvalue, and the secondary eigenvalues of the quadratic matrix $\mathbf{Q}_{2N-1}$ in (8) are

$$\lambda_0 = a + (N-1)N,$$

and

$$\lambda_k = a - \frac{1 - (-1)^k \cos \frac{\pi k}{2N-1}}{2 \sin^2 \frac{\pi k}{2N-1}}, \quad k = 1, \dots, 2N-2$$

respectively.

- 2) The constrained quadratic function $Q_{2N-1}(\mathbf{w}, a)$ is convex if all the secondary eigenvalues $\lambda_k \geq 0$, of which the global minimum is achieved by $\mathbf{w} = \frac{1}{2N-1}(1, 1, \dots, 1)$.

It is clear that the minimization of the fractional quadratic function in (33) can be explicitly obtained by $\mathbf{w} = \frac{1}{2N-1}(1, 1, \dots, 1)$ if $Q_{2N-1}(\mathbf{w}, a)$ is convex, this is because the denominator term, $1 - \frac{1}{K} \sum_{i=0}^{2N-2} w_i^2$, is concave under the constraint in (7). Therefore, based on the second conclusion of Berlekamp, the Welch bound for QCSSs cannot be improved if

$$\begin{aligned}
& \min_{1 \leq k \leq 2N-2} \lambda_k = \lambda_1 = \lambda_{2N-2} \\
&= \frac{(MN-1)N}{K} - \frac{1}{4 \sin^2 \frac{\pi}{2(2N-1)}} \\
&\geq 0,
\end{aligned} \tag{54}$$

which is equivalent to

$$K \leq \bar{K} = \left\lfloor 4(MN-1)N \sin^2 \frac{\pi}{2(2N-1)} \right\rfloor. \tag{55}$$

For (55), in order to get the tightest upper bound of K (or minimum of $\bar{K}$), it is wise to see how $\bar{K}$ changes with N . To this end, let $N = x$, and define

$$f(x) = \frac{4x(Mx-1)}{K} \sin^2 \frac{\pi}{2(2x-1)} - 1, \tag{56}$$

with

$$\begin{aligned}
g(x) &= \frac{K}{4(2Mx-1) \sin \frac{\pi}{2x-1}} \frac{\partial f(x)}{\partial x} \\
&= \tan \frac{\pi}{2(2x-1)} - \frac{2Mx-2}{2Mx-1} \frac{\pi x}{(2x-1)^2},
\end{aligned} \tag{57}$$

where $\frac{\partial f(x)}{\partial x}$ denotes the derivative of $f(x)$. We claim herein that $f(x)$ is monotonically decreasing for $M \geq 2$, and equivalently, $g(x)$ is constantly less than 0. Now we present below the proof.

Since the small angle approximation of the tangent function [33] is

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots, \quad \text{with } 0 \leq x \leq 1 \tag{58}$$

it is clear that

$$\tan x \leq x + \frac{2x^3}{3}, \quad \text{for } 0 \leq x \leq 1. \tag{59}$$

By (59), we can see that $g(x)$ in (57) is upper bounded by $\frac{\pi}{2(2x-1)^3} h(x)$, where

$$\begin{aligned}
h(x) &= \frac{\pi^2}{6} + (2x-1) \left(\frac{1}{M} - 1 + \frac{1}{M(2Mx-1)} \right) \\
&< \frac{\pi^2}{6} + (2x-1) \left(\frac{1}{M} - 1 \right) + \frac{1}{M^2} \\
&\leq \frac{\pi^2}{6} + \frac{3}{4} - x.
\end{aligned} \tag{60}$$

Thus, by (60), to have $g(x) < 0$ (or equivalently, $h(x) < 0$), it is required that

$$x > \frac{\pi^2}{6} + \frac{3}{4} \approx 2.3949, \tag{61}$$

which is for sure for $x \geq 3$ and $M \geq 2$. In the case of $x = 2$, by (57), some simple calculations suffice to show that $g(2) < 0$.

