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We propose to generate an artificial non-Abelian U(3) gauge field by using a two-tripod scheme, namely, two tripod configurations sharing a common ground-state level and driven by resonant one-photon transitions. Using an appropriate combination of four Laguerre-Gauss and two Hermite-Gauss laser beams, we are able to produce a U(3) monopole and a U(3) spin-orbit coupling for both alkali-metal and alkaline-earth-metal atoms. This two-tripod scheme could open the way to the study of interacting spinor condensates subjected to U(3) monopoles.

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I. INTRODUCTION

Within less than a decade, ultracold quantum gases have successfully pervaded many fields of physics. Indeed, they provide a rather unique testing bed where theorists' dreams can be turned into carefully designed experimental situations. This is particularly true in the condensed matter realm where they became a key player in many-body physics [1–3]. Quantum Hall effects did not escape the trend. The catch, however, is that atoms are neutral and one thus needs to implement an artificial gauge field acting on the atoms that would give rise to a strong enough effective magnetic field. One of the first ideas was to set quantum gases into rapid rotation [4]. Since then, more versatile and promising schemes have been introduced, some even realized, all based on light-atom interactions [5–11]. These light-induced artificial gauge fields, encompassing Abelian and non-Abelian situations, have opened the door to a whole class of model Hamiltonians [12–15] and are addressing diverse physical situations ranging from artificial Dirac monopoles [16] to spin-orbit (SO) coupling [17–19] and topological phases [20,21], non-Abelian particles [22], and mixed-dimensional systems [23–27].

In this paper, we discuss a proposal to generate artificial non-Abelian U(3) gauge fields. Our scheme, based on a single-particle approach, is a straightforward generalization of the tripod scheme discussed in [28]. It is based on three space-dependent dark states (DSs) arising from the coupling with resonant one-photon transitions between Zeeman sublevels belonging to different hyperfine states of an alkali atom, such as ⁸⁷Rb, subjected to a magnetic field. In the following, we first introduce the laser scheme we propose and work out the general expressions for both the effective vector and the effective scalar fields. We next discuss two specific laser configurations: the first one gives rise to a non-Abelian U(3) monopole, while the second one gives rise to a non-Abelian SO-like coupling. Finally, we discuss alkaline-earth-metal atoms, taking the fermionic isotope of strontium as a paradigmatic example. In this case however, because the Zeeman shifts of the lowest hyperfine states ¹S₀ are negligible, a slightly different laser configuration is required to appropriately couple the electronic levels.

In both situations, monopole and SO coupling, working with a gauge group larger than U(2) brings potentially more interesting physics, for two main reasons. First, the number of species being larger, new and nontrivial many-body phenomena can emerge. For example, it has been found that a three-color fermionic system with attractive interactions gives rise to a nontrivial trionic ground state separating the usual BEC and BCS phases [29]. Second, the gauge group being larger, it contains more subgroups and these subgroups can still have a nontrivial structure. Therefore, in the presence of interactions, a larger gauge group allows for different symmetry-breaking scenarios for the ground state. For instance, in the situation of a Higgs field coupled to a U(3) gauge field, i.e., extending the 't Hooft monopole to a larger group, it has been found that the U(3)-monopole solutions can have two different kinds of topology, depending on the subgroup, leaving the ground state invariant [30]. Similarly, it has been shown, that, on a square lattice, a system with a U(3) SO coupling, even in the noninteracting regime, has topologically nontrivial states, in contrast to the U(2) case [19].

II. TWO-TRIPOD SCHEME

We consider the two-tripod coupling scheme depicted in Fig. 1. It is based on two usual tripod schemes. One couples ground states $|1\rangle$, $|2\rangle$, $|3\rangle$ to excited level $|6\rangle$. The other couples ground states $|3\rangle$, $|4\rangle$, $|5\rangle$ to the excited level $|7\rangle$. As one can see these two tripod schemes are not independent since they share one common ground-state level, namely, level $|3\rangle$. This very situation can be implemented with alkali atoms, for instance, by considering the D_1 line of ⁸⁷Rb atoms. In this case, one first applies a magnetic field to split the Zeeman structure of both the ground $F_g = 2$ and the excited $F_e = 1$ states and one then shines six suitably polarized resonant laser beams to produce the desired two-tripod coupling scheme shown in Fig. 1. Section V below gives more details about the experimental realization and its limitations.

Since the two-tripod scheme involves five ground states coupled to two excited states, one expects three degenerate

FIG. 1. Two-tripod scheme for the D_1 line of ^{87}Rb . An external magnetic field, chosen as the quantization axis, is applied to the atoms and lifts the Zeeman degeneracy of the ground-state (with total spin $F_g = 2$) and excited-state (with total spin $F_e = 1$) manifolds. Six resonant laser beams then illuminate the atoms, their polarization state being appropriately chosen to address the π and $\sigma_{\pm}$ transitions shown. States $|i\rangle$ ($1 \leq i \leq 5$) refer to the Zeeman levels $|2, m_g\rangle$ ($|m_g| \leq 2$) in the ground-state manifold. States $|i\rangle$ ($i = 6, 7$) refer to the Zeeman levels $|1, m_e = \pm 1\rangle$ of the excited-state manifold. All other excited levels (three of them indicated by horizontal dashed lines) are not addressed since they are far off-resonance from any of the six laser beams. This two-tripod scheme gives rise to three dark states with vanishing energy.

