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2014

La, Q. D., Chew, Y. H., & Soong, B. H. (2014). Oligopolistic spectrum allocation game via market competition under spectrum broker. *Computer Networks*, 70, 225-239.

<https://hdl.handle.net/10356/104988>

<https://doi.org/10.1016/j.comnet.2014.05.009>

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Oligopolistic Spectrum Allocation Game via Market Competition under Spectrum Broker

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Abstract

For future dynamic spectrum access (DSA) systems, new spectrum management methodologies will be adopted such that the licensed primary spectrum holders can reallocate part of their available bandwidth to the unlicensed secondary service providers for profits, based on market-driven mechanisms. This is known as spectrum market or spectrum trading. This paper aims to propose a dynamic spectrum market model where each spectrum holder has a limited amount of spectrum and is allowed to enter a portion of its available bandwidth into the market managed by a spectrum broker, besides its primary services. To model the price dynamics, a continuous-time price adjustment process governed by a differential equation is considered. Subsequently, we show that the problem is a dynamic N -player oligopoly differential game, subject to the bandwidth constraint. We analyze the feedback Nash equilibrium (NE) solutions for the general game and provide a complete, closed-form solution for the special symmetric case. The solution can be characterized into three distinct regions and transitions between these regions may occur as time evolves, which will be thoroughly investigated. In addition, we propose a discrete-time price adjustment implemented at the spectrum broker. Extensive numerical studies are provided to investigate various aspects of the proposed competition.

Keywords:

Spectrum allocation, spectrum market, differential game, Markov Nash equilibrium, market equilibrium

1. Introduction

Traditionally, the radio spectrum has been licensed to operators via a static allocation approach. However, it is also a fact that the occasional and intermittent nature of such primary transmission has led to the under-utilization of spectrum at any given location and time and resulted in the presence of spectrum holes. Dynamic spectrum access (DSA) [1] is one recent

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concept brought up in order to improve the efficiency of spectrum usage and relieve wireless users from spectrum shortage. It promotes spectrum sharing through the development of new wireless network protocols and new business models to enhance spectrum utilization. The development of cognitive radio technology [2] serves as an ideal platform for realizing DSA, which allows for smart, autonomous spectrum access protocols to be integrated into the cognitive radios besides their standard communication interface. Under the DSA paradigm, the concept of spectrum markets [3] has been proposed, whereby the existing licensed primary spectrum holders (PSHs) are allowed to reallocate its excessive bandwidth by selling or leasing to secondary service providers (SSPs) for monetary gains¹, which at the same time improves the overall spectrum efficiency. The study by Yoon *et al.* [5] in fact suggested that economic welfare for both spectrum sellers and buyers can be increased with secondary spectrum trading. As a price is charged for the rights to use the licensed spectrum, the establishment of a spectrum pricing model and market mechanism is of key consideration. Therefore, micro-economic models and game-theoretic techniques can be useful for the analysis of market-driven spectrum allocation.

1.1. Related Work

A number of approaches to spectrum market have been proposed based on well-known economic models. Niyato *et al.* [6] examined a scenario where multiple SSPs and one PSH played a game subject to market rules under the Cournot model. The oligopoly market among multiple PSHs was studied via a dynamic Bertrand game in [7] and [21]. Another classic economic model, the Stackelberg leader-follower game, was adopted by Wang *et al.* [9]. There is a rich literature in auction-based spectrum sharing (e.g., [10, 11, 12, 14, 15] and references therein), which is another notable economics-driven approach in which the resources are regarded as divisible goods which can be charged upon allocation via auction. In addition, Byun *et al.* [13] employed an inventory model to help a PSH to decide on the optimal amount of spectrum to trade in order to minimize its economic costs.

A crucial aspect of spectrum markets is how the spectrum can be priced among the competing players. Three different pricing models, including market-equilibrium pricing, competitive

¹The benefits are most often in terms of monetary gains, although there are exceptions. For example, in [4], the PSHs agreed to lease spectrum to the SSPs as long as these unlicensed operators were committed to cooperate with them in relaying their transmission.

pricing and cooperative pricing, were studied in [8]. Ileri *et al.* [16] suggested that the players offered spectrum through demand-responsive pricing based on an acceptance probability. Here, the authors also employed the concept of a spectrum server that mediates spectrum transactions. On the other hand, Xing *et al.* [17] investigated the price dynamics for two different buyer types, i.e., the quality-sensitive and the price-sensitive. Furthermore, Isiklar *et al.* [18] assumed that spectrum price and demand could be modeled based on the spatial distribution of the SSPs on a unit interval. Meanwhile, Kasbekar *et al.* [19] investigated a spectrum pricing game with valuation uncertainty from the buyers.

Most of the aforementioned schemes focused on the secondary markets among licensed primary users and unlicensed cognitive users. In [3], Berry *et al.* discussed the possible emergence of a two-tier market structure, with the involvement of an upper-tier trading among spectrum owners to reallocate spectrum resources. A relatively similar model was considered in [20], in which the service providers go through a spectrum acquisition process, prior to the actual duopoly to attract end-users. While the large bodies of literature seem to emphasize the competition for bandwidth directly among primary and secondary cognitive radios, few works have examined the “upper” tier involving spectrum owners and secondary service providers which is the focus of this paper. In such a two-tier approach, after a decision at the upper tier has been made, the secondary or primary operator can then perform optimization independently on their individual network which is the design objective of the lower tier. Optimization in the lower tier is a separate problem from the upper tier and is outside the scope of this paper.

It is also noticed that prior works relied on discrete-time update rules to model the price dynamics. Although discrete-time processes can be easily implemented in real-time, they do not describe the exact time-varying behaviors of price change due to the dynamics of competition and interaction in the spectrum market. In order to characterize the dynamics in a deterministic manner, one may assume a continuous-time model where the market’s movement is akin to a dynamical control system, with market price as the state variable, governed by a differential equation. The dynamic sticky-price oligopoly [26] is one of such models that could be useful in formulating the spectrum market and would be adopted for our model.

1.2. Paper Contributions and Outline

As mentioned earlier, this paper investigates the spectrum allocation among licensed and unlicensed providers via market competition, i.e., the upper-tier market in [3]. The multiple PSHs are the players of the game who try to allocate portions of their bandwidth to other SSPs by participating in the secondary market. Unlike some existing schemes where the players propose their own prices (e.g., [7, 16, 19, 20, 21, 22]), we consider an alternative market model in which the players compete non-cooperatively and simultaneously in terms of their output levels (i.e., offered bandwidth) under a common market price moderated by the spectrum broker. Such competition is believed to be more suitable for capacity-constrained players [23]. Moreover, based on the stickiness assumption² [26], we are able to capture the exact continuous-time behaviors of the price dynamics, which is an advantage of such formulation compared to discrete-time counterparts. As such, the oligopolistic competition is an N -player differential game [24] and we can characterize its solution by the feedback Markov NE. In practice, such dynamics can also be approximated by discrete-time price adjustment, which can be easily implemented at the spectrum broker.

In summary, the paper's key contributions are as follows.

- Market-based spectrum allocation formulated as an oligopolistic differential game is presented. We discuss the general heterogeneous N -player game and illustrate how to derive the closed-form NE by using the symmetric N -player scenario. The equilibrium may lie in several regions depending on the price level and transitions may occur as time elapses, which will be thoroughly analyzed.
- Discrete-time price mechanism and protocol are proposed for the purpose of implementing the game in practice. The conditions for the adjustment interval will be derived.
- Numerical studies are carried out to gain insight into the system behaviors. The impacts of parameters such as the number of players or maximum bandwidth are addressed.

The rest of this paper is organized as follows. Section 2 introduces the market model and the spectrum allocation game. In Section 3, we discuss the general outcomes of the N -player

²It is assumed that the price changes continuously and smoothly as a function over time, i.e., price is said to be *sticky*.

Figure 1: The spectrum market model, with multiple PSHs and multiple SSPs in the presence of a spectrum broker.

oligopoly. Next, Section 4 gives the complete analytical solution to the symmetric N -player case. Section 5 focuses on the discrete-time price adjustment. Simulation results are provided in Section 6. Finally, Section 7 concludes the paper.

