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**THE INFLUENCE OF PRESTRESSING FORCE ON
NATURAL FREQUENCIES OF BEAMS**

**GAN BING-ZHENG
SCHOOL OF CIVIL AND ENVIRONMENTAL ENGINEERING**

2019

THE INFLUENCE OF PRESTRESSING FORCE ON NATURAL FREQUENCIES OF BEAMS

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A thesis submitted to the Nanyang Technological University
in partial fulfilment of the requirement for the degree of
Doctor of Philosophy

2019

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research, is free of plagiarised materials, and has not been submitted for a higher degree to any other University or Institution.

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This thesis contains material from 1 paper published in the following peer-reviewed journal in which I am listed as an author.

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The contributions of the co-authors are as follows:

- Prof Chiew and Prof Lu provided the initial project direction and edited the manuscript drafts.
- I prepared the manuscript drafts. The manuscript was revised by Prof Lu and Prof Fung.
- I conducted the theoretical investigation. Prof Fung checked the derivation.

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TABLE OF CONTENTS

<i>ACKNOWLEDGEMENT</i>	<i>i</i>
<i>TABLE OF CONTENTS</i>	<i>ii</i>
<i>SUMMARY</i>	<i>v</i>
<i>LIST OF TABLES</i>	<i>vii</i>
<i>LIST OF FIGURES</i>	<i>ix</i>
<i>NOTATION</i>	<i>xii</i>
<i>CHAPTER 1</i>	<i>1</i>
1.1 RESEARCH BACKGROUND	1
1.2 OBJECTIVES AND SCOPE	5
1.3 CONTRIBUTIONS AND ORIGINALITY	6
1.4 ORGANIZATION	7
<i>CHAPTER 2</i>	<i>9</i>
2.1 THEORETICAL ANALYSIS ON THE EFFECT OF PRESTRESSING FORCE ON NATURAL FREQUENCIES	9
2.1.1 Timoshenko theory	14
2.1.2 Hamed and Frostig's theory	16
2.2 EXPERIMENTAL INVESTIGATION ON THE EFFECT OF PRESTRESSING FORCE ON NATURAL FREQUENCIES	17

CHAPTER 3	23
3.1 CLASSIFICATION OF THE DIFFERENT TYPE OF PRESTRESSED BEAMS	23
3.2 MATHEMATICAL DEDUCTION INDICATING DIFFERENCES BETWEEN TIMOSHENKO AND HAMED & FROSTIG SCENARIOS	28
 CHAPTER 4	 36
4.1 THEORETICAL INVESTIGATION	36
4.1.1 Timoshenko scenario	37
4.1.2 Hamed and Frostig scenario	40
4.1.3 Beam with external centered straight tendon partly attached to the beam	43
4.1.4 Beam with external trapezoidal tendon	47
4.2 NUMERICAL SIMULATION COMPARED WITH THEORETICAL SOLUTION	58
4.2.1 Modeling of prestress effect and model validation	58
4.2.2 Beam with external straight tendon partly attached to the beam	61
4.2.3 Beam with external trapezoidal tendon	66
4.3 ANALYSIS STUDIES OF RELATED REFERENCES	70
4.3.1 Introduction of related research study	70
4.3.2 Comparison of proposed theoretical solution and related reference	71
4.4 APPLICATION OF THEORETICAL MODEL OF PRESTRESSED BEAM WITH EXTERNAL TRAPEZOIDAL TENDON ON PRESTRESSED TRUSS	74
4.4.1 Beam analogy	74
4.4.2 Comparison of truss simulation, beam simulation and theoretical model	76

CHAPTER 5	79
5.1 EFFECT OF FRACTURES ON NATURAL FREQUENCIES AND ITS CLOSURE DUE TO PRESTRESSING FORCE	79
5.2 INITIAL CONCRETE MATERIAL STATE AND SHRINKAGE CRACKS	81
5.3. NUMERICAL SIMULATION ON EFFECT OF SHRINKAGE CRACKS IN PRESTRESSED RC BEAMS	84
5.3.1 General model considerations	84
5.3.2 Modeling cracks of concrete in FE model	84
5.3.3 Modeling of shrinkage cracks and model validation	86
5.4. SIMULATION OF TWO SETS OF PREVIOUS EXPERIMENTS AND COMPARISONS	89
5.4.1 Noble et al.'s experiment	90
5.4.2 Jang et al.'s experiment	94
5.4.3 Prestressing application on concrete beams	98
5.4.4 Summary of experiments of Noble et al. and Jang et al.	100
CHAPTER 6	101
6.1 INTRODUCTION	101
6.2 CONCLUSIONS	104
6.2.1 The classification standard of prestressed beams of different types	104
6.2.2 The influence of prestressing force on the fundamental natural frequency of typical beams of Hybrid Type	104
6.2.3 Contradiction between the theoretical solution and experimental investigations	105
6.3 RECOMMENDATIONS FOR FUTURE RESEARCH WORK	106
REFERENCES	108

SUMMARY

This thesis investigates the effect of prestressing force on the natural frequencies of beams. Two representative theories were proposed long time ago. One of them suggested that the natural frequencies would decrease with prestress due to the “compression (P- Δ) effect” induced by the prestressing force. The other indicated that the natural frequencies would not be affected by the prestressing force since the prestressing force did not change the overall stiffness property of the system. Although there are discrepancies in the conclusion, it is widely accepted now that both of their conclusions are correct but they should be used in different situations. However, there are some misunderstandings on their scopes of applications. The scenarios explained by these two theories are just two extreme examples and there are many prestressed beams with external tendons between them whose natural frequencies cannot be predicted by either of them. These prestressed beams are not considered in the discussions mentioned above. Hence a comprehensive conclusion about this problem including all the scenarios is needed. In addition, the effect of a particular form of the external prestressing tendon (trapezoidal profile tendon) on the fundamental natural frequency is investigated by the energy method in this thesis. As a widely used tendon profile and a typical example of the scenarios (Manisekar et al. 2018, Kianmofrad et al. 2017, El-Zohairy and Salim, 2017, Cui et al., 2017) which cannot be fitted into the two theories as mentioned, no quantitative estimation of the effect of prestressing force can be found by the existing theories. The mathematical solution is provided and this new method can be extended to calculate the natural frequencies of other prestressed beams with external tendons. After synthesizing the theories from a purely theoretical point of view involving no ambiguity, some inconsistent experiments results got from tests conducted on prestressed concrete beams need to be considered. These tests showed an increasing trend of the natural

frequencies with the prestressing force which cannot be explained by either of the theories. Some researchers believed that the theoretical solution could not be applied to real beams given the diverging experiment evidence. One of the research targets in this thesis is to provide a systematic explanation of the reasons causing the contradictions. Firstly, the existing experiments on prestressed concrete beams are reviewed, and experimental data are discussed in light of the effect of shrinkage cracks. Then, numerical simulations using finite element models are carried out to simulate the influence of the prestressing force on natural frequencies with the existence of these shrinkage cracks. The results demonstrate that such shrinkage-type cracks inside the concrete indeed tend to close when the prestress is applied, and this, in turn, increases the bending stiffness and consequently results in an increase of the natural frequencies of the beams. Finally, a coherent conclusion on the effect of prestressing force on natural frequencies of beams is proposed.

LIST OF TABLES

	Page
Table 4-1. Comparison of numerical simulation and theoretical solution	60
Table 4-2. Difference induced by the mesh of numerical simulation (compared to the model with the mesh size of 16.7mm)	61
Table 4-3. Difference induced by the mesh of finite element model of the prestressed beam of Hybrid Type (external straight tendon partly attached to the beam)	62
Table 4-4. Comparison of numerical simulation and theoretical solution on the effect of prestressing force on the fundamental natural frequency of a beam of Hybrid Type (external straight tendon partly attached to the beam)	63
Table 4-5. The fundamental natural frequency of prestressed beams of different types	65
Table 4-6. Difference induced by the mesh of finite element model of the prestressed beam of Hybrid Type (external trapezoidal tendon partly attached to the beam)	67
Table 4-7. Comparison of numerical simulation and theoretical solution on the effect of prestressing force on the fundamental natural frequency of a beam of Hybrid Type (external trapezoidal tendon partly attached to the beam)	67
Table 4-8. Comparison of test results and theoretical solutions of Miyamoto and this paper	72
Table 4-9. Change of fundamental natural frequencies of different models under prestressing force	77

Table 5-1. Comparison of Noble's test and simulation	93
Table 5-2. Comparison of Jang's test and simulation	97

LIST OF FIGURES

	Page
Figure 1-1. High-rise buildings used the prestressing mechanism in early days	2
Figure 1-2. Content of thesis	6
Figure 2-1. Vibrating beam under axial tension (Timoshenko et al., 1974)	14
Figure 2-2. Substructure layout and internal stress resultants of a typical prestressed beam: (a) prestressed beam; (b) concrete beam component; and (c) tensioned cable component (Hamed and Frostig, 2006)	16
Figure 3-1. Scenarios with different interaction of the prestressing tendon with the primary structure	24
Figure 3-2. The simple understanding of the classification of (a) Timoshenko scenario and (b) Hamed & Frostig scenario	27
Figure 3-3. Basic parameters of the beam	29
Figure 3-4. Free body diagram of a slice of the beam in Timoshenko scenario (Type-A)	30
Figure 3-5. Free body diagram of a slice of the beam in Hamed and Frostig scenario (Type-B)	32
Figure 3-6. The relative position of tendon and beam	34
Figure 4-1. Timoshenko scenario for energy method	37

Figure 4-2. Relationship between axial tension and transverse loading intensity on tendon	40
Figure 4-3. Beam with external centered straight tendon partly attached to the beam	44
Figure 4-4. Beam with external trapezoidal tendon	47
Figure 4-5. Deformation of beam	49
Figure 4-6. Configuration of prestressed beam	59
Figure 4-7. Finite element model of prestressed beam	59
Figure 4-8. Comparison of numerical simulation and theoretical solution of the effect of prestressing force on the fundamental natural frequency of a beam of Hybrid Type (external straight tendon partly attached to the beam)	64
Figure 4-9. Comparison of the reduction of the fundamental natural frequency of prestressed beams of different types	65
Figure 4-10. Comparison of numerical simulation and theoretical solution of the effect of prestressing force on the fundamental natural frequency of a beam of Hybrid Type (external trapezoidal tendon partly attached to the beam)	68
Figure 4-11. Dimensions and arrangement of external tendons of test specimen (Miyamoto et al., 2000)	71
Figure 4-12. Prestressed truss with external trapezoidal tendon	74
Figure 4-13. Change of fundamental natural frequencies of different models under prestressing force	78
Figure 5-1. Simplified model of concrete beam, reinforcing bars and the connection between them (Chen et al., 2004)	82

Figure 5-2. Fracture zone and Fictitious Crack Model (Petersson, 1981)	85
Figure 5-3. Test of Shardakov et al. (2016)	88
Figure 5-4. The pattern of crack in numerical simulation	89
Figure 5-5. Tensile stress accumulated in concrete	91
Figure 5-6. Cracks due to shrinkage in Noble's specimen	92
Figure 5-7. Sketch of a test of Jang et al. (2009)	95
Figure 5-8. Tensile stress accumulated in concrete	96
Figure 5-9. Comparison of Jang's tests and simulations	97

NOTATION

Δl_h	Length change of horizontal part of prestressing tendon
Δl_d	Length change of diagonal part of prestressing tendon
λ_1	Vertical components of interaction between prestressing tendon and concrete beam
λ_2	Longitudinal components of interaction between prestressing tendon and concrete beam
π	Archimedes' constant
ρ	Density
φ	Shape function of the bending beam
ω	Fundamental natural frequency
ω_i	The i th natural frequency
A	The area of the beam section
A_t	The area of beam section of tendon
a	Constant in shape function of the beam during vibration
b	Change of distance between two supports of the beam
b_f	Distance between left end and deviator when beam is bending
C_1	Constant in shape function of the beam during vibration

C_2	Constant in shape function of the beam during vibration
d	Vertical displacement at midspan in the 3-point bending test
dx	Thickness of the slice taken at an arbitrary point
E	Young's modulus
E_t	Young's modulus of tendon
F	Vertical load at midspan in the 3-point bending test
f	Distance between left end of beam and deviator F
f_c	Compressive strength of concrete
f_t	Tensile strength of concrete
G_F	Fracture energy
g	Distance between left end of the beam and deviator G
h	Height of beam
I	Moment of inertia
K	Kinetic energy of bending beam
L	Original distance between two supports of the beam
l	Length of beam
l_d	Length of diagonal part of tendon
l_h	Length of horizontal part of tendon
M	Bending moment on an arbitrary point of the beam

M_{xx}	Bending moment stress resultants in the concrete beam in Hamed and Frostig calculation
m	Mass per unit length of the beam
N	Normal force on an arbitrary point of the beam
N_{xx}	Stress resultants in the longitudinal direction in the concrete beam in Hamed and Frostig calculation
P	Axial force
P_h	Horizontal force transferred from tendon on the beam
P_{h0}	Initial horizontal force transferred from tendon on the beam
P_t	Axial force needed to maintain a bending beam bending as curve $y_t(x)$
P_v	Vertical force transferred from tendon on the beam
P_{v0}	Initial vertical force transferred from tendon on the beam
q	Transverse loading intensity on beam
q_t	Transverse loading intensity needed to maintain a beam bending as curve $y_t(x)$
S	Total force on an arbitrary point of tendon
s_f	Length of beam between left end and deviator
S_x	Horizontal force on an arbitrary point of tendon
S_y	Vertical force on an arbitrary point of tendon
s	Length of beam

t	Time
U	Strain energy of bending beam
U_b	Strain energy of beam apart from tendon
U_t	Strain energy of tendon
V	Shear force on an arbitrary point of the beam
W	Potential energy induced by axial force
W_1	Potential energy induced by axial force
W_2	Potential energy induced by transverse interaction between beam and tendon
w	Shape function of beam varying with time
w_0	Constant in shape function of the beam during vibration
w_c	Critical displacement of the fracture
X	Shape function of the beam during vibration
X_i	The i th shape function of the beam during vibration
x	Distance between the left end of the beam and the arbitrary point
x_f	Horizontal displacement of deviator F
x_g	Horizontal displacement of deviator G
x_L	Horizontal displacement of the right end of the beam
y	Transverse displacement of the arbitrary point
y_f	Vertical displacement of deviator F

z_{cab} Tendon eccentricity

CHAPTER 1

INTRODUCTION

1.1 RESEARCH BACKGROUND

Prestress is widely used in civil engineering nowadays because of its effect on reducing the deflection of long-span structure and preventing cracks in concrete. The influence of prestress on the ultimate load carrying capacity was studied by many scholars (Mattock et al., 1971, Ramos and Aparcio, 1996, Ariyawardena and Ghali, 2002). The concept of prestressed concrete appeared more than 100 years ago. It is not available at first due to prestress loss. With the development of material science and design idea, high strength steel is used to solve the problem of prestress loss and the prestressing technique began to be used in concrete members (Gasparini et al., 2003, 2006). Numerous research studies on prestressed concrete have been done about the 1970's. The prestressing tendons are seldom used in steel structures except for some projects to strengthen existing steel bridges before 2000 (Belena, 1977, Triostky, 1990, Chon, 2001, Ronghe and Gupta, 2002). Chon (2001) also mentioned that some leaning high-rise buildings including Puerta de Europa in Madrid and the Leaning Tower of Pizza in Ann Arbor used the prestressing mechanisms due to the high difficulty of construction.

As buildings with a peculiar shape, extremely large span or height are more common these years, the application of prestressed steel structure is more common for its light weight and the ability of deflection control. These structures could be slenderer since excess deflection is controlled by prestressing tendons. However, the dynamic response of these slender structures becomes quite important criteria. In some

scenarios, the natural frequencies of the structure will change with prestress. If it falls to the range of frequencies of periodical load exerted on the structure, resonance will occur and the dynamic response could be huge. Hence the natural frequencies, as the fundamental parameter reflecting the dynamic properties of a structure, should be studied for these prestressed structures.

(a) Puerta de Europa in Madrid (b) Leaning Tower of Pizza in Ann Arbor

Figure 1-1. High-rise buildings used the prestressing mechanism in early days

The fundamental mechanics involved in typical prestressed conditions are well understood and their effect on the natural frequencies of the components can be established without ambiguities. Two extreme scenarios can be well explained by the theoretical models proposed by Timoshenko et al. (1974) and Hamed & Frostig (2006), respectively.

Timoshenko et al. (1974) studied a simply supported beam under an externally applied axial compression force and proved that the natural frequencies of the beam would decrease with the increase of the compression force. In the context of prestressed beams, this theory would strictly apply only in the prestressing condition where the tendon is anchored at the beam ends, and apart from this, the tendon is completely separated from the main beam. The prestressing tendon is equivalent to external compression force at the beam ends when the beam vibrates in this condition. For easy reference, we shall call this scenario as Type-A prestressed condition.

Hamed and Frostig (2006) considered a prestressed beam with prestressing tendons in curved profiles. It is concluded that the natural frequencies will not be affected by the prestressing force in such cases. Also, for convenience, we shall call this scenario as the Type-B prestressed condition. In their derivation, vibration in transverse direction is considered. The transverse deflection of prestressing tendon and adjacent main beam was identical at every point no matter the prestressed beam is bonded or unbonded, e.g., the duct and prestressing tendon fit well and the tendon moves with the duct during vibration even they are not grouted. This contact condition will be referred as “perfectly attached” or “fully attached” in the foregoing discussion in this thesis. If the prestressing tendon attached to the main beam at some points (except two ends) instead of every point, it is referred as “partly attached” which will be elaborated in Section 3.1. For the scenario of Type-A introduced in previous paragraph, it is referred as “perfectly detached” or “fully detached”. A prestressed beam where the tendon is bonded to the main beam clearly belongs to “perfectly attached”, although bonding (grouting) is not a necessary condition. The contact condition of “perfectly attached”, “partly attached”, “fully detached” are noted as “real interaction” between tendon and main beam in this thesis which is different from the definition of widely used technical terms “bonded”, “unbonded” and “grouted”. The application of these new terms will be elaborated in Chapter 3.

Jaiswal (2008) observed that Timoshenko’s theory applied well to the prestressed beams with unbonded tendons while Hamed and Frostig’s theory applied well to prestressed beams with bonded tendons. Wang et al. (2013) made similar observations. However, as mentioned above the fundamental difference between the two ideal conditions is not bonded or unbonded tendons. Timoshenko’s theory is derived on a regular beam under external axial force (no prestressing force involved). Hence it could only be applied to the prestressed beams under similar condition. In other words, the prestressing tendon must be straight, unbonded, connected to the main beam only at both ends and does not move with main beam in transverse direction during vibration. For unbonded prestressed beam that every point of prestressing tendon moves with main beam in transverse direction during vibration, Hamed and Frostig’s theory should be used instead of Timoshenko’s theory. Naturally, prestressing tendon moves with main beam for bonded prestressed beams,

hence Hamed and Frostig's theory can be used for bonded prestressed beam too. For example, a prestressed beam with a tendon in curved profiles should be fitted into Hamed and Frostig's theory no matter it is bonded or unbonded because the prestressing tendon will deflect with the primary structure even it is unbonded.

The misunderstanding described above could occur because there is no comprehensive research that compares the effect of prestressing force on natural frequencies of different types of structures (Type-A and Type-B) which will be done in this thesis.

Besides, the theories of Timoshenko and Hamed & Frostig just represent two extreme scenarios (fully separated and perfectly attached); there is no satisfactory quantitative solution for the situations between them and most of the external prestressing tendons in steel structure should be fitted into this Hybrid Type, e.g., external tendons with the trapezoidal profile which are attached to the primary structure only at the ends of the beam and two deviators. As a typical form of prestressing tendon, the influence of the prestressing force in it on the natural frequencies should be studied. Miyamoto et al. (1995, 2000) have done a series of research on the prestressed beams with external tendons. However, their theoretical solution and tests results did not fit very well and the reason of the divergence was not figured out clearly. This part of work can be used to guide the design of many external prestressed structures which is necessary.

Although there is a misunderstanding on the application scope of two basic theories, the theories themselves are well understood. The natural frequencies of structures will decrease, more or less, with the prestressing force regarding the connection between the main structure and tendon. However, many tests conducted on concrete beams (Saiidi et al., 1994, Jang et al., 2010) showed an increasing trend which cannot be explained by either of the theories. This contradiction should be explained otherwise the applicability of theories is doubtful because of these diverging evidences shown in tests.

1.2 OBJECTIVES AND SCOPE

Based on the discussion above, this study focuses on the influence of prestressing force on natural frequencies of a beam. The scopes of applications of two representative theories are clarified so they could be used correctly in different situations. Essentially, they are distinguished by the real contact (definition of “real contact” can be found in Section 1.1) between tendon and main beam instead of prestressing technique (bonded or unbonded). Hybrid types of prestressed structures that is in between two extreme scenarios (Type-A and Type-B), such as prestressed beams with external trapezoidal tendons, are studied by the mathematical method. Both the “P- Δ ” effect of axial compression and the transverse interaction at two deviators are taken into consideration in this calculation. The contradiction between theories and tests are explained by considering the effect of crack closure since most tests are conducted on concrete and shrinkage cracks are not considered.

The main objectives of this study are as follows:

a) To clarify the scopes of applications of theories of two extreme scenarios so they could be used in corresponding types of prestressed structures correctly, both theories are reviewed and analysed. Their essential difference is explained by theoretical study. It should be noted that cracks are not considered in either of them and the conclusion is applicable to steel and concrete beams without cracks.

b) To study the effect of prestressing force on natural frequencies of prestressed structure with external trapezoidal tendon which is a typical example that between two extreme scenarios, energy method is taken to provide a mathematical solution of it. This energy method can be extended to other tendon profiles.

c) To prove the contradiction between theories and some experiments are caused by the effect of shrinkage cracks in concrete beams which are not considered by the experiment designer, some numerical models are established to simulate some tests in published papers with or without the influence of shrinkage cracks. Hence, the theories are proved to be applicable in real projects as long as the effect of concrete material for the prestressed concrete member is considered.

Figure 1-2. Content of thesis

1.3 CONTRIBUTIONS AND ORIGINALITY

The main contribution of the present study is to provide an in-depth understanding of the influence of prestressing force on natural frequencies in a different situation depending on how the prestressing tendon is connected to the primary structure.