DSs with vanishing energy. Indeed, in the rotating wave approximation, the two-tripod Hamiltonian reads

$$H_0 = -\hbar(\Omega_1^l|6\rangle\langle 1| + \Omega_2^l|6\rangle\langle 2| + \Omega_3^l|6\rangle\langle 3| + \Omega_3^r|7\rangle\langle 3| + \Omega_2^r|7\rangle\langle 4| + \Omega_1^r|7\rangle\langle 5|) + \text{H.c.} \quad (1)$$

We now parametrize the position-dependent Rabi frequencies as [12,28]

$$\begin{aligned} \Omega_1^a &= \Omega_a \sin \theta_a \cos \phi_a e^{iS_1^a}, \\ \Omega_2^a &= \Omega_a \sin \theta_a \sin \phi_a e^{iS_2^a}, \\ \Omega_3^a &= \Omega_a \cos \theta_a e^{iS_3^a}, \end{aligned} \quad (2)$$

where $a = l, r$. The twelve different quantities Ω_i^a , θ_a , ϕ_a , and S_i^a ($i = 1, 2, 3$) are generally space dependent. It is then straightforward to compute the three orthonormal DSs of the two-tripod scheme:

$$|D_l\rangle = \sin \phi_l e^{iS_{31}^l} |1\rangle - \cos \phi_l e^{iS_{32}^l} |2\rangle, \quad (3a)$$

$$|D_r\rangle = \sin \phi_r e^{iS_{31}^r} |5\rangle - \cos \phi_r e^{iS_{32}^r} |4\rangle, \quad (3b)$$

$$\begin{aligned} |D_0\rangle &= \frac{1}{\alpha_0} [\cot \theta_l \cos \phi_l e^{iS_{31}^l} |1\rangle + \cot \theta_l \sin \phi_l e^{iS_{32}^l} |2\rangle - |3\rangle \\ &\quad + \cot \theta_r \cos \phi_r e^{iS_{31}^r} |5\rangle + \cot \theta_r \sin \phi_r e^{iS_{32}^r} |4\rangle], \\ \alpha_0 &= (1 + \cot^2 \theta_l + \cot^2 \theta_r)^{1/2}, \end{aligned} \quad (3c)$$

where $S_{ij}^a = S_i^a - S_j^a$ ($a = l, r$). One may note that when $\Omega_3^r = 0$, corresponding to $\theta_r = \pi/2$, then the two-tripod scheme breaks up into a U(2) tripod configuration coupling states $|1\rangle, |2\rangle, |3\rangle$ to $|6\rangle$ and an independent U(1) Λ configuration coupling states $|4\rangle, |5\rangle$ to $|7\rangle$. Then state $|D_r\rangle$ identifies with the DS of the Λ configuration, while $|D_l\rangle$ and the corresponding $|D_0\rangle$ state identify with the two DSs of the

left-tripod configuration. The same type of considerations can be made if $\Omega_3^l = 0$, corresponding to $\theta_l = \pi/2$. In other words, the states $|D_a\rangle$ ($a = l, r$) are DSs for the left and right U(2) tripod configuration as well as DSs for the Λ configuration. The remaining state $|D_0\rangle$, embodying all five Zeeman ground-state levels, reflects the coupling of the two U(2) tripod configurations when both Ω_3^l and Ω_3^r are nonzero. It boils down to the missing tripod DS when Ω_3^l or Ω_3^r vanishes.

From the DS expressions, (3), one can derive the vector and scalar potentials associated with the two-tripod scheme. The vector potential $\vec{A}$ is now a 3×3 Hermitian matrix with entries

$$\begin{aligned} \vec{A}_{11} &= \cos^2 \phi_l \vec{\nabla} S_{23}^l + \sin^2 \phi_l \vec{\nabla} S_{13}^l, \\ \vec{A}_{33} &= \cos^2 \phi_r \vec{\nabla} S_{23}^r + \sin^2 \phi_r \vec{\nabla} S_{13}^r, \\ \vec{A}_{22} &= \frac{1}{\alpha_0^2} [\cot^2 \theta_l (\cos^2 \phi_l \vec{\nabla} S_{13}^l + \sin^2 \phi_l \vec{\nabla} S_{23}^l) \\ &\quad + \cot^2 \theta_r (\cos^2 \phi_r \vec{\nabla} S_{13}^r + \sin^2 \phi_r \vec{\nabla} S_{23}^r)], \\ \vec{A}_{21} &= \vec{A}_{12}^* = \frac{\cot \theta_l}{\alpha_0} \left(\frac{1}{2} \sin(2\phi_l) \vec{\nabla} S_{12}^l + i \vec{\nabla} \phi_l \right), \\ \vec{A}_{23} &= \vec{A}_{32}^* = \frac{\cot \theta_r}{\alpha_0} \left(\frac{1}{2} \sin(2\phi_r) \vec{\nabla} S_{12}^r + i \vec{\nabla} \phi_r \right), \\ \vec{A}_{31} &= \vec{A}_{13}^* = 0, \end{aligned} \quad (4)$$

where the asterisk denotes complex conjugation. The scalar potential expression is rather involved and is given in Appendix A for the sake of completeness.