2. System Model and Problem Formulation

First of all, we describe the spectrum market model under investigation, as depicted in Fig. 1. The system consists of N PSHs currently deploying wireless services in the same geographical area, e.g., operators providing wireless access to a group of subscribers, which we refer to as their primary spectrum usage. The PSHs are allowed to lease portions of their available bandwidth to other unlicensed service providers (i.e., SSPs) in exchange for monetary profits. By entering this spectrum market, the PSHs become oligopolists (i.e., sellers in a market dominated by a small number of firms) and compete among themselves to maximize monetary profits.

2.1. The Spectrum Broker

In the system model, we assume the presence of a spectrum broker. In practical deployment, the spectrum broker could be an authorized agent from the regulatory bodies in a particular geographical region. It could be set up as a centralized platform running the appropriate brokering mechanisms and protocols that monitor the transactions in the market. The advantages of having such an entity in the system are evident:

- The broker can act as a mediator between the PSHs and the SSPs, synchronizing and coordinating the activities of the spectrum market.
- The broker can also announce the spectrum price governed by the market demand (resulting from all SSPs) and supply (resulting from all PSHs) to the players and adjust it over time, thus functioning as a market controller.
- Equivalently, it is a resource allocator which manages the spectrum assets via market-driven mechanism. Network protocols and algorithms can be executed at the broker.

The underlying principles of the Cournot oligopoly [23], i.e., the oligopolists producing homogeneous goods which are sold at a common market price, can be expected to hold for this model. This is especially true if the PSHs all own the same types of infrastructures, such as cellular operators. The SSPs are acquiring the spectrum from the PSHs in order to provide their own wireless services. Thus, they are only interested in the physical bandwidths offered by the PSHs. If the infrastructures of all the PSHs are assumed to be capable of meeting the SSPs' QoS requirements, then the SSPs have no incentives to differentiate among the spectrum offered by different PSHs. Such equal preferences from the buyers in spectrum markets have also been assumed by existing works, e.g., in [18]. Under these conditions, the SSPs will behave like price-takers who go along with the market price, and the market can always be cleared (i.e., selling out all offered bandwidth) by the spectrum broker. As such, the SSPs play a passive role in this model and can be collectively represented by the spectrum broker.

2.2. Dynamic Oligopoly

Under the previous conditions, we assume that each PSH i has a limited amount of bandwidth B_i , from which a portion b_i can be offered to the secondary market, and $B_i - b_i$ units are

allocated to primary services. A unit bandwidth is sold at market price p , which is monitored by the spectrum broker. Then, each oligopolist i determines its strategy in terms of its offered bandwidth or output b_i , which in turn controls the market price level, according to

$$p = f(b_1, b_2, \dots, b_N) \quad (1)$$

where the function f is called the *inverse demand function*, reflecting the cumulative need of the SSPs, which should be continuous and differentiable with respect to all b_i . Player i is assumed to maximize the net profit $U_i = R_i - C_i$, where $R_i(p, b_i)$ and $C_i(p, b_i)$ are respectively its revenue and cost functions, dependent on both the market price as well as the player's own strategy, which we will define later.

The model described above is a simple static market mechanism. Static market model has its shortcomings as it does not address the dynamic process in which market price changes, but only the outcomes before and after the changes. In reality, the price, and hence the players' bandwidth supplies, hardly change abruptly, which necessitates the use of stickiness assumption in the dynamic oligopoly model [26]. In dynamic markets, the competition takes place over time. The system as a whole can be modeled as a differential game [24] where the time-dependent price $p(t)$ acts as a *state variable* and the outputs b_i , which can be dependent on both time t and state p , interact dynamically with the system to control the movement of $p(t)$. In differential games, the general strategy b_i can be categorized into several types which may or may not be dependent of the current price or state variable. For autonomous games with an infinite horizon [24] such as this game, we are interested in the *stationary Markov* strategy, defined as follows.

Definition 1. The strategy function b_i of player i can be classified as stationary Markov if $b_i \equiv b_i(p(t))$, which is solely a function of the current state.

At time t , the inverse demand function (1) leads to a desirable price level

$$\hat{p}(t) = f(b_1(p(t)), b_2(p(t)), \dots, b_N(p(t))). \quad (2)$$

However, this quantity is generally not equal to the current price $p(t)$. As a result, $p(t)$

adjusts towards $\hat{p}(t)$, not in an abrupt manner but gradually over time, in response to the difference $\Delta p(t) = \hat{p}(t) - p(t)$, following the trajectory equation

$$\dot{p}(t) = \frac{dp(t)}{dt} = G(\Delta p(t)), \quad p(0) = p_0 \quad (3)$$

where $G(\cdot)$ is a function of Δp , which must satisfy $G(0) = 0$ and $dG(u)/du > 0, \forall u$ [26].

A player's instantaneous payoff can be given by $U_i(p(t), b_i(p(t)), b_{-i}(p(t)))$ where b_{-i} denotes the joint strategies of players other than i . However, its objective now is to maximize the accumulated payoff, discounted over time by a discount rate $r \geq 0$ (which signifies diminishing valuation of the payoff over time), i.e.,

$$J_i(p(t), b_i(p(t)), b_{-i}(p(t))) = \int_0^\infty e^{-rt} U_i(p(t), b_i(p(t)), b_{-i}(p(t))) dt. \quad (4)$$

2.3. Linear Constrained Sticky-price Oligopoly Model

A general framework for dynamic oligopoly has been presented. Next, we define the utility function and the actual price dynamics that will be used to study the dynamic spectrum market.

The utility function of each player should represent the satisfaction of a PSH in terms of monetary payoff, resulting from allocating its bandwidth to both its primary services and the secondary market. As such, the following revenue and cost functions for player i are proposed, i.e.,

$$R_i = \underbrace{\pi_i(B_i - b_i(p(t)))}_{R_{i,1}} + \underbrace{p(t)b_i(p(t))}_{R_{i,2}}, \quad C_i = \alpha_i b_i(p(t)) + \beta_i b_i^2(p(t)). \quad (5)$$

Here, the revenues of player i includes $R_{i,1}$ from the primary services and $R_{i,2}$ from the secondary market, since $B_i - b_i(p(t))$ and $b_i(p(t))$ are the current amount of bandwidth to be allocated to the primary services and the spectrum market, respectively. The existing primary subscribers pay a fixed fee, and one can assume that each unit bandwidth from a PSH can sufficiently serve a fixed number of subscribers, which translates into a constant return rate π_i dollars per unit bandwidth allocated to the primary service. Note that $-\pi_i b_i(p(t))$ can also be regarded as the opportunity cost from not leasing the bandwidth. On the other hand, the revenue from secondary market depends directly on the price p .

The cost $\alpha_i b_i(p(t))$ accounts for the commission paid to the spectrum broker and miscella-

Figure 2: A quadratic profit function. Profit maximization occurs at the saturation point, where marginal revenue (MR) equals marginal cost (MC).

neous cost, which are assumed to be linearly proportional to the amount of bandwidth leased; while $\beta_i b_i^2(p(t))$ is proportional to the square of the primary service's bandwidth loss, which indicates that a larger penalty should be imposed to compensate for the performance degradation of the primary service. The reason is simply because the more bandwidth a PSH offers to the secondary market, it is more likely that the service quality to primary subscribers will be degraded, and an increasing penalty is used to avoid over-sacrificing the service quality of primary subscribers. Here, coefficients α_i and β_i are positive constants. Quadratic cost has been adopted frequently in microeconomics, as well as in market-based DSA schemes for cognitive radio networks [7, 6]. Reasons for adopting such cost in the utility function are because it is the most common concave function which can be used to best represent the saturation of player's utility (see Fig. 2); and at the same time it is analytically tractable which can sufficiently provide useful insights into the system.

For the price dynamics, we assume that $G(\cdot)$ in (3) is a linear function in Δp , i.e.,

$$G(\Delta p(t)) = k(\hat{p}(t) - p(t)) \quad (6)$$

where the constant $k > 0$ is called the price adjustment speed. The inverse demand function in (1) is a commonly adopted function in Cournot oligopoly, given by

$$\hat{p}(t) = f(b_1(p(t)), \dots, b_N(p(t))) \triangleq a - \lambda \sum_{i=1}^N b_i(p(t)), \quad (7)$$

where constants $a, \lambda > 0$ are the intercept and slope of the inverse demand curve, respectively.

