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HIGH DIMENSIONAL FINITE ELEMENTS FOR MULTISCALE WAVE EQUATIONS*

BINGXING XIA[†] AND VIET HA HOANG[†]

Abstract. For locally periodic multiscale wave equations in $\mathbb{R}^d$ that depend on a macroscopic scale and n microscopic separated scales, we solve the high dimensional limiting multiscale homogenized problem that is posed in $(n+1)d$ dimensions and is obtained by multiscale convergence. We consider the full and sparse tensor product finite element methods, and analyze both the spatial semidiscrete and the fully (both temporal and spatial) discrete approximating problems. With sufficient regularity, the sparse tensor product approximation achieves a convergence rate essentially equal to that for the full tensor product approximation, but requires only an essentially equal number of degrees of freedom as for solving an equation in $\mathbb{R}^d$ for the same level of accuracy. For the initial condition $u(0, x) = 0$, we construct a numerical corrector from the finite element solution. In the case of two scales, we derive an explicit homogenization error which, together with the finite element error, produces an explicit rate of convergence for the numerical corrector. Numerical examples for two- and three-scale problems in one or two dimensions confirm our analysis.

Key words. multiscale wave equations, homogenization, finite element, high dimension, sparse tensor product, corrector

AMS subject classifications. 35B27, 35L20, 65N30, 74Q10

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1. Introduction. Wave propagation in heterogeneous media arises in many important practical and engineering areas. As for other multiscale problems, a direct numerical simulation requires resolving all the fine scales and needs a mesh size that is at most of the order of the smallest scale; therefore it is prohibitively expensive. The theory of homogenization studies the limit when the fine scales converge to zero which approximates the solution to the original multiscale problem macroscopically. To establish the limiting homogenized equation, we need the solutions of local cell problems (see, for example, Benssousan, Lions, and Papanicolaou [4]). The total number of degrees of freedom and floating point operations needed for solving these cell problems can be exceedingly large, especially when the multiscale problem is only locally periodic and when it depends on many microscopic scales.

For multiscale wave equations, there have been some attempts to address this issue though when comparing to elliptic and parabolic equations, the literature is still quite limited. Owhadi and Zhang [22] proposed a general method for d dimensional problems that solves d multiscale elliptic problems and uses their solutions to transform a macroscopic scale basis to a multiscale one. Though general, the complexity of establishing this basis is comparable to that for directly solving the multiscale problem. In [18], Jiang, Efendiev, and Ginting studied a numerical procedure based on the Multiscale Finite Element (FE) method [17] to solve wave equations with continuum spatial scales assuming limited global information. The method is quite general, but the complexity of constructing the multiscale FE basis can be rather high. Using the

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Heterogeneous Multiscale Method [9], Engquist, Holst, and Runborg [10] developed a finite difference method for multiscale wave equations. They also performed computation for large time that shows dispersive behavior similar to that in Santosa and Symes [23]. The Heterogeneous Multiscale Method for multiscale wave equations was developed by Abdulle and Grote in [1] to compute the solution of the homogenized equation. They show that the number of degrees of freedom required is independent of the scale but it grows superlinearly when the discretizing mesh size decreases.

In this paper, we develop the high dimensional approach for locally periodic wave equations that depend on multiple separated scales. The method is based on the sparse high dimensional finite element method developed in Hoang and Schwab [15] for elliptic equations (see also [14] and [13]) which solves the multiscale homogenized problems obtained via multiscale convergence [21, 2, 3]. These problems are posed in a tensorized domain and depend on $n + 1$ d dimensional variables when the multiscale problems are posed in $\mathbb{R}^d$ and depend on one macroscopic scale and n microscopic scales. In comparison to the methods mentioned above, the number of degrees of freedom required grows only log-linearly when the discretizing mesh decreases while the rate of convergence is essentially equal to that for the full tensor product finite elements, whose complexity grows superlinearly. Apart from the log term, the rate of convergence and the complexity are uniform with respect to the number of scales. Further, as shown in [15], the resulting linear system at each time step can be solved by the conjugate gradient method that requires only a log-linear number of floating point operations and a log-linear number of memory units with respect to the number of degrees of freedom. We obtain the solution of the homogenized equation which describes the multiscale solution macroscopically, and the scale interacting terms which provide the microscopic behavior. Restricting our consideration to the case where the initial condition g_0 in (2.5) equals 0, we construct a numerical corrector from the finite element approximations of these scale interacting terms for the multiscale problem. The method thus provides a feasible approach to compute numerically both the solution to the homogenized equation and the corrector. For two-scale problems, we deduce a homogenization rate of convergence, which, together with the finite element rate of convergence, implies an explicit rate of convergence for the numerical corrector. For general initial conditions, as shown in [5], it is more complicated to construct correctors for multiscale wave equations than for elliptic equations as the energy of the multiscale wave equations does not always converge to the energy of the homogenized equations. However, these scale interacting terms always form a part of the corrector, albeit another term which requires that solving a separate multiscale problem is needed (see [5]).

This paper is organized as follows. In the next section, we define the multiscale wave equation and recall the concept of multiscale convergence applying to time dependent problems. We then deduce the high dimensional limiting multiscale homogenized equation of the wave equation. We study the approximation of the multiscale homogenized equation by using general finite element spaces in section 3 where we consider both the spatial semidiscrete and the fully discrete problems. For the convergence of the general approximating schemes in section 3, and to obtain the convergence rates for the particular cases of full and sparse tensor FE approximations in section 5, we require sufficient regularity of the solution of the high dimensional multiscale homogenized problem. We show that the regularity can be achieved with sufficiently regular coefficients and domains in section 4. In section 5, we consider the full tensor product and the sparse tensor product approximations. When the solution is sufficiently regular, we derive explicit rates of convergence for the finite element

approximations. In particular, we show that while the number of degrees of freedom for the sparse tensor product approximation is essentially (within a log term) equal to that for solving an equation that depends only on the slow variable, it achieves essentially equal accuracy to that of the full tensor product approximation. To get the rate of convergence for the numerical corrector constructed in section 6, we need a homogenization rate of convergence for (2.5). For the two-scale case, when the initial condition g_0 in (2.5) equals 0, adapting the procedure of Jikov, Kozlov, and Oleĭnik [19] for elliptic equations we deduce the $O(\varepsilon^{1/2})$ homogenization rate of convergence in terms of the microscopic scale ε , although we need to overcome some technical difficulty associated with wave equations. To the best of our knowledge, such a rate has not been established for the homogenization of wave equations. For multiscale problems, a homogenization rate is not available. However, in section 6 we can still deduce a general corrector for the multiscale homogenized problem. In section 7, we construct a numerical corrector using the finite element solution of the high dimensional multiscale homogenized problem, and for the case of two scales, we deduce an explicit rate of convergence for the corrector in terms of both the homogenization rate of convergence and the finite element error. Finally, in section 8, we present some numerical examples for two- and three-scale problems in one and two dimensions that confirm our analysis. We then summarize the paper with some conclusions on the complexity of the method. This paper is concluded with some appendices that present the long proofs of and remarks on some results in the previous sections.

Throughout this paper, by c we denote a generic constant that does not depend on the scales and the approximating parameters. For a Banach space X , we denote by $L^p(0, T; X)$ the Bochner spaces of p -summable functions from $(0, T)$ to X . By ∇ , without specifying explicitly the variable, we denote the gradient with respect to x . For a function $q(t, x)$ that depends on t and x , when we only wish to emphasize the t dependence, we write it as $q(t)$. The symbol $\#$ denotes spaces of periodic functions with respect to the unit cube Y in $\mathbb{R}^d$; in particular, $H_{\#}^k(Y)$ denotes the spaces of Sobolev spaces of periodic functions in Y and $C_{\#}^k(Y)$ denotes the spaces of periodic functions that are k times continuously differentiable.

2. Multiscale wave equation. We now introduce the multiscale wave equation for which we study the homogenization limit via multiscale convergence. We first present the setting up for the problem. We then review the concept of multiscale convergence and apply it to our equation.

2.1. Problem setting. Let $D \in \mathbb{R}^d$ be a bounded domain. Let $Y = (0, 1)^d$ be the unit cube in $\mathbb{R}^d$. For an integer n that denotes the number of microscopic scales that the problem depends on, let $Y_1, Y_2, \dots, Y_n$ be n copies of Y ; and by $\mathbf{Y}$ the space of all nd dimensional vectors $\mathbf{y} = (y_1, y_2, \dots, y_n)$ such that $y_i \in Y_i$ for $i = 1, \dots, n$. We denote by A a mapping from the product space $D \times \mathbf{Y}$ to the space $\mathbb{R}_{sym}^{d \times d}$ of real symmetric matrices such that for all $x \in D$ and $\mathbf{y} \in \mathbf{Y}$, $A(x, \mathbf{y})$ satisfies the boundedness condition

$$(2.1) \quad |A(x, \mathbf{y})\xi \cdot \zeta| \leq \beta|\xi||\zeta|$$

and the coerciveness condition

$$(2.2) \quad \alpha|\xi|^2 \leq A(x, \mathbf{y})\xi \cdot \xi$$

for all vectors $\xi, \zeta \in \mathbb{R}^d$ where α and β are positive constants. We assume that the coefficient A is periodic with respect to y_i with the period being Y_i ($i = 1, \dots, n$) and

is sufficiently smooth:

$$(2.3) \quad A(x, \mathbf{y}) \in C^1(\bar{D}, C_{\#}^1(\bar{Y}_1, \dots, C_{\#}^1(\bar{Y}_n) \dots)),$$

where $C_{\#}^1(\bar{Y}_i)$ denotes the space of continuously differentiable functions that are periodic in Y_i . Let $\varepsilon > 0$ be a small quantity. Let $\varepsilon_1(\varepsilon), \dots, \varepsilon_n(\varepsilon)$ be n positive functions of ε that denote the n microscopic scales that the problem depends on. These scales converge to 0 when $\varepsilon \rightarrow 0$. We assume that they satisfy the scale separation assumption, i.e., for $i = 1, \dots, n-1$,

$$(2.4) \quad \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon_{i+1}(\varepsilon)}{\varepsilon_i(\varepsilon)} = 0.$$

Without loss of generality, we let $\varepsilon_1 = \varepsilon$. The multiscale coefficient $A^\varepsilon : D \rightarrow \mathbb{R}_{sym}^{d \times d}$ is defined as

$$A^\varepsilon(x) = A\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}\right).$$

We denote $V = H_0^1(D)$ and $H = L^2(D)$. Let $g_0 \in V$ and $g_1 \in H$. For $T > 0$, let $f(t, x) \in L^2(0, T; H)$. We consider the wave equation in D with Dirichlet boundary condition: Find $u^\varepsilon(t, x) \in L^2(0, T; V) \cap H^1(0, T; H)$ such that

$$(2.5) \quad \begin{aligned} \frac{\partial^2 u^\varepsilon}{\partial t^2} - \nabla \cdot (A^\varepsilon(x) \nabla u^\varepsilon) &= f, \\ u^\varepsilon(0, \cdot) &= g_0, \quad \frac{\partial u^\varepsilon}{\partial t}(0, \cdot) = g_1, \\ u^\varepsilon(t, x) &= 0, \quad \text{for } x \in \partial D. \end{aligned}$$

Denoting by $(\cdot, \cdot)_H$ the inner product in H , extended to the duality pairing $\langle V', V \rangle$, we consider the weak form of (2.5) as

$$(2.6) \quad \left(\frac{\partial^2 u^\varepsilon}{\partial t^2}, \phi \right)_H + \int_D A^\varepsilon(x) \nabla u^\varepsilon(t, x) \cdot \nabla \phi(x) dx = \int_D f(t, x) \phi(x) dx \quad \forall \phi \in V.$$

From conditions (2.1) and (2.2), we have the following existence and uniqueness result.

PROPOSITION 2.1. *Problem (2.6) has a unique solution $u^\varepsilon \in L^2(0, T; V) \cap H^1(0, T; H)$ such that*

$$(2.7) \quad \|u^\varepsilon\|_{L^2(0, T; V) \cap H^1(0, T; H)} \leq c(\|f\|_{L^2(0, T; H)} + \|g_0\|_V + \|g_1\|_H),$$

where the constant c only depends on α , β and T .

The proof of this proposition is standard; see, for example, Wloka [24]. Indeed, it can be shown that $u^\varepsilon \in C([0, T]; V) \cap C([0, T]; H)$. In this paper, we study the asymptotic behavior of u^ε when $\varepsilon \rightarrow 0$ via multiscale convergence method. We first extend the standard notions ofNguetseng [21], Allaire [2], and Allaire and Briane [3] to functions that depend on time variable t .

2.2. Multiscale convergence. We first formulate the concept of multiscale convergence for functions that depend on t . The definition is a simple extension of that by Allaire and Briane [3].

DEFINITION 2.2. A sequence of functions $u^\varepsilon \in L^2(0, T; H)$ is said to $(n+1)$ -scale converge to a function $u_0 \in L^2(0, T; D \times \mathbf{Y})$ if for all smooth functions $\phi(t, x, \mathbf{y})$ which are Y_i periodic with respect to y_i ,

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_D u^\varepsilon(t, x) \phi\left(t, x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}\right) dx dt = \int_0^T \int_D \int_{\mathbf{Y}} u_0(t, x, \mathbf{y}) \phi(t, x, \mathbf{y}) d\mathbf{y} dx dt.$$

This definition makes sense because of the following result.

PROPOSITION 2.3. From a bounded sequence u^ε in $L^2(0, T; H)$, there is a subsequence (not renumbered) that $(n+1)$ -scale converges to a function $u_0 \in L^2(0, T; D \times \mathbf{Y})$.

For a bounded sequence $\{u^\varepsilon\}_\varepsilon \subset L^2(0, T; V)$ we have the following proposition.

PROPOSITION 2.4. Let $\{u^\varepsilon\}_\varepsilon \subset L^2(0, T; V)$ be a bounded sequence. There is a subsequence (not renumbered), a function $u_0 \in L^2(0, T; V)$, and n functions $u_i \in L^2((0, T) \times D \times Y_1 \times \dots \times Y_{i-1}, H_{\#}^1(Y_i)/\mathbb{R})$ such that $u^\varepsilon \rightharpoonup u_0$ in $L^2(0, T; V)$ and

$$\nabla u^\varepsilon \xrightarrow{(n+1)\text{-scale}} \nabla u_0 + \nabla_{y_1} u_1 + \dots + \nabla_{y_n} u_n.$$

The proofs of Propositions 2.3 and 2.4 are almost identical to those in Allaire [2] and Allaire and Briane [3].

2.3. Multiscale limit of the wave equation. We study the multiscale convergence of the solution to the multiscale wave equation (2.6). For conciseness, for $i = 1, \dots, n$, we denote by

$$V_i := L^2(D \times Y_1 \times \dots \times Y_{i-1}, H_{\#}^1(Y_i)/\mathbb{R}),$$

and by

$$\mathbf{V} := V \times V_1 \times \dots \times V_n.$$

The space $\mathbf{V}$ is equipped with the norm: for $\mathbf{v} = (v_0, v_1, \dots, v_n) \in \mathbf{V}$,

$$\|\mathbf{v}\|_{\mathbf{V}} = \|v_0\|_V + \sum_{i=1}^n \|v_i\|_{V_i} = \|\nabla v_0\|_H + \sum_{i=1}^n \|\nabla_{y_i} v_i\|_{L^2(D \times Y_1 \times \dots \times Y_i)}.$$

We then have

$$L^2(0, T; V_i) = L^2((0, T) \times D \times Y_1 \times \dots \times Y_{i-1}, H_{\#}^1(Y_i)/\mathbb{R}),$$

and

$$L^2(0, T; \mathbf{V}) = L^2(0, T; V) \times \prod_{i=1}^n L^2((0, T) \times D \times Y_1 \times \dots \times Y_{i-1}, H_{\#}^1(Y_i)/\mathbb{R}).$$

With the bilinear form $B : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$ defined by

$$B(\mathbf{w}, \mathbf{v}) = \int_D \int_{\mathbf{Y}} A(x, \mathbf{y}) (\nabla w_0 + \nabla_{y_1} w_1 + \dots + \nabla_{y_n} w_n) \cdot (\nabla v_0 + \nabla_{y_1} v_1 + \dots + \nabla_{y_n} v_n) d\mathbf{y} dx$$

for all $\mathbf{w} = (w_0, w_1, \dots, w_n)$, $\mathbf{v} = (v_0, v_1, \dots, v_n)$ in $\mathbf{V}$, we have the following result.

THEOREM 2.5. *The solution u^ε of (2.6) converges weakly to u_0 in $L^2(0, T; V)$ and ∇u^ε ($n+1$)-scale converges to $\nabla u_0 + \nabla_{y_1} u_1 + \dots + \nabla_{y_n} u_n$ where $u_i \in L^2(0, T; V_i)$; $\mathbf{u} = (u_0, u_1, \dots, u_n)$ is the unique solution of the problem*

$$(2.8) \quad \left(\frac{\partial^2 u_0(t)}{\partial t^2}, v_0 \right)_H + B(\mathbf{u}, \mathbf{v}) = \int_D f(t, x) v_0(x) dx$$

for all $\mathbf{v} = (v_0, v_1, \dots, v_n) \in \mathbf{V}$, with the initial condition

$$(2.9) \quad u_0(0, x) = g_0(x), \quad \frac{\partial u_0}{\partial t}(0, x) = g_1(x).$$

Proof. From (2.7), we deduce that there exists $\mathbf{u} \in \mathbf{V}$ such that $u^\varepsilon \rightharpoonup u_0$ in $L^2(0, T; V)$ and ∇u^ε ($n+1$)-scale converges to $\nabla u_0 + \nabla_{y_1} u_1 + \dots + \nabla_{y_n} u_n$. Let $\psi_0 \in C_0^\infty(D)$; and for $i = 1, \dots, n$, $\psi_i \in C_0^\infty(D, C_\#^\infty(Y_1, \dots, C_\#^\infty(Y_{i-1}), C_\#^\infty(Y_i))) \dots$ and $q(t) \in C_0^\infty((0, T))$. Let the test function ϕ in (2.6) be

$$\phi(t, x) = \left(\psi_0(x) + \sum_{i=1}^n \varepsilon_i \psi_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) \right) q(t).$$

We then obtain

$$\begin{aligned} & \int_0^T \left(u_0(t), \psi_0 + \sum_{i=1}^n \varepsilon_i \psi_i \left(\cdot, \frac{\cdot}{\varepsilon_1}, \dots, \frac{\cdot}{\varepsilon_i} \right) \right)_H \frac{d^2 q(t)}{dt^2} dt \\ & + \int_0^T \int_D A^\varepsilon(x) \nabla u^\varepsilon(x) \cdot \left(\nabla \psi_0 + \sum_{i=1}^n \varepsilon_i \left(\nabla_x \psi_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) + \sum_{j=1}^i \frac{1}{\varepsilon_j} \nabla_{y_j} \psi_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) \right) \right) q(t) dx dt \\ & = \int_0^T \int_D f(t, x) \left(\psi_0(x) + \sum_{i=1}^n \varepsilon_i \psi_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) \right) q(t) dx dt. \end{aligned}$$

From (2.4), in the limit $\varepsilon \rightarrow 0$, we obtain

$$\int_0^T \left(u_0(t, \cdot), \psi_0(\cdot) \right)_H \frac{d^2 q(t)}{dt^2} dt + \int_0^T B(\mathbf{u}(t), \boldsymbol{\psi}) q(t) dt = \int_0^T \int_D f(t, x) \psi_0(x) q(t) dx dt.$$

As the previous equations holds for all $q(t) \in C_0^\infty(0, T)$, we deduce

$$\left(\frac{\partial^2 u_0(t)}{\partial t^2}, \psi_0 \right)_H + B(\mathbf{u}(t), \boldsymbol{\psi}) = \int_D f(t, x) \psi_0(x) dx$$

in the sense of distribution, where

$$\boldsymbol{\psi} = (\psi_0, \psi_1, \dots, \psi_n).$$

Using a density argument, we deduce (2.8). Initial conditions (2.9) come from the fact that

$$\frac{\partial u^\varepsilon}{\partial t} \rightharpoonup \frac{\partial u_0}{\partial t} \text{ in } L^2(0, T; H) \quad \text{and} \quad \frac{\partial^2 u^\varepsilon}{\partial t^2} \rightharpoonup \frac{\partial^2 u_0}{\partial t^2} \text{ in } L^2(0, T; V').$$

We now show that problem (2.8) has a unique solution, i.e., problem (2.8) with zero initial conditions and with $f = 0$ only has zero solution. Following the same procedure as in Wloka [24, Theorem 29.1], fixing $s \in (0, T)$, we define for each $i = 0, 1, \dots, n$

$$w_i(t) = \begin{cases} -\int_t^s u_i(\sigma) d\sigma & \text{for } t < s, \\ 0 & \text{for } t \geq s. \end{cases}$$

Let $\mathbf{w} = (w_0, w_1, \dots, w_n)$. We have

$$\frac{d}{dt} \left(\frac{\partial u_0(t)}{\partial t}, w_0(t) \right)_H = \left(\frac{\partial^2 u_0(t)}{\partial t^2}, w_0(t) \right)_H + \left(\frac{\partial u_0(t)}{\partial t}, \frac{\partial w_0(t)}{\partial t} \right)_H.$$

Letting $\mathbf{v} = \mathbf{w}$ in (2.8), we have

$$\int_0^s \left(B(\mathbf{u}(t), \mathbf{w}(t)) + \frac{d}{dt} \left(\frac{\partial u_0(t)}{\partial t}, w_0(t) \right)_H - \left(\frac{\partial u_0(t)}{\partial t}, \frac{\partial w_0(t)}{\partial t} \right)_H \right) dt = 0.$$

As

$$B(\mathbf{u}(t), \mathbf{w}(t)) = \frac{1}{2} \frac{d}{dt} B(\mathbf{w}(t), \mathbf{w}(t)) \quad \text{and} \quad \left(\frac{\partial u_0(t)}{\partial t}, \frac{\partial w_0(t)}{\partial t} \right)_H = \frac{1}{2} \frac{d}{dt} (u_0(t), u_0(t))_H,$$

from the fact that $\partial u_0 / \partial t(0) = 0$, $u_0(0) = 0$, and $\mathbf{w}(s) = 0$, we have

$$B(\mathbf{w}(0), \mathbf{w}(0)) + \|u_0(s)\|_H^2 = 0.$$

This implies that $u_0(s) = 0$ and $\int_0^s \mathbf{u}(\sigma) d\sigma = 0$ for all s . From this, we deduce that $\mathbf{u}(s) = 0$. $\square$

To illustrate how the solution u^ε of the multiscale problem (2.5) can be approximated by the solution $\mathbf{u}$ of problem (2.8), we mention the following result for the case of two scales ($n = 1$). The proof for it can be found, for example, in [5].