III. U(3) MONOPOLE

In the following, we use the spherical coordinate system (r, θ, φ) about axis Oz to parametrize a point $M(x, y, z)$ in space. Then, from Eq. (4), one can check that a gauge field corresponding to a U(3) monopole can be generated by using the Rabi frequencies

$$\begin{aligned} \Omega_{1,2}^l &= \Omega_0 \frac{\rho}{R} e^{i(kz \mp \varphi)}, \quad \Omega_3^l = \Omega_0 \frac{z}{R} e^{ikx}, \\ \Omega_{1,2}^r &= \Omega_0 \frac{\rho}{R} e^{-i(kz \pm \varphi)}, \quad \Omega_3^r = \Omega_0 \frac{z}{R} e^{-ikx}, \end{aligned} \quad (5)$$

where $\rho = \sqrt{x^2 + y^2} = r \sin \theta$. The corresponding laser beam configuration is shown in Fig. 2, the quantization axis being along axis Oy . The three beams addressing the left tripod configuration consist of two linearly polarized copropagating (along axis Oz) Laguerre-Gauss beams with orbital angular momentum $\pm \hbar$ and of a linearly polarized Hermite-Gauss beam propagating along axis Ox . The three remaining beams are just “reflection images” of the previous beams and address the right tripod configuration. The potential vector then reads

$$\begin{aligned} \vec{A} &= -\hbar \frac{\cos \theta}{\sqrt{2} \sin \theta} \frac{\hat{e}_\varphi}{r} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &\quad - \hbar k \sin^2 \theta (\hat{e}_z - \hat{e}_x) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\quad + \hbar k (\hat{e}_z - \hat{e}_x) \mathbb{1}. \end{aligned} \quad (6)$$

FIG. 2. (Color online) Laser beam configuration giving rise to a non-Abelian U(3)-monopole with unit charge and associated with the generator J_x of the SO(3) subgroup. The horizontal beams are Laguerre-Gauss modes carrying opposite orbital angular momentum $\pm\hbar$, shown by the curved (green) arrows. Vertical laser beams are $n = 1$ Hermite-Gauss modes. The bias magnetic field lifting the Zeeman degeneracies in Fig. 1 defines the quantization axis. One possible configuration consists in choosing the magnetic field along Oy . Then all beams are linearly polarized (thin black arrows) and can selectively address their allocated transitions since their polarization states have a nonvanishing projection on the desired $\sigma_{\pm}$ and π transitions.

The last term is inessential: it is a constant gradient term proportional to the unit matrix that can be gauged away through a U(1) transformation. The second term depends on the wave number k and is similar to the k -dependent terms found in the U(2) monopole case [12,28]. It is not singular and leads to nonmonopole terms. We do not discuss it in the following, though it can play an important role in the dynamics [20]. Finally, the first term can be rewritten as $\vec{A}_m = \vec{a}_m J_x$, where $\vec{a}_m = -\cos\theta\hat{e}_\varphi/(r\sin\theta)$. It corresponds to a non-Abelian U(3) monopole with unit effective magnetic charge $Q = 1$ coupled to the spin-1 operator

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (7)$$

Indeed, the corresponding non-Abelian magnetic field $\vec{B}_m = \vec{\nabla} \times \vec{A}_m + \vec{A}_m \times \vec{A}_m / (i\hbar)$ reads $\vec{B}_m = (\vec{\nabla} \times \vec{a}_m) J_x$. Let us first consider a general Abelian vector field of the form $\vec{a} = g(\theta)\hat{e}_\varphi/(r\sin\theta)$. Then the corresponding Abelian magnetic field reads

$$\vec{B} = \vec{\nabla} \times \vec{a} = \frac{g'(\theta)}{\sin\theta} \frac{\hat{e}_r}{r^2} - 2\pi\phi(z)\delta(x)\delta(y)\hat{e}_z, \quad (8)$$

$$\phi(z) = g(0)\Theta(z) + g(\pi)\Theta(-z), \quad (9)$$

where $\Theta(u)$ is the step function. It consists of a monopole contribution B_m given by the first term on the right-hand side of Eq. (8) and of a Dirac-like string contribution with flux $-2\pi\phi(z)$. The magnetic charge Q_m associated with the monopole field is computed with the help of Gauss theorem.

FIG. 3. (Color online) Laser beam configuration generating a non-Abelian U(3) monopole with unit charge and associated with a 3×3 matrix which does not belong to the SO(3) subgroup. It is obtained by flipping the sign of the orbital angular momentum carried by the right horizontal beams. The conventions and polarizations of the beams are the same as in Fig. 2.

It reads

$$Q_m = \frac{1}{4\pi} \int_S \vec{B} \cdot d\vec{S} = \frac{g(\pi) - g(0)}{2}, \quad (10)$$

where S is the sphere of radius r and $d\vec{S} = r^2 \sin\theta d\theta d\varphi \hat{e}_r$ [32]. Since we have $g(\theta) = -\cos\theta$ in our two-tripod situation, we thus get a genuine monopole field with unit magnetic charge coupled to J_x . It is worth noting that, in high-energy physics, it has been established that *all* non-Abelian monopoles are simply obtained as Abelian monopoles times a *constant* (charge) matrix Q [33]. The string is undetectable when $\exp(4\pi i Q/\hbar) = \mathbb{1}$. This is also what we get here.

