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Momentum-space dynamics of Dirac quasiparticles in correlated random potentials: Interplay between dynamical and Berry phases

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We consider Dirac quasiparticles, as realized with cold atoms loaded in a honeycomb lattice or in a π -flux square lattice, in the presence of a weak correlated disorder such that the disorder fluctuations do not couple the two Dirac points of the lattices. We numerically and theoretically investigate the time evolution of the momentum distribution of such quasiparticles when they are initially prepared in a quasimonochromatic wave packet with a given mean momentum. The parallel transport of the pseudospin degree of freedom along scattering paths in momentum space generates a geometrical phase which alters the interference associated with reciprocal scattering paths. In the massless case, a well-known dip in the momentum distribution develops at backscattering (relative to the Dirac point considered) around the transport mean free time. This dip later vanishes in the honeycomb case because of trigonal warping. In the massive case, the dynamical phase of the scattering paths becomes crucial. Its interplay with the geometrical phase induces an additional transient broken reflection symmetry in the momentum distribution. The direction of this asymmetry is a property of the Dirac point considered, independent of the energy of the wave packet. These Berry-phase effects could be observed in current cold-atom lattice experiments.

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I. INTRODUCTION

Topology and gauge fields, as exemplified by Chern numbers [1], Berry phases and curvatures [2], are key concepts at the heart of many high-energy or condensed-matter phenomena [3,4]. For solid-state systems, the main reason is that Bloch's theorem introduces wave numbers $\mathbf{k}$ belonging to a parameter space with the topology of a torus, the Brillouin zone, and a set of periodic wave functions depending parametrically on $\mathbf{k}$. Together they offer natural settings for bundles and connections [5]. Besides high-energy and condensed-matter systems, ultracold-atomic systems have progressively and successfully confirmed their ability to mimic or emulate some paradigmatic phenomena of condensed-matter systems, including topological effects. Indeed, one can now load independent or interacting bosons and/or fermions into two-dimensional, perfectly controlled, optical lattices [6–8] and one can generate carefully designed synthetic magnetic fields [9–12]. Last but not least, powerful imaging techniques allow one to study the real-space or momentum-space time dynamics of the simulated system; see, for instance, Refs. [13,14], where the coherent backscattering effect was addressed. These advances pave the way to accurate cold-atom experiments targeting topology-related effects such as defects [15], color superfluidity [16], momentum-space Berry curvatures [17], or quantum Hall states with strong effective magnetic fields [18–21].

In this work, we are interested in the influence of topological effects on the quantum interference of waves propagating in a disordered system. We propose to combine existing cold-atom experiments on artificial gauge field [8] and quantum transport [13,14] to study the momentum-space dynamics

of Dirac quasiparticles [22–24], as found, for example, in graphene sheets [25] or in topological insulators [26–28], in the presence of a correlated and weak on-site disorder. A semiclassical approach shows that the dynamics of these quasiparticles is unaffected by the topology of the band structure as long as a net effective electric field is absent [4,29]. We go beyond this semiclassical approach to address the competition happening at short times between Berry-phase effects induced by the Bloch-state topology and dynamical coherent corrections to transport due to the interference associated with reciprocal scattering paths [30–32]. Usually, these loop interferences are constructive and enhance the return probability of a propagating wave, giving rise to weak-localization corrections [30]. However, it is long known that they are destructive for magnetic or spin-orbit scattering since a rotation of 2π of the electron spins gives rise to a sign flip of the interference terms. In this case, one gets weak-*anti* localization effects instead, as revealed by magnetoresistance measurements and suppression of the coherent backscattering (CBS) effect [30,33]. The very same situation happens in graphene [34–37] where the band structure induces two Berry monopoles of opposite charge located at the Dirac points, hence an effective magnetic field in momentum space. The theoretical calculation of the loop interference terms is generally done in the diffusion approximation [32], i.e., at long times. As a consequence, its dominant contribution is expected to happen in the vicinity of the CBS momentum even in the presence of a Berry phase. We investigate here the short-time regime and show that the momentum-space dynamics exhibits different features, in particular when the Dirac quasiparticles are massive. In this case, the dynamics in momentum space is no longer reflection symmetric and an observable fringe

pattern develops at a momentum away from the CBS one and moves towards it as time increases. This reflection symmetry breaking is similar to the shift of interference fringes in a certain direction observed for Aharonov-Bohm experiments in real space. A key aspect of this symmetry breaking is the subtle interplay between dynamical and Berry phases which impacts the *transient* dynamics where off-shell effects contribute significantly, which is a situation markedly different from the usual steady-state regime. The impact of the dynamical phase is clearly missed in the diffusion approximation. However, momentum-space measurements are challenging in condensed-matter systems. From this point of view, cold-atom systems offer an interesting alternative, as witnessed by the recent observation of CBS with matter waves [13,14,38]. Furthermore, graphenelike experiments can be easily performed with noninteracting cold atoms loaded on a honeycomb [8,39–42] or on a π -flux square [9,11,43] optical lattice at an energy close to the charge neutrality point $E = 0$ (Dirac regime).

II. MODEL

In the following, we consider a cold-atom experiment setup that combines those of Refs. [8,11,13,14]. Because of the recent scientific interest in topological insulators and graphene, we focus on the π -flux square lattice and on the honeycomb one; see Appendix A. Each lattice has two inequivalent Dirac points located at $\mathbf{K} = 0$ and $\mathbf{K}' = \frac{\pi}{a}\hat{x}$ (π -flux square lattice) and at $\mathbf{K} = 4\pi/(3\sqrt{3}a)\hat{x}$ and $\mathbf{K}' = -\mathbf{K}$ (honeycomb lattice), where a is the lattice constant. Their common energy is chosen to be $E = 0$ (charge neutrality point). For each of these lattices, we start with an initial quasimonochromatic Gaussian wave packet $|\psi_0\rangle$ with a mean momentum $\mathbf{k}_0 = \mathbf{K} + \mathbf{q}_0$ close to the Dirac point at $\mathbf{K}$ ($q_0 a \ll 1$) and with a momentum spread $\sigma_0 \ll |\mathbf{k}_0|$. This initial state is essentially a plane-wave state $|\mathbf{k}_0\rangle$ and we have checked that decoherence effects due to the finite width of the wave packet [38] are negligible over the times scales considered here. In the following, we conveniently choose $\mathbf{q}_0$ to lie on the negative x axis and the initial energy E_0 of $|\mathbf{k}_0\rangle$ to be negative (lower band).