PROPOSITION 2.6. *For $n = 1$, assume that the initial condition $g_0 = 0$. Assume further that the solution $w_i(x, y)$ of cell problems (6.1) are in $C(\bar{D}, W^{1,\infty}(Y))$. The following corrector result holds*

$$\lim_{\varepsilon \rightarrow 0} \left\| \nabla u^\varepsilon(\cdot, \cdot) - \left(\nabla u_0(\cdot, \cdot) + \nabla_y u_1 \left(\cdot, \cdot, \frac{\cdot}{\varepsilon} \right) \right) \right\|_{L^\infty(0,T;H)} = 0.$$

We will study in more details correctors for the homogenization problem (2.8) in section 6.

3. Finite element discretization. In this section, we consider finite element discretization of the high dimensional limiting problem (2.8). To study the general framework, we assume a general nested sequence of finite element spaces for approximating $\mathbf{u} \in \mathbf{V}$ without specifying these spaces explicitly. We will study particular finite element spaces in the later sections. To approximate $u_0 \in V$, we consider a hierarchy of finite element subspaces of V ,

$$V^1 \subset V^2 \subset \dots \subset V$$

such that $\bigcup_{L=1}^\infty V^L$ is dense in V ; and for $i = 1, \dots, n$, to approximate u_i , we consider the finite element subspaces of V_i

$$V_i^1 \subset V_i^2 \subset \dots \subset V_i$$

such that $\bigcup_{L=1}^\infty V_i^L$ is dense in V_i . We first consider the spatial semidiscrete problem of (2.8).

3.1. Spatial semidiscrete problem. With the spaces $V^L \subset V$ and $V_i^L \subset V_i$ above, we define by

$$\mathbf{V}^L = V^L \times V_1^L \times \cdots \times V_n^L \subset \mathbf{V}.$$

We consider the spatial semidiscrete approximating problem: Find $\mathbf{u}^L = (u_0^L, u_1^L, \dots, u_n^L) \in \mathbf{V}^L$ such that

$$(3.1) \quad \left(\frac{\partial^2 u_0^L}{\partial t^2}, v_0^L \right)_H + B(\mathbf{u}^L, \mathbf{v}^L) = (f, v_0^L)_H$$

for all $\mathbf{v}^L = (v_0^L, v_1^L, \dots, v_n^L) \in \mathbf{V}^L$, with the initial condition $u_0^L(0, \cdot) = g_0^L \in V^L$ and $\frac{\partial u_0^L}{\partial t}(0, \cdot) = g_1^L \in V^L$. The initial conditions g_0^L and g_1^L are chosen so that they approximate g_0 and g_1 in V and H , respectively, e.g., as the orthogonal projections of g_0 and g_1 to V^L .

PROPOSITION 3.1. *Problem (3.1) has a unique solution.*

Proof. Let $\mathbf{M}$ be the Gram matrix for the basis of V^L . For the bilinear form $B(\mathbf{u}^L, \mathbf{v}^L)$, let $\mathbf{A}_{00}$ be the matrix describing the interaction of the basis functions of V^L , let $\mathbf{A}_{01}$ be the matrix describing the interaction of the basis functions of V^L with the basis functions of $V_1^L \times V_2^L \times \cdots \times V_n^L$, and let $\mathbf{A}_{11}$ be the matrix describing the interaction of the basis functions of $V_1^L \times V_2^L \times \cdots \times V_n^L$ with themselves. Let $\mathbf{F}$ be the column matrix describing the interaction of f and the basis functions of V^L . Let $\mathbf{c}_0$ be the coefficient vector in the expansion of u_0^L with respect to the basis of V^L . Let $\mathbf{c}_1$ be the coefficient vector in the expansion of $(u_1^L, \dots, u_n^L)$ with respect to the basis of $V_1^L \times \cdots \times V_n^L$. We have

$$\begin{aligned} \mathbf{M} \frac{d^2 \mathbf{c}_0}{dt^2} + \mathbf{A}_{00} \mathbf{c}_0 + \mathbf{A}_{01} \mathbf{c}_1 &= \mathbf{F}, \\ \mathbf{A}_{01}^\top \mathbf{c}_0 + \mathbf{A}_{11} \mathbf{c}_1 &= 0. \end{aligned}$$

As $\mathbf{A}_{11}$ is positively definite, we have

$$\mathbf{M} \frac{d^2 \mathbf{c}_0}{dt^2} + (\mathbf{A}_{00} - \mathbf{A}_{01} \mathbf{A}_{11}^{-1} \mathbf{A}_{01}^\top) \mathbf{c}_0 = \mathbf{F}.$$

Since $\det \mathbf{M} \neq 0$, this equation has a unique solution. $\square$

We analyze problem (3.1) following the approach of Dupont [8]. For each $t \in [0, T]$, let $\mathbf{w}^L(t) = (w_0^L, w_1^L, \dots, w_n^L) \in \mathbf{V}^L$ be the solution of the problem

$$(3.2) \quad B(\mathbf{w}^L(t) - \mathbf{u}(t), \mathbf{v}^L) = 0$$

for all $\mathbf{v}^L = (v_0^L, \dots, v_n^L) \in \mathbf{V}^L$.

For $i = 0, \dots, n$, let

$$(3.3) \quad \eta_i^L = w_i^L - u_i.$$

We denote by $\boldsymbol{\eta}^L = (\eta_0^L, \eta_1^L, \dots, \eta_n^L) \in \mathbf{V}$. We then have from (2.8) and (3.2)

$$(3.4) \quad B(\mathbf{w}^L, \mathbf{v}^L) = \left(f - \frac{\partial^2 u_0}{\partial t^2}, v_0^L \right)_H$$

in the distribution sense for all $\mathbf{v}^L = (v_0^L, v_1^L, \dots, v_n^L) \in \mathbf{V}^L$.

We have the following estimates for $\|\boldsymbol{\eta}\|_{\mathbf{V}}$.

LEMMA 3.2. For the solution $\boldsymbol{w}^L$ of (3.2),

$$(3.5) \quad \|\boldsymbol{\eta}^L(t)\|_{\mathbf{V}} \leq c \inf_{\boldsymbol{v}^L \in \mathbf{V}^L} \|\boldsymbol{u}(t) - \boldsymbol{v}^L\|_{\mathbf{V}}.$$

Proof. From (3.2), we have

$$B(\boldsymbol{w}^L - \boldsymbol{u}, \boldsymbol{w}^L - \boldsymbol{u}) = B(\boldsymbol{w}^L - \boldsymbol{u}, \boldsymbol{v}^L - \boldsymbol{u})$$

for all $\boldsymbol{v}^L \in \mathbf{V}^L$. From (2.1) and (2.2), we get the conclusion. $\square$

For the development of the finite element convergence in the later sections, we develop the following estimates for the time derivatives of $\boldsymbol{w}^L$.

LEMMA 3.3. If $\frac{\partial \boldsymbol{u}}{\partial t} \in C([0, T]; \mathbf{V})$, then

$$\left\| \frac{\partial \boldsymbol{\eta}^L}{\partial t} \right\|_{L^\infty(0, T; \mathbf{V})} \leq c \sup_{t \in [0, T]} \inf_{\boldsymbol{v}^L \in \mathbf{V}^L} \left\| \frac{\partial \boldsymbol{u}}{\partial t}(t) - \boldsymbol{v}^L \right\|_{\mathbf{V}}.$$

If $\frac{\partial^2 \boldsymbol{u}}{\partial t^2} \in L^2(0, T; \mathbf{V})$, then

$$\left\| \frac{\partial^2 \boldsymbol{\eta}^L}{\partial t^2} \right\|_{L^2(0, T; \mathbf{V})} \leq c \inf_{\boldsymbol{v}^L \in L^2(0, T; \mathbf{V}^L)} \left\| \frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \boldsymbol{v}^L \right\|_{L^2(0, T; \mathbf{V})}.$$

Proof. If $\frac{\partial \boldsymbol{u}}{\partial t} \in C([0, T]; \mathbf{V})$, from (3.2) we have

$$(3.6) \quad B\left(\frac{\partial(\boldsymbol{w}^L - \boldsymbol{u})}{\partial t}, \boldsymbol{v}^L\right) = 0$$

for all $\boldsymbol{v}^L \in \mathbf{V}^L$. We then proceed as in the proof of Lemma 3.2 to show the first inequality. The proof for the second inequality is similar. $\square$

Let

$$(3.7) \quad \boldsymbol{\zeta}^L = \boldsymbol{u}^L - \boldsymbol{w}^L.$$

We have from (3.1) and (3.4)

$$(3.8) \quad \left(\frac{\partial^2 \boldsymbol{\zeta}_0^L}{\partial t^2}, \boldsymbol{v}_0^L \right)_H + B(\boldsymbol{\zeta}^L, \boldsymbol{v}^L) = - \left(\frac{\partial^2 \boldsymbol{\eta}_0^L}{\partial t^2}, \boldsymbol{v}_0^L \right)_H$$

for all $\boldsymbol{v}^L = (v_0^L, v_1^L, \dots, v_n^L) \in \mathbf{V}^L$. We then have the following estimate.

LEMMA 3.4. Assume that $\boldsymbol{u} \in H^1(0, T; \mathbf{V})$ and $\frac{\partial^2 \boldsymbol{\eta}_0^L}{\partial t^2} \in L^2(0, T; H)$. Then

$$(3.9) \quad \left\| \frac{\partial \boldsymbol{\zeta}_0^L}{\partial t} \right\|_{L^\infty(0, T; H)}^2 + \|\boldsymbol{\zeta}^L\|_{L^\infty(0, T; \mathbf{V})}^2 \leq c \left[\|\nabla \boldsymbol{\zeta}_0^L(0)\|_H^2 + \left\| \frac{\partial \boldsymbol{\zeta}_0^L}{\partial t}(0) \right\|_H^2 + \left\| \frac{\partial^2 \boldsymbol{\eta}_0^L}{\partial t^2} \right\|_{L^2(0, T; H)}^2 \right].$$

Proof. As $\boldsymbol{u} \in H^1(0, T; \mathbf{V})$, $\frac{\partial \boldsymbol{u}}{\partial t}$ is well defined and so are $\frac{\partial \boldsymbol{w}^L}{\partial t}$ and $\frac{\partial \boldsymbol{\zeta}^L}{\partial t}$. Let $\boldsymbol{v}^L = \frac{\partial \boldsymbol{\zeta}^L}{\partial t}$ in (3.8),

$$\left(\frac{\partial^2 \boldsymbol{\zeta}_0^L}{\partial t^2}, \frac{\partial \boldsymbol{\zeta}_0^L}{\partial t} \right)_H + B\left(\boldsymbol{\zeta}^L, \frac{\partial \boldsymbol{\zeta}^L}{\partial t}\right) = - \left(\frac{\partial^2 \boldsymbol{\eta}_0^L}{\partial t^2}, \frac{\partial \boldsymbol{\zeta}_0^L}{\partial t} \right)_H,$$

then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left[\left(\frac{\partial \zeta_0^L}{\partial t}, \frac{\partial \zeta_0^L}{\partial t} \right)_H + B(\zeta^L, \zeta^L) \right] &= - \left(\frac{\partial^2 \eta_0^L}{\partial t^2}, \frac{\partial \zeta_0^L}{\partial t} \right)_H \\ &\leq \frac{1}{2\gamma} \left\| \frac{\partial^2 \eta_0^L}{\partial t^2} \right\|_H^2 + \frac{\gamma}{2} \left\| \frac{\partial \zeta_0^L}{\partial t} \right\|_H^2, \end{aligned}$$

where $\gamma > 0$ is chosen to be sufficiently small. Integrating on $(0, t)$, from (2.1) and (2.2) we have

$$\begin{aligned} &\sup_{t \in (0, T)} \left\{ \left\| \frac{\partial \zeta_0^L}{\partial t}(t) \right\|_H^2 + \alpha \|\zeta^L(t)\|_{\mathbf{V}} \right\} \\ &\leq \frac{1}{2\gamma} \left\| \frac{\partial^2 \eta_0^L}{\partial t^2} \right\|_{L^2(0, T; H)}^2 + \frac{\gamma T}{2} \sup_{t \in (0, T)} \left\| \frac{\partial \zeta_0^L}{\partial t}(t) \right\|_H^2 + \beta \|\zeta^L(0)\|_{\mathbf{V}}^2 + \left\| \frac{\partial \zeta_0^L}{\partial t}(0) \right\|_H^2. \end{aligned}$$

For $t = 0$, let $v_0^L = 0$ and $v_i^L = \zeta_i^L(0)$ in (3.8). We then have

$$\begin{aligned} &\int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) (\nabla_x \zeta_0^L(0) + \nabla_{y_1} \zeta_1^L(0) + \cdots + \nabla_{y_n} \zeta_n^L(0)) \\ &\quad \cdot (\nabla_{y_1} \zeta_1^L(0) + \cdots + \nabla_{y_n} \zeta_n^L(0)) d\mathbf{y} dx = 0. \end{aligned}$$

From this, (2.1), and (2.2), we deduce that

$$\alpha \|\nabla_{y_1} \zeta_1^L(0) + \cdots + \nabla_{y_n} \zeta_n^L(0)\|_{L^2(D \times \mathbf{Y})} \leq \beta \|\nabla \zeta_0^L(0)\|_H.$$

Letting γ be sufficiently small, we get the conclusion. $\square$

We then have the following convergence result for the semidiscrete problem (3.1).

PROPOSITION 3.5. Assume that $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2(0, T; \mathbf{V})$, and that

$$(3.10) \quad \lim_{L \rightarrow \infty} \|g_0^L - g_0\|_V = 0 \quad \text{and} \quad \lim_{L \rightarrow \infty} \|g_1^L - g_1\|_H = 0.$$

Then

$$\lim_{L \rightarrow \infty} \left\{ \left\| \frac{\partial(u_0^L - u_0)}{\partial t} \right\|_{L^\infty(0, T; H)} + \|\mathbf{u}^L - \mathbf{u}\|_{L^\infty(0, T; \mathbf{V})} \right\} = 0.$$

Proof. From Lemma 3.4, as $\mathbf{u}^L - \mathbf{u} = \zeta^L + \eta^L$, we have

$$\begin{aligned} &\left\| \frac{\partial(u_0^L - u_0)}{\partial t} \right\|_{L^\infty(0, T; H)}^2 + \|\mathbf{u}^L - \mathbf{u}\|_{L^\infty(0, T; \mathbf{V})}^2 \\ &\leq c \left[\|\nabla \zeta_0^L(0)\|_H^2 + \left\| \frac{\partial \zeta_0^L}{\partial t}(0) \right\|_H^2 + \left\| \frac{\partial^2 \eta_0^L}{\partial t^2} \right\|_{L^2(0, T; H)}^2 \right] + \left\| \frac{\partial \eta_0^L}{\partial t} \right\|_{L^\infty(0, T; H)}^2 \\ (3.11) \quad &+ c \|\eta^L\|_{L^\infty(0, T; \mathbf{V})}^2. \end{aligned}$$

We show that $\lim_{L \rightarrow \infty} \|\eta^L\|_{L^\infty(0, T; \mathbf{V})} = 0$. As $\mathbf{u} \in C([0, T], \mathbf{V})$ $\mathbf{u}$ is uniformly continuous as a function from $[0, T]$ to $\mathbf{V}$. For $\epsilon > 0$, there is a piecewise constant (with respect to t) function $\tilde{\mathbf{u}} \in L^\infty(0, T; \mathbf{V})$ such that $\|\mathbf{u} - \tilde{\mathbf{u}}\|_{L^\infty(0, T; \mathbf{V})} < \epsilon$. As $\tilde{\mathbf{u}}(t)$

obtains only a finite number of $\mathbf{V}$ -values, there is an L and $\mathbf{v}^L \in L^\infty(0, T; \mathbf{V}^L)$ such that $\|\tilde{\mathbf{u}} - \mathbf{v}^L\|_{L^\infty(0, T; \mathbf{V})} < \epsilon$. Thus

$$\lim_{L \rightarrow \infty} \sup_{t \in (0, T)} \inf_{\mathbf{v}^L \in \mathbf{V}^L} \|\mathbf{u}(t) - \mathbf{v}^L\|_{\mathbf{V}} = 0.$$

We then apply Lemma 3.2.

Similarly, we have

$$\lim_{L \rightarrow \infty} \left\| \frac{\partial \eta_0^L}{\partial t} \right\|_{L^2(0, T; H)} = 0 \quad \text{and} \quad \lim_{L \rightarrow \infty} \left\| \frac{\partial^2 \eta_0^L}{\partial t^2} \right\|_{L^2(0, T; H)}^2 = 0.$$

Further, we have that

$$\|\nabla \zeta_0^L(0)\|_H \leq \|\nabla u_0^L - \nabla u_0\|_H + \|\nabla u_0 - \nabla w_0^L\|_H,$$

which converges to 0 when $L \rightarrow \infty$ due to (3.10) and Lemma 3.2. Similarly, we have

$$\lim_{L \rightarrow \infty} \left\| \frac{\partial \zeta_0^L}{\partial t}(0) \right\|_H = 0.$$

From these, we get the conclusion. $\square$

3.2. Fully discrete problems. In this subsection we consider time discretization of the Galerkin approximating problem (3.1). We follow the approach and notation of Dupont [8] for a stable time discretizing scheme.

Let M be a positive integer, and let $\Delta t = T/M$. Let $t_m = m\Delta t$. For a Banach space X , for any functions $r(t, x) \in C([0, T]; X)$, we denote by $r_m = r(t_m, \cdot)$. Following Dupont [8], we employ the following notation:

$$\begin{aligned} r_{m+1/2} &= (1/2)(r_{m+1} + r_m), & r_{m,\theta} &= \theta r_{m+1} + (1-2\theta)r_m + \theta r_{m-1}, \\ \partial_t r_{m+1/2} &= (r_{m+1} - r_m)/\Delta t, & \partial_t^2 r_m &= (r_{m+1} - 2r_m + r_{m-1})/(\Delta t)^2, \\ \delta_t r_m &= (r_{m+1} - r_{m-1})/(2\Delta t). \end{aligned}$$

Assuming that $f \in C([0, T]; H)$, for problem (2.8), with the finite element space $\mathbf{V}^L$ in the previous subsection, we define the following fully discrete problem: For $m = 1, \dots, M$, find $\mathbf{u}_m^L = (u_{0,m}^L, u_{1,m}^L, \dots, u_{n,m}^L) \in \mathbf{V}^L$ such that for $m = 1, \dots, M-1$,

$$(3.12) \quad (\partial_t^2 u_{0,m}^L, v_0^L)_H + B(\mathbf{u}_{m,1/4}^L, \mathbf{v}^L) = (f_{m,1/4}, v_0^L)_H$$

for all $\mathbf{v}^L = (v_0^L, v_1^L, \dots, v_n^L) \in \mathbf{V}^L$.

As in [8], we choose $\theta = 1/4$ for the scheme to be stable and achieve the minimal time truncation error when the solution is sufficiently smooth.

For a continuous function $r : [0, T] \rightarrow X$, we denote by

$$\|r\|_{\tilde{L}^\infty(0, T; X)} = \max_{0 \leq m < M} \|r_{m+1/2}\|_X.$$

We also denote by

$$\|\partial_t r\|_{\tilde{L}^\infty(0, T; X)} = \max_{0 \leq m < M} \|\partial_t r_{m+1/2}\|_X.$$

For the solution of problem (3.12), we define ζ_m^L as

$$(3.13) \quad \zeta_m^L = \mathbf{u}_m^L - \mathbf{w}_m^L,$$

where $\mathbf{w}^L$ is defined in (3.2). We then have the following result.

LEMMA 3.6. Assume that $\mathbf{u} \in H^1(0, T; \mathbf{V})$ and $\partial^2 \eta_0^L / \partial t^2 \in L^2(0, T; H)$. If $\partial^3 u_0 / \partial t^3 \in L^2(0, T; H)$, then there exists a constant c (independent of Δt and $\mathbf{u}$) such that

$$\begin{aligned} & \|\partial_t \zeta_0^L\|_{\tilde{L}^\infty(0, T; H)} + \|\zeta^L\|_{\tilde{L}^\infty(0, T; \mathbf{V})} \\ & \leq c \left[\|\zeta_{1/2}^L\|_{\mathbf{V}} + \|\partial_t \zeta_{0,1/2}^L\|_H + \left\| \frac{\partial^2 \eta_0^L}{\partial t^2} \right\|_{L^2(0, T; H)} + \left\| \frac{\partial^3 u_0}{\partial t^3} \right\|_{L^2(0, T; H)} \Delta t \right], \end{aligned}$$

and if $\partial^4 u_0 / \partial t^4 \in L^2(0, T; H)$, then there exists a constant c such that

$$\begin{aligned} & \|\partial_t \zeta_0^L\|_{\tilde{L}^\infty(0, T; H)} + \|\zeta^L\|_{\tilde{L}^\infty(0, T; \mathbf{V})} \\ & \leq c \left[\|\zeta_{1/2}^L\|_{\mathbf{V}} + \|\partial_t \zeta_{0,1/2}^L\|_H + \left\| \frac{\partial^2 \eta_0^L}{\partial t^2} \right\|_{L^2(0, T; H)} + \left\| \frac{\partial^4 u_0}{\partial t^4} \right\|_{L^2(0, T; H)} (\Delta t)^2 \right]. \end{aligned}$$

The proof of this lemma is long; we present it in Appendix A.