If, starting from the laser configuration shown in Fig. 2, one just flips the sign of the orbital angular momentum carried by each of the right Laguerre-Gauss fields, i.e.,

$$\Omega_{1,2}^r = \Omega_0^r \frac{\rho}{R} e^{-i(kz \mp \varphi)}, \quad (11)$$

while keeping all the other fields unaffected (see Fig. 3), then only the monopole part of the full U(3) vector potential is modified, and it now reads $\vec{A}_m = \vec{a}_m \tilde{J}_x$, where

$$\tilde{J}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = S J_x S, \quad (12)$$

where S is the diagonal matrix with entries $(1, 1, -1)$ and representing a reflection about the plane (Ox, Oy) . The Abelian monopole with unit charge described by $\vec{a}_m$ is now coupled to the new matrix $\tilde{J}_x$. This monopole is thus associated with a Hermitian matrix which turns out to be a generator, like J_x is, of the rotation subgroup SO(3) of U(3) (see later discussion). One may note that $J_x = \sqrt{2}\hbar(g_1 + g_6)$, while $\tilde{J}_x = \sqrt{2}\hbar(g_1 - g_6)$, where the matrices $g_{1,6}$ are Gell-Mann matrices [31]. For the sake of completeness, we give the eight Gell-Mann matrices g_i ($1 \leq i \leq 8$) in Appendix B. By changing the laser beam configuration, a U(3) monopole associated with another combination of Gell-Mann matrices could be obtained in principle.

Similarly to the U(2) situation studied in [20], understanding the topological properties [30] of the ground state and excitations of interacting particles subjected to a U(3) monopole field would certainly lead to new and interesting physics that could be targeted in cold-atom experiments. More precisely, since the U(3) gauge group is larger than U(2), one expects that different kinds of topological charges show up, depending on the symmetry of the ground state.

IV. SPIN-ORBIT COUPLING

A non-Abelian SO coupling can alternatively be achieved by considering the laser beam configuration shown in Fig. 2 but where all six laser beams are now linearly polarized plane waves with the following Rabi frequencies:

$$\begin{aligned}\Omega_{1,2}^l &= \frac{\Omega_l}{\sqrt{2}} \sin \theta_l e^{\pm i \vec{k}_l \cdot \vec{r}}, & \Omega_3^l &= \Omega_l \cos \theta_l e^{-i \vec{k}_3 \cdot \vec{r}}, \\ \Omega_{1,2}^r &= \frac{\Omega_r}{\sqrt{2}} \sin \theta_r e^{\pm i \vec{k}_r \cdot \vec{r}}, & \Omega_3^r &= \Omega_r \cos \theta_r e^{i \vec{k}_3 \cdot \vec{r}}\end{aligned}\quad (13)$$

($\varphi_r = \varphi_l = \pi/4$). Here we have $\vec{k}_3 \perp \vec{k}_{l,r}$, these vectors being in the plane (Ox, Oz) orthogonal to the quantization axis Oy and the polarization states being the same as in Fig. 2. Simplifying further to the case $\theta_r = \theta_l = \Theta$, the effective vector potential reads

$$\vec{A} = \frac{1}{2} \vec{k}_3 J_z + \frac{2\hbar}{\sqrt{2 + \tan^2 \Theta}} (\vec{k}_l g_1 + \vec{k}_r g_6) \quad (14)$$

$$= \frac{1}{2} \vec{k}_3 J_z + \frac{2\hbar \vec{k}_l}{\sqrt{2 + \tan^2 \Theta}} (g_1 \pm g_6), \quad (15)$$

where the last equality has been obtained by also assuming $\vec{k}_r = \pm \vec{k}_l$. One can generate different types of SO coupling, in the spirit of what is done in [35] to induce a Rashba or a Linear Dresselhaus SO coupling. For example, from the expansion of $(\vec{p} - \vec{A})^2/(2m)$, we get the SO-coupling terms $V_{SO} = \hbar k/(2m) v_{SO}$, where

$$\begin{aligned}v_{SO} &= (J_z/\hbar) p_x + 2\beta(\Theta)(g_1 \pm g_6) p_z, \\ v_{SO} &= 2\beta(\Theta)(g_1 \pm g_6) p_x + (J_z/\hbar) p_z\end{aligned}\quad (16)$$

for $(\vec{k}_3, \vec{k}_l) = (k\hat{e}_x, k\hat{e}_z)$ and for $(\vec{k}_3, \vec{k}_l) = (k\hat{e}_z, k\hat{e}_x)$. Here $\beta(\Theta) = 2/\sqrt{2 + \tan^2 \Theta}$ and equals unity if $\tan \Theta = \sqrt{2}$, i.e., if all left (resp. right) Rabi frequencies are equal. One should note that, contrary to the U(2) scheme, the expansion features the non-Abelian potential $\vec{A} \cdot \vec{A}/(2m)$, which adds up to the non-Abelian potential Φ . As one can easily check, this term is not proportional to the identity and therefore should play a role in the dynamics and in the ground-state properties of an interacting system. For $\tan \Theta = \sqrt{2}$, it reads

$$\frac{\vec{A} \cdot \vec{A}}{2m} = \frac{\hbar^2 k^2}{4m} (1 - g_4). \quad (17)$$

