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Large Circular Loops in the Estimation of an Incident Emitter's Direction-of-Arrival or Polarization

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Abstract—Magnetically *small* loops have been the focus of the research literature on loop-antenna signal processing, but such magnetically small loop-antennas are electromagnetically inefficient. Magnetically *large* loop-antennas will instead be analyzed here in this paper, in the context of estimating an incident source's direction-of-arrival or polarization. Specifically, three large loop-antennas here are collocated and are oriented orthogonally, in order to measure all three Cartesian components of the incident magnetic field simultaneously all at one specific spatial position. This orthogonal triad offers azimuth-elevation bivariate directivity despite the three loops' spatial collocation. For such a triad of magnetically large loops, this paper (first in the open literature) formulates the array manifold, develops the corresponding algorithms in closed form to estimate an incident source's azimuth-elevation direction-of-arrival or polarization, and demonstrates these proposed estimators' precision as close to the Cramér-Rao bounds.

Index Terms—antenna arrays, antenna measurements, antenna radiation patterns, antennas, array signal processing, arrays, direction-of-arrival estimation, directive antennas, loop antenna arrays, loop antennas, parameter estimation, polarimetry, polarization, polarization estimation, radio direction finding, signal processing antennas.

I. INTRODUCTION

Please consider three magnetic loops in spatial collocation and in perpendicular orientation among themselves. These loops' spatial collocation realizes physical compactness, thus ease for transportation and deployment, whereas the three loops' orthogonal orientation facilitates the measurement of the incident magnetic-field vector by all its three distinct Cartesian components.

For *small* loops, such a triad has already been considered in the open literature for estimating incident sources' azimuth-elevation directions-of-estimation and polarizations, for example in [8], [12]–[14], [21], [25]. Among these references, [8], [14], [21] offer closed-form algorithms. Such estimates' error variance, Cramér-Rao bounds and other lower bounds have been derived in closed forms in [13], [25]. Nonetheless, all aforementioned references implicitly presume the loops

to be magnetically *small*, such that the loop-triad's 3×1 array manifold would simply equal the incident wavefield's magnetic-field vector expressed in Cartesian coordinates. This simplifying assumption would be approximately valid for circular loops with a radius of $R < \frac{\lambda}{20\pi}$.

Such a small loop, however, is *ineffective* as a radiator and is barely effective as a receiver, due to the small ratio of radiation-to-loss resistance [11, p. 231]¹. To be a practical antenna, the loop radius R should substantially exceeds $\frac{\lambda}{20\pi}$. Such magnetically *large* loops have nonetheless been entirely overlooked by the signal-processing literature on direction finding or polarization estimation.

On the other hand, for any circular loop-antenna with a circumference of $2\pi R \geq 3.8317\lambda$ (3.8317 equals the smallest positive zero of the Bessel function of the first kind and order 1), the loop-antenna's beam-pattern would suffer from sidelobes [11, p. 249]. Indeed, for $2\pi \frac{R}{\lambda}$ taking on any value near any zero of the Bessel function (like $2\pi \frac{R}{\lambda} \sim 3.8317$), the loop antenna would give a very small voltage, (i.e., would become electromagnetically inefficient), hence should be avoided.

Given the two preceding paragraphs' considerations on $\frac{R}{\lambda}$, real-world loop-antennas often have a wavelength-normalized circumference of $2\pi \frac{R}{\lambda} \in (0.1, 3.8317)$.² There are available [6], [7] hardware implementations of a triad of magnetically large loops in collocation and orthogonal orientation.^{3,4,5} For such magnetically “large” circular-loop antennas, this paper (i) will formulate a new array manifold for three magnetically large circular loops that are collocated and orthogonally oriented, (ii) will pioneer closed-form signal-processing algorithms for the abovementioned triad to estimate the incident

¹ Based on equations (2-90b), (5-24), and (5-64a) in [11], the radiation efficiency is only 4.5% for a loop with a circumference of $2\pi R = \frac{\lambda}{10}$, 42% if $2\pi R = \frac{\lambda}{4}$, 63% if $2\pi R = \frac{\lambda}{3}$, but 97% if $2\pi R = \lambda$.

² A circular loop with circumference equal to one wavelength is commonly known as a resonant loop and is considered as fundamental as the half-wavelength dipole (pp. 249-250, Section 5.3.2 of [11]). Such a loop is classified as an end-fire antenna, as the maximum radiation occur along the axis of loop, therefore making it useful in directional antenna arrays, for example in Yagi-Uda arrays. The resonant loop is significant also because its radiation resistance approximately equals 100Ω , which can be matched easily to a 50Ω or a 75Ω transmission line.

³ Such a triad's antenna electromagnetics has been analyzed in [9].

⁴ Such a triad of magnetically *small* loops has been physically implemented in [24].

⁵ An orthogonal pair of electrically large loops of size up to $2\pi \frac{R}{\lambda} = \frac{1}{2}$ has been implemented in hardware in [23].

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signal's azimuth-elevation direction-of-arrival and polarization, and (iii) will show that these newly proposed estimators can offer an error variance approaching the corresponding Cramér-Rao lower bounds.