LEMMA 3.7. There is a constant c independent of the finite element spaces $\mathbf{V}^L$ and Δt such that

$$(3.14) \quad \|\boldsymbol{\eta}^L\|_{\tilde{L}^\infty(0, T; \mathbf{V})} \leq c \sup_{0 \leq m < M} \inf_{\mathbf{v}^L \in \mathbf{V}^L} \|\mathbf{u}_{m+1/2} - \mathbf{v}^L\|_{\mathbf{V}}.$$

Further, if $\partial u_0 / \partial t \in L^\infty(0, T; V)$, then there exists a constant c such that

$$\|\partial_t \eta_0^L\|_{\tilde{L}^\infty(0, T; V)} \leq c \sup_{0 \leq m < M} \inf_{\mathbf{v}^L \in \mathbf{V}^L} \|\partial_t \mathbf{u}_{m+1/2} - \mathbf{v}^L\|_{\mathbf{V}}.$$

Proof. Averaging (3.2) at t_{m+1} and t_m with weights 1/2 and 1/2, we obtain

$$B(\mathbf{w}_{m+1/2}^L - \mathbf{u}_{m+1/2}, \mathbf{v}^L) = 0$$

for all $\mathbf{v}^L \in \mathbf{V}^L$. We, therefore, have

$$B(\mathbf{w}_{m+1/2}^L - \mathbf{u}_{m+1/2}, \mathbf{w}_{m+1/2}^L - \mathbf{u}_{m+1/2}) = B(\mathbf{w}_{m+1/2}^L - \mathbf{u}_{m+1/2}, \mathbf{v}^L - \mathbf{u}_{m+1/2})$$

for all $\mathbf{v}^L \in \mathbf{V}^L$. Using (2.1) and (2.2), we have

$$\|\mathbf{w}_{m+1/2}^L - \mathbf{u}_{m+1/2}\|_{\mathbf{V}} \leq c \|\mathbf{v}^L - \mathbf{u}_{m+1/2}\|_{\mathbf{V}},$$

for all $\mathbf{v}^L \in \mathbf{V}^L$. From this, we get the conclusion. The proof for the estimate of $\|\partial_t \eta_0^L\|_{\tilde{L}^\infty(0, T; V)}$ is similar. $\square$

PROPOSITION 3.8. Assume that $\mathbf{u} \in H^1(0, T; \mathbf{V})$ and $\partial^2 \eta_0^L / \partial t^2 \in L^2(0, T; H)$. If $\partial^3 u_0 / \partial t^3 \in L^2(0, T; H)$, then there exists a constant c such that

$$\begin{aligned} & \|\partial_t u_0^L - \partial_t u_0\|_{\tilde{L}^\infty(0, T; H)} + \|\mathbf{u}^L - \mathbf{u}\|_{\tilde{L}^\infty(0, T; \mathbf{V})} \\ & \leq c \left[\Delta t \left\| \frac{\partial^3 u_0}{\partial t^3} \right\|_{L^2(0, T; H)} + \left\| \frac{\partial^2 \eta_0^L}{\partial t^2} \right\|_{L^2(0, T; H)} + \|\partial_t \zeta_{0,1/2}^L\|_H + \|\zeta_{1/2}^L\|_{\mathbf{V}} \right] \\ (3.15) \quad & + \|\partial_t \eta_0^L\|_{\tilde{L}^\infty(0, T; H)} + \|\boldsymbol{\eta}^L\|_{\tilde{L}^\infty(0, T; \mathbf{V})}. \end{aligned}$$

If $\partial^4 u_0 / \partial t^4 \in L^2(0, T; H)$, then there exists a constant c such that

$$\begin{aligned} & \|\partial_t u_0^L - \partial_t u_0\|_{\tilde{L}^\infty(0, T; H)} + \|\mathbf{u}^L - \mathbf{u}\|_{\tilde{L}^\infty(0, T; \mathbf{V})} \\ & \leq c \left[(\Delta t)^2 \left\| \frac{\partial^4 u_0}{\partial t^4} \right\|_{L^2(0, T; H)} + \left\| \frac{\partial^2 \eta_0^L}{\partial t^2} \right\|_{L^2(0, T; H)} + \|\partial_t \zeta_{0, 1/2}^L\|_H + \|\zeta_{1/2}^L\|_{\mathbf{V}} \right] \\ (3.16) \quad & + \|\partial_t \eta_0^L\|_{\tilde{L}^\infty(0, T; H)} + \|\boldsymbol{\eta}^L\|_{\tilde{L}^\infty(0, T; \mathbf{V})}. \end{aligned}$$

In particular, if $\partial^2 \mathbf{u} / \partial t^2 \in L^2(0, T; \mathbf{V})$ and $\partial^3 u_0 / \partial t^3 \in L^2(0, T; H)$, when $\mathbf{u}_0^L$ and $\mathbf{u}_1^L$ are chosen so that

$$\lim_{L \rightarrow \infty} \|\partial_t \zeta_{0, 1/2}^L\|_H = 0, \quad \lim_{L \rightarrow \infty} \|\zeta_{1/2}^L\|_{\mathbf{V}} = 0,$$

then

$$(3.17) \quad \lim_{\substack{L \rightarrow \infty \\ \Delta t \rightarrow 0}} \|\partial_t u_0^L - \partial_t u_0\|_{\tilde{L}^\infty(0, T; H)} + \|\mathbf{u}^L - \mathbf{u}\|_{\tilde{L}^\infty(0, T; \mathbf{V})} = 0.$$

Proof. The proof of (3.15) and (3.16) follows from the definition of ζ^L and $\mathbf{w}^L$ and Lemma 3.6. The limit (3.17) then follows from Lemmas 3.2 and 3.7. We note that as $\mathbf{u} \in H^2(0, T; \mathbf{V})$, from Lemma 3.3

$$\lim_{L \rightarrow \infty} \left\| \frac{\partial^2 \eta_0^L}{\partial t^2} \right\|_{L^2(0, T; H)} = 0.$$

Further, as $\frac{\partial \mathbf{u}}{\partial t} \in C([0, T]; \mathbf{V})$ we have

$$\sup_{0 \leq m < M} \inf_{\mathbf{v}^L \in \mathbf{V}^L} \|\partial_t \mathbf{u}_{m+1/2} - \mathbf{v}^L\|_{\mathbf{V}} \leq \sup_{0 \leq t \leq T} \inf_{\mathbf{v}^L \in \mathbf{V}^L} \left\| \frac{\partial \mathbf{u}}{\partial t} - \mathbf{v}^L \right\|_{\mathbf{V}} \rightarrow 0 \text{ when } L \rightarrow \infty.$$

The proof for this uses the fact that $\frac{\partial \mathbf{u}}{\partial t}$ is uniformly continuous. The argument is similar to that in the proof of Proposition 3.5. $\square$

4. Regularity for the solution $\mathbf{u}$ of problem (2.8). To obtain the estimates and convergence in the previous section and the rates of convergence in section 5, it is essential that the solution $\mathbf{u}$ of (2.8) possesses the sufficient temporal and spatial regularity. We thus establish in this section the smoothness condition on the data that guarantees the required regularity.

We first deduce the homogenized problem for u_0 from (2.8). We have from (2.8) that

$$u_n = w_{nl} \left(\frac{\partial u_0}{\partial x_l} + \frac{\partial u_1}{\partial y_{1l}} + \cdots + \frac{\partial u_{n-1}}{\partial y_{(n-1)l}} \right),$$

where $w_{nl} \in L^2(D \times Y_1 \times \cdots \times Y_{n-1}, H_{\#}^1(Y_n)/\mathbb{R})$ satisfies the cell problem

$$\int_D \int_{Y_1} \cdots \int_{Y_n} A(e_l + \nabla_{y_n} w_{nl}) \cdot \nabla_{y_n} \phi_n d\mathbf{y} dx = 0$$

for all $\phi_n \in L^2(D \times Y_1 \times \cdots \times Y_{n-1}, H_{\#}^1(Y_n)/\mathbb{R})$, where e_l is the l th unit vector in $\mathbb{R}^d$. Using this, we have from (2.8) that

$$\begin{aligned} & \int_D \int_{Y_1} \cdots \int_{Y_{n-1}} \int_{Y_n} A(I + \nabla_{y_n} w_n) \left(\nabla_x u_0 + \sum_{k=1}^{n-1} \nabla_{y_k} u_k \right) \\ & \cdot \nabla_{y_{n-1}} \phi_{n-1} dy_1 \cdots dy_{n-1} dy_n dx = 0, \end{aligned}$$

where w_n denotes the vector function $(w_{n1}, \dots, w_{nd})$. Denoting by

$$A^{n-1}(x, y_1, \dots, y_{n-1}) = \int_{Y_n} A(I + \nabla_{y_n} w_n) dy_n = \int_{Y_n} A(I + \nabla_{y_n} w_n) \cdot (I + \nabla_{y_n} w_n) dy_n,$$

we have that

$$u_{n-1} = w_{(n-1)l} \left(\frac{\partial u_0}{\partial x_l} + \frac{\partial u_1}{\partial y_{1l}} + \dots + \frac{\partial u_{n-2}}{\partial y_{(n-2)l}} \right),$$

where $w_{(n-1)l} \in L^2(D \times Y_1 \times \dots \times Y_{n-1}, H_{\#}^1(Y_n)/\mathbb{R})$ satisfies the problem

$$\int_D \int_{Y_1} \dots \int_{Y_{n-1}} A^{n-1}(e_l + \nabla_{y_{n-1}} w_{(n-1)l}) \cdot \nabla_{y_{n-1}} \phi_{n-1} dy_1 \dots dy_{n-1} dx = 0,$$

for all $\phi_{n-1} \in L^2(D \times Y_1 \times \dots \times Y_{n-2}, H_{\#}^1(Y_{n-1})/\mathbb{R})$. Letting $A^n = A$, we then have, recursively,

$$u_i = w_{il} \left(\frac{\partial u_0}{\partial x_l} + \dots + \frac{\partial u_{i-1}}{\partial y_{(i-1)l}} \right),$$

where $w_{il} \in L^2(D \times Y_1 \times \dots \times Y_{i-1}, H_{\#}^1(Y_i)/\mathbb{R})$ satisfies the problem

$$(4.1) \quad \int_D \int_{Y_1} \dots \int_{Y_i} A^i(e_l + \nabla_{y_i} w_{il}) \cdot \nabla_{y_i} \phi_i dx dy_1 \dots dy_i = 0$$

for all $\phi_i \in L^2(D \times Y_1 \times \dots \times Y_{i-1}, H_{\#}^1(Y_i)/\mathbb{R})$. The matrix A^i is defined as

$$(4.2) \quad A^i(x, y_1, \dots, y_i) = \int_{Y_{i+1}} A^{i+1}(I + \nabla_{y_{i+1}} w_{i+1}) \cdot (I + \nabla_{y_{i+1}} w_{i+1}) dy_{i+1}$$

for $i < n$. Continuing this process, we finally get the homogenized coefficient $A^0(x)$ as

$$A^0(x) = \int_{Y_1} A^1(x, y_1)(I + \nabla_{y_1} w_1) \cdot (I + \nabla_{y_1} w_1) dy_1.$$

The homogenized problem is then

$$(4.3) \quad \frac{\partial^2 u_0}{\partial t^2} - \nabla \cdot (A^0(x) \nabla u_0) = f(t, x), \quad u_0(0, \cdot) = g_0, \quad \frac{\partial u_0}{\partial t}(0, \cdot) = g_1.$$

The solution $\mathbf{u}$ of (2.8) is written in terms of u_0 as

$$(4.4) \quad u_i = w_i \cdot (I + \nabla_{y_{i-1}} w_{i-1}) \dots (I + \nabla_{y_1} w_1) \nabla u_0.$$

As $A \in C^1(\bar{D}, C_{\#}^1(\bar{Y}_1, (\dots C_{\#}^1(\bar{Y}_n) \dots)))$, we deduce that $w_{il} \in C^1(\bar{D}, C_{\#}^1(\bar{Y}_1 \dots H_{\#}^1(Y_i) \dots))$. From which we have that $A^i \in C^1(\bar{D}, C_{\#}^1(\bar{Y}_1, \dots, C_{\#}^1(\bar{Y}_i) \dots))$ and $A^0 \in C^1(\bar{D})$. For the convergence in Propositions 3.5 and 3.8, we need regularity of $\mathbf{u}$ with respect to t . We now establish these.

PROPOSITION 4.1. Assume that

$$(4.5) \quad \begin{aligned} f &\in H^2(0, T; H), \\ g_1 &\in V, \\ f(0) - \nabla \cdot (A^0(x) \nabla g_0) &\in V, \\ \frac{\partial f}{\partial t}(0) - \nabla \cdot (A^0(x) \nabla g_1) &\in H, \end{aligned}$$

then

$$(4.6) \quad \frac{\partial^2 u_0}{\partial t^2} \in L^2(0, T; V) \quad \text{and} \quad \frac{\partial^3 u_0}{\partial t^3} \in L^2(0, T; H).$$

Assume

$$(4.7) \quad \begin{aligned} & f \in H^3(0, T; H), \\ & g_1 \in V, \\ & f(0) - \nabla \cdot (A^0(x) \nabla g_0) \in V, \\ & \frac{\partial f}{\partial t}(0) - \nabla \cdot (A^0(x) \nabla g_1) \in V, \\ & \frac{\partial^2 f}{\partial t^2}(0) + \nabla \cdot (A^0(x) \nabla (f(0) - \nabla \cdot (A^0(x) \nabla g_0))) \in H, \end{aligned}$$

then

$$(4.8) \quad \frac{\partial^3 u_0}{\partial t^3} \in L^2(0, T; V) \quad \text{and} \quad \frac{\partial^4 u_0}{\partial t^4} \in L^2(0, T; H).$$

Proof. The proof follows directly from the regularity of the wave equation (4.3) where (4.5) and (4.7) are the compatibility conditions that guarantee the regularities (4.6) and (4.8) (see, e.g., Wloka [24, Chapter 5]). From (4.5), we have that

$$(4.9) \quad \frac{\partial^2}{\partial t^2} \left(\frac{\partial u_0}{\partial t} \right) - \nabla \cdot \left(A^0 \nabla \frac{\partial u_0}{\partial t} \right) = \frac{\partial f}{\partial t},$$

with the compatibility initial conditions

$$\frac{\partial u_0}{\partial t}(0, \cdot) = g_1(\cdot) \in V, \quad \frac{\partial}{\partial t} \frac{\partial u_0}{\partial t}(0, \cdot) = f(0, \cdot) - \nabla(A^0(\cdot) \nabla g_0(\cdot)) \in H.$$

We have further that

$$(4.10) \quad \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 u_0}{\partial t^2} \right) - \nabla \cdot (A^0 \nabla \frac{\partial^2 u_0}{\partial t^2}) = \frac{\partial^2 f}{\partial t^2},$$

with the compatibility initial conditions

$$\frac{\partial^2 u_0}{\partial t^2}(0, \cdot) = f(0, \cdot) - \nabla(A^0(\cdot) \nabla g_0(\cdot)) \in V$$

and

$$\frac{\partial}{\partial t} \frac{\partial^2 u_0}{\partial t^2}(0, \cdot) = \frac{\partial f}{\partial t}(0, \cdot) - \nabla(A^0(\cdot) \nabla g_1(\cdot)) \in H.$$

We then deduce (4.6). Regularity condition (4.8) is proved similarly. $\square$

Remark 4.2. In section 6, where we establish the corrector for the homogenized problem (4.3), we restrict our consideration to the case where $g_0 = 0$. In this case, condition (4.5c) holds when $f(0) \in V$ and (4.5d) holds when $g_1 \in V \cap H^2(D)$. Condition (4.7d) holds when $g_1 \in H_0^3(D)$ and $\partial f / \partial t(0) \in V$, and condition (4.7e) holds when $\partial^2 f / \partial t^2(0) \in H$ and $f(0) \in H^2(D)$.

For the finite element rates of convergence in section 5, we need spatial regularity of $\mathbf{u}$. To obtain explicit rates of convergence, we define the regularity spaces $\bar{\mathcal{H}}$ and $\hat{\mathcal{H}}$ as follows.

We define by $\bar{\mathcal{H}}_i$ the spaces of functions $w(x, y_1, \dots, y_i) \in L^2(D \times Y_1 \times \dots \times Y_{i-1}, H_{\#}^2(Y_i)/\mathbb{R})$ such that $w \in L^2(Y_1 \times \dots \times Y_i, H^1(D))$ and $w \in L^2(D \times \prod_{j \neq k} Y_j, H_{\#}^1(Y_k)/\mathbb{R})$ for all $k = 1, \dots, i-1$. The space $\bar{\mathcal{H}}_i$ is equipped with the norm

$$\|w\|_{\bar{\mathcal{H}}_i} = \|w\|_{L^2(D \times Y_1 \times \dots \times Y_{i-1}, H_{\#}^2(Y_i)/\mathbb{R})} + \|w\|_{L^2(Y_1 \times \dots \times Y_i, H^1(D))} + \sum_{k=1}^{i-1} \|w\|_{L^2(D \times \prod_{j \neq k} Y_j, H_{\#}^1(Y_k)/\mathbb{R})}.$$

We define by $\bar{\mathcal{H}}$ the space

$$(4.11) \quad \bar{\mathcal{H}} = \{\mathbf{w} = (w_0, w_1, \dots, w_n) \in \mathbf{V} : w_0 \in H^2(D), w_i \in \bar{\mathcal{H}}_i, i = 1, \dots, n\},$$

which is equipped with the norm

$$\|\mathbf{w}\|_{\bar{\mathcal{H}}} = \|w_0\|_{H^2(D)} + \sum_{i=1}^n \|w_i\|_{\bar{\mathcal{H}}_i}.$$

For the sparse tensor product approximation in subsection 5.3, we define the regularity space $\hat{\mathcal{H}} \subset \bar{\mathcal{H}}$. First, we define the spaces $\hat{\mathcal{H}}_i$ of functions $w(x, y_1, \dots, y_i)$ that are Y_j -periodic with respect to y_j for $j = 1, \dots, i$ such that, for all $(\alpha_0, \dots, \alpha_i) \in (R^d)^{i+1}$ with $0 \leq |\alpha_j| \leq 1$ for $j = 0, \dots, i-1$ and $|\alpha_i| \leq 2$,

$$\frac{\partial^{|\alpha_0|+\dots+|\alpha_i|} w}{\partial^{\alpha_0} x \partial^{\alpha_1} y_1 \dots \partial^{\alpha_i} y_i} \in L^2(D \times Y_1 \times \dots \times Y_i).$$

We equip the space $\hat{\mathcal{H}}_i$ with the norm

$$\|\mathbf{w}\|_{\hat{\mathcal{H}}_i} = \sum_{\substack{1 \leq |\alpha_i| \leq 2 \\ 0 \leq |\alpha_j| \leq 1 (0 \leq j \leq i-1)}} \left\| \frac{\partial^{|\alpha_0|+\dots+|\alpha_i|} w}{\partial^{\alpha_0} x \partial^{\alpha_1} y_1 \dots \partial^{\alpha_i} y_i} \right\|_{L^2(D \times Y_1 \times \dots \times Y_i)}.$$

In other words, it is the norm of the space $H^1(D, H_{\#}^1(Y_1, \dots, H_{\#}^1(Y_{i-1}, H_{\#}^2(Y_i)/\mathbb{R}) \dots))$.

We define the space

$$(4.12) \quad \hat{\mathcal{H}} = \{\mathbf{w} = (w_0, w_1, \dots, w_n) \in \mathbf{V} : w_0 \in H^2(D), w_i \in \hat{\mathcal{H}}_i, i = 1, \dots, n\},$$

which is equipped with the norm

$$\|\mathbf{w}\|_{\hat{\mathcal{H}}} = \|w_0\|_{H^2(D)} + \sum_{i=1}^n \|w_i\|_{\hat{\mathcal{H}}_i}.$$

We have the following results.

PROPOSITION 4.3. Assume that $f \in H^1(0, T; H)$, $g_1 \in V$, $g_0 \in V \cap H^2(D)$, and the domain D is convex. Then $\mathbf{u} \in L^\infty(0, T; \hat{\mathcal{H}})$.

Proof. As $A(x, \mathbf{y}) \in C^1(\bar{D}, C_{\#}^1(\bar{Y}_1, \dots, C_{\#}^1(\bar{Y}_n) \dots))$, $A^0(\cdot) \in C^1(\bar{D})$. From (4.9), we deduce that

$$\frac{\partial u_0}{\partial t} \in L^\infty(0, T; V) \quad \text{and} \quad \frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H).$$

Thus

$$(4.13) \quad -\nabla \cdot (A^0 \nabla u_0) = f - \frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H).$$

Due to the convexity of the domain D , from Theorem 3.2.1.2 of Grisvard [12], we deduce that $u_0 \in L^\infty(0, T; H^2(D))$. From (4.1), we have that

$$-\nabla_{y_i}(A(x, y_1, \dots, y_i) \nabla_{y_i} w_{il}) = \nabla_{y_i}(A(x, y_1, \dots, y_i) e_i).$$

From (2.3), we then have

$$(4.14) \quad w_{il} \in C^1(\bar{D}, C^1(\bar{Y}_1, \dots, H_{\#}^2(Y_i)/\mathbb{R} \dots)).$$

Therefore, from (4.4) we deduce that $u_i \in \hat{\mathcal{H}}_i$. $\square$

5. Full and sparse tensor finite element methods. As u_i is posed in the product domain $D \times Y_1 \times \dots \times Y_i$, it is natural to define the finite element approximating spaces V_i^L in the previous section as tensor product spaces. In particular, we use full tensor product spaces and sparse tensor product spaces that are constructed from increment or detailed spaces of a nested sequence. We, therefore, first define a hierarchy of finite element spaces from which the tensor product spaces are constructed.

5.1. Hierarchical finite element spaces. Let D be a polyhedron with plane sides. We divide D into simplices recursively. The set of simplices $\mathcal{T}^l$ of mesh size $h_l = O(2^{-l})$ is obtained by dividing each simplex in $\mathcal{T}^{l-1}$ into four congruent triangles for $d = 2$ or 8 tetrahedra for $d = 3$. The cube Y is divided into sets of simplices $\mathcal{T}_{\#}^l$ which are periodically distributed in a similar manner. We define the following finite element spaces:

$$\begin{aligned} V^l &= \{u \in H^1(D) : u|_K \in \mathcal{P}_1(K) \ \forall K \in \mathcal{T}^l\}, \\ V_0^l &= \{u \in H_0^1(D) : u|_K \in \mathcal{P}_1(K) \ \forall K \in \mathcal{T}^l\}, \\ V_{\#}^l &= \{u \in H_{\#}^1(Y) : u|_K \in \mathcal{P}_1(K) \ \forall K \in \mathcal{T}_{\#}^l\}, \end{aligned}$$

where $\mathcal{P}_1(K)$ denotes the set of linear polynomials in K . We recall the following approximating properties for functions with sufficient regularity (see, e.g., Ciarlet [6]). For $s \geq 0$

$$\inf_{v \in V^l} \|u - v\|_{L^2(D)} \leq ch_l^{1+s} \|u\|_{H^{1+s}(D)}$$

for all $u \in H^{1+s}(D)$, and

$$\inf_{v \in V_0^l} \|u - v\|_{H^1(D)} \leq ch_l^s \|u\|_{H^{1+s}(D)}$$

for $u \in H^{1+s}(D) \cap V$. For periodic functions on Y we have

$$\begin{aligned} \inf_{v \in V_{\#}^l/\mathbb{R}} \|u - v\|_{H_{\#}^1(Y)/\mathbb{R}} &\leq ch_l^s \|u\|_{H_{\#}^{1+s}(Y)/\mathbb{R}}, \\ \inf_{v \in V_{\#}^l} \|u - v\|_{L^2(Y)} &\leq ch_l^{1+s} \|u\|_{H_{\#}^{1+s}(Y)} \end{aligned}$$

for all periodic functions $u \in H_{\#}^{1+s}(Y)$.