At time $t = 0$, $|\psi_0\rangle$ is subjected to a spatially correlated disordered potential $V(\mathbf{r})$ with Gaussian statistics, vanishing mean value $\bar{V} = 0$, and Gaussian two-point correlator $F(\mathbf{r}_i - \mathbf{r}_j) = \overline{V(\mathbf{r}_i)V(\mathbf{r}_j)} = W^2 \exp[-\frac{(\mathbf{r}_i - \mathbf{r}_j)^2}{2\zeta^2}]$. Here the overbar denotes the ensemble average, W is the disorder fluctuations strength, and ζ is the disorder correlation length. We then numerically compute the wave function $|\Psi(t)\rangle$ at time t in the case where $k_0\zeta \gg 1$. Hereafter, we focus on the diffuse component $\rho_D(t) = |\delta\Psi(t)\rangle\langle\delta\Psi(t)|$ where $|\delta\Psi(t)\rangle = |\Psi(t)\rangle - |\bar{\Psi}(t)\rangle$, which is the dominant contribution at times larger than the scattering mean free time $\tau_s(E_0) \equiv \tau_s$ [44]. We have typically used 10^4 disorder configurations to extract $\rho_D(t)$. The potential fluctuations can only efficiently couple momenta $\mathbf{k}$, which are typically within ζ^{-1} away from each other. In particular, when $\zeta \gg a$, intervalley scattering is suppressed and the neighborhoods of the two inequivalent Dirac points of the clean lattices are then uncoupled [37]. Writing the scattered momenta as $\mathbf{k} = \mathbf{K} + \mathbf{q}$, we thus have $qa \ll 1$.

At lowest order in $\mathbf{q}$ (see Appendix A), the dynamics is then governed by the effective Dirac spinor Hamiltonian $H = H_0 + V(\mathbf{r})$, where $H_0 = -c(p_x\sigma_x + p_y\sigma_y) + mc^2\sigma_z$, featuring the momentum operator $\mathbf{p} = -i\hbar\nabla$, the Dirac quasiparticles mass m [45], their velocity c , and the usual Pauli matrices $\sigma_x, \sigma_y, \sigma_z$. In momentum space and matrix form, H_0 reads

$$H_0 = \hbar c (-q_x\sigma_x - q_y\sigma_y + q_m\sigma_z) \equiv \hbar c \mathbf{Q} \cdot \boldsymbol{\sigma} \\ = \underbrace{\sqrt{\hbar^2 q^2 c^2 + m^2 c^4}}_{\varepsilon_q = \hbar c Q^2} \begin{pmatrix} \cos\theta_q & -\sin\theta_q e^{-i\varphi_q} \\ -\sin\theta_q e^{i\varphi_q} & -\cos\theta_q \end{pmatrix}, \quad (1)$$

where $q_m = mc/\hbar$ is the Dirac particles' Compton wave number, ε_q is their energy dispersion relation, and $\boldsymbol{\sigma}$ is the Pauli vector operator. The spherical angles of $\mathbf{Q} = (-q_x, -q_y, q_m)$ are θ_q and $\varphi_q + \pi$, with φ_q being the polar angle of $\mathbf{q}$. The spinorial nature of the Dirac Hamiltonian directly reflects the bipartite nature of the graphene lattice. The corresponding spin-up and spin-down basis vectors $|\uparrow\rangle$ and $|\downarrow\rangle$ represent Bloch waves on either one of the two graphene sublattices ($\theta_q = 0, \pi$). As one may note, the mass term q_m , acting as a Zeeman field, lifts the energy degeneracy between these two spin states. More crucially, it also modifies the Berry curvature and leads to the main results of this work (see later discussion).

III. NUMERICAL PARAMETERS

We give here the typical numerical parameters that we have used in our simulations (as shown in Fig. 1) and we compare them to their experimental counterparts, as found in Refs. [8,11,13,14]. In our lattice simulation, we have used a tunneling amplitude $J/\hbar \approx 600$ Hz, as typically found in Refs. [8,11]. The mass gaps used in our simulations are typically in the range $2mc^2/\hbar \sim 180$ –300 Hz, while the tunable mass gap achieved in Ref. [8] is around 20 Hz. The interesting dynamics in our problem typically occurs at times $t \sim 20$ –60 $\hbar/J$. This would mean experimental observation times in the range 200 to 600 μ s. In our simulations, the disorder correlation length is $\zeta = 5a$. In Refs. [13,14], one finds $\zeta \sim 200$ –540 nm. Since a is determined by the laser wavelength used to create the lattice, and is typically in the range 700–800 nm [8,11], this would mean $\zeta \sim 0.3$ –0.7 a . However, producing speckle patterns with larger correlation lengths is not difficult. Finally, the disorder strength used in our simulations is $W/\hbar \approx 120$ Hz, while those reported in Refs. [13,14] are in the range 1–3 kHz. Here again, producing a weaker disorder strength is not an experimental issue.

IV. MAIN RESULTS

For weak disorder ($\tau_s \gg \hbar/|E_0|$), interband transitions are strongly suppressed, as explained from the scattering vertex properties (see Appendix B). The band index $s = \pm$ becomes a good quantum number and the dynamics is essentially restricted to the initial energy band. Choosing $E_0 < 0$, the spinor wave associated with the dynamics is then $|u_q^-\rangle = \sin\frac{\theta_q}{2}e^{-i\varphi}|\uparrow\rangle + \cos\frac{\theta_q}{2}|\downarrow\rangle$, with eigenenergy $-\varepsilon_q$. We have duly numerically checked that, with our parameters, the population in the positive-energy band is less than three

FIG. 1. (Color online) Snapshots of the diffuse momentum-space density $n_D(\mathbf{q}, t)$ (normalized to its maximum value) obtained at different times when Dirac quasiparticles are subjected to a correlated disorder with strength W and correlation length ζ . The initial momentum (white disks) is $\mathbf{k}_0 = \mathbf{K} + \mathbf{q}_0$ with $\mathbf{q}_0 = -q_0\hat{e}_x$, $q_0 > 0$, and the initial energy is chosen in the lower band, $E_0 = -\varepsilon_{q_0}$. The black and white dotted lines mark the energy shell, but leave out the region around $-\mathbf{q}_0$ for clarity. (a) π -flux square lattice with $m = 0$, $q_0\zeta = 2$, $q_0a = 0.41$, $W = 0.29|E_0|$, $|E_0|\tau_s/\hbar = 3.1$, and $|E_0|\tau_B/\hbar = 25$. The energy shell is filled symmetrically with respect to the x axis and develops a CBS dip at $-\mathbf{q}_0$ due to a π Berry phase. (b) π -flux square lattice with $mc^2 = 0.53|E_0|$, $q_0\zeta = 2$, $q_0a = 0.41$, $W = 0.24|E_0|$, $|E_0|\tau_s/\hbar = 2.4$, and $|E_0|\tau_B/\hbar = 21$. The energy shell is filled anisotropically at short times due to the interplay between dynamical and geometrical phase factors. (c) Graphene lattice with $mc^2 = 0.53|E_0|$, $q_0\zeta \approx 1.27$, $q_0a = 0.25$, $W = 0.37|E_0|$, $|E_0|\tau_s/\hbar = 2.35$, and $|E_0|\tau_B/\hbar = 5.9$. Due to trigonal warping, the energy shell is no longer inversion symmetric: interfering reciprocal paths are always off-shell and the observed fringe structure disappears in time. (d) Same as (c), except that $\mathbf{q}_0 = q_0\hat{e}_y$ where $q_0\zeta = 1.34$ and $q_0a = 0.27$. It shows that the transient asymmetry observed at a given energy around a given Dirac point does not depend on the polar angle of the initial momentum.