The rest of this paper is organized as follows: Section II will first develop the notation. Section III will derive a magnetically large circular-loop antenna's "effective length", and will then formulate the loop-triad's array manifold. Section IV will propose new eigen-based estimates of an incident source's arrival direction nor polarization without any prior knowledge of the loops' radius or of the incident signal's wavelength. Section V will present Monte Carlo simulations to verify these new estimators' efficacy as well as statistical closeness to the Cramér-Rao lower bounds, and to show these new estimators as robust against imperfect prior information. Section VI will conclude the entire paper.

II. REVIEW OF BACKGROUND PRELIMINARIES

A. Spatial Coordinates for the Subsequent Analysis

To facilitate the subsequent exposition, the following spatial angles are defined graphically in Figure 1:

- 1) $\theta_x \in [0, \pi]$ is the angle between the positive x -axis and the incident source's propagation direction,
- 2) $\theta_y \in [0, \pi]$ is the angle between the positive y -axis and the incident source's propagation direction,
- 3) $\theta_z \in [0, \pi]$ is the polar angle (a.k.a. the zenith angle) between the positive z -axis and the incident source's propagation direction,
- 4) $\phi_x \in [-\pi, \pi]$ is the (azimuth) angle between the projection of the positive x -axis and the incident source's propagation direction on the x - o - y plane,
- 5) $\phi_y \in [-\pi, \pi]$ is the angle between the projection of the positive y -axis and the incident source's propagation direction onto the y - o - z plane,
- 6) $\phi_z \in [-\pi, \pi]$ is the angle between projection of the positive z -axis and the incident source's propagation direction on the x - o - z plane.

The spatial angles are interrelated:

$$\begin{aligned}
 \sin(\theta_x) &= \sqrt{\sin^2(\theta_z) \sin^2(\phi_x) + \cos^2(\theta_z)}, \\
 \cos(\theta_x) &= \sin(\theta_z) \cos(\phi_x), \\
 \sin(\theta_y) &= \sqrt{\sin^2(\theta_z) \cos^2(\phi_x) + \cos^2(\theta_z)}, \\
 \cos(\theta_y) &= \sin(\theta_z) \sin(\phi_x), \\
 \sin(\phi_y) &= \frac{\cos(\theta_z)}{\sqrt{\sin^2(\theta_z) \sin^2(\phi_x) + \cos^2(\theta_z)}}, \\
 \cos(\phi_y) &= \frac{\sin(\theta_z) \sin(\phi_x)}{\sqrt{\sin^2(\theta_z) \sin^2(\phi_x) + \cos^2(\theta_z)}}, \\
 \sin(\phi_z) &= \frac{\sin(\theta_z) \cos(\phi_x)}{\sqrt{\sin^2(\theta_z) \cos^2(\phi_x) + \cos^2(\theta_z)}}, \\
 \cos(\phi_z) &= \frac{\cos(\theta_z)}{\sqrt{\sin^2(\theta_z) \cos^2(\phi_x) + \cos^2(\theta_z)}}.
 \end{aligned}$$

B. Vector Representation of the Electromagnetic Field at One Location.

Consider a fully polarized plane-wave of unit power per unit area ⁶, with an auxiliary polarization angle of $\gamma \in [0, \frac{\pi}{2}]$ and a polarization phase difference of $\eta \in [-\pi, \pi)$. Suppose this electromagnetic wave impinges upon the Cartesian origin from a polar angle of $\theta_z \in [0, \pi]$ and an azimuth angle of $\phi_x \in [-\pi, \pi)$. The electric-field vector may then be represented in Cartesian coordinates as

$$\begin{aligned}
 \mathbf{e} &:= e_x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e_y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + e_z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} \\
 &= \underbrace{e_{\theta_z}}_{\sin(\gamma)e^{j\eta}} \underbrace{\begin{bmatrix} \cos(\phi_x) \cos(\theta_z) \\ \sin(\phi_x) \cos(\theta_z) \\ -\sin(\theta_z) \end{bmatrix}}_{\mathbf{u}_{\theta_z}} \\
 &\quad + \underbrace{e_{\phi_x}}_{\cos(\gamma)} \underbrace{\begin{bmatrix} -\sin(\phi_x) \\ \cos(\phi_x) \\ 0 \end{bmatrix}}_{\mathbf{u}_{\phi_x}} \\
 &= \begin{bmatrix} \cos(\phi_x) \cos(\theta_z) & -\sin(\phi_x) \\ \sin(\phi_x) \cos(\theta_z) & \cos(\phi_x) \\ -\sin(\theta_z) & 0 \end{bmatrix} \begin{bmatrix} e_{\theta_z} \\ e_{\phi_x} \end{bmatrix}. \quad (1)
 \end{aligned}$$

⁶The definitions here are consistent with the notion of a "unit power per unit area" in the electromagnetic wave, if a) the units of electric field are volts per meter, b) a unit area is one square meter, and c) the free space impedance is normalized to 1 ohm.

Fig. 1: Different spatial angles inter-relationship.