5.2. Full tensor product finite elements. As $u_i \in L^2(D \times Y_1 \times \cdots \times Y_{i-1}, H_{\#}^1(Y_i)/\mathbb{R}) \cong L^2(D) \otimes L^2(Y_1) \otimes \cdots \otimes L^2(Y_{i-1}) \otimes H_{\#}^1(Y_i)/\mathbb{R}$, we choose $\bar{V}_i^L$ as

$$\bar{V}_i^L = V^L \otimes \underbrace{V_{\#}^L \otimes \cdots \otimes V_{\#}^L}_{i-1 \text{ times}} \otimes V_{\#}^L / \mathbb{R}$$

to approximate u_i . We define

$$\bar{\mathbf{V}}^L = \{(\bar{v}_0^L, \{\bar{v}_i^L\}) : \bar{v}_0^L \in V_0^L, \bar{v}_i^L \in \bar{V}_i^L\}.$$

For these spaces, when w is in the regularity space $\bar{\mathcal{H}}_i$ defined in section 4, we have the following approximating properties.

LEMMA 5.1. For $w \in \bar{\mathcal{H}}_i$,

$$\inf_{v^L \in V_i^L} \|w - v^L\|_{L^2(D \times Y_1 \times \cdots \times Y_{i-1}, H_{\#}^1(Y_i)/\mathbb{R})} \leq ch_L \|w\|_{\bar{\mathcal{H}}_i}.$$

We refer to Hoang and Schwab [15] for a proof. For the space $\bar{\mathcal{H}}$ defined in (4.11), we deduce that for $\mathbf{w} \in \bar{\mathcal{H}}$,

$$\inf_{\mathbf{v}^L \in \bar{\mathbf{V}}^L} \|\mathbf{w} - \mathbf{v}^L\|_{\mathbf{V}} \leq ch_L \|\mathbf{w}\|_{\bar{\mathcal{H}}}.$$

The spatial semidiscrete problem (3.1) is now written as follows: Find $\bar{\mathbf{u}}^L = (\bar{u}_0^L, \bar{u}_1^L, \dots, \bar{u}_n^L)$ such that

$$(5.1) \quad \left(\frac{\partial^2 \bar{u}_0^L}{\partial t^2}, \bar{v}_0^L \right)_H + B(\bar{\mathbf{u}}^L, \bar{\mathbf{v}}^L) = (f, \bar{v}_0^L)_H \quad \forall \bar{\mathbf{v}}^L \in \bar{\mathbf{V}}^L$$

with $\bar{u}_0^L(0) = g_0^L$ and $\frac{\partial \bar{u}_0^L}{\partial t}(0) = g_1^L$. Equation (3.7) now becomes

$$\zeta^L = \bar{\mathbf{u}}^L - \mathbf{w}^L.$$

We then have the following result for the continuous time approximation.

PROPOSITION 5.2. Assume that condition (4.5) holds and the domain D is convex and that g_0^L and g_1^L are chosen so that

$$(5.2) \quad \|g_0^L - g_0\|_V \leq ch_L \quad \text{and} \quad \|g_1^L - g_1\|_H \leq ch_L,$$

then

$$\left\| \frac{\partial(\bar{u}_0^L - u_0)}{\partial t} \right\|_{L^\infty(0, T; H)} + \|\bar{\mathbf{u}}^L - \mathbf{u}\|_{L^\infty(0, T; \mathbf{V})} \leq ch_L.$$

Proof. We show that the right-hand side of (3.11) is smaller than ch_L^2 where c is independent of L . With condition (4.5), from (4.10), we deduce that $\frac{\partial^2 u_0}{\partial t^2} \in L^2(0, T; V)$. From (4.4) and (4.14), we have $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2(0, T; \mathbf{V})$. From (4.13), we have that $u_0 \in L^\infty(0, T; H^2(D))$ so $\mathbf{u} \in L^\infty(0, T; \bar{\mathcal{H}})$.

From Lemmas 3.2 and 5.1, we deduce that for all $t \in [0, T]$

$$\|\boldsymbol{\eta}^L(t)\|_{\mathbf{V}} \leq ch_L \|\mathbf{u}(t)\|_{\bar{\mathcal{H}}}.$$

We have further that

$$\|\nabla \zeta_0^L(0)\|_H \leq \|\nabla u_0^L(0) - \nabla u_0(0)\|_H + \|\nabla u_0(0) - \nabla w_0^L(0)\|_H \leq ch_L,$$

due to (5.2) and $\|\boldsymbol{\eta}(0)\|_{\mathbf{V}} \leq ch_L$. We now show that for all $t \in [0, T]$

$$\left\| \frac{\partial \eta_0^L}{\partial t} \right\|_{L^\infty(0, T; H)} = \left\| \frac{\partial u_0}{\partial t} - \frac{\partial w_0^L}{\partial t} \right\|_{L^\infty(0, T; H)} \leq ch_L,$$

where c is independent of L . We consider $\phi(t) \in \mathbf{V}$ as the unique solution of the elliptic problem

$$(5.3) \quad B(\phi(t), \mathbf{v}) = \int_D \frac{\partial(u_0 - w_0^L)}{\partial t}(t, x) v_0(x) dx$$

for all $\mathbf{v} \in \mathbf{V}$. This problem has a unique solution as $\frac{\partial(u_0 - w_0^L)}{\partial t}(t, \cdot) \in H$. We have that ϕ_0 is the solution of the elliptic problem

$$-\nabla \cdot (A^0(x) \nabla \phi_0(t, x)) = \frac{\partial(u_0 - w_0^L)}{\partial t}.$$

As D is convex,

$$\|\phi_0(t)\|_{H^2(D)} \leq c \left\| \frac{\partial(u_0 - w_0^L)}{\partial t}(t) \right\|_H.$$

Further, similarly to (4.4), we have

$$\phi_i(t) = w_i \cdot (I + \nabla_{y_{i-1}} w_{i-1}) \dots (I + \nabla_{y_1} w_1) \nabla \phi_0(t),$$

which implies

$$\|\phi(t)\|_{\mathcal{H}} \leq c \left\| \frac{\partial(u_0 - w_0^L)}{\partial t}(t, \cdot) \right\|_H.$$

Let $\mathbf{v} = \frac{\partial(\mathbf{u} - \mathbf{w}^L)}{\partial t}(t) \in \mathbf{V}$. We have

$$B\left(\phi(t), \frac{\partial(\mathbf{u} - \mathbf{w}^L)}{\partial t}(t)\right) = \left\| \frac{\partial(u_0 - w_0^L)}{\partial t}(t) \right\|_H^2.$$

From (3.6), due to the symmetry of B , for all $\mathbf{v}^L \in V^L$

$$B\left(\phi(t) - \mathbf{v}^L, \frac{\partial(\mathbf{u} - \mathbf{w}^L)}{\partial t}(t)\right) = \left\| \frac{\partial(u_0 - w_0^L)}{\partial t}(t) \right\|_H^2.$$

We, therefore, have

$$\begin{aligned} \left\| \frac{\partial(u_0 - w_0^L)}{\partial t}(t) \right\|_H^2 &\leq \beta \inf_{\mathbf{v}^L \in \mathbf{V}^L} \|\phi(t) - \mathbf{v}^L\|_{\mathbf{V}} \left\| \frac{\partial(\mathbf{u} - \mathbf{w}^L)}{\partial t}(t) \right\|_{\mathbf{V}} \\ &\leq ch_L \|\phi(t)\|_{\mathcal{H}} \left\| \frac{\partial(\mathbf{u} - \mathbf{w}^L)}{\partial t}(t) \right\|_{\mathbf{V}} \\ &\leq ch_L \left\| \frac{\partial(u_0 - w_0^L)}{\partial t}(t) \right\|_H \left\| \frac{\partial(\mathbf{u} - \mathbf{w}^L)}{\partial t}(t) \right\|_{\mathbf{V}}. \end{aligned}$$

From Lemma 3.3, as $\frac{\partial \mathbf{u}}{\partial t} \in L^\infty(0, T; \mathbf{V})$, we deduce that $\|\frac{\partial(\mathbf{u}-\mathbf{w}^L)}{\partial t}(t)\|_{\mathbf{V}}$ is bounded uniformly for all t . Hence

$$\left\| \frac{\partial(u_0 - w_0^L)}{\partial t}(t) \right\|_H \leq ch_L.$$

In particular, for $t = 0$, we have

$$\left\| \frac{\partial \zeta_0^L}{\partial t}(0, \cdot) \right\|_H \leq \|g_1^L - g_1\|_H + \left\| \frac{\partial u_0}{\partial t}(0) - \frac{\partial w_0^L}{\partial t}(0) \right\|_H \leq ch_L.$$

Similarly, by considering the problem

$$B(\phi(t), \mathbf{v}) = \int_D \frac{\partial^2(u_0 - w_0^L)}{\partial t^2}(t, x) v_0(x) dx,$$

using the fact that $\|\frac{\partial^2(\mathbf{u}-\mathbf{w}^L)}{\partial t^2}\|_{L^2(0, T; \mathbf{V})}$ is bounded with respect to L , we deduce that

$$\left\| \frac{\partial^2(u_0 - w_0^L)}{\partial t^2} \right\|_{L^2(0, T; H)} \leq ch_L.$$

From this, we get the conclusion. $\square$

The fully discrete problem is now written as follows: Find $\bar{\mathbf{u}}_m^L = (\bar{u}_{0,m}^L, \bar{u}_{1,m}^L, \dots, \bar{u}_{n,m}^L) \in \bar{\mathbf{V}}^L$ such that

$$(5.4) \quad (\partial_t^2 \bar{u}_{0,m}^L, \bar{v}_0^L)_H + B(\bar{\mathbf{u}}_{m,1/4}^L, \bar{\mathbf{v}}^L) = (f_{m,1/4}, \bar{v}_0^L)_H$$

for all $\bar{\mathbf{v}}^L = (\bar{v}_0^L, \bar{v}_1^L, \dots, \bar{v}_n^L) \in \bar{\mathbf{V}}^L$.

We have the following result.

PROPOSITION 5.3. Assume that condition (4.5) holds and that D is a convex domain. If

$$\|\partial_t \zeta_{0,1/2}^L\|_H \leq c(\Delta t + h_L), \quad \|\zeta_{1/2}^L\|_{\mathbf{V}} \leq c(\Delta t + h_L),$$

then

$$\|\partial_t \bar{u}_0^L - \partial_t u_0\|_{\tilde{L}^\infty(0, T; H)} + \|\bar{\mathbf{u}}^L - \mathbf{u}\|_{\tilde{L}^\infty(0, T; \mathbf{V})} \leq c(\Delta t + h_L).$$

If (4.7) holds, and

$$\|\partial_t \zeta_{0,1/2}^L\|_H \leq c((\Delta t)^2 + h_L), \quad \|\zeta_{1/2}^L\|_{\mathbf{V}} \leq c((\Delta t)^2 + h_L),$$

then

$$\|\partial_t \bar{u}_0^L - \partial_t u_0\|_{\tilde{L}^\infty(0, T; H)} + \|\bar{\mathbf{u}}^L - \mathbf{u}\|_{\tilde{L}^\infty(0, T; \mathbf{V})} \leq c((\Delta t)^2 + h_L).$$

Proof. We use (3.15) and (3.16). As shown in the proof of Proposition 5.2, we have

$$\left\| \frac{\partial^2 \eta_0^L}{\partial t^2} \right\|_{L^2(0, T; H)} \leq ch_L.$$

From Lemma 3.2 and the regularity $\mathbf{u} \in L^\infty(0, T; \bar{\mathcal{H}})$, we have

$$\|\boldsymbol{\eta}^L\|_{\tilde{L}^\infty(0, T; \mathbf{V})} \leq ch_L.$$

It remains to show that $\|\partial_t \eta_0^L\|_{\tilde{L}^\infty(0, T; H)} \leq ch_L$. We proceed similarly as for the proof of Proposition 5.2. We consider $\phi_m \in \mathbf{V}$ as the solution of the problem

$$B(\phi_m, \mathbf{v}) = \int_D \frac{\eta_{0, m+1}^L - \eta_{0, m}^L}{\Delta t} v_0 dx$$

for all $\mathbf{v} = (v_0, v_1, \dots, v_n) \in \mathbf{V}$. As $\frac{\eta_{0, m+1}^L - \eta_{0, m}^L}{\Delta t} \in H$, we deduce that $\phi_m \in \bar{\mathcal{H}}$ and that

$$\|\phi_m\|_{\bar{\mathcal{H}}} \leq c \left\| \frac{\eta_{0, m+1}^L - \eta_{0, m}^L}{\Delta t} \right\|_H.$$

Let $\mathbf{v} = \frac{\eta_{m+1}^L - \eta_m^L}{\Delta t} \in \mathbf{V}$, we have that

$$B\left(\phi_m, \frac{\eta_{m+1}^L - \eta_m^L}{\Delta t}\right) = \left\| \frac{\eta_{0, m+1}^L - \eta_{0, m}^L}{\Delta t} \right\|^2.$$

From (3.2), we have

$$B\left(\phi_m - \mathbf{v}^L, \frac{\eta_{m+1}^L - \eta_m^L}{\Delta t}\right) = \left\| \frac{\eta_{0, m+1}^L - \eta_{0, m}^L}{\Delta t} \right\|^2$$

for all $\mathbf{v}^L \in \mathbf{V}^L$. We then deduce

$$\begin{aligned} \left\| \frac{\eta_{0, m+1}^L - \eta_{0, m}^L}{\Delta t} \right\|_H^2 &\leq \beta \inf_{\mathbf{v}^L \in \mathbf{V}^L} \|\phi_m - \mathbf{v}^L\|_{\mathbf{V}} \left\| \frac{\eta_{m+1}^L - \eta_m^L}{\Delta t} \right\|_{\mathbf{V}} \\ &\leq ch_L \left\| \frac{\eta_{0, m+1}^L - \eta_{0, m}^L}{\Delta t} \right\|_H \left\| \frac{\eta_{m+1}^L - \eta_m^L}{\Delta t} \right\|_{\mathbf{V}}. \end{aligned}$$

As $\mathbf{u} \in H^2(0, T; \mathbf{V})$ so $\boldsymbol{\eta}^L \in H^2(0, T; \mathbf{V})$. We have that

$$\left\| \frac{\eta_{m+1}^L - \eta_m^L}{\Delta t} \right\|_{\mathbf{V}} \leq c \sup_{t \in (0, T)} \left\| \frac{\partial \boldsymbol{\eta}^L}{\partial t}(t) \right\|_{\mathbf{V}},$$

which is uniformly bounded for all Δt due to Lemma 3.3. From this we have

$$\left\| \frac{\eta_{0, m+1}^L - \eta_{0, m}^L}{\Delta t} \right\|_H \leq ch_L.$$

We then get the conclusion. $\square$

5.3. Sparse tensor product finite elements. The dimension of the full tensor product space $\bar{\mathbf{V}}^L$ is $O(2^{ndL})$ which is prohibitively large when n and L are large for solving problems (5.1) and (5.4). In this section, we develop the sparse tensor product FE spaces with much reduced dimension but approximate both the spatial semidiscrete problem and the fully discrete problem with essentially equal accuracy as the full tensor product FEs. We first define the following orthogonal projections in the norm of $L^2(D)$, $L^2(Y)$, and $H_{\#}^1(Y)$:

$$P^{l0} : L^2(D) \rightarrow V^l, \quad P_{\#}^{l0} : L^2(Y) \rightarrow V_{\#}^l, \quad P_{\#}^{l1} : H_{\#}^1(Y)/\mathbb{R} \rightarrow V_{\#}^l/\mathbb{R}.$$

The increment spaces are defined as

$$W^l = (P^{l0} - P^{(l-1)0})V^l, \quad W_{\#}^{l0} = (P_{\#}^{l0} - P_{\#}^{(l-1)0})V_{\#}^l, \quad W_{\#}^{l1} = (P_{\#}^{l1} - P_{\#}^{(l-1)1})V_{\#}^l/\mathbb{R}$$

with the convention that $P^{-10} = 0$, $P_{\#}^{-10} = 0$, and $P_{\#}^{-11} = 0$. We then have

$$V^l = \bigoplus_{0 \leq i \leq l} W^i, \quad V_{\#}^l = \bigoplus_{0 \leq i \leq l} W_{\#}^{i0}, \quad V_{\#}^l/\mathbb{R} = \bigoplus_{0 \leq i \leq l} W_{\#}^{i1}.$$

The full tensor product space $\bar{V}_i^L$ can be written as

$$\bar{V}_i^L = \bigoplus_{\substack{0 \leq j_k \leq L \\ k=0,1,\dots,i}} W^{j_0} \otimes W_{\#}^{j_1 0} \otimes \dots \otimes W_{\#}^{j_{i-1} 0} \otimes W_{\#}^{j_i 1}.$$

We define the sparse tensor product spaces as

$$\hat{V}_i^L = \bigoplus_{0 \leq j_0 + j_1 + \dots + j_i \leq L} W^{j_0} \otimes W_{\#}^{j_1 0} \otimes \dots \otimes W_{\#}^{j_{i-1} 0} \otimes W_{\#}^{j_i 1},$$

and the space

$$\hat{\mathbf{V}}^L = \{(\hat{u}_0^L, \{\hat{u}_i^L\}) : \hat{u}_0^L \in V_0^L, \hat{u}_i^L \in \hat{V}_i^L\}.$$

For functions in the regularity spaces $\hat{\mathcal{H}}_i$ defined in section 4, we have the following estimate.

LEMMA 5.4. For $w \in \hat{\mathcal{H}}_i$,

$$\inf_{v^L \in \hat{V}_i^L} \|w - v^L\|_{V_i} \leq cL^{i/2} h_L \|w\|_{\hat{\mathcal{H}}_i}.$$

For $\mathbf{w} \in \hat{\mathcal{H}}$, we then have

$$\inf_{\mathbf{v}^L \in \hat{\mathbf{V}}^L} \|\mathbf{w} - \mathbf{v}^L\|_{\mathbf{V}} \leq cL^{n/2} h_L \|\mathbf{w}\|_{\hat{\mathcal{H}}}.$$

The proof of this lemma can be found in Hoang and Schwab [15].

The spatial semidiscrete problem (3.1) now becomes the following: find $\hat{\mathbf{u}}^L = (\hat{u}_0^L, \hat{u}_1^L, \dots, \hat{u}_n^L)$ such that

$$(5.5) \quad \left(\frac{\partial^2 \hat{u}_0^L}{\partial t^2}, \hat{v}_0^L \right)_H + B(\hat{\mathbf{u}}^L, \hat{\mathbf{v}}^L) = (f, \hat{v}_0^L)_H \quad \forall \hat{\mathbf{v}}^L \in \hat{\mathbf{V}}^L$$

with $\hat{u}_0^L = g_0^L$ and $\frac{\partial \hat{u}_1^L}{\partial t} = g_1^L$. We then have the following result for the semidiscrete problem.

PROPOSITION 5.5. Assume that condition (4.5) holds and the domain D is convex and that g_0^L and g_1^L are chosen so that

$$(5.6) \quad \|g_0^L - g_0\|_V \leq cL^{n/2}h_L \quad \text{and} \quad \|g_1^L - g_1\|_H \leq cL^{n/2}h_L,$$

then

$$\left\| \frac{\partial(\hat{u}_0^L - u_0)}{\partial t} \right\|_{L^\infty(0,T;H)} + \|\hat{\mathbf{u}}^L - \mathbf{u}\|_{L^\infty(0,T;\mathbf{V})} \leq cL^{n/2}h_L.$$

The proof of the proposition is identical to that for Proposition 5.2. Indeed, the regularity $\phi_0 \in H^2(D)$ implies that $\phi(t)$ defined in (5.3) is in $\hat{\mathcal{H}}$ and that

$$\|\phi(t)\|_{\hat{\mathcal{H}}} \leq c \left\| \frac{\partial(u_0 - w_0^L)}{\partial t}(t) \right\|_H.$$

From this, by a similar argument as in the proof of Proposition 5.2, we get

$$\left\| \frac{\partial(u_0 - w_0^L)}{\partial t}(t) \right\|_H \leq ch_L L^{n/2}. \quad \square$$

The fully discrete problem (3.12) becomes the following: Find $\hat{\mathbf{u}}_m^L = (\hat{u}_{0,m}^L, \hat{u}_{1,m}^L, \dots, \hat{u}_{n,m}^L) \in \hat{\mathbf{V}}^L$ such that

$$(5.7) \quad (\partial_t^2 \hat{u}_{0,m}^L, \hat{v}_0^L)_H + B(\hat{\mathbf{u}}_{m,1/4}^L, \hat{\mathbf{v}}^L) = (f_{m,1/4}, \hat{v}_0^L)_H$$

for all $\hat{\mathbf{v}}^L = (\hat{v}_0^L, \hat{v}_1^L, \dots, \hat{v}_n^L) \in \hat{\mathbf{V}}^L$.

We have the following result.

PROPOSITION 5.6. Assume that condition (4.5) holds and that D is a convex domain. Assume further that

$$\|\partial_t \zeta_{0,1/2}^L\|_H \leq c(\Delta t + h_L L^{n/2}), \quad \|\zeta_{1/2}^L\|_{\mathbf{V}} \leq c(\Delta t + h_L L^{n/2}).$$

Then

$$\|\partial_t \hat{u}_0^L - \partial_t u_0\|_{\tilde{L}^\infty(0,T;H)} + \|\hat{\mathbf{u}}^L - \mathbf{u}\|_{\tilde{L}^\infty(0,T;\mathbf{V})} \leq c(\Delta t + h_L L^{n/2}).$$

If condition (4.7) holds and

$$\|\partial_t \zeta_{0,1/2}^L\|_H \leq c((\Delta t)^2 + h_L L^{n/2}), \quad \|\zeta_{1/2}^L\|_{\mathbf{V}} \leq c((\Delta t)^2 + h_L L^{n/2}),$$

then

$$\|\partial_t \hat{u}_0^L - \partial_t u_0\|_{\tilde{L}^\infty(0,T;H)} + \|\hat{\mathbf{u}}^L - \mathbf{u}\|_{\tilde{L}^\infty(0,T;\mathbf{V})} \leq c((\Delta t)^2 + h_L L^{n/2}).$$

The proof of this proposition is similar to that for Proposition 5.3.