percent of the total population. Figure 1 shows our main numerical results for the momentum-space diffuse density $n_D(\mathbf{q}, t) = \langle \mathbf{k} | \rho_D(t) | \mathbf{k} \rangle$ ($\mathbf{k} = \mathbf{K} + \mathbf{q}$) of a coherent spinorial wave packet propagating in a disordered potential with *long-range* spatial correlation. An important contribution to the density $\rho_D(t)$ comes from the interference between amplitudes

FIG. 2. (Color online) (a) Pictorial interpretation of Figs. 1(b) and 1(c) in terms of real-space scatterings. There is a higher chance for the incoming massive Dirac quasiparticles to be scattered through the bottom (solid larger arrows) half-circle path than the upper (dashed smaller arrows) half-circle path. (b) Reciprocal scattering paths connecting on-shell momentum $\mathbf{q}_0$ to final momentum $\mathbf{q}$ and contributing to the maximally crossed series n_C . The dotted circle represents the energy shell, assumed here to be inversion symmetric with respect to the Dirac point for simplicity. Given the direct path (solid red), its reciprocal partner (dashed red) is obtained by inversion about $(\mathbf{q} + \mathbf{q}_0)/2$ (point I), implying $\mathbf{q}_i \rightarrow \mathbf{q}'_i = \mathbf{q} + \mathbf{q}_0 - \mathbf{q}_i$. Both paths cannot be simultaneously on-shell, except when $\mathbf{q} + \mathbf{q}_0 = 0$.

associated to counterpropagating scattering paths in real space. These correspond to a closed loop in momentum space and the parallel transport of the spin degree of freedom along this closed loop gives rise to a Berry phase entering the interference contribution. For massless Dirac particles living on the π -flux square lattice, the dynamics symmetrically fills the energy contours and one witnesses, at backscattering, the known destructive interference observed in magnetoresistance measurements [30] because of a π Berry phase. For massive Dirac particles, the dynamics exhibits a *transient* broken reflection symmetry driven by the interplay between dynamical and geometric phases acquired by scattering paths in momentum space. We illustrate the meaning of this transient broken symmetry in terms of real-space scatterings in Fig. 2(a): the incoming Dirac quasiparticles are backscattered by preferentially following clockwise or anticlockwise U-turn scattering paths, depending on the Dirac point considered. The same numerical observations are made for the honeycomb lattice with the important difference that trigonal warping blurs the interference effects as time increases since the energy contours around a given Dirac point are no longer inversion symmetric. Our theoretical analysis is based on the usual diagrammatic perturbation theory [32], but with the important difference that we account for dynamical phases as well as for off-energy-shell scattering paths. Both ingredients are missed in the usual diffusion approximation.

V. THEORY

At weak disorder, diagrammatic perturbation theory [32] predicts three main contributions to $n_D(\mathbf{q}, t)$: the single scattering background $n_S(\mathbf{q}, t)$ (which we will not address here), the diffusive background $n_L(\mathbf{q}, t)$ associated with multiple scattering path sequences $\Gamma_N = \{\mathbf{q}_0 \rightarrow \mathbf{q}_1 \rightarrow (\dots) \rightarrow \mathbf{q}_{N-1} \rightarrow \mathbf{q}\}$, and the maximally crossed contribution $n_C(\mathbf{q}, t)$ arising from the interference between Γ_N and its reciprocal

partner $\tilde{\Gamma}_N = \{q_0 \rightarrow q'_1 \rightarrow (\dots) \rightarrow q'_{N-1} \rightarrow q\}$ where $q'_i = -q_i + q_0 + q$ (see Appendix B for derivation); see Fig. 2(b). The contribution of N th-order scattering paths ($N \geq 1$) to $n_D(q, t)$ is

$$n_L^{(N)}(q, t) = \int [d\mu] e^{-\frac{t}{\tau_s}} |\Psi_{\Gamma_N}(t)|^2, \quad (2)$$

$$n_C^{(N)}(q, t) = \int [d\mu] e^{-\frac{t}{\tau_s}} \text{Re}[\Psi_{\Gamma_N}^*(t) \Psi_{\tilde{\Gamma}_N}(t)], \quad (3)$$

where $[d\mu] = P(q_0 - q_1) \prod_{i=1}^{N-1} \frac{dq_i}{(2\pi)^2} P(q_i - q_{i+1})$ is the integration measure with the convention $q_N = q$, and $P(q) = 2\pi\zeta^2 W^2 e^{-\frac{\zeta^2 q^2}{2}}$ is the Fourier transform of $F(\mathbf{r})$. Note that the correlation functions P in $[d\mu]$ typically enforce small momentum changes after each scattering, i.e., $|q_{i+1} - q_i| \lesssim 1/\zeta$. Defining $\omega_i = \varepsilon_{q_i}/\hbar$, the amplitude $\Psi_{\Gamma_N}(t)$ associated with the scattering path Γ_N is

$$\Psi_{\Gamma_N}(t) = \underbrace{\left[\int \dots \int_0^t \left(\prod_{i=0}^{i=N} dt_i e^{-i\omega_i t_i} \right) \delta \left(\sum_{j=0}^{j=N} t_j - t \right) \right]}_{\text{dynamical factor}} \times \underbrace{\left[\prod_{i=0}^{i=N-1} \langle u_{q_{i+1}}^- | u_{q_i}^- \rangle \right]}_{\text{spinor factor}}. \quad (4)$$

To get further physical insight into the problem, we assume $q_0\zeta \gg 1$, though this limit is not strictly satisfied for all the data presented here. In this case, the momentum changes satisfy $|q_{i+1} - q_i| \ll |q_0|$ and the spinor factor can be conveniently recast into a geometrical phase factor $\prod_{i=0}^{i=N-1} \langle u_{q_{i+1}}^- | u_{q_i}^- \rangle \approx \exp(i \int d\mathbf{q} \cdot \mathbf{A}_q^-)$, where the Berry connection $\mathbf{A}_q^- = i \langle u_q^- | \nabla_q u_q^- \rangle$ is the momentum-space analog of a vector potential. We have numerically checked that the dynamical factor vanishes at long times on a time scale set by the spread of intermediate energies (see Appendix C). The phase of this dynamical factor is well approximated by $(-E_N t/\hbar)$, where $E_N = \frac{1}{N+1} \sum_{i=0}^{i=N} \varepsilon_{q_i}$ is the average energy along the scattering path. The amplitude of the dynamical factor forces the various energies contributing to the scattering path to lie close to the energy shell. It can be easily checked that this result is exact for single scattering or in the long-time limit where all intermediate energies are equal.