The corresponding magnetic-field vector equals

$$\begin{aligned}
 \mathbf{h} &:= \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} \\
 &= [\mathbf{u}_{\phi_x}, -\mathbf{u}_{\theta_z}] \begin{bmatrix} e_{\theta_z} \\ e_{\phi_x} \end{bmatrix} \\
 &= \begin{bmatrix} -\sin(\phi_x) & -\cos(\phi_x)\cos(\theta_z) \\ \cos(\phi_x) & -\sin(\phi_x)\cos(\theta_z) \\ 0 & \sin(\theta_z) \end{bmatrix} \begin{bmatrix} \sin(\gamma) e^{j\eta} \\ \cos(\gamma) \end{bmatrix}. \quad (2)
 \end{aligned}$$

III. THE NEWLY FORMULATED ARRAY MANIFOLD OF A TRIAD OF MAGNETICALLY LARGE LOOPS

A. The z -oriented loop-antenna's voltage

Consider a circular loop-antenna of radius R , lying on the x - y Cartesian plane, symmetrically about the Cartesian origin. This loop's normal vector points along the z -axis, thus labeled as “ z -oriented”. Onto this loop, suppose an electromagnetic wave impinges from the far field at a polar angle of θ_z and an azimuth angle of ϕ_x .

This loop-antenna's open-circuit voltage equals (see equation (4-20) of [4], or equation (2-93) of [11])

$$v_z = \mathbf{e} \cdot \ell_z^{(\frac{R}{\lambda})}(\theta_z, \phi_x), \quad (3)$$

where the vector $\ell_z^{(\frac{R}{\lambda})}(\theta_z, \phi_x)$ denotes the loop-antenna's “effective length” (a.k.a. the “effective height”), and $\cdot$ symbolizes the vector dot product.

This “effective length”, despite its name, refers to a 3×1 vector that varies non-monotonically with the loop's physical size (R). This “effective length”, also despite its name, varies with the incident signal's direction-of-arrival (θ_z, ϕ_x); hence, the “effective length” is a property *not only* of the loop-antenna itself, but also depends on the incident source. Therefore, though the “effective length” has a unit in length, it is not a “length” per se and not a constant scalar, but a “pattern” to be multiplied into the incident electric-field vector (as in (3)) to give the antenna's voltage. The “effective length” for the loop could indeed be negative, corresponding to a change of voltage polarity at the terminals.

If a uniform current⁷ is achieved along a z -oriented circular loop, the “effective length” of (3) may be found by comparing (2-92) [11, p. 88] and (5-54b) in [11, p. 248], to give the vector,

$$\ell_z^{(\frac{R}{\lambda})}(\theta_z, \phi_x) = j2\pi R J_1 \left(2\pi \frac{R}{\lambda} \sin(\theta_z) \right) \mathbf{u}_{\phi_x}, \quad (4)$$

where $J_1(\cdot)$ represents the Bessel function of the first kind and order 1.

⁷“Uniform” here means that the current's magnitude and phase are constant along the length of the loop at any one time instant. However, that magnitude and that phase may vary over time. Please refer to [19], [15], [20], [1], [16], [22], [17]. A uniform current can be achieved over the loop's entire circumference, by subdividing the loop into sections/arcs and using different feed lines (fed from common feed source) to feed each section separately [11, p. 249].

The “effective length” vector in (4) has these properties:

- The “effective length” is proportional to $J_1 \left(2\pi \frac{R}{\lambda} \sin(\theta_z) \right)$, whereas $J_1(\kappa_o) = 0$ at $\kappa_o \in \{3.8317, 7.0156, 10.1735, 13.3237, 16.4706, \dots\}$ [18, Section 2.7.2, p. 71]. Each κ_o has its corresponding values of $(\frac{R}{\lambda}, \theta_z)$.
- The “effective length” becomes zero along the loop axis (i.e., the axis perpendicular to the plane on which the loop lies), irrespective of the loop radius R . This is because $\theta_z = 0$ implies that (4) would degenerate to $j2\pi R J_1(0) = j2\pi R \times 0 = 0$.

Returning to the voltage v_z : (1), (3), and (4) together give

$$\begin{aligned}
 v_z &= j2\pi R J_1 \left(2\pi \frac{R}{\lambda} \sin(\theta_z) \right) \cos(\gamma) \\
 &= \ell_z^{(\frac{R}{\lambda})}(\theta_z, \phi_x) h_z, \quad (5)
 \end{aligned}$$

with the scalar,

$$\ell_z^{(\frac{R}{\lambda})}(\theta_z, \phi_x) := j2\pi R \frac{J_1 \left(2\pi \frac{R}{\lambda} \sin(\theta_z) \right)}{\sin(\theta_z)}. \quad (6)$$