The dimension of the sparse tensor product space $\hat{\mathbf{V}}^L$ is $O(h_L^{-d} L^n)$, which is much less than the dimension of $\bar{\mathbf{V}}^L$. The error we obtain in Proposition 5.6 is essentially equal to that obtained in Proposition 5.3. We will make some remarks on the choice of $\hat{\mathbf{u}}_0^L$ and $\hat{\mathbf{u}}_1^L$ in Appendix B.

6. Corrector for the homogenization problem (2.8). For two-scale elliptic problems it is well known that with sufficient regularity for the solutions $w_i(x, y)$ of cell problems (6.1), there is always a corrector for the weak convergence $u^\varepsilon \rightharpoonup u_0$ (see, e.g., [4]). However, a similar corrector result for the wave equation (2.5) does not always hold as the energy of the multiscale problem (2.5) only converges to the energy of the homogenized equation when the initial condition g_0 satisfies special conditions. Brahim-Otsmane, Francfort, and Murat [5] proved a corrector result for wave equation (2.5) with another term v^ε which satisfies the multiscale wave equation (2.5) with zero right-hand side and special initial conditions.

In this section, we restrict our attention to the case where the initial condition is $g_0 = 0$. This allows us to deduce corrector results similar to those for elliptic equations, and to deduce a homogenization rate of convergence for the two-scale case. These results will be used to deduce the convergence of the finite element solution $\mathbf{u}^L$ in the physical variable x .

6.1. Corrector for two-scale problem. For two-scale problems ($n = 1$), we deduce in this section a homogenization rate of convergence. We denote the coefficient $A(x, \mathbf{y})$ in this case as $A(x, y)$.

For $i = 1, \dots, d$, we let $w_i(x, y) \in L^2(D; H_{\#}^1(Y)/\mathbb{R})$ be the solution of the cell problem

$$(6.1) \quad -\nabla_y \cdot (A(x, y)(e_i + \nabla_y w_i(x, y))) = 0,$$

where e_i is the i th unit vector in $\mathbb{R}^d$. The homogenized coefficient $A^0(x)$ is defined as

$$(6.2) \quad A^0(x) = \int_Y A(x, y)(\mathbf{I} + \nabla_y w) \cdot (\mathbf{I} + \nabla_y w) dy,$$

where $\mathbf{I}$ is the identity matrix and $w = (w_1, w_2, \dots, w_d) \in (L^2(D, H_{\#}^1(Y)/\mathbb{R}))^d$. Since $A(x, y) \in C^1(\bar{D}, C_{\#}^1(\bar{Y}))$, $w_i(x, y) \in C^1(\bar{D}, H_{\#}^1(Y))$, and, therefore, $A^0(x) \in (C^1(\bar{D}))_{sym}^{d \times d}$. We then have the following result on the homogenization rate of convergence.

PROPOSITION 6.1. *For two-scale problems, assume that*

$$g_0 = 0, \quad g_1 \in H^2(D) \cap V, \quad f \in H^2(0, T; H), \quad f(0) \in V$$

and D is a convex domain. There exists a constant c that does not depend on ε such that

$$\left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t} \right\|_{L^\infty(0, T; H)} + \left\| \nabla u^\varepsilon - \left[\nabla u_0 + \nabla_y u_1 \left(\cdot, \cdot, \frac{\cdot}{\varepsilon} \right) \right] \right\|_{L^\infty(0, T; H)} \leq c\varepsilon^{1/2}.$$

We present the proof of this proposition in Appendix C.

6.2. Corrector for multiscale problem. For general multiscale problems, we are not aware of any analytic homogenization rates of convergence, even for elliptic problems. However, for the case where the scales are such that $\varepsilon_i/\varepsilon_{i+1}$ is an integer for all $i = 1, \dots, n-1$, a corrector for the elliptic problem is deduced in Hoang and Schwab [15] following the earlier work by Cioranescu, Damlamian, and Griso [7]. We now deduce a corresponding corrector for the general multiscale wave equation (2.5). We will use the following operator.

DEFINITION 6.2. The $(n+1)$ -scale “unfolding” operator $\mathcal{T}_n^\varepsilon : L^1(D) \rightarrow L^1(D \times \mathbf{Y})$ is defined as

$$\mathcal{T}_n^\varepsilon(\phi)(x, \mathbf{y}) = \phi\left(\varepsilon_1 \left[\frac{x}{\varepsilon_1}\right] + \varepsilon_2 \left[\frac{y_1}{\varepsilon_2/\varepsilon_1}\right] + \cdots + \varepsilon_n \left[\frac{y_{n-1}}{\varepsilon_n/\varepsilon_{n-1}}\right] + \varepsilon_n y_n\right),$$

where the function $\phi \in L^1(D \times Y)$ is understood as 0 outside D ; $[\cdot]$ denotes the “integer” part with respect to Y .

Let D^ε be the 2ε neighborhood of D , we have

$$(6.3) \quad \int_D \phi dx = \int_{D^\varepsilon} \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(\phi) d\mathbf{y} dx.$$

A simple proof for this can be found in, e.g., [7]. It can be shown that

$$(6.4) \quad \mathcal{T}_n^\varepsilon(\nabla u^\varepsilon) \rightharpoonup \nabla u_0 + \nabla_{y_1} u_1 + \cdots + \nabla_{y_n} u_n \text{ in } L^2(D \times \mathbf{Y}).$$

Following [7], we define the “folding” operator $\mathcal{U}_n^\varepsilon$ as follows.

DEFINITION 6.3. For $\Phi \in L^1(D \times \mathbf{Y})$ (understood to be zero when $x \notin D$) and for $\varepsilon > 0$ sufficiently small, the “folding” operator $\mathcal{U}_n^\varepsilon(\Phi) \in L^1(D)$ is defined as

$$(6.5) \quad \begin{aligned} \mathcal{U}_n^\varepsilon(\Phi)(x) = & \int_{Y_1} \cdots \int_{Y_n} \Phi\left(\varepsilon_1 \left[\frac{x}{\varepsilon_1}\right] + \varepsilon_1 t_1, \frac{\varepsilon_2}{\varepsilon_1} \left[\frac{\varepsilon_1}{\varepsilon_2} \left\{\frac{x}{\varepsilon_1}\right\}\right] + \frac{\varepsilon_2}{\varepsilon_1} t_2, \dots, \right. \\ & \left. \frac{\varepsilon_n}{\varepsilon_{n-1}} \left[\frac{\varepsilon_{n-1}}{\varepsilon_n} \left\{\frac{x}{\varepsilon_{n-1}}\right\}\right] + \frac{\varepsilon_n}{\varepsilon_{n-1}} t_n, \left\{\frac{x}{\varepsilon_n}\right\}\right) dt_n \cdots dt_1 \end{aligned}$$

where $\{\cdot\} = \cdot - [\cdot]$.

For $\Phi \in L^1(D \times \mathbf{Y})$ which are understood as 0 when $x \notin D$,

$$\int_{D^\varepsilon} \mathcal{U}_n^\varepsilon(\Phi)(x) dx = \int_D \int_{\mathbf{Y}} \Phi d\mathbf{y} dx$$

(see [7]). We have the following results.

PROPOSITION 6.4. For the multiscale wave equation (2.5), assuming that $g_0 = 0$, $g_1 \in V \cap H^2(D)$, and $f \in H^1(0, T; H)$, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t} \right\|_{L^\infty(0, T; H)} \\ + \|\nabla u^\varepsilon(t) - \mathcal{U}_n^\varepsilon(\nabla u_0(t) + \nabla_{y_1} u_1(t) + \cdots + \nabla_{y_n} u_n(t))\|_{L^\infty(0, T; H)} = 0. \end{aligned}$$

We present the proof of this proposition in Appendix D.

7. Convergence in the physical variable x . We now derive numerical correctors from the finite element approximation of problem (2.8) for the multiscale solution u^ε .

7.1. Convergence in the physical variable for two-scale problem. In this section, we establish the convergence in the physical variables of the solutions of problems (3.1), (3.12) for both the full and sparse tensor approximations. In the two scale case, the operator $\mathcal{U}_n^\varepsilon$ defined in (6.5) becomes

$$(7.1) \quad \mathcal{U}^\varepsilon(\Phi)(x) = \int_Y \Phi\left(\varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}\right) dz.$$

Let D^ε be a 2ε neighborhood of D . Regarding Φ as zero when x is outside D , we have

$$(7.2) \quad \int_{D^\varepsilon} \mathcal{U}^\varepsilon(\Phi)(x) dx = \int_D \int_Y \Phi(x, y) dx dy.$$

We then have the following result.

LEMMA 7.1. Assume that $g_0 = 0$, $g_1 \in V$, $f \in H^1(0, T; H)$, and the domain D is convex. Then

$$\sup_{t \in (0, T)} \int_D \left| \nabla_y u_1 \left(t, x, \frac{x}{\varepsilon} \right) - \mathcal{U}^\varepsilon(\nabla_y u_1(t, \cdot, \cdot))(x) \right|^2 dx \leq c\varepsilon^2.$$

Proof. We note from (4.9) that $\partial^2 u_0 / \partial t^2 \in L^\infty(0, T; H)$. Therefore, from (4.3), we have

$$-\nabla \cdot (A^0 \nabla u_0) = f - \frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H).$$

Thus $u_0 \in L^\infty(0, T; H^2(D))$. Since $A \in C^1(\bar{D}, C_\#^1(\bar{Y}))$, $w_l(x, y) \in C^1(\bar{D}, C_\#^1(\bar{Y}))$. The proof then follows from that of Lemma 5.5 of Hoang and Schwab [16]. $\square$

We then have the following result on the semidiscrete approximation (3.1).

THEOREM 7.2. Assume that $f \in H^1(0, T; H)$, $g_0 = 0$, $g_1 \in V \cap H^2(D)$, $f(0) \in V$, the domain D is convex, and the initial conditions are chosen so that $g_0^L = 0$ and $\|g_1^L - g_1\|_H \leq ch_L$. Then the solution of the semidiscrete problem (5.1) for the full tensor product FE spaces satisfies

$$\left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial \bar{u}_0^L}{\partial t} \right\|_{L^\infty(0, T; H)} + \|\nabla u^\varepsilon - [\nabla \bar{u}_0^L + \mathcal{U}^\varepsilon(\nabla_y \bar{u}_1^L)]\|_{L^\infty(0, T; H)} \leq c(h_L + \varepsilon^{1/2}),$$

and the solution of problem (5.5) for the sparse tensor product FE spaces satisfies

$$\left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial \hat{u}_0^L}{\partial t} \right\|_{L^\infty(0, T; H)} + \|\nabla u^\varepsilon - [\nabla \hat{u}_0^L + \mathcal{U}^\varepsilon(\nabla_y \hat{u}_1^L)]\|_{L^\infty(0, T; H)} \leq c(h_L L^{n/2} + \varepsilon^{1/2}).$$

Proof. With the hypothesis of the theorem, condition (4.5) holds. The regularity required for Propositions 5.2, 5.5, and 6.1 hold. For all $\Phi \in L^2(D \times Y)$, we have

$$(\mathcal{U}^\varepsilon(\Phi)(x))^2 \leq \mathcal{U}^\varepsilon(\Phi^2)(x).$$

Therefore, for $t \in [0, T]$

$$\|\mathcal{U}^\varepsilon(\nabla_y u_1(t)) - \mathcal{U}^\varepsilon(\nabla_y \bar{u}_1^L(t))\|_H \leq \|\nabla_y u_1(t) - \nabla_y \bar{u}_1^L(t)\|_{L^2(D \times Y)}.$$

From Propositions 5.2, 5.5, and 6.1 we get the conclusion for the full tensor product approximation. The result for sparse tensor product approximation can be derived similarly. $\square$

For the fully discrete problems (5.4) and (5.7) we have the following result.

THEOREM 7.3. Assume that $g_0 = 0$ and the domain D is convex. If

$$(7.3) \quad f \in H^2(0, T; H), \quad g_1 \in V \cap H^2(D), \quad f(0) \in V,$$

then when

$$\|\partial_t \zeta_{0,1/2}^L\|_H \leq c(\Delta t + h_L), \quad \|\zeta_{1/2}^L\|_V \leq c(\Delta t + h_L)$$

for the full tensor product approximating problem (5.4) we have

$$(7.4) \quad \max_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t \bar{u}_{0,m+1/2}^L \right\|_H + \|\nabla u^\varepsilon - [\nabla \bar{u}_0^L + \mathcal{U}^\varepsilon(\nabla_y \bar{u}_1^L)]\|_{\tilde{L}^\infty(0,T;H)} \leq c(\Delta t + h_L + \varepsilon^{1/2})$$

and when

$$\|\partial_t \zeta_{0,1/2}^L\|_H \leq c(\Delta t + h_L L^{1/2}), \quad \|\zeta_{1/2}^L\|_{\mathbf{V}} \leq c(\Delta t + h_L L^{1/2}),$$

for the sparse tensor product approximating problem (5.7) we have

$$(7.5) \quad \max_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t \hat{u}_{0,m+1/2}^L \right\|_H + \|\nabla u^\varepsilon - [\nabla \hat{u}_0^L + \mathcal{U}^\varepsilon(\nabla_y \hat{u}_1^L)]\|_{\tilde{L}^\infty(0,T;H)} \leq c(\Delta t + h_L L^{1/2} + \varepsilon^{1/2}).$$

If we assume further that

$$(7.6) \quad f \in H^3(0, T; H), \quad g_1 \in H_0^3(D), \quad \text{and } f(0) \in H^2(D),$$

then when

$$\|\partial_t \zeta_{0,1/2}^L\|_H \leq c((\Delta t)^2 + h_L), \quad \|\zeta_{1/2}^L\|_{\mathbf{V}} \leq c((\Delta t)^2 + h_L),$$

for the full tensor approximating problem (5.4) we have

$$(7.7) \quad \Delta t \max_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t \bar{u}_{0,m+1/2}^L \right\|_H + \|\nabla u^\varepsilon - [\nabla \bar{u}_0^L + \mathcal{U}^\varepsilon(\nabla_y \bar{u}_1^L)]\|_{\tilde{L}^\infty(0,T;H)} \leq c((\Delta t)^2 + h_L + \varepsilon^{1/2})$$

and when

$$\|\partial_t \zeta_{0,1/2}^L\|_H \leq c((\Delta t)^2 + h_L L^{1/2}), \quad \|\zeta_{1/2}^L\|_{\mathbf{V}} \leq c((\Delta t)^2 + h_L L^{1/2}),$$

for the sparse tensor product approximating problem (5.7)

$$(7.8) \quad \Delta t \max_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t \hat{u}_{0,m+1/2}^L \right\|_H + \|\nabla u^\varepsilon - [\nabla \hat{u}_0^L + \mathcal{U}^\varepsilon(\nabla_y \hat{u}_1^L)]\|_{\tilde{L}^\infty(0,T;H)} \leq c((\Delta t)^2 + h_L L^{1/2} + \varepsilon^{1/2}).$$

Proof. With $A(x, y) \in C^1(\bar{D}, C_\#^1(\bar{Y}))$, $g_0 = 0$ and the domain D is convex, and with either (7.3) or (7.6), the conditions of Proposition 6.1 holds. We note that

$$\frac{1}{\Delta t}(u_{0,m+1} - u_{0,m}) - \frac{\partial u_0}{\partial t}(t_m) = \frac{\partial u_0}{\partial t}(\tau) - \frac{\partial u_0}{\partial t}(t_m) = \int_{t_m}^\tau \frac{\partial^2 u_0}{\partial t^2}(\tau') d\tau'$$

for $t_m \leq \tau \leq t_{m+1}$. From (4.9), we deduce that $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H)$. Therefore,

$$\sup_{0 \leq m < M} \left\| \partial_t u_{0,m+1/2} - \frac{\partial u_0}{\partial t}(t_m) \right\|_H \leq c\Delta t.$$

With (7.3), (4.5) holds. From Propositions 5.3 and 5.6, we deduce (7.4) and (7.5). The two estimates (7.7) and (7.8) are deduced similarly as (7.6) leads to (4.7). $\square$

7.2. Convergence in the physical variable in the multiscale case. For general multiscale problems, we deduce in this section the convergence of the solution of both the semidiscrete problem (3.1) and the fully discrete problem (3.12). As we do not have an explicit homogenization rate of convergence we will not distinguish the two cases of full and sparse tensor products. We thus use our general notation for general finite element spaces in section 3. The results in this section hold for both full and sparse tensor product spaces. For problem (3.1), we have the following result.

THEOREM 7.4. *Assume that $g_0 = 0$, $g_1 \in V$, $f \in H^2(0, T; H)$, $f(0) \in V$, and $g_0^L = 0$, $\|g_1^L - g_1\|_H \rightarrow 0$ when $L \rightarrow \infty$. Then for the solution of the semidiscrete problem (3.1) we have*

$$\lim_{L \rightarrow \infty} \left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0^L}{\partial t} \right\|_{L^\infty(0, T; H)} + \|\nabla u^\varepsilon - \mathcal{U}_n^\varepsilon(\nabla u_0^L + \nabla_{y_1} u_1^L + \cdots + \nabla_{y_n} u_n^L)\|_{L^\infty(0, T; H)} = 0.$$

Proof. With the hypothesis of the theorem, condition (4.5) holds. We, therefore, have $\frac{\partial^2 u_0}{\partial t^2} \in L^2(0, T; V)$. Thus from (4.4), $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2(0, T; \mathbf{V})$.

We have further that

$$\begin{aligned} & \|\mathcal{U}_n^\varepsilon(\nabla u_0(t) + \nabla_{y_1} u_1(t) + \cdots + \nabla_{y_n} u_n(t)) - \mathcal{U}_n^\varepsilon(\nabla u_0^L(t) \\ & \quad + \nabla_{y_1} u_1^L(t) + \cdots + \nabla_{y_n} u_n^L(t))\|_H \\ & \leq \|\mathbf{u}(t) - \mathbf{u}^L(t)\|_{\mathbf{V}}. \end{aligned}$$

The result then follows from Propositions 3.5 and 6.4. $\square$

Similarly, we have the following result for the fully discrete problem (3.12).

THEOREM 7.5. *Assume that $g_0 = 0$, $g_1 \in V$, $f \in H^2(0, T; H)$ and $f(0) \in V$, and $g_0^L = 0$. If $\mathbf{u}_0^L$ and $\mathbf{u}_1^L$ are chosen so that*

$$\lim_{L \rightarrow \infty} \|\partial_t \zeta_{0,1/2}^L\|_H = 0, \quad \lim_{L \rightarrow \infty} \|\zeta_{1/2}^L\|_{\mathbf{V}} = 0,$$

then

$$\begin{aligned} & \lim_{\substack{L \rightarrow \infty \\ \Delta t \rightarrow 0}} \sup_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t u_{0,m+1/2}^L \right\|_H \\ & \quad + \|\nabla u^\varepsilon - \mathcal{U}_n^\varepsilon(\nabla u_0^L + \nabla_{y_1} u_1^L + \cdots + \nabla_{y_n} u_n^L)\|_{\tilde{L}^\infty(0, T; H)} = 0. \end{aligned}$$

Proof. From the hypothesis of the theorem, conditions (4.5) hold. Thus convergence (3.17) holds. From Propositions 3.8 and 6.4 we get the result. $\square$

8. Numerical results. We present in this section some numerical examples for two- and three-scale problems that confirm our analysis above. We solve the fully discretized Galerkin problem (3.12). At each time step m , we solve for $\mathbf{u}_{m+1}^L$ in full or sparse tensor finite element product spaces as developed in subsections 5.2 and 5.3. The discretized problem is formulated as a linear system of equations for $\mathbf{c}_m$ that represents components of $\mathbf{u}_m^L$ in the basis of $\mathbf{V}^L$. The linear system we solve is

$$\begin{aligned} \left(\frac{1}{(\Delta t)^2} \mathbf{M} + \frac{1}{4} \mathbf{S} \right) \mathbf{c}_{m+1} &= \frac{1}{4} \mathbf{F}_{m+1} + \frac{1}{2} \mathbf{F}_m + \frac{1}{4} \mathbf{F}_{m-1} \\ &+ \frac{2}{(\Delta t)^2} \mathbf{M} c_{0,m} - \frac{1}{(\Delta t)^2} \mathbf{M} c_{0,m-1} - \frac{1}{2} \mathbf{S} \mathbf{c}_m - \frac{1}{4} \mathbf{S} \mathbf{c}_{m-1}. \end{aligned}$$

Here $c_{0,m}$ denotes the $\dim \mathbf{V}^L \times 1$ column vector whose first $\dim V^L$ components are those of $u_{0,m}^L$ in the basis of V^L and all the other components are 0, $\mathbf{M}$ is the

Gram matrix for the basis functions of V^L in the H norm extending to 0 to form a $\dim \mathbf{V}^L \times \dim \mathbf{V}^L$ matrix. The stiffness matrix $\mathbf{S}$ is derived from $\mathbf{A}_{00}$, $\mathbf{A}_{01}$, $\mathbf{A}_{11}$ in Proposition 3.1. On the right-hand side, $\mathbf{F}_m$ is the time dependent column vector whose first $\dim V^L$ components form the column vector that describes the interaction of $f(t_m)$ and the basis functions of V^L in the H norm and all other components are 0. The problem is then solved by conjugate gradient method with full tensor FE basis in subsection 5.2 and sparse tensor FE basis in subsection 5.3.

To construct the sparse tensor finite element product spaces, it is convenient to use the wavelet preconditioning. We thus briefly present the assumptions on the finite element spaces on D and Y in the next subsection.

8.1. Multilevel FE spaces and preconditioning. To construct the increment spaces W^l , $W_{\#}^{l0}$, and $W_{\#}^{l1}$ in section 5.3, we assume the following assumptions.