As phases factors do not impact $|\Psi_{\Gamma_N}(t)|^2$, $n_L(q, t)$ symmetrically fills the energy shell with respect to q_0 in the course of time (see Fig. 1). A fully isotropic background $n_{bg}(q)$ is obtained after time scales set by the Boltzmann transport time τ_B [46]. In the long-time limit, this background is given by

$$n_{bg}(q) = \int \frac{dE}{2\pi} \frac{\mathcal{A}(q, E) \mathcal{A}(q_0, E)}{2\pi \nu(E)}, \quad (5)$$

where $\mathcal{A}(q, E) = 2\pi \langle \mathbf{K} + \mathbf{q} | \delta(E - H) | \mathbf{K} + \mathbf{q} \rangle$ is the spectral function and $\nu(E) = \int \frac{d\mathbf{q}}{(2\pi)^2} \mathcal{A}(q, E)$ is the density of states (DoS). Energy domains where $\nu(E)$ is small contribute more to the background. In our case, it corresponds to states with small $|q|$. The maximum of the background is thus slightly shifted from the energy shell $\varepsilon_q = E_0$ towards the Dirac point,

as witnessed in Figs. 1 and 8. On the other hand, we see from Eq. (3) that Γ_N and $\tilde{\Gamma}_N$ combine to form a *closed* trajectory $\mathcal{C}$ in momentum space with finite area [Fig. 2(b)]. In the large- N limit, this closed trajectory develops a dynamical phase $\phi_d = \Delta t/\hbar$ where

$$\Delta = \lim_{N \rightarrow \infty} (E'_N - E_N) = \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{i=0}^{i=N} (\varepsilon_{q'_i} - \varepsilon_{q_i}), \quad (6)$$

while the geometric phase is

$$\phi_g = \oint d\mathbf{q} \cdot \mathbf{A}_q^- = \pm \iint_S \Omega_q^- dS, \quad (7)$$

where the second integral is over a surface S in momentum space with boundary $\mathcal{C}$. The Berry curvature $\Omega_q^- = \nabla_q \times \mathbf{A}_q^- = \Omega_q^- \mathbf{e}_z$ is the momentum-space analog of a magnetic field [24]. The sign of ϕ_g is positive for anticlockwise paths and negative for the clockwise paths shown in Fig. 2. As a result, $n_L + n_C$ contains modulation terms such as $[1 + \cos(\phi_d + \phi_g)]$.

VI. MASSLESS PARTICLES: ANTI-CBS EFFECT

For massless Dirac particles ($m = 0$), $\Omega_q^- = \pi \delta(q)$ and $\phi_g = \pm\pi$ for all paths that enclose the Dirac point. Furthermore, at exact backscattering ($q + q_0 = 0$), ϕ_d vanishes identically and n_C always gives a destructive contribution resulting in a density dip at $-q_0$ (anti-CBS effect); see Figs. 1(a), 3, and 8. Constructive interference peaks can be inferred from the condition $\cos \phi_d = -1$. They are located symmetrically with respect to the x axis and move towards $-q_0$ in the course of time; see Figs. 1(a) and 3.

FIG. 3. (Color online) Normalized momentum-space density $n_D(q, t)/n_L(q_0, t)$ of massless Dirac particles on a π -flux square lattice as a function of the polar angle φ along the circle $\mathbf{q} = q_0(\cos \varphi, \sin \varphi)$ at three different times. The simulation parameters are the same as for Fig. 1(a). To extract the background (red dashes), we use the quadratic fit $[1 + C(t) \times \varphi^2]$ and find $C(t)$ by matching the wings of $n_D(q, t)$ in the angular range $0.2 < |\varphi|/\pi < 0.5$. The (blue) vertical line marks the backscattering direction. The dip at backscattering reflects the destructive interference of reciprocal path amplitudes due to the π Berry phase.

FIG. 4. (Color online) Normalized momentum-space density $n_D(\mathbf{q}, t)/n_L(\mathbf{q}_0, t)$ of massive Dirac particles on a π -flux square lattice as a function of the polar angle φ along the circle $\mathbf{q} = \mathbf{q}_0(\cos \varphi, \sin \varphi)$ with $|\mathbf{q}_0|a = 0.41$ at three different times. The rest mass energy is $mc^2 = 0.3|E_0|$ (equivalently, $m = 0.6m^*$ where $m^*c^2 = |E_0|/2$; see text), while the other simulation parameters are the same as for Fig. 1(a). The scattering time is $\tau_s = 2.5\hbar/|E_0|$. To extract the background (red dashes), we use the quadratic fit $[1 + C(t) \times \varphi^2]$ and find $C(t)$ by matching the wings of $n_D(\mathbf{q}, t)$ in the angular range $0.2 < |\varphi|/\pi < 0.5$. The blue vertical line marks the backscattering direction. The (blue) dotted horizontal lines give the magnitude of the density $(1 + \cos \phi_g)$ that is expected in the long-time limit at backscattering. As $m < m^*$, $(1 + \cos \phi_g) < 0$ and the interference between reciprocal path amplitudes is destructive, giving rise to a dip at backscattering in the long-time limit.

VII. MASSIVE PARTICLES: TRANSIENT BROKEN SYMMETRY

For massive Dirac particles ($m \neq 0$), the picture is different. Now the Berry curvature reads [24]

$$\Omega_q^- = \frac{\cos \theta_q}{2Q^2} = \frac{q_m}{2(q^2 + q_m^2)^{3/2}}. \quad (8)$$

The reflection symmetry is broken and the interference peak on clockwise paths is much greater than the other one; see Figs. 1(b) and 1(c), as well as Figs. 4–6 for the density as a function of the polar angle on the energy shell. This asymmetry is easily understood since clockwise and anticlockwise paths have same (positive) dynamical phases but opposite geometrical phases. Thus, as long as $\Omega_0 \neq 2\pi$ (massive case), $\cos(\phi_d - |\phi_g|) > \cos(\phi_d + |\phi_g|)$, which explains the observed asymmetry. One may note that the asymmetry pattern does not depend on the sign of E_0 and is thus the same for the two bands above and below the charge neutrality point. For the graphene lattice, the dynamics around the two inequivalent Dirac points display opposite asymmetries because the associated Berry curvatures are opposite. We stress that the observed asymmetry cannot be explained if the interfering scattering paths are both on-shell, as the dynamic phase of the closed trajectory would identically vanish leaving the interference contribution insensitive to the sign of ϕ_g .

FIG. 5. (Color online) Same as Fig. 4, but for a rest mass energy $mc^2 = 0.5|E_0|$ ($m = m^*$; see text). In the long-time limit, we expect the interference pattern to disappear since $1 + \cos \phi_g = 0$.