The main originality of the work includes the following:

a) Theoretical analysis of both theories (Timoshenko theory and Hamed & Frostig theory) in a comparable situation. The difference between two theories is not large when the prestressing force is low. However, the mechanism of two scenarios are totally different and the error could be large with the increase of prestressing force. The ambiguity on the scope of application of two existing theories should be clarified so they could be applied correctly. It is necessary to inform engineers that the natural frequencies of some structures could decrease with the increase of prestressing force although it is negligible in normal level of prestressing force now. They would check the natural frequencies when prestressing force is high. It could be dangerous if they

simply believe that the natural frequencies would not be affected by prestressing force in any circumstances which seems to be correct in projects now.

Besides, these two existing theories just represent two extreme scenarios. Many scenarios in between which are seldom considered are included in the conclusion of this thesis.

b) The theoretical investigation and numerical study of the prestressed beam with the external trapezoidal tendon. As a typical scenario in between two theories as mentioned above, it cannot be calculated by either of the theories. A theoretical model is developed for it which provides the quantitative estimation of the influence of prestressing force on the fundamental natural frequency. This energy method can be used to build models for other scenarios.

c) The numerical simulation of prestressed concrete members with shrinkage cracks. The contradiction between some experiments and theories can be explained by the closure of cracks due to prestressing force and this process is proven by these numerical simulations. A comprehensive coherent conclusion on the effect prestressing force on natural frequencies is provided.

1.4 ORGANIZATION

The contents of this study are arranged as follows:

Chapter 2 is the literature review. The representative theories and experiments will be introduced.

Chapter 3 covers the theoretical analysis comparing the influence of prestressing force on natural frequencies in two extreme scenarios. More discussion will be conducted based on the comparison, so the misunderstanding on the scopes of applications of both theories will be clarified. A comprehensive conclusion of this problem from a pure prestressing loading point of view is proposed.

Chapter 4 demonstrates the theoretical investigation and numerical simulation on a prestressed beam (without cracks) with external trapezoidal tendon studying the

effect of prestressing force on the fundamental natural frequencies. The theoretical solution and finite element model results will be compared. The energy method of calculating natural frequencies of prestressed beams is introduced in detail.

Chapter 5 presents the numerical simulation of some tests in published papers that contradicts theories. The effect of shrinkage cracks will be considered in the model. Some literature on modeling of cracks will be reviewed and some validation models will be developed to support the modeling technique. A coherent conclusion considering the cracks in concrete is proposed.

Chapter 6 summarizes the results and observations in this thesis. Recommendations for future work are also given for future works.

CHAPTER 2

LITERATURE REVIEW

Prestress is widely used as an effective technique on controlling the deflection nowadays (Iori, 2003, Marrey and Grote, 2003). With the aid of prestressing tendons on the deflection control, structures or members can be slenderer. However, the dynamic response may not necessarily be improved by prestressing tendons. What is more, the natural frequencies could be reduced due to prestress in some scenarios and fall in the range of frequencies of the periodical load exerted on the structure, and this could cause resonance. Hence, excessive dynamic response can be a potential risk in these slender structures, and it is therefore significant to estimate the effect of prestressing force on natural frequencies (Shi et al., 2014, Noble et al., 2015, Shin et al., 2016, Li, 2016). As mentioned in Chapter 1, there are two representative theories showing that the natural frequencies of beams will decrease or remain stable in different scenarios when the prestressing force increases. On the other hand, the results of many experiments indicate that the natural frequencies tend to increase with prestressing force. These research studies will be investigated in Chapter 2.

2.1 THEORETICAL ANALYSIS ON THE EFFECT OF PRESTRESSING FORCE ON NATURAL FREQUENCIES

The effect of external axial force on the natural frequencies is studied a long time ago and can be found in the books of Timoshenko et al. (1974) and Tse et al. (1978). The “compression (P- Δ) effect” described in their books is the fundamental cause of the decrease of natural frequencies of prestressed beams. “Compression (P- Δ) effect” in this thesis is solely geometric non-linearity which is used to describe the influence of

axial force on bending beam. The axis of application of the axial force remains unchanged while the beam deflects due to bending. Hence an additional moment on the section in span will be induced by this axial force which reduces the equivalent flexural stiffness and natural frequencies of the beam. As a typical point of view on this problem, this theory will be introduced in Section 2.1.1 in details.

Raju and Rao (1986) investigated the effect of prestress along with the influence of shear and rotatory inertia on the natural frequencies of a simply supported prestressed beam. The prestressing force is regarded as external axial compressive load directly in their paper consequently the “compression ($P-\Delta$) effect” took effect and reduced the natural frequencies. Besides, Alkhairi and Naaman (1994), Abraham et al. (1995), Law and Lu (2005) got similar conclusions.

Diverging evidence is found in the tests conducted by Saiidi et al. in 1994. These tests once caused heated discussion on this topic (Dall’Asta and Dezi, 1996b, Deak, 1996, Jain and Goel, 1996). Dall’Asta and Dezi (1996b) proposed a linear model and concluded that the effect of prestressing force on natural frequencies was negligible in most circumstances. However, the “compression ($P-\Delta$) effect” is caused by the nonlinearity of the prestressed beam as can be seen in the deduction in Timoshenko’s book. Hence this effect is not considered by Dall’Asta and Dezi because it is beyond the capabilities of this linear model (Hamed and Frostig, 2006). However, Dall’Asta et al. (1996a, 1998, 1999, 2005, 2007) took $P-\Delta$ effect into consideration in a series of following research studies.

Jain and Goel (1996) pointed out that the Timoshenko theory is deduced on the beam under external forces and should only be applied to this scenario. They believed that once the prestressing tendon was fixed to the beam, the prestressing force became an internal force and cannot affect the natural frequencies of the beam. This conclusion is questionable because the straight prestressing tendon which is perfectly detached from the main beam can be replaced by a pair of external axial forces at both ends of the beam and the “compression ($P-\Delta$) effect” would take effect in this situation.

Deak (1996) indicated that the Timoshenko theory can only be used in the beams under external axial forces or prestressed beams similar to them. This description is

more rigorous than Jain and Goel. However, Deak also concluded that the natural frequencies would not be affected by the prestressing force in other scenarios which is not entirely correct because the scenarios of Hybrid Type are not considered. Besides, no solid mathematical evidence was provided in this discussion.

The discussion is followed by more research studies. Miyamoto et al. (2000) studied prestressed beams with external trapezoidal tendons. Unlike Raju et al. (1986) regarding the prestressing force as constant external force, the change of the magnitude of force in the tendon and the beam due to flexural vibration was considered. They concluded that both the magnitude of the initial prestressing force and the arrangement of the prestressing tendon would affect the natural frequencies of the beam. When the influence of the prestressing force is predominant, the natural frequencies would decrease as the increase of prestressing force which means the “compression (P- Δ) effect” was taking effect. However, as the external trapezoidal tendon was fixed to the main beam at both deviators, the interactions at deviators should be taken into consideration which was ignored in this paper.

Similarly, Chan and Yung (2000) emphasized the “compression (P- Δ) effect” of the prestressing force on the natural frequencies of prestressed bridges. The numerical model of the prestressed bridge was developed and its flexural stiffness was compared with the original bridge without prestress. It was demonstrated in the results that the flexural stiffness of the prestressed bridge decreased due to prestressing force.

Contrary to the research studies mentioned above, Kim et al. (2004) proposed a theoretical analysis showing that the natural frequencies increased with prestressing force. As being demonstrated in Timoshenko’s book (1974), the natural frequencies (and flexural stiffness) of a wire under external axial tension would increase as the opposite of “compression (P- Δ) effect”. Hence the flexural stiffness of prestressing tendon was supposed to increase and this increment was added directly to the equivalent flexural stiffness of the prestressed beam in the paper of Kim et al. However, the prestressing tendons would exert axial compression to the beam which means the “compression (P- Δ) effect” would take effect at the same time but it was

not considered in their paper. In other words, tensioned tendon could promote equivalent flexural stiffness of beam while compression transferred from tendon to beam could reduce equivalent flexural stiffness of beam due to “compression (P- Δ) effect”. It could be proved mathematically that these two effect will balance each other (Section 3.2) which means the natural frequencies would not be affected by the prestressing force in this situation. Similar to Kim et al. (2004), Kato and Shimada (1986), Mirza et al. (1990), Singh (1991), Mo and Hwang (1996) and Liu et al. (2013) tried to use the change of natural frequencies to detect prestress loss.

Then, Hamed and Frostig (2006) conducted rigorous mathematical deduction on a prestressed beam with prestressing tendons in arbitrary curved profiles and concluded that the natural frequencies would not change with the prestressing force in such cases. Some misunderstandings are eliminated by their work. For example, researchers had different opinions whether the “compression (P- Δ) effect” existed in prestressed beams and what was its source. Some scholars simply regarded the prestressing force as internal force and ignore this effect while some others proposed a linear model which was not capable to take this effect into consideration. Hamed and Frostig replaced the prestressing tendon with some external forces and the “compression (P- Δ) effect” was included in their equations. They also pointed out that the geometry nonlinearity must be considered in order to estimate the “compression (P- Δ) effect” hence linear models are not capable enough for this problem. On the other hand, some researchers considered “compression (P- Δ) effect” but neglected the interaction between tendon and beam in the span which could balance out this “compression (P- Δ) effect” as mentioned in previous paragraph. Hamed and Frostig indicated that their results were due to the fact that the change in the tendon eccentricity due to vibration had been ignored which was caused by the interaction mentioned. In the comprehensive deduction of Hamed and Frostig, this interaction is considered properly too.

Their research study was widely cited (Bedon and Morassi, 2014, Whelan et al., 2014, Kenna and Basu, 2015, Yan et al., 2017, Orlowska et al., 2018) as the solid evidence of the point of view that the prestressing force would not influence natural frequencies and it will be elaborated in Section 2.1.2. It should be noted that their theory is based

on the assumption that the tendon is perfectly attached to main beam (the definition of “perfectly attached” can be found in Section 1.1). It cannot be used in some scenarios that the tendon and main beam are fully (or partly) separated. The reason will be demonstrated in detail in Chapter 3.

Many experiments and numerical simulations are conducted to investigate this problem afterward and some attempts were made to explain their results. Jaiswal (2008) developed a series of finite element models of beam with straight prestressing tendon. It was observed that the natural frequencies of the prestressed beams with unbonded tendon would reduce as the prediction of Timoshenko et al. while the natural frequencies of the prestressed beams with bonded tendon were not changed as the conclusion of Hamed and Frostig. Hence Jaiswal concluded that the Timoshenko theory should be used in unbonded prestressed beams while Hamed and Frostig theory should be used in bonded prestressed beams.

Wang et al. (2013) reviewed research of Dall’Asta and Leoni (1999) who found that natural frequency would decrease with increase of prestressing force for unbonded prestressed beams. Then, Wang et al. (2013) reviewed studies of Hamed and Frostig (2006) and Breccolotti et al. (2009) and concluded that their theories were correct for bonded prestressed beams. In summary, they believed that “compression ($P-\Delta$) effect” was prominent for unbonded prestressed beams but not for bonded prestressed beams. Besides, Wang et al. (2013) conducted some experiments but the results were not conclusive.

The conclusion of Jaiswal (2008) and Wang et al. (2013) is a general misunderstanding to classify the scenarios of prestressed beams because the fundamental difference of the two extreme scenarios (Timoshenko scenario and Hamed & Frostig scenario) is the real interaction between the tendon and the main beam in span instead of the prestress technique. This conclusion will be elaborated and deduced mathematically in Chapter 3.

2.1.1 Timoshenko theory

Timoshenko et al. (1974) derived the natural frequencies of a beam under axial compression which showed that it would decrease with the increase of the compression. The similar deduction can be found in the book of Tse et al. (1978). For generality, Timoshenko et al. set tension as positive and did their derivation on a beam under tension in the beginning. Then, they simply replaced P with $-P$ to get the conclusion for beam under compression. The simply supported beam under tension at both ends is calculated as Fig. 2-1:

Figure 2-1. Vibrating beam under axial tension (Timoshenko et al., 1974)

The external axial force P is exerted at both ends of a simply supported beam. Let P be positive when it is tension for generality. There is inertial force distributed in the whole beam during vibration and the inertial force per unit length is represented as transverse loading intensity q in Fig. 2-1. The moment induced by this inertial force is noted as M . A random slice of the beam is taken out whose length is dx and the distance between the left end of the beam and this slice is recorded as x . The transverse displacement of this slice is noted as y . The differential equation of the deflection curve is:

$$EI \frac{\partial^2 y}{\partial x^2} = M + Py \quad (2-1)$$

It should be noted that the second derivative of the bending moment M equals the transverse loading intensity q which is the inertial force per unit length in this free vibration case. It can be represented as $-\rho A \frac{\partial^2 y}{\partial t^2}$.

Double differentiating Eqn. 2-1 with respect to x and substituting the expression of inertial force into Eqn. 2-1, the following equation is found:

$$EI \frac{\partial^4 y}{\partial x^4} = -\rho A \frac{\partial^2 y}{\partial t^2} + P \frac{\partial^2 y}{\partial x^2} \quad (2-2)$$

Eqn. 2-2 is the force equilibrium in the transverse direction. The influence of external axial force is represented by the third term considering geometry nonlinearity, which is the key that changes the natural frequencies of the beam.

Take the solution of Eqn. 2-2 in the form:

$$y(x, t) = X(C_1 \cos pt + C_2 \sin pt) \quad (2-3)$$

Substitute Eqn. 2-3 to Eqn. 2-2:

$$EI \frac{\partial^4 X}{\partial x^4} = \rho A p^2 X + P \frac{\partial^2 X}{\partial x^2} \quad (2-4)$$

Assume the shape function is $X_i(x) = \sin \frac{i\pi x}{l}$ for the simply supported beams and substitute it into Eqn. 2-4, the corresponding angular frequencies of vibration can be found as

$$\omega_i = \frac{i^2 \pi^2}{l^2} \sqrt{\frac{EI}{\rho A}} \sqrt{1 + \frac{l^2 P}{i^2 \pi^2 EI}} \quad (2-5)$$

When the axial force is compression, we can replace P with $-P$ (with P being the absolute value) and the expression of the frequencies becomes

$$\omega_i = \frac{i^2 \pi^2}{l^2} \sqrt{\frac{EI}{\rho A}} \sqrt{1 - \frac{l^2 P}{i^2 \pi^2 EI}} \quad (2-6)$$

Note that the angular frequencies of vibration for the beam without axial force are

$\omega_i = \frac{i^2 \pi^2}{l^2} \sqrt{\frac{EI}{\rho A}}$. Therefore, it can be seen that the axial compression will reduce the natural frequency of the beam.

2.1.2 Hamed and Frostig's theory

Hamed and Frostig (2006) studied the effects of prestressing force on the natural frequencies in both bonded and unbonded conditions. It should be noted that in both situations the tendons are assumed to deflect with the main beam at every point. Under such a condition, they proved with a mathematical method that the prestressing force would not affect the natural frequencies of a beam.

The model used by the authors is illustrated in Fig. 2-2.

Figure 2-2. Substructure layout and internal stress resultants of a typical prestressed beam: (a) prestressed beam; (b) concrete beam component; and (c) tensioned cable component (Hamed and Frostig, 2006)

The complex prestressed beam is divided into two substructures: concrete beam and tensioned cable. The interaction between the concrete beam and tensioned cable is recorded as q_{eq} whose vertical component and longitudinal component are represented as λ_1 and λ_2 , respectively in the further calculation. Then, Hamilton's principle is used to get the force equilibrium in the vertical direction at every point of the beam (excluding tendon).

$$\frac{\partial(N_{xx}\frac{\partial y}{\partial x})}{\partial x} + \frac{\partial^2 M_{xx}}{\partial x^2} = m \frac{\partial^2 y}{\partial t^2} + \lambda_1 + \frac{\partial(\lambda_2 z_{cab})}{\partial x} \quad (2-7)$$

where m and y are the mass per unit length and the vertical displacement of the concrete beam, respectively, N_{xx} and M_{xx} are the stress resultants in the longitudinal direction and bending moment stress resultants in concrete beam, respectively, λ_1 and λ_2 are the vertical and longitudinal components of interaction between prestressing tendon and concrete beam while z_{cab} is the tendon eccentricity.

Eqn. 2-7 represents the force equilibrium of the beam (excluding tendon) in a transverse direction like Eqn. 2-2. The left-hand side of the equation is an internal force while the right-hand side of the equation is external force including inertial force and interaction transferred from the prestressed tendon. Since Hamed and Frostig considered tendons with a different profile such as parabolic tendon, the initial tendon eccentricity (z_{cab}) at a different point is a function of x depending on the arrangement of prestressing tendon. For prestressed beams with straight tendons, this initial tendon eccentricity is constant independent of x and can be taken out from the derivative $\frac{\partial(\lambda_2 z_{cab})}{\partial x}$ as $\frac{\partial \lambda_2}{\partial x} z_{cab}$.

Introducing the appropriate boundary and continuity conditions, Hamed and Frostig re-wrote the equation and solved it by Multiple Shooting method. Because of the complexity of the equations, no explicit solution of closed form can be found. However, it was proved that the equation of motion was a linear differential equation and superposition principle was available. Hence the prestressing force can be deleted from the beam without changing the system property including the natural frequencies. They used this indirect way to prove that the prestressing force will not affect the natural frequencies of the beam.

2.2 EXPERIMENTAL INVESTIGATION ON THE EFFECT OF PRESTRESSING FORCE ON NATURAL FREQUENCIES

Except for the theoretical investigations, a number of experiments have been conducted in an attempt to provide insight into the phenomenon (Saiidi et al., 1994, Jang et al., 2010, Wang et al., 2013, Noble et al., 2016).

Saiidi et al. (1994) conducted dynamic tests on an actual concrete bridge and a laboratory specimen. The bridge was a 47.2m long, simply supported post-tensioned concrete box girder structure. A field test was conducted on the bridge. The natural frequencies of the structure were extracted from the dynamic data by FFT (Fast Fourier Transformation). It was found that the natural frequencies of real bridge decreased with the prestress loss. During the period of testing, the prestressing force dropped from 53492kN to 49440kN (a reduction of 7.57%) while the natural frequency of the fundamental bending mode decreased from 2.028Hz to 2.011Hz (a reduction of 0.84%).

Since this phenomenon contradicted the expectation from Timoshenko theory, Saiidi et al. carried out more tests on a simply supported post-tensioned concrete beam in the laboratory. The laboratory specimen was 3658mm long and had a cross-section of 102mm $\times$ 127mm. A 12.7mm diameter tendon was centrally placed in the specimen through a 25mm diameter duct. The strength of concrete was 20.3MPa and the reinforcements were Grade 60 bars. The tendon was Grade 250. During the tests, the prestressing force was increased from 0 to 131.3kN which corresponded to a compressive stress ratio of 0.5 ($N/f_c A = 0.5$). The estimated axial buckling load for the beam was 174.8kN and this maximum of prestressing force was 75% of the buckling load.

Both static tests and dynamic tests were conducted. In the static tests, the nearly concentrated load was exerted at midspan and the vertical displacements of the beam under different prestressing force were measured. The influence of prestressing force on the flexural stiffness could be demonstrated in these static tests. The tests were repeated with different lateral load at midspan (126kN and 247kN) and both of the results showed that the flexural stiffness increased with the prestressing force.

In the dynamic tests, the prestressed beam under different prestressing force was excited by impact and dynamic data was collected during the free vibration. The tests were repeated four times with different impacts, two of them are applied at midspan and the other two were at quarter point. The fundamental bending mode frequency was extracted by FFT and increased from 11.41Hz to 15.07Hz when the prestressing

force climbed from 0 to 131.3kN. Then it decreased to 12.09Hz when the prestressing force dropped to 15.5kN.

Both the static tests and dynamic tests conducted on the concrete beam, along with the field tests on the concrete bridge indicated that the natural frequencies increased with the prestressing force. It should be noted that the maximum displacement measured in static tests was about 1mm which was much smaller than the gap (6.15mm) between the tendon and the duct and the duct is ungrouted. Hence if the duct was straight enough and the installation of the tendon had good accuracy, the tendon was not supposed to touch the duct during the vibration which means the tests of Saiidi et al. should be classified as Type-A and Timoshenko theory is applicable. However, the tests showed contrary results with theory.

They mentioned that the specimen developed a small crack at midspan under its own weight during handling. They pointed out that it could be the reason causing the discrepancy between their tests and Timoshenko theory. However, no solid evidence was found in their paper.

As mentioned in Section 2.1, the tests of Saiidi et al. aroused heated discussion. Scholars argued which type of theoretical model (Type-A or Type-B) should be used to estimate the natural frequencies of prestressed beams. In fact, neither of the theoretical solutions could explain the increasing trend of natural frequencies when the prestressing force goes up. It will be demonstrated in Chapter 5 that the discrepancy revealed in these tests is caused by the shrinkage cracks in concrete while the material is assumed to be homogenous in both Timoshenko theory and Hamed & Frostig theory. Hence, a comprehensive and coherent conclusion is necessary for this problem.

Instead of unbonded prestressed beams in the tests of Saiidi' et al., Jang et al. (2009) conducted dynamic and static tests on simply supported prestressed concrete beams with the bonded centric tendon. Since the magnitude of the prestressing force cannot be changed for bonded prestressed beams, they made six scale-model specimens with different prestressing forces for the tests. All six specimens were 8m long and had a cross-section of 300mm $\times$ 300mm. Three strands with a diameter of 15.2mm were

arranged in the center of the beam as prestressing tendon. The Young's Modulus of the concrete was not given in Jang's paper. The buckling load of the beam is 3123kN if Young's Modulus of concrete is assumed to be 30GPa. The maximum of prestressing force among these six beams was 523kN which was 16.7% of the buckling load.

Both impact test and SIMO sine sweep test as dynamic tests were conducted separately. The fundamental natural frequencies got from two test methods fitted well. The difference ratio of their results was no more than 0.7% indicating that both test methods were reliable. The fundamental natural frequency of the prestressed beams increased from 7.567Hz to 8.757Hz when the prestressing force increased from 0 to 523kN. The change ratio was up to 15.7% which demonstrated a significant increment on the natural frequency.