It is assumed that $a \gg \alpha + \pi$. From (3), (6) and (7), the price trajectory is

$$\dot{p}(t) = G(\Delta p(t)) = k \left[a - \lambda \sum_{i=1}^N b_i(p(t)) - p(t) \right], \quad p(0) = p_0. \quad (8)$$

Note that the dynamic oligopoly is a variant of the sticky-price duopoly [26, 27], which was used originally to study a market of two players. The model is also one of linear-quadratic control models (LQCM), which are very important in control theory as well as economics. Its popularity is partially due to the tractability and uniqueness of analytical solutions.

Under this formulation, the problem is modeled as a differential game and can be stated as the following distributed optimization, i.e., for each i ,

$$\begin{aligned} & \max_{b_i} J_i(p(t), b_i(p(t)), b_{-i}(p(t))) \\ &= \int_0^\infty e^{-rt} [p(t)b_i(p(t)) + \pi_i(B_i - b_i(p(t))) - \alpha_i b_i(p(t)) - \beta_i b_i^2(p(t))] dt \\ & \text{s.t.} \begin{cases} \dot{p}(t) = k \left[a - \lambda \sum_{i=1}^N b_i(p(t)) - p(t) \right], \quad p(0) = p_0. \\ p(t) \geq 0, \quad 0 \leq b_i(p(t)) \leq B_i \quad \forall t, i. \end{cases} \end{aligned} \quad (9)$$

Due to the limited available bandwidth of the PSHs, each player is also constrained by a maximum bandwidth B_i and $b_i(p(t)) \in [0, B_i]$ for all $p(t)$. Thus, a differential game $\mathcal{G}$ has been formulated for the N PSHs over the infinite time horizon $[0, \infty)$. Differential games [24] are characterized by the existence of the state variables, which evolve dynamically and continuously over time following some differential equations. In $\mathcal{G}$, the state is the price $p(t)$ and the state trajectory is described by (8). The state space is the domain of $p(t)$, i.e., $\mathbb{X} = [0, \infty)$. Each player i maximizes its accumulated utility function $J_i(p(t), b_i(p(t)), b_{-i}(p(t))) : \mathbb{X} \times \mathbb{S} \mapsto \mathbb{R}$. Here $\mathbb{S} = \times_{i=1}^N [0, B_i]$ is the game's strategy space.

From differential game theory [24], one can see that in such games, if everyone uses stationary Markov strategies, then each player faces an optimal control problem. Solutions of the optimal control problems correspond to the Markov NEs of the game. It is known that stationary Markov feedback NEs are subgame-perfect. The definition of such NEs is as follows.

Definition 2. The strategy profile $(b_1^*, b_2^*, \dots, b_N^*)$, $b_i^* \equiv b_i^*(p(t))$ is a stationary Markov feed-

back NE if for any player i , any $p \in \mathbb{X}$ and at any time t ,

$$J_i(p(t), b_i^*(p(t)), b_{-i}^*(p(t))) \geq J_i(p(t), b'_i(p(t)), b_{-i}^*(p(t))), \quad \forall b'_i \neq b_i^*. \quad (10)$$

Using techniques from optimal control theory, a characterization of the stationary Markov feedback NEs can be obtained. In particular, for infinite-horizon games, the conditions for such NEs are stated in the following theorem [25].

Theorem 1. *The differential game discussed herein admits a stationary Markov strategy profile $(b_1^*, b_2^*, \dots, b_N^*)$, $b_i^* \equiv b_i^*(p)$ as a stationary Markov feedback NE if for any i , there exists a continuously differentiable function $V_i(p) : \mathbb{X} \mapsto \mathbb{R}$ that satisfies ³*

$$\begin{aligned} rV_i(p) &= \max_{b_i} \left\{ U_i(p, b_i(p), b_{-i}^*(p)) + \frac{\partial V_i(p)}{\partial p} \cdot G(p, b_i(p), b_{-i}^*(p)) \right\} \\ &= U_i(p, b_i^*(p), b_{-i}^*(p)) + \frac{\partial V_i(p)}{\partial p} \cdot G(p, b_i^*(p), b_{-i}^*(p)). \end{aligned} \quad (11)$$

PROOF. See [25], pp. 34. □

In (11), recall that r is the time discount rate and $G(\cdot)$ is the state trajectory function defined by (8). Eq. (11) is known as the Hamilton-Jacobi-Bellman (HJB) equation. Its solution $V_i(p)$ (called the *value function*) is an important function in optimal control theory, which represents the maximum achievable value for player i starting from state p following the optimal strategy.

3. General Solution to the Dynamic Oligopoly

In this section, we give an outline of the general solution to the proposed dynamic oligopoly in (9) and discuss the challenges in obtaining the complete solution. By applying Theorem 1, one can write a set of N simultaneous HJB equations. In particular, for each player i , there exists a corresponding value function $V_i(p) : [0, \infty) \mapsto \mathbb{R}$, which is everywhere continuous and

³From here onwards, in dealing with the HJB equation and its subsequent analysis, it is understood that p is treated as a variable and the results should hold for all t ; so p , $b_i(p)$ and $V_i(p)$ can be used instead of $p(t)$, $b_i(p(t))$ and $V_i(p(t))$.

differentiable, such that

$$rV_i(p) = \max_{b_i} \left\{ (p b_i + \pi_i(B_i - b_i) - \alpha_i b_i - \beta_i b_i^2) + \frac{\partial V_i(p)}{\partial p} \cdot k \left[a - \lambda b_i - \lambda \sum_{j=1, j \neq i}^N b_j - p \right] \right\}. \quad (12)$$

As the maximand in (12) is quadratic in b_i , one can carry out the maximization by taking the partial derivative with respect to b_i . Hence, we define $\Phi_i, \forall i = 1, \dots, N$ such that

$$\begin{aligned} \Phi_i &= \frac{\partial}{\partial b_i} \left(U_i(p, b_i, b_{-i}) + \frac{\partial V_i}{\partial p} \cdot G(p, b_i, b_{-i}) \right) \\ &= p - (\alpha_i + \pi_i) - 2\beta_i b_i - k\lambda \frac{\partial V_i}{\partial p}. \end{aligned} \quad (13)$$

Because of the bandwidth constraint $0 \leq b_i \leq B_i$, the solution to $\Phi_i(b_i) = 0$ will be the optimal b_i only when the constraint is satisfied; otherwise the optimal b_i should occur at the boundaries, i.e., either at 0 or B_i , conditioned on the sign of Φ_i . In summary,

$$b_i^* = \begin{cases} \frac{1}{2\beta_i} \left[p - (\alpha_i + \pi_i) - k\lambda \frac{\partial V_i(p)}{\partial p} \right] & \Phi_i = 0, \\ 0 & \Phi_i < 0, \\ B_i & \Phi_i > 0. \end{cases} \quad (14)$$

The interpretation of the conditions are as follows.

- If for a player, $\Phi_i < 0$ implies $p < (\alpha_i + \pi_i) + 2\beta_i b_i + k\lambda \frac{\partial V_i}{\partial p}$. The spectrum price p can be seen as the MR (earnings from leasing one more unit of bandwidth in the market). The right-hand side of the above inequality is the MC [26] (additional cost of one more unit bandwidth). As $\text{MR} < \text{MC}$, the player gains no profits and will drop out of the market.
- Similarly, if $\Phi_i > 0$, $\text{MR} > \text{MC}$. Thus, the revenue from the secondary market will be so high that player i will output maximum quantities in the market.
- If $\Phi_i = 0$, $\text{MR} = \text{MC}$. Thus, player i faces perfect competition and will offer an output that maximizes his profits. This corresponds to the Markov NE strategy.

Table 1: Nine possible outcomes of the dynamic duopoly.