B. The x -oriented loop-antenna's voltage

Consider a circular loop lying instead on the y - z plane, i.e. with a normal vector pointing along the x -axis. This loop's effective length vector equals

$$\ell_x^{(\frac{R}{\lambda})}(\theta_z, \phi_x) = j2\pi R J_1 \left(2\pi \frac{R}{\lambda} \sin(\theta_x) \right) \mathbf{u}_{\phi_y}, \quad (7)$$

where

$$\mathbf{u}_{\phi_y} := [0, -\sin(\phi_y), \cos(\phi_y)]^T. \quad (8)$$

This loop's corresponding voltage equals

$$v_x = \mathbf{e} \cdot \ell_x^{(\frac{R}{\lambda})}(\theta_z, \phi_x) = \ell_x^{(\frac{R}{\lambda})}(\theta_z, \phi_x) h_x, \quad (9)$$

with the scalar,

$$\ell_x^{(\frac{R}{\lambda})}(\theta_z, \phi_x) := j2\pi R \frac{J_1 \left(2\pi \frac{R}{\lambda} \sqrt{\sin^2(\theta_z) \sin^2(\phi_x) + \cos^2(\theta_z)} \right)}{\sqrt{\sin^2(\theta_z) \sin^2(\phi_x) + \cos^2(\theta_z)}}. \quad (10)$$

C. The y -oriented loop-antenna's voltage

Consider a circular loop lying now on the x - z plane, i.e. with a normal vector pointing along the y -axis. This loop's effective length vector is

$$\ell_y^{(\frac{R}{\lambda})}(\theta_z, \phi_x) = j2\pi R J_1 \left(2\pi \frac{R}{\lambda} \sin(\theta_y) \right) \mathbf{u}_{\phi_z}, \quad (11)$$

where

$$\mathbf{u}_{\phi_z} := [\cos(\phi_z), 0, -\sin(\phi_z)]^T. \quad (12)$$

This loop's corresponding voltage equals

$$v_y = \mathbf{e} \cdot \ell_y^{(\frac{R}{\lambda})}(\theta_z, \phi_x) = \ell_y^{(\frac{R}{\lambda})}(\theta_z, \phi_x) h_y, \quad (13)$$

with the scalar,

$$\ell_y^{\left(\frac{R}{\lambda}\right)}(\theta_z, \phi_x) := j2\pi R \frac{J_1\left(2\pi \frac{R}{\lambda} \sqrt{\sin^2(\theta_z) \cos^2(\phi_x) + \cos^2(\theta_z)}\right)}{\sqrt{\sin^2(\theta_z) \cos^2(\phi_x) + \cos^2(\theta_z)}}. \quad (14)$$

D. The loop-triad's new 3×1 array manifold

Sections III-A to III-C have derived the voltage of a magnetically large loop at various Cartesian orientations.

Now, consider a triad of such large loops, all with a radius R , co-centered, but orthogonally oriented relative to each other. Such a triad has a 3×1 array manifold⁸ of

$$\mathbf{a}^{\left(\frac{R}{\lambda}\right)}(\theta_z, \phi_x, \gamma, \eta) := \begin{bmatrix} \ell_x^{\left(\frac{R}{\lambda}\right)}(\theta_z, \phi_x) \\ \ell_y^{\left(\frac{R}{\lambda}\right)}(\theta_z, \phi_x) \\ \ell_z^{\left(\frac{R}{\lambda}\right)}(\theta_z, \phi_x) \end{bmatrix} \circ \mathbf{h}(\theta_z, \phi_x, \gamma, \eta), \quad (15)$$

where $\circ$ denotes element-by-element multiplication.

E. The Special Case of Magnetically “Small” Loops

The label of “small” is applied to circular loops with a wavelength-normalized circumference of $2\pi \frac{R}{\lambda} < \frac{1}{10}$. (Please see [11], Section 5.1 and p. 231)⁹

For such magnetically “small” loops: (5), (9), and (13) can be simplified as follows: The argument inside $J_1(\cdot)$ becomes $2\pi \frac{R}{\lambda} \sin(\theta) < \frac{1}{10} \sin(\theta) \leq \frac{1}{10} \ll \sqrt{2}, \forall \theta \in [0, \pi]$. However, for any positive $\kappa \ll \sqrt{2}$ ([2, p. 360]), $J'_0(\kappa) = J_1(\kappa) \approx \frac{\kappa}{2}$. Hence,

$$\begin{aligned} \ell_x^{\left(\frac{R}{\lambda} < \frac{1}{20\pi}\right)}(\theta_z, \phi_x) &\approx j \frac{2\pi^2 R^2}{\lambda} \sin(\theta_x) \mathbf{u}_{\phi_y}, \\ \ell_z^{\left(\frac{R}{\lambda} < \frac{1}{20\pi}\right)}(\theta_z, \phi_x) &\approx j \frac{2\pi^2 R^2}{\lambda} \sin(\theta_z) \mathbf{u}_{\phi_x}, \\ \ell_y^{\left(\frac{R}{\lambda} < \frac{1}{20\pi}\right)}(\theta_z, \phi_x) &\approx j \frac{2\pi^2 R^2}{\lambda} \sin(\theta_y) \mathbf{u}_{\phi_z}. \end{aligned}$$

Using the above results for $R < \frac{\lambda}{20\pi}$,

$$\mathbf{a}^{\left(\frac{R}{\lambda} < \frac{1}{20\pi}\right)}(\theta, \phi, \gamma, \eta) = j \frac{2\pi^2 R^2}{\lambda} \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix}, \quad (16)$$

with an “effective length” of $[1, 1, 1]^T$.