The asymmetry is best seen at times t comparable to τ_B , when the interference peak can be resolved from the diffusive background. Figure 7(a) shows the numerically extracted peak position as a function of time, while Fig. 7(b) shows a rough estimate obtained from the “bright fringe” condition $\cos(\phi_d - |\phi_g|) = 1$, where ϕ_d and ϕ_g are computed along the path maximizing the product of the disorder correlation function and of the dynamical factor with respect to the intermediate momenta (see Appendix D). This bright fringe

FIG. 6. (Color online) Same as Fig. 4 but for a rest mass energy $mc^2 = 0.75|E_0|$ ($m = 1.5m^*$; see text). The (blue) dotted horizontal lines give the magnitude of the density $(1 + \cos \phi_g)$ that is expected in the long-time limit at backscattering. As $m > m^*$, $(1 + \cos \phi_g) > 0$ and the interference between reciprocal path amplitudes is now constructive, giving rise to a peak at backscattering in the long-time limit.

FIG. 7. (Color online) Time evolution of the azimuthal angle φ (in units of π) of the interference peak observed in momentum space for the π -flux square lattice when $q_0\zeta = 2$, $q_0a = 0.41$. Starting from $\varphi = \pi$ (initial momentum $\mathbf{q}_0$), it decreases and reaches $\varphi = 0$ (CBS momentum $-\mathbf{q}_0$) in the long-time limit. The full symbols in (a) show the numerical values, while the corresponding open symbols in (b) give the theoretical values derived from the “bright fringe” condition $\cos(\phi_d - |\phi_g|) = 1$. Blue circles: $W = 0.29|E_0|$ and $m = 0$. In this case, there is another interference peak at $-\varphi$. Green squares: $W = 0.27|E_0|$ and $mc^2 = 0.3|E_0|$. Red diamonds: $W = 0.24|E_0|$ and $mc^2 = 0.53|E_0|$. Dashed lines are the algebraic predictions computed with Eq. (9).

estimate is backed up by an analytical prediction [47]:

$$\varphi(t) = \frac{\pi^2}{2(1 + \cos \theta_{q_0})} \frac{\hbar}{|E_0|t}. \quad (9)$$

The agreement with numerics is qualitatively good (see Fig. 7), with discrepancies of less than a factor of 2 at most.

VIII. MASSIVE PARTICLES: INTERFERENCE CORRECTIONS AT BACKSCATTERING

At backscattering, $\phi_g = \pm\Omega_0/2$, where $\Omega_0 = 2\pi(1 - \cos \theta_{q_0})$ is the solid angle of the cone subtended by the vector $\mathbf{Q}_0 = (-\mathbf{q}_0, q_m)$ around the z axis. At a sufficiently long time ($t \gg \tau_B$), the diffusion approximation applies and the diffusive background becomes angularly flat, $n_L = n_{bg}(q_0)$. The interference contribution turns constructive when $\Omega_0 \leq \pi$, i.e., for rest mass energies larger than $m_*c^2 = |E_0|/2$ (equivalently for Compton wave numbers larger than $m_*c/\hbar = \sqrt{3}q_0$). We thus observe a density *dip* at $\varphi = 0$ for $m < m^*$ (see Fig. 4) but a *peak* when $m > m^*$ (see Fig. 6), both with a magnitude of $1 + \cos \phi_g$ relative to n_L . At intermediate time ($t \lesssim \tau_B$), we always see a density dip located in the negative-angle sector (and, respectively, a density peak located in the positive-angle sector) moving towards $\varphi = 0$ as time increases. This observation is consistent with our analysis of the transient dynamics using the optimized path that maximizes the product of the disorder correlation function and of the dynamical factor (see Appendix D). However, our optimized-path approach incorrectly predicts that the density at backscattering ($\varphi = 0$) should have a relative magnitude of $(1 + \cos \phi_g)$, which is

not what is observed at intermediate times; see, e.g., the top two panels in Fig. 4. The reason behind the discrepancy is the contribution of other scattering paths. In particular, when $m < m^*$, the optimized path gives a negative contribution to n_C due to a geometrical phase $\phi_g \approx \pm\Omega_0/2$. However, the straight line in momentum space connecting $\mathbf{q}_0$ to $-\mathbf{q}_0$ is another extremal path and it has zero dynamical and geometric phases. This path gives thus a positive contribution to n_C and impacts the transient dynamics by reducing the magnitude of n_C at backscattering. Surprisingly, these additional scattering paths dress the density dip at negative φ with the relative magnitude $(1 + \cos \phi_g)$, even at intermediate times (see Fig. 4). The moving dip thus features the expected correct dip at backscattering in the long-time limit. We have also checked these observations for $m = 0.8m^*$ (data not shown).

IX. EFFECT OF TRIGONAL WARPING ON A HONEYCOMB LATTICE

An astute reader may have noticed the triangular shape of the energy contour in Fig. 1(c) for the graphene case. This trigonal warping [35,48] breaks the inversion symmetry of the energy contours around each Dirac point and prevents

FIG. 8. (Color online) Diffuse momentum-space density $n_D(\mathbf{q}, t)$ along the x axis for massless Dirac particles on the honeycomb lattice in the presence of trigonal warping. The times (in units of $\hbar/|E_0|$) are shown next to the symbols in the insets. Left panel: $q_0\zeta = 1.27$, $q_0a = 0.25$, $W = 0.44|E_0|$, $|E_0|\tau_s/\hbar = 2$, and $|E_0|\tau_B/\hbar = 9$. The three vertical dashed lines, from left to right, mark the initial momentum $\mathbf{q}_0$, the CBS momentum $-\mathbf{q}_0$ and the on-shell momentum $\mathbf{q}'_0$ opposite to $\mathbf{q}_0$ with respect to the Dirac point. The CBS dip is centered at $-\mathbf{q}_0$ but not at $\mathbf{q}'_0$ as marked by the dashed vertical lines. Right panel: same as left panel but for $q_0\zeta = 1.86$, $q_0a = 0.37$. One has $|E_0|\tau_s/\hbar = 3$ and $|E_0|\tau_B/\hbar = 15.6$. In this case, the effect of trigonal warping is stronger: $\mathbf{q}'_0$ is further away from $-\mathbf{q}_0$ but the CBS dip remains centered at $-\mathbf{q}_0$. Note that, as it should, the background maximum is located at a momentum slightly lower than $\mathbf{q}'_0$ but larger than $-\mathbf{q}_0$ due to DoS effects (see text).

q_0 and $-q_0$ from lying on the same energy shell. No direct and reverse paths can then be simultaneously on-shell and the CBS dip observed for $m = 0$ becomes a transient effect, disappearing in the course of time (see Fig. 8). The situation gets worse as the initial energy E_0 is chosen further away from the charge neutrality point. Surprisingly enough, this transient CBS dip still occurs at $-q_0$ and not at the on-shell point q'_0 opposite to q_0 . This is confirmed by a path-optimization procedure (see Appendix E). However, a clear understanding of this observation is still missing.