Case	$b_1^*(p)$	$b_2^*(p)$	Conditions	Player 1	Player 2
i	0	0	$\Phi_1 < 0, \Phi_2 < 0$	Out	Out
ii	0	$X_2p + Y_2$	$\Phi_1 < 0, \Phi_2 = 0$	Out	Monopolist
iii	$X_3p + Y_3$	0	$\Phi_1 = 0, \Phi_2 < 0$	Monopolist	Out
iv	0	B_2	$\Phi_1 < 0, \Phi_2 > 0$	Out	Saturation
v	B_1	0	$\Phi_1 > 0, \Phi_2 < 0$	Saturation	Out
vi	$X_6p + Y_6$	B_2	$\Phi_1 = 0, \Phi_2 > 0$	Monopolist	Saturation
vii	B_1	$X_7p + Y_7$	$\Phi_1 > 0, \Phi_2 = 0$	Saturation	Monopolist
viii	B_1	B_2	$\Phi_1 > 0, \Phi_2 > 0$	Saturation	Saturation
ix	$X_9p + Y_9$	$X'_9p + Y'_9$	$\Phi_1 = 0, \Phi_2 = 0$	Duopolist	Duopolist

3.1. Outcomes of Duopoly

We first investigate the simplest case where there are only $N = 2$ players (duopoly). Due to the 3 possible conditions for each Φ_i , the complete solutions for this dynamic duopoly can involve $3^2 = 9$ different outcomes. The dynamic sticky-price duopoly with output constraints was first investigated by Simaan *et al.* [26], albeit for different utility functions and equilibrium concepts. Nevertheless, by following the same analysis, one can obtain the final equilibrium solutions for each of the 9 different cases. The results are summarized in Table 1 below.

Therefore, each PSH may end up either: (a) allocating all the bandwidth to its primary service (i.e., staying out of the market); (b) offering all the bandwidth to the market (i.e., output saturation), or (c) balancing the primary and secondary spectrum allocation. Possibility (a) may occur if a PSH finds that it does not accumulate any profit by entering the secondary market, perhaps due to limited available spectrum, high cost incurred by giving up bandwidths used for its primary services, or whether the price offered by the market does not generate revenue. On the other hand, possibility (c) is the opposite, i.e., a highly profitable market which allows a PSH to collect maximum revenues by saturating its output. In (b) where the PSH balances its spectrum allocation between its primary services and the secondary market, the exact amount of bandwidth that it should offer to the market is then determined as a linear feedback function $X_jp + Y_j$ of the current market price level p , where X_j and Y_j can be obtained as functions of $r, k, \lambda, \alpha_1, \alpha_2, \beta_1, \beta_2, \pi_1, \pi_2$. In case (ii) and (iii), it is profitable for only one PSH to enter the market; thus, it acts as a monopolist without any competition from the other PSH. In case (vi) and (vii), one PSH supplies all its spectrum and the other supplies the remaining need of the market in the same manner as a monopolist.

The final stable outcome that the market operates in depends on various parameters, such as the players' cost and profit factors ($\alpha_{i,j}$, $\beta_{i,j}$ and $\pi_{i,j}$), as well as the market inherent attributes (a , k and λ). From the system's point of view, possibility (a) may not be encouraged as one would like to promote mutual spectrum access; and so is possibility (b) since the primary service should be guaranteed priority access to spectrum. As such, the authority might be interested in designing the market so that the *true duopoly* case will occur, i.e., case (ix) in Table 1. Under such a free competition scenario, at the Markov NE, it is observed that both players adapt their strategies as a linear function of the price p .

3.2. General Oligopoly

For N players, the three possible outcomes (i.e., staying out, free competition and saturation) will also happen to each player, either at the steady state or during the transient period. As such, a complete analysis of the equilibrium solution must account for all 3^N different scenarios. In general, one may tabulate the results for N -player in a fashion similar to Table 1. A player in his "oligopolist" mode is also predicted to have a feedback strategy as a linear function of the current market price, i.e., $b_i^* = X_j p + Y_j$. However, obtaining a closed form for X_j and Y_j , as well as their existence conditions, is often technically challenging. To demonstrate, let us take a look at the case where all players are oligopolists (i.e., $b_i = \frac{1}{2\beta_i} [p - (\alpha_i + \pi_i) - k\lambda \frac{\partial V_i(p)}{\partial p}]$, $\forall i$). Substituting this into (12) gives

$$\begin{aligned} 2\beta_i r V_i(p) &= (p - (\alpha_i + \pi_i)) \left(p - (\alpha_i + \pi_i) - k\lambda \frac{\partial V_i(p)}{\partial p} \right) \\ &\quad - \frac{1}{2} \left(p - (\alpha_i + \pi_i) - k\lambda \frac{\partial V_i(p)}{\partial p} \right)^2 + 2\beta_i \pi_i B_i \\ &\quad + k \frac{\partial V_i(p)}{\partial p} \left[2\beta_i (a - p) + \lambda \sum_{j=1}^N \frac{\beta_i}{\beta_j} \left(p - (\alpha_i + \pi_i) - k\lambda \frac{\partial V_j(p)}{\partial p} \right) \right], \quad \forall i. \end{aligned} \quad (15)$$

Eq. (15) is a system of N simultaneous nonlinear, coupled partial differential equations (PDE) in terms of $V_i(p)$, $i = 1, \dots, N$, where there is no direct solving method. Similar to the 2-player model [26, 27], one might assume a value function such that $\partial V_i / \partial p = D_i p - E_i$ and subsequently, $V_i = \frac{1}{2} D_i p^2 - E_i p + F_i$, $\forall i$. Hence, player i 's feedback strategy is indeed of the form $b_i^* = X_j p + Y_j$ where $X_j = 1 - k\lambda D_i$ and $Y_j = k\lambda E_i - \alpha_i - \pi_i$. Next, in order to find

the unknowns D_i and E_i , we substitute $V_i(p), \forall i$ into (15) and group terms of similar powers of p together to obtain $A_{i1}p^2 + A_{i2}p + A_{i3} = 0$ which must hold for all values of p . Thus, it is required that A_{i1} , A_{i2} and A_{i3} must be simultaneously 0, $\forall i$, which yields

$$-k^2\lambda^2 D_i^2 + \left(k\lambda \sum_{j=1}^N \frac{\beta_i}{\beta_j} (k\lambda D_j - 1) - 2\beta_i(1+r) \right) D_i + 1 = 0, \forall i \quad (16a)$$

$$(kD_i\Theta_i + (\alpha_i + \pi_i)(k\lambda D_i - 1)) + (k\Theta_i - 2\beta_i r)E_i = 0, \forall i \quad (16b)$$

$$4\beta_i r F_i - (4\beta_i \pi_i B_i + (\alpha_i + \pi_i)^2 - kE_i\Theta_i - (\alpha_i + \pi_i)k\lambda E_i) = 0, \forall i \quad (16c)$$

where $\Theta_i = 2\beta_i a - \lambda \sum_{j \neq i} \frac{\beta_i}{\beta_j} (k\lambda E_j - \alpha_j - \pi_j)$. Thus, D_i will be obtained from (16a); E_i will be obtained from (16b) which depends on the solution to (16a); and F_i will be obtained from (16c) which depends on the solution to (16b). However, both (16a) and (16b) are systems of N coupled nonlinear, asymmetric algebraic equations in D_i and E_i , $i = 1, \dots, N$ respectively. Obtaining closed forms for D_i and E_i and their existence conditions is generally mathematically challenging.

Note that this is only for one out of the 3^N scenarios. Similar technical challenges occur when analyzing the other regions of the oligopoly. Moreover, even if the NE is obtained for each separate region, the game may start in one of these regions and cross into another as the price varies. Therefore, identifying all the strategy transitions is often not tractable due to the possible 3^N scenarios, which adds further difficulties in the analysis. Such *strategy transition* was also not addressed adequately in existing dynamic duopoly models [26, 27].

In order to demonstrate a complete Markov NE solution and thoroughly investigate the transition of strategy across different regions, in the following section, we will demonstrate a special case in which players are assumed to be homogeneous.