This (16) agrees with [5], [21]. This (16) has been used in most signal-processing references for direction finding or

⁸ An array’s “array manifold”, roughly speaking, spans the set of all possible “steering vectors” obtainable from that array. The present triad’s “array manifold” depends on the incident signal’s parameters of $\theta_z, \phi_x, \gamma, \eta$. The triad’s “array manifold” then corresponds to $\{\mathbf{a}^{\left(\frac{R}{\lambda}\right)}(\theta_z, \phi_x, \gamma, \eta) \mid \forall \theta_z \in [0, \pi], \forall \phi_x \in [0, 2\pi], \forall \gamma \in [0, \frac{\pi}{2}], \forall \eta \in [-\pi, \pi]\}$.

⁹ Such a magnetically small loop’s radiation pattern is independent of the loop’s shape. Hence, their “effective lengths” would remain the same for any non-circular shape (like square, elliptical) and would depend only on the loop’s enclosed area [11, p. 231].

polarization estimation, thereby assuming (sometimes only implicitly) that the loops are magnetically “small”, despite being electrically inefficient. In contrast, the earlier Sections III-A to III-D address magnetically large loops that are electrically efficient.

IV. NEW CLOSED-FORM ESTIMATES OF AN INCIDENT SOURCE’S AZIMUTH-ELEVATION DIRECTION-OF-ARRIVAL OR POLARIZATION

Eigen-based direction-finding algorithms¹⁰ characteristically involve an intermediate step, that estimates each incident source’s steering vector, but correct only to within a complex-value scalar c , which is unknown to the algorithm. That is, available is the 3×1 estimate

$$\hat{\mathbf{a}} \approx c \mathbf{a}^{\left(\frac{R}{\lambda}\right)}(\theta_z, \phi_x, \gamma, \eta), \quad (17)$$

from which the direction-of-arrival or the polarization of the impinging electromagnetic wave are to be estimated. (This approximation becomes equality in noiseless or asymptotic cases.)

The unknown complex phase ($\angle c$), can be eliminated from the third entry of $\hat{\mathbf{a}}$ as follows:

$$\begin{aligned} \bar{\mathbf{a}} &:= \hat{\mathbf{a}} e^{-j\angle[\hat{\mathbf{a}}]_3} \\ &\approx |c| \begin{bmatrix} |\ell_x^{\left(\frac{R}{\lambda}\right)}(\theta_z, \phi_x)| \\ |\ell_y^{\left(\frac{R}{\lambda}\right)}(\theta_z, \phi_x)| \\ |\ell_z^{\left(\frac{R}{\lambda}\right)}(\theta_z, \phi_x)| \end{bmatrix} \\ &\quad \odot \begin{bmatrix} -\sin(\phi_x) \sin(\gamma) \cos(\eta) - \cos(\phi_x) \cos(\theta_z) \cos(\gamma) \\ \cos(\phi_x) \sin(\gamma) \cos(\eta) - \sin(\phi_x) \cos(\theta_z) \cos(\gamma) \\ \sin(\theta_z) \cos(\gamma) \end{bmatrix} \\ &\quad + j|c| \begin{bmatrix} |\ell_x^{\left(\frac{R}{\lambda}\right)}(\theta_z, \phi_x)| \\ |\ell_y^{\left(\frac{R}{\lambda}\right)}(\theta_z, \phi_x)| \\ |\ell_z^{\left(\frac{R}{\lambda}\right)}(\theta_z, \phi_x)| \end{bmatrix} \odot \begin{bmatrix} -\sin(\phi_x) \sin(\gamma) \sin(\eta) \\ \cos(\phi_x) \sin(\gamma) \sin(\eta) \\ 0 \end{bmatrix}. \end{aligned} \quad (18)$$

Here, (18) provides 5 real-value constraints from which the bivariate direction-of-arrival or the bivariate polarization can be estimated. There are 5 real-value constraints, because there

¹⁰ The following offers a brief introduction to eigen-based parameter-estimation algorithms.

Suppose a wireless signal $\{s(m), \forall m = 1, 2, \dots, M\}$ reaches L number of sensors, whose measured data equals an $L \times 1$ vector of $\mathbf{z}(m) = s(m)\mathbf{a} + \mathbf{n}(m)$ at the m th time sample. Here, the $L \times 1$ vector $\mathbf{a}$ symbolizes the L sensors’ array manifold, and $\{\mathbf{n}(m), \forall m = 1, 2, \dots, M\}$ represents the additive noise at the L sensors and is spatially uncorrelated across the L sensors. From M such time samples, an eigen-based parameter-estimation algorithm would form an $L \times L$ data covariance matrix of $\hat{\mathbf{C}} := \frac{1}{M} \sum_{m=1}^M \mathbf{z}(t_m) [\mathbf{z}(t_m)]^H$, where the superscript H symbolizes the Hermitian operator. Suppose further that $\{s(t), \forall t\}$ and $\{\mathbf{n}(t), \forall t\}$ are each temporally stationary, and not cross-correlated between them. Then, $\hat{\mathbf{C}} \approx \mathbf{C} = P_s \mathbf{a} \mathbf{a}^H + P_n \mathbf{I}$, where P_s denotes the incident signal’s power, P_n refers to each antenna’s noise power, and $\mathbf{I}$ signifies an $L \times L$ identity matrix. The $L \times L$ matrix of $\hat{\mathbf{C}}$ is Hermitian, and asymptotically approaches $\mathbf{C}$ as $M \rightarrow \infty$. Moreover, $\hat{\mathbf{C}}$ has a principal eigenvector asymptotically equal to $\alpha \mathbf{a}$, where α can be any complex-value number that has a magnitude of $1/\|\mathbf{a}\|$ and that is algebraically independent of $\mathbf{a}$. Hence, the principal eigenvector of the sampled data-covariance matrix $\hat{\mathbf{C}}$ is approximately $\alpha \mathbf{a}$.