ASSUMPTION 8.1. (i) For each $j \in \mathbb{N}_0^d$, there exists a set of indices $I^j \subset \mathbb{N}_0^d$ and a set of basis functions $\phi^{jk} \in H^1(D)$, $k \in I^j$, such that $V^l = \text{span}\{\phi^{jk} : |j|_{\infty} \leq l\}$. There are constants $c_2 > c_1 > 0$ such that if $\phi = \sum_{|j|_{\infty} \leq l, k \in I^j} \phi^{jk} c_{jk} \in V^l$, then the norm equivalence holds:

$$c_1 \sum_{\substack{|j|_{\infty} \leq l \\ k \in I^j}} 2^{2s|j|_{\infty}} |c_{jk}|^2 \leq \|\phi\|_{H^s(D)}^2 \leq c_2 \sum_{\substack{|j|_{\infty} \leq l \\ k \in I^j}} 2^{2s|j|_{\infty}} |c_{jk}|^2, \quad s = 0, 1.$$

(ii) For the space $L^2(Y)$, for each $j \in \mathbb{N}_0^d$, there exists a set of indices $I_0^j \subset \mathbb{N}_0^d$ and a set of basis functions $\phi_0^{jk} \in L^2(Y)$, $k \in I_0^j$, such that $V_{\#}^l = \text{span}\{\phi_0^{jk} : |j|_{\infty} \leq l\}$. There are constants $c_4 > c_3 > 0$ such that if $\phi = \sum_{|j|_{\infty} \leq l, k \in I_0^j} \phi_0^{jk} c_{jk} \in V^l$, then

$$c_3 \sum_{\substack{|j|_{\infty} \leq l \\ k \in I_0^j}} |c_{jk}|^2 \leq \|\phi\|_{L^2(Y)}^2 \leq c_4 \sum_{\substack{|j|_{\infty} \leq l \\ k \in I_0^j}} |c_{jk}|^2.$$

(iii) For each $j \in \mathbb{N}_0^d$, there exists a set of indices $I_1^j \subset \mathbb{N}_0^d$ and a set of basis functions $\phi_1^{jk} \in H_{\#}^1(Y)/\mathbb{R}$, $k \in I_1^j$, such that $\{\phi_1^{jk} : |j|_{\infty} \leq l\}$ is a basis of $V_{\#}^l/\mathbb{R}$. There are constants $c_6 > c_5 > 0$ such that if $\phi = \sum_{|j|_{\infty} \leq l, k \in I_1^j} \phi_1^{jk} c_{jk}$, then

$$c_5 \sum_{\substack{|j|_{\infty} \leq l \\ k \in I_1^j}} |c_{jk}|^2 \leq \|\phi\|_{H_{\#}^1(Y)/\mathbb{R}}^2 \leq c_6 \sum_{\substack{|j|_{\infty} \leq l \\ k \in I_1^j}} |c_{jk}|^2.$$

We consider the following examples of hierarchical FE bases in one dimension.

Example. (i) For $D = (0, 1)$ we define the hierarchical base for $L^2(D)$ as follows. At level 0, we choose three piecewise linear functions: ϕ^{01} takes obtains values (1, 0) at (0, 1/2) and is 0 in (1/2, 1), ϕ^{02} is continuous piecewise linear and obtains values (0, 1, 0) at (0, 1/2, 1), and ϕ^{03} obtains values (0, 1) at (1/2, 1) and is 0 in (0, 1/2). For each level, we choose piecewise wavelet function ϕ that takes values (0, -1, 2, -1, 0) at (0, 1/2, 1, 3/2, 2). The left boundary function ϕ^{left} takes values (-2, 2, -1, 0) at (0, 1/2, 1, 3/2), and the right boundary function ϕ^{right} takes values (0, -1, 2, -2) at (1/2, 1, 3/2, 2). For levels $j \geq 1$, the index set $I^j = \{1, 2, \dots, 2^j\}$. The wavelet functions are defined as $\phi^{j1}(x) = 2^{-j/2} \phi^{left}(2^j x)$, $\phi^{jk}(x) = 2^{-j/2} \phi(2^j x - k + 3/2)$ for $k = 2, \dots, 2^j - 1$ and $\phi^{j2^j} = \phi^{right}(2^j x - 2^j + 2)$. This base satisfies Assumption 8.1 (i).

FIG. 1. The full and sparse tensor energy error for a one-dimensional two-scale problem.

(ii) For $Y = (0, 1)$, we define the base of $H_{\#}^1(Y)/\mathbb{R}$ as in (i) but exclude ϕ^{01} , ϕ^{03} . At other levels, all the functions ϕ^{left} and ϕ^{right} are replaced by the piecewise linear functions that take values $(0, 2, -1, 0)$ at $(0, 1/2, 1, 3/2)$ and values $(0, -1, 2, 0)$ at $(1/2, 1, 3/2, 2)$, respectively.

When $D = (0, 1)^d$, the basis functions can be constructed by taking the tensor products of the basis functions in $(0, 1)$. They satisfy Assumption 8.1 after appropriate scaling; see [11].

Alternatively we can use generating systems, also called multilevel frames for discretization, as considered in [11] or [13].

8.2. One-dimensional two-scale problem. We first consider a one-dimensional two-scale problem where $A(x, y) = (2/3)(1 + x)(1 + \cos^2(2\pi y))$ in the domain $D = (0, 1)$. The two-scale problem is

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} - \frac{\partial}{\partial x} \left(\frac{2}{3}(1+x) \left(1 + \cos^2 \left(2\pi \frac{x}{\varepsilon} \right) \right) \frac{\partial u^\varepsilon}{\partial x} \right) = t^3 + 9t \left(x - \frac{\log(1+x)}{\log 2} \right) \quad \text{in } D,$$

$$u^\varepsilon(t, 0) = u^\varepsilon(t, 1) = 0, \quad u^\varepsilon(0, x) = 0, \quad \frac{\partial u^\varepsilon}{\partial t}(0, x) = 0.$$

The two-scale limiting equation has the exact homogenized solution

$$u_0(t, x) = \frac{3}{2\sqrt{2}} \left(x - \frac{\log(1+x)}{\log 2} \right) t^3$$

and the scale interaction term $u_1(x, y)$ (with an additive constant C),

$$u_1(t, x, y) = \frac{3}{2\sqrt{2}} \left(1 - \frac{1}{(1+x)\log 2} \right) \left(\frac{1}{2\pi} \tan^{-1} \left(\frac{\tan 2\pi y}{\sqrt{2}} \right) - y + C \right) t^3.$$

In Figure 1, we plot the error $\|\mathbf{u} - \bar{\mathbf{u}}^L\|_{\mathbf{V}}$ and the error $\|\mathbf{u} - \hat{\mathbf{u}}^L\|_{\mathbf{V}}$ versus the mesh size h_L for $t = 1$ for the full and sparse tensor product approximations. We choose $\Delta t = h_L^{1/2}/2$ or the closest value so that $1/\Delta t$ is an integer. In particular, we plot for $(h_L, \Delta t) = (1/4, 1/4), (1/8, 1/6), (1/16, 1/8), (1/32, 1/12)$, and $(1/64, 1/16)$.

We observe that the full and sparse tensor product FE errors are essentially equal despite the fact that the number of degrees of freedom we used for the sparse tensor approximation is much less than that for the full tensor product spaces. This confirms the results of Proposition 5.6 that the sparse tensor FE method provides a solution with essentially equal accuracy as the full tensor product FEs, using a much reduced number of degrees of freedom. Figure 1 shows clearly that both the errors of sparse and full FEs behave as $O(h_L)$. In the log – log coordinates the multiplying factor $|\log h_L|^{1/2}$ in the error of the sparse tensor FE does not show any effect. As we choose the time step Δt to be of the order $h_L^{1/2}$, Figure 1 also shows that we achieve the error $O((\Delta t)^2)$ with respect to t when the solution is sufficiently smooth, as established in Proposition 5.6.

8.3. One-dimensional three-scale problem. We then consider a one-dimensional three-scale problem where $A(x, y_1, y_2) = (4/9)(1+x)(1+\cos^2(2\pi y_1))(1+\cos^2(2\pi y_2))$. The problem is

$$\begin{aligned} \frac{\partial^2 u^\varepsilon}{\partial t^2} - \frac{\partial}{\partial x} \left(\frac{4}{9}(1+x) \left(1 + \cos^2 \left(2\pi \frac{x}{\varepsilon_1} \right) \right) \left(1 + \cos^2 \left(2\pi \frac{x}{\varepsilon_2} \right) \right) \frac{\partial u^\varepsilon}{\partial x} \right) \\ = t^3 + \frac{27t}{4} \left(x - \frac{\log(1+x)}{\log 2} \right) \text{ in } D, \end{aligned}$$

$$u^\varepsilon(t, 0) = u^\varepsilon(t, 1) = 0, \quad u^\varepsilon(0, x) = 0, \quad \frac{\partial u^\varepsilon}{\partial t}(0, x) = 0.$$

The three-scale limiting problem has the exact homogenized solution

$$u_0(t, x) = \frac{9}{8} \left(x - \frac{\log(1+x)}{\log 2} \right) t^3$$

and the scale interaction terms $u_1(x, y_1)$ and $u_2(x, y_1, y_2)$ (C_1 and C_2 are constants)

$$u_1(t, x, y_1) = \frac{9}{8} \left(1 - \frac{1}{(1+x)\log 2} \right) \left(\frac{1}{2\pi} \tan^{-1} \left(\frac{\tan 2\pi y_1}{\sqrt{2}} \right) - y_1 + C_1 \right) t^3,$$

$$\begin{aligned} u_2(t, x, y_1, y_2) \\ = \frac{9}{8} \left(1 - \frac{1}{(1+x)\log 2} \right) \left(\frac{1}{2\pi} \tan^{-1} \left(\frac{\tan 2\pi y_2}{\sqrt{2}} \right) - y_2 + C_2 \right) \frac{\sqrt{2}}{1 + \cos^2 2\pi y_1} t^3. \end{aligned}$$

In Figure 2 the error $\|\mathbf{u} - \hat{\mathbf{u}}^L\|_{\mathbf{V}}$ is plotted versus the mesh size for the sparse tensor product for $t = 1$. The mesh size h and the time step Δt are chosen as for the two-scale problem above. This example confirms our theoretical results in Proposition 5.6 for three scale problems. In terms of the finite element mesh h_L , in the log – log coordinates, the error $\|\mathbf{u} - \hat{\mathbf{u}}^L\|_{\mathbf{V}}$ versus the mesh size h_L is a straight line of slope 1, i.e., the $O(h_L)$ behavior is dominant. Although the theoretical result of Proposition 5.6 states that the error is now of the order $L^{3/2}h_L$, i.e., the effect of the log multiplying factor should be stronger than in the previous two-scale example, it still does not show any significant effects in this figure. As we choose $\Delta t = O(h_L^{1/2})$, the example shows that the error $O((\Delta t)^2)$ is achieved with respect to the time step when the solution is sufficiently regular.

FIG. 2. The sparse tensor energy error for a one-dimensional three-scale problem.

8.4. Two-dimensional two-scale problem. We consider in $D = (0, 1) \times (0, 1)$, the two-scale problem

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} - \nabla_x \cdot (a^\varepsilon(x) \nabla_x u^\varepsilon) = f(t, x) \quad \text{in } D,$$

$$u^\varepsilon(t, \cdot) = 0 \quad \text{on } \partial D, \quad u^\varepsilon(0, x) = 0, \quad \frac{\partial u^\varepsilon}{\partial t}(0, x) = 0,$$

where

$$a^\varepsilon(x) = (1 + x_1)(1 + x_2) \left(1 + \cos^2 \left(2\pi \frac{x_1}{\varepsilon} \right) \right) \left(1 + \cos^2 \left(2\pi \frac{x_2}{\varepsilon} \right) \right),$$

$$f(t, x) = 6t(x_1 - x_1^2)(x_2 - x_2^2) - \nabla \cdot \left(\frac{3}{\sqrt{2}}(1 + x_1)(1 + x_2) \nabla ((x_1 - x_1^2)(x_2 - x_2^2)) \right) t^3.$$

Following Jikov, Kozlov, and Oleĭnik [19, page 17], the homogenized coefficient can be computed exactly. This problem has the exact homogenized solution

$$u_0(t, x) = (x_1 - x_1^2)(x_2 - x_2^2)t^3$$

and the scale interaction terms

$$\begin{aligned} u_1(t, x, y) = & (1 - 2x_1)(x_2 - x_2^2) \left(\frac{1}{2\pi} \tan^{-1} \left(\frac{\tan 2\pi y_1}{\sqrt{2}} \right) - y_1 + C_1 \right) t^3 \\ & + (x_1 - x_1^2)(1 - 2x_2) \left(\frac{1}{2\pi} \tan^{-1} \left(\frac{\tan 2\pi y_2}{\sqrt{2}} \right) - y_2 + C_2 \right) t^3. \end{aligned}$$

In Figure 3 the error $\|\mathbf{u} - \hat{\mathbf{u}}^L\|_{\mathbf{V}}$ is plotted versus the mesh size for the sparse tensor product for $(\Delta t, h) = (1/4, 1/4)$, $(1/8, 1/6)$, and $(1/16, 1/8)$ for $t = 1$. The straight line of slope 1 once more confirms the theoretical results in Proposition 5.6. The predominant behavior of the error is $O(h_L)$ in terms of the finite element mesh and is $O((\Delta t)^2)$ in terms of the time step, as we choose $\Delta t = O(h_L^{1/2})$. This is achieved as the solution is sufficiently regular.

FIG. 3. The sparse tensor energy error for a two-dimensional two-scale problem.

FIG. 4. The analytical and numerical approximation to $\frac{\partial u^\varepsilon}{\partial x_2}$.

For this problem, we plot in Figure 4 the finite element approximation for $\frac{\partial u^\varepsilon}{\partial x_2}$ in Theorem 7.3 at $t = 1$

$$\frac{\partial \hat{u}_0^L}{\partial x_2} + \mathcal{U}^\varepsilon \left(\frac{\partial \hat{u}_1^L}{\partial y_2} \right)$$

and compare it to the analytical approximation

$$\frac{\partial u_0(x)}{\partial x_2} + \frac{\partial u_1}{\partial y_2} \left(x, \frac{x}{\varepsilon} \right)$$

in Proposition 6.1.

In Figure 4, the picture on the left is the analytical approximation $\frac{\partial u_0}{\partial x_2}(x) + \frac{\partial u_1}{\partial y_2}(x, \frac{x}{\varepsilon})$ and the picture on the right is its numerical approximation $\frac{\partial \hat{u}_0^L}{\partial x_2} + \mathcal{U}^\varepsilon \left(\frac{\partial \hat{u}_1^L}{\partial y_2} \right)$. We choose $h = 1/8$ and $\varepsilon = 1/64$. The surface of the right picture correctly represents the one shown in the left, which is the analytical approximation of the derivative $\frac{\partial u^\varepsilon}{\partial x_1}$. This shows that the numerical corrector constructed in section 7 for the two-scale problem, in particular in Theorem 7.3, using the operator $\mathcal{U}^\varepsilon$, provides an accurate description of the behavior of the multiscale solution u^ε in the $H^1(D)$ norm. We note

FIG. 5. Zoomed in Figure 4.

two features. In the numerical approximation $\frac{\partial \hat{u}_0^L}{\partial x_2} + \mathcal{U}^\varepsilon\left(\frac{\partial \hat{u}_1^L}{\partial y_2}\right)$, the piecewise constant behavior of $\frac{\partial \hat{u}_0^L}{\partial x_2}$ is quite clear due to the piecewise linear finite element approximation of the solution u_0 , although globally the numerical corrector still represents correctly the exact solution. In Figure 5 we show a “zoomed in” portion of Figure 4 where the left picture is the analytical approximation in Proposition 6.1 and the right picture is the numerical approximation in Theorem 7.3. Both pictures show clearly the locally oscillating behavior of the exact and the approximated solution constructed from the operator $\mathcal{U}^\varepsilon$.

9. Conclusions. In this paper, we propose the full and sparse tensor product finite element methods for multiscale wave equations. With sufficient regularity, we achieve the rate of convergence $O(\Delta t)^2 + O(h_L |\log h_L|^{n/2})$ in Proposition 5.6 for the sparse tensor approximation. The number of degrees of freedom used is $\hat{N}_L = O(h_L^d |\log h_L|^n)$. For solving problem (5.7), the term $(\partial_t^2 \hat{u}_{0,m}^L, \hat{v}_0^L)$ only contributes a low dimension matrix. Solving the system by conjugate gradient method, the main cost is for performing matrix-vector multiplication of the stiffness matrix of the bilinear form B . As shown in Hoang and Schwab [15], with an appropriate matrix vector multiplication algorithm, the linear problem in each step can be solved with a total of $O(\hat{N}_L)(\log \hat{N}_L)^{n+d(n+1)}$ floating point operations. Letting $\Delta t = O(h_L^{1/2})$ the number of time steps required is $O(h_L^{-1/2})$, which is $O(\hat{N}_L^{1/(2d)} (\log \hat{N}_L)^{-n/(2d)})$. Thus the total number of floating point operations needed is $O(\hat{N}_L^{1/(2d)+1} (\log \hat{N}_L)^{-n/(2d)+n+d(n+1)})$. Apart from the log factor, the rate of convergence and the complexity involved are independent of the number of scales. The numerical examples for two- and three-scale problems confirm the analysis. Our method not only produces the solution u_0 of the homogenized equation but also the full corrector for a class of problems and a part of the corrector for the general problems. Using them we can get the local behavior of u^ε by computing an approximation for ∇u^ε . We have run a simulation to compute this for the whole two dimensional domain $(0, 1)^2$, even though this is the local behavior and only needs to be computed at points of interest.

Appendix A. In this appendix, we present the proof of Lemma 3.6.

Proof. Averaging equation (3.4) at time steps t_{m+1} , t_m , and t_{m-1} with weights $1/4$, $1/2$, and $1/4$ we get

$$B(\mathbf{w}_{m,1/4}^L, \mathbf{v}^L) = \left(f_{m,1/4} - \frac{\partial^2 u_{0,m,1/4}}{\partial t^2}, v_0^L \right)_H.$$

From this and (3.12) we have

$$(\partial_t^2 u_{0,m}^L, v_0^L)_H + B(\boldsymbol{\zeta}_{m,1/4}^L, \mathbf{v}^L) = \left(\frac{\partial^2 u_{0,m,1/4}}{\partial t^2}, v_0^L \right)_H.$$

Therefore,

$$(\partial_t^2 \zeta_{0,m}^L, v_0^L)_H + B(\boldsymbol{\zeta}_{m,1/4}^L, \mathbf{v}^L) = \left(-\partial_t^2 w_{0,m}^L + \frac{\partial^2 u_{0,m,1/4}}{\partial t^2}, v_0^L \right)_H.$$

Since $w_0^L = u_0 + \eta_0^L$, we deduce that

$$(\partial_t^2 \zeta_{0,m}^L, v_0^L)_H + B(\boldsymbol{\zeta}_{m,1/4}^L, \mathbf{v}^L) = \left(-\partial_t^2 \eta_{0,m}^L - \partial_t^2 u_{0,m} + \frac{\partial^2 u_{0,m,1/4}}{\partial t^2}, v_0^L \right)_H.$$

Choose $\mathbf{v}^L = \delta_t \boldsymbol{\zeta}_m^L$, we then have

$$(\partial_t^2 \zeta_{0,m}^L, \delta_t \zeta_{0,m}^L)_H + B(\boldsymbol{\zeta}_{m,1/4}^L, \delta_t \boldsymbol{\zeta}_m^L) = \left(-\partial_t^2 \eta_{0,m}^L - \partial_t^2 u_{0,m} + \frac{\partial^2 u_{0,m,1/4}}{\partial t^2}, \delta_t \zeta_{0,m}^L \right)_H.$$

We have the following relationships:

$$\begin{aligned} \partial_t^2 r_m &= \frac{1}{\Delta t} (\partial_t r_{m+1/2} - \partial_t r_{m-1/2}), \quad r_{m,1/4} = \frac{1}{2}(r_{m+1/2} + r_{m-1/2}), \\ \delta_t r_m &= \frac{1}{2} (\partial_t r_{m+1/2} + \partial_t r_{m-1/2}) = \frac{1}{\Delta t} (r_{m+1/2} - r_{m-1/2}). \end{aligned}$$

Let $s_m = \frac{\partial^2 u_{0,m,1/4}}{\partial t^2} - \partial_t^2 u_{0,m}$, we get

$$\begin{aligned} & \frac{1}{2\Delta t} \left[\left(\partial_t \zeta_{0,m+1/2}^L - \partial_t \zeta_{0,m-1/2}^L, \partial_t \zeta_{0,m+1/2}^L + \partial_t \zeta_{0,m-1/2}^L \right)_H \right. \\ & \quad \left. + B(\boldsymbol{\zeta}_{m+1/2}^L + \boldsymbol{\zeta}_{m-1/2}^L, \boldsymbol{\zeta}_{m+1/2}^L - \boldsymbol{\zeta}_{m-1/2}^L) \right] \\ &= \frac{1}{2} \left(s_m - \partial_t^2 \eta_{0,m}^L, \partial_t \zeta_{0,m+1/2}^L + \partial_t \zeta_{0,m-1/2}^L \right)_H. \end{aligned}$$

As $B(\cdot, \cdot)$ is a symmetric bilinear form, we deduce

$$\begin{aligned} & \frac{1}{2\Delta t} \left[\left\| \partial_t \zeta_{0,m+1/2}^L \right\|_H^2 - \left\| \partial_t \zeta_{0,m-1/2}^L \right\|_H^2 + B(\boldsymbol{\zeta}_{m+1/2}^L, \boldsymbol{\zeta}_{m+1/2}^L) - B(\boldsymbol{\zeta}_{m-1/2}^L, \boldsymbol{\zeta}_{m-1/2}^L) \right] \\ & \leq \frac{1}{2} \|s_m - \partial_t^2 \eta_{0,m}^L\|_H \left\| \partial_t \zeta_{0,m+1/2}^L + \partial_t \zeta_{0,m-1/2}^L \right\|_H \\ & \leq \frac{1}{2\gamma} \left(\|s_m\|_H^2 + \|\partial_t^2 \eta_{0,m}^L\|_H^2 \right) + \frac{\gamma}{2} \left(\left\| \partial_t \zeta_{0,m+1/2}^L \right\|_H^2 + \left\| \partial_t \zeta_{0,m-1/2}^L \right\|_H^2 \right) \end{aligned}$$

for all $\gamma > 0$. For $1 \leq j < M$, summing this up over $m = 1, \dots, j$, we deduce that

$$\begin{aligned} & \left[\left\| \partial_t \zeta_{0,j+1/2}^L \right\|_H^2 - \left\| \partial_t \zeta_{0,1/2}^L \right\|_H^2 + B(\zeta_{j+1/2}^L, \zeta_{j+1/2}^L) - B(\zeta_{1/2}^L, \zeta_{1/2}^L) \right] \\ & \leq \frac{1}{\gamma} \Delta t \sum_{m=1}^M \left(\|s_m\|_H^2 + \|\partial_t^2 \eta_{0,m}^L\|_H^2 \right) + 2\gamma T \max_{1 \leq m < M} \left\| \partial_t \zeta_{0,m+1/2}^L \right\|_H^2 \\ & \quad + \gamma \Delta t \|\partial_t \zeta_{0,1/2}^L\|_H^2. \end{aligned}$$