4. Solution to the Symmetric Dynamic Oligopoly

In the following discussion, we consider the special case of symmetric oligopoly, i.e., $\pi_i = \pi$, $\alpha_i = \alpha$, $\beta_i = \beta$ and $B_i = B$, $\forall i$. As mentioned before, common market price holds if the competition takes place among operators with the same type of infrastructures and the SSPs have equal bandwidth preference. Furthermore, many of the parameters from multiple players

can be not too far away from each other, e.g., operators having the same amount of spectrum assigned by the authority which leads to the assumption of identical bandwidth constraint; and most of the time, the subscribers' rates π_i for different operators are roughly similar that one can approximate them to be identical for theoretical analysis. Thus, some useful insights can be gained from analysis of the symmetric oligopoly. Next, we look at its actual solution. It is assumed that the slope of the inverse demand function $\lambda = 1$ for simplicity. Thus, eq. (15) becomes

$$\begin{aligned}
2\beta r V_i(p) = & (p - (\alpha + \pi)) \left(p - (\alpha + \pi) - k \frac{\partial V_i(p)}{\partial p} \right) \\
& - \frac{1}{2} \left(p - (\alpha + \pi) - k \frac{\partial V_i(p)}{\partial p} \right)^2 + 2\beta \pi B \\
& + k \frac{\partial V_i(p)}{\partial p} \left[2\beta a - (2\beta + N)p + N(\alpha + \pi) + k \sum_{j=1}^N \frac{\partial V_j(p)}{\partial p} \right], \quad \forall i.
\end{aligned} \tag{17}$$

Similarly, if we also let $V_i(p)$ be of the form $\frac{1}{2}D_i p^2 - E_i p + F_i$, then (17) consists of N symmetric ordinary differential equations in $V_i(p)$. The asymmetric solution where $D_i \neq D_j$, $E_i \neq E_j$ and $F_i \neq F_j$, can be shown to be asymptotically unstable (i.e., it may occur that $\lim_{t \rightarrow \infty} p(t) = \infty$). This leaves us to investigate the symmetric case where $D_i = D_j$, $E_i = E_j$, $F_i = F_j$, and subsequently $V_i = V_j = V$, $\forall i, j$. Hence, the optimal strategies will be symmetric where

$$b_i^*(p) = \tilde{b} = \frac{1}{2\beta} \left[p - (\alpha + \pi) - k \frac{\partial V(p)}{\partial p} \right], \quad \forall i \tag{18}$$

if there are no bandwidth constraints. However, when coupled with the constraint $0 \leq b(p) \leq B$, three possibilities can occur:

1. *Case A:* $\tilde{b} < 0$. The actual strategy taken by the players will be $b^*(p) = 0$ regardless of p .
2. *Case B:* $0 \leq \tilde{b} < B$. The players follow the feedback strategy $b^*(p) = \tilde{b}$.
3. *Case C:* $\tilde{b} \geq B$. Here, the strategy will saturate at $b^*(p) = B$ regardless of p .

Next, we derive the solutions for the three cases in sections 4.1-4.3. We address the transitions from one region to another and complete the equilibrium analysis in section 4.4. In addition, section 4.5 graphically illustrates the results.

4.1. Solution for Case A

In Case A, $b^*(p) = 0$ means that all PSHs supply zero outputs to the market. It can be implied that the price level may drop too low for any of the players to gain positive profits, so PSHs dropped out of the competition. The solution to this case is straightforward.

First, we find the value function $V(p)$ by substituting $b = 0$ in the HJB equation (12) to obtain $rV(p) = k(a - p)\frac{\partial V(p)}{\partial p} + \pi B$, which is a standard ordinary differential equation (ODE) in $V(p)$. Its solution is

$$V(p) = \frac{C}{(a - p)^{r/k}} + \frac{\pi B}{r}, \quad C = \text{const.} \quad (19)$$

Next, we compute the trajectory of $p(t)$. With $b = 0$, eq. (8) becomes $\dot{p}(t) = k(a - p(t))$, whose general solution is

$$p(t) = a - C e^{-kt}, \quad C = \text{const.} \quad (20)$$

If the game begins in Case A, then one can apply initial condition $p(0) = p_0$ to get $C = p_0 - a$. Moreover, since the exponential term $C e^{-kt}$ vanishes as $t \rightarrow \infty$, and the price converges asymptotically to the steady-state value $p = a$ if it lies in the region of Case A (to be determined in Section 4.4). This steady-state price is known the *market equilibrium*, where the supplies and demands are balanced. This is not the same as the other equilibrium concept, i.e., the Markov NE in terms of the strategy of the players. As the trajectory describes a linear system, the steady state is globally stable.

In summary, Case A's outcome is stated as follows.

Proposition 1. *In Case A, the players' NE strategy is $b^*(p) = 0$ for all p in the region. The market equilibrium in this case is given by $p = a$.*

PROOF. Evident from the previous analysis. □

4.2. Solution for Case B

Case B represents the true oligopoly scenario where all PSHs dynamically optimize their strategies in the market. Similarly, we first find the value function $V(p)$. The result is stated in the lemma below.

Lemma 1. *The value function $V(p)$ and its derivative $\frac{\partial V(p)}{\partial p}$ in Case B are of the following forms*

$$\frac{\partial V(p)}{\partial p} = Dp - E, \quad (21a)$$

$$V(p) = \frac{1}{2}Dp^2 - Ep + F, \quad (21b)$$

where

$$D = \frac{(2\beta + N)k + \beta r - \sqrt{((2\beta + N)k + \beta r)^2 - (2N - 1)k^2}}{(2N - 1)k^2}, \quad (21c)$$

$$E = \frac{2\beta kDa + (NkD - 1)(\alpha + \pi)}{(2N - 1)k^2D - (2\beta + N)k - 2\beta r}, \quad (21d)$$

$$F = \frac{(\alpha + \pi)^2 + (2N - 1)k^2E^2 - 2NkE(\alpha + \pi) - 4\beta kEa}{4\beta r} + \frac{\pi B}{r}. \quad (21e)$$

PROOF. By following the same steps as in the general oligopoly, one can substitute $V(p)$ in (21b) back to (17) and simplify, yielding $A_1p^2 + A_2p + A_3 = 0$ which must hold for all p . We require A_1 , A_2 and A_3 are simultaneously 0. In particular, $A_1 = 0$ gives

$$(2N - 1)k^2D^2 - 2((2\beta + N)k + \beta r)D + 1 = 0 \quad (22)$$

which is quadratic in D . By taking the smaller root, D can be obtained as in (21c).

Additionally, it can be shown that $D < \frac{1}{k}$, as after some algebraic manipulation, this will be equivalent to $2k\beta(r + 2k)(2N - 1) > 0$ which is true. Choosing such D will ensure that the price is convergent, as will be shown later on.

Next, A_2 and A_3 are linear in E and F , respectively. Therefore, E and F as in (21d) and (21e) can be obtained readily, the details of which are neglected. $\square$

Then, we proceed to find the stationary Markov feedback NE and the steady-state price level as follows. With $\frac{\partial V(p)}{\partial p}$ given in (21b), the stationary Markov feedback NE strategy of players in Case B has the final form

$$b^*(p) = \frac{1}{2\beta} [(1 - kD)p + (kE - \alpha - \pi)]. \quad (23)$$

Substituting (23) into (8) gives

$$\dot{p}(t) = k \left(a - \frac{N [(1 - kD)p(t) + (kE - \alpha - \pi)]}{2\beta} - p(t) \right). \quad (24)$$

The general solution to this is

$$p(t) = \Gamma + C e^{-k \left(1 + \frac{N(1-kD)}{2\beta} \right) t}, \quad C = \text{const}, \quad (25)$$

with

$$\Gamma = \frac{2\beta a + N(\alpha + \pi - kE)}{2\beta + N(1 - kD)}. \quad (26)$$

In (25), the constant C can be found with some initial condition. In particular, if Case B takes place at the starting point of the game $p(0) = p_0$ then $C = p_0 - \Gamma$. Moreover, since $D < \frac{1}{k}$ as previously stated, the exponent $-k(1 + \frac{N(1-kD)}{2\beta})t$ is negative. Hence, the exponential term here also vanishes as $t \rightarrow \infty$, and the price reaches the asymptotically stable steady state Γ , provided that Γ lies in the region for Case B.

Proposition 2. *In Case B, the players' NE strategy $b^*(p)$ is given in (23) for all p in the region. The market equilibrium in this case is given by Γ in (26).*

PROOF. Evident from the previous analysis. $\square$

4.3. Solution for Case C

Case C is the direct opposite of Case A, where the PSHs have no incentives to provide primary services and they are now leasing all their available bandwidth B . Therefore, by applying the same analysis, similar results can be obtained.

The HJB equation in Case C gives

$$k(p - a + NB) \frac{\partial V(p)}{\partial p} + rV(p) = Bp - \alpha B - \beta B^2, \quad (27)$$

whose solution is

$$V(p) = Rp + S + \frac{C}{(a - NB - p)^{r/k}}, \quad C = \text{const}, \quad (28)$$

where

$$R = \frac{B}{r + k}, \quad (29a)$$

$$S = \frac{1}{r} \left[\frac{kB}{r + k} (a - NB) - \alpha B - \beta B^2 \right]. \quad (29b)$$

Next, we solve the trajectory $\dot{p}(t) = k(a - NB - p(t))$. Its solution is

$$p(t) = a - NB - C e^{-kt}, \quad C = \text{const}, \quad (30)$$

which also converges to $a - NB$ if the point lies in the region for Case C. Also, if the game begins in Case C, then the constant $C = p_0 - (a - NB)$.