exist three non-zero components in the real-value part of (18) and there exist two non-zero components in the imaginary-value part of (18).

1) *Estimation of the Incident Source's Azimuth-Elevation Direction-of-Arrival, (θ_z, ϕ_x)* : The following new algorithm estimates an incident source's direction-of-arrival (θ_z, ϕ_x) , assuming that incident source's polarization as prior known. For such a scenario in real-world applications, please refer to footnote #12 in [26].

From (18), define

$$\alpha_x := \frac{\Im\{\{\bar{\mathbf{a}}\}_1\}}{\Re\{\{\bar{\mathbf{a}}\}_1\} \sin(\eta) - \Im\{\{\bar{\mathbf{a}}\}_1\} \cos(\eta)} \quad (19)$$

$$= \frac{\tan(\phi_x) \tan(\gamma)}{\cos(\theta_z)}, \quad (20)$$

$$\alpha_y := \frac{\Im\{\{\bar{\mathbf{a}}\}_2\}}{\Im\{\{\bar{\mathbf{a}}\}_2\} \cos(\eta) - \Re\{\{\bar{\mathbf{a}}\}_2\} \sin(\eta)} \quad (21)$$

$$= \frac{\cot(\phi_x) \tan(\gamma)}{\cos(\theta_z)}, \quad (22)$$

where $\Re\{\cdot\}$ refers to the real-value part and $\Im\{\cdot\}$ symbolizes the imaginary-value part of the entity inside the curly bracket.

The incident source's bivariate direction-of-arrival may then be estimated as

$$\hat{\phi}_x = \begin{cases} \arctan\left(\left|\frac{\alpha_x}{\alpha_y}\right|^{1/2}\right), & \text{if } \phi_x \in [0, \frac{\pi}{2}), \\ \pi + \arctan\left(-\left|\frac{\alpha_x}{\alpha_y}\right|^{1/2}\right), & \text{if } \phi_x \in [\frac{\pi}{2}, \pi), \\ \pi + \arctan\left|\frac{\alpha_x}{\alpha_y}\right|^{1/2}, & \text{if } \phi_x \in [\pi, \frac{3\pi}{2}), \\ 2\pi + \arctan\left(-\left|\frac{\alpha_x}{\alpha_y}\right|^{1/2}\right), & \text{if } \phi_x \in [\frac{3\pi}{2}, 2\pi); \end{cases} \quad (23)$$

$$\hat{\theta}_z = \arccos\left(\frac{\tan(\gamma) \tan(\hat{\phi}_x)}{\alpha_x}\right). \quad (24)$$

No prior knowledge is needed by these new estimation formulas of the large circular loops' radius R or the incident signal's wavelength λ .

2) *Estimation of the Incident Source's Polarization, (γ, η)* : The new algorithm below estimates an impinging source's bivariate polarization (γ, η) , presuming the incident source's direction-of-arrival (θ_z, ϕ_x) as prior known. For such a scenario in real-world applications, please refer to footnote #15 in [26].

Taking the ratio of the real and imaginary parts of the first entry of (18) gives

$$\frac{\Re\{\{\bar{\mathbf{a}}\}_1\}}{\Im\{\{\bar{\mathbf{a}}\}_1\}} = \frac{\cos(\theta_z)}{\tan(\phi_x) \tan(\gamma) \sin(\eta)} + \frac{1}{\tan(\eta)}. \quad (25)$$

Taking the ratio of the real and imaginary parts of the second entry of (18) gives

$$\frac{\Re\{\{\bar{\mathbf{a}}\}_2\}}{\Im\{\{\bar{\mathbf{a}}\}_2\}} = -\frac{\tan(\phi_x) \cos(\theta_z)}{\tan(\gamma) \sin(\eta)} + \frac{1}{\tan(\eta)}. \quad (26)$$

Next, relate (25) and (26) as follows, in order to retain only the one unknown of η ,

$$\begin{aligned} & \frac{\Re\{\{\bar{\mathbf{a}}\}_1\}}{\Im\{\{\bar{\mathbf{a}}\}_1\}} \tan(\phi_x) + \frac{\Re\{\{\bar{\mathbf{a}}\}_2\}}{\Im\{\{\bar{\mathbf{a}}\}_2\}} \cot(\phi_x) \\ &= \frac{1}{\tan(\eta)} (\tan(\phi_x) + \cot(\phi_x)). \end{aligned} \quad (27)$$