From the above analysis, one can see that $f(x)$ is indeed monotonically decreasing for $M \geq 2$, and therefore, $\bar{K}$ in (55) will decrease and converge to a fixed value as N tends to positive infinity. This fixed value, i.e., $\bar{K}|_{N=+\infty}$, can be obtained by considering the fact that

$$\min_{x \geq 2} f(x) = f(+\infty) = \frac{\pi^2 M}{4K} - 1 \geq 0. \tag{62}$$

Hence, provided that

$$K \leq \bar{K}|_{N=+\infty} = \left\lfloor \frac{\pi^2 M}{4} \right\rfloor, \tag{63}$$

the Welch bound for QCSSs cannot be improved by the derived lower bound for any $M \geq 2$ and $N \geq 2$. ■

REFERENCES

- [1] M. J. E. Golay, "Complementary series," *IRE Trans. Inf. Theory*, vol. IT-7, pp. 82-87, Apr. 1961.
- [2] C. Tseng and C. Liu, "Complementary sets of sequences," *IEEE Trans. Inf. Theory*, vol. IT-18, pp. 644-665, Sep. 1972.
- [3] N. Suehiro and M. Hatori, "N-Shift cross-orthogonal sequences," *IEEE Trans. Inf. Theory*, vol. IT-34, pp. 143-146, Jan. 1988.