Case C's outcome is summarized as follows.

Proposition 3. *In Case C, the players' NE strategy is $b^*(p) = B$ for all p in the region. The market equilibrium in this case is given by $p = a - NB$.*

PROOF. Evident from the previous analysis. □

4.4. Transition Between Different Market Scenarios

In the previous analysis, one has obtained the solutions for different possible market situations, i.e., cases A, B and C. However, it is still necessary to determine the required conditions and define the exact regions in which each of these cases holds. It is possible that the price trajectory may reach a boundary point and enter another region over time and a strategy transition is said to take place.

First, let us look for these boundary values of the price p . Notice that in Case B, $b(p)$ is not only a linear function in p , but also bounded in $[0, B]$. Thus, $p(t)$ must also be bounded in $[p_1, p_2]$ which defines the boundaries for Case B. By solving $b(p) = 0$ and $b(p) = B$, respectively, with $b(p)$ in (23), one can obtain

$$p_1 = \frac{\alpha + \pi - kE}{1 - kD}, \quad (31a)$$

$$p_2 = \frac{\alpha + \pi - kE + 2\beta B}{1 - kD}. \quad (31b)$$

From this result, it is clear that the region for Case A to occur is $p(t) \in [0, p_1)$ and for Case C $p(t) \in [p_2, \infty)$. Therefore, at any time instant t , the current market scenario and players' strategies depend on current state $p(t)$ and which interval it is in. Specifically, at the beginning $t = 0$, the value of p_0 will determine the starting scenario of the market.

Next, the transition mechanisms are examined. Suppose that at a certain time $t = \hat{t}$, $p(t)$ reaches a boundary value p_u (i.e., either p_1 or p_2) and is about to cross over to the neighboring region. Note that $\hat{t}$ can easily be calculated by solving $p(t) = p_u$ with $p(t)$ in either (25), (20) or (30). At $\hat{t}$, the strategy $b(t)$, the trajectory $p(t)$ and the value function $V(p)$ take on new shapes as a transition occurs. Here, according to Theorem 1, it is mandatory that $V(p)$ is continuous and differentiable. Moreover, the "stickiness" assumption requires $p(t)$ to be continuous.

The continuity of $p(t)$ can be satisfied if the new initial condition $p(\hat{t}) = p_u$, $u = 1, 2$ is enforced when re-solving the state equation. For example, if a switch from Case A or Case C to Case B happens, solving (24) with the new initial condition yields a new shape of $p(t)$ as given by

$$p(t) = \Gamma + (p_i - \Gamma) e^{-k \left(1 + \frac{N(1-kD)}{2\beta}\right) (t - \hat{t})}, \quad \text{for } t \geq \hat{t}. \quad (32)$$

On the other hand, $V(p)$ is comprised of several piecewise smooth functions in $[0, p_1)$, $[p_1, p_2)$ and $[p_2, \infty)$. In order to ensure that it is continuous at p_1 , one can let $V_1 = \frac{1}{2}Dp_1^2 - Ep_1 + F$ and solve (19) with the boundary condition $V(p_1) = V_1$. Similarly, one can let $V_2 = \frac{1}{2}Dp_1^2 - Ep_1 + F$ and incorporate $V(p_2) = V_2$ in (28). Hence, the complete value function is given by

$$V(p) = \begin{cases} (V_1 - \frac{\pi B}{r}) \frac{(a - p_1)^{r/k}}{(a - p)^{r/k}} \\ \frac{1}{2}Dp^2 - Ep + F \\ Rp + S + (V_2 - Rp_2 - S) \frac{(a - NB - p_2)^{r/k}}{(a - NB - p)^{r/k}} \end{cases} \quad (33)$$

corresponding to the intervals $p \leq p_1$, $p_1 \leq p < p_2$ and $p \geq p_2$, respectively.

Finally, it remains to verify that

$$\begin{cases} \lim_{p \rightarrow p_1^+} \frac{\partial V(p)}{\partial p} = \lim_{p \rightarrow p_1^-} \frac{\partial V(p)}{\partial p} \\ \lim_{p \rightarrow p_2^+} \frac{\partial V(p)}{\partial p} = \lim_{p \rightarrow p_2^-} \frac{\partial V(p)}{\partial p} \end{cases} \quad (34)$$

which can be shown to be true (details omitted). With that, the complete solution for the proposed differential game is fully characterized. It is summarized by the following theorem.

Theorem 2. *With the value function in (33), the symmetric differential game has a symmetric stationary Markov feedback NE in*

$$b^*(p) = \begin{cases} 0, & p < p_1 \\ \frac{1}{2\beta} [(1 - kD)p + (kE - \alpha - \pi)], & p_1 \leq p < p_2 \\ B, & p \geq p_2. \end{cases} \quad (35)$$

with p_1 and p_2 in (31). Moreover, it will result in an asymptotically stable market equilibrium at either $p = a$, $p = \Gamma$ defined in (26) or $p = a - NB$. Additionally, the behaviors of the players' strategy depend on the region of the current state $p(t)$ and transition between regions can occur at boundary points in a manner similar to (32).

PROOF. Evident from the previous analysis. $\square$

4.5. A Visualization of the Market Regions

It has been shown that the symmetric oligopoly's Markov NE may occur in three regions. In order to gain a better understanding of these market scenarios and transitions, Fig. 3 (not to scale) is used to help illustrate the results. It depicts the different regions in the $p-\pi$ plane, which offers a look into the spectrum market in terms of comparative market price. Naturally, π and p can be understood as the price of spectrum for the primary services and the secondary market, respectively. Thus, at a given time, the point (p, π) in the $p-\pi$ plane determines which region the game currently is in (see Fig. 3). The three regions are separated by the two boundary lines $p = p_1(\pi)$ and $p = p_2(\pi)$, as p_1 and p_2 in (31) can be shown to be linear in terms of π ; and these two lines can be proven to be parallel. Consequently, if p is too low compared

Figure 3: Three regions on the p - π plane.

to π , the secondary market investment is worse than the primary investment, which leads to Case A. Meanwhile, an appropriate range of p for a given π can result in true oligopoly; and Case C occurs when p is much higher than π .

The dotted lines $p = a$, $p = \Gamma(\pi)$ and $p = a - NB$ in Fig. 3 are the steady-state prices for the three cases. Besides, the group of lines $p = a$, $p = \Gamma(\pi)$ and $p = p_2(\pi)$ are shown to be concurrent (i.e., they meet at a common point); and so are $p = a - NB$, $p = \Gamma(\pi)$ and $p = p_1(\pi)$, as depicted in the figure, which can be mathematically proven. As such, the actual steady-state price is represented by the segments of these three lines that lie within their corresponding regions, represented by arrows; and as π increases, the actual steady-state follows the continuous path highlighted by these arrows in Fig. 3. Moreover, for a given π and initial price p_0 , a trajectory of $p(t)$ can be depicted as a vertical straight arrow pointing towards the steady-state path. For example, in Fig. 3, for the initial price p_{02} and given π_2 , they correspond to the trajectory that starts from the point (p_{02}, π_2) within Case C. This trajectory later crosses into Case B until it reaches the steady-state path, which is the arrow segment of $p = \Gamma(\pi)$ within Case B. It should be noted that Fig. 3 depicts the behaviors of the game as a whole, not the time-varying behaviors of a particular variable. Such behaviors will be numerically studied in Section VI.

5. Discrete-time Price Adjustment

Under the previous analysis, theoretically, the PSHs in the market adapt their offered bandwidth continuously as a function of the current market price, which is not practical since the process requires price feedback and players' decisions almost instantaneously. Therefore, to implement the dynamic spectrum market, we propose a discrete-time price adjustment protocol, in which the price is adjusted dynamically in equally time-slotted manner. The protocol can be effectively implemented via the spectrum broker and should be agreed upon by all participating oligopolists in the spectrum market. The basic procedures are as follows.

