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A Bayesian Multivariate Risk-Neutral Method for Pricing Reverse Mortgages

Atsuyuki Kogure, Jackie Li, and Shinichi Kamiya

Abstract

In this paper, we propose a Bayesian multivariate framework to price reverse mortgages which involve several risks in both insurance and financial sectors (e.g. mortality rates, interest rates, and house prices). Our method is a multivariate extension of the Bayesian risk-neutral method developed by Kogure and Kurachi (2010). We apply the proposed method to Japanese data to examine the possibility for a successful introduction of reverse mortgages into Japan. The results suggest some promising future for this new market.

1 Introduction

A reverse mortgage has long been considered as an effective way for an individual to manage longevity risk, especially in rapidly aging countries like Japan. This market has become particularly important under the continuing trend of moving away from traditional defined benefit (DB) retirement plans to defined contribution (DC) plans. Simply speaking, a reverse mortgage allows a retiree to borrow against the value of his or her home property. The loan is then repaid when the borrower dies and the property is sold. Usually, the loan is non-recourse, which means that the lender cannot claim the borrower's assets other than the mortgaged property.

In Japan, the major source of post-retirement income is public pensions (Fujisawa and Li 2012). As the pension system is pay-as-you-go and the old-age dependency ratio is expected to rise significantly from 38% in 2010 to 80% in 2050 (OECD 2009), whether the system can sustain in the long term has become a serious concern. Since housing wealth constitutes about half of private total assets (Creighton et al. 2005), a natural solution to the ageing problem is to unlock the housing equity. It thus appears that there is a very large potential market for reverse mortgages.

However, a reverse mortgage involves several risks, most notably longevity, house prices, and interest rates. Recently, these risks have been increasingly uncertain, which may partly explain the present staggering state of development of the reverse mortgage market. In

particular, it is not a straightforward exercise to price the non-recourse provision under these uncertainties. In order to price reverse mortgage plans appropriately, a multivariate framework which can allow for multiple risks simultaneously is required. In this paper, we propose a Bayesian multivariate method by generalizing the univariate Bayesian method suggested in Kogure and Kurachi (2010) and apply this new method to examine the possibility for a successful introduction of reverse mortgages into Japan. For this paper we assume that interest rates are fixed and consider the joint distribution of house prices and mortality rates. Some earlier studies on pricing reverse mortgages include Hosty et al. (2008), Li et al. (2010), Ji et al. (2012), and Lee et al. (2012), which adopted a univariate risk-neutral method for one particular risk. In contrast, we undertake a multivariate risk-neutral approach for both house price risk and longevity risk under a Bayesian setting.

There have been several pricing methods proposed in the literature, e.g. Wang et al. (2008) and Chen et al. (2010). Our method differs from the existing ones in that we take a fully Bayesian approach. As discussed in Li (2013), there are a number of distinct advantages of using the Bayesian approach. First, certain prior or reference information can be incorporated into the modeling process in a formal manner. Second, different model structures can be taken into account coherently within the same framework. Moreover, this approach allows for process error, parameter error, and possibly model error in pricing. Specifically, we obtain the distributions of the risk factors and then translate them into a risk-neutral form by applying the maximum entropy principle. Note that the expected return in a risk-neutral world is equal to the risk-free interest rate. See Cairns et al. (2006) for some discussion on parameter uncertainty and risk-neutral valuation in evaluating longevity risk.

The rest of the paper is organized as follows. In Section 2, we present a method to convert the joint distribution to its risk-neutral form by applying the minimum cross-entropy principle. The risk-neutral distribution thus obtained is deduced to factor into the product of two marginal distributions. After a short explanation of reverse mortgage plans in Section 3, we fit a Bayesian Lee-Carter model in Section 4 and a time series model in Section 5 to Japanese data and deduce the risk-neutral distribution of each risk factor. By combining these results, we examine the feasibility for an introduction of reverse mortgages into Japan in Section 6. The figures suggest some promising future for the introduction. Section 7 concludes the paper.

2 Bayesian risk-neutral method

As ways to hedge risks such as stock price movements or accidental deaths, financial options and insurance products are widely used. Such hedging instruments are contingent contracts which pay off a certain amount of money dependent on observed risk factors such as stock prices or deaths, and are thus called derivatives. In this section, we give pricing principles for derivatives based on Bayesian predictive pricing.