Apart from the dynamic tests, the authors also performed static bending tests. The displacement of the beam decreased with the increase of prestressing force indicating the increase of flexural stiffness.

Both the results of dynamic tests and static tests demonstrated an increasing trend of flexural stiffness consequently natural frequencies. Since the specimens were bonded prestressed beams, assuming the bond between the tendon and main beam was perfect at every point, the Hamed and Frostig theory was applicable in this circumstance which indicated that the natural frequencies should remain unchanged. However, diverging evidence was provided in their experiments and needed to be explained.

Wang et al. (2013) conducted both dynamic and static tests on five simply supported prestressed beams. All beams were 3500mm long and had a cross-section of 300mm $\times$ 160mm. Two wires with a diameter of 7mm were placed in a 50mm diameter duct as prestressing tendon in each beam. Three of the prestressed beams had parabolic tendons with 31.4kN, 39.3kN and 47.1kN and were named P500, P625 and P750, respectively. The other two prestressed beams had straight tendon with a constant eccentricity of 80mm and the prestressing force was 31.4kN and 47.1kN. They were named S500 and S750, respectively. The Young's Modulus of the concrete is 23GPa in their tests and the buckling load of the beam can be calculated as 6666kN. Hence

the magnitude of the prestressing force of the beams was in the range of 0.47% to 0.71% of the buckling load.

The tendons in their tests were not bonded at first. They measured the natural frequencies of the beam when the prestressing force was 1) not applied, 2) applied but tendon was not grouted to concrete and 3) applied and grouted. These measured natural frequencies are recorded in their Table 1 and no clear trend can be found. One of the possible reasons is that the applied prestressing force was too small. Take the tests of S750 as an example, even if the “compression (P- Δ) effect” described in Timoshenko theory took effect, the fundamental natural frequency would reduce by 0.35% which was difficult to be monitored. Hence the deviation of the frequencies in the table may be caused by an error of handling in the tests.

Noble et al. (2016) tested 9 prestressed simply supported beams. These beams were 2m long and had a cross-section of 200mm $\times$ 150mm. The 15.7mm diameter post-tensioned unbonded tendons were threaded through a 20mm diameter duct in a concrete beam. The prestressing force increased from 0 to 200kN in 20kN increments. The Young’s Modulus was 26.88GPa hence the buckling load can be calculated which was about 6632kN. The maximum of prestressing force was 3% of Euler’s critical load and the fundamental natural frequency was supposed to decrease by about 1.5% according to Timoshenko theory. They reported no structural cracks due to the small deflections in the static and dynamic tests. The dynamic tests were performed using an impact hammer and the vibration signals were then analysed to obtain the natural frequencies directly. In the static tests, the beams were placed under 3-point bending and the displacement was recorded to calculate the flexural stiffness and this was then used to calculate the natural frequencies.

It was illustrated in Fig. 29 of their paper that the fundamental natural frequency of the prestressed beam with centric tendon increased from 68.66Hz to 69.32Hz when the prestressing force went up from 0 to 200kN according to the dynamic tests results. More significant increment was found in their static tests. The fundamental natural frequency increased from about 69Hz to about 100Hz. Although they did not match each other well and Noble et al. concluded that no statistically significant relationship

between prestressing force and natural frequencies is found, neither of test results demonstrated decrease trend as the prediction of Timoshenko theory. As mentioned in Section 2.1, if the prestressing tendon was not attached to the main beam, Hamed & Frostig theory should not be used in this scenario.

Most of the experimental investigation reviewed in this section demonstrated that the natural frequencies of prestressed beams tended to increase with prestressing force (Hop, 1991, Saiidi et al., 1994, Zhang, 2007, Jang et al., 2010) while the tests of Wang et al. (2013) showed no significant trend. However, no satisfactory theoretical solution explaining the increasing trend of natural frequencies has been proposed now.

As to some unbiased factors such as ambient vibration in lab or installation error in test (position of anchorage of tendon could be changed after every tensioning process, et al.), they would affect the accuracy of experiments but they are not supposed to change the trend completely. In other words, the measured natural frequencies in tests might fluctuate violently due to these factors but should not show a significant increase trend (when prestressing force increase) while existing theories predict that natural frequencies should decrease or unchanged.

Some other factors such as difference between support in tests and theoretical models or immediate prestressing force loss are not discussed in this thesis. Because these factors are not affected by prestressing force and consequently cannot make the natural frequencies increase with prestressing force.

There are two factors would affect the relationship between natural frequencies and prestressing force and this thesis would focus on them. Firstly, the actual installation and arrangement of tendons could result in the interaction of prestressing tendon with primary structure. This could transform the beam from Type-A to Hybrid Type (or even Type-B) and make the natural frequencies reduction much smaller. The prestressed beams of Hybrid Type will be discussed in Chapter 3 and Chapter 4. Secondly, concrete properties are prestress-dependent, for example with an increase in the flexural rigidity (equivalent EI) due to crack closure. This will add further complexity to the final outcome and will be discussed in Chapter 5.

CHAPTER 3

ANALYTICAL STUDY ON SCOPE OF APPLICATION OF EXISTING THEORIES

3.1 CLASSIFICATION OF THE DIFFERENT TYPE OF PRESTRESSED BEAMS

Both the theories of Timoshenko et al. and Hamed & Frostig are deduced rigorously in their particular mathematical models. However, how to classify the real beams into the right mathematical models is still not clear. The theories of Hamed and Frostig is more general because the profile of prestressing tendon is arbitrary, and they did their mathematical deduction for both bonded and unbonded prestressing tendons and got the same results. On the other hand, Timoshenko's theory is based on the assumption that the prestressing tendon can be represented by a pair of external forces (Fig. 3-1a) which makes it restricted to one particular situation (Type-A) as shown in Fig. 3-1b. It should be noted that the tendon cannot be in contact with the main beam in span even during vibration. The main contribution and originality of this Chapter is to reveal the essential difference between Timoshenko scenario and Hamed & Frostig scenario and clarify their scope of application.

(a) Beam under externally applied axial compression force-Timoshenko scenario

(b) Post-tensioned beam with tendon detached from beam other than at two ends-Timoshenko scenario

(c) Post-tensioned beam with tendon constrained to adjacent beam nodes in transverse direction (in addition to fixed at two ends)-Hamed and Frostig scenario

Figure 3-1. Scenarios with different interaction of the prestressing tendon with the primary structure

It seems that Timoshenko theory is nothing else but explaining the mechanism of a specific scenario and Hamed & Frostig theory is applicable to the rest. However, Timoshenko theory is actually more important than it seems because it represents one extreme condition that the tendon is completely separated from the main beam. In this scenario, Hamed & Frostig theory is not applicable and the natural frequencies will decrease with the increase of prestress. It reveals that there are a series of circumstances that the prestressing tendons are not perfectly attached to the main beam, the Hybrid Type as mentioned in Chapter 1, in which Hamed & Frostig theory does not cover. In addition, the reduction of natural frequencies due to prestressing force in these beams of Hybrid Type cannot be larger than Timoshenko scenario.

Hamed and Frostig assumed that the profile of prestressing tendon was arbitrary in their unbonded scenario. Hence the tendon would deflect with the main beam naturally even in their unbonded scenario which excludes the typical Timoshenko

scenario automatically. In other words, it is not true to conclude that the natural frequencies are not affected by prestressing force in general. In fact, typical Hamed & Frostig theory can only be applied in the prestressed beams that the tendons are fully attached to the beam (Type-B). Type-A as a scenario that the tendons fully detached to the beam is surely excluded from Hamed & Frostig theory. Meanwhile, it indicates that Hamed & Frostig theory is not applicable in the Hybrid Type with tendons partly attached to the beam either. The term “partly attached” means that the tendon touches and has identical transverse deflection with main beam at some points (e.g. deviators). It is a term that contrast with “fully attached” whose definition is given in Section 1.1. It should be noted that the bonding (grouting) at these points is not a necessary condition.

Instead of a singular situation as Type-A, many prestressed beams should be classified as this Hybrid Type, e.g., beams with external tendons which are fixed to the main beam at a few points (deviators), beams with unbonded straight tendons which have real interaction with the main beam at some points due to the inaccuracies of installation and arrangement of tendon. Neither existing theory is available in this Hybrid Type. The reduction of natural frequencies due to the prestress in these situations falls between Type-A and Type-B and needs specific calculation model. Hence it is necessary to propose a systematic conclusion on the effect of prestressing force on natural frequencies in all scenarios and a practical way to classify the real prestressed beam into these three types with solid evidence.

Since Timoshenko theory and Hamed & Frostig theory are regarded as two extreme scenarios, they should be comparable and the scope of application of them should be clarified rigorously. There are some misunderstandings on this problem now. Jaiswal (2008) observed that Timoshenko’s theory applied well to the prestressed beams with unbonded tendons while Hamed and Frostig’s theory applied well to prestressed beams with bonded tendons. Wang et al. (2013) made similar observations. However, as mentioned above, the unbonded prestressing tendons could be fully attached to the main beam especially when they have a curved profile because the tendon touches the main beam after tensioning and they should be classified as Type-B instead of Type-A. What is more, an unbonded prestressed beam of typical Type-A in small

vibration will be transferred to a Hybrid Type when the vibration is large enough that the tendon touches the main beam. In other words, the essential difference of them should be the real interaction of tendons and beams during vibration rather than the prestress technique.

As Hamed and Frostig did their theoretical analysis in a general condition and their conclusion is drawn indirectly, it is necessary to take a simple example (Fig. 3-1c) similar to Timoshenko scenario (Fig. 3-1b). In this simple example, the theoretical solution of the natural frequencies of the bonded prestressed beam can be found directly. The essential difference between the two scenarios is clearer in the following deduction because these two examples are comparable. All the disturbances are eliminated in this comparison. For example, the Type-A here is a beam with a straight tendon which does not touch the main beam in the span during the vibration. It is not the external axial forces in the original Timoshenko theory so the influence on natural frequencies induced by transferring the external forces to prestressing tendon is excluded. In addition, it can be seen if the tube in the unbonded prestressed beam in Fig. 3-1b is small enough which makes the tendon deflect with the main beam, the forces in the beam are actually identical to the bonded prestressed beam shown in Fig. 3-2b instead of Fig. 3-2a.

The difference between them could be roughly deduced by the general idea as shown below:

Figure 3-2. The simple understanding of the classification of (a) Timoshenko scenario and (b) Hamed & Frostig scenario

Cut the prestressed beam under free vibration from an arbitrary section and indicate all forces and moments transferred from support, adjacent main beam or tendon. The force diagram of Type-A (Timoshenko scenario) and Type-B (Hamed and Frostig scenario) is illustrated in Fig. 3-2. $M(x)$, $V(x)$ and N are moment, shear force and axial force acting on this section from adjacent main beam, respectively. S in Fig. 3-2a is tension in tendon while $S_x(x)$ and $S_y(x)$ in Fig. 3-2b are horizontal and vertical component of tension in tendon. It should be noted that the tendon and main beam are considered as one object and the interaction between them will not be illustrated in Fig. 3-2, hence only vertical support reaction is shown at the left end. Besides, since the beam is under free vibration, there is inertial force all the time which is not illustrated in Figure. 3-2.

It can be seen that the prestressing force applies an additional moment in Timoshenko scenario and making no moment in Hamed and Frostig scenario. This additional

moment will aggravate the bending which is the source of the “compression (P- Δ) effect” and the essential difference between the two theories.

This deduction just give the reader an intuition of “compression (P- Δ) effect” in prestressed beams and is not rigorous enough because the magnitude of $M(x)$ and $V(x)$ in Fig. 3-2a and Fig. 3-2b may not be identical and the prestressing force is not the only one variable needed to be considered. It induces ambiguity to the deduction here and a more rigorous derivation is proposed in Section 3.2.

3.2 MATHEMATICAL DEDUCTION INDICATING DIFFERENCES BETWEEN TIMOSHENKO AND HAMED & FROSTIG SCENARIOS

The difference between Timoshenko and Hamed & Frostig theories are explained briefly in Section 3.1. The original deductions of them as demonstrated in Chapter 2, they seem not comparable as so many different factors in their deduction, e.g., the difference between externally applied axial force and the prestressing tendon, the difference between the straight tendon and curved profile tendon, and so on so forth.

It would be better to draw the conclusion that the essential differences between these two scenarios are the additional moment induced by prestressing force (or externally applied axial force) after providing solid evidence when other factors are eliminated because this conclusion is not widely admitted yet. As mentioned before, Jaiswal (2008) and Wang et al. (2013) concluded that they are differentiated by construction technology of prestressing tendons instead of the contact conditions which indicates that there is still some confusion in distinguishing two scenarios.

Although a simple and clear deduction shown in Section 3.1 can be used to indicate the essential difference between them, it is not rigorous enough. Hence a more rigorous mathematical deduction will be demonstrated in this section to support our view. In order to eliminate other factors that suspected to affect the results, both beams have straight unbonded prestressing tendon and the only difference between them is that the tendon deflects with main beams in Hamed and Frostig scenario while

the tendon is detached to the main beam in Timoshenko scenario as we did in section 3.1.

Free body diagrams of two scenarios are drawn and analysed. The fundamental parameters and the coordinate system of these simply supported beams are illustrated as Fig. 3-3:

Figure 3-3. Basic parameters of the beam

According to the Euler-Bernoulli beam theory, the equation of motion representing the force equilibrium in the transverse direction of a beam without prestressing force is shown as Eqn. 3-1.

$$EI \frac{\partial^4 y}{\partial x^4} = -\rho A \frac{\partial^2 y}{\partial t^2} \quad (3-1)$$

Where EI is the flexural stiffness of the beam, ρA is the mass of unit length of the beam, $y(x)$ is the deflection of the beam along the length. Introducing the mode shape of the beam as the deduction of Timoshenko theory, the fundamental natural frequency of the beam can be calculated as $\frac{\pi^2}{l^2} \sqrt{\frac{EI}{\rho A}}$.

Firstly, the Timoshenko scenario (Type-A) is studied. An arbitrary slice of the beam (including the tendon) is taken out and the forces acting on it are drawn in the free body diagram illustrated in Fig. 3-4.

Figure 3-4. Free body diagram of a slice of the beam in Timoshenko scenario
(Type-A)

Since the tendon is detached to the main beam in span, there is no interaction between them in both vertical and horizontal direction. As no other force in the horizontal direction is involved, the axial force in tendon and beams (S and N , respectively) remain constant through out their whole length. The shear force $V + \frac{\partial V}{\partial x} dx$ in the beam is changed due to the inertial force $\rho A dx \frac{\partial^2 y}{\partial t^2}$ in the vertical direction. No shear force is involved in tendon and it remains straight as its original position.

The force equilibrium of the beam slice (Fig. 3-4) in the vertical direction is demonstrated as:

$$V = V + \frac{\partial V}{\partial x} dx + \rho A dx \frac{\partial^2 y}{\partial t^2} \quad (3-2)$$

The moment equilibrium of the free body slice about its center is represented as:

$$M + V dx + \frac{1}{2} \frac{\partial V}{\partial x} dx dx + N dy = M + \frac{\partial M}{\partial x} dx \quad (3-3)$$

Since the third term (dx^2) in Eqn. 3-3 is infinitesimal of a higher order of other terms, it can be neglected. In addition, divide both sides of Eqn. 3-2 by dx as dx is a non-zero value. Rewrite Eqn. 3-2 and Eqn. 3-3 as following:

$$\frac{\partial V}{\partial x} + \rho A \frac{\partial^2 y}{\partial t^2} = 0 \quad (3-4)$$

$$Vdx + Ndy - \frac{\partial M}{\partial x} dx \approx 0 \quad (3-5)$$

Substitute Eqn. 3-5 into Eqn. 3-4:

$$\frac{\partial^2 M}{\partial x^2} - N \frac{\partial^2 y}{\partial x^2} = -\rho A \frac{\partial^2 y}{\partial t^2} \quad (3-6)$$

For the Euler-Bernoulli beam in small deflection, $\frac{\partial^2 M}{\partial x^2}$ equals to $EI \frac{\partial^4 y}{\partial x^4}$ and Eqn. 3-6 is rewritten as:

$$EI \frac{\partial^4 y}{\partial x^4} = -\rho A \frac{\partial^2 y}{\partial t^2} + N \frac{\partial^2 y}{\partial x^2} \quad (3-7)$$

The equation of motion deduced by free body diagram analysis is the same as original Timoshenko's equations (Eqn. 2-2). Two objectives are achieved. Firstly, this result is deduced on a beam with prestressing tendons instead of externally applied axial force which proved that Timoshenko theory can be applied to prestressed beams. Secondly, Timoshenko scenario and Hamed & Frostig scenario are discussed in a comparable situation with same mathematical method which reveals the essential difference between them. A brief deduction is introduced in Section 3.1 to indicate the essential difference between two existing theories. We concentrated on the influence of prestressing force in that deduction. However, the deduction is ambiguous and need further explanation because the other forces and moments should be considered too. Hence more rigorous mathematical derivation is proposed in Section 3.2. In this derivation, they are considered and the results are identical to Timoshenko theory.

Then, the free body diagram of Hamed & Frostig scenario (Type-B) is considered. A random slice of the beam and the forces on it are illustrated in Fig. 3-5.

Figure 3-5. Free body diagram of a slice of the beam in Hamed and Frostig scenario (Type-B)

In Hamed and Frostig model, the tendon deflects with the main beam at every point. The tendon is no longer at its original position and the horizontal and vertical component of tendon force is recorded as S_x and S_y in Fig. 3-5. What is more important, the acting point of the force in tendon moves with the main beam during vibration. Hence the moment induced by the prestressing force and the axial force in the main beam is balanced out in Type-B ($Ndy = S_x dy$ in Eqn. 3-9). The other forces in the free body of Type-B are identical to Type-A.

The force equilibrium of the beam slice (Fig. 3-5) in the vertical direction is demonstrated as:

$$V + S_y = V + \frac{\partial V}{\partial x} dx + S_y + \frac{\partial S_y}{\partial x} dx + \rho A dx \frac{\partial^2 y}{\partial t^2} \quad (3-8)$$

The moment equilibrium of the free body slice about its center is represented as:

$$M + Vdx + S_y dx + \frac{1}{2} \frac{\partial V}{\partial x} dx dx + \frac{1}{2} \frac{\partial S_y}{\partial x} dx dx + Ndy = M + \frac{\partial M}{\partial x} dx + S_x dy \quad (3-9)$$

Divide both sides of Eqn. 3-8 and Eqn. 3-9 by dx as dx is a non-zero value. Since the fourth term and fifth term (dx^2) in Eqn. 3-9 are infinitesimal of higher order of other terms, they could be neglected. As the axial forces in the beam (N) and tendon (S_x) are a pair of action and reaction forces in the prestressed beam, the terms Ndy and

$S_x dy$ in Eqn. 3-9 canceled out. Then, Eqn. 3-8 and Eqn. 3-9 are rewritten to get Eqn. 3-10 and Eqn. 3-11, respectively. The terms $\frac{1}{2} \frac{\partial V}{\partial x} dx$ and $\frac{1}{2} \frac{\partial S_y}{\partial x} dx$ are small compared to other terms (V , S_y and $\frac{\partial M}{\partial x}$). Hence they are ignored in Eqn. 3-11:

$$\frac{\partial V}{\partial x} + \frac{\partial S_y}{\partial x} + \rho A \frac{\partial^2 y}{\partial t^2} = 0 \quad (3-10)$$

$$V + S_y - \frac{\partial M}{\partial x} \approx 0 \quad (3-11)$$

Substitute Eqn. 3-11 to Eqn. 3-10

$$\frac{\partial^2 M}{\partial x^2} = -\rho A \frac{\partial^2 y}{\partial t^2} \quad (3-12)$$

For the Euler-Bernoulli beam in small deflection, $\frac{\partial^2 M}{\partial x^2}$ equals to $EI \frac{\partial^4 y}{\partial x^4}$ and Eqn. 3-12 is rewritten as:

$$EI \frac{\partial^4 y}{\partial x^4} = -\rho A \frac{\partial^2 y}{\partial t^2} \quad (3-13)$$

All the terms containing prestressing force are eliminated which makes Eqn. 3-13 identical to Eqn. 3-1 and they will get the same fundamental natural frequency. Hence the prestressed beam classified as Type-B is identical to a regular beam without prestressing force or external axial force. It can be seen from the deduction of this scenario that if the tendon is unbonded, as long as it deflects with the beam, the prestressing force in it will not affect the natural frequencies.

From the mathematical deduction in this section, it can be seen that the fundamental difference between Type-A and Type-B is the additional moment induced by the prestressing force in Type-A. When the acting point of prestressing force (relative position of the tendon in every section of the main beam) changes with the main beam, this addition moment is zero consequently no influence of the prestressing force on natural frequencies is found.

From the discussion above, it can be concluded that the classification of Type-A and Type-B depends on the relative position of the tendon and beam which is decided by the real interaction between them no matter bonded or unbonded. Every point of the tendon in Type-A stays at its original position and the effect of prestressing force on natural frequencies follows the prediction of Timoshenko theory. Meanwhile, the prestressing force will not affect the natural frequencies as Hamed and Frostig theory when every point of the tendon deflects with the main beam. Between these two extreme conditions, there is a Hybrid Type such as prestressed beams with external tendon which is attached to the beam at a few points as illustrated in Fig. 3-6. The relative position of the tendon and the beam remains unchanged at certain sections (A, B, C and D) as Type-B. The other points of the tendon are not attached to the beam directly and their relative position to the corresponding beam section is changing during the vibration which will induce the “compression ($P-\Delta$) effect” as the beams of Type-A. For example, we take out an arbitrary section from a static prestressed beam and the tendon locates at point P of the beam section. When this beam is under free vibration, as the external tendon is not connected to main beam at this section and the relative location of tendon at this section is no longer point P. In this circumstance, “compression ($P-\Delta$) effect” is induced.

Figure 3-6. The relative position of tendon and beam

Therefore, neither Timoshenko theory nor Hamed and Frostig theory can be used to estimate the reduction of natural frequencies of the beams of Hybrid Type. Their natural frequencies will decrease with the increase of prestressing force but not as much as the prediction of Type-A. Specific models are needed for different tendon

profiles. The external trapezoidal prestressing tendon as a common tendon arrangement is studied in Chapter 4 as an example of Hybrid Type.