1. Initially, the spectrum broker sets initial price $p[t_0] = p_0$.
2. At time t_{n+1} , the spectrum broker announces an updated price $p[t_{n+1}]$ according to

$$p[t_{n+1}] = p[t_n] + h \times k(a - p[t_n] - \sum_{i=1}^N b_i[t_n]). \quad (36)$$

3. Each player adapts its strategy $b[t_{n+1}] = b^*(p[t_{n+1}])$ with $b^*(.)$ defined in (35).
4. The adjustment continues until price reaches a stable point, e.g., when $|p[t_{n+1}] - p[t_n]| < p_{tol}$ where $p_{tol} > 0$ is a tolerance value.

Eq. (36) is in fact the gradient update method in numerical analysis [28]. Parameter h is the stepsize or the adjustment interval, given by $h = t_{n+1} - t_n$. An appropriate choice for h is required to ensure its convergence.

Lemma 2. *Given the game's parameters, in order for the update process to converge, the stepsize h should not exceed a maximum value $h_{\max} = \frac{2}{k + \frac{kN(1-kD)}{2\beta}}$.*

PROOF. The stability of the numerical method (36) implies $|1 + hJ| < 1$ [28]. In other words, the Jacobian $J = \frac{dG}{dp}$ of the price trajectory has eigenvalues of magnitudes less than 1. Recall that $G(.)$ is given by (8). Evaluating J yields

$$J = \begin{cases} k \left(1 + \frac{N(1-kD)}{2\beta} \right), & p \in [p_1, p_2], \\ k, & p \notin [p_1, p_2]. \end{cases} \quad (37)$$

Table 2: Parameter settings for the spectrum market.

Parameters	Values
Number of PSHs, N	4
Total bandwidth of each PSH, B	3000 kHz
Intercept of inverse demand curve, a	11000 \$/kHz
Slope of inverse demand curve, λ	1 \$/(kHz) ²
Price adjustment speed, k	0.5
Discount rate, r	0.3
Income rate of primary services, π	800 \$/kHz
First-order cost coefficient, α	400 \$/kHz
Second-order cost coefficient, β	0.5 \$/(kHz) ²

Thus, $|1 + hJ| < 1$ gives $h < h_{\max} = \min(\frac{2}{k}, \frac{2}{k(1 + \frac{N(1-kD)}{2\beta})})$. □

6. Numerical Results

In this section, numerical examples and analysis are presented in order to further investigate the properties and behaviors of the proposed game. Furthermore, computer simulations are also done to study the convergence of the discrete-time price adjustment method.

6.1. An Example

First of all, let us observe the theoretical behaviors of some important functions in the game and the strategy transitions over the time horizon through a specific example. The settings may not reflect all the real-time data of an actual spectrum market; but the example could be useful to illustrate the model's behaviors. In this example, the various game parameters are chosen as given in Table 2, which gives a price trajectory equation $\dot{p}(t) = 0.5[11000 - 4b(t) - p(t)]$ and an accumulated revenue of $\int_0^\infty e^{-0.3t}[p(t)b(t) + 800(3000 - b(t)) - (400b(t) + 0.5b^2(t))]dt$. Thereafter, for each of three different values of the initial spectrum price, $p_0 = 50, 2500$, and 7000 \$/kHz, the actual price $p(t)$, the Markov NE strategy $b(p(t))$, the value function $V(p(t))$ and its derivative $\partial V/\partial p$ are tracked following their derived theoretical values according to Theorem 2 and plotted against time t , as shown in Fig. 4.

As verified by the previous analysis, the price $p(t)$ is asymptotically stable and converges to the market equilibrium as $t \rightarrow \infty$; and so will $b(p(t))$ converge to its final value. This market equilibrium price $\Gamma \approx 3508.7$ \$/kHz lies in Case B, within $p_1 \approx 1425.3$ \$/kHz and $p_2 \approx 4762.6$

Figure 4: An example of the oligopoly.

\$/kHz; so the players all balance their spectrum allocation to both primary services and the spectrum market. In this case, approximately $b^* \approx 1873$ kHz is to be offered to the secondary market. It is clear that the unique equilibrium point is independent of the initial state p_0 , which is due to the global stability of the system. $V(p)$ is shown to be continuous. Furthermore, the continuity of $\partial V / \partial p$ confirms the differentiability of $V(p(t))$.

The value of p_0 affects the starting region and the subsequent shapes of $p(t)$, $b(p(t))$ and $V(p(t))$ (see Fig. 4). For a very low value $p_0 = 50$ \$/kHz $< p_1$, there is at first no incentive for the PSHs to allocate any bandwidth to the SSPs due to negative profit, so they stay out of the market (i.e., Case A). At this point, the inverse demand law gives a desirable price $\hat{p}(0) = a > p(0)$, which implies that the demands exceed the supplies and will drive $p(t)$ up. As time t elapses, $p(t)$ increases and reaches p_1 at $\hat{t}_1 \approx 0.27$ time unit (this point is marked by an asterisk in Fig. 4), where a transition to Case B occurs and the players start to offer positive outputs to the market. The game then proceeds towards the NE. Similarly, for $p_0 = 7000$ \$/kHz $> p_2$, the game starts in Case C and switches midway to Case B at $\hat{t}_2 \approx 0.66$ time unit. Case B occurs from the beginning for $p_0 = 2500$ \$/kHz.

Figure 5: Convergence of the discrete-time method towards the theoretical solution for (a) Price and (b) Strategy.

6.2. Convergence of the Discrete-Time Method

The discrete-time price adjustment scheme was simulated for the game where the parameters were set similar to the previous example (see Table 2), except for the initial price $p_0 = 7500$ \$/kHz. As such, $p_0 > p_2 = 4762.6$ \$/kHz and one would expect $p(t)$ to start in Case C, then cross into Case B midway and converge to the market equilibrium $\Gamma \approx 3508.7$ \$/kHz. For this simulation, two different stepsizes were selected, i.e., $h_1 = 0.5h_{\max}$ and $h_2 = 0.2h_{\max}$, where the maximum allowable interval is estimated to be $h_{\max} \approx 0.87$ according to Lemma 2. The tolerance threshold is $p_{tol} = 0.01$. The two discrete-time processes were plotted in Fig. 5, together with the theoretical $p(t)$.

Clearly, with a properly chosen interval below the maximum allowable value, the discrete-time dynamics of $p[t]$ and $b[t]$ converged to the theoretical equilibrium after a period of time, as shown in Fig. 5. However, the convergence speed depends on how small the stepsize is. Also, with h_2 , the discrete solution approximates the theoretical solution with less errors than with h_1 . However, there is a tradeoff between accuracy and computational cost. A longer interval means less computations on the spectrum broker, but gives a larger error which translates into higher profit loss between the actual and the optimal value. Such profit loss can be on the PSHs (negative errors) or the SSPs (positive errors). Therefore, the realization of such market

Figure 6: Comparison of (a) Price and (b) Accumulated profit for different discount rates.

protocol should be agreed among the PSHs, SSPs and the spectrum broker so that the costs for all sides are within acceptable margins.

6.3. Impacts of Discount Rate

We also study the impacts of discount rate r on the price dynamics and the PSH's accumulated payoff function in Fig. 6, using the previous parameter settings. At time t , the value of J_i evaluated up to time t , i.e., $J'_i(t) = \int_{\tau=0}^t e^{-r\tau} U_i d\tau$, is plotted. We observe that although r does not have a significant impact on the equilibrium market price (which falls in a narrow range from 3400 to 3600 \$/kHz), it does affect the payoff function. J_i tends to grow quicker for small r and diverges if $r \rightarrow 0$, while it converges faster for larger r . In fact, the discount factor e^{-rt} captures the time preference of an impatient player, i.e., one who usually places a higher valuation on his rewards at an earlier point in time than that at a future point. Thus, a player with a higher r focuses more on his immediate revenues while one with lower r places more emphasis on profits in the long run.

6.4. Impacts of Number of Players and Maximum Bandwidth

Next, the steady-state behaviors of the game, particularly of the spectrum price $p(t)$ and the optimal bandwidth output $b(p(t))$ are studied when the number of players N and the total

Figure 7: Steady-state values of (a) Price and (b) Strategy versus the number of players N .

bandwidth B are varied. For the previous parameter settings as in Table 2, the steady-state values (i.e., limits as $t \rightarrow \infty$) of $p(t)$ and $b(p(t))$ are plotted against the number of players N for three different values of the maximum bandwidth $B = 1500, 2000$ and 3000 kHz in Fig. 7. Some observations can be made as follows.