2.1 Risk-neutral distributions

For ease of explanation, we consider a single period framework with current time 0 and a time point $T > 0$ in the future. Let X (a continuous random variable) denote the value of a risk factor at time T and write $F(x)$ for the distribution function of X . We wish to price a derivative which pays off $C(X)$ at time T dependent on X . Assuming that the risk-free interest rate r is constant over the whole investment period, the expected present value of the derivative at time 0 can be given by the formula:

$$V = (1+r)^{-T} \int C(x) dF^*(x) , \quad (1)$$

where $F^*(x)$ is so-called the risk-neutral distribution function of X , which can be expressed as:

$$dF^*(x) = p(x) dF(x) ,$$

via a state price density $p(x)$. Note that all the integrals in this paper cover the entire range of the variable of interest.

The state price density $p(x)$ represents a transformation from $F(x)$ to $F^*(x)$. In the standard option pricing theory, the concept of completeness is evoked under the no-arbitrary principle to determine the state price density $p(x)$ and thus the risk-neutral distribution $F^*(x)$. The market is ‘complete’ if there are many securities traded in the market and any cash flow can be replicated via dynamic hedging strategies. However, it would be unrealistic to assume such completeness for non-financial risks such as mortality risk or natural hazard risk, as the market of securitization of these risks is currently far from having sufficient liquidity. In the insurance risk theory, the state price density is then chosen based on the particular problem being investigated. Two principles have been widely applied. One is the Esscher transform, in which the state price density takes the form:

$$p(x) = \frac{e^{\gamma x}}{E(e^{\gamma X})} .$$

Here $E(\bullet)$ is the expectation under $F(x)$ and γ is a constant to be determined consistently with some market conditions. The other is the Wang transform, where the state price density is of the form:

$$p(x) = \frac{\phi(\xi(x) - \gamma)}{\phi(\xi(x))}, \quad \xi(x) = \Phi^{-1}(F(x)),$$

where $\Phi(\bullet)$ denotes the standard normal distribution function and $\phi(\bullet)$ denotes its density. See Gerber and Shiu (1994) for discussions on the Esscher transform and Wang (2000) on the Wang transform. On the other hand, Li (2010) argued that for the Wang transform, subjective decisions are often required in determining the market prices of risk, and it is not straightforward to include parameter error. Moreover, Kogure and Kurachi (2010) commented that the Esscher transform can usually incorporate only one constraint for a market price. Comparatively, our Bayesian maximum entropy approach set forth in Section 2.3 is more flexible and does not have these limitations.

2.2 Bayesian risk-neutral distributions

In practice, the distribution function $F(x)$ is unknown and we set up a certain statistical model $\{f(x|\theta), \theta \in \Theta\}$ to which $f(x|\theta) = \frac{d}{dx}F(x|\theta)$ is assumed to belong. Then the formula (1) turns into:

$$V(\theta) = (1+r)^{-T} \int C(x) f^*(x|\theta) dx, \quad (2)$$

with $f^*(x|\theta) = p(x)f(x|\theta)$.

In the standard frequentist approach, the maximum likelihood estimator $\hat{\theta}$ of θ is obtained based on data D_0 and plugged into (2) to yield:

$$V(\hat{\theta}) = (1+r)^{-T} \int C(x) f^*(x|\hat{\theta}) dx. \quad (3)$$

However, when the specification $\{f(x|\theta), \theta \in \Theta\}$ involves many parameters, parameter uncertainty in (3) becomes a real concern. Then it would be more desirable to take the Bayesian approach. With a posterior density $g(\theta|D_0)$ of θ given data D_0 , the Bayesian pricing formula becomes:

$$\begin{aligned} \tilde{V} &= \int V(\theta) g(\theta|D_0) d\theta \\ &= (1+r)^{-T} \int C(x) \int f^*(x|\theta) g(\theta|D_0) d\theta dx \\ &= (1+r)^{-T} \int C(x) f^*(x|D_0) dx. \end{aligned}$$

Here,

$$f^*(x|D_0) = \int f^*(x|\theta) g(\theta|D_0) d\theta = p(x) f(x|D_0), \quad (4)$$

with $f(x|D_0) = \int f(x|\theta) g(\theta|D_0) d\theta$. Thus $f^*(x|D_0)$ is the risk-neutral version of the Bayesian predictive density $f(x|D_0)$. MCMC simulation can then be used to draw samples from the predictive density. For more information about the MCMC algorithms, see Kogure and Kurachi (2010) and Li (2013).