V. EXPERIMENTAL REALIZATION AND LIMITATIONS

In Sec. III, we have shown that non-Abelian U(3)-monopole contributions can be obtained using specific laser beam configurations. More precisely, the common ground state shared by the two tripods, namely, ground-state level $|3\rangle$,

should be coupled to excited states with Hermite-Gauss laser beams propagating along the same axis. The other ground states are coupled to excited states with Laguerre-Gauss beams propagating perpendicularly to the Hermite-Gauss beams [see Eqs. (5) and (11) and Figs. 2 and 3]. Because of these constraints, one cannot solely rely on the laser polarization degrees of freedom and dipole selection rules to independently and selectively address the different transitions of the two-tripod scheme. One also needs to apply a strong magnetic field to lift the Zeeman degeneracy in the ground-state and excited-state manifolds to get well-separated transitions and avoid spurious spontaneous emission processes from unwanted transitions.

For the D_1 line of ^{87}Rb that we used as an example, this Zeeman degeneracy lifting is even favored by the opposite signs of the Landé factors in the ground state ($F_g = 2$) and in the excited state ($F_e = 1$). Regardless of any possible technical issues in generating the required magnetic field, its maximum value is limited by the hyperfine splitting $\Delta_{\text{hs}} \approx 142\Gamma$ of the excited state ($\Gamma = 2\pi \times 5.8$ MHz is the natural line width of the transition); otherwise the coupling to the other hyperfine manifold $F_e = 2$ will start playing a non-negligible role. In addition, since light-induced gauge potentials originate from photon momentum exchanges with the atoms, their energy scale is thus of the order of the recoil energy, $E_R = \hbar\omega_R \approx 0.6 \cdot 10^{-3} \hbar\Gamma$. Therefore, the rate of any residual spontaneous emission from the off-resonant transitions should be lower than $\hbar^{-1} E_R$, which, in the case of ^{87}Rb , might be difficult to achieve. Along the same line of thought, it becomes even more challenging to address the effect of the gauge potential on the atom dynamics over time scales in the millisecond range and beyond [6].

From this point of view, fermionic isotopes of alkaline-earth-metal atoms provide interesting alternatives to rubidium atoms. In particular, the hyperfine splitting of the 3P_1 excited state of the strontium isotope ^{87}Sr is $\Delta_{\text{hs}} \approx 10^5 \Gamma$, at least three orders of magnitude larger than that of alkali atoms. In this case, the left and right tripod configurations can be driven independently (see Fig. 4) and should be well protected from spurious spontaneous emission due to unwanted off-resonant transitions. Furthermore, the narrow line width ($\Gamma = 7.4$ kHz) of the intercombination line at 689 nm leads to a large Zeeman shift (compared to Γ) with reasonable magnetic fields of a few tens of gauss. However, the 1S_0 ground state carries no electronic spin, only a nuclear one, $I = 9/2$. Regarding our laser coupling scheme, the Zeeman shift in the ground state is thus essentially negligible and one now has to rely solely on the polarization states of the beams to address independently and selectively the left and right tripod transitions. More precisely, the Laguerre-Gauss beams addressing the left (resp. right) tripod, and propagating along the Oz axis, must have opposite circular polarizations and should now drive the $1 \leftrightarrow 6$ and $3 \leftrightarrow 6$ $\sigma_{\pm}$ transitions (resp. the $3 \leftrightarrow 7$ and $5 \leftrightarrow 7$ $\sigma_{\pm}$ transitions). The Hermite-Gauss beams, propagating along the Ox axis, must have a linear polarization, along the Oz axis, and address the $2 \leftrightarrow 6$ and $4 \leftrightarrow 7$ π transitions. One immediately sees that the laser beam configuration proposed in Sec. III, and generating a U(3) monopole, fails to meet these polarization constraints. Figure 4 shows the new laser beam configuration with the appropriate quantization axis and polarization states. In the following, Sec. VI, we discuss in some

FIG. 4. (Color online) (a) Two-tripod scheme in the case of ^{87}Sr atoms. An external magnetic field is still applied but, for a reasonable magnetic strength, only the Zeeman degeneracy of the excited state manifold is lifted (the ground-state manifold carries no electronic spin). Now, contrary to the rubidium case, the laser fields must have exactly the polarization required by the transition they need to address. This imposes the Zeeman field to be applied along the quantization axis Oz and the left and right Laguerre-Gauss beams to propagate along Oz with opposite circular polarizations (shown by thin black arrows) to address the $\sigma_{\pm}$ transitions. (b) The vertical Hermite-Gauss beams propagate along Ox and are linearly polarized along Oz to address the π transitions.

detail the properties of the resulting new gauge potentials. In particular, we show that a U(3) monopole with a nonzero charge can still be recovered by using an appropriate gauge transformation.

One may also note that the SO-coupling scheme, depicted in Sec. IV, can be extended to the case of the alkaline-earth-metal atoms, again using appropriate polarization states for the laser beams to address the proper transitions. One finds SO-coupling terms similar to those given in Eq. (16).

