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Analysis of one-third harmonic generation in waveguides

Yunxu Sun,^{1,2,4} Xuguang Shao,^{1,*} Tianye Huang,¹ Zhifang Wu,¹ Timothy Lee,³
Shum Ping Perry,¹ and Gilberto Brambilla³

¹*School of Electrical and Electronics Engineering, Nanyang Technological University,
50 Nanyang Avenue, 639798, Singapore*

²*Key Laboratory of Network Oriented Intelligent Computation, School of Electronic and Information Engineering,
Shenzhen Graduate School, Harbin Institute of Technology, Shenzhen 518055, China*

³*Optoelectronics Research Centre, University of Southampton, Southampton, UK*

⁴*e-mail: sunyunxu@hitsz.edu.cn*

**Corresponding author: XGShao@ntu.edu.sg*

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This paper investigates one-third harmonic generation (OTHG) by analytical methods with a nondepletion approximation and with an exact solution in continuous wave conditions. The nondepletion method shows that OTHG with a small initial power is confined in a very small range except that the overlapping integrals of the pump and signal follow a certain relation. The efficiency depends only on the initial conditions. Increasing pump power only shortens the interaction length to reach the maximum conversion. Furthermore, we exactly explore OTHG by the elliptic functions whose expressions depend on the types of roots derived from the integral and the initial power. We find that the output power level is limited by the initial conditions and the structure of the waveguide, while the pump power only determines the period. There is a constant value Γ that is determined by U_0 , θ_0 , and the overlap integrals, where U_0 and θ_0 are the initial power of the signal and the initial phase difference between pump and signal, respectively. We found that a highly efficient conversion only occurs when Γ is larger than a specific value Γ_c , called a critical value. A Γ_c provides a relation between U_0 and θ_0 . So a set of critical conditions of U_0 and θ_0 is obtained. A highly efficient conversion may be supported if U_0 is larger than the power in this set. We investigated some typical structure parameters and found the minimum initial power supporting high conversion efficiency. In OTHG, the variation curve has a sharp peak pattern, which means that a variation of the initial phase difference leads to a great change of the conversion. We established a way to get the smallest initial power with a large phase tolerance. Finally, we find a relation among the overlapping integrals and phase mismatching that can support a high conversion efficiency with a small initial power. This study gives valuable suggestions on the experimental design. © 2014 Optical Society of America

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1. INTRODUCTION

One-third harmonic generation (OTHG), also called third-order parametric fluorescence, is a challenging area that is related to the generation of entangled photon triplets. OTHG is the inversion process of third-harmonic generation (THG) that has been studied in detail and optimized [1]. However, an OTHG experiment is considered significantly more complicated because of the small value of third-order susceptibilities and the phase detuning by Kerr effects [2]. Spontaneous OTHG, in which the initial power of the signal is zero, is a quantum process that generates three entangled photons from vacuum. It has been studied in detail theoretically but has not been demonstrated in experiments [3–6]. An experiment with nonzero initial signals, which is a stimulated process, has been demonstrated in KTP crystal. It was claimed that classical theory can be used to describe the process if the input power of the initial signal is high enough [4,7]. Afshar *et al.* have studied the efficiencies of THG and OTHG and observed that high conversion efficiency only occurred when the ratio of initial power and pump power was larger than a specific value, i.e., 2% [8]. However, they did not provide the

reason for their observation, nor did they provide a way to obtain the ratio. Therefore, more work is necessary to clarify the process of OTHG, such as the effects of the initial phase and the requirement on initial power to support highly efficient conversion and the optimum experiment conditions.

Waveguides may be good devices to demonstrate OTHG in that the overlapping integrals of the signal and pump alter according to the waveguide structure. The continuous wave (CW) condition is a usual experimental condition for OTHG and THG [8,9]. Here, we focus on the CW OTHG process based on waveguides with a small initial signal power using classical theory in CW conditions. By classical coupled equations, the property of signal light evolution is discussed to give valuable suggestions for the experiment design. First, the nondepletion method is used to give a clear concept of OTHG with a very low initial power. Further, based on the former works of the exact solution for four-wave mixing [10], complete analytical solutions are also obtained to study possible high conversion with a very low initial condition. Finally, the influence of the waveguide structure and the initial conditions on OTHG will

be discussed. A set of optimum experimental conditions will be suggested.