Firstly, as N increases, the steady-state price $p(\infty)$ and offered bandwidth $b(p(\infty))$ tend to drop for the same B . This agrees with the economic principle that increasing supplies lead to reduction of the equilibrium market price, hence the decrease in $p(\infty)$. At the same time, the more competitors, the smaller share of the spectrum market a PSH will get, which explains the drop in $b(p(\infty))$.

Secondly, the decline in $p(\infty)$ is initially linear with respect to N , corresponding to Case C's equilibrium price $p = a - NB$. This is particularly evident in the shape of $p(t)$ in Fig. 7 for $B = 1500$ kHz where $N \leq 5$. However, when $N > 5$, it is implied that Case C's equilibrium no longer holds and the trajectory switches midway to Case B where its equilibrium price Γ given in (26) becomes nonlinear in N . On the other hand, $b(p(\infty))$ is seen to remain at its maximum bandwidth level B in Case C for $N \leq 5$, and then also drop nonlinearly with respect to N in Case B for $N > 5$. Similar observations are also made for other values of B , where $p(\infty)$ is linear and $b(p(\infty))$ is constant for small N ; and they drop nonlinearly for sufficiently large N .

Interestingly, it can be verified that as $N \rightarrow \infty$, $\lim_{N \rightarrow \infty} \Gamma = \alpha + \pi$, $\lim_{N \rightarrow \infty} p_1 = \alpha + \pi$, $\lim_{N \rightarrow \infty} p_2 = \alpha + \pi + 2\beta B$ and $\lim_{N \rightarrow \infty} b(\Gamma) = 0$. That is, for infinitely many players, the Case B steady-state output approaches 0 as the Case B equilibrium price Γ approaches the first-order cost factor.

Furthermore, it is seen that as B increases, for the same N , both $p(\infty)$ and $b(p(\infty))$ will converge to Γ and $b(\Gamma)$ in Case B if N is large. Note that Γ in (26) is independent of B . Theoretically, one can show that if $B > b(p_0)$ with $b(p)$ in (23), it no longer has any effects on $p(\infty)$ and $b(p(\infty))$. This implies that once the PSHs have an abundant amount of unused bandwidth, they will ultimately benefit from participating in the secondary spectrum market and hence achieve better utilization of spectrum. However, due to the fixed demand from the SSPs, the optimal offered bandwidth to the market will stabilize even if the maximum available bandwidth gets excessively large.

6.5. Number of Players under Bounded Total Bandwidth

In this section, we also study the behaviors of price and strategies of the game under varying N and B , under an additional constraint of bounded total bandwidth, i.e., $NB = \mathcal{B} = \text{const.}$ This special case has a practical implication: in a certain geographical location, there may exist many licensed PSHs but the wireless bandwidths are limited by a fixed amount of $\mathcal{B}$. Thus, each PSH shares an equal amount of available spectrum $B = \mathcal{B}/N$ which is to be allocated to both their primary services and the secondary spectrum market.

Let us look at the behaviors of the dynamic oligopoly under this particular constraint through a numerical example. One can assume that the system parameters also follow Table 2 except for N and B , and additionally, $p_0 = 1500$ \$/kHz and $NB = \mathcal{B} = 7200$ kHz. As such, Fig. 8(a) and Fig. 8(b) show the various trajectories of $p(t)$ and their corresponding outputs $b(p(t))$ over time t for different values of N . As seen in Fig. 8(a), we see that the steady-state price is decreasing for relatively small N (2 and 3); but it ceases to drop and stays constant after N exceeds 4. Furthermore, it is indicated in Fig. 8(b) that the steady-state strategy falls within Case B for small N , and Case C for $N \geq 4$. The reason that steady-state price becomes constant at 3800 (\$/kHz) is because in Case C, all bandwidths are allocated to the secondary market and the total supplied bandwidths are saturated at $NB = \mathcal{B}$; so the steady-state price becomes $a - NB = 3800$. The steady-state strategy in Fig. 8(b) is still decreasing for $N \geq 4$ because more PSHs means less bandwidth per player, inversely proportional to N according

Figure 8: Comparison of (a) Price and (b) Strategy when $NB = \mathcal{B} = \text{const.}$

to $B = \mathcal{B}/N$. Thus, in this example, $N = 3$ is the largest number of players at which true oligopoly is still in effect, while $N^* = 4$ is the critical point where Case C occurs and beyond which steady-state price saturates.

From this numerical example, one naturally asks whether the critical number of players N^* exists in the general case and how to estimate N^* . In fact, we can show that N^* is the value of N that satisfies $\Omega(N-1) \leq \mathcal{B} < \Omega(N)$, where $\Omega(N) = \frac{a(1-kD) - (\alpha + \pi - kE)}{(1-kD) + (2\beta/N)}$; and N^* exists when $\Omega(2) \leq \mathcal{B} \leq a - \alpha - \pi$. The proof of this is given in Appendix A. In Fig. 9, the function $\Omega(N)$ is plotted for $N \geq 2$ which shows the existence region of N^* . As a consequence, two extreme cases are revealed. Firstly, when $\mathcal{B} < \Omega(2)$, any number of players will always result in saturation (Case C), implying that the system suffers from total bandwidth shortage which will affect the primary services. Secondly, when $\mathcal{B} > \Omega(\infty) = a - \alpha - \pi$, Case B will occur for any number of players, which means that the total bandwidths are abundant enough to allow for free oligopolistic competition.

7. Conclusions

New spectrum allocation schemes such as spectrum markets are currently being researched in order to improve the utilization of the scarce spectrum resources. This paper studies one such market-driven model where the licensed holders of spectrum allocate their fixed amount

Figure 9: Plot of the function $\Omega(N)$, showing the existence region of N^* .

of bandwidth both to primary service and the spectrum market, managed by the spectrum broker. We formulate the oligopolistic competition as a sticky-price differential game under constraints. As such, the game is shown to have multiple outcomes even in the 2-player case. For the N -player scenario, we derive the Markov feedback NE when the game is symmetric and properly address the strategy transition. An important design issue is to select proper parameters so that the costs incurred by all participants are minimized, while maintaining free competition (true oligopoly) so as to encourage mutual sharing of spectrum. Via our numerical and theoretical analysis, we weigh the impact of parameters such as the number of player and the total bandwidth, as well as their desirable values. We also propose a discrete-time adjustment method which is run by the spectrum broker.

Appendix A. Critical Number of Players in Section 6.5

We derive the number of players N^* and its existence condition.

Proposition 4. *The critical number of players N^* in Section 6.5 exists and is the one that satisfies $\Omega(N-1) \leq \mathcal{B} < \Omega(N)$ if $\Omega(2) \leq \mathcal{B} \leq a - \alpha - \pi$, where $\Omega(N) = \frac{a(1-kD) - (\alpha + \pi - kE)}{(1-kD) + (2\beta/N)}$.*

PROOF. We notice that when N^* exists, for N up to N^*-1 , steady state in Case B occurs which requires that $\Gamma \in [p_1, p_2]$. At N^* , Case C takes place and starts to violate the condition $\Gamma \leq p_2$.

By equating $\Gamma = p_2$ while noting that $B = \mathcal{B}/N$, one can obtain $\mathcal{B} = \frac{a(1-kD) - (\alpha + \pi - kE)}{(1-kD) + (2\beta/N)} = \Omega(N)$, which is a function of N . If N is relaxed to take real values in $[2, +\infty)$, then it can be shown that $\Omega(N)$ is monotonically increasing in N (as graphically shown in Fig. 9). Since $\lim_{N \rightarrow \infty} D = \lim_{N \rightarrow \infty} E = 0$, this implies $\lim_{N \rightarrow \infty} \Omega(N) = a - \alpha - \pi$, i.e., $\Omega(N)$ is upper-bounded by $a - \alpha - \pi$. Moreover, as N is the number of players, only values of $N \geq 2$ are valid. Therefore, for given values of $\mathcal{B}$ such that $\Omega(2) \leq \mathcal{B} \leq a - \alpha - \pi$, there exists a unique integer N^* such that $\Omega(N^* - 1) \leq \mathcal{B} < \Omega(N^*)$ which is the critical N . $\square$

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