2.3 Risk-neutralization of predictive distributions

To obtain the state price density $p(x)$ for (4), we apply a non-parametric technique proposed by Stutzer (1996) instead of using ‘parametric’ methods such as the Esscher or Wang transform. Suppose that there are m securities. For each $1 \leq i \leq m$, let $h_i(X)$ and v_i denote the payoff function at time T and the market value at time 0 of the i th security. Under the risk-neutral predictive density $f^*(x|D_0)$, we have the moment conditions:

$$E^*[h_i(X)|D_0] = \int h_i(x) f^*(x|D_0) dx = (1+r)^T v_i, \quad (5)$$

for $i = 1, 2, \dots, m$. Stutzer (1996) suggested using $f^*(x|D_0)$ that minimizes the Kullback-Leibler information or the cross-entropy:

$$\int f^*(x|D_0) \ln \left(\frac{f^*(x|D_0)}{f(x|D_0)} \right) dx,$$

subject to the moment conditions (5) and also the probability density constraint:

$$\int f^*(x|D_0) dx = 1. \quad (6)$$

The resultant minimum cross-entropy density is shown to be of the form:

$$f^*(x|D_0) = f(x|D_0) \exp \{ \gamma_0 + \gamma_1 h_1(x) + \dots + \gamma_m h_m(x) \}, \quad (7)$$

where the constants $\gamma_0, \gamma_1, \dots, \gamma_m$ are determined by (5) and (6). This approach was taken in Kogure and Kurachi (2010) to price longevity risk in Japan.

For some applications, several risk factors are involved and we need to consider the risk-neutralization of a multivariate predictive density. Suppose that, in addition to X , we have another risk factor Y . Let $f(x, y|D_0)$ denote the two-dimensional predictive density. Assume that there are m securities which in general depend on both X and Y . For each $1 \leq i \leq m$, let $h_i(X, Y)$ and v_i denote the payoff function at time T and the market value at time 0 of the i th security respectively. Then, following the idea of Stutzer (1996), we wish to minimize the

two-dimensional cross-entropy:

$$\iint f^*(x, y | D_0) \ln \left(\frac{f^*(x, y | D_0)}{f(x, y | D_0)} \right) dx dy ,$$

with respect to $f^*(x, y | D_0)$, subject to the moment conditions:

$$E^*[h_i(X, Y) | D_0] = \iint h_i(x, y) f^*(x, y | D_0) dx dy = (1+r)^T v_i , \quad (8)$$

for $i = 1, 2, \dots, m$ and the constraint:

$$\iint f^*(x, y | D_0) dx dy = 1 . \quad (9)$$

The resulting two-dimensional minimum cross-entropy density can be deduced as:

$$f^*(x, y | D_0) = f(x, y | D_0) \exp \{ \gamma_0 + \gamma_1 h_1(x, y) + \dots + \gamma_m h_m(x, y) \} , \quad (10)$$

where the constants $\gamma_0, \gamma_1, \dots, \gamma_m$ are determined by (8) and (9). If X and Y are independent under $f(x, y | D_0)$, they are also independent under the risk-neutral distribution (10) under some technical conditions (e.g. Kapur 1993). Since the reverse mortgage market in Japan is still in its infancy stage, for computation convenience, we follow some earlier studies (e.g. Lee, Wang, and Huang 2012) in assuming that the two risks are independent in the risk-neutral world.

3 Reverse mortgages

3.1 The non-recourse provision

A reverse mortgage loan is non-recourse in the sense that the property owner's obligation to repay the loan is limited to the proceeds from the sale of the property when the loan is terminated. Suppose that an individual aged x enters a reverse mortgage contract at time 0. Denote by L_t the outstanding balance of the loan and HP_t the value of the mortgaged property at time t . If the borrower dies at time t , the loan is terminated at that time and the lender will receive:

$$\min(L_t, HP_t) = L_t - \max(L_t - HP_t, 0) . \quad (11)$$

We assume here that L_t is given deterministically as $L_t = L_0(1 + r_L)^t$ with r_L being the interest rate charged on the reverse mortgage loan. The second term on the right hand side of (11) represents a cash flow from the non-recourse provision. It may be regarded as a put option on the house price HP_t with a strike price L_t . Thus the non-recourse provision is the borrower's right to exercise a put option on his death. For other types of reverse mortgages, see the Equity Release Report (2005).

3.2 Evaluation of reverse mortgages

We wish to evaluate the reverse mortgage loan for a cohort group of age x at time 0. Let τ denote the highest attained age and put $T = \tau - x$. Furthermore, let I_t denote an index which represents the proportion of the cohort group who will die from time $t - 1$ to t . The per capita cash flow from the reverse mortgage is given as:

$$(L_t - \max(L_t - HP_t, 0)) \times I_t \quad \text{for } t = 1, 2, \dots, T.$$

We assume that I_t and HP_t are uncertain, but discount rates are fixed. Let $d(t) = (1 + r)^{-t}$ be the t -year discount factor. We also assume that mortality experience and house prices are homogeneous within the group. At each time t , a certain portion of the group will die, which will terminate some loans and lead to some cash flows. The expected present value at time 0 of each reverse mortgage loan of the group (on average) is therefore:

$$E^* \left[\sum_{t=1}^T d(t) (L_t - \max(L_t - HP_t, 0)) I_t \right] = \sum_{t=1}^T d(t) L_t E^*[I_t] - \sum_{t=1}^T d(t) E^*[\max(L_t - HP_t, 0) I_t]. \quad (12)$$

The term $E^*(\bullet)$ is the expectation under the risk-neutral measure. This expected present value must be more than L_0 for the reverse mortgage to be provided in a sustainable manner.