2. NONDEPLETION MODEL

CW OTHG is governed by the coupled equations that are derived from classical Helmholtz equations along the z direction by the slowly varying envelope approximation:

$$\frac{\partial A_1}{\partial z} = i\gamma_0[(J_1|A_1|^2 + 2J_2|A_3|^2)A_1 + J_3(A_1^*)^2 A_3 e^{i\delta\beta z}], \quad (1)$$

$$\frac{\partial A_3}{\partial z} = i\gamma_0[(6J_2|A_1|^2 + 3J_5|A_3|^2)A_3 + J_3 A_1^3 e^{-i\delta\beta z}], \quad (2)$$

where the subscripts, 1 and 3, indicate the signal wave with a frequency of ω and the pump wave with frequency of 3ω , respectively. A is the complex amplitude and $\gamma_0 = 2\pi n^{(2)}/\lambda$, $\delta\beta = \beta(3\omega) - 3\beta(\omega)$. J_1 , J_2 , J_3 , and J_5 are field overlap integrals determined by the waveguide structure [8,11], which correspond to the terms for signal self-phase modulation (SPM), cross-phase modulation, pump-signal overlap, and pump SPM [12].

When the nondepletion assumption is implemented, $|A_1|^2 \ll |A_3|^2$, and Eqs. (1) and (2) can be simplified as

$$\frac{\partial A_1}{\partial z} = i\gamma_0[2J_2|A_3|^2 A_1 + J_3(A_1^*)^2 A_3 e^{i\delta\beta z}], \quad (3)$$

$$\frac{\partial A_3}{\partial z} = i3\gamma_0 J_5 |A_3|^2 A_3. \quad (4)$$

Here, the complex amplitudes are decomposed into their absolute amplitudes and phase terms as $A_1 = \rho_1 e^{i\phi_1}$ and $A_3 = \sqrt{P_3} e^{i\phi_3}$, where ρ_1 is the amplitude of the signal wave, P_3 is the pump power with $\rho_1 \ll \sqrt{P_3}$, and ϕ_1 and ϕ_3 are the phase terms of ω and 3ω , respectively. Simple manipulations show that $\phi_3 = 3\gamma_0 J_5 P_3 z$, and the nondepletion coupled equations transform into a set of equations governing ρ_1 and $\theta = \delta\beta z - 3\phi_1 + \phi_3$:

$$\frac{d\rho_1}{dz} \approx -\gamma_0 J_3 \rho_1^2 \sqrt{P_3} \sin \theta, \quad (5)$$

$$\begin{aligned} \frac{d\theta}{dz} &= P_3 \left(\frac{\delta\beta}{P_3} + 3\gamma_0 J_5 - 6\gamma_0 J_2 - 3\gamma_0 J_3 \frac{\rho_1}{\sqrt{P_3}} \cos \theta \right) \\ &\approx P_3 \left(\frac{\delta\beta}{P_3} + 3\gamma_0 (J_5 - 2J_2) \right) = P_3 \delta\beta', \end{aligned} \quad (6)$$

where $\delta\beta' = \delta\beta/P_3 + 3\gamma_0(J_5 - 2J_2)$. The solution is obtained by integration as

$$\begin{aligned} \rho_1(z) &= \frac{P_3 \delta\beta' \rho_1(0)}{P_3 \delta\beta' - \gamma_0 J_3 \rho_1(0) \sqrt{P_3} [\cos(P_3 \delta\beta' z + \theta_0) - \cos \theta_0]} \\ &= \frac{\rho_1(0)}{1 - \frac{\gamma_0 J_3 \rho_1(0)}{\sqrt{P_3} \delta\beta'} [\cos(P_3 \delta\beta' z + \theta_0) + \cos \theta_0]}, \end{aligned} \quad (7)$$

where $\theta_0 = \phi_3(0) - 3\phi_1(0)$. The amplitude is a periodic function with period $2\pi/\delta\beta'$. If $\delta\beta' < 0$, the maximum value of ρ_1 is

$$\rho_1\left(\frac{\pi}{\delta\beta'}\right) = \frac{\rho_1(0)}{1 - \frac{2\gamma_0 J_3 \rho_1(0)}{\sqrt{P_3} \delta\beta'}}. \quad (8)$$

when $\theta_0 = 0$.