Choosing γ sufficiently small, we deduce that

$$\begin{aligned} \max_{1 \leq j < M} \left[\left\| \partial_t \zeta_{0,j+1/2}^L \right\|_H^2 + B(\zeta_{j+1/2}^L, \zeta_{j+1/2}^L) \right] & \leq c \Delta t \sum_{m=1}^M \left(\|s_m\|_H^2 + \|\partial_t^2 \eta_{0,m}^L\|_H^2 \right) \\ & \quad + c \|\partial_t \zeta_{0,1/2}^L\|_H^2 + c \|\zeta_{1/2}^L\|_{\mathbf{V}}^2. \end{aligned}$$

Following Dupont [8], using the integral formula of the remainder of Taylor expansion, we write

$$\partial_t^2 \eta_{0,m}^L = (\Delta t)^{-2} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|) \frac{\partial^2 \eta_0^L}{\partial t^2} (t_m + \tau) d\tau;$$

this implies that

$$\sum_{m=1}^M \|\partial_t^2 \eta_{0,m}^L\|_H^2 \Delta t \leq \frac{4}{3} \left\| \frac{\partial^2 \eta_0^L}{\partial t^2} \right\|_{L^2(0,T;H)}^2.$$

For s_m , we have

$$\begin{aligned} s_m &= \frac{1}{4} \int_0^{\Delta t} \left(1 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^3 u_0}{\partial t^3} (t_m + \tau) d\tau \\ & \quad - \frac{1}{4} \int_{-\Delta t}^0 \left(1 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^3 u_0}{\partial t^3} (t_m + \tau) d\tau. \end{aligned}$$

Therefore,

$$\|s_m\|_H^2 \leq c \Delta t \int_{t_{m-1}}^{t_m} \left\| \frac{\partial^3 u_0}{\partial t^3}(\tau) \right\|_H^2 d\tau.$$

From (2.1) and (2.2), we get

$$\begin{aligned} & \left\| \partial_t \zeta_0^L \right\|_{\tilde{L}^\infty(0,T;H)}^2 + \|\zeta^L\|_{\tilde{L}^\infty(0,T;\mathbf{V})}^2 \\ & \leq c \left[(\Delta t)^2 \left\| \frac{\partial^3 u_0}{\partial t^3} \right\|_{L^2(0,T;H)}^2 + \left\| \frac{\partial^2 \eta_0^L}{\partial t^2} \right\|_{L^2(0,T;H)}^2 + \|\partial_t \zeta_{0,1/2}^L\|_H^2 + \|\zeta_{1/2}^L\|_{\mathbf{V}}^2 \right]. \end{aligned}$$

If $\partial^4 u_0 / \partial t^4 \in L^2(0, T; H)$, we have

$$s_m = \frac{1}{12} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|) \left(3 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^4 u_0}{\partial t^4} (t_m + \tau) d\tau.$$

From this, we get

$$\|s_m\|_H^2 \leq c(\Delta t)^3 \int_{t_{m-1}}^{t_{m+1}} \left\| \frac{\partial^4 u_0}{\partial t^4} \right\|_H^2 d\tau.$$

From (2.1) and (2.2) we get

$$\begin{aligned} & \|\partial_t \zeta_0^L\|_{\tilde{L}^\infty(0,T;H)}^2 + \|\zeta^L\|_{\tilde{L}^\infty(0,T;\mathbf{V})}^2 \\ & \leq c \left[(\Delta t)^4 \left\| \frac{\partial^4 u_0}{\partial t^4} \right\|_{L^2(0,T;H)}^2 + \left\| \frac{\partial^2 \eta_0^L}{\partial t^2} \right\|_{L^2(0,T;H)}^2 + \|\partial_t \zeta_{0,1/2}^L\|_H^2 + \|\zeta_{1/2}^L\|_{\mathbf{V}}^2 \right]. \quad \square \end{aligned}$$

Appendix B. We make some remarks on the choice of $\mathbf{u}_0^L$ and $\mathbf{u}_1^L$ so that the conditions of Proposition 5.6 are satisfied.

We assume conditions (4.7) and that the domain is convex. We find $\hat{u}_1^L \in \hat{V}_1^L, \dots, \hat{u}_n^L \in \hat{V}_n^L$ such that

$$\int_D \int_{\mathbf{Y}} A(x, \mathbf{y}) (\nabla g_0 + \nabla_{y_1} \hat{u}_1^L + \dots + \nabla_{y_n} \hat{u}_n^L) \cdot (\nabla_{y_1} \hat{v}_1^L + \dots + \nabla_{y_n} \hat{v}_n^L) dx d\mathbf{y} = 0$$

for all $\hat{v}_1^L \in \hat{V}_1^L, \dots, \hat{v}_n^L \in \hat{V}_n^L$. We have

$$\|u_1 - \hat{u}_1^L\|_{V_1} + \dots + \|u_n - \hat{u}_n^L\|_{V_n} \leq ch_L L^{n/2}.$$

We find $\hat{u}_0^L \in V^L$ by projection such that $\|\hat{u}_0^L - g_0\|_V \leq ch_L$ and set $\hat{\mathbf{u}}_0^L = (\hat{u}_0^L, \hat{u}_1^L, \dots, \hat{u}_n^L)$. The functions $\hat{u}_{1t}^L \in \hat{V}_1^L, \dots, \hat{u}_{nt}^L \in \hat{V}_n^L$ are found by solving

$$\int_D \int_{\mathbf{Y}} A(x, \mathbf{y}) (\nabla g_1 + \nabla_{y_1} \hat{u}_{1t}^L + \dots + \nabla_{y_n} \hat{u}_{nt}^L) \cdot (\nabla_{y_1} \hat{v}_1^L + \dots + \nabla_{y_n} \hat{v}_n^L) dx d\mathbf{y} = 0$$

for all $\hat{v}_1^L \in \hat{V}_1^L, \dots, \hat{v}_n^L \in \hat{V}_n^L$. From (4.7), $\frac{\partial u_0}{\partial t}(0) \in H^2(D)$ so $\frac{\partial u_i}{\partial t} \in \hat{\mathcal{H}}_i$. We then have

$$\|\hat{u}_{1t}^L - \frac{\partial u_1}{\partial t}(0)\|_{V_1} + \dots + \|\hat{u}_{nt}^L - \frac{\partial u_n}{\partial t}(0)\|_{V_n} \leq ch_L L^{n/2}.$$

Letting $g_1^L \in V^L$ be such that $\|g_1^L - g_1\|_V \leq ch_L$, we then set $\hat{u}_{0t}^L = g_1^L$, $\hat{\mathbf{u}}_{0t}^L = (\hat{u}_{0t}^L, \hat{u}_{1t}^L, \dots, \hat{u}_{nt}^L)$. We find $\hat{u}_{0tt}^L \in V^L$ such that $\|\hat{u}_{0tt}^L - \frac{\partial^2 u_0}{\partial t^2}\|_H \leq ch_L$ and $\|\hat{u}_{0tt}^L\|_V \leq \|\frac{\partial^2 u_0}{\partial t^2}\|_V$. We choose $\hat{\mathbf{u}}_1^L$ as

$$\hat{\mathbf{u}}_1^L = \hat{\mathbf{u}}_0^L + \Delta t \hat{\mathbf{u}}_{0t}^L + \frac{1}{2}(\Delta t)^2 \hat{u}_{0tt}^L.$$

We have

$$\begin{aligned} \partial_t \zeta_{0,1/2}^L &= \frac{1}{\Delta t} ((\hat{u}_{0,1}^L - \hat{u}_{0,0}^L - (w_{0,1}^L - w_{0,0}^L))) \\ &= \hat{u}_{0t}^L + \frac{1}{2} \Delta t \hat{u}_{0tt}^L - \frac{\partial w_0^L}{\partial t}(0) - \frac{1}{2} \Delta t \frac{\partial^2 w_0^L}{\partial t^2}(0) + X, \end{aligned}$$

where $\|X\|_H \leq c(\Delta t)^2$. We note that

$$\left\| \hat{u}_{0t}^L - \frac{\partial w_0^L}{\partial t}(0) \right\|_H \leq \left\| \hat{u}_{0t}^L - \frac{\partial u_0}{\partial t}(0) \right\|_H + \left\| \frac{\partial u_0}{\partial t}(0) - \frac{\partial w_0^L}{\partial t}(0) \right\|_H \leq ch_L L^{n/2}$$

due to Lemma 3.3. Similarly, we have

$$\left\| \hat{u}_{0tt}^L - \frac{\partial^2 w_0^L}{\partial t^2}(0) \right\|_H \leq \left\| \hat{u}_{0tt}^L - \frac{\partial^2 u_0}{\partial t^2}(0) \right\|_H + \left\| \frac{\partial^2 u_0}{\partial t^2}(0) - \frac{\partial^2 w_0^L}{\partial t^2}(0) \right\|_H \leq ch_L L^{n/2}.$$

Therefore, $\|\partial_t \zeta_{0,1/2}^L\|_H \leq c((\Delta t)^2 + h_L L^{n/2})$. Further,

$$\begin{aligned} \|\zeta_{1/2}^L\|_{\mathbf{V}} &\leq \frac{1}{2}(\|\zeta_0^L\|_{\mathbf{V}} + \|\zeta_1^L\|_{\mathbf{V}}) \\ &\leq \frac{1}{2}\|\hat{u}_0^L - \mathbf{u}_0\|_{\mathbf{V}} + \frac{1}{2}\|\mathbf{u}_0 - \mathbf{w}_0^L\|_{\mathbf{V}} + \frac{1}{2}\left\| \hat{u}_0^L + \Delta t \hat{u}_{0t}^L - \mathbf{u}_0 - \Delta t \frac{\partial \mathbf{u}}{\partial t}(0) \right\|_{\mathbf{V}} \\ &\quad + \frac{1}{2}\left\| \mathbf{u}_0 + \Delta t \frac{\partial \mathbf{u}}{\partial t}(0) - \mathbf{w}_0^L - \Delta t \frac{\partial \mathbf{w}^L}{\partial t}(0) \right\|_{\mathbf{V}} + c(\Delta t)^2 \\ &\leq c((\Delta t)^2 + h_L L^{n/2}). \end{aligned}$$

If only (4.5) holds, we require $O(\Delta t) + h_L L^{n/2}$ error and, therefore, do not need to consider $\frac{\partial^2 u_0}{\partial t^2}(0)$. Otherwise, to compute $\frac{\partial^2 u_0}{\partial t^2}(0)$, we note that

$$\begin{aligned} \frac{\partial^2 u_0}{\partial t^2}(0) &= f(0) - \nabla \cdot \int_{\mathbf{Y}} A(x, \mathbf{y})(\nabla u_0(0) + \nabla_{y_1} u_1(0) + \cdots + \nabla_{y_n} u_n(0)) d\mathbf{y} \\ &= f(0) - \int_{\mathbf{Y}} \nabla A(x, \mathbf{y})(\nabla u_0(0) + \nabla_{y_1} u_1(0) + \cdots + \nabla_{y_n} u_n(0)) d\mathbf{y} \\ &\quad - \int_{\mathbf{Y}} A(x, \mathbf{y}) \nabla((\nabla u_0(0) + \nabla_{y_1} u_1(0) + \cdots + \nabla_{y_n} u_n(0))) d\mathbf{y}. \end{aligned}$$

The spatial derivative of $\frac{\partial^2 u_0}{\partial t^2}(0)$ can be computed as follows. We note that for almost all $x \in D$, for all $\phi_1 \in V_1, \dots, \phi_n \in V_n$ which does not depend on x , we have

$$\int_{\mathbf{Y}} A(x, \mathbf{y})(\nabla u_0(0) + \nabla_{y_1} u_1(0) + \cdots + \nabla_{y_n} u_n(0)) \cdot (\nabla_{y_1} \phi_1 + \cdots + \nabla_{y_n} \phi_n) d\mathbf{y} = 0.$$

Therefore,

$$\begin{aligned} &\int_{\mathbf{Y}} A(x, \mathbf{y}) \frac{\partial}{\partial x_k} (\nabla u_0(0) + \nabla_{y_1} u_1(0) + \cdots + \nabla_{y_n} u_n(0)) \cdot (\nabla_{y_1} \phi_1 + \cdots + \nabla_{y_n} \phi_n) d\mathbf{y} \\ &= - \int_{\mathbf{Y}} \frac{\partial A(x, \mathbf{y})}{\partial x_k} (\nabla u_0(0) + \nabla_{y_1} u_1(0) + \cdots + \nabla_{y_n} u_n(0)) \cdot (\nabla_{y_1} \phi_1 + \cdots + \nabla_{y_n} \phi_n) d\mathbf{y}. \end{aligned}$$

Thus for all $\phi_1 \in V_1, \dots, \phi_n \in V_n$,

$$\begin{aligned} &\int_D \int_{\mathbf{Y}} A(x, \mathbf{y}) \left(\nabla \frac{\partial}{\partial x_k} u_0(0) + \nabla_{y_1} \frac{\partial}{\partial x_k} u_1(0) + \cdots + \nabla_{y_n} \frac{\partial}{\partial x_k} u_n(0) \right) \\ &\quad \cdot (\nabla_{y_1} \phi_1 + \cdots + \nabla_{y_n} \phi_n) d\mathbf{y} dx \\ &= - \int_D \int_{\mathbf{Y}} \frac{\partial A(x, \mathbf{y})}{\partial x_k} (\nabla u_0(0) + \nabla_{y_1} u_1(0) + \cdots + \nabla_{y_n} u_n(0)) \\ &\quad \cdot (\nabla_{y_1} \phi_1 + \cdots + \nabla_{y_n} \phi_n) d\mathbf{y} dx. \end{aligned}$$

Once the approximation $\hat{\mathbf{u}}^L(0)$ has been computed, we can then compute an approximation for $\frac{\partial \mathbf{u}}{\partial x_k}(0)$ in $\mathbf{V}$. Other spatial derivatives of $\mathbf{u}(0)$ can be computed similarly.

From this, we can approximate $\nabla \frac{\partial^2 u_0}{\partial t^2}$ and find $\hat{u}_{0tt}^L$.

If $g_0 = 0$ as we consider in the next section, the problem becomes much easier as $\mathbf{u}(0) = 0$. Note that if $g_0 \neq 0$, in order for condition (4.7c) to hold, $g_0 \in H_0^3(D)$ and for (4.7e) to hold, $A^0(x)$ needs to be in $W^{2,\infty}(D)$, i.e., $A(x, \mathbf{y})$ needs to be in $W^{2,\infty}$ with respect to x .

Appendix C. We now present the proof of Proposition 6.1.

Proof. First, we establish the regularity

$$(C1) \quad u_0 \in H^2(0, T; V) \cap L^\infty(0, T; H^2(D)), \quad \frac{\partial u_0}{\partial t} \in L^\infty(0, T; V \cap H^2(D)).$$

As $g_0 = 0$, condition (4.5) holds. From (4.10), we have $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; V)$ and $\frac{\partial^3 u_0}{\partial t^3} \in L^\infty(0, T; H)$. From (4.9), we have

$$-\nabla \cdot \left(A^0 \nabla \frac{\partial u_0}{\partial t} \right) = \frac{\partial f}{\partial t} - \frac{\partial^3 u_0}{\partial t^3} \in L^\infty(0, T; H).$$

We, therefore, deduce that $\frac{\partial u_0}{\partial t} \in L^\infty(0, T; H^2(D))$.

For u^ε , we have that

$$(C2) \quad \frac{\partial^2}{\partial t^2} \left(\frac{\partial u^\varepsilon}{\partial t} \right) - \nabla \cdot \left(A^\varepsilon \nabla \frac{\partial u^\varepsilon}{\partial t} \right) = \frac{\partial f}{\partial t},$$

with the compatibility condition

$$\frac{\partial u^\varepsilon}{\partial t}(0) = g_1 \in V, \quad \frac{\partial}{\partial t} \frac{\partial u^\varepsilon}{\partial t}(0) = f(0) - \nabla \cdot (A^\varepsilon(\cdot) \nabla g_0(\cdot)) = f(0) \in H.$$

Thus

$$(C3) \quad \frac{\partial u^\varepsilon}{\partial t} \in L^2(0, T; V) \cap H^1(0, T; H).$$

Let

$$u_1^\varepsilon(t, x) = u_0(t, x) + \varepsilon u_1\left(t, x, \frac{x}{\varepsilon}\right).$$

We first show that

$$\|\nabla \cdot (A^\varepsilon \nabla u_1^\varepsilon) - \nabla \cdot (A^0 \nabla u_0)\|_{L^\infty(0, T; H^{-1}(D))} \leq c\varepsilon.$$

For elliptic problems, Jikov, Kozlov, and Oleĭnik [19] shows this for the case where $A(x, y) = A(y)$ and for smooth u_0 . However, the result also holds for the case where $A(x, y)$ depends on x and for the weaker regularity $u_0 \in H^2(D)$. This is done in detail in Hoang and Schwab [16]. We thus only present the main ideas of the proof here.

We note that

$$\begin{aligned} (A^\varepsilon \nabla u_1^\varepsilon)_i(t, x) &= A_{ij}^0(x) \frac{\partial u_0}{\partial x_j}(t, x) \\ &\quad + g_i^j\left(x, \frac{x}{\varepsilon}\right) \frac{\partial u_0}{\partial x_j}(t, x) + \varepsilon w_k\left(x, \frac{x}{\varepsilon}\right) A_{ij}\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 u_0}{\partial x_j \partial x_k}(t, x) \\ &\quad + \varepsilon A_{ik}\left(x, \frac{x}{\varepsilon}\right) \frac{\partial w_j}{\partial x_k}\left(x, \frac{x}{\varepsilon}\right) \frac{\partial u_0}{\partial x_j}(t, x), \end{aligned}$$

where

$$g_i^j(x, y) = A_{ij}(x, y) + A_{ik}(x, y) \frac{\partial w_j}{\partial y_k}(x, y) - A_{ij}^0(x).$$

From (6.1) and (6.2), for every $x \in D$, we have

$$\int_Y g_i^j(x, y) dy = 0 \quad \text{and} \quad \frac{\partial}{\partial y_i} g_i^j(x, y) = 0.$$

Adapting the proof of Jikov, Kozlov, and Oleĭnik [19] for the case $A(x, y) = A(y)$, Hoang and Schwab [16] show that there are functions $\alpha_{ij}^k(x, y)$ which are Y -periodic with respect to y such that $\alpha_{ij}^k = -\alpha_{ji}^k$ and

$$g_i^k(x, y) = \frac{\partial}{\partial y_j} \alpha_{ij}^k(x, y),$$

and that when $A(x, y) \in (C^1(\bar{D}, C_\#^1(\bar{Y})))^{d \times d}$, $\alpha_{ij}^k(x, y) \in C^1(\bar{D}, C_\#(\bar{Y}))$. We therefore have

$$(A^\varepsilon \nabla u_1^\varepsilon(t, x) - A^0(x) \nabla u_0(t, x))_i = \varepsilon \frac{\partial}{\partial x_j} \left(\alpha_{ij}^k \left(x, \frac{x}{\varepsilon} \right) \frac{\partial u_0}{\partial x_k}(t, x) \right) + (r_\varepsilon)_i(t, x),$$

where

$$\begin{aligned} (r_\varepsilon)_i(t, x) = & -\varepsilon \frac{\partial \alpha_{ij}^k(x, y)}{\partial x_j} \Big|_{y=x/\varepsilon} \frac{\partial u_0(t, x)}{\partial x_k}(t, x) - \varepsilon \alpha_{ij}^k \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_0}{\partial x_k \partial x_j}(t, x) \\ & + \varepsilon w_k \left(x, \frac{x}{\varepsilon} \right) A_{ij} \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_0}{\partial x_k \partial x_j}(t, x) \\ & + \varepsilon A_{ik} \left(x, \frac{x}{\varepsilon} \right) \frac{\partial w_j}{\partial x_k} \left(x, \frac{x}{\varepsilon} \right) \frac{\partial u_0}{\partial x_j}(t, x). \end{aligned}$$

As $\alpha_{ij}^k(x, y) \in C^1(\bar{D}, C(\bar{Y}))$, we deduce from (C1) that $\|(r_\varepsilon)_i(\cdot)\|_{L^\infty(0, T; H)} \leq c\varepsilon$. Using $\alpha_{ij}^k = -\alpha_{ji}^k$ we get

$$(C4) \quad \|\nabla \cdot (A^\varepsilon \nabla u_1^\varepsilon(t, x)) - \nabla \cdot (A^0 \nabla u_0(t, x))\|_{L^\infty(0, T; H^{-1}(D))} \leq c\varepsilon.$$

Let $\tau^\varepsilon \in C_0^\infty(D)$ be such that $\tau^\varepsilon = 1$ outside an ε neighborhood of ∂D and $\varepsilon |\nabla \tau^\varepsilon(x)| \leq c$ for all $\varepsilon > 0$. Let

$$\begin{aligned} (C5) \quad w_1^\varepsilon(t, x) &= u_0(t, x) + \varepsilon \tau^\varepsilon(x) w^k \left(x, \frac{x}{\varepsilon} \right) \frac{\partial u_0}{\partial x_k}(t, x) \\ &= u_1^\varepsilon(t, x) - \varepsilon (1 - \tau^\varepsilon(x)) w^k \left(x, \frac{x}{\varepsilon} \right) \frac{\partial u_0}{\partial x_k}(t, x). \end{aligned}$$

We have that

$$\begin{aligned} (C6) \quad \nabla(u_1^\varepsilon - w_1^\varepsilon)(t, x) &= -\varepsilon \nabla \tau^\varepsilon(x) w^k \left(x, \frac{x}{\varepsilon} \right) \frac{\partial u_0}{\partial x_k}(t, x) \\ &\quad + (1 - \tau^\varepsilon(x)) \nabla_y w^k \left(x, \frac{x}{\varepsilon} \right) \frac{\partial u_0}{\partial x_k}(t, x) \\ &\quad + \varepsilon (1 - \tau^\varepsilon(x)) \nabla w^k \left(x, \frac{x}{\varepsilon} \right) \frac{\partial u_0}{\partial x_k}(t, x) \\ &\quad + \varepsilon (1 - \tau^\varepsilon(x)) w^k \left(x, \frac{x}{\varepsilon} \right) \nabla \frac{\partial u_0}{\partial x_k}(t, x). \end{aligned}$$