CHAPTER 4

THEORETICAL MODEL OF PRESTRESSED BEAM WITH EXTERNAL TRAPEZOIDAL TENDONS

Two typical theories studying the effect of prestressing force on natural frequencies in two extreme conditions are reviewed and well understood now. However, as mentioned in Chapter 3, many external prestressing tendons are partly attached to the main beam which means they are between Type-A (perfectly detached) and Type-B (fully attached) and no existing theory is found for these Hybrid Type scenarios. The original contribution of this Chapter is to propose theoretical solution for prestressed beam with external trapezoidal tendon which is a typical example of Hybrid Type. The conclusion is verified by comparing with numerical models and experiments in reference. The theoretical solution could be used directly to prestressed beam with external trapezoidal tendon in project to quantify the influence of prestressing force on the fundamental natural frequency. Besides, the energy method used in this Chapter can be applied in the analysis of prestressed beam with other tendon profiles.

4.1 THEORETICAL INVESTIGATION

The energy method is used to study the influence of prestressing force on natural frequencies step by step. Firstly, the fundamental natural frequency of Timoshenko scenario (Type-A) and typical Hamed and Frostig scenario (Type-B) are deduced for verification. Before heading to the external trapezoidal prestressing tendon, a prestressed beam with external straight tendon with four points fixed to the beam is studied first in Section 4.1.3. As a simple example of Hybrid Type, it has identical tendon profile with Type-A except two more points in the span are fixed to the beam.

Hence the reduction of their fundamental natural frequency is comparable, and the influence of the two fixing points is emphasized. Finally, the effect of prestressing force on the fundamental natural frequency of prestressed beam with external trapezoidal tendons is estimated in Section 4.1.4.

4.1.1 Timoshenko scenario

Before studying the external trapezoidal prestressing tendon, it is useful to reiterate the basic theories (Timoshenko theory and Hamed & Frostig theory) using this energy method. The new method can be verified by comparing its results with the classic theories. In addition, some expressions in these basic scenarios can be used directly in the following complex situation like prestressed beam with trapezoidal tendon.

A simply supported prestressed beam subjected to axial compression is illustrated in Fig. 4-1:

Figure 4-1. Timoshenko scenario for energy method

The original distance between the two supports of the beam is L before the external axial force is exerted. When the beam is under compression and bends as shown in Fig. 4-1, this distance varies with time and is noted as b . The length of the beam is s . The distance from the left end of the beam to an arbitrary point is recorded as x while the vertical deformation at this point is y . Hence the differential of the beam curve can be expressed as:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} dx \quad (4-1)$$

Total length of the beam s can be represented by integrating ds in the whole curve:

$$L \approx s = \int_0^b ds = \int_0^b \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} dx \approx \int_0^b \left[1 + \frac{1}{2} \left(\frac{\partial y}{\partial x}\right)^2\right] dx = b + \frac{1}{2} \int_0^b \left(\frac{\partial y}{\partial x}\right)^2 dx \quad (4-2)$$

Strictly speaking, s becomes smaller and does not equal to L when the beam is under axial force because of axial deformation due to compression, but this deformation is small compared to distance change of support due to bending (from L to b) so it is assumed that L equals to s .

Then, the work done by the axial force P can be expressed. The potential energy W induced by P is represented as Eqn. 4-3.

$$W = -P(L - b) = -\frac{P}{2} \int_0^b \left(\frac{\partial y}{\partial x}\right)^2 dx \approx -\frac{P}{2} \int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx \quad (4-3)$$

The strain energy of the bending beam U is expressed as:

$$U = \frac{EI}{2} \int_0^L \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx \quad (4-4)$$

The kinetic energy of the bending beam K is expressed as:

$$K = \frac{\rho A}{2} \int_0^L \left(\frac{\partial y}{\partial t}\right)^2 dx \quad (4-5)$$

Assume $y(x, t) = w(t)\varphi(x)$, $w(t) = w_0 \sin \omega t$, $\varphi(x) = a \sin \frac{\pi x}{L}$. Substitute it into Eqn. 4-3, Eqn. 4-4 and Eqn. 4-5:

$$W = -\frac{Pw^2(t)a^2\pi^2}{2L^2} \int_0^L \cos^2 \frac{\pi x}{L} dx = -\frac{Pw^2(t)a^2\pi^2}{2L^2} \left(\frac{L}{2} + \frac{1}{2} \int_0^L \cos \frac{2\pi x}{L} dx\right) = -\frac{Pw^2(t)a^2\pi^2}{4L} = -\frac{Pw_0^2a^2\pi^2}{4L} \sin^2 \omega t \quad (4-6)$$

$$U = \frac{EIw^2(t)a^2\pi^4}{2L^4} \int_0^L \sin^2 \frac{\pi x}{L} dx = \frac{EIw^2(t)a^2\pi^4}{2L^4} \left(\frac{L}{2} - \frac{1}{2} \int_0^L \cos \frac{2\pi x}{L} dx\right) = \frac{EIw^2(t)a^2\pi^4}{4L^3} = \frac{EIw_0^2a^2\pi^4}{4L^3} \sin^2 \omega t \quad (4-7)$$

$$K = \frac{\rho A}{2} \left(\frac{dw}{dt} \right)^2 \int_0^L \varphi^2(x) dx = \frac{\rho A a^2}{2} \left(\frac{dw}{dt} \right)^2 \int_0^L \sin^2 \frac{\pi x}{L} dx = \frac{\rho A a^2}{2} \left(\frac{dw}{dt} \right)^2 \left(\frac{L}{2} - \frac{1}{2} \int_0^L \cos \frac{2\pi x}{L} dx \right) = \frac{\rho A a^2 L}{4} \left(\frac{dw}{dt} \right)^2 = \frac{\rho A a^2 L w_0^2 \omega^2}{4} \cos^2 \omega t \quad (4-8)$$

Assume the beam is not deformed when $t=0$, the potential energy W induced by P , the strain energy U and the kinetic energy K are calculated as:

$$W(0) = 0 \quad (4-9)$$

$$U(0) = 0 \quad (4-10)$$

$$K(0) = \frac{\rho A a^2 L w_0^2 \omega^2}{4} \quad (4-11)$$

When $t=\pi/2\omega$, the potential energy W induced by P , the strain energy U and the kinetic energy K are calculated as:

$$W\left(\frac{\pi}{2\omega_n}\right) = -\frac{P w_0^2 a^2 \pi^2}{4L} \quad (4-12)$$

$$U\left(\frac{\pi}{2\omega_n}\right) = \frac{EI w_0^2 a^2 \pi^4}{4L^3} \quad (4-13)$$

$$K\left(\frac{\pi}{2\omega_n}\right) = 0 \quad (4-14)$$

According to the energy conservation law, the total energy is unchanged from $t=0$ to $t=\pi/2\omega$, hence

$$W(0) + U(0) + K(0) = W\left(\frac{\pi}{2\omega}\right) + U\left(\frac{\pi}{2\omega}\right) + K\left(\frac{\pi}{2\omega}\right) \quad (4-15)$$

Substitute Eqn. 4-9 – Eqn. 4-14 into Eqn. 4-15:

$$\frac{\rho A a^2 L w_0^2 \omega^2}{4} = -\frac{P w_0^2 a^2 \pi^2}{2L} + \frac{EI w_0^2 a^2 \pi^4}{4L^3} \quad (4-16)$$

Rewrite Eqn. 4-16:

$$\rho A L^4 \omega^2 = -P \pi^2 L^2 + EI \pi^4 \quad (4-17)$$

The fundamental natural frequency of the beam is represented as Eqn. 4-18:

$$\omega^2 = \frac{\pi^4 EI}{L^4 \rho A} - \frac{\pi^2 P}{L^2 \rho A} = \frac{\pi^4 EI}{L^4 \rho A} \left(1 - \frac{PL^2}{EI\pi^2}\right) \quad (4-18)$$

It can be seen that this result is identical to the Timoshenko theory (Eqn. 2-6). The existence of potential energy W induced by axial force P represents “compression (P- Δ) effect” which makes the fundamental natural frequency decrease. Hence this energy method and these assumptions are available in this circumstance.

4.1.2 Hamed and Frostig scenario

Hamed and Frostig theory is deduced in a very general condition and no closed form solution can be found due to its complexity. A simply supported beam with centered straight prestressing tendon which is fully attached to the main beam is taken as an example here for simplicity. The similarity of the models in Section 4.1.2 and Section 4.1.1 can also assure that they are comparable and the essential difference between them is emphasized.

The simply supported prestressed beam with centered straight tendon which deflects with the main beam can be divided into two substructures as Hamed and Frostig did. Only the main beam is considered while the tendon is taken out and replaced by the interaction between them.

Similar to the deduction in Section 4.1.1, the definitions of L , s and b are not changed so the deduction and result of the strain energy of the beam are the same. An identical coordinate system as Section 4.1.1 is set in this model too.

Figure 4-2. Relationship between axial tension and transverse loading intensity on tendon

Firstly, the tendon as a stretched wire without flexural stiffness is considered. Axial tension is added to the tendon. In order to keep the wire deforming as the curve $y_t(x)$, the transverse loading intensity $q_t(x)$ is needed (Fig. 4-2). According to the theory of Timoshenko (1974), the relationship among the transverse loading intensity q_t , axial force P_t and the deflection $y_t(x)$ can be represented as Eqn. 4-19:

$$q_t = -P_t \frac{\partial^2 y_t}{\partial x^2} \quad (4-19)$$

As action and reaction forces, the axial force P and transverse loading intensity q with opposite direction are exerted to the main beam from the tendon as illustrated in Fig. 4-2. The work done by these two forces are recorded as W_1 and W_2 , respectively.

In Type-B, every point of the tendon deflects with the main beam hence their deflection curves $y_t(x)$ and $y(x)$ are identical. Hence the transverse loading intensity acting on the beam can be expressed as:

$$q = -P \frac{\partial^2 y}{\partial x^2} \quad (4-20)$$

Take a small segment dx in the beam. The work done by the transverse load q in a short period of time dt can be represented as:

$$dW_2 = q dx dy \quad (4-21)$$

It should be noted that the dy here means the increment of deflection y in this period of time dt .

Assume $y(x, t) = w(t)\varphi(x)$, $w(t) = w_0 \sin \omega t$, $\varphi(x) = a \sin \frac{\pi x}{L}$ so q and dy can be represented as:

$$dy = w_0 \omega a \sin \frac{\pi x}{L} \cos \omega t dt \quad (4-22)$$

$$q = P w_0 a \frac{\pi^2}{L^2} \sin \frac{\pi x}{L} \sin \omega t \quad (4-23)$$

Substituting Eqn. 4-22 and Eqn. 4-23 into Eqn. 4-21:

$$dW_2 = Pw_0^2 a^2 \omega \frac{\pi^2}{L^2} \sin^2 \frac{\pi x}{L} \sin \omega t \cos \omega t dx dt \quad (4-24)$$

Integrating Eqn. 4-24 to get the total work done by the transverse loading intensity q :

$$W_2 = Pw_0^2 a^2 \omega \frac{\pi^2}{L^2} \int_0^L \sin^2 \frac{\pi x}{L} dx \int_0^t \sin \omega t \cos \omega t dt \quad (4-25)$$

Rewriting Eqn. 4-25, one obtains:

$$W_2 = \frac{Pw_0^2 a^2 \pi^2}{4L} \sin^2 \omega t \quad (4-26)$$

Similar to Timoshenko scenario in Section 4.1.1, the potential energy W_1 induced by axial force P , the strain energy U and the kinetic energy K are calculated as

$$W_1 = -\frac{Pw_0^2 a^2 \pi^2}{4L} \sin^2 \omega t \quad (4-27)$$

$$U = \frac{EIw_0^2 a^2 \pi^4}{4L^3} \sin^2 \omega t \quad (4-28)$$

$$K = \frac{\rho A a^2 L w_0^2 \omega^2}{4} \cos^2 \omega t \quad (4-29)$$

According to the energy conservation law, total energy is unchanged from $t=0$ to $t=\pi/2\omega$, hence

$$W_1(0) + W_2(0) + U(0) + K(0) = W_1\left(\frac{\pi}{2\omega}\right) + W_2\left(\frac{\pi}{2\omega}\right) + U\left(\frac{\pi}{2\omega}\right) + K\left(\frac{\pi}{2\omega}\right) \quad (4-30)$$

As it can be seen, $W_1 + W_2 = 0$, substitute it into the Eqn. 4-30:

$$U(0) + K(0) = U\left(\frac{\pi}{2\omega}\right) + K\left(\frac{\pi}{2\omega}\right) \quad (4-31)$$

Substitute Eqn. 4-28 and Eqn. 4-29 into Eqn. 4-31:

$$\frac{\rho A a^2 L w_0^2 \omega^2}{4} = \frac{EIw_0^2 a^2 \pi^4}{4L^3} \quad (4-32)$$

$$\rho A L^4 \omega^2 = EI \pi^4 \quad (4-33)$$

The fundamental natural frequency of this prestressed beam is calculated as:

$$\omega^2 = \frac{\pi^4 EI}{L^4 \rho A} \quad (4-34)$$

This result is identical to the fundamental natural frequency of beam without prestressing force which conforms to the conclusion of Hamed and Frostig that the natural frequencies are not changed by prestressing force. For prestressed beams of Type-A, the main beam is exerted only by the axial force which induces the “compression (P- Δ) effect” and reduces its natural frequencies. On the other hand, the tendon deflects with main beam and the transverse interaction induce potential energy W_2 . As mentioned in Section 4.1.1, the potential energy induced by axial force (W_1) represents “compression (P- Δ) effect” and makes fundamental natural frequency of beam decrease. In Type-B, $W_1 + W_2 = 0$. In other words, “compression (P- Δ) effect” is balanced out by the transverse interaction between the tendon and main beam. Therefore, the natural frequencies of the beams of Type-B are not affected.

4.1.3 Beam with external centered straight tendon partly attached to the beam

Before investigating the trapezoidal tendon, a straight tendon with a similar configuration is considered first. This centered straight tendon is fixed to the beam at both ends (E and H) and two points in span (F and G). The length and height of the beam are L and h , respectively. The distance between E and F (equals the distance between G and H due to symmetry) is recorded as f while the length of EG is noted as g (Fig. 4-3). The prestressing force in the tendon is P .

The essential difference among this Hybrid Type and two extreme scenarios (Type-A and Type-B) can be revealed clearly with this model because all three models have centered straight tendon. All parameters are identical except the connection condition between the tendon and main beam (partly fixed, fully detached and perfectly attached, respectively).

Figure 4-3. Beam with external centered straight tendon partly attached to the beam

Similar to Type-B, the forces transferred from the prestressing tendons include two parts: the axial force and the transverse force. The work done by them are represented as W_1 and W_2 , respectively.

According to the geometric relation, the magnitude of the transverse force transferred from tendon on the beam is:

$$P_v = \frac{y_f}{\sqrt{f^2 + y_f^2}} P \quad (4-35)$$

where y_f is the vertical displacement of point F. Since y_f^2 is small compared to f^2 , it could be ignored, and Eqn. 4-35 is rewritten as:

$$P_v = \frac{y_f}{f} P \quad (4-36)$$

Assume $y(x, t) = w(t)\varphi(x)$, $w(t) = w_0 \sin \omega t$, $\varphi(x) = a \sin \frac{\pi x}{L}$. Substitute $x=f$ into this equation:

$$y_f = w_0 a \sin \frac{\pi f}{L} \sin \omega t \quad (4-37)$$

Differentiate Eqn. 4-37 with respect to t :

$$dy_f = w_0 a \omega \sin \frac{\pi f}{L} \cos \omega t dt \quad (4-38)$$

The work done by this transverse load from point F and point G in a short period of time dt can be represented as:

$$dW_2 = 2P_v dy_f \quad (4-39)$$

Substitute Eqn. 4-36, Eqn. 4-37 and Eqn. 4-38 into Eqn. 4-39:

$$dW_2 = \frac{2Pw_0^2 a^2 \omega}{f} \sin^2 \frac{\pi f}{L} \sin \omega t \cos \omega t dt \quad (4-40)$$

Integrate dW_2 to calculate the total work done by P_v :

$$W_2 = \frac{2Pw_0^2 a^2 \omega}{f} \sin^2 \frac{\pi f}{L} \int_0^t \sin \omega t \cos \omega t dt = \frac{Pw_0^2 a^2}{f} \sin^2 \frac{\pi f}{L} \sin^2 \omega t \quad (4-41)$$

The strain energy U , kinetic energy K and the work done by prestressing force in horizontal direction W_1 are identical to Hamed and Frostig scenario in Eqn. 4-28, Eqn. 4-29 and Eqn. 4-27, respectively.

Similar to Eqn. 4-30 in Section 4.1.2:

$$W_1(0) + W_2(0) + U(0) + K(0) = W_1\left(\frac{\pi}{2\omega}\right) + W_2\left(\frac{\pi}{2\omega}\right) + U\left(\frac{\pi}{2\omega}\right) + K\left(\frac{\pi}{2\omega}\right) \quad (4-42)$$

Substitute Eqn. 4-27, Eqn. 4-28, Eqn. 4-29 and Eqn. 4-41 into Eqn. 4-42:

$$\frac{\rho A a^2 L w_0^2 \omega^2}{4} = -\frac{P w_0^2 a^2 \pi^2}{4L} + \frac{P}{f} w_0^2 a^2 \sin^2 \frac{\pi f}{L} + \frac{EI w_0^2 a^2 \pi^4}{4L^3} \quad (4-43)$$

The fundamental natural frequency of the prestressed beam is calculated:

$$\omega^2 = \frac{\pi^4 EI}{L^4 \rho A} - \frac{\pi^2 P}{L^2 \rho A} + \frac{4P}{f} \sin^2 \frac{\pi f}{L} \quad (4-44)$$

There are a few ways to compare Timoshenko scenario, Hamed and Frostig scenario and this hybrid type scenario.

Firstly, when double differentiate Eqn. 2-1 to get Eqn. 2-2, the effect of axial force is in a form of transverse load distributed along the beam. It can be seen that when axial

force is in tension, this transverse load is towards equilibrium position and when axial force is compression, this transverse load is away from equilibrium position. This phenomenon indicates how axial force affects equivalent flexural stiffness of the beam which also called “P- Δ effect”.

In Timoshenko scenario, the tendon is straight and does not touch main beam. So it could be replaced by a pair of axial compression which makes the equivalent flexural stiffness of beam decrease. In Hamed and Frostig scenario, there is transverse interaction between tendon and main beam and its can be calculated by analysing the tendon itself. Eqn. 2-2 is used except changing compression to tension. Hence its magnitude is identical to the effect of axial compression and they just cancel out each other because they have different direction. In other words, the equivalent flexural stiffness of Hamed and Frostig scenario will not be changed. In hybrid type scenario, the beam in Section 4.1.3 is taken as example because its tendon is straight. The prestressed beam with trapezoidal tendon cannot be compared directly to Timoshenko scenario because they have different prestressing tendon profile. In this case, the tendon is replaced by axial compression and two concentrated interaction force at point F and G. The axial compression reduces natural frequency as Timoshenko scenario, but the effect of interaction at point F and G cannot be represented as the form in Eqn. 2-2 hence cannot cancel out the “P- Δ effect” induced by axial compression. So their contributions are demonstrated by the second and third term separately in Eqn. 4-44.

Secondly, for Timoshenko scenario, from Eqn. 4-3, it can be seen that work done by the axial force is non-zero and the second term in Eqn. 4-18 representing second order geometric effect comes from this work. For Hamed and Frostig scenario, the prestressing tendon is replaced by axial compression and transverse interaction. Their contributions are calculated as W_1 (Eqn. 4-26) and W_2 (Eqn. 4-27), respectively. It can be seen that $W_1 + W_2 = 0$ which means their contribution cancel out. Similarly, the contribution of axial compression and interaction in hybrid type scenario are calculated as W_1 (Eqn. 4-26) and W_2 (Eqn. 4-41) and $W_1 + W_2 \neq 0$. This type of prestressed beams is called hybrid type because they are between Timoshenko scenario and Hamed and Frostig scenario.

4.1.4 Beam with external trapezoidal tendon

In this section, a prestressed beam with external trapezoidal tendon is considered as shown in Fig. 4-4. The tendon is put through two deviators at point F and G with two ends fixed at point E and H which makes it a trapezoidal profile. The other points of the tendon are not attached to the beam.

Figure 4-4. Beam with external trapezoidal tendon

The vertical distance between anchor point E and deviator F is h which equals to the height of the beam in this case. The horizontal distance from the left end of the beam E to the left deviator at F is noted as f while the horizontal distance from E to G is g . Due to the symmetry of the beam, the length of EF equals to GH.

Assume the total force of diagonal tendon and the horizontal tendon is identical which has acceptable accuracy for external tendons with frictionless connection at deviators (Pisani, 2018). In the diagonal tendon, the horizontal and vertical component of prestressing force (P_h and P_v , respectively) are applied at both deviators.

The contribution of prestressing tendon in terms of energy contains two parts: 1) the work done by pure prestressing force including horizontal component and vertical component; 2) the strain energy of prestressing tendons. The magnitude of internal force in prestressing tendons would change during vibration (due to length change of tendon) which is considered in the calculation of strain energy of prestressing tendons. The magnitude of the total prestressing force is regarded as unchanged because no prestress loss occurs in this process.

Considering the fundamental natural frequency whose mode shape is symmetric, similar to the model in Section 4.1.1, the equation of motion of the deformed beam is assumed to be:

$$y(x, t) = w(t)\varphi(x) \quad (4-45)$$

where $w(t)$ represents the motion of beam and $\varphi(x)$ is shape function of beam:

$$w(t) = w_0 \sin \omega t \quad (4-46)$$

$$\varphi(x) = a \sin \frac{\pi x}{L} \quad (4-47)$$

w_0 and a are constants representing the amplitude of vibration. ω and L are fundamental natural frequency and length of the beam, respectively. x is the horizontal distance between reference point and left end of beam while t is time.

The displacement of deviator F is represented by vertical component y_f and horizontal component x_f .

Vertical component of displacement of deviator F (y_f) could be obtained by simply substituting horizontal coordinator to equation of motion (Eqn. 4-45):

$$y_f = y(f, t) = w_0 a \sin \frac{\pi x}{L} \sin \omega t \quad (4-48)$$

Horizontal component of displacement of deviator F (x_f) could be obtained similar to the derivation in Section 4.1.1 when we get $L - b = \frac{1}{2} \int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx$ in Eqn. 4-3.