Meanwhile, $\rho_1(0)$ is the same maximum when $\theta_0 = \pi$. If $(2\gamma_0 J_3 \rho_1(0))/(\sqrt{P_3} \delta\beta') \ll 1$,

$$\rho_1\left(\frac{\pi}{\delta\beta'}\right) \approx \rho_1(0) + \frac{2\gamma_0 J_3}{\sqrt{P_3} \delta\beta'} \rho_1(0)^2. \quad (9)$$

The maximum conversion is inversely proportional to $\delta\beta'$ and $\sqrt{P_3}$. From Eq. (8), the maximum value will decrease by improving the pump power, which is quite different from other nonlinear phenomena. Accordingly, the maximum conversion efficiency is achieved for longer interaction lengths.

Therefore, we can get some conclusions from the nondepletion result that the highest conversion is determined by θ_0 and is inversely proportional to $\sqrt{P_3}$. A large J_3 or a small $J_5 - 2J_2$ will result in a high conversion. However, the nondepletion assumption is not valid when $2\gamma_0 J_3 \rho_1(0)/\sqrt{P_3} \delta\beta' = 1$, which is a singular point in Eq. (8).

If perfect phase matching is achieved, we can get a relation of overlap integrals with a specific conversion rate, which is $\mu = (\rho_1(\pi/P_3 \delta\beta'))/\rho_1(0)$. Therefore, the structure parameter can be written as

$$\frac{J_5 - 2J_2}{J_3} \approx \frac{4}{3} \frac{1}{1 - \frac{1}{\mu}} \frac{\rho}{\sqrt{P_3}}. \quad (10)$$

This equation can give a guideline for the experimental design. If a high conversion is expected, we have $1/\mu \ll 1$ that leads to

$$\frac{J_5 - 2J_2}{J_3} \approx \frac{4}{3} \frac{\rho}{\sqrt{P_3}}. \quad (11)$$

We also note that the period is proportional to the inverse of $J_5 - 2J_2/J_3$, which means a long interaction is required to get higher conversion efficiencies. If the perfect phase matching condition cannot be attained, high conversion can also be achieved by

$$\frac{\delta\beta}{P_3 J_3} + \gamma_0 \frac{3(J_5 - 2J_2)}{J_3} = \frac{1}{1 - \frac{1}{\mu}} \frac{\rho_1(0)}{\sqrt{P_3}}. \quad (12)$$

So if the structure is determined, Eq. (12) can be satisfied to get a high conversion by changing $\delta\beta$. The above conclusions give us a suggestion of the experimental conditions.

3. EXACT SOLUTION OF OTHG

To get an exact solution of the OTHG equations, the following substitutions are adopted: $A_1 = \sqrt{UP} e^{i\phi_1}$ and $A_3 = \sqrt{(1-U)P} e^{i\phi_3}$, where P is the total power of ω_1 and ω_2 , and U is the ratio of signal power to P , which is called the power for convenience. By definitions $\theta = \delta\beta z - 3\phi_1 + \phi_3$ and $\zeta = \gamma_0 J_3 P z$, the coupled equations transform into

$$\frac{dU}{d\zeta} = -2\sqrt{U^3(1-U)} \sin \theta, \quad (13)$$

$$\frac{d\theta}{d\zeta} = (\Delta S + b) + \frac{\cos \theta \frac{d\sqrt{U^3(1-U)}}{d\zeta}}{\sin \theta \sqrt{U^3(1-U)}} + (a - b)U, \quad (14)$$

where ΔS , a , and b are defined as

$$\Delta S = \frac{\delta\beta}{\gamma_0 J_3 P}, \quad (15)$$

$$a = \frac{3(2J_2 - J_1)}{J_3}, \quad (16)$$

$$b = \frac{3(J_5 - 2J_2)}{J_3}. \quad (17)$$

Equation (14) is solved by straight integration leading to

$$\Gamma = \sqrt{U^3(1-U)} \cos \theta - \frac{a-b}{4} U^2 - \frac{b+\Delta S}{2} U, \quad (18)$$