Thus

$$\begin{aligned}
 & \nabla \cdot (A^\varepsilon (\nabla(u_1^\varepsilon - w_1^\varepsilon))) \\
 &= \varepsilon \nabla \cdot \left(A^\varepsilon(x) (1 - \tau^\varepsilon(x)) \nabla w^k \left(x, \frac{x}{\varepsilon} \right) \frac{\partial u_0}{\partial x_k}(t, x) \right) \\
 & \quad + \varepsilon \nabla \cdot \left(A^\varepsilon(x) (1 - \tau^\varepsilon(x)) w^k \left(x, \frac{x}{\varepsilon} \right) \nabla \frac{\partial u_0}{\partial x_k}(t, x) \right) \\
 (C7) \quad & + \nabla \cdot \left(A^\varepsilon(x) \left(-\varepsilon \nabla \tau^\varepsilon(x) w^k \left(x, \frac{x}{\varepsilon} \right) + (1 - \tau^\varepsilon(x)) \nabla_y w^k \left(x, \frac{x}{\varepsilon} \right) \right) \frac{\partial u_0}{\partial x_k}(t, x) \right).
 \end{aligned}$$

Since $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; V)$, we deduce

$$(C8) \quad \left\| \frac{\partial^2 w_1^\varepsilon}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} \right\|_{L^\infty(0, T; H)} \leq c\varepsilon.$$

Using

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} - \nabla \cdot (A^\varepsilon \nabla u^\varepsilon) = \frac{\partial^2 u_0}{\partial t^2} - \nabla \cdot (A^0 \nabla u_0),$$

from (C4), (C7), and (C8) we deduce that

$$\begin{aligned}
 & \frac{\partial^2 (u^\varepsilon - w_1^\varepsilon)}{\partial t^2} - \nabla \cdot (A^\varepsilon \nabla (u^\varepsilon - w_1^\varepsilon)) \\
 &= \frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 w_1^\varepsilon}{\partial t^2} + \nabla \cdot (A^\varepsilon \nabla (w_1^\varepsilon - u_1^\varepsilon)) + \nabla \cdot (A^\varepsilon \nabla u_1^\varepsilon) - \nabla \cdot (A^0 \nabla u_0) \\
 &= \frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 w_1^\varepsilon}{\partial t^2} + \nabla \cdot (A^\varepsilon \nabla u_1^\varepsilon) - \nabla \cdot (A^0 \nabla u_0) \\
 & \quad - \varepsilon \nabla \cdot \left(A^\varepsilon (1 - \tau^\varepsilon) \nabla w^k \left(\cdot, \frac{\cdot}{\varepsilon} \right) \frac{\partial u_0}{\partial x_k} \right) - \varepsilon \nabla \cdot \left(A^\varepsilon (1 - \tau^\varepsilon) w^k \left(\cdot, \frac{\cdot}{\varepsilon} \right) \nabla \frac{\partial u_0}{\partial x_k} \right) \\
 & \quad - \nabla \cdot \left(A^\varepsilon \left(-\varepsilon \nabla \tau^\varepsilon w^k \left(\cdot, \frac{\cdot}{\varepsilon} \right) + (1 - \tau^\varepsilon) \nabla_y w^k \left(\cdot, \frac{\cdot}{\varepsilon} \right) \right) \frac{\partial u_0}{\partial x_k} \right) \\
 &= X - \nabla \cdot \left(A^\varepsilon \left(-\varepsilon \nabla \tau^\varepsilon w^k \left(\cdot, \frac{\cdot}{\varepsilon} \right) + (1 - \tau^\varepsilon) \nabla_y w^k \left(\cdot, \frac{\cdot}{\varepsilon} \right) \right) \frac{\partial u_0}{\partial x_k} \right),
 \end{aligned}$$

where X denotes the sum of the first six terms. From (C1), (C4), and (C8) we deduce that $\|X\|_{L^\infty(0, T; H^{-1}(D))} \leq c\varepsilon$. Since $\frac{\partial u_0}{\partial t} \in L^\infty(0, T; V \cap H^2(D))$, we have that $\frac{\partial w_1^\varepsilon}{\partial t} \in L^\infty(0, T; V)$ and $\sup_{t \in (0, T)} \left\| \frac{\partial w_1^\varepsilon}{\partial t}(t) \right\|_V < c$ for c independent of ε . Therefore,

$$\begin{aligned}
 & \int_0^t \left(\frac{\partial^2 (u^\varepsilon - w_1^\varepsilon)}{\partial s^2}(s, \cdot), \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial s}(s, \cdot) \right)_H ds \\
 & \quad + \int_0^t \int_D A^\varepsilon(x) \nabla (u^\varepsilon(s, x) - w_1^\varepsilon(s, x)) \cdot \frac{\partial}{\partial s} \nabla (u^\varepsilon(s, x) - w_1^\varepsilon(s, x)) dx ds \\
 &= \int_0^t \left(X(s, \cdot), \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial s}(s, \cdot) \right)_H ds \\
 & \quad + \int_0^t \int_D A^\varepsilon(x) \left(-\varepsilon \nabla \tau^\varepsilon(x) w^k \left(x, \frac{x}{\varepsilon} \right) + (1 - \tau^\varepsilon(x)) \nabla_y w^k \left(x, \frac{x}{\varepsilon} \right) \right) \\
 & \quad \cdot \frac{\partial u_0}{\partial x_k}(s, x) \frac{\partial}{\partial s} \nabla (u^\varepsilon(s, x) - w_1^\varepsilon(s, x)) dx ds
 \end{aligned}$$

$$\begin{aligned} &\leq c \|X\|_{L^2(0,T;H^{-1}(D))} \left(\left\| \frac{\partial u^\varepsilon}{\partial s} \right\|_{L^2(0,T;V)} + \left\| \frac{\partial w_1^\varepsilon}{\partial s} \right\|_{L^2(0,T;V)} \right) \\ &\quad + \int_0^t \int_D A^\varepsilon(x) \left(-\varepsilon \nabla \tau^\varepsilon(x) w^k \left(x, \frac{x}{\varepsilon} \right) + (1 - \tau^\varepsilon(x)) \nabla_y w^k \left(x, \frac{x}{\varepsilon} \right) \right) \\ &\quad \cdot \left[\frac{\partial}{\partial s} \left(\frac{\partial u_0}{\partial x_k}(s, x) \nabla(u^\varepsilon(s, x) - w_1^\varepsilon(s, x)) \right) - \frac{\partial^2 u_0}{\partial s \partial x_k}(s, x) \nabla(u^\varepsilon(s, x) - w_1^\varepsilon(s, x)) \right] dx ds. \end{aligned}$$

Let $\hat{D}^\varepsilon \subset D$ be an ε neighborhood of ∂D outside which $\tau^\varepsilon(x) = 1$. As $u^\varepsilon(0, x) = 0$ and $w_1^\varepsilon(0, x) = 0$, and

$$\left\| \frac{\partial u^\varepsilon}{\partial t}(0) - \frac{\partial w_1^\varepsilon}{\partial t}(0) \right\|_H = \left\| \frac{\partial u_0}{\partial t}(0, \cdot) - \frac{\partial w_1^\varepsilon}{\partial t}(0) \right\|_H \leq c\varepsilon \left\| \frac{\partial}{\partial t} \nabla u_0(0) \right\|_H,$$

we deduce that

$$\begin{aligned} &\left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H^2 + \|\nabla(u^\varepsilon(t) - w_1^\varepsilon(t))\|_H^2 \leq c\varepsilon \\ &\quad + c \left\| \frac{\partial u_0}{\partial x_k}(t) \right\|_{L^2(\hat{D}^\varepsilon)} \|\nabla(u^\varepsilon(t) - w_1^\varepsilon(t))\|_H \\ &\quad + c \int_0^T \left\| \frac{\partial^2 u_0}{\partial t \partial x_k}(s) \right\|_{L^2(\hat{D}^\varepsilon)} ds \sup_{0 \leq t \leq T} \|\nabla(u^\varepsilon(t) - w_1^\varepsilon(t))\|_H. \end{aligned}$$

Following Hoang and Schwab [16], as ∂D is Lipschitz, for all functions $\phi \in C^\infty(\bar{D})$ we have $\|\phi\|_{L^2(\hat{D}^\varepsilon)}^2 \leq c\varepsilon^2 \|\phi\|_{H^1(D)}^2 + c\varepsilon \|\phi\|_{L^2(\partial D)}^2$, so

$$(C9) \quad \|\phi\|_{L^2(\hat{D}^\varepsilon)} \leq c\varepsilon^{1/2} \|\phi\|_{H^1(D)} \quad \forall \phi \in H^1(D).$$

We, therefore, have

$$\left\| \frac{\partial u_0}{\partial x_k}(t, \cdot) \right\|_{L^2(\hat{D}^\varepsilon)} \leq c\varepsilon^{1/2} \left\| \frac{\partial u_0}{\partial x_k}(t, \cdot) \right\|_{H^1(D)}$$

and

$$\left\| \frac{\partial^2 u_0}{\partial t \partial x_k}(t, \cdot) \right\|_{L^2(\hat{D}^\varepsilon)} \leq c\varepsilon^{1/2} \left\| \frac{\partial^2 u_0}{\partial t \partial x_k}(t, \cdot) \right\|_{H^1(D)}.$$

From (C1), we deduce that for all $t \in (0, T)$

$$\begin{aligned} &\left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H^2 + \|\nabla(u^\varepsilon - w_1^\varepsilon)(t)\|_H^2 \leq c\varepsilon + c\varepsilon^{1/2} \|\nabla(u^\varepsilon - w_1^\varepsilon)(t)\|_H \\ &\quad + c\varepsilon^{1/2} \sup_{t \in (0, T)} \|\nabla(u^\varepsilon - w_1^\varepsilon)(t)\|_H, \end{aligned}$$

which implies

$$\left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H + \|\nabla(u^\varepsilon - w_1^\varepsilon)(t)\|_H \leq c\varepsilon^{1/2}.$$

From (C1) and (C6) we have that

$$\left\| \frac{\partial(u_1^\varepsilon - w_1^\varepsilon)}{\partial t} \right\|_{L^\infty(0, T; H)} \leq c\varepsilon.$$

Further, from (C1) and (C7) by using (C9) we get

$$\|\nabla(u_1^\varepsilon - w_1^\varepsilon)\|_{L^\infty(0,T;H)} \leq c\varepsilon^{1/2}.$$

We then get the conclusion. $\square$

Appendix D. We prove Proposition 6.4 in this appendix.

We consider the expression

$$\begin{aligned} E^\varepsilon(t) &= \int_D \left(\frac{\partial u^\varepsilon}{\partial t}(t) - \frac{\partial u_0}{\partial t}(t) \right)^2 dx \\ &\quad + \int_D \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(A^\varepsilon)(\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon(t)) - (\nabla u_0(t) + \nabla_{y_1} u_1(t) + \cdots + \nabla_{y_n} u_n(t))) \\ &\quad \cdot (\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon(t)) - (\nabla u_0(t) + \nabla_{y_1} u_1(t) + \cdots + \nabla_{y_n} u_n(t))) d\mathbf{y} dx. \end{aligned}$$

From (6.3) and the conservation of energy of wave equations (see Lions and Margenes [20], also Brahim-Otsmane, Francfort, and Murat [5]) we have

$$\begin{aligned} \int_D \left(\frac{\partial u^\varepsilon}{\partial t} \right)^2 dx + \int_D \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(A^\varepsilon) \mathcal{T}_n^\varepsilon(\nabla u^\varepsilon(t)) \cdot \mathcal{T}_n^\varepsilon(\nabla u^\varepsilon(t)) d\mathbf{y} dx \\ = \int_D \left(\frac{\partial u^\varepsilon}{\partial t} \right)^2 dx + \int_D A^\varepsilon \nabla u^\varepsilon(t) \cdot \nabla u^\varepsilon(t) dx \\ = \int_D g_1^2 dx + 2 \int_0^t \int_D f \frac{\partial u^\varepsilon}{\partial t} dx dt \\ \rightarrow \int_D g_1^2 dx + 2 \int_0^t \int_D f \frac{\partial u_0}{\partial t} dx dt. \end{aligned}$$

From this and (6.4), we have that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} E^\varepsilon(t) &= \int_D g_1^2 dx + 2 \int_0^t \int_D f \frac{\partial u_0}{\partial t} dx dt \\ &\quad - \left[\int_D \left(\frac{\partial u_0}{\partial t} \right)^2 dx + \int_D \int_{\mathbf{Y}} A(x, \mathbf{y}) (\nabla u_0(t) + \nabla_{y_1} u_1(t) + \cdots + \nabla_{y_n} u_n(t)) \right. \\ &\quad \left. \cdot (\nabla u_0(t) + \nabla_{y_1} u_1(t) + \cdots + \nabla_{y_n} u_n(t)) d\mathbf{y} dx \right]. \end{aligned}$$

Since A^0 is the homogenized coefficient of the multiscale coefficient $A(x, \mathbf{y})$, we have

$$\begin{aligned} \int_D \int_{\mathbf{Y}} A(x, \mathbf{y}) (\nabla u_0(t) + \nabla_{y_1} u_1(t) + \cdots + \nabla_{y_n} u_n(t)) \\ \cdot (\nabla u_0(t) + \nabla_{y_1} u_1(t) + \cdots + \nabla_{y_n} u_n(t)) d\mathbf{y} dx \\ = \int_D A^0 \nabla u_0(t) \cdot \nabla u_0(t) dx. \end{aligned}$$

From (4.3), we deduce that

$$\int_D \left(\frac{\partial u_0}{\partial t} \right)^2 dx + \int_D A^0 \nabla u_0 \cdot \nabla u_0 dx = \int_D g_1^2 dx + 2 \int_0^t \int_D f \frac{\partial u_0}{\partial t} dx dt.$$

Thus $\lim_{\varepsilon \rightarrow 0} E^\varepsilon(t) = 0$.

Next, we show that the convergence is uniform. From (C2) in Appendix C, we find that $\frac{\partial u^\varepsilon}{\partial t}$ is uniformly bounded in $H^1(0, T; H) \cap L^2(0, T; V)$ with respect to ε and $\frac{\partial u_0}{\partial t}$ is in $H^1(0, T; H) \cap L^2(0, T; V)$. Also from (4.9), $\frac{\partial u_0}{\partial t} \in L^2(0, T; V)$ so $\frac{\partial \mathbf{u}}{\partial t} \in L^2(0, T; \mathbf{V})$.

We show that $E^\varepsilon(t)$ is equicontinuous with respect to ε . Considering the first term, we have for $s < t$

$$\begin{aligned} & \left| \int_D \left[\left(\frac{\partial u^\varepsilon}{\partial t}(t) - \frac{\partial u_0}{\partial t}(t) \right)^2 - \left(\frac{\partial u^\varepsilon}{\partial t}(s) - \frac{\partial u_0}{\partial t}(s) \right)^2 \right] dx \right| \\ &= \left| \int_D \left(\left(\frac{\partial u^\varepsilon}{\partial t}(t) - \frac{\partial u^\varepsilon}{\partial t}(s) \right) - \left(\frac{\partial u_0}{\partial t}(t) - \frac{\partial u_0}{\partial t}(s) \right) \right) \right. \\ & \quad \cdot \left. \left(\frac{\partial u^\varepsilon}{\partial t}(t) + \frac{\partial u^\varepsilon}{\partial t}(s) - \frac{\partial u_0}{\partial t}(t) - \frac{\partial u_0}{\partial t}(s) \right) dx \right| \\ &\leq \left(\left\| \frac{\partial u^\varepsilon}{\partial t}(t) - \frac{\partial u^\varepsilon}{\partial t}(s) \right\|_H + \left\| \frac{\partial u_0}{\partial t}(t) - \frac{\partial u_0}{\partial t}(s) \right\|_H \right) \\ & \quad \cdot \left(\left\| \frac{\partial u^\varepsilon}{\partial t}(t) \right\|_H + \left\| \frac{\partial u^\varepsilon}{\partial t}(s) \right\|_H + \left\| \frac{\partial u_0}{\partial t}(t) \right\|_H + \left\| \frac{\partial u_0}{\partial t}(s) \right\|_H \right). \end{aligned}$$

From the regularity $\frac{\partial u^\varepsilon}{\partial t} \in H^1(0, T; H)$ in (C3), we have that $\left\| \frac{\partial u^\varepsilon}{\partial t}(t) \right\|_H$ is uniformly bounded for all t . Similarly, $\left\| \frac{\partial u_0}{\partial t}(t) \right\|_H$ is uniformly bounded for all t . Therefore,

$$\begin{aligned} & \left| \int_D \left[\left(\frac{\partial u^\varepsilon}{\partial t}(t) - \frac{\partial u_0}{\partial t}(t) \right)^2 - \left(\frac{\partial u^\varepsilon}{\partial t}(s) - \frac{\partial u_0}{\partial t}(s) \right)^2 \right] dx \right| \\ &\leq c \left(\left\| \int_s^t \frac{\partial^2 u^\varepsilon}{\partial t^2}(\tau) d\tau \right\|_H + \left\| \int_s^t \frac{\partial^2 u_0}{\partial t^2}(\tau) d\tau \right\|_H \right) \\ &\leq c \int_s^t \left(\left\| \frac{\partial^2 u^\varepsilon}{\partial t^2}(\tau) \right\|_H + \left\| \frac{\partial^2 u_0}{\partial t^2}(\tau) \right\|_H \right) d\tau \\ &\leq c(t-s)^{1/2} \left(\int_s^t \left(\left\| \frac{\partial^2 u^\varepsilon}{\partial t^2}(\tau) \right\|_H^2 + \left\| \frac{\partial^2 u_0}{\partial t^2}(\tau) \right\|_H^2 \right) d\tau \right)^{1/2} \leq c(t-s)^{1/2} \end{aligned}$$

due to $\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{H^1(0, T; H)}$ is uniformly bounded with respect to ε and $\frac{\partial u_0}{\partial t} \in H^1(0, T; H)$.

We can deal with the second term in the expression of $E^\varepsilon(t)$ in a similar manner, using the fact that u^ε is uniformly bounded in $H^1(0, T; V)$ and $\mathbf{u} \in H^1(0, T; \mathbf{V})$. We then get the conclusion from the Arzelà–Ascoli theorem. $\square$

REFERENCES

- [1] A. ABDULLE AND M. J. GROTE, *Finite element heterogeneous multiscale method for the wave equation*, Multiscale Model. Simul., 9 (2011), pp. 766–792.
- [2] G. ALLAIRE, *Homogenization and two-scale convergence*, SIAM J. Math. Anal., 23 (1992), pp. 1482–1518.
- [3] G. ALLAIRE AND M. BRIANE, *Multiscale convergence and reiterated homogenisation*, Proc. Roy. Soc. Edinburgh Sect. A, 126 (1996), pp. 297–342.
- [4] A. BENSOUSSAN, J.-L. LIONS, AND G. PAPANICOLAOU, *Asymptotic Analysis for Periodic Structures*, Stud. Math. Appl. 5, North-Holland, Amsterdam, 1978.
- [5] S. BRAHIM-OTSMANE, G. A. FRANCFORT, AND F. MURAT, *Correctors for the homogenization of the wave and heat equations*, J. Math. Pures Appl. (9), 71 (1992), pp. 197–231.

- [6] P. CIARLET, *The Finite Element Method for Elliptic Problems*, Stud. Math. Appl. 4, North-Holland, Amsterdam, 1978.
- [7] D. CIORANESCU, A. DAMLAMIAN, AND G. GRISO, *The periodic unfolding method in homogenization*, SIAM J. Math. Anal., 40 (2008), pp. 1585–1620.
- [8] T. DUPONT, *L^2 -estimates for Galerkin methods for second order hyperbolic equations*, SIAM J. Numer. Anal., 10 (1973), pp. 880–889.
- [9] W. E AND B. ENGQUIST, *The heterogeneous multiscale methods*, Commun. Math. Sci., 1 (2003), pp. 87–132.
- [10] B. ENGQUIST, H. HOLST, AND O. RUNBORG, *Multiscale methods for the wave propagation in heterogeneous media*, Commun. Math. Sci., 9 (2011), pp. 33–56.
- [11] M. GRIEBEL AND P. OSWALD, *Tensor product type subspace splittings and multilevel iterative methods for anisotropic problems*, Adv. Comput. Math., 4 (1995), pp. 171–206.
- [12] P. GRISVARD, *Elliptic Problems in Nonsmooth Domains*, Monogr. Stud. Math. 24, Pitman (Advanced Publishing Program), Boston, 1985.
- [13] H. HARBRECHT AND C. SCHWAB, *Sparse tensor finite elements for elliptic multiple scale problems*, Comput. Methods Appl. Mech. Engrg., 200 (2011), pp. 3100–3110.
- [14] V. H. HOANG, *Sparse finite element method for periodic multiscale nonlinear monotone problems*, Multiscale Model. Simul., 7 (2008), pp. 1042–1072.
- [15] V. H. HOANG AND CH. SCHWAB, *High-dimensional finite elements for elliptic problems with multiple scales*, Multiscale Model. Simul., 3 (2005), pp. 168–194.
- [16] V. H. HOANG AND CH. SCHWAB, *Analytic regularity and polynomial approximation of stochastic, parametric elliptic multiscale PDEs*, Anal. Appl. (Singap.), 11 (2013), 1350001.
- [17] T. Y. HOU AND X.-H. WU, *A multiscale finite element method for elliptic problems in composite materials and porous media*, J. Comput. Phys., 134 (1997), pp. 169–189.
- [18] L. JIANG, Y. EFENDIEV, AND V. GINTING, *Analysis of global multiscale finite element methods for wave equations with continuum spatial scales*, Appl. Numer. Math., 60 (2010), pp. 862–876.
- [19] V. V. JIKOV, S. M. KOZLOV, AND O. A. OLEĖNIK, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, 1994, translated from the Russian by G. A. Yosifian [G. A. Iosifyan].
- [20] J.-L. LIONS AND E. MAGENES, *Non-homogeneous Boundary Value Problems and Applications*, Vol. I, Die Grundlehren der mathematischen Wissenschaften, Band 181, Springer-Verlag, New York, 1972; translated from the French by P. Kenneth.
- [21] G. NGUETSENG, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal., 20 (1989), pp. 608–623.
- [22] H. OWHADI AND L. ZHANG, *Numerical homogenization of the acoustic wave equations with a continuum of scales*, Comput. Methods Appl. Mech. Engrg., 198 (2008), pp. 397–406.
- [23] F. SANTOSA AND W. SYMES, *A dispersive effective medium for wave propagation in periodic composites*, SIAM J. Appl. Math., 51 (1991), pp. 984–1005.
- [24] J. WLOKA, *Partial Differential Equations*, Cambridge University Press, Cambridge, 1987; translated from the German by C. B. Thomas and M. J. Thomas.