As illustrated in Fig. 4-8, the horizontal distance between left end and point F (deviator) is f when the beam is straight. This distance is changed to b_f when the beam bends (point F moves to F' when the beam bends). The length of this part of beam is s_f .

Figure 4-5. Deformation of beam

The differential of the beam curve can be expressed as:

$$ds_f = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} dx \quad (4-49)$$

The length s_f can be calculated by integrate Eqn. 4-49:

$$s_f = \int_0^{b_f} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} dx \approx \int_0^{b_f} \left[1 + \frac{1}{2} \left(\frac{\partial y}{\partial x}\right)^2\right] dx = b_f + \frac{1}{2} \int_0^{b_f} \left(\frac{\partial y}{\partial x}\right)^2 dx \quad (4-50)$$

Strictly speaking, s_f is smaller than f because the beam is under axial compression. However, this deformation is small compared to the horizontal distance change due to bending (from left end to point F). So it is assumed that s_f equals to f and s_f is used to replace f in Eqn. 4-51.

The horizontal component of displacement of point F (x_f) is $f - b_f$:

$$x_f = f - b_f \approx s_f - b_f = \frac{1}{2} \int_0^{b_f} \left(\frac{\partial y}{\partial x}\right)^2 dx \approx \frac{1}{2} \int_0^f \left(\frac{\partial y}{\partial x}\right)^2 dx$$

$$x_f = \frac{w_0^2 a^2 \pi^2}{4L^2} \left(f + \frac{L}{2\pi} \sin \frac{2\pi f}{L}\right) \sin^2 \omega t \quad (4-51)$$

Similar to the derivative in Section 4.1.3, the kinetic energy, strain energy and potential energy induced by prestressing force are calculated in the following sub-Sections respectively.

4.1.4.1 Strain energy of both tendon and main beam

Firstly, the strain energy of tendon should be derived. The stretched tendon would deform due to the vibration. This length change of the tendon could be represented by geometry parameters of vibration and is independent of the magnitude of prestressing force.

The length change of horizontal tendon (FG part) Δl_h is calculated as:

$$\Delta l_h = [(g + x_g) - (f + x_f)] - (g - f) = x_g - x_f \quad (4-52)$$

where x_g is the horizontal displacement of point G. Similarly, the length change of EF and GH in the horizontal direction is represented as $x_f - 0$ and $x_L - x_g$, respectively. According to the symmetry of the beam, they should be identical:

$$x_f = x_L - x_g \quad (4-53)$$

where x_L is the horizontal displacement of right end of beam and can be obtained similar to x_f shown in Eqn. 4-51:

$$x_L = \frac{1}{2} \int_0^L \left(\frac{\partial y}{\partial x} \right)^2 dx = \frac{w_0^2 a^2 \pi^2}{4L} \sin^2 \omega t \quad (4-54)$$

Substitute Eqn. 4-51, Eqn. 4-53 and Eqn. 4-54 into Eqn. 4-52:

$$\Delta l_h = x_L - 2x_f = \frac{w_0^2 a^2 \pi^2}{4L^2} (L - 2f - \frac{L}{\pi} \sin \frac{2\pi f}{L}) \sin^2 \omega t \quad (4-55)$$

As to the diagonal tendon (EF part), its length change depends on the displacement of both ends. Since the displacement is small, the effect of rotation of beam section at E and F is ignored. Hence the horizontal displacement of point F is x_f while horizontal displacement of E is zero.

The length change of diagonal tendon (Δl_d) is calculated:

$$\Delta l_d = \sqrt{(f + x_f)^2 + (h + y_f)^2} - \sqrt{f^2 + h^2} \quad (4-56)$$

The total strain energy of tendon should be calculated including horizontal part and diagonal part. The length of horizontal part l_h and diagonal part l_d are:

$$l_h = g - f \quad (4-57)$$

$$l_d = \sqrt{f^2 + h^2} \quad (4-58)$$

The strain energy of tendon could be represented as:

$$U_t = 2 \cdot \frac{1}{2} \frac{E_t A_t}{l_d} \Delta l_d^2 + \frac{1}{2} \frac{E_t A_t}{l_h} \Delta l_h^2 \quad (4-59)$$

Substitute Eqn. 4-55, Eqn. 4-56, Eqn. 4-57 and Eqn. 4-58 into Eqn. 4-59:

$$U_t = E_t A_t \left(\frac{2f^2 + 2h^2 + 2fx_f + 2hy_f + x_f^2 + y_f^2}{\sqrt{f^2 + h^2}} - 2\sqrt{f^2 + h^2 + 2fx_f + 2hy_f + x_f^2 + y_f^2} \right) + \frac{1}{2} \frac{E_t A_t}{g-f} (x_L - 2x_f)^2 \quad (4-60)$$

Since x_f^2 and y_f^2 are small value compared to f^2 , h^2 , fx_f and hy_f , they could be ignored:

$$U_t = E_t A_t \left(\frac{2f^2 + 2h^2 + 2fx_f + 2hy_f}{\sqrt{f^2 + h^2}} - 2\sqrt{f^2 + h^2 + 2fx_f + 2hy_f} \right) + \frac{1}{2} \frac{E_t A_t}{g-f} (x_L - 2x_f)^2$$

$$= 2E_t A_t \left[\sqrt{f^2 + h^2} + \frac{fx_f + hy_f}{\sqrt{f^2 + h^2}} - \sqrt{f^2 + h^2 + 2fx_f + 2hy_f} + \frac{(x_L - 2x_f)^2}{4(g-f)} \right] \quad (4-61)$$

Eqn. 4-61 can be transformed to:

$$U_t = 2E_t A_t \left[\frac{-2fx_f - 2hy_f}{\sqrt{f^2 + h^2} + \sqrt{f^2 + h^2 + 2fx_f + 2hy_f}} + \frac{fx_f + hy_f}{\sqrt{f^2 + h^2}} + \frac{(x_L - 2x_f)^2}{4(g-f)} \right]$$

$$= 2E_t A_t \left\{ \left[\frac{\sqrt{f^2 + h^2 + 2fx_f + 2hy_f} - \sqrt{f^2 + h^2}}{\sqrt{f^2 + h^2}(\sqrt{f^2 + h^2} + \sqrt{f^2 + h^2 + 2fx_f + 2hy_f})} \right] (fx_f + hy_f) + \frac{(x_L - 2x_f)^2}{4(g-f)} \right\}$$

$$\begin{aligned}
&= 2E_t A_t \left\{ \left[\frac{2(fx_f + hy_f)^2}{\sqrt{f^2 + h^2}(\sqrt{f^2 + h^2} + \sqrt{f^2 + h^2 - 2fx_f - 2hy_f})^2} \right] + \frac{(x_L - 2x_f)^2}{4(g-f)} \right\} \\
&= 2E_t A_t \left\{ \left[\frac{2f^2 x_f^2 + 4fhx_f y_f + 2h^2 y_f^2}{\sqrt{f^2 + h^2}(\sqrt{f^2 + h^2} + \sqrt{f^2 + h^2 - 2fx_f - 2hy_f})^2} \right] + \frac{4x_f^2 - 4x_f x_L + x_L^2}{4(g-f)} \right\} \quad (4-62)
\end{aligned}$$

The strain energy of the main part of the beam is represented similarly to Eqn. 4-28 in Section 4.1.2:

$$U_b = \frac{E_b I_b w_0^2 a^2 \pi^4}{4L^3} \sin^2 \omega t \quad (4-63)$$

The total strain energy of beam and tendon is the sum of U_b and U_t :

$$\begin{aligned}
U &= \frac{E_b I_b w_0^2 a^2 \pi^4}{4L^3} \sin^2 \omega t + 2E_t A_t \left\{ \frac{4x_f^2 - 4x_f x_L + x_L^2}{4(g-f)} + \right. \\
&\quad \left. \left[\frac{2f^2 x_f^2 + 4fhx_f y_f + 2h^2 y_f^2}{\sqrt{f^2 + h^2}(\sqrt{f^2 + h^2} + \sqrt{f^2 + h^2 - 2fx_f - 2hy_f})^2} \right] \right\} \quad (4-64)
\end{aligned}$$

The expression of y_f , x_f and x_L can be found in Eqn. 4-48, Eqn. 4-51 and Eqn. 4-54, respectively. Since $y(x, t) = w_0 a \cdot \sin \omega_n t \cdot \sin \frac{\pi x}{L}$ and the displacement is small, so $w_0 a$ is small. It can be seen that the strain energy of beam U_b is a quadratic term of $w_0 a$ while some terms in the strain energy of tendon U_t are a higher order of small displacement quantities (x_f^2 , x_L^2 , $x_f x_L$ and $x_f y_f$ terms) and can be ignored.

The total strain energy is simplified as:

$$U \approx \frac{E_b I_b w_0^2 a^2 \pi^4}{4L^3} \sin^2 \omega t + 2E_t A_t \left\{ \left[\frac{2h^2 y_f^2}{\sqrt{f^2 + h^2}(\sqrt{f^2 + h^2} + \sqrt{f^2 + h^2 - 2fx_f - 2hy_f})^2} \right] \right\} \quad (4-65)$$

x_f and y_f are small compared to f and h . Hence the term $\sqrt{f^2 + h^2 - 2fx_f - 2hy_f}$ in the equation above could be rewritten as $\sqrt{f^2 + h^2}$ for simplicity. Then, the equation is shown as:

$$U \approx \frac{E_b I_b w_0^2 a^2 \pi^4}{4L^3} \sin^2 \omega t + \frac{E_t A_t h^2 w_0^2 a^2}{(f^2 + h^2)^{1.5}} \sin^2 \frac{\pi f}{L} \sin^2 \omega t \quad (4-66)$$

4.1.4.2 Potential energy induced by prestressing force

Then, the potential energy induced by prestressing force should be calculated. The prestressing force is P . Let $\frac{w_0^2 a^2 \pi^2}{4L^2} \left(f + \frac{L}{2\pi} \sin \frac{2\pi f}{L} \right) = B$ and $w_0 a \sin \frac{\pi f}{L} = C$, hence $x_f = B \sin^2 \omega t$ and $y_f = C \sin \omega t$. The initial vertical component and horizontal component of the prestressing force in diagonal tendons are noted as P_{v0} and P_{h0} , respectively. The relation between them are demonstrated as Eqn. 4-67 and Eqn. 4-68 according to the geometry.

$$P_{v0} = \frac{h}{\sqrt{f^2 + h^2}} P \quad (4-67)$$

$$P_{h0} = \frac{f}{\sqrt{f^2 + h^2}} P \quad (4-68)$$

The forces P_v and P_h fluctuate with time. The relation between them is represented as:

$$\frac{h+y_f}{f+x_f} = \frac{P_v}{P_h} \quad (4-69)$$

Rewrite Eqn. 4-69:

$$P_h h - P_v f + P_v \left(\frac{P_h}{P_v} y_f - x_f \right) = 0 \quad (4-70)$$

Since the magnitude of P_h and P_v is in the same level in real projects ($\frac{P_h}{P_v}$ is not a small value) while x_f is infinitesimal of higher order compared with y_f , it is reasonable to

assume that x_f is infinitesimal of higher order compared with $\frac{P_h}{P_v} y_f$ too. Hence x_f term could be neglected, the equation is simplified:

$$P_h h + P_h y_f - P_v f = 0 \quad (4-71)$$

$$P_v = \frac{h+y_f}{f} P_h \quad (4-72)$$

Besides, the relation between horizontal component of prestressing force (varying with time) and total prestressing force is shown as Eqn. 4-73:

$$\frac{(f+x_f)^2}{(f+x_f)^2 + (h+y_f)^2} = \frac{P_h^2}{P^2} \quad (4-73)$$

Hence

$$P_h^2 (f^2 + 2fx_f + x_f^2) + P_h^2 (h^2 + 2hy_f + y_f^2) - P^2 (f^2 + 2fx_f + x_f^2) = 0 \quad (4-74)$$

Neglect the infinitesimal of higher order, Eqn. 4-74 is rewritten as:

$$P_h^2 f^2 + P_h^2 (h^2 + 2hy_f) - P^2 f^2 = 0 \quad (4-75)$$

Hence

$$P_h = P \frac{f}{\sqrt{f^2 + h^2 + 2hy_f}} \quad (4-76)$$

Assume the vertical distance between E and F is smaller than half of the distance between beam end and deviator ($2h < f$). As this arrangement of tendon profile is effective on controlling the deflection consequently widely used in real projects, this assumption is reasonable. Then, $f^2 + h^2 + 2hy_f = 2h(\frac{f}{2h}f + y_f + \frac{h}{2})$. In the meantime, y_f is small compared to f and $h/2$. Noted that $\frac{f}{2h}f > f$, hence y_f is small compared to $\frac{f}{2h}f$ too. Then the term y_f could be neglected:

$$P_h \approx P \frac{f}{\sqrt{f^2+h^2}} = P_{h0} \quad (4-77)$$

$$P_v = \frac{h+y_f}{f} P_h = P \frac{h+y_f}{\sqrt{f^2+h^2}} = P \frac{h+w_0 a \sin \frac{\pi f}{L} \sin \omega t}{\sqrt{f^2+h^2}} \quad (4-78)$$

The additional work done by the vertical component of prestressing force is:

$$dW_2 = 2(P_v - P_{v0})dy_f \quad (4-79)$$

Substitute Eqn. 4-67 and Eqn. 4-78 into Eqn. 4-79:

$$dW_2 = 2Pw_0 a \sin \frac{\pi f}{L} \frac{\sin \omega t}{\sqrt{f^2+h^2}} dy_f \quad (4-80)$$

As dy_f is $dy(f,t)$, the increment of y_f in a short period of time t is demonstrated as:

$$dy(f,t) = w_0 \omega a \sin \frac{\pi f}{L} \cos \omega t dt \quad (4-81)$$

Substitute Eqn. 4-81 into Eqn. 4-80:

$$dW_2 = \frac{2P\omega w_0^2 a^2}{\sqrt{f^2+h^2}} \sin^2 \frac{\pi f}{L} \sin \omega t \cdot \cos \omega t dt \quad (4-82)$$

Integrate dW_2 in a period of time t to get the total work done:

$$W_2 = \frac{2P\omega w_0^2 a^2}{\sqrt{f^2+h^2}} \sin^2 \frac{\pi f}{L} \int_0^t \sin \omega t \cdot \cos \omega t dt = \frac{Pw_0^2 a^2}{\sqrt{f^2+h^2}} \sin^2 \frac{\pi f}{L} \sin^2 \omega t \quad (4-83)$$

The work done by horizontal force P_h (W_1) is calculated separately for three segments (EF, FG and GH). For the fundamental symmetric mode, the work done in EF part equals to GH part.

EF part:

$$W_{1EF} = -P_h x_f \quad (4-84)$$

Substitute Eqn. 4-51 and Eqn. 4-77 into Eqn. 4-84:

$$W_{1EF} = -\frac{Pw_0^2 a^2 \pi^2}{2L^2} \frac{f}{\sqrt{f^2+h^2}} \left(\frac{f}{2} + \frac{L}{4\pi} \sin \frac{2\pi f}{L} \right) \sin^2 \omega t \quad (4-85)$$

FG part:

$$W_{1FG} = -P\Delta l_h = -P(x_g - x_f) \quad (4-86)$$

x_f is calculated by Eqn. 4-51 and x_g can be calculated by similar method. Substitute them into Eqn. 4-86:

$$W_{1FG} = -\frac{Pw_0^2 a^2 \pi^2}{4L^2} \left[g - f + \frac{L}{2\pi} \left(\sin \frac{2\pi g}{L} - \sin \frac{2\pi f}{L} \right) \right] \sin^2 \omega t \quad (4-87)$$

Hence the total potential energy induced by horizontal force is:

$$W_1 = W_{1FG} + 2W_{1EF} = -\frac{Pw_0^2 a^2 \pi^2}{2L^2} \sin^2 \omega t \left(\frac{g-f}{2} + \frac{f^2}{\sqrt{f^2+h^2}} + \frac{L}{4\pi} \sin \frac{2\pi g}{L} + \frac{L}{4\pi} \frac{2f - \sqrt{f^2+h^2}}{\sqrt{f^2+h^2}} \sin \frac{2\pi f}{L} \right) \quad (4-88)$$

4.1.4.3 Kinetic energy of main beam

Similar to Hamed scenario in Section 4.1.2, the kinetic energy of the beam is represented:

$$K = \frac{\rho A a^2 L w_0^2 \omega^2}{4} \cos^2 \omega t \quad (4-89)$$

4.1.4.4 Total energy of prestressed beam

The total energy of the prestressed beam with external trapezoidal tendon can be calculated:

$$W_1(0) + W_2(0) + U(0) + K(0) = W_1 \left(\frac{\pi}{2\omega} \right) + W_2 \left(\frac{\pi}{2\omega} \right) + U \left(\frac{\pi}{2\omega} \right) + K \left(\frac{\pi}{2\omega} \right) \quad (4-90)$$

Substitute Eqn. 4-66, Eqn. 4-83, Eqn. 4-88 and Eqn. 4-89 into Eqn. 4-90:

$$\begin{aligned} \frac{\rho A a^2 L w_0^2 \omega^2}{4} = & -\frac{P w_0^2 a^2 \pi^2}{2L^2} \left(\frac{g-f}{2} + \frac{f^2}{\sqrt{f^2+h^2}} + \frac{L}{4\pi} \sin \frac{2\pi g}{L} + \frac{L}{4\pi} \frac{2f-\sqrt{f^2+h^2}}{\sqrt{f^2+h^2}} \sin \frac{2\pi f}{L} \right) + \\ & \frac{P w_0^2 a^2}{\sqrt{f^2+h^2}} \sin^2 \frac{\pi f}{L} + \frac{E_b I_b w_0^2 a^2 \pi^4}{4L^3} + \frac{E_t A_t h^2 w_0^2 a^2}{(f^2+h^2)^{1.5}} \sin^2 \frac{\pi f}{L} \end{aligned} \quad (4-91)$$

Let

$$D = \frac{2}{L} \left(\frac{g-f}{2} + \frac{f^2}{\sqrt{f^2+h^2}} + \frac{L}{4\pi} \sin \frac{2\pi g}{L} + \frac{L}{4\pi} \frac{2f-\sqrt{f^2+h^2}}{\sqrt{f^2+h^2}} \sin \frac{2\pi f}{L} \right) \quad (4-92)$$

Hence

$$\begin{aligned} \frac{\rho A a^2 L w_0^2 \omega^2}{4} = & -\frac{P w_0^2 a^2 \pi^2 D}{4L} + \frac{P w_0^2 a^2}{\sqrt{f^2+h^2}} \sin^2 \frac{\pi f}{L} + \frac{E_b I_b w_0^2 a^2 \pi^4}{4L^3} + \frac{E_t A_t h^2 w_0^2 a^2}{(f^2+h^2)^{1.5}} \sin^2 \frac{\pi f}{L} \end{aligned} \quad (4-93)$$

$$\begin{aligned} \omega^2 = & -\frac{P \pi^2 D}{\rho A L^2} + \frac{4P}{\rho A L \sqrt{f^2+h^2}} \sin^2 \frac{\pi f}{L} + \frac{E_b I_b \pi^4}{\rho A L^4} + \frac{4E_t A_t h^2}{\rho A L (f^2+h^2)^{1.5}} \sin^2 \frac{\pi f}{L} \\ = & \frac{\pi^4}{L^4} \frac{E_b I_b}{\rho A} - \frac{\pi^2}{L^2} \frac{P}{\rho A} D + \frac{4P}{\rho A L \sqrt{f^2+h^2}} \sin^2 \frac{\pi f}{L} + \frac{4E_t A_t h^2}{\rho A L (f^2+h^2)^{1.5}} \sin^2 \frac{\pi f}{L} \end{aligned} \quad (4-94)$$

As it can be seen, the first term of Eqn. 4-94 represents the fundamental natural frequency of beam without the effect prestressing force similar to Type-B in Section 4.1.2 (Eqn. 4-34).

The second term of Type-A in Section 4.1.1 (Eqn. 4-18) and Eqn. 4-94 represents the “compression (P-Δ) effect” of prestressing force. They have similar form except for Eqn. 4-94 has one more parameter D representing the arrangement of the tendon. It is reasonable because the profile of tendon in two models is different..

The third term is induced by the connection between the tendon and the beam at the locations of two deviators, similar to the third term of the previous example of Hybrid Type in Section 4.1.3.

The fourth term indicates the contribution of the tendon itself which is ignored in the beams with straight tendons (Section 4.1.1, Section 4.1.2 and Section 4.1.3).

It should be noted that the rotation of the beam section will induce additional displacement at the points E, F, G and H which is ignored when calculating their displacement for simplicity. This could induce inaccuracy which is illustrated in Section 4.2.3.

This theoretical model of prestressed beam with external trapezoidal tendon is practical in real projects since trapezoidal tendon is widely used. Engineers could substitute the dimension of main beam, arrangement of tendon and magnitude of prestressing force into Eqn. 4-94 to find out the influence of prestressing force on fundamental natural frequency. It is useful for them to check whether the natural frequency is acceptable under the prestressing force in their design.

4.2 NUMERICAL SIMULATION COMPARED WITH THEORETICAL SOLUTION

The deduction in Section 4.1.1 and Section 4.1.2 have identical results with the corresponding theories (Timoshenko theory and Hamed & Frostig theory, respectively) while no existing theory is found on the prestressed beams of Hybrid Type discussed in Section 4.1.3 and Section 4.1.4. A series of finite element models are developed to compare with these theoretical models for verification. The simulations of prestressed beams of Type-A and Type-B are done first to validate the modeling technique and make sure that these numerical models can be used to simulate the effect of prestressing force on the natural frequencies of beams in different scenarios. Then, the finite element models of prestressed beams of Hybrid Type are developed to validate the theoretical solution proposed in Section 4.1.3 and Section 4.1.4.

4.2.1 Modeling of prestress effect and model validation

The modeling process for the prestress installation is described in this Section. After validating this modeling technique by comparing their results with existing theories, the same procedure is applied in all following analysis simulating prestress in this thesis.