- [4] H. H. Chen, J. F. Yeh, and N. Suehiro, "A multicarrier CDMA architecture based on orthogonal complementary codes for new generations of wideband wireless communications," *IEEE Commun. Mag.*, vol. 39, pp.126-135, Oct. 2001.
- [5] H. H. Chen, *The Next Generation CDMA Technologies*, 1st Edition, Jul. 2007, Hardcover, John Wiley & Sons.
- [6] J. A. Davis and J. Jedwab, "Peak-to-mean power control in OFDM, Golay complementary sequences, and Reed-Muller codes," *IEEE Trans. Inf. Theory*, vol. 45, pp. 2397-2417, Nov. 1999.
- [7] K. G. Paterson, "Generalized Reed-Muller codes and power control in OFDM modulation," *IEEE Trans. Inf. Theory*, vol. 46, pp. 104-120, Jan. 2000.
- [8] A. Rathinakumar and A. K. Chaturvedi, "Complete mutually orthogonal Golay complementary sets from Reed-Muller codes," *IEEE Trans. Inf. Theory*, vol. 54, pp. 1339-1346, Mar. 2008.
- [9] P. Spasojevic and C. N. Georgiades, "Complementary sequences for ISI channel estimation," *IEEE Trans. Inf. Theory*, vol. 47, pp. 1145-1152, Mar. 2001.
- [10] S. Wang, A. Abdi, "MIMO ISI channel estimation using uncorrelated Golay complementary sets of polyphase sequences," *IEEE Trans. Veh. Tech.*, vol. 56, pp. 3024-3040, Sep. 2007.
- [11] A. Pezeshki, A. R. Calderbank, W. Moran, and S. D. Howard, "Doppler resilient Golay complementary waveforms," *IEEE Trans. Inf. Theory*, vol. 54, pp. 4254-4266, Sep. 2008.
- [12] P. Fan, N. Suehiro, N. Kuroyanagi, and X. M. Deng, "A class of binary sequences with zero correlation zone," *IEE Electron. Lett.*, vol. 35, pp. 777-779, May 1999.
- [13] R. Appuswamy and A. K. Chaturvedi, "A new framework for constructing mutually orthogonal complementary sets and ZCZ sequences," *IEEE Trans. Inf. Theory*, vol. 52, pp. 3817-3826, Aug. 2006.
- [14] X. Tang, P. Fan and J. Lindner, "Multiple binary ZCZ sequence sets with good cross-correlation property based on complementary sequence sets," *IEEE Trans. Inf. Theory*, vol. 56, pp. 4038-4045, Aug. 2010.
- [15] L. R. Welch, "Lower bounds on the maximum cross-correlation of signals," *IEEE Trans. Inf. Theory*, vol. IT-20, pp. 397-399, May 1974.
- [16] V. I. Levenshtein, "New lower bounds on aperiodic crosscorrelation of binary codes," *IEEE Trans. Inf. Theory*, vol. 45, pp. 284-288, Jan. 1999.
- [17] S. Boztas, "New lower bounds on aperiodic cross-correlation of codes over n th roots of unity," Research Report 13, Department of Mathematics, Royal Melbourne Institute of Technology, Australia, 1998.
- [18] D. Peng and P. Fan, "Generalised Sarwate bounds on the aperiodic correlation of sequences over complex roots of unity," *IEE Proc.-Commun.*, vol. 151, pp. 375-382, Aug. 2004.
- [19] Z. Liu, Y. L. Guan, B. C. Ng, and H. H. Chen, "Correlation and set size bounds of complementary sequences with low correlation zones," *IEEE Trans. Commun.*, vol. 59, Iss. 12, pp. 3285-3289, Dec. 2011.
- [20] Z. Liu, U. Parampalli, Y. L. Guan and S. Boztas, "Constructions of optimal and near-optimal quasi-complementary sequence sets from Singer difference sets," Accepted by *IEEE Wireless Commun. Letters*, May 2013.
- [21] W. H. Mow, *Sequence design for spread spectrum*, The Chinese University Press (Chinese University of Hong Kong, Shatin, Hong Kong), 1995.
- [22] J. L. Massey, "On Welch's bound for correlation of a sequence set," *Proc. 1991 IEEE Int. Symposium on Information Theory (ISIT'1991)*, pp. 385, Jun. 1991, Budapest, Hungary.
- [23] J. L. Massey and T. Mittelholzer, "Welch's bound and sequence sets for code-division multiple-access systems," in *Sequences II: Methods in Communication, Security, and Computer Science*, R. Capocelli, et al., Eds. New York: Springer-Verlag, 1993.
- [24] M. Rupp and J. L. Massey, "Optimum sequence multisets for synchronous code-division multiple-access channels," *IEEE Trans. Inf. Theory*, vol. 40, pp. 1261-1266, July 1994.
- [25] W. H. Mow, "Optimal sequence sets meeting Welch's lower bound," *Proc. 1995 IEEE Int. Symposium on Information Theory (ISIT'1995)*, pp. 90, Sep. 1995, Whistler, British Columbia, Canada.
- [26] Dilip V. Sarwate, "Meeting the Welch bound with equality, sequences and their Applications: Proceedings of SETA'98", ed. C. Ding, T. Hellesest, and H. Niederreiter, DMTCS Series, Springer-Verlag, 1999.
- [27] P. Viswanath and V. Anantharam, "Optimal sequences and sum capacity of synchronous CDMA systems," *IEEE Trans. Inf. Theory*, vol. 45, pp. 1984-1993, Sep. 1999.
- [28] S. Ulukus and R. D. Yates, "Iterative construction of optimum signature sequence sets in synchronous CDMA systems," *IEEE Trans. Inf. Theory*, vol. 47, pp. 1989-1998, Jul. 2001.
- [29] S. Ulukus and R. D. Yates, "User capacity of asynchronous CDMA Systems with matched filter receivers and optimum signature sequences," *IEEE Trans. Inf. Theory*, vol. 50, pp. 903-909, May 2004.
- [30] P. Cota, "On the optimal sequences and total weighted square correlation of synchronous CDMA systems in multipath channels," *IEEE Trans. Veh. Tech.*, vol. 56, pp. 2063-2072, Jul. 2007.
- [31] Z. Liu and Y. L. Guan, "Meeting the Levenshtein bound with weighted-correlation complementary set," *Proc. 2012 IEEE Int. Symposium on Information Theory (ISIT'2012)*, pp. 1010-1013, Jul. 2012, Boston, US.
- [32] E. R. Berlekamp, *Algebraic Coding Theory*, New York: McGraw-Hill, 1968.
- [33] "<http://en.wikipedia.org/wiki/Small-angle-approximation>".

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