where Γ is a constant determined by the initial conditions. Equation (18) is substituted into Eq. (13) to get

$$\zeta = \int_{U_0}^{U(\zeta)} \frac{1}{\sqrt{f(U)}} dU, \quad (19)$$

$$f(U) = 4(1-U)U^3 - 4\left(\frac{1}{4}(a-b)U^2 + \Gamma + \frac{1}{2}U(b+\Delta S)\right)^2. \quad (20)$$

Equation (19) is an elliptic integral, which is dependent on the four roots of $f(U) = 0$. The sign of the integral is determined by the opposite sign of $\sin \theta_0$. The roots may be all real or contain two complex conjugates and two real numbers, which depend on the value of Γ and a , b . As defined, U is in the interval $[0, 1]$.

From the integration of Eq. (19), we can understand why spontaneous OTHG is not supported in classical theory. If the initial power, U_0 , is zero, the solution is not allowed. So spontaneous OTHG is not supported by classical theory and must be described by quantum mechanics.

From the property of Eq. (19) and its physical limitations, U varies in an interval determined by the two real roots, where U_0 locates. Accordingly, there are two conditions for the roots of $f(u)$: (1) two of them are complex conjugate, whereas the other two are real and (2) all four roots are real. In the former condition, $\eta_1 = \eta_2^*$, $\eta_3 < U_0 < \eta_4$, the solution is given as

$$U = \frac{\eta_4 N + \eta_3 M - (\eta_4 M - \eta_3 N) \operatorname{cn}\left(\frac{\zeta \pm \zeta_0}{\zeta_c}, k\right)}{M + N + (M - N) \operatorname{cn}\left(\frac{\zeta \pm \zeta_0}{\zeta_c}, k\right)}, \quad (21)$$

where $\operatorname{cn}(x, k)$ is the Jacobi elliptic function and other items are defined as

$$m = \operatorname{Re}\left(\frac{\sqrt{-(\eta_1 - \eta_2)^2}}{2}\right), \quad (22)$$

$$n = \operatorname{Re}\left(\frac{\eta_1 + \eta_2}{2}\right), \quad (23)$$

$$M^2 = \frac{(\eta_4 - n)^2}{2} + m^2, \quad (24)$$

$$N^2 = (\eta_3 - n)^2 + m^2, \quad (25)$$

$$g = \frac{1}{\sqrt{MN}}, \quad (26)$$

$$k^2 = \frac{(\eta_4 - \eta_3)^2 - (M_N)^2}{4MN}, \quad (27)$$

$$c_0^2 = 4 + \frac{a^2}{4} - \frac{ab}{2} + \frac{b^2}{4}, \quad (28)$$

$$\zeta_c = \frac{g}{c_0}, \quad (29)$$

$$\zeta_0 = \zeta_c F\left(\cos^{-1}\left(\frac{(\eta_4 - U_0)N - (U_0 - \eta_3)M}{(\eta_4 - U_0)N + (U_0 - \eta_3)M}\right), k\right). \quad (30)$$

When all four roots are real, the solution is expressed by Jacobi elliptic function cn . The roots are $\eta_1 < \eta_2 < \eta_3 < \eta_4$. If $\eta_1 < U_0 \leq \eta_2$, the solution is given as

$$U(\zeta) = \frac{\eta_1 \eta_4 - \eta_2 \eta_1 + (\eta_4 \eta_2 - \eta_1 \eta_4) \operatorname{sn}^2\left(\frac{\zeta \pm \zeta_0}{\zeta_c}, k\right)}{\eta_4 - \eta_2 + (\eta_2 - \eta_1) \operatorname{sn}^2\left(\frac{\zeta \pm \zeta_0}{\zeta_c}, k\right)}, \quad (31)$$

where sn is the Jacobi elliptic function; c_0 and ζ_c are the same as Eqs. (28) and (29); and ζ_0 , g , and k are redefined as

$$\zeta_0 = \zeta_c F\left(\sin^{-1}\sqrt{\frac{(\eta_4 - \eta_2)(U_0 - \eta_1)}{(\eta_2 - \eta_1)(\eta_4 - U_0)}}, k\right), \quad (32)$$

$$g = \frac{2}{\sqrt{(\eta_4 - \eta_2)(\eta_3 - \eta_1)}}, \quad (33)$$

$$k = \sqrt{\frac{(\eta_4 - \eta_3)(\eta_2 - \eta_1)}{(\eta_4 - \eta_2)(\eta_3 - \eta_1)}}. \quad (34)$$

g is the same for all the solutions with four real roots.