X. DISCUSSION AND CONCLUSION

We have highlighted the nontrivial interplay between dynamical and topological phases showing up in the interference corrections to transport of Dirac particles subjected to long-range correlated spatial disorder. Since the disorder is correlated, intervalley scattering is suppressed and the dynamics in momentum space takes place around one given Dirac point. At short times, this interplay leads to a backscattering process where incoming particles preferentially follow a clockwise or an anticlockwise U-turn path. At longer times, the backscattering probability is either enhanced or suppressed, depending on the Berry phase accumulated along scattering paths circling the Dirac point once. This accumulated Berry phase depends on the mass of the Dirac particles. In the case of a honeycomb lattice, trigonal warping turns this modification of the backscattering probability into a transient effect. This interplay of dynamical and geometrical phases could be experimentally addressed using cold atoms loaded in a π -flux square or a graphenelike optical lattice. In the context of solid-state devices, detection of an asymmetric scattering effect depending on the valley index could foster applications in valleytronics [49].

In the graphene community, it is known that the absence of intervalley scattering implies the absence of localization effects; see [37] and references therein. This is also what we find here for the honeycomb lattice. This can simply be understood because suppression of bulk transport imposes zero momentum on average, which cannot be the case if the dynamics is restricted to a single Dirac cone. However, since intervalley is not totally suppressed, Dirac cones can couple at much longer times and one might expect Anderson localization (AL) to take place in the end. Previous works [50–54] have numerically demonstrated AL on a honeycomb lattice when the disorder potential is δ correlated, in which case a single-scattering event is sufficient to backscatter an incoming Dirac particle to the opposite Dirac cone. The situation for correlated potentials may prove, however, more subtle, in particular for the π -flux square lattice. As such, an interesting extension of our present work would be to study the coherent forward scattering (CFS) effect [55,56] in systems where lattice effects are crucial. In bulk systems, the CFS effect arises if wave motion is bounded in space by AL. Importantly, it is shown [57] that the CFS effect in quasi-one-dimensional systems survives a magnetic field, whereas the CBS peak does not because of broken time-reversal symmetry. This implies that to address the fate of AL localization, the CFS should be a better indicator of AL than CBS.

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APPENDIX A: CLEAN LATTICE HAMILTONIANS

Our numerical simulations have been carried out using a tight-binding model on either a π -flux square lattice or on the honeycomb lattice; see Fig. 9 for our conventions. These two lattice models lead to the same low-energy effective Dirac Hamiltonian given by Eq. (1) in the main text. The tight-binding Hamilton operator is given by

$$H_{\text{lat}} = -J \sum_{\langle i,j \rangle} (e^{iA_{ij}} c_i^\dagger c_j + \text{H.c.}) + \delta \left(\sum_{i \in \text{A}} n_i - \sum_{j \in \text{B}} n_j \right), \quad (\text{A1})$$

where $J \geq 0$ is the hopping amplitude and 2δ is the on-site energy imbalance between nearest-neighbor sites. Here, $c_i^\dagger$ (c_i) is the creation (annihilation) operator at lattice site i , $n_i = c_i^\dagger c_i$ is the corresponding number operator, H.c. means Hermitian conjugate, and $\langle i,j \rangle$ restricts the summation to nearest-neighbor pairs. A_{ij} is identically zero for the honeycomb

FIG. 9. (Color online) (a) Honeycomb lattice and its hexagonal and rectangular (in red) Brillouin zone. The two inequivalent Dirac points are located at $\mathbf{K} = (\mathbf{b}_1 + \mathbf{b}_2)/3 = \frac{4\pi}{3\sqrt{3}a}\hat{\mathbf{e}}_x$ and at $\mathbf{K}' = -\mathbf{K}$ where the $\mathbf{b}_i$'s are the reciprocal lattice vectors and a is the lattice constant. (b) π -flux square lattice with its accumulated phase factors around a plaquette and its square Brillouin zone. There is a Dirac point located at the band center Γ at $\mathbf{K} = 0$ and at the band edge at $\mathbf{K}' = (\mathbf{b}_1 + \mathbf{b}_2)/2 = \frac{\pi}{a}\hat{\mathbf{e}}_x$. Both lattices are bipartite with A-type (full circles) and B-type (open circles) sites, each sublattice being a Bravais lattice.

lattice, whereas $\sum_{\square} A_{ij} = \pi$ for the π -flux square lattice, where $\square$ denotes an elementary anticlockwise plaquette.

Defining the creation operator $f_{k,\sigma}^\dagger$ of Bloch states living on sublattices $\sigma = A, B$ through

$$f_{k\sigma}^\dagger = \frac{1}{\sqrt{N_c}} \sum_{i \in \sigma} e^{i\mathbf{k} \cdot \mathbf{r}_i} c_i^\dagger, \quad (\text{A2a})$$

$$c_{i \in \sigma}^\dagger = \frac{1}{\sqrt{N_c}} \sum_{\mathbf{k} \in \Omega} e^{-i\mathbf{k} \cdot \mathbf{r}_i} f_{k\sigma}^\dagger, \quad (\text{A2b})$$

where N_c is the number of unit cells and Ω is the Brillouin zone, the lattice Hamilton operator can be recast as

$$H_{\text{lat}} = \sum_{\mathbf{k} \in \Omega} (f_{k,A}^\dagger, f_{k,B}^\dagger) \begin{pmatrix} \delta & Z_k \\ Z_k^* & -\delta \end{pmatrix} \begin{pmatrix} f_{k,A} \\ f_{k,B} \end{pmatrix}, \quad (\text{A3})$$

where

$$Z_k = -J e^{-i \frac{k_y a}{2}} \left[e^{i \frac{3k_y a}{2}} + 2 \cos \left(\frac{\sqrt{3} k_x a}{2} \right) \right] \quad (\text{A4})$$

for the honeycomb lattice and

$$Z_k = -2J [\sin(k_x a) - i \sin(k_y a)] \quad (\text{A5})$$

for the π -flux square lattice. The two inequivalent Dirac points are found by solving $Z_k = 0$. They are $\mathbf{K} = \frac{4\pi}{3\sqrt{3}a} \hat{\mathbf{e}}_x$ and $\mathbf{K}' = -\mathbf{K}$ for the honeycomb lattice and $\mathbf{K} = 0$ and $\mathbf{K}' = \frac{\pi}{a} \hat{\mathbf{e}}_x$ for the π -flux square lattice. Irrespective of the lattice, $|Z_k|$ is always inversion symmetric under $\mathbf{k} \rightarrow -\mathbf{k}$. However, one may note that when the inversion symmetry is made about any of the Dirac points, e.g., $(\mathbf{K} + \mathbf{q}) \rightarrow (\mathbf{K} - \mathbf{q})$, this “local” property only keeps true for the π -flux square lattice.