The initial preparation of the atoms in a specific DS is rather simple. Starting from a fully spin-polarized sample, all the lasers connecting the empty Zeeman substates of the ground level should be initially switched on. Then a slow ramping of the remaining laser fields guarantees a full transfer into a given DS. The final detection can be done by abruptly turning off the laser fields. In this case, the DSs are mapped onto the bare spin states. The “flavor” texture of the gas can then be measured by using spin-dependent imaging techniques (see Refs. [36] and

[37], for instance). Interestingly, we note that the bare-state basis has a larger dimension than the DS basis. This means that useful information on the relative phase of the DSs could be extracted as well.

We close this section by remembering that the gauge-field description of the atomic dynamics relies on the assumption of an adiabatic evolution which should be fulfilled at any time. This means that the mean Rabi frequency should be much higher than any other characteristic frequencies governing the dynamics of the atomic external degrees of freedom. In particular, one should prevent atoms from going too close to the monopole singularity, where all laser fields vanish, because the adiabatic assumption would break.

VI. ALKALINE-EARTH-METAL ATOMS

A. U(3) monopole

Taking into account the polarization constraints that alkaline-earth-metal atoms bring into the game, a laser configuration that corresponds to a realistic experimental situation is the following:

$$\Omega_{1,3}^l = \Omega_0 \frac{\rho}{R} e^{i(kz \mp \varphi)}, \quad \Omega_2^l = \Omega_0 \frac{z}{R} e^{ikx}, \quad (18)$$

$$\Omega_{1,3}^r = \Omega_0 \frac{\rho}{R} e^{-i(kz \mp \varphi)}, \quad \Omega_2^r = \Omega_0 \frac{z}{R} e^{-ikx}. \quad (19)$$

The corresponding non-Abelian vector potential is

$$\vec{A} = \hbar \frac{\cos \theta \sin \theta}{\sqrt{1 + 2 \sin^2 \theta}} \frac{\hat{e}_{\varphi}}{r} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} + \hbar \frac{(2 - \sin^2 \theta)}{\sin \theta} \frac{\hat{e}_{\varphi}}{r} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + (\dots), \quad (20)$$

where $(\dots)$ represents terms which do not contribute to the singular part of the radial magnetic field such as (nonsingular) k -dependent terms or the $(\varphi$ -independent) $\hat{e}_{\theta}$ component. Each term explicitly written in Eq. (20) corresponds to a non-Abelian monopole-like magnetic field, the first one coupled to $2\hbar(g_1 - g_6)$ and the second one to J_z [which is also a linear combination of the Gell-Mann matrices of the SU(3) group]. However, each of these terms alone is such that $g(\pi) = g(0)$. They thus each carry a vanishing magnetic charge according to Eq. (10) and the present configuration seems not, strictly speaking, to produce a true magnetic monopole. However, let us apply the gauge transformation $U = e^{i\theta \tilde{J}_y / \hbar}$, where

$$\tilde{J}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix} = S J_y S. \quad (21)$$

Since $\tilde{J}_a = S J_a S$ ($a = x, y, z$) satisfies the usual angular momentum commutation relations $[\tilde{J}_a, \tilde{J}_b] = i\hbar \epsilon_{abc} \tilde{J}_c$ (note that $\tilde{J}_z = S J_z S = J_z$), one can use the rotation algebra and easily compute the transformed monopole gauge field

$$\vec{A}' = U \vec{A} U^\dagger + i\hbar U \vec{\nabla} U^\dagger,$$

$$\begin{aligned} \vec{A}' = & (2 - \sin^2 \theta)(\tilde{J}_x + \cot \theta J_z) \frac{\hat{e}_\varphi}{r} + \frac{\cos \theta \sin^2 \theta}{\sqrt{1 + 2 \sin^2 \theta}} \\ & \times (\cot \theta \tilde{J}_x + \tilde{J}_y - J_z) \frac{\hat{e}_\varphi}{r} + (\dots), \end{aligned} \quad (22)$$

where again $(\dots)$ represents the terms which do not contribute to the singular part of the radial magnetic field. The only term describing a true U(3) monopole is

$$\vec{A}'_m = \frac{2 \cos \theta}{\sin \theta} \frac{\hat{e}_\varphi}{r} J_z, \quad (23)$$

$$\vec{B}'_m = -2 \frac{\hat{e}_r}{r^2} J_z, \quad (24)$$

with nonvanishing charge $Q = -2$.

B. Gauge transformations and magnetic charge

The previous situation is similar to the relationship between the U(2) 't Hooft–Polyakov monopole and the U(1) Dirac monopole [34]. A non-Abelian gauge transformation can be used to fully remove the string singularity of the Dirac monopole along the negative z axis by transferring it to the associated Higgs field. In the process, it is the total charge of the monopole and of the Higgs field that is conserved. For example, in Ref. [20], the authors compute the ground-state properties of a Bose-Einstein condensate subjected to a U(2) monopole (namely, an Abelian one coupled to σ_x). They apply a gauge transformation to remove the string singularity (along the full z axis) to get a new non-Abelian gauge field but with a vanishing magnetic charge. This is the kind of situation we face here, where we find a U(3) monopole with a vanishing charge in one gauge and with a nonvanishing charge in another gauge.