If $\eta_2 \leq U_0 < \eta_3$, the solution is given as

$$U = \frac{\eta_4 \eta_3 - \eta_3 \eta_2 + (\eta_4 \eta_2 - \eta_4 \eta_3) \operatorname{sn}^2\left(\frac{\zeta \pm \zeta_0}{\zeta_c}, k\right)}{\eta_4 - \eta_3 + (\eta_2 - \eta_3) \operatorname{sn}^2\left(\frac{\zeta \pm \zeta_0}{\zeta_c}, k\right)}, \quad (35)$$

where ζ_0 , k , and ζ_c are defined as

$$k = \sqrt{\frac{(\eta_3 - \eta_2)(\eta_4 - \eta_1)}{(\eta_4 - \eta_2)(\eta_3 - \eta_1)}}, \quad (36)$$

$$\zeta_0 = \zeta_c F\left(\sin^{-1} \sqrt{\frac{(\eta_4 - \eta_2)(\eta_3 - U_0)}{(\eta_3 - \eta_2)(\eta_4 - U_0)}}, k\right). \quad (37)$$

If $\eta_3 \leq U_0 < \eta_4$, the solution is given as

$$U = \frac{\eta_4 \eta_1 - \eta_4 \eta_3 + (\eta_3 \eta_1 - \eta_4 \eta_1) \operatorname{sn}^2\left(\frac{\zeta \pm \zeta_0}{\zeta_c}, k\right)}{\eta_1 - \eta_3 + (\eta_3 - \eta_4) \operatorname{sn}^2\left(\frac{\zeta \pm \zeta_0}{\zeta_c}, k\right)}, \quad (38)$$

where ζ_0 and k are defined as

$$k = \sqrt{\frac{(\eta_4 - \eta_3)(\eta_2 - \eta_1)}{(\eta_4 - \eta_2)(\eta_3 - \eta_1)}}, \quad (39)$$

$$\zeta_0 = \zeta_c F\left(\sin^{-1} \sqrt{\frac{(\eta_3 - \eta_1)(\eta_4 - U_0)}{(\eta_4 - \eta_1)(U_0 - \eta_1)}}, k\right). \quad (40)$$

4. DISCUSSION

A. OTHG in Certain Structure

OTHG can reach a high conversion only when the initial power is larger than a certain value, which is also reported by Ref. [8]. Here, the initial conditions supporting high conversion under certain waveguide structures will be discussed by the results in Section 3. As shown previously, the conversion process is related to initial conditions.

We assume the field overlapping integrals as $J_5 = 6.0879 \times 10^{10} \text{ m}^{-2}$, $J_1 = 3.3468 \times 10^{10} \text{ m}^{-2}$, $J_2 = 3.2300 \times 10^{10} \text{ m}^{-2}$, and $J_3 = 2.9317 \times 10^{10} \text{ m}^{-2}$, which leads to $a = 3.18$ and $b = -0.38$. The four roots depend on the initial value, Γ , that is determined by U_0 and $\cos \theta_0$. As U_0 is between 0 and 1 and $\cos \theta_0$ is between -1 and 1, Γ is between -0.831 and 0.130. So the four roots are the functions of Γ by the general root expression of $f(u)$. There are two complex conjugate roots unless Γ is between 0 and 0.0008. The two real roots and the absolute of imaginary of the complex roots are plotted in Fig. 1. We found that there are four real roots only when Γ is in the interval $[0, 0.0008]$. By studying Eq. (18), we found that U_0 is between the two lower roots or between the two upper roots.