Defining $\Phi_k = -\text{Arg}(Z_k)$, the eigenenergies are $\varepsilon_{ks} = s \varepsilon_k = s \sqrt{\delta^2 + |Z_k|^2}$ (where $s = \pm$ is the band index) and the corresponding eigenvectors are

$$|k+\rangle = \cos(\theta_k/2) |kA\rangle + e^{i\Phi_k} \sin(\theta_k/2) |kB\rangle, \quad (\text{A6})$$

$$|k-\rangle = -e^{-i\Phi_k} \sin(\theta_k/2) |kA\rangle + \cos(\theta_k/2) |kB\rangle, \quad (\text{A7})$$

with $\cos \theta_k = \delta/\varepsilon_k$. Identifying the Bloch waves $|kA\rangle$ and $|kB\rangle$ as up and down components $|\uparrow\rangle$ and $|\downarrow\rangle$, we define the pseudospin spinors as

$$|u_{k+}\rangle = \begin{pmatrix} \cos(\theta_k/2) \\ \sin(\theta_k/2) e^{i\Phi_k} \end{pmatrix}, \quad (\text{A8})$$

$$|u_{k-}\rangle = \begin{pmatrix} -\sin(\theta_k/2) e^{-i\Phi_k} \\ \cos(\theta_k/2) \end{pmatrix}, \quad (\text{A9})$$

with $\langle u_{ks} | u_{ks'} \rangle = \delta_{ss'}$. One should note that the eigenenergies of the π -flux square lattice always display the inversion symmetry $\mathbf{k} \rightarrow -\mathbf{k}$, a feature which proves central when discussing the anti-CBS effect at long times.

Expanding $\mathcal{H}_k$ around the Dirac points, i.e., $\mathbf{q} = \mathbf{k}$ for the π -flux square lattice and $\mathbf{q} = \mathbf{k} - \mathbf{K}$ for the honeycomb lattice ($qa \ll 1$), we recover the Dirac Hamiltonian [Eq. (1) in main text] at lowest order in qa , with $c = 2Ja/\hbar$ and $c = 3Ja/(2\hbar)$ for the square and honeycomb lattice, respectively. The mass of the Dirac particles is defined as $\delta = mc^2$ and the corresponding spinor waves are obtained by setting $\Phi_k = \varphi_q + \pi$ where φ_q is the polar angle of $\mathbf{q}$. The dispersion relation becomes $\varepsilon_q = \hbar c \sqrt{q^2 + q_m^2}$ with

$q_m = mc/\hbar$. When higher-order terms are considered, trigonal warping sets in for the honeycomb case and destroys the inversion symmetry $\mathbf{q} \rightarrow -\mathbf{q}$ (equivalently, $\varphi_q \rightarrow \varphi_q + \pi$), implying $\varepsilon_{-\mathbf{q}} \neq \varepsilon_{\mathbf{q}}$, while this inversion symmetry keeps exact for the π -flux square lattice. This is clearly seen in the Taylor expansion of $|Z_k|$ when the next-order term is considered:

$$|Z_k/J| \approx 2qa - \frac{\cos^4 \varphi_q + \sin^4 \varphi_q}{3} (qa)^3 \quad (\text{A10})$$

for the π -flux square lattice, while

$$|Z_k/J| \approx \frac{3qa}{2} + \frac{3(\cos \varphi_q \sin^2 \varphi_q - \cos^3 \varphi_q)}{8} (qa)^2 \quad (\text{A11})$$

for the honeycomb lattice.

APPENDIX B: SCATTERING VERTEX AND MAXIMALLY CROSSED SERIES

The disorder potential reads $V = \sum_{i \in A} n_i + \sum_{i \in B} n_i$. Since we assume Gaussian statistics, all its odd-order moments vanish and its even-order moments boil down, through Wick's theorem, to products of two-point correlators. Thus the two-point correlator is the only key ingredient in the diagrammatic expansion of the Dyson and Bethe-Salpeter equations.

A straightforward calculation show that

$$\overline{\langle k_2 s_2 | V | k_1 s_1 \rangle \langle k_3 s_3 | V | k_4 s_4 \rangle} = \delta_{k_1+k_4, k_2+k_3} P(\mathbf{k}_1 - \mathbf{k}_2) \langle u_{k_2 s_2} | u_{k_1 s_1} \rangle \langle u_{k_3 s_3} | u_{k_4 s_4} \rangle, \quad (\text{B1})$$

where P is the Fourier transform of the disorder two-point spatial correlator. Since we assume here that the disorder correlation length is much larger than the lattice constant ($\zeta \gg a$), P is peaked around 0 and $\mathbf{k}_1 \approx \mathbf{k}_2$. In turn, because of momentum conservation, $\mathbf{k}_3 \approx \mathbf{k}_4$. This enforces $\langle u_{k_2 s_2} | u_{k_1 s_1} \rangle \langle u_{k_3 s_3} | u_{k_4 s_4} \rangle \approx \delta_{s_1 s_2} \delta_{s_3 s_4}$, implying that the band index is an approximate good quantum number. Similar considerations show that the self-energy is also quasidiagonal in the band index, at least at the Born approximation.

As an illustrative example, Fig. 10 shows the third-order term in the maximally crossed series for incoming and outgoing Dirac momenta $\mathbf{q}$ and $\mathbf{q}'$. The band index being approximately conserved here, it is not specified in the figure. Applying momentum conservation at each scattering vertex, it is easy to show that an intermediate Dirac momentum $\mathbf{q}_i$ in one scattering path is mapped onto $\mathbf{q}'_i = \mathbf{q} + \mathbf{q}' - \mathbf{q}_i$ in the reciprocal path. Reciprocal scattering paths connecting $\mathbf{q}$ to $\mathbf{q}'$ in momentum space are images of each other through the inversion about $(\mathbf{q} + \mathbf{q}')/2$; see Fig. 2 in main text.

APPENDIX C: DYNAMICAL FACTOR

With the change of variables $t_i \rightarrow y_i = t_i/t$, the dynamical factor [Eq. (4) in main text] is written as

$$D_N(t) = t^N \int \cdots \int_0^1 \left(\prod_{i=0}^{i=N} dy_i \right) \times \delta(\sqrt{N+1} \mathbf{v} \cdot \mathbf{y} - 1) e^{-i\mathbf{x} \cdot \mathbf{y}}, \quad (\text{C1})$$

FIG. 10. Third-order maximally crossed diagrams for incoming and outgoing wave vectors $\mathbf{q}$ and $\mathbf{q}'$. Full circles connected by a dashed line correspond to a scattering vertex. Solid lines between two consecutive full circles represent average Green's function $\bar{G}$ solving Dyson equation (retarded for the upper lines, advanced for the lower lines). At each vertex, momentum conservation is applied. As one can easily see, if $\mathbf{q}_i$ ($i = 1, 2$ here) is an intermediate momentum for the upper line, then the corresponding momentum in the lower line is $\mathbf{q}'_i = \mathbf{q} + \mathbf{q}' - \mathbf{q}_i$. This means that reciprocal paths connecting $\mathbf{q}$ to $\mathbf{q}'$ are thus images of each other through an inversion about $(\mathbf{q} + \mathbf{q}')/2$.