We further assume that I_t and HP_t are independent under the risk-neutral predictive distribution. Then the second term on the right hand side of (12) can be written as:

$$\sum_{t=1}^T d(t) E^*[I_t] E^*[\max(L_t - HP_t, 0)], \quad (13)$$

and so all the valuation comes down to the calculation of $E^*[I_t]$ and $E^*[\max(L_t - HP_t, 0)]$.

4 Bayesian pricing of I_t

In the following, we set:

$$I_t = (1 - q_x(0)) \times (1 - q_{x+1}(1)) \times \dots \times (1 - q_{x+t-2}(t-2)) \times q_{x+t-1}(t-1),$$

where $q_x(t)$ is the probability that an individual aged x at time t will die within one year. Accordingly, we consider modeling of $q_x(t)$.

4.1 The Lee-Carter model

Following Lee and Carter (1992), the force of mortality $\mu_x(t)$ is modeled as:

$$\mu_x(t) = \exp\{\alpha_x + \beta_x \kappa_t\},$$

where α_x , β_x , and κ_t are parameters to be estimated. The time-specific parameter κ_t represents a systematic effect influencing all ages; the age-specific parameter β_x represents the

percentage change of mortality in response to an infinitesimal change in κ_t for each age x ; the other age-specific parameter α_x represents an intrinsic effect of age x . We assume that:

$$\mu_{x+s}(t+u) = \mu_x(t), \quad (14)$$

for integers x and t , and for all $0 \leq s, u < 1$. Then, under the Lee-Carter framework, the death probability $q_x(t)$ is given as:

$$q_x(t) = 1 - \exp\{-\exp\{\alpha_x + \beta_x \kappa_t\}\}.$$

4.2 Bayesian modeling of mortality

To obtain the predictive distribution of I_t , we consider a statistical modeling of mortality rates.

Let m_{xt} be the crude central death rate:

$$m_{xt} = \text{Death}_{xt} / \text{Exposure}_{xt}, \quad (15)$$

where Death_{xt} is the number of deaths recorded at age x in year t , and Exposure_{xt} is the matching exposure to the risk of death.

As a statistical model, the standard Lee-Carter methodology assumes that $y_{xt} = \ln m_{xt}$, the natural logarithm of the observed mortality rate, follows a regression model:

$$y_{xt} = \alpha_x + \beta_x \kappa_t + \varepsilon_{xt} \quad \text{for } x = x_{\min}, x_{\min} + 1, \dots, x_{\max} \text{ and for } t = t_{\min}, t_{\min} + 1, \dots, t_{\max}, \quad (16)$$

where the error terms ε_{xt} 's are distributed independently and identically as the normal distribution with zero mean and a constant variance. Since the regression equation (16) is bilinear in β_x and κ_t , this model is not identified. To make it identifiable, the parameters $\{\beta_x\}$ and $\{\kappa_t\}$ are restricted such that:

$$\sum_{x=x_{\min}}^{x_{\max}} \beta_x = 1 \text{ and } \sum_{t=t_{\min}}^{t_{\max}} \kappa_t = 0. \quad (17)$$

For each t , we define $\mathbf{y}_t$ to be a vector:

$$\mathbf{y}_t = (y_{x_{\min}t}, \dots, y_{x_{\max}t})',$$

consisting of the log mortality rates for all ages observed in year t . Here $'$ denotes the transpose operation. The statistical framework of the standard Lee-Carter methodology can be set up as a state-space model:

$$\begin{aligned} \text{observation equation:} \quad & \mathbf{y}_t = \boldsymbol{\alpha} + \boldsymbol{\beta} \kappa_t + \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \sim \mathbf{N}_M(\mathbf{0}_M, \sigma_\varepsilon^2 \mathbf{I}_M), \\ \text{state equation:} \quad & \kappa_t = \lambda + \kappa_{t-1} + \omega_t, \quad \omega_t \sim \mathbf{N}(0, \sigma_\omega^2), \end{aligned} \quad (18)$$

where $\boldsymbol{\alpha} = (\alpha_{x_{\min}}, \dots, \alpha_{x_{\max}})'$, $\boldsymbol{\beta} = (\beta_{x_{\min}}, \dots, \beta_{x_{\max}})'$, $M = x_{\max} - x_{\min} + 1$, $\mathbf{I}_M$ is the identity matrix of dimension $M \times M$, λ is the drift term of the random walk, and $\boldsymbol{\varepsilon}_t$ and ω_t are independent.