As the direction-of-arrival angle ϕ_x has been assumed as prior known, η may be estimated from (27) as

$$\hat{\eta} = \arctan\left\{\frac{\tan(\phi_x) + \cot(\phi_x)}{\frac{\Re\{\{\bar{\mathbf{a}}\}_1\}}{\Im\{\{\bar{\mathbf{a}}\}_1\}} \tan(\phi_x) + \frac{\Re\{\{\bar{\mathbf{a}}\}_2\}}{\Im\{\{\bar{\mathbf{a}}\}_2\}} \cot(\phi_x)}\right\}. \quad (28)$$

From the ratios of the real and imaginary parts of the first and second entries of (18), one additional constraint can be obtained:

$$\frac{\Re\{\{\bar{\mathbf{a}}\}_1\}}{\Im\{\{\bar{\mathbf{a}}\}_1\}} - \frac{\Re\{\{\bar{\mathbf{a}}\}_2\}}{\Im\{\{\bar{\mathbf{a}}\}_2\}} = \frac{\cos(\theta_z)}{\tan(\gamma) \sin(\eta)} (\cot(\phi_x) + \tan(\phi_x)). \quad (29)$$

As η is already estimated, multiply $\sin(\hat{\eta})$ to both sides of (29), in order to obtain

$$\sin(\hat{\eta}) \left(\frac{\Re\{\{\bar{\mathbf{a}}\}_1\}}{\Im\{\{\bar{\mathbf{a}}\}_1\}} - \frac{\Re\{\{\bar{\mathbf{a}}\}_2\}}{\Im\{\{\bar{\mathbf{a}}\}_2\}} \right) \quad (30)$$

$$= \frac{\cos(\theta_z)}{\tan(\gamma)} [\cot(\phi_x) + \tan(\phi_x)]. \quad (31)$$

With θ_z and ϕ_x presumed known, γ can be estimated from (30) as

$$\hat{\gamma} = \arctan\left\{\frac{[\cot(\phi_x) + \tan(\phi_x)] \cos(\theta_z)}{\sin(\hat{\eta}) \left[\frac{\Re\{\{\bar{\mathbf{a}}\}_1\}}{\Im\{\{\bar{\mathbf{a}}\}_1\}} - \frac{\Re\{\{\bar{\mathbf{a}}\}_2\}}{\Im\{\{\bar{\mathbf{a}}\}_2\}} \right]}\right\}. \quad (32)$$

No prior knowledge is needed by these new estimation formulas of the large circular loops' radius R or the incident signal's wavelength λ .

V. MONTE CARLO SIMULATIONS

To verify the efficacy of the eigen-based estimators proposed in Section IV, Monte Carlo simulations are conducted in this section.

To focus on the influence of the large loop-antenna's physical radius R , a simple data model will be used below: A zero-mean complex-value Gaussian noise signal $\{s(m), \forall m = 1, 2, \dots, M\}$, impinges upon a triad of large loops which have been described earlier. At the m th time-instant, the triad collects the 3×1 data-vector,

$$\mathbf{z}(m) = \mathbf{a}^{(\frac{R}{\lambda})}(\theta_z, \phi_x, \gamma, \eta) s(m) + \mathbf{n}(m). \quad (33)$$

Moreover, the additive noise $\mathbf{n}(m)$ denotes a 3×1 vector of zero-mean complex-value Gaussian noise of a prior known variance of σ_s^2 , circularly symmetric in distribution on the complex plane, statistically independent from all other entities, while spatio-temporally uncorrelated over time and across the loops, hence with a deterministic covariance matrix of $\mathbf{\Gamma}_0 = \text{diag}(\sigma_n^2, \sigma_n^2, \sigma_n^2)$, where σ_n^2 represents the noise variance at each loop.

TABLE I: How a larger loop would improve the precision in the direction-of-arrival estimation. (Same simulation setting as the black dotted curve in Figure 2.)

	at $2\pi\frac{R}{\lambda} = 0.5$	at $2\pi\frac{R}{\lambda} = 3$	error reduction
$\sqrt{\text{CRB}}(\theta_z)$	6.15	0.36	94%
$\sqrt{\text{CRB}}(\phi_x)$	7.14	0.37	95%

With M number of time-samples, the collected data-set may be represented as a $3M \times 1$ matrix:

$$\zeta = [(\mathbf{z}(1))^T, \dots, (\mathbf{z}(M))^T]^T. \quad (34)$$

In all subsequent numerical examples: $M = 100$, $\phi_x = 45^\circ$, $\theta_z = 60^\circ$, $\gamma = 45^\circ$, and $\eta = 60^\circ$. Each icon on every graph represents 1,000 independent Monte Carlo experiments.