where $\mathbf{x} = (\omega_0 t, \dots, \omega_N t)$ and $\mathbf{v} = \frac{1}{\sqrt{N+1}}(1, \dots, 1)$ is the hypercube diagonal unit vector. We write $\mathbf{y} = y_{\parallel} \mathbf{v} + \mathbf{r}$ and $\mathbf{x} = x_{\parallel} \mathbf{v} + \mathbf{x}_{\perp}$, where $\mathbf{r}$ and $\mathbf{x}_{\perp}$ are orthogonal to $\mathbf{v}$. We have $x_{\parallel} = \sqrt{N+1} E_N t / \hbar$ where $E_N = \frac{1}{N+1} \sum_{j=0}^N \varepsilon_{q_j}$. Integrating out the δ distribution, one easily gets

$$D_N(t) = \frac{t^N V_N e^{-i E_N t / \hbar}}{\sqrt{N+1}} \int_{\Omega_N} \frac{(d\mathbf{r})}{V_N} e^{-i \mathbf{x}_{\perp} \cdot \mathbf{r}}. \quad (\text{C2})$$

The integral runs over the regular N -simplex Ω_N obtained as the intersection of the $(N+1)$ hypercube with the hyperplane orthogonal to $\mathbf{v}$ located at $y_{\parallel} = \frac{1}{\sqrt{N+1}}$, $(d\mathbf{r})$ is the N -volume element, while $V_N = \frac{\sqrt{N+1}}{N!}$ is the volume of the N simplex.

For $\mathbf{x}_{\perp}^2 \ll (N+1)(N+2)$, we numerically find that

$$\int_{\Omega_N} \frac{(d\mathbf{r})}{V_N} e^{-i \mathbf{x}_{\perp} \cdot \mathbf{r}} \approx e^{-\frac{\mathbf{x}_{\perp}^2}{2(N+1)(N+2)}}. \quad (\text{C3})$$

Since $\mathbf{x}_{\perp}^2 = \mathbf{x}^2 - x_{\parallel}^2 = (N+1)\sigma_E^2 t^2 / \hbar^2$, we get

$$D_N(t) \approx \frac{t^N}{N!} e^{-i E_N t / \hbar} e^{-\frac{\sigma_E^2 t^2}{2\hbar^2(N+2)}}, \quad (\text{C4})$$

where the energy spread reads

$$\sigma_E^2 = \frac{1}{N+1} \sum_{i=0}^N \varepsilon_{q_i}^2 - E_N^2. \quad (\text{C5})$$

We conclude that, for a given scattering path, (i) the dynamical phase is approximated by $(-E_N t / \hbar)$, and (ii) the dynamical factor decays at a rate determined by the spread of the intermediate energies.

APPENDIX D: OPTIMIZATION PROCEDURE

Within the previous approximation, we get

$$n_C(\mathbf{q}, t) \approx e^{-\frac{t}{\tau_s}} \sum_N \frac{t^{2N}}{(N!)^2} I_N(\mathbf{q}, t), \quad (\text{D1})$$

FIG. 11. (Color online) Plot of the minimized function $F_{\min}$ as a function of the final momentum along the x axis (in units of $1/\zeta$) for the honeycomb lattice and at $q_0 \zeta = 1.27$. The orange vertical dashed lines mark the momenta $\mathbf{q} = -\mathbf{q}_0$ and $\mathbf{q} = \mathbf{q}'_0$ (on-shell momentum opposite to $\mathbf{q}_0$). Noticeably, the minimum of $F_{\min}$ is located at $\mathbf{q} = -\mathbf{q}_0$, not at $\mathbf{q}'_0$, and is also the momentum where a density dip develops for $n_D(\mathbf{q}, t)$; see left panels of Fig. 4 in the main text.

$$I_N(\mathbf{q}, t) = \int [d\mu] e^{-\frac{(\sigma_E^2 + \sigma_F^2) \mu^2}{2(N+2)\hbar^2}} \cos(\Delta_N t - \Phi_g^{(N)}). \quad (\text{D2})$$

We infer the interference peak location from the condition $\cos(\Delta_N t - \Phi_g^{(N)}) = 1$ obtained with the path $\{\mathbf{q}_0, \mathbf{q}_1, \dots, \mathbf{q}_{N-1}, \mathbf{q}_N = \mathbf{q}\}$ maximizing the weight of the interference term with respect to the intermediate momenta $\mathbf{q}_i$ ($i = 1, \dots, N-1$). This is achieved by maximizing the product F of disorder correlation functions in the integration

FIG. 12. (Color online) Same as Fig. 11, but for $q_0 \zeta = 1.86$. In this case, the energy shell is further away from the Dirac point and the trigonal warping is a more prominent effect. Here again the minimum is located at $-\mathbf{q}_0$, which is where the density dip is witnessed; see the right panels of Fig. 4 in the main text.

measure $[d\mu]$ with the Gaussian factor related to the energy spreads:

$$F = \frac{\zeta^2}{2} \sum_{i=0}^{i=N-1} (\mathbf{q}_{i+1} - \mathbf{q}_i)^2 + \frac{(\sigma_E^2 + \sigma_E'^2)t^2}{2(N+2)\hbar^2}. \quad (\text{D3})$$

Here, σ_E and σ_E' are the energy spreads along the direct path and its reciprocal partner; see Eq. (C5). As we only expect a crude estimate for the peak position, we have further assumed $\mathbf{q}$ to lie on the energy shell. Results presented in Fig. 3 of the main text have been obtained using $N \approx t/\tau_s$, where the scattering mean free time τ_s is estimated from the time decay of the ballistic component. For smaller times, the path would be pulled slightly towards the Dirac point near $\mathbf{q}_0$ and $\mathbf{q}$ because

the disorder correlation function then plays a more significant role.

APPENDIX E: CBS POINT AND TRIGONAL WARPING

For each $\mathbf{q}$ lying on the x axis, we minimize the function F , defined in Eq. (D3), with respect to the intermediate momenta $\mathbf{q}_i$, as explained in the previous section. For the parameters investigated in this work, the minimized function $F_{\min}$ always shows a minimum at $\mathbf{q} = -\mathbf{q}_0$; see Figs. 11 and 12. Since the density dip in n_D also occurs at $\mathbf{q} = -\mathbf{q}_0$, we conclude that, even in the presence of trigonal warping, $-\mathbf{q}_0$ remains the CBS momentum. We also note that $F_{\min}$ assumes larger values when $q_0\zeta$ gets larger. This observation is consistent with Fig. 4 in the main text, which shows that the CBS dip vanishes at a faster rate for $q_0\zeta = 1.86$ than for $q_0\zeta = 1.27$.

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