Let us illustrate a bit further the modification of the effective magnetic charge of the non-Abelian monopole when a unitary change of the DS basis set is performed by discussing the usual tripod scheme studied in [12] and [28]. It is obtained from our two-tripod scheme by setting $\Omega'_i = 0$. One can build a DS on the Zeeman states $|1\rangle$ and $|2\rangle$ coupled to the excited state $|6\rangle$ by the σ_+ and π Laguerre-Gauss beams. The DS basis set is then

$$\begin{aligned} |D_1\rangle &= \frac{1}{\sqrt{2}} [e^{iS_{31}} |1\rangle - e^{iS_{32}} |2\rangle], \\ |D_2\rangle &= \frac{1}{\sqrt{2}} \cos \theta [e^{iS_{31}} |1\rangle + e^{iS_{32}} |2\rangle] - \sin \theta |3\rangle. \end{aligned} \quad (25)$$

Dropping the k -dependent part for simplicity, these DSs generate the usual U(2)-monopole gauge field, $\cos \theta \sigma_x \hat{e}_\phi / (r \sin \theta)$, which has an effective magnetic charge $Q = -1$.

If, on the contrary, one chooses to build a DS on the Zeeman states $|1\rangle$ and $|3\rangle$ coupled to the excited state by the σ_+ Laguerre-Gauss beam and by the σ_- Hermite-Gauss beam, one instead gets

$$\begin{aligned} |D'_1\rangle &= \frac{1}{\sqrt{1 + \cos^2 \theta}} [\sqrt{2} \cos \theta e^{iS_{21}} |1\rangle - \sin \theta e^{iS_{23}} |3\rangle], \\ |D'_2\rangle &= \frac{1}{2\sqrt{1 + \cos^2 \theta}} [\sqrt{2} \sin^2 \theta e^{iS_{21}} |1\rangle \\ &+ 2 \sin \theta \cos \theta e^{iS_{23}} |3\rangle - \sqrt{2}(1 + \cos^2 \theta) |2\rangle]. \end{aligned} \quad (26)$$

Dropping again the terms which do not contribute to the singular part of the radial magnetic field, the new vector potential is

$$\vec{A}' = \left[\mathbb{1} + \frac{\cos \theta \sin^2 \theta}{1 + \cos^2 \theta} \sigma_x + \frac{2 \cos^2 \theta}{1 + \cos^2 \theta} \sigma_z \right] \frac{\hbar \hat{e}_\phi}{r \sin \theta}, \quad (27)$$

where σ_x and σ_z are the standard Pauli matrices. These expressions bear some resemblance to the U(3) situation [see Eq. (20)]. Here, too, each term gives rise to a vanishing magnetic charge $Q = 0$. On the other hand, the transformation from the $|D'_a\rangle$ basis to the $|D_a\rangle$ basis ($a = 1, 2$) is given by the unitary matrix

$$U = \frac{e^{iS_{23}}}{\sqrt{1 + \cos^2 \theta}} \begin{pmatrix} \cos \theta & 1 \\ 1 & -\cos \theta \end{pmatrix}. \quad (28)$$

This shows that the gauge fields $\vec{A}'$ and $\vec{A}$ are simply related by the gauge transformation $\vec{A}' = U \vec{A} U^\dagger + i\hbar U \vec{\nabla} U^\dagger$. Therefore, the two gauge fields correspond to the same physical situation, a U(2) monopole, but with $\vec{B}' = U \vec{B} U^\dagger$, despite the fact that the effective magnetic charges are different.

This is somehow what we have obtained for the U(3) case above, i.e., the physics of a U(3) monopole but from an unusual gauge perspective. However it is important to note that, contrary to the U(2) case, the U(3) gauge potentials obtained with the laser configuration used for the alkaline-earth-metal atoms and the gauge potentials obtained with the laser configuration used for the rubidium case are *not* related by a gauge transformation. Indeed the DSs obtained in one scheme are linearly independent from the DSs obtained with the other scheme since they span different subspaces of the full Hilbert space. Therefore they cannot be related by a (position-dependent) 3×3 unitary matrix, as required for a proper gauge transformation; only a larger one in the full Hilbert space can.

VII. CONCLUSION

In this paper we have proposed a workable experimental scheme, the so-called two-tripod configuration, to generate non-Abelian U(3) artificial static gauge fields for cold atomic gases. Our scheme only relies on one-photon resonant transitions and gives rise to three degenerate DSs. We have given the laser beam configurations for both alkali-metal and alkaline-earth-metal atoms and explained how to generate a U(3) monopole or a U(3) SO coupling. Future work includes the study of the properties of the ground state and excitations of spinor condensates subjected to such U(3) gauge potentials, in particular, from a topological point of view (spin textures, etc).