Figure 4-6. Configuration of prestressed beam

Two FE models of simply supported prestressed beams are developed. The length, height and width of both beams are 4m, 100mm and 75mm, respectively. The material of the main body of beams is concrete in which Young's Modulus is 26.88GPa and density is 2300kg/m³. A straight steel prestressing tendon with 8mm diameter is arranged at the center of each beam and is simulated by beam element. The Young's Modulus of the steel is 210GPa and the density is 7900kg/m³. The tendon in the first model is detached from the beam to simulate Timoshenko scenario. It is only tied to the beam at both ends. The tendon in the second model is tied to the beam at every node to simulate Hamed and Frostig scenario. Lateral deflection of the beams is restricted to prevent the vibration in the horizontal direction.

Figure 4-7. Finite element model of prestressed beam

The Euler's critical load of the beam can be calculated by $P_{cr} = \frac{\pi^2 EI}{L^2}$ (simply supported). Substitute the dimensions of the beam and P_{cr} can be calculated as 103.7kN. The maximum of prestressing force reaches 90kN which is 86.9% of the Euler's critical load to make sure the change of natural frequencies of the beam under

high level of prestressing force is investigated. The beam under prestressing force of 0N and 45kN (43.4% of Euler's critical load) is also simulated.

The prestressing force is applied with temperature field. A temperature decline of 100°C in the tendon is applied while the expansion coefficient of tendon material is set to be 4.699×10^{-5} (45kN) or 9.399×10^{-5} (90kN). The main beam is not affected by the temperature field thus no shrinkage occurs in it. However, the main beam will be shorter due to the compression exerted by the prestressing tendon consequently causing prestress loss in the tendon. This deformation of the main beam is considered when calculating the expansion coefficient. In other words, the final prestressing force in the tendon after anchorage is 45kN or 90kN, respectively. It is shown in the models that the compressive stress in concrete is 6MPa (12MPa) in the prestressed beam with 45kN (90kN) prestressing force which confirms that the prestressing force is applied properly.

A vertical load is applied to the midspan of the beam in the first step. Then it is deactivated, and the beam is under free vibration. The deflection of the center point at midspan is recorded and analysed using FFT function in Matlab to get the fundamental natural frequency of the beam.

Three groups of models are developed to verify the mesh convergence. The average mesh sizes are 16.7mm, 12.5mm and 6.25mm, respectively. The percentage of difference of the fundamental natural frequency due to mesh is illustrated in Table 4-2. They fit well which indicates that the mesh resolution (16.7mm) is good enough and its results are shown in Table 4-1.

Table 4-1. Comparison of numerical simulation and theoretical solution

The magnitude of prestressing force (kN)	Timoshenko scenario		Hamed and Frostig scenario	
	Theoretical solution (Hz)	Numerical simulation (Hz)	Theoretical solution (Hz)	Numerical simulation (Hz)
0	9.688	9.719	9.688	9.740

45	7.287	7.353	9.688	9.740
90	3.513	3.659	9.688	9.740

Table 4-2. Difference induced by the mesh of numerical simulation (compared to the model with the mesh size of 16.7mm)

Magnitude of prestressing force (kN)	Timoshenko scenario			Hamed and Frostig scenario		
	Mesh size 16.7mm	Mesh size 12.5mm	Mesh size 6.25mm	Mesh size 16.7mm	Mesh size 12.5mm	Mesh size 6.25mm
0	0%	0.33%	0.66%	0%	0.44%	0.87%
45	0%	0.53%	1.06%	0%	0.44%	0.87%
90	0%	0.74%	1.48%	0%	0.44%	0.87%

As it can be seen in Table 4-1, the results of the numerical model and theoretical solution have a good agreement which indicates that the modeling technique can be used to calculate the natural frequencies of prestressed beams with acceptable precision.

4.2.2 Beam with external straight tendon partly attached to the beam

As the finite element models are verified in Section 4.2.1 by comparing the results of Type-A and Type-B with Timoshenko theory and Hamed & Frostig theory, respectively, the same modeling technique is used to simulate a prestressed beam of Hybrid Type whose theoretical estimation on the influence of prestressing force on fundamental natural frequency is proposed in Section 4.1.3.

A 4m beam with a cross-section of 200mm × 400mm is simulated. An 8mm diameter tendon is centrally placed in the beam. Similar to the description in Section 4.1.3, the tendon is fixed to the beam at both ends (E and H) and two points in span (F and G) and the distance between E and F (G and H) is 1m. It is identical to a beam of Type-A except for two points of the tendon (F and G) are attached to the main beam.

The material of the beams is concrete whose Young's Modulus is 30GPa and density is 2500kg/m³. The Young's Modulus of steel tendon is 200GPa. The prestressing force increases from 0 to 15000kN. It is calculated that the Euler's critical load of the beam is 19739kN. The maximum prestressing force in the simulation is 76% of Euler's critical load. This model is developed to simulate the beam shown in Fig. 4-3 in Section 4.1.3.

The mesh convergence of the finite element model in Section 4.2.2 is verified. The average element size of the original model is 66.7mm and two more models with smaller element sizes are developed to study the influence of the mesh on the results of the models. The two verifying models have the average element size of 40mm and 20mm, respectively. The fundamental natural frequency of them under different levels of prestressing force is calculated and compared with the corresponding results of the original model as shown in Table 4-3.

Table 4-3. Difference induced by the mesh of finite element model of the prestressed beam of Hybrid Type (external straight tendon partly attached to the beam)

Magnitude of prestressing force (kN)	Difference		
	Mesh size 66.7mm	Mesh size 40mm	Mesh size 20mm
0	0%	0.78%	1.16%
500	0%	0.80%	1.18%
1500	0%	0.79%	1.17%
5000	0%	0.89%	1.18%
10000	0%	0.73%	1.02%
15000	0%	0.36%	0.46%

As can be seen in Table 4-3, the difference induced by mesh size is less than 1.2%. Hence the mesh used in the original model is fine enough and the mesh convergence study of this model is done.

The fundamental natural frequency of the beam under different magnitudes of prestressing force is shown in Table 4-4 and Fig. 4-8:

Table 4-4. Comparison of numerical simulations and theoretical solutions on the effect of prestressing force on fundamental natural frequency of a beam of Hybrid Type (external straight tendon partly attached to the beam)

Magnitude of prestressing force (kN)	Fundamental natural frequency (Hz)		Error (%)
	Numerical simulation	Theoretical solution	
0	39.14	39.27	0.33
500	39.06	39.18	0.30
1500	38.91	38.99	0.20
5000	38.25	38.32	0.19
10000	37.22	37.34	0.32
15000	36.12	36.33	0.59

Figure 4-8. Comparison of numerical simulation and theoretical solution of the effect of prestressing force on the fundamental natural frequency of a beam of Hybrid Type (external straight tendon partly attached to the beam)

It can be seen that the theoretical solution and numerical simulation has a good agreement as the errors are less than 1%. Both of them demonstrate a slightly decreasing trend. The fundamental natural frequency of the beam reduces from about 39.2Hz to 36.2Hz. The reduction reaches 7.7% of the original fundamental natural frequency which is quite a significant change and cannot be neglected like beams of Type-B. On the other hand, this decrease is much smaller than the beams of Type-A with identical parameters shown in Table 4-5 and Fig. 4-9. It should be noted that two FE models are developed in Section 4.2.1 to simulate prestressed beams of Type-A and Type-B and the results and theoretical solution have good agreement. Hence no specific FE models are made for them in this Section. The theoretical solution of Type-A and Type-B are used in Table 4-5 and Fig. 4-9.

Table 4-5. The fundamental natural frequency of prestressed beams of different types

Magnitude of prestressing force (kN)	Fundamental natural frequency (Hz)			
	Hybrid Type (external straight tendon partly attached to main beam)		Type-A	Type-B
	Numerical simulation	Theoretical solution	Theoretical solution	Theoretical solution
0	39.14	39.27	39.27	39.27
500	39.06	39.18	38.77	39.27
1500	38.91	38.99	37.75	39.27
5000	38.25	38.32	33.93	39.27
10000	37.22	37.34	27.58	39.27
15000	36.12	36.33	19.24	39.27

Figure 4-9. Comparison of the reduction of the fundamental natural frequency of prestressed beams of different types

Three prestressed beams (Type-A, Type-B and Hybrid Type) have identical dimensions, material properties and boundary conditions. The only difference among them is the connection between the tendon and the main beam. It can be seen that a considerable decrease of fundamental natural frequency of the beams of Hybrid Type is observed and it does not follow the prediction of Timoshenko theory. Hence it is necessary to develop specific models for the Hybrid Type prestressed beams to estimate their natural frequency reduction due to prestressing force.

4.2.3 Beam with external trapezoidal tendon

The theoretical model of prestressed beams with external straight tendons is validated by numerical simulation in Section 4.2.2. In this Section, the same modeling technique will be used to develop finite element models to verify the theory proposed for prestressed beams with external trapezoidal tendon.

The concrete beam simulated in this Section has identical parameters with the model in Section 4.2.1. The length of the beam is 4m and the dimension of the beam section is 200mm $\times$ 400mm. The Young's Modulus of the material of the beam and tendon is 30GPa and 200GPa, respectively. The density of the beam is 2500 kg/m³.

The tendon in this Section has a trapezoidal profile as shown in Fig. 4-4. The tendon is fixed to the main beam at two ends (E and H) and two deviators (F and G). The horizontal distance between E and F (G and H) is 1m while the length of the horizontal segment of the tendon (FG) is 2m. Three segments of the tendon (EF, FG and GH) work as a whole and the friction at the deviators is ignored hence the total magnitude of the prestressing force in three segments is the same. The prestressing force increases from 0 to 15000kN (76% of Euler's critical load of the main beam).

The mesh convergence of the model is verified. The average element size of the original model is 66.7mm and two more models with smaller element sizes (40mm and 20mm, respectively) are developed for the mesh convergence verification. The fundamental natural frequencies of the beams under different levels of prestressing force are calculated and compared with the original model. The difference dividing the corresponding result of the original model is indicated in Table 4-6. For example,

when prestressing force is 500kN, the natural frequency of the original model (mesh size 66.7mm) is 39.60Hz while the natural frequency of the model with 20mm mesh size is 39.91Hz, the difference in Table 4-6 is $(39.91-39.60)/39.60=0.78\%$.

Table 4-6. Difference induced by the mesh of finite element model of the prestressed beam of Hybrid Type (external trapezoidal tendon partly attached to the beam)

Magnitude of prestressing force (kN)	Difference		
	Mesh size 66.7mm	Mesh size 40mm	Mesh size 20mm
0	0%	0.36%	0.76%
500	0%	0.39%	0.78%
1500	0%	0.38%	0.75%
5000	0%	0.34%	0.24%
10000	0%	0.05%	0.21%
15000	0%	0.49%	1.08%

The maximum of difference is 1.08% when the prestressing force is 15000kN. This difference is acceptable, and the element size of the original model is small enough.

The fundamental natural frequency of the beam under different magnitude of prestressing force is shown in Table 4-7 and Fig. 4-10:

Table 4-7. Comparison of numerical simulation and theoretical solution on the effect of prestressing force on the fundamental natural frequency of a beam of Hybrid Type (external trapezoidal tendon partly attached to the beam)

Magnitude of prestressing force (kN)	Fundamental natural frequency (Hz)		
	Theoretical solution	Numerical simulation	Error (%)

0	40.29	40.47	0.43
500	40.20	40.39	0.46
1500	40.02	40.24	0.54
5000	39.37	39.59	0.57
10000	38.42	38.38	0.11
15000	37.45	36.92	1.41

Figure 4-10. Comparison of numerical simulation and theoretical solution of the effect of prestressing force on the fundamental natural frequency of a beam of Hybrid Type (external trapezoidal tendon partly attached to the beam)

The errors demonstrated in Fig. 4-10 is larger than Fig. 4-8. As mentioned in Section 4.1.4, additional displacement at deviators and beam ends will be induced by the rotation of beam section which is ignored in this calculation. Since the tendon is centrally placed in Section 4.1.3 (Fig. 4-8), the displacement of two points fixing prestressing tendon is not affected by this section rotation. Hence ignoring it would

not influence the accuracy in this scenario. It can induce error to the beam with trapezoidal tendon especially when the rotation is large.

However, almost all the errors between the theoretical solution and numerical simulation are below 1% which are still acceptable. Good agreement is observed in both Section 4.2.2 and Section 4.2.3 which indicates that this energy method can be used to predict the reduction of fundamental natural frequency due to the prestressing force in these circumstances. What is more, it should be applicable to many other beams of Hybrid Type such as beams with external prestressing tendons with other profiles.

The comprehensive conclusion of the influence of prestressing force on natural frequencies is proposed in Chapter 3. Most types of prestressed beams are considered and can be classified into Type-A, Type-B and Hybrid Type defined in this thesis. The theoretical solutions of Type-A and Type-B have already been given by Timoshenko et al. in 1974 and Hamed & Frostig in 2006, respectively. However, no satisfactory theory is found to study the Hybrid Type which includes almost all the prestressed beams with external tendons.

The qualitative conclusion can be drawn that the natural frequencies of beams of Hybrid Type would decrease in some level since they are between Type-A (decrease trend) and Type-B (no change). Because the different arrangement of tendons and their attachment to the main beams can affect the influence of prestressing force on the natural frequencies, specific models are needed for the quantitative solution of different situations.

The prestressed beam with external trapezoidal tendon as a widely used type of beam is studied as an example. The quantitative estimation of the effect of prestressing force on fundamental natural frequency is given by the energy method. The theoretical solution is verified by some numerical simulations.

Besides, a prestressed beam with a partly attached external straight tendon is studied and compared with beams of Type-A and Type-B with identical dimensions. The reduction of their fundamental natural frequency is demonstrated in Fig. 4-9. It can

be seen that the beams of Hybrid Type have a significant difference from the other two types and should be considered separately.

4.3 ANALYSIS STUDIES OF RELATED REFERENCES

Miyamoto et al. (2000) studied prestressed beams with external trapezoidal tendons and their paper was widely cited (Park et al., 2005, Xiong and Zhang, 2008, Tuttipongsawat et al., 2018, Wang et al., 2018). They proposed theoretical solutions on the influence of external prestressing tendon on natural frequencies of beam and did experiments to verify their results.

4.3.1 Introduction of related research study

In the analytical studies of Miyamoto et al. (2000), the change of the magnitude of force in the tendon and the beam due to flexural vibration was considered. They concluded that both the magnitude of the initial prestressing force and the arrangement of the prestressing tendon would affect the natural frequencies of the beam. The increase of prestressing force would reduce the natural frequencies as so called “compression (P- Δ) effect”.

However, there are two points in their deduction which are questionable. Firstly, the forces transferred from the external trapezoidal tendon to the main beam should include forces at two ends and two deviators because the trapezoidal tendon was attached to the main beam at these four points. However, only the interaction at two ends were considered in their paper. The interactions at deviators should be taken into consideration which was ignored.

Besides, as mentioned above, they emphasized the influence of the change of magnitude of internal forces in prestressing tendons on natural frequencies which was assumed to be simply proportional to the vibration amplitude. Actually, the angle of the diagonal part of tendon would change during vibration which can change the horizontal and vertical component of internal forces in tendon which was ignored.

They conducted experiments to verify their theoretical model. External prestressing tendons were cast on four beams as illustrated in Fig. 4-11.

Figure 4-11. Dimensions and arrangement of external tendons of test specimen (Miyamoto et al., 2000)

The first two beams had smaller eccentricity while the third and fourth had larger eccentricity. Different tendon used in specimens are indicated in Fig. 4-11 too. Impact hammer tests were conducted on the beams before prestressing and the fundamental natural frequencies were found. Then, the flexural stiffness of the composite beam was calculated with the natural frequencies and was substituted to their theoretical models to calculate the natural frequencies of beams with prestressing.

4.3.2 Comparison of proposed theoretical solution and related reference

The parameters needed in the theoretical proposed in this thesis can be found in the paper of Miyamoto et al. (2000) except the flexural stiffness. As mentioned in Section 4.3.2, the flexural stiffness used in their calculation was obtained by Impact hammer tests conducted on beams before tensioning. They can be calculated from their results

and substituted to the analytical solution proposed in this thesis. The results are listed in Table 4-8.

Table 4-8. Comparison of test results and theoretical solutions of Miyamoto and this paper

Specimen	Prestressing force (kN)	Fundamental natural frequency (Hz)		
		Measured	Miyamoto theoretical solution	Proposed theoretical solution
No. 1	0	56.32	56.32	56.5121
	9.8	56.72	57.48	56.5099
	29.4	54.42	57.44	56.5055
No. 2	0	53.40	53.40	53.6728
	19.6	53.45	54.35	53.6681
	39.2	53.22	54.31	53.6636
	58.8	53.10	54.27	53.6589
No. 3	0	52.18	52.18	52.4089
	14.7	54.38	55.60	52.4053
	19.6	54.06	55.59	52.4042
	24.5	54.47	55.58	52.4030
No. 4	0	51.60	51.60	51.8947
	14.7	56.72	54.35	51.8911
	19.6	56.41	54.34	51.8899
	24.5	56.72	54.33	51.8887

Both theoretical solutions of Miyamoto and this thesis indicated that the natural frequencies would decrease with the increase of prestressing force after the tensioning

of the tendon. According to the results of this thesis, the change in natural frequency should be very small under this level of prestressing force. As illustrated in Section 4.2, this change would be more noticeable when prestressing force is in high level compared to the Euler's critical load of the beam. No significant trend was demonstrated in the measured fundamental natural frequencies which make sense from the prediction of this thesis. The difference of the measured natural frequencies could be caused by errors induced in the process of tensioning or measurement.

According to the criteria mentioned in the paper of Nowak and El-Hor (1995), the allowable tensile and compressive stresses in concrete under prestressing force and other load are: 1) $3\sqrt{f'_c}$ (tensile stress, in psi instead of MPa); 2) $0.4f'_c$ (compressive stress). For concrete whose compressive strength (f'_c) is 30MPa (4351psi), the allowable tensile stress is 1.36 MPa (197.88psi) and allowable compressive stress is 12MPa.

According to the FE model of Specimen No. 2 (same dimension with No. 1 but higher prestressing force, 58.8kN), the maximum compressive stress in concrete is 2.06MPa. The maximum tensile stress in concrete is 0.56MPa. Both of them are less than allowable stress.

The maximum compression in steel beam under concrete slab is 18.98 MPa. Since the flange is classified as compact flange (it can be calculated that $b/t < \lambda_p$), local buckling will not occur. Besides, global buckling does not likely to occur to this composite beam. Hence it is reasonable to conclude that the magnitude of prestressing force is governed by the stress in concrete.

When the magnitude of prestressing force increases from 58.8kN to 135.3kN, the maximum compressive stress in concrete is 4.75MPa and the maximum tensile stress in concrete is 1.3MPa which is close to the allowable tensile stress. Hence the maximum of practical prestressing force in this situation should be about 135.3kN.

Similar check has been done to Specimen No. 3 and No. 4. When the prestressing force is 34.5kN, the tensile stress in concrete is 1.3MPa which is close to allowable stress.

It can be concluded that within the practical range, the effect of external prestressing force on natural frequency of this composite beam is negligible. Since the pre-camber caused by prestressing tendon would induce tensile stress to concrete slab and the allowable tensile stress is very small. Actually, this model is more practical to steel structure which will be introduced in the following Section 4.4.

4.4 APPLICATION OF THEORETICAL MODEL OF PRESTRESSED BEAM WITH EXTERNAL TRAPEZOIDAL TENDON ON PRESTRESSED TRUSS

As external prestressing tendons are widely used in steel mega-truss (Ng et al., 2011, Aydin and Cakir, 2015), the results demonstrated in Section 4.1.4 is more useful for these trusses. Hence, the technique of beam analogy transforming the truss into a homogeneous beam with acceptable accuracy is needed. With the beam analogy and the estimation of the effect of prestressing force, the change of natural frequencies of prestressed trusses can be calculated which is very useful in real projects.

4.4.1 Beam analogy

In the theoretical model of prestressed beam proposed in Section 4.1.4, the key parameters of the beam including the arrangement of the tendon, the dimensions, mass and equivalent flexural stiffness of the main beam. The aim of Section 4.4.1 is to build a rectangle section prestressed beam to simulate a prestressed steel truss. A prestressed truss with external trapezoidal tendon is taken as an example as illustrated in Fig. 4-12:

Figure 4-12. Prestressed truss with external trapezoidal tendon

The length and height of the truss are 4m and 0.4m, respectively. An external trapezoidal prestressing tendon is fixed on the truss at two ends and two deviators in the span. The distance between deviator and the adjacent end is 1m for both deviators. The hollow rectangular section is applied to chords and braces. The section of the two chords is 90mm × 90mm × 9mm. The section of the brace is 50mm × 50mm × 4mm. The vertical member at both ends has a solid rectangular section of 80mm × 80mm. The prestressing tendon has a circular section with a diameter of 8mm. Both truss members and tendon are steel whose Young's Modulus is 200GPa and density is 7900kg/m³.

In the process of beam analogy, the length, height of the main truss and the arrangement of prestressing force are unchanged. In order to find out the equivalent flexural stiffness of the truss, a finite element model of the truss is developed. A 3-point bending test on the truss model is simulated. A concentrated load 48kN is applied to the midspan of truss in vertical direction. The vertical deflection of the midspan is 2.02mm. The flexural stiffness of the beam should be identical to the truss; hence it can be calculated by Eqn. 4-83:

$$EI = \frac{FL^3}{48d} \quad (4-83)$$

Where F is the vertical load, L is the length of the truss (beam), d is the displacement of the truss at midspan. Assume the material of the solid beam is similar to concrete whose Young's Modulus is 30GPa. The height of the beam is the same with the truss (0.4m). Hence the width of the beam can be calculated as 0.2m.

An FE model of the solid beam with a rectangular section of 0.2m × 0.4m is developed for verification. Its length is 4m and the arrangement of prestressing tendon is identical to the prestressed truss. Similar 3-point bending test is simulated on it too. The displacement at midspan is 2.01mm under vertical loading of 48000kN. As it can be seen, the equivalent flexural stiffness of the truss and beam are similar.

With the dimension of the beam, its volume can be calculated. As the mass of this solid beam and truss is supposed to be the same, the density of the material of this solid beam can be calculated as 832kg/m^3 .