The point at which η_2 and η_3 converge is a critical point that determines the critical value, Γ_c . If Γ is larger than Γ_c or negative, U_0 locates between the two real roots, $\eta_3 < \eta_4$. As U varies just between η_3 and η_4 , a set of roots with small η_3 and large η_4 results in a high conversion efficiency with a small initial power. For clarity, the range of Γ and the roots are shown in Table 1.

Table 1 gives the range of Γ and U_0 for simplicity.

If U_0 is between the lower roots, a high conversion efficiency is not supported because the highest conversion limit is η_2 , which is always small, shown as the gray area in Fig. 1. By Eq. (18), Γ_c can determine a set of U_0 with θ_0 shown in Fig. 2(a). U_0 is in the interval $[0.0035, 0.013]$. Accordingly, the minimum initial power, U_0 , supporting a high conversion efficiency is derived as 0.003 with $\theta_0 = 0$, where the roots are $(0.01270 - i0.0018, 0.01270 + i0.0018, 0.0035, 0.7178)$. So U varies between 0.0035 and 0.7178. A high conversion may be expected. As shown in Ref. [8], a high conversion is

Fig. 1. Two real roots of $f(u) = 0$, $\eta_3 < \eta_4$, are plotted depending on Γ . There is an interruption of η_3 in the dashed area that is plotted in the inset graph. There are four real roots when γ is between zero and the critical value, highlighted in the inset graph. The distribution of corresponding initial power, shown as the shaded area, is in the interval $[\eta_1, \eta_2]$, which means only a low conversion is supported in this interval.

supported only when U_0 is larger than 0.02. Here, we can extend the high conversion range of U_0 to 0.0035. However, the initial phase, θ_0 , plays a key role in the conversion type when U_0 is between 0.0035 and 0.013.

The first three roots with the cosine of different θ_0 are plotted in Figs. 2(b)–2(d). The points in Figs. 2(b)–2(d) are the critical points. There are four real roots if U_0 is smaller than the corresponding U_0 of the points, whereas two real roots are there when U_0 is larger. At these points, Γ is calculated as 0.0008.

If the θ_0 slightly changes from 0, the critical initial power becomes large, which means that Γ , larger than Γ_c , does not support high conversion anyway. So the phase difference between pump and signal must be controlled strictly, which is surely difficult. Therefore, a relatively large value of U_0 between 0.0035 and 0.013 is chosen for a reasonable phase tolerance.

By calculating with the results in Section 3, Fig. 3(a) shows the evolution of U on the conditions as $U_0 = 0.012$, and θ_0 equals $\pi, 0, \pi/2, 2\pi/3, \pi/3$, respectively. The positive θ_0 shares the same evolution function as the negative one after shifting of ζ becomes ζ_c from $-\zeta_c$. We can see that the initial phase difference influences the position of highest conversion. Usually, the phase difference between two lasers is stochastic. So the position of highest conversion is random at a large distance. This suggests that the phase difference be controlled strictly to get a high conversion with a specific interaction length. The corresponding solutions, simulating

Table 1. U_0 and Roots

Γ	Roots	U_0
$-0.5 < \Gamma < 0$	$\eta_1 = \eta_2^*, \eta_3 < \eta_4$	$\eta_3 < U_0 < \eta_4$
$0 < \Gamma < \Gamma_c$	$\eta_1 < \eta_2 < \eta_3 < \eta_4$	$\eta_1 < U_0 < \eta_2$
		$\eta_3 < U_0 < \eta_4$
$\Gamma_c < \Gamma < 0.2$	$\eta_1 = \eta_2^*, \eta_3 < \eta_4$	$\eta_3 < U_0 < \eta_4$

Fig. 2. Critical value, Γ_c , determines a set of U_0 and θ_0 , shown in (a), from which we obtain the range of the corresponding U_0 . The first three roots depending on initial power U_0 are plotted with different θ_0 as π , $\pi/2$, and 0 , shown in (b), (c), and (d), respectively. When U_0 is larger than the value determined on the line in (a), a highly efficient conversion is supported. Critical U_0 increases with the increase of θ_0 from 0 to π .

the exact coupled wave equations, as Eqs. (1) and (2) by the ordinary differential equations (ODE) numerical method, are shown in Fig. 3(b), which is consistent with Fig. 3(a).