From (18), the likelihood function is given as:

$$L(y | \alpha, \beta, \kappa, \lambda, \sigma_\varepsilon^2, \sigma_\omega^2) = \prod_{t=t_{\min}}^{t_{\max}} \prod_{x=x_{\min}}^{x_{\max}} f(y_{xt} | \alpha_x, \beta_x, \kappa_t, \sigma_\varepsilon^2) \\ \propto \left(\frac{1}{\sigma_\varepsilon} \right)^{JM} \exp \left\{ - \frac{\sum_{t=t_{\min}}^{t_{\max}} \sum_{x=x_{\min}}^{x_{\max}} (y_{xt} - (\alpha_x + \beta_x \kappa_t))^2}{2\sigma_\varepsilon^2} \right\}, \quad (19)$$

where $J = t_{\max} - t_{\min} + 1$ and $\kappa = (\kappa_{t_{\min}}, \dots, \kappa_{t_{\max}})'$.

The prior distributions for the parameters α and β in the observation equation are chosen as:

$$\alpha \sim N_M(\mathbf{0}_M, \underline{\sigma}_\alpha^2 \mathbf{I}_M), \quad \beta \sim N_M((1/M)\mathbf{1}_M, \underline{\sigma}_\beta^2 \mathbf{I}_M),$$

where $\mathbf{0}_M$ and $\mathbf{1}_M$ are M -dimensional vectors of all zeros and all ones respectively. Here, the underlined characters stand for hyperparameters, which are parameters of the prior distributions. The prior for σ_ε^2 is set up as:

$$\sigma_\varepsilon^2 \sim \text{IG}(\underline{a}_\varepsilon, \underline{b}_\varepsilon),$$

where $\text{IG}(a, b)$ denotes the inverse Gamma distribution with shape parameter a and scale parameter b . The priors for λ and σ_ω^2 in the state equation are taken as:

$$\lambda \sim N(\underline{\lambda}_0, \underline{\sigma}_\lambda^2), \quad \sigma_\omega^2 \sim \text{IG}(\underline{a}_\omega, \underline{b}_\omega).$$

4.3 Risk-neutral predictive distribution

Suppose that we generate N paths in the MCMC sampling, each of which is distributed as the process $\{m_{xt}, t = 1, 2, \dots, T\}$, and denote them by:

$$\{(m_{x1}^{(j)}, m_{x2}^{(j)}, \dots, m_{xT}^{(j)}), j = 1, 2, \dots, N\}.$$

Now consider a simple life annuity in which an individual receives a constant payment, which is set to one monetary unit for simplicity, at the end of each year over T years as long as he or she is alive. The present value of this annuity for an individual aged x at time 0 is defined to be:

$$a_x = \sum_{t=1}^T d(t)(1 - q_x(0)) \dots (1 - q_{x+t-1}(t-1)). \quad (20)$$

Let a_x^{market} be the market value of the life annuity at time 0. For a probability π^* over $\{m_{xt}^{(j)}\}$ to be risk-neutral, it must satisfy:

$$\sum_{j=1}^N a_x^{(j)} \pi_j^* = a_x^{\text{market}}, \quad (21)$$

with $a_x^{(j)}$ calculated by $\{m_{xt}^{(j)}\}$. Denote by π the empirical distribution of the N paths from the MCMC sampling, which typically puts an equal mass of $1/N$ on each path. The maximum

entropy principle stipulates that the risk-neutral distribution π^* should minimize the Kullback-Leibler information divergence:

$$\sum_{j=1}^N \pi_j^* \ln(\pi_j^* / \pi_j), \quad (22)$$

subject to (21) and:

$$\sum_{j=1}^N \pi_j^* = 1 \text{ and } \pi_j^* > 0 \quad \text{for } j = 1, 2, \dots, N. \quad (23)$$

It is well known that the solution to this constrained minimization problem is given as:

$$\hat{\pi}_j^* = \frac{\pi_j \exp(\gamma a_x^{(j)})}{\sum_{j=1}^N \pi_j \exp(\gamma a_x^{(j)})} \quad \text{for } j = 1, 2, \dots, N, \quad (24)$$

where γ is a constant obtained by solving the equation:

$$a_x^{\text{market}} = \frac{\sum_{j=1}^N a_x^{(j)} \exp(\gamma a_x^{(j)})}{\sum_{j=1}^N \exp(\gamma a_x^{(j)})}.$$