Hence the procedure of the beam analogy can be summarized as:

1. Simulate a 3-point bending test on the prestressed truss and record the displacement at the midspan;
2. Calculate the equivalent stiffness of the prestressed truss with the result of the 3-point bending test;
3. In order to retain the same arrangement of prestressing tendon, the length and height of the idealized prestressed beam are identical to the prestressed truss. As the flexural stiffness is calculated, the width of the beam can be found.
4. Get the mass of the prestressed truss and calculate the volume of the idealized beam with its length, height and width. Then the density of the material of the idealized beam can be found.

With this procedure, the prestressed beam can be used to simulate the steel truss and all necessary parameters such as flexural stiffness, density, dimensions of beam and arrangement of prestressing tendon are determined for the theoretical model. Substitute these parameters and the magnitude of prestressing force into Eqn. 4-94 and the change of fundamental natural frequency can be evaluated.

4.4.2 Comparison of truss simulation, beam simulation and theoretical model

It has been demonstrated in Section 4.2.3 that the fundamental natural frequency of the prestressed beam would decrease with the increase of prestressing force. The parameters of the solid beam are substituted into the theoretical model and the reduction of fundamental natural frequency under different magnitudes of prestressing force is estimated. In the meanwhile, the FE models of prestressed truss and prestressed “equivalent” solid beam are developed and fundamental natural

frequencies of them are evaluated to compare with the theoretical solution for validation.

The dimensions of the truss and “equivalent” beam can be found in Section 4.4.1. Prestress is applied with temperature drop in the simulation similar to models in Section 4.2. It should be noted that the lateral displacement of every joint of the truss is restricted to prevent buckling in this direction. In real project, there should be a series of trusses connected with links at the joints. Hence this boundary condition (restricting lateral displacement) is reasonable.

Table 4-9. Change of fundamental natural frequencies of different models under prestressing force

Magnitude of prestressing force (kN)	Truss (Simulation)		Beam (Simulation)		Beam (Theory)	
	Natural frequency (Hz)	Change rate (%)	Natural frequency (Hz)	Change rate (%)	Natural frequency (Hz)	Change rate (%)
0	69.14	0	69.48	0	69.85	0
2500	68.69	0.65	68.65	1.19	69.05	1.14
5000	67.95	1.72	67.82	2.39	68.24	2.30
7500	67.27	2.70	67.05	3.49	67.43	3.47
12000	65.92	4.65	65.05	6.37	65.93	5.61

Figure 4-13. Change of fundamental natural frequencies of different models under prestressing force

As can be seen from Fig. 4-13, the fundamental natural frequencies change in similar trend and magnitude. The theoretical model is deduced on a solid beam and it is verified by comparing with numerical simulations of the prestressed beam in both Section 4.2.3 and Section 4.4.2. According to the results illustrated in Table 4-9 and Fig. 4-13, this theoretical model can be used on the prestressed truss.

Firstly, a solid beam is established by conducting beam analogy on the truss as indicated in Section 4.4.1. Key parameters including the equivalent flexural stiffness, the width of beam section and the density of the material are found in this process.

Then substitute the parameters of this solid beam into the theoretical model. The fundamental natural frequencies of the prestressed truss under different magnitudes of prestressing force can be estimated with acceptable accuracy.

CHAPTER 5

EXPERIMENT DESCRIPTION AND EXPLANATION ON CONTRADICTION BETWEEN THEORY AND TEST

5.1 EFFECT OF FRACTURES ON NATURAL FREQUENCIES AND ITS CLOSURE DUE TO PRESTRESSING FORCE

Since prestressing is efficient in concrete structures, many research studies were done on prestressed concrete (Motavalli et al., 2010, Xue et al., 2010, Akguzel and Pampanin, 2011, Hajihashemi et al., 2011, Marti et al., 2014, Chen et al., 2015). As mentioned in the literature review in Chapter 2, most of the tests studying the influence of prestressing force on natural frequencies were conducted on prestressed RC beams too. The experiments conducted by Jang et al. (2010) and Saiidi et al. (1994) showed that the natural frequencies increased with prestressing force, and such results cannot be explained by either Timoshenko's model or Hamed and Frostig's model. According to the theories, the natural frequencies are supposed to reduce by a varying degree (or unchanged) with the presence of prestressing force and should not increase. To the knowledge of the author, there has been no satisfactory theory that can explain an increase in the natural frequencies in a prestressed beam.

A logical explanation is in the direction that somehow the material of the beam has become stiffer with the application of prestress. This judgement would eliminate many factors which could induce error to those tests mentioned above. For example, if the supports were not truly pinned as the idealized model, the measured natural frequencies might be higher or lower than expected. However, this error should be a

constant when prestressing force is changing because no evidence is found that the application of prestress could affect the condition of the roller. In other words, it is independent of prestressing force.

Similar conclusion could be drawn to many other factors. For example, the roller is usually under the beam in tests while they should be at the centroid of end section of beam in idealized model. The prestressing force is usually smaller than expected because of immediate prestressing force loss when the tendon is anchored to the beam end after tensioning. These factors would induce a constant (or nearly constant) error to the test and they will not be changed by the application of prestress.

Besides, there are some unbiased factors. For example, it is difficult to ensure that the position of anchorage of tendon are exactly the same after every tensioning process. Besides, there might be other installation error in test. They would affect the accuracy of experiments but they are not supposed to change the trend completely. In other words, the measured natural frequencies in tests might fluctuate violently due to these factors but should not show a significant increase trend (when prestressing force increase) while existing theories predict that natural frequencies should decrease or unchanged.

Considering the fact that all the tests mentioned above were done on prestressed reinforced concrete beams, the effect of fracture should not be overlooked. However, many of the researchers did not mention the existence of cracks except Saiidi et al. (1994) and Noble et al. (2016). Saiidi et al. (1994) mentioned that a small crack is found in the midspan during handling in their laboratory specimen. Noble et al. (2016) reported that no visible crack was found in their specimens. Nevertheless, cracks could still have existed internally within the concrete of the RC beams, escaping detection from the surface. In fact, the well-known shrinkage cracks tend to occur near the steel bars inside concrete (Chen et al., 2004), as will also be demonstrated in the numerical simulations in Section 5.4 in this paper. These cracks can reduce the natural frequencies of an RC beam in the “original” (un-prestressed) state. Once the prestress is applied, the pre-existing shrinkage type of internal cracks in concrete

would close up, causing the stiffness of the concrete beam to increase and therefore increase the natural frequencies of the beam in a different direction.

In order to verify the above hypothesis, numerical models have been developed to simulate the experiments by Noble et al. (2006) and Jang et al (2010) as their tests are representative examples of prestressed beams of Type-A and Type-B, respectively. Since few details about the curing of their concrete beams were mentioned in their papers, it is assumed that their concrete beams were cured in the regular process in the laboratory. The initial state of the concrete beams, especially the severity and distribution of the shrinkage cracks, is essential for simulating their tests reasonably. To this end, a review of relevant research studies concerning the formation and propagation of shrinkage cracks is carried out, and the results are applied in the numerical simulation.

5.2 INITIAL CONCRETE MATERIAL STATE AND SHRINKAGE CRACKS

In order to investigate the potential influence of prestress on the state of concrete material and the subsequent effect on natural frequency, it is essential to describe initial cracking that could exist in unloaded concrete in a reinforced concrete beam in quantitative terms. In Section 5.2, related research studies are reviewed to estimate the spacing of initial shrinkage cracks. The estimation will be used in the subsequent simulations.

The formation of shrinkage cracks requires energy which can be used to calculate the range of the shrinkage crack spacing (Leonhardt, 1977, Chen et al., 2004). In actual specimens, the final spacing of cracks has some uncertainties due to random defects in concrete materials. For concrete beams with transverse reinforcement, the stirrups induce discontinuity in the concrete and as a result, the shrinkage cracks tend to form near the locations of stirrups (Rizkalla et al., 1982). After the spacing of shrinkage cracks is determined, the depth of cracks may be calculated by the Fictitious Crack Model (Hillerborg et al., 1976). The state of shrinkage cracks is described with spacing and depth calculated.

Leonhardt (1977) proposed a method to calculate the minimum spacing of cracks. Leonhardt pointed out that concrete starts to crack when the load reaches tensile strength. The stress in steel will suddenly increase because of cracking of concrete. Bond-slip could occur if the increase is large enough. It was assumed that the minimum value for the average spacing of crack was the sum of the length with active bond stress and half of the length with almost no bond stress. The value of these two lengths could be calculated by the stress in the steel at the crack immediately after cracking, the diameter of longitudinal reinforcement, the percentage of steel reinforcement and a factor depending on the concrete cover and the spacing between longitudinal bars.

Chen et al. (2004) proposed a theoretical method to predict the shrinkage crack spacing of concrete pavement. They simulated the concrete as an elastic bar which was restrained by reinforcing bars or subgrade as illustrated in Fig. 5-1. The initial strain caused by shrinkage was assumed as constant and the stress in concrete was caused by reinforcing bars or subgrade which prevent the concrete from free shrinkage. The shear force restraining concrete was simulated by a series of springs. The stiffness of these distributed springs was a constant determined by the property of the interface of concrete and reinforcement.

Figure 5-1. Simplified model of concrete beam, reinforcing bars and the connection between them (Chen et al., 2004)

The calculation procedure is as follows. The displacement, stress and strain of every point of the elastic bar can be found so the strain energy of this elastic bar U_B and the energy stored in the distributed springs U_S can be represented separately. Besides, the energy needed for cracking U_C can be evaluated by Fictitious Crack Model (Fig. 5-

2c) which will be introduced in detail in Section 5.3. The displacement of the elastic bar after first cracking can be found considering unloading near the crack. Then the redistributed stress of the elastic bar after first cracking can be calculated. There could be another crack if this redistributed stress reaches tensile strength of concrete. In this way, the spacing of cracks can be predicted.

With this theoretical model, the stress in concrete can be calculated by the distance of the point to the adjacent crack. When it reaches the tensile strength of concrete, the crack tends to form at this point and this distance is the minimum spacing of shrinkage cracks.

Rizkalla et al. (1982) first reviewed many papers on the spacing of cracks and compared their results with experiments. They concluded that predictions using the method by Leonhardt (1977) tended to match test results well when no transverse reinforcements existed in the specimens. They then conducted their own experiments with nine specimens with transverse reinforcement, i.e. stirrups. It was found that in these tests the cracks always propagated from the location of the stirrups. In the specimens where the spacing of stirrups was much larger than the crack spacing predicted using Leonhardt's method, additional cracks would form between stirrups.

The above tests indicated that while Leonhardt's method is generally applicable, the discontinuity induced by transverse reinforcement could disrupt and control the spacing of cracks. It should be noted that the tests by Rizkalla et al. (1982) were conducted on reinforced concrete slabs in tension; however, the influence of transverse reinforcement on the spacing of cracks was similar to the situation with shrinkage cracks.

Besides, it should be clarified that transverse reinforcements do not encourage shrinkage cracks. According to the model of Leonhardt (1977) and Chen et al. (2004), the minimum spacing of shrinkage cracks can be calculated. Shrinkage cracks tend to form at some locations even no transverse reinforcement is put there. For these shrinkage cracks which are supposed to occur anyway, they are more likely to occur at the location of transverse reinforcements.

On the basis of the above review, in the numerical simulations which would be described in Section 5.3, the spacing of shrinkage cracks will be predicted first using both the models by Leonhardt (1977) and Chen et al. (2004). The final spacing will be determined considering the predictions and the spacing of stirrups. When the predictions are close to the spacing of stirrups or multiple of the spacing of stirrups, cracks would form from the location of these stirrups.

5.3. NUMERICAL SIMULATION ON EFFECT OF SHRINKAGE CRACKS IN PRESTRESSED RC BEAMS

5.3.1 General model considerations

This Section presents the numerical simulation of prestressed RC beams to investigate the natural frequencies due to the effect of 1) shrinkage cracks and 2) the closure of cracks in concrete due to prestressing.

The finite element models used in the simulation are developed with ABAQUS. The simulation process is carried out with the following steps. Firstly, shrinkage cracks in the reinforced concrete beams before prestressing are simulated. Then the prestressing tendon is tensioned. The shrinkage cracks will close to different degrees due to different levels of prestressing force. The natural frequencies of the beams before and after the application of the prestressing force are compared, and the results are discussed in conjunction with the experimental observations.

5.3.2 Modeling cracks of concrete in FE model

Many crack theories have been studied. The Fictitious Crack Model proposed by Hillerborg et al. (1976), Modeer (1979), Petersson (1981) is widely used (Dong et al., 2016). According to linear elastic theory, there is infinite stress at the tip of a crack, but this is not the case for concrete because the material in the crack tip zone would be partly damaged before a high level of stress could develop. The Fictitious Crack Model, therefore, defines a fracture zone at the crack tip. Numerous micro-cracks exist in this fracture zone and this zone actually shows plastic behaviour. In the analysis, the fracture zone is replaced by a slit that is able to transfer tensile stress (Fig. 5-2b) and the stress transferring capability depends on the width of the slit in

the Fictitious Crack Model as illustrated in Fig. 5-2c. The width of the slit w varies along the fracture zone and consequently, the stress transferring capability changes at different points in the slit as shown in Fig. 5-2c.

Figure 5-2. Fracture zone and Fictitious Crack Model (Petersson, 1981)

The crack will propagate when the stress transferring capability at the tip falls to zero. The displacement w is noted as w_c when the tensile stress transferring capability falls to zero. This parameter w_c can be calculated using fracture energy G_F by Eqn. 5-1 (Petersson, 1981). G_F is the area of the triangle illustrated in Fig. 5-2c. Both w_c and G_F are material properties of concrete.

$$w_c = \frac{2G_F}{f_t} \quad (5-1)$$

In the finite element analysis, shrinkage of concrete is simulated by a temperature drop. The temperature field is only applied to the concrete while reinforcement bars are not affected. Hence the bars will prevent the concrete from shrinkage. With the restraint from reinforcement bars, tensile stress will occur in concrete, similar to the model of Chen et al. (2004). When the tensile stress is large enough, shrinkage cracks will form and propagate.

The process of crack propagation is simulated by setting the material of concrete. The damage evolution type is displacement and the displacement at failure is w_c which means whether the crack propagates or not is controlled by the relative displacement

near the tip of the crack. When this displacement reaches w_c , the crack will propagate to this point and its capacity of transferring tensile stress will drop to zero.

The fracture energy G_F for most regular concrete is 70-140 N/m (Petersson, 1981) while tensile strength of concrete f_t is usually set as 3MPa, so the critical value w_c can be calculated by Eqn. 5-1 (4.67×10^{-5} - 9.33×10^{-5} m). In the numerical simulation of this paper, w_c is set as 5×10^{-5} m (ABAQUS, 2009). With these settings, a fictitious crack would appear at those locations where tensile stress reaches tensile strength. The tensile stress transferring capability of the fictitious crack keeps decreasing with displacement. When the displacement reaches w_c , it drops to zero and the fictitious crack at this point becomes the real crack.

The shrinkage strain of a concrete beam largely depends on the exposed drying surface area-to-volume ratio and it varies from 100×10^{-6} (all sealed) to 500×10^{-6} (not sealed) (Zhou et al., 2013). It was not mentioned by most researchers doing prestressed beam tests because they did not think shrinkage cracks would affect their results. The shrinkage strain in numerical simulation are taken as 200×10^{-6} (3 sides sealed) for all models because this is common for concrete casting in laboratory and it is shown that this amount of shrinkage strain along with the defects induced by the stirrups outside steel bars is large enough for the formation and propagation of cracks in concrete. By setting the expansion coefficient and a temperature drop in concrete, this shrinkage process can be simulated precisely.

5.3.3 Modeling of shrinkage cracks and model validation

In the present numerical simulation, the extended finite element method (XFEM) is employed to simulate the formation and propagation of “shrinkage cracks” in RC beams.

The enriched shape functions with special characteristics are used in XFEM to bring the discontinuous information to the computational field and it is an efficient numerical method to solve problems with discontinuities (Xu et al., 2016, ABAQUS, 2009). It allows cracks to form and propagate inside the element.

For simplicity, in the present simulation of shrinkage-induced cracks, the stirrups are not explicitly included but they are represented by some defects at corresponding locations of stirrups. The defects replacing stirrups are simulated by an array of shell elements embedded in the model. They act as slits which cannot transfer tensile stress and can transfer compressive stress when setting them as “cracks” in the interaction module in ABAQUS. The depth of each crack is set to be identical to the diameter of the stirrups. Hence the effect of stirrups on concrete continuity is simulated by these slits. When the temperature of concrete drops and the shear force transferred from reinforcement bars prevents the shrinkage of concrete, cracks propagate from some of these slits.

Technically, the cracks should have initiated automatically when using XFEM in ABAQUS. However, after developing many trial FE models, it turns out that the software cannot initiate cracks properly in complicated models. No convergent results can be obtained if the slits representing stirrups are not embedded properly.

Since Rizkalla et al. (1982) have proved the importance of stirrups on crack initiation, it is reasonable to induce these defects representing stirrups. Their conclusion is verified by these trial models because the cracks always tend to initiate from these defects when their spacing is close to the predictions proposed by Leonhardt (1977) and Chen et al. (2004). This indicates that their predictions are reliable. In some circumstances that the spacing of stirrups and the predicted spacing of cracks vary widely. It is probable that no convergent results of these FE models can be found.

To verify the modeling of cracks, a simply supported concrete beam tested by Shardakov et al. (2016) is simulated. The beam is 1200mm long and the distance between the two supports is 1100mm. The section of the beam is 120mm × 220mm. The material is set to be concrete whose Young's Modulus is 35GPa, density is 2400kg/m³ and tensile strength is 3MPa.

a) Dimension of specimen

b) Pattern of crack

Figure 5-3. Test of Shardakov et al. (2016)

Shardakov et al. (2016) conducted 4-point bending on the beam illustrated in Fig. 5-3a. A crack occurred and propagated with the increase of load as shown in Fig. 5-3b. The fundamental natural frequency of the beam without crack is 2069Hz. When the crack propagated under a bending moment of 5kNm, the fundamental natural frequency decreased by 148Hz. The authors also developed a numerical model and the change of fundamental natural frequency with cracks is 109Hz in their model.

In this Section, a similar crack modeling technique introduced in Section 5.3.2 is used. A 1mm depth defect is embedded in the mid-span at the bottom. A bending moment

of 5kNm is added to the beam as the tests of Shardakov and the crack in midspan propagates as illustrated in Fig. 5-4. A small concentrated force in the midspan is added and then removed to induce vibration. The vertical displacement of the centre point of the beam is recorded and analysed by FFT function in Matlab to obtain the fundamental natural frequency. The fundamental natural frequency of the beam before cracking is 2041Hz and the fundamental natural frequency decreases by 181Hz after cracking.

Figure 5-4. The pattern of crack in numerical simulation

The change of fundamental natural frequency due to cracking in the model is compared with tests and they agree well. Hence, the crack modeling technique used in this Chapter can reasonably evaluate the effect of cracks on natural frequencies.

5.4. SIMULATION OF TWO SETS OF PREVIOUS EXPERIMENTS AND COMPARISONS

The tests done by Noble et al. (2016) and Jang et al. (2009) are simulated in this Section. The prestressing tendon is detached to the beam in tests of Noble et al. which makes it similar to Type-A. Meanwhile, the prestressing tendon is fully bonded to the beam in Jang et al.'s tests. Thus, their results will be compared with Type-B model

prediction. Both of their test results contradict the corresponding theories and they will be simulated considering the effect of shrinkage cracks to account for the discrepancy in each case.

5.4.1 Noble et al.'s experiment

Noble et al. (2016) tested 9 concrete beams using both static method and dynamic method. All the beams are simply supported and have prestressing tendons with different eccentricities. Since their static tests were conducted to measure flexural stiffness and then obtained the natural frequency, their dynamic tests result was more straightforward because the natural frequency was directly measured in the tests. Among the 9 specimens of the tests of Noble et al., the one with the centric prestressing tendon is simulated in this thesis. The length of the beam is 2m and the height and width of the beam section are 200mm and 150mm, respectively. The concrete beam is reinforced by two D8 bars at the top and two D12 bars at the bottom. There are 11 H8 stirrups in the beam having an interval of 200mm. A 15.7mm diameter prestressing tendon is placed in a 20mm diameter tube in the center of concrete beam and the prestressing force increases gradually from 0 to 200kN at 20kN increment.

As mentioned in Section 5.3, slits are embedded at reasonable positions for the initiation of cracks in the numerical models. Otherwise, the calculation cannot proceed properly. Hence it is essential to use the theories of Chen et al. (2004), Leonhardt (1977) and Rizkalla et al. (1982) to predict the spacing of shrinkage cracks and embed defects. After the initiation of cracks, the propagation of cracks is calculated by the software automatically following the theory of Fictitious Crack Model proposed by Hillerborg et al. (1976). This process of crack modeling is verified by the finite element model in Section 5.3.3 by comparing with the experiments done by Shardakov et al. in 2016.

According to the theory of Chen et al., tensile stress occurs when the reinforcement bars prevent the shrinkage of concrete. Tensile stress will be released at the location of cracks and accumulates in the concrete. Cracks can only initiate when tensile stress

reaches tensile strength. The accumulated tensile stress of this beam can be calculated using the method proposed by Chen et al. and is illustrated in Fig. 5-5.

Figure 5-5. Tensile stress accumulated in concrete

The vertical axis represents tensile stress in concrete while the horizontal axis represents distance between cracks. It can be seen that tensile stress reaches tensile strength (3MPa) when the distance is 210mm which means the minimum spacing of cracks is 210mm.

Then the theory of Leonhardt is used for verification. The average crack spacing of Noble's specimen is 184mm according to Leonhardt's equation. It should be noted that the minimum of crack spacing calculated in the models of Chen et al. and Leonhardt is based on the assumption that no transverse reinforcement is involved. Since Rizkalla et al. concluded that the transverse reinforcements had an essential influence on cracking and cracks always initiate from those reinforcements when possible, the position of stirrups should be considered when determining the final results of crack spacing.

The spacing of stirrup is 200mm, close to the predicted spacing of cracks. It is reasonable that cracks initiate from every stirrup. The spacing of embedded defects introduced in Section 5.4.1 is 200mm. As mentioned in Section 5.3.2, applying 200

$\times 10^{-6}$ (3 sides sealed) shrinkage strain by defining the temperature drop and the expansion coefficient of concrete, some cracks are found to propagate from the stirrups as shown in Fig. 5-6.