If the initial power is very small, such as 0.0001 , the solution of $f(u)$ has four real roots. As shown in Fig. 1, U_0 locates between the two smallest roots. Accordingly, the variation of signal is denoted by Eq. (31). Figure 4(a) shows that the variation pattern changes with θ_0 . When θ_0 is zero, the power of the signal will transfer into the pump, which is a THG process rather than OTHG. This is similar to the result derived from the nondepletion method. Figure 4(b) gives the corresponding

Fig. 3. (a) Initial power is put at 0.012 for a tolerance of phase. The variations of U with different θ_0 are plotted. It is shown that power of the signal decreases when $\theta_0 = \pi$, which is a THG process. Other values of θ_0 support high efficiency. But the variation curves are shifted by the change of θ_0 . So the initial phase must be controlled strictly in order to get a stable high signal output with a specific interaction length. (b) A standard ODE numerical method is used to evaluate the exact coupled wave equations [Eqs. (1) and (2)] at the corresponding conditions in (a). The results in (a) and (b) are consistent.

result by ODE numerical simulation of the exact equations (1) and (2).

In summary, for a specific structure that leads to a and b , the critical value is obtained by analyzing the roots of $f(u) = 0$. By Eq. (18), a set of U_0 and θ_0 is determined. For a good tolerance of phase shifting, a medium value should be selected as the initial power.

B. Optimization of Experiment

Here, we begin to discuss the way to decrease the critical value in order to support highly efficient conversion.

As shown in Section 2 by the nondepletion method, we find that the value of $J_5 - 2J_2/J_3$, proportional to b as shown in Eq. (17), plays a key role in OTHG with low initial power. If it gets a small initial value, a high conversion may be obtained with a large interaction length. In particular, the conversion is infinity when it is near zero or near $-(\delta\beta/J_3\sqrt{P_3})$, which is beyond the approximation of the nondepletion method. However, we can also get information that the conversion efficiency may be higher if b is near zero when a perfect phase match is achieved as $\Delta S = 0$.

If $b = 0$, $f(u)$ transforms to

$$f(U) = 4(1 - U)U^3 - \left(\frac{a}{4}U^2 + \Gamma\right)^2. \quad (41)$$

Furthermore, we consider the situation in which a is also zero. Γ is zero by the initial phase difference as $\pi/2$; the solution is obtained as

Fig. 4. (a) Extremely small initial power is put as 10^{-4} . θ_0 is chosen as π , $\pi/2$, and 0 . $\theta_0 = 0$ leads to high efficiency, whereas $\theta_0 = \pi$ leads to reduction of power. (b) A standard ODE numerical method is used to evaluate the exact coupled wave equations [Eqs. (1) and (2)] at the corresponding conditions in (a). The results in (a) and (b) are consistent.

$$U(\zeta) = \frac{4}{4 + (\zeta_0 - \zeta)^2}, \quad (42)$$

where $\zeta_0 = 2\sqrt{1/U_0 - 1}$. ζ_0 is the position of maximum conversion, which is inversely related to U_0 .

If $\Gamma \neq 0$, it follows the procedure in Section 3. We use the small initial power in Section 3 as $U_0 = 0.0001$. By Eq. (41), we can get the power evolution with a maximum conversion up to 100% with a long distance and a sharp peak. If $a \neq 0$, it also follows the procedure in Section 3. If a varies between -8 and 8 , and Γ varies between -2.06024 and 2.06024 , the two roots are functions of a and Γ , as shown in Fig. 5.

We can find that there is a large difference between η_3 and η_4 when $a < 1$, which implies that an efficient conversion may be obtained in this region. Using the same initial condition as above, we can find the maximum of the third root is much smaller than the initial value, U_0 , which means U can reach a large value. A smaller a leads to a small period. Figure 6(b) gives two examples with a as 8 and 0.1 , respectively, which are evaluated by the exact coupled wave equations [Eqs. (1) and (2)] with the ODE numerical method. Figure 6(a) is η_4 depending on a . As discussed above, η_4 is the upper limit of U , so a large value of a will lead to a small conversion.