4.4 Evaluation of $E^*[I_t]$ for Japanese data

The data set used are the death rates of Japan for ages 65 to 99 over the period from 1973 to 2008 for both males and females, taken from the Human Mortality Database (HMD 2013). In the MCMC simulation, we discarded the first 5,000 iterations to eliminate initial effects and used the 1,000 iterations afterwards for sampling. We calculated $E^*[I_t]$ under the setup below:

$$\begin{aligned} \text{age of cohort:} & \quad x = 65, 70, 75; \\ \text{discount factor:} & \quad d(t) = (1 + r)^{-t}, r = 0.005. \end{aligned}$$

As of 1 April 2013, the 10-year Japan government bond yield was merely 0.57% p.a. The low interest rate environment is expected to continue under Japan's recent decision to use a very loose monetary policy to lift the economy out of deflation. Hence we took 0.5% p.a. as the risk-free interest rate r in our illustration.

In Japan, valuation assumptions are generally specified by the regulator. To determine the market price, a_x^{market} , we used the death rates for annuity products in the 'life insurance standard table 2007', which was constructed by the Institute of Actuaries of Japan. Most Japanese life insurance companies price their insurance products using this standard table. We also used the specified valuation discount rate of 1.5% p.a. in computing the market price. Figure 1 depicts the computed values of $E^*[I_t]$ as well as $E[I_t]$.

Figure 1: Computed $E^*[I_t]$ (solid lines) and $E[I_t]$ (dashed lines) for Japanese males (left) and females (right)

5 Bayesian pricing of the non-recourse provision

In this section, we evaluate $E^*[\max(L_t - HP_t, 0)]$ for Japanese data. We first find a suitable time series model and deduce its predictive distribution. We then transform it into the risk-neutral version to calculate $E^*[\max(L_t - HP_t, 0)]$.

5.1 Modeling house price index

We realize there are strong autocorrelations and changing volatility in the log returns of our

house price index data (see Section 5.3). Accordingly, as in Chen et al. (2010) and Li et al. (2010), we consider the ARIMA(p, d, q)-GARCH(u, v) models. Let r_t be the log return at time t and the general model is specified as:

$$\begin{aligned}\Delta^d r_t &= \phi_0 + \sum_{i=1}^p \phi_i \Delta^d r_{t-i} + \delta_t + \sum_{j=1}^q \theta_j \delta_{t-j} , \\ \sigma_t^2 &= \gamma_0 + \sum_{i=1}^u \gamma_i \delta_{t-i}^2 + \sum_{j=1}^v \eta_j \sigma_{t-j}^2 ,\end{aligned}\tag{25}$$

where $\delta_t \sim N(0, \sigma_t^2)$. Following Rosenberg and Young (1999), we assume that the prior distribution for the unknown parameters is multivariate normal. (Note that p, d , and q here are orders of the ARIMA model, and are not related to the mortality data in Section 4 or the discount factor.)

5.2 Risk-neutral predictive distribution

Once we have the predictive distribution of r_t , we can change it to the risk-neutral form exactly the same way as in Section 4.3. Suppose that we generate N paths in the MCMC sampling, each of which is distributed as the process $\{r_t, t = 1, 2, \dots, T\}$. Denote them by:

$$\{(r_1^{(j)}, r_2^{(j)}, \dots, r_T^{(j)}), j = 1, 2, \dots, N\} ,$$

and set:

$$HP_t^{(j)} = HP_{t-1}^{(j)} \exp(r_t^{(j)}) \quad \text{for } t = 1, 2, \dots, T.$$

For a probability π^* over $\{HP_t^{(j)}\}$ to be risk-neutral, it must satisfy:

$$d(t) \sum_{j=1}^N HP_t^{(j)} \pi_j^* = HP_0 .\tag{26}$$

According to the minimum cross-entropy principle discussed earlier, the risk-neutral distribution π^* should minimize (22), subject to (23) and (26). The solution is given as:

$$\hat{\pi}_j^* = \frac{\pi_j \exp(\gamma_t d(t) HP_t^{(j)})}{\sum_{j=1}^N \pi_j \exp(\gamma_t d(t) HP_t^{(j)})} \quad \text{for } j = 1, 2, \dots, N,$$

where γ_t is a constant obtained by solving the equation:

$$HP_0 = d(t) \frac{\sum_{j=1}^N HP_t^{(j)} \exp(\gamma_t d(t) HP_t^{(j)})}{\sum_{j=1}^N \exp(\gamma_t d(t) HP_t^{(j)})} .$$

5.3 House prices in Japan

We used the Japanese version of the S&P / Case-Shiller Home Price Indices as a representative for the value of the mortgaged property. The data consists of monthly prices from January 1995 to December 2008. Figure 2 shows the values of the S&P / Case-Shiller Home Price Indices and the log returns for Japan as well as the US.