Figure 5-6. Cracks due to shrinkage in Noble's specimen

STATUSXFEM represents the state of the element. STATUSXFEM of an element is 1 when this element is completely cracked. If an element contains no crack, STATUSXFEM of this element is 0. The value of STATUSXFEM lies between 0 and 1. It can be seen in Fig. 5-6 that some elements are labelled as red and their STATUSXFEM is 1 which means they are completely cracked. The crack length is up to 21mm. On the other hand, the blue part contains no crack.

Two prestressing cases are simulated when the beam is under 200kN prestressing force and when there is no prestressing force. The fundamental natural frequency of the beam without prestressing force is 68.49Hz, while it is 69.44Hz when prestressing force increases to 200kN. The change rate is 1.39% while it is 0.96% in the dynamic test of Noble et al. (2016). The results are listed in Table 5-1:

Table 5-1. Comparison of Noble's test and simulation

Noble's test result			Simulation results	
Prestressing force (kN)	Magnitude (Hz)	Change rate (%)	Magnitude (Hz)	Change rate (%)
0	68.66	0	68.49	0
200	69.32	0.96	69.44	1.39

The simulation showed reasonable agreement with tests results which indicated that the slightly increasing trend of natural frequency demonstrated in tests of Noble et al. could be caused by the effect of closure of shrinkage cracks. This influence is not considered in Timoshenko's theoretical solution which could be the reason of their contradiction.

This difference (0.96%) is too small to conclude that the natural frequencies increase with prestressing force. However, as mentioned in Section 5.4, Noble's tests should be classified as Type-A and it can be calculated that the natural frequencies should decrease 1.80% when prestressing force increases from 0 to 200kN. In summary, the discrepancy between test and theory is actually about 2.76%. As discussed at the beginning of Chapter 5, many factors could influence the accuracy of test results but none of them could affect the changing trend of natural frequencies and a change of this amount should be noticeable.

In the paper of Noble et al., 30 signals for each post-tensioning load level were presented. Mean value was calculated and regression analysis was conducted hence the error of their tests result was reduced to a reasonable level. When the natural frequency increases about 3% (2Hz in this case) compared to the prediction of theory, we cannot simply conclude that it is due to the error in tests.

In order to verify this results, one more validation numerical model is developed. In this validation model, the shrinkage cracks are not simulated. Both of them (68.66Hz

and 68.49Hz) are less than the results of this validation model without shrinkage cracks (69.93Hz). It should be noted that all conditions of validation model (without crack) and original numerical models (with cracks) are exactly the same and the results should be accurate. Hence it is reasonable to conclude that cracks could deteriorate flexural stiffness of beam, and natural frequencies of beam will increase when some of these cracks are closed by prestressing force.

The conclusion in this Section may not be practical because the final change of natural frequencies is not significant. Actually, these tests conducted by Noble et al. are just used to study the effect of prestressing force on concrete prestressed beams and the specimens themselves are not very practical (straight tendon which are fully detached to main beam). However, as the typical example of Type-A, the mechanism should be studied. Their natural frequencies would decrease due to “compression (P- Δ) effect” induced by prestressing force and natural frequencies would increase because the prestressing force could close some of the cracks in concrete. If one of the effect predominates, the natural frequencies could change with noticeable amount.

5.4.2 Jang et al.’s experiment

Jang et al. (2009) conducted experiments on bonded prestressed concrete beams in order to investigate the relation between prestressing force and natural frequencies. Since the magnitude of prestressing force cannot be changed once the bonded concrete beam is cast, 6 beams with identical dimensions and material properties are tested.

They are 8m long simply supported beams with a beam section of 300mm $\times$ 300mm. They are reinforced by a D16 bar at each corner. D10 stirrups are arranged with 100mm interval at two ends and 150mm interval at mid-span. The lengths of the strengthen ends and midspan are illustrated in Fig. 5-7. The tendon consists of three strands with a diameter of 15.2mm and is placed at the center of beam section. The sketch of their test is illustrated in Fig. 5-7.

Figure 5-7. Sketch of a test of Jang et al. (2009)

They are pre-tensioned first and then grouting material is injected to the duct containing the tendons. The prestressing forces are 0kN, 146kN, 264kN, 356kN, 465kN and 523kN in 6 beams and the corresponding natural frequencies are 7.567Hz, 8.190Hz, 8.498Hz, 8.672Hz, 8.690Hz and 8.757Hz, respectively.

It can be seen that their fundamental natural frequency showed an increase of 15.7% which is larger compared to tests of Noble et al.

It should be noted that the natural frequency increased by 14.6% when the prestressing force increased from 0 to 356kN and it just increased another 1.1% when the prestressing force is kept increasing to 523kN. It is obvious that the increment of natural frequency is much smaller after the prestressing force reaches 356kN. Hence 356kN as a special point along with 0kN and 523kN are taken as three prestressing load cases simulated in this thesis.

Similar to the tests of Noble et al., the tensile stress in concrete is calculated with the theory of Chen et al. and the results are illustrated in Fig. 5-8.

Figure 5-8. Tensile stress accumulated in concrete

The tensile stress reaches 3MPa in the vertical axis when the distance to adjacent crack is 300mm in the horizontal axis. Hence the minimum spacing of cracks is 300mm in the prediction of Chen et al. Parameters are substituted to reflect Leonhardt's theory and the average crack spacing is calculated as 319mm. Again, the position of stirrups should be considered according to the research of Rizkalla et al. (1982). Since the spacing of stirrup is 100mm at both ends of beam and 150mm in midspan, it is reasonable that cracks propagate at every three stirrups at the beam ends and they propagate at every two stirrups at the midspan.

As mentioned in Section 5.3.2, 200×10^{-6} (3 sides sealed) shrinkage strain is applied to the beam with temperature field. The fundamental natural frequency of the beam is 7.81Hz, 8.93Hz and 8.93Hz when the prestressing force is 0, 356kN and 523kN, respectively. The change rate is 14.3% when the prestressing force increases from 0 to 356kN and it remains stable after it. The results are listed in Table 5-2 and Fig. 5-9:

Table 5-2. Comparison of Jang's test and simulation

Jang's test result			Simulation results	
Prestressing force (kN)	Magnitude (Hz)	Change rate (%)	Magnitude (Hz)	Change rate (%)
0	7.567	0	7.81	0
356	8.672	14.6	8.93	14.3
523	8.757	15.7	8.93	14.3

Figure 5-9. Comparison of Jang's tests and simulations

When the prestressing force increases from 0 to 356kN which is 68% of the maximum prestressing force (523kN), the natural frequency increases 1.105Hz which is 92.9% of the total natural frequency change. After this point, when the prestressing force keeps increasing, the natural frequency remains stable. Both tests results and numerical simulation showed the same trend. This phenomenon could be explained that almost all the cracks are closed when the prestressing force reaches 356kN.

Hence, keep increasing the prestressing force will not increase natural frequency after all cracks are closed. Again, the results of experiments and simulation fit well.

One more validation model is made for tests of Jang et al. too. No crack is simulated in this validation model, and the prestressing force is zero. The other parameters are identical to the numerical model simulating the specimen of Jang et al. in this Section. The fundamental natural frequency is 8.93Hz which is equal to the cracked beam with a high level of prestressing force. Hence, it is reasonable to conclude that the increasing trend of natural frequencies is caused by the closure of shrinkage cracks inside the concrete which was not considered by Hamed and Frostig.

5.4.3 Prestressing application on concrete beams

5.4.3.1 Range of prestressing force magnitude

In real constructions, the magnitude of prestressing force should be in a range considering both the service limit state (SLS) and ultimate limit state (ULS). In other words, the pre-camber induced by the prestressing force should not cause cracks in concrete and the concrete in compression zone should not be damaged by the prestressing force.

For SLS, the tensile stress induced by the prestressing force should be less than $3\sqrt{f'_c}$ (Nowak and El-Hor, 1995) where f'_c is the compressive strength of concrete in psi instead of MPa. For example, if the compressive strength of concrete is 40MPa (5801.51psi), the allowable tensile stress is 228.50psi (1.58MPa).

For ULS, the compressive stress induced by prestressing force should be less than $0.4f'_c$ to prevent concrete from crushing according to the paper of Nowak and El-Hor (1995).

Take the test specimens of Jang et al. as an example. They are 8m long simply supported beams with a beam section of 300mm $\times$ 300mm. Assuming the compressive strength of concrete is 40MPa, the allowable tensile stress is 1.58MPa and allowable compressive stress is 16MPa. Since the prestressing tendons are in the

centre of section, no tensile stress is induced and the magnitude of prestressing force should not exceed $16\text{MPa} \times 300\text{mm} \times 300\text{mm} = 1440\text{kN}$. The magnitude of prestressing force in tests would satisfy this requirement.

However, it is not realistic to locate prestressing tendon in centre of the beam. If the prestressing force is located at the bottom of this beam and the eccentricity is 100mm, the moment (M) induced by the prestressing force equals to Pe where P is magnitude of the prestressing force and e is the eccentricity. The maximum tensile stress in concrete is $\frac{Pe h}{2I} - \frac{P}{A} = \frac{P}{0.09}$ which should be less than 1.58MPa. Hence the maximum prestressing force should be 142kN according to SLS. Meanwhile, the maximum compressive stress in concrete is $\frac{Pe h}{2I} + \frac{P}{A} = \frac{P}{0.03}$ which should be smaller than 16MPa. The prestressing force should be less than 480kN according to ULS. In conclusion, the maximum prestressing force should be 142kN in this circumstance. This example indicates that the prestressing force in real structures would not reach such a high level as in the experiments and their influence would be smaller accordingly.

5.4.3.2 Prestress losses

Prestress losses are another factor that could affect the practicality of the conclusion of Jang's test. If the change of prestressing force is too large, the change of natural frequency could be induced by this inaccuracy.

Prestress losses include short term and long term losses. Short term losses are usually caused by elastic shortening of beam, friction and anchorage slip while long term losses are induced by creep and shrinkage of concrete and relaxation of steel.

Take Jang's test specimens as examples. The tendon consists of three strands with a diameter of 15.2mm. When the prestressing force is 523kN, the tensile stress in tendon is 960MPa. According to AASHTO, the estimate lump sum prestress loss is 221MPa when the compressive strength of concrete is 27.6MPa. In other words, the prestress loss is about 23% in this case. Most of them are long term prestress losses and will not be found in the tests. Hence the test results were not significantly affected by prestress losses.

However, the prestress losses in real prestressed beam could be about 20% which makes the conclusion of Jang's tests inaccurate. According to the test results, the natural frequency increased 15.7% when prestressing force increased to 356kN. In real project, the actual prestressing force would be about 20% off and the natural frequency will increase $15.7 \times 0.8 = 12.56\%$ (it is assumed to be linear before all cracks closed according to Fig. 5-9). It should be noted that this amount of increase is still large and should be considered.

In summary, the prestress losses could affect the accuracy of Jang's conclusion in real project, but the change of natural frequency are not negligible. It is necessary to conduct dynamic tests on those critical members especially when its prestressing force is large.

5.4.4 Summary of experiments of Noble et al. and Jang et al.

Noble's tests and Jang's tests as two representative experiments corresponding to Type-A (Timoshenko theory) and Type-B (Hamed & Frostig theory) are chosen in Section 5.4. As both theoretical solutions underestimated the natural frequencies of prestressed beams, it is reasonable to deduce that there might be factors which were ignored by both theories. Cracks in concrete could be one of them because material is assumed to be homogeneous in both theories.

Besides, the magnitude of increment of fundamental natural frequency in these two tests differ greatly from each other (0.96% in Noble's tests and 15.7% in Jang's tests) which indicates that the change of natural frequencies of prestressed beam could be significant in some beams. It is negligible in some cases but this does not mean that it could be ignored in all beams.

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

6.1 INTRODUCTION

In this study, the influence of prestressing force on natural frequencies of beams is investigated. It can be seen in the literature review that there are three different opinions on this problem (natural frequencies tend to decrease, no change or increase). Two representative theories proposed by Timoshenko et al. (1974) and Hamed & Frostig (2006) are deduced mathematically. Timoshenko et al. studied the effect of external axial force on the natural frequencies of beams. This scenario described the “compression (P- Δ) effect” which decreased the natural frequencies of beams. Their work is widely cited to support the view that the natural frequencies of prestressed beams would decrease. Hamed and Frostig (2006) did theoretical investigations on prestressed beams in more general situations and concluded that the natural frequencies would not be affected by prestressing force. Their conclusion is also widely accepted based on the rigorous theoretical deduction they proposed. Besides, many experiments on concrete prestressed beams indicated an increasing trend of natural frequencies with an increase of prestressing force which contradicted either of the mentioned theory. The task of the research is to propose a coherent conclusion on this problem, harmonize the existing theories and experiment phenomenon and explain all these contradictions. In order to fulfill these objectives, this study is carried out in three phases.

Firstly, the theories of Timoshenko et al. and Hamed and Frostig are integrated and their scopes of application are classified from a prestressing point of view. It is widely

accepted that both theories are correct in their respective circumstances. However, a common misunderstanding lying in the scope of applications of the two theories is that Timoshenko theory is applicable to unbonded prestressed beams while Hamed and Frostig theory is used in bonded prestressed beams. Actually, it should be the real interaction (the definition of “real interaction” can be found in Section 1.1 of this thesis) between the tendon and main beam in span (perfectly separated or fully attached) instead of the prestress technique (bonded or unbonded) that distinguish the beams of different types. With this classification method, the prestressed beams can be placed into different categories using corresponding theories with no ambiguity. So the two theories are harmonized and this is explained by some rigorous theoretical investigation in Chapter 3.

As the proposal of a new standard of classification of beams, many prestressed beams with external tendons cannot be classified into either of the two theories. The interaction between the tendon and main beam in the span in the two existing theories indicates two extreme scenarios; the tendon is fully detached from the main beam in Timoshenko theory (Type-A) while the tendon is perfectly attached to the main beam in Hamed and Frostig theory (Type-B). Many of the external prestressing tendons are partly attached to the main beam. For example, the external trapezoidal tendon is attached to the main beam at two deviators while the other points of the tendons are separated from the beam. This Hybrid Type of prestressed beam is seldom considered in existing research studies and theoretical models are needed.

Hence the study on these beams of Hybrid Type is conducted in the second phase of this research. As the influence of the prestressing force on natural frequencies differs for a different arrangement of prestressing tendons, the model of a typical example (prestressed beams with external trapezoidal tendon) is investigated in Chapter 4. Both theoretical investigation and numerical simulations are carried out. The energy method used in Chapter 4 can be used in many other prestressed beams with different arrangements of tendons, so it is the key to study the prestressed beams of Hybrid Type.

A special case of this Hybrid Type (prestressed beams with external straight prestressing tendon partly attached to the main beam) is studied by this method in Chapter 4. An expression is proposed which shows that the fundamental natural frequency of prestressed beams of this configuration would decrease but the change is not as much as the prediction of Timoshenko et al. In other words, the reduction of prestressed beams of Hybrid Type is between Type-A and Type-B.

Hence, a coherent and comprehensive conclusion on prestressed beams of different types is drawn from a prestressing point of view. The standard to classify different types of them is proposed. Besides, the Hybrid Type, a series of scenarios seldom considered before, is studied.

Finally, the contradiction between the theories and experiments is explained. According to theoretical solutions, when prestressing force increases, natural frequencies of prestressed beams of Type-B are unchanged, while natural frequencies of those beams of Type-A and Hybrid Type decrease in different levels. However, the experiments showed an increasing trend of natural frequencies which cannot be explained by any of the theories. The cause of the diverging experiment evidence is found to be the existence of shrinkage cracks. Some finite element models are developed to simulate the experiments done by Noble et al. (2016) and Jang et al. (2009). According to the description of their tests, Noble's tests should be classified as Type-A while Jang's tests should be Type-B. In other words, the measured natural frequencies in Noble's tests should decrease while they should be unchanged in Jang's tests. However, no statically significant relationship between prestressing force and natural frequencies was found in the former while the natural frequencies increased with prestressing force in the latter. Neither of them can be explained by corresponding existing theories. The closure of the shrinkage cracks due to prestressing force is simulated in these models and the results show good agreement with their tests. The material nonlinearity is not considered in these theories which could explain these diverging tests results.

6.2 CONCLUSIONS

6.2.1 The classification standard of prestressed beams of different types

Since the Timoshenko theory is deduced in prestressed beams under external axial force, while Hamed and Frostig theory is considered in prestressed beams with random tendon profiles, they are not directly comparable. In order to emphasize the essential differences between the prestressed beams of these two types, a typical Timoshenko beam and an example of Hamed and Frostig beam are studied. Both of them are unbonded prestressed beams with straight tendon. However, one of them has the tendon fully detached from the main beam while the other one has the tendon perfectly attached to the main beam. Solid evidence has been provided from the analytical investigation that the key character distinguishing prestressed beams of two types is the real interaction between the tendon and the main beam instead of prestress technique (bonded or unbonded) as commonly believed. Hence, the prestressed beams can be assorted into Type-A, Type-B and Hybrid Type with this classification standard.

The classification method not only eliminates misunderstanding on the scopes of application of existing theories but also proposes the definition of prestressed beams of Hybrid Type. The Hybrid Type is a group of prestressed beams with tendons partly attached to the main beam. Neither of the mainstream theory is applicable to the prestressed beams of this type and no satisfactory model is found for them.

6.2.2 The influence of prestressing force on the fundamental natural frequency of typical beams of Hybrid Type

Generally speaking, the connection between the tendon and main beam of Hybrid Type is stronger than Type-A but is weaker than Type-B. Hence it is reasonable to give a qualitative conclusion that the natural frequencies of prestressed beams of Hybrid Type will decrease but not as much as Type-A prestressed beams with identical parameters. This conclusion is validated by the analytical model of a prestressed beam with straight tendon partly attached to the main beam in Chapter 4. Besides, the external trapezoidal tendon as a widely used profile of prestressing tendons is studied and the analytical solution of the fundamental natural frequency

reduction is given. Both of the beams are investigated by energy method which can be used in many other prestressed beams with different profiles of prestressing tendons.

Finite element models are developed to verify the beams of Hybrid Type studied in Section 4.1.3 and Section 4.1.4. Both of them are 4m beams with the cross-section of $200\text{mm} \times 400\text{mm}$. The diameter of prestressing tendon is 8mm. It is a straight tendon with two points (F and G) fixed to the main beam in the former beam while it is an external trapezoidal tendon fixed at two deviators in the latter beam. Good agreement is found when the theoretical solution and numerical simulation of the beams are compared.

Till now, the two existing theories are harmonized and some typical examples of the new category are studied. Finally, the coherent and systematic conclusion is given.

6.2.3 Contradiction between the theoretical solution and experimental investigations

Experiments have been done to study the influence of prestressing force on natural frequencies and many of them indicated that the natural frequencies tend to increase with the increase of prestressing force which cannot be explained by theories. Most experiments on this problem are conducted on concrete prestressed beams. As the material of beams is simply considered as homogeneous in the theoretical investigation and cracks are very common in concrete material, these cracks in the concrete are one potential reason causing this contradiction between theories and tests.

In order to verify this hypothesis, two representative experiments done by Noble et al. in 2016 and Jang et al. in 2009 are simulated in Section 5.4. In Noble's test, the prestressing tendons are completely separated from the main beams in the span. On the other hand, the prestressing tendons are fully attached to the main beam in Jang's tests. Hence they should be assorted as Type-A and Type-B, respectively. Both of their test results demonstrate an increasing trend of natural frequencies when the prestressing force increases which contradicts corresponding theoretical solution.

The tests of Noble et al. and Jang et al. are simulated by numerical models considering the existence of shrinkage cracks. It is shown in the finite element models that these cracks are closed as the prestressing force increases consequently the natural frequencies of the beam increase. In the experiment of Noble et al., the fundamental natural frequency increased by 0.96% when the prestressing force reached 200kN. This increase of fundamental natural frequency is 1.39% in the numerical simulation. In the Jang et al.'s tests, the fundamental natural frequency increased 14.6% when the prestressing force increased from 0kN to 356kN. Then the prestressing force kept increasing to 523kN, but the fundamental natural frequency remained stable (increased to 15.7%). A similar situation is found in the finite element model. An increase of 14.3% of the fundamental natural frequency is found when the prestressing force reaches 356kN. Then the fundamental natural frequency is unchanged. This phenomenon indicates that the cracks in the concrete are completely closed.

Hence the conclusion about this problem can be proposed now. When the main beam is made of steel or material which can be regarded as homogeneous, the natural frequencies of beams will decrease or stay unchanged due to prestressing force depending on the real interaction between tendons and the main beam. When the material is not homogeneous such as concrete with shrinkage cracks, the natural frequencies may increase because of the closure of cracks. Specific numerical models are needed for these concrete prestressed beams for exact results.

6.3 RECOMMENDATIONS FOR FUTURE RESEARCH WORK

1. Since external prestressing tendons are widely used in steel mega-truss instead of solid beams, the results demonstrated in Chapter 4 is more useful for these trusses. Although a simple method of beam analogy transferring the truss into a homogeneous beam with acceptable accuracy is provided in Section 4.4, the behaviour of local members under a high level of stress may influence the natural frequencies too. This change of natural frequencies of prestressed trusses can be studied which is very useful in real projects.

2. More specific theoretical models predicting the natural frequencies of the prestressed beams with different tendon profiles of Hybrid Type as Chapter 4 can be developed. Besides, the model can be improved by considering the rotation of the beam section to obtain better accuracy.

3. The experiments done by Noble et al. and Jang et al. can be improved. Two series of tests can be done in future work. Firstly, the concrete prestressed beams without reinforcement bars can be tested. Without the restriction from reinforcement bars, the beams can shrink freely so no shrinkage cracks will occur. In addition, the section of these beams should be small, and the concrete is cured properly to prevent the formation of other potential cracks. In this circumstance, the influence of prestressing force on natural frequencies can be measured without the disturbance of cracks. Secondly, a series of full-sized concrete prestressed beams can be tested to study the effect of prestressing force in real projects. Parametric study can be conducted based on the tests results. The length, section and reinforcement ratio are key parameters affecting the formation of cracks and should be studied.

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