According to the result in Eq. (11), a high conversion may be expected when b equals $-\sqrt{U_0}$. Accordingly, we get

Fig. 5. Value of two real roots, η_3 and η_4 , depending on a and Γ , is shown in (a) and (b), respectively. It is found that the interval in which U varies has a large upper limit and a small lower limit at the circle region, which means a possible high conversion with small initial power. In the rest region, the difference between the upper and lower limits is small and does not lead to highly efficient conversion.

$b = -0.01$ as the initial power 0.0001 . a is assumed as 1 . As shown in Fig. 7, the variation of signal is plotted by the ODE numerical method, in which we confirm that a highly efficient conversion occurs.

However, to design a waveguide satisfying the requirement is difficult. From Eq. (18), if ΔS is not zero, dependence of Γ

Fig. 6. (a) η_4 is the upper limit of the conversion. A large a causes a small upper limit so that the maximum of conversion is small. (b) Variations with a as 0.1 and 1 , simulated by the exact coupled wave equations [Eqs. (1) and (2)]. A large a leads to a long length with a higher efficiency and a smaller maximum value. So a small value of a is desirable.

Fig. 7. Highly efficient conversion of OTHG is supported when b equals $-\sqrt{U_0}$, evaluated by the ODE numerical method by the exact coupled wave equations [Eqs. (1) and (2)].

Fig. 8. OTHG process discussed on the condition of $a = 1$, $U_0 = 10^{-4}$, and $\theta_0 = 0.5$. (a) When $b = -\Delta S$ is satisfied, a high conversion can be obtained. A larger value of b leads to a longer length to get the maximum conversion. (b) When $b = -0.1$ is supposed, there are only two real roots at this condition. The smaller one, η_3 , is plotted depending on ΔS . It is found that η_3 is larger than U_0 when ΔS is in the interval $[0.095, 0.13]$, which means that a high conversion is supported in this interval. (c) Variations of U with different ΔS , as 0.095, 0.105, and 0.115, respectively, are plotted. The lengths to get the maximum conversion are different according to the values of ΔS . (a) and (c) are simulated by the exact coupled wave equations [Eqs. (1) and (2)].

on $\Delta S + b$ is similar to dependence of Γ on b with ΔS being zero. If ΔS is the negative of b , we can get the result that b is zero. We set $a = 1$ and initial conditions as $U_0 = 10^{-4}$ and $\theta_0 = 0.5$. When $b = \Delta S$ is satisfied, a high conversion can be obtained.

Numerical simulation of Eqs. (1) and (2) confirms that a highly efficient conversion is supported by both values of b as 0.1 and 4, shown in Fig. 8. However, a larger value of b leads to a longer length to get the maximum conversion. A small value of b is desired in a variation with short length. To get a highly efficient OTHG experiment, we should let the sum of ΔS and b be as small as possible. In Fig. 8(b), the smaller real root is plotted with dependence of ΔS . Highly efficient conversion may be supported at the interval of ΔS as $[0.075, 0.14]$ in that η_3 is smaller than U_0 . The variation patterns of U with different ΔS is shown in Fig. 8(c). It is obvious that highly efficient conversion is supported by all these values of ΔS .

For an experimental demonstration, an initial signal power and a pump power are determined. The waveguide should be designed to let the value of b be as small as possible. And the mismatching parameter, $\delta\beta$, should be tuned to equal b to support a highly efficient conversion during the experiment. So we suggest a guideline to construct the structure parameter supporting a high conversion OTHG from a low initial power.

a and b determined by J_1 to J_5 should be designed as small as possible. The mismatching factor is considered to be around the negative value of b .

5. CONCLUSION

Using the nondepletion and exact solution, we analyze the OTHG process to find the dependence on the initial condition and structure parameter. The critical value Γ_c that derives a set of critical U_0 and θ_0 is found. High conversion can be achieved only from the initial power larger than the specific ratio of the whole power for a specific structure, whereas the conversion is very low. $\Delta S + b \approx 0$ causes the critical value Γ_c to be low, which supports a high conversion efficiency with a small initial phase. This work can give a useful direction to the experimental demonstration of OTHG.

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