Figure 2: S&P / Case-Shiller Home Price Indices (left) and log returns (right) for Japan (solid lines) and US (dashed lines)

Figure 3 shows that there are significant autocorrelations in the log returns, which suggest that ARIMA models are potential candidates for modeling the house price returns. The Portmanteau statistic of the log returns gives a p -value of 0.0000. As noted in Rosenberg and Young (1999), AR models are more convenient to work with in a Bayesian setting than MA models, and if a sufficiently large number of AR terms are included, the effects are similar to a given ARIMA model. As such, after inspecting the sample partial autocorrelation functions, we fitted an ARIMA(6, 0, 0) model (i.e. AR(6) model) to the log returns. The left panel of Figure 4 reveals that the fitted model captures the autocorrelations reasonably well. The sample autocorrelation functions of the residuals are largely within the 95% confidence intervals (dotted lines), and the Portmanteau statistic gives a p -value of 0.7778.

The right panel of Figure 4, however, indicates that there seems to be some extent of conditional heteroskedasticity in the data. The Portmanteau statistic of the squared residuals shows a p -value of 0.0787, which is just a little higher than the 5% significance level. This feature may point to the use of a GARCH-type model. After examining the sample partial autocorrelation functions of the squared residuals and testing a number of models, we selected an ARIMA(6, 0, 0)-GARCH(2, 0) model for the log returns. This model is relatively parsimonious, and as reflected in Figure 5, it provides a reasonable allowance for both the

autocorrelations and the heteroskedasticity property generally. The Portmanteau statistic of the squared standardized innovations shows a p -value of 0.5709, which reassures that the model fit is satisfactory.

Figure 3: Sample autocorrelation functions of log returns

Figure 4: Sample autocorrelation functions of residuals (left) and squared residuals (right) of fitted ARIMA(6, 0, 0) model

Figure 5: Sample autocorrelation functions of standardized innovations (left) and squared standardized innovations (right) of fitted ARIMA(6, 0, 0)-GARCH(2, 0) model

We ran 10,000 steps of MCMC simulation. We discarded the first 5,000 steps to remove the effects of the starting values and used the next 5,000 steps for sampling. Figures 6 and 7 display the autocorrelations of the successive simulated samples and the sampling history respectively for the model parameters. (In the figures, p[1] to p[9] stand for ϕ_0 to ϕ_6 , γ_0 , and γ_2 respectively.) It can clearly be seen that the autocorrelations are minimal and the level of convergence is satisfactory (i.e. there are no particular trends in the sampling history), which means that the simulation process has been carried out appropriately.

Figure 6: Autocorrelation plots of MCMC samples

5.4 Evaluation of $E^*[\max(L_t - HP_t, 0)]$

We convert the predictive distributions of future house prices into the risk-neutral form as explained in Section 5.2 and then evaluate $E^*[\max(L_t - HP_t, 0)]$ in (13). We set up the inputs as follows:

$$\begin{aligned} HP_0 &= 40 \text{ (million yen),} \\ L_0 &= 20, 24, 28, 32, 36, 40 \text{ (million yen),} \\ L_t &= L_0 (1 + r_L)^t, \\ d(t) &= (1 + r)^{-t}, \quad r = 0.005. \end{aligned}$$

The results for $r_L = 0.02, 0.03, 0.04, 0.05, 0.06$ (with $L_0 = 20$) are shown in Figure 8.

Figure 7A: History plots of MCMC samples

Figure 7B: History plots of MCMC samples

Figure 8: Computed $E^*[\max(L_t - HP_t, 0)]$ (solid lines) and $E[\max(L_t - HP_t, 0)]$ (dashed lines)

6 Pricing reverse mortgages in Japan

We finally proceed to pricing reverse mortgages in Japan by combining the results in Sections 4 and 5. We add $\tau = 100$ to the inputs set up in Section 3. Table 1 gives the values of (12) for $L_0 = 20, 24, 28, 32, 36, 40$, $r_L = 0.02, 0.03, 0.04, 0.05, 0.06$, and $x = 65, 70, 75$. We notice the following:

- (a) The expected present value of the reverse mortgage for the male cohort is larger than that for the corresponding female cohort.