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APPENDIX A: GENERAL EXPRESSION FOR THE SCALAR POTENTIAL

The entries for the scalar potential Hermitian matrix Φ read

$$\begin{aligned}
\Phi_{11} &= \frac{1 + \cot^2 \theta_r}{1 + \cot^2 \theta_l + \cot^2 \theta_r} \left[\frac{1}{4} \sin^2(2\phi_l) (\vec{\nabla} S_{12}^l)^2 + (\vec{\nabla} \phi_l)^2 \right], \\
\Phi_{33} &= \frac{1 + \cot^2 \theta_l}{1 + \cot^2 \theta_l + \cot^2 \theta_r} \left[\frac{1}{4} \sin^2(2\phi_r) (\vec{\nabla} S_{12}^r)^2 + (\vec{\nabla} \phi_r)^2 \right], \\
\Phi_{22} &= \frac{\cot^2 \theta_l (1 + \cot^2 \theta_r)}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^2} (\cos^2 \phi_l \vec{\nabla} S_{13}^l + \sin^2 \phi_l \vec{\nabla} S_{23}^l)^2 + \frac{\cot^2 \theta_r (1 + \cot^2 \theta_l)}{(1 + \cot^2 \theta_r + \cot^2 \theta_l)^2} (\cos^2 \phi_r \vec{\nabla} S_{13}^r + \sin^2 \phi_r \vec{\nabla} S_{23}^r)^2 \\
&\quad - 2 \frac{\cot^2 \theta_l \cot^2 \theta_r}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^2} (\cos^2 \phi_r \vec{\nabla} S_{13}^r + \sin^2 \phi_r \vec{\nabla} S_{23}^r) (\cos^2 \phi_l \vec{\nabla} S_{13}^l + \sin^2 \phi_l \vec{\nabla} S_{23}^l) \\
&\quad + \left(\vec{\nabla} \frac{\cot \theta_l}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^{1/2}} \right)^2 + \left(\vec{\nabla} \frac{\cot \theta_r}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^{1/2}} \right)^2 + \left(\vec{\nabla} \frac{1}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^{1/2}} \right)^2, \\
\Phi_{13} &= - \frac{\cot \theta_l \cot \theta_r}{1 + \cot^2 \theta_l + \cot^2 \theta_r} \left(\frac{1}{2} \sin(2\phi_l) \vec{\nabla} S_{12}^l - i \vec{\nabla} \phi_l \right) \left(\frac{1}{2} \sin(2\phi_r) \vec{\nabla} S_{12}^r + i \vec{\nabla} \phi_r \right), \\
\Phi_{12} &= i \vec{\nabla} \left(\frac{\cot \theta_l}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^{1/2}} \right) \left(\frac{1}{2} \sin(2\phi_l) \vec{\nabla} S_{12}^l - i \vec{\nabla} \phi_l \right) + \frac{\cot \theta_l (1 + \cot^2 \theta_r)}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^{3/2}} \left(\frac{1}{2} \sin(2\phi_l) \vec{\nabla} S_{12}^l - i \vec{\nabla} \phi_l \right) \\
&\quad \times (\cos^2 \phi_l \vec{\nabla} S_{13}^l + \sin^2 \phi_l \vec{\nabla} S_{23}^l) - \frac{\cot \theta_l \cot^2 \theta_r}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^{3/2}} \left(\frac{1}{2} \sin(2\phi_l) \vec{\nabla} S_{12}^l - i \vec{\nabla} \phi_l \right) (\cos^2 \phi_r \vec{\nabla} S_{13}^r + \sin^2 \phi_r \vec{\nabla} S_{23}^r), \\
\Phi_{32} &= i \vec{\nabla} \left(\frac{\cot \theta_r}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^{1/2}} \right) \left(\frac{1}{2} \sin(2\phi_r) \vec{\nabla} S_{12}^r - i \vec{\nabla} \phi_r \right) + \frac{\cot \theta_r (1 + \cot^2 \theta_l)}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^{3/2}} \left(\frac{1}{2} \sin(2\phi_r) \vec{\nabla} S_{12}^r - i \vec{\nabla} \phi_r \right) \\
&\quad \times (\cos^2 \phi_r \vec{\nabla} S_{13}^r + \sin^2 \phi_r \vec{\nabla} S_{23}^r) - \frac{\cot \theta_r \cot^2 \theta_l}{(1 + \cot^2 \theta_r + \cot^2 \theta_l)^{3/2}} \left(\frac{1}{2} \sin(2\phi_r) \vec{\nabla} S_{12}^r - i \vec{\nabla} \phi_r \right) (\cos^2 \phi_l \vec{\nabla} S_{13}^l + \sin^2 \phi_l \vec{\nabla} S_{23}^l).
\end{aligned}$$

APPENDIX B: THE GELL-MANN MATRICES

The set of Gell-Mann matrices g_i ($1 \leq i \leq 8$) is one possible representation of the infinitesimal generators of the special unitary group SU(3). An important representation features the 3×3 Hermitian λ matrices $g_i = \lambda_i/2$ with

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned}$$

APPENDIX C: SCALAR POTENTIAL FOR THE U(3) CASE

The scalar potential for the laser scheme described by Eq. (5) (+ sign) or by Eq. (11) (− sign) reads

$$\begin{aligned}
\Phi_{\pm} &= \frac{\hbar^2}{2m} \left[\left(\frac{1}{2r^2 \sin^2 \theta} + \frac{1}{2r^2} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \left(\frac{1}{r^2} + \frac{k^2}{2} \sin^2 2\theta \right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right. \\
&\quad \left. \mp \frac{\cos^2 \theta}{2r^2 \sin^2 \theta} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \frac{\sqrt{2}k}{4r} \sin 2\theta \sin \varphi \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \end{pmatrix} \right].
\end{aligned} \tag{C1}$$

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