- (b) For both sexes, the expected present value of the reverse mortgage tends to decrease with age.
- (c) For both sexes, the expected present value of the reverse mortgage increases with the loan interest rate.
- (d) A particular attention should be paid on whether each expected present value is more than L_0 , as it indicates that the reverse mortgage is provided in a sustainable manner. Those cases with the value lower than L_0 (i.e. not sustainable) are highlighted in Table 1. It can be seen that a higher loan interest rate allows a higher loan to value (LTV) ratio. As of November 2004, UK loan interest rates were around 7% p.a. (Equity Release Report 2005), 10-year bond yield was about 4.5% p.a., and so the premium embedded was 2.5% p.a. Applying this figure to our case leads to a loan interest rate of 3% p.a., which implies a maximum LTV ratio of about 80% for males and 60% for females, based on Table 1. Moreover, as the Japan market is relatively new and is not as saturated as the UK market, the Japanese insurers can potentially charge higher interest rates initially, which then allows a higher LTV ratio, even up to 80%-90% as in Table 1. These observations appear to support a successful introduction of reverse mortgages into Japan.

7 Concluding remarks

In this paper, we have proposed a Bayesian multivariate framework to price reverse mortgages and applied the proposed method to Japanese data in order to examine the feasibility for an introduction of reverse mortgages into Japan. We have obtained desirable empirical results, indicating some promising future for reverse mortgages in Japan.

Nevertheless, due to the scope of the paper, there are certain areas that are not covered, including stochastic modeling of interest rates, joint lives, having more than one decrement, model error, and other product designs. Further research is required to address these issues.

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Table 1: The expected present values of the reverse mortgages $L_0 = 20$ (LTV = 50%)

$x \setminus r_L$	Males					Females				
	0.02	0.03	0.04	0.05	0.06	0.02	0.03	0.04	0.05	0.06
65	22.7	25.5	28.1	30.5	32.8	21.1	23.9	26.5	28.9	31.2
70	22.2	24.8	27.2	29.4	31.3	20.1	22.6	24.9	27.0	29.0
75	21.4	23.7	25.8	27.8	29.5	18.9	21.1	23.2	25.1	26.7

 $L_0 = 24$ (LTV = 60%)

$x \setminus r_L$	Males					Females				
	0.02	0.03	0.04	0.05	0.06	0.02	0.03	0.04	0.05	0.06
65	25.8	28.4	30.9	33.1	35.1	23.6	26.2	28.7	31.0	33.2
70	25.5	28.0	30.2	32.1	33.9	22.8	25.1	27.3	29.2	30.9
75	24.9	27.1	29.0	30.8	32.3	21.8	23.9	25.8	27.4	28.8

 $L_0 = 28$ (LTV = 70%)

$x \setminus r_L$	Males					Females				
	0.02	0.03	0.04	0.05	0.06	0.02	0.03	0.04	0.05	0.06
65	28.3	30.8	33.0	35.1	36.7	25.6	28.1	30.4	32.6	34.5
70	28.3	30.5	32.5	34.2	35.6	25.0	27.2	29.1	30.9	32.2
75	27.9	29.9	31.6	33.1	34.2	24.2	26.1	27.7	29.1	30.2

 $L_0 = 32$ (LTV = 80%)

$x \setminus r_L$	Males					Females				
	0.02	0.03	0.04	0.05	0.06	0.02	0.03	0.04	0.05	0.06
65	30.4	32.6	34.7	36.5	37.7	27.3	29.6	31.8	33.9	35.3
70	30.5	32.5	34.3	35.7	36.6	26.7	28.7	30.5	32.0	33.0
75	30.3	32.1	33.6	34.7	35.4	26.1	27.8	29.2	30.3	31.0

 $L_0 = 36$ (LTV = 90%)

$x \setminus r_L$	Males					Females				
	0.02	0.03	0.04	0.05	0.06	0.02	0.03	0.04	0.05	0.06
65	32.0	34.1	36.0	37.4	38.1	28.6	30.8	33.0	34.7	35.5
70	32.3	34.1	35.6	36.7	37.2	28.1	30.0	31.6	32.8	33.3
75	32.3	33.7	34.9	35.7	36.1	27.6	29.1	30.3	31.1	31.4

 $L_0 = 40$ (LTV = 100%)

$x \setminus r_L$	Males					Females				
	0.02	0.03	0.04	0.05	0.06	0.02	0.03	0.04	0.05	0.06
65	33.3	35.2	36.9	38.0	38.2	29.7	31.8	33.8	35.3	35.5
70	33.6	35.2	36.4	37.2	37.4	29.2	30.9	32.3	33.2	33.4
75	33.6	34.8	35.7	36.2	36.4	28.7	30.0	30.9	31.4	31.5

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