A. Estimation of the Incident Source's Azimuth-Elevation Direction-of-Arrival, (ϕ_x, θ_z)

Using the new algorithm proposed earlier in Section IV-1 for direction finding, Monte Carlo simulation results are presented in Figure 2, along with the corresponding Cramér-Rao bounds (CRB) plotted for comparison. These Cramér-Rao bounds have been derived in Appendix A. (Recall that the Cramér-Rao bound lower-bounds the estimation error variance of any unbiased estimator; and the Cramér-Rao bound itself is independent of the specific estimator employed.) Figure 2 indicates that the estimation method proposed in Section IV-1 performs close to the Cramér-Rao bounds.

Recall that the “effective length” affects the loop’s gain, nonlinearly and non-monotonically in (4-11) through Bessel functions. Figure 2 and Table I further suggest that the larger the loop, the more accurate the estimates would be.

The above presumes the proposed estimators of a perfect prior knowledge of the incident source’s polarization. This presumption is relaxed in the simulations presented in Figure 3. There, the polarization is assumed only approximately known, with random uncertainties of Δ_γ and Δ_η , respectively from the true γ and η , where Δ_γ and Δ_η are each presumed as Gaussian distributed, with a zero mean and a standard deviation of σ_γ and σ_η , respectively. Figure 3 presents Monte Carlo simulations at $2\pi\frac{R}{\lambda} = 1$ and at a 11dB signal-to-noise power ratio, showing that a $\sigma_\gamma = \sigma_\eta = 3^\circ$ uncertainty would raise the RMSE’s by only 1%, and that a $\sigma_\gamma = \sigma_\eta = 5^\circ$ by 3%-4%. Therefore, the estimators proposed in Section IV-1 can be robust against imprecise prior information of the incident source’s polarization.

B. Estimation of the Incident Source's Polarization, (γ, η)

Using the new algorithms proposed earlier in Section IV-2 for polarization estimation, Monte Carlo simulation results are presented in Figure 4, along with the corresponding Cramér-Rao bounds (CRB) for comparison. These Cramér-Rao bounds have been derived in Appendix B.

Fig. 2: RMSE and $\sqrt{\text{CRB}}$ for $\hat{\theta}_z$ and $\hat{\phi}_x$, at SNR = 10, 20, 35dB.

TABLE II: How a larger loop would improve the precision in the polarization estimation. (The same simulation setting as the black dotted curve in Figure 4.)

	at $2\pi\frac{R}{\lambda} = 0.5$	at $2\pi\frac{R}{\lambda} = 3$	error reduction
$\sqrt{\text{CRB}}(\gamma)$	4.27	0.30	93%
$\sqrt{\text{CRB}}(\eta)$	8.55	0.44	95%

Figure 4 indicates that the estimators proposed in Section IV-2 also perform near the respective Cramér-Rao bounds. Figure 4 and Table II further reinforce the point that the larger the loop, the more precise the estimates would become.

The above presumes the proposed estimators of a perfect prior knowledge of the incident source’s direction-of-arrival. This presumption is relaxed in the simulations presented in Figure 5. There, the direction-of-arrival is assumed only approximately known, with random uncertainties of Δ_{θ_z} and Δ_{ϕ_x} , respectively from the true θ_z and ϕ_x , where Δ_{θ_z} and Δ_{ϕ_x} are each presumed as Gaussian distributed, with a zero mean and a standard deviation of σ_{θ_z} and σ_{ϕ_x} , respectively. Figure 5 presents Monte Carlo simulations at $2\pi\frac{R}{\lambda} = 1$ and at a 11dB signal-to-noise power ratio. There, a $\sigma_{\theta_z} = \sigma_{\phi_x} = 3^\circ$ uncertainty would raise the RMSE’s by under 0.5%; and a

Fig. 3: How the proposed direction-of-arrival estimators ($\hat{\theta}_z$ and $\hat{\phi}_x$) would be degraded by an imperfect fore-knowledge of the incident source's polarization (γ, η).

$\sigma_{\theta_z} = \sigma_{\phi_x} = 5^\circ$ uncertainty would raise the RMSE's by about 1%; Therefore, the polarization estimators proposed in Section IV-2 can be robust against imprecise prior information of the incident source's direction-of-arrival.

VI. CONCLUSION

Magnetically large loops are considered for the first time in the open literature (to the present authors' best knowledge) to develop signal-processing algorithms to estimate incident emitters' direction-of-arrival or polarization without any prior knowledge of the large circular loop's radius nor of the incident signal's wavelength. Despite that the magnetically large circular loops' gain pattern involves the Bessel function, closed-form estimation algorithms are developed here, not requiring any iterative search. These proposed estimation algorithms can approach the Cramér-Rao lower bounds. Moreover, these proposed estimators can be robust against imprecise prior information about the incident source.

Fig. 4: RMSE and $\sqrt{\text{CRB}}$ for $\hat{\gamma}$ and $\hat{\eta}$, at SNR = 10, 20, 35dB.

APPENDIX A: THE CRAMÉR-RAO BOUND FOR DIRECTION FINDING

The algorithm newly proposed in Section IV-1 requires no prior knowledge of $\frac{R}{\lambda}$ or σ_n^2 . Hence, these are considered here as nuisance parameters.

Re-write the $3M \times 1$ dataset of (34) as

$$\zeta = \mathbf{s} \otimes \mathbf{a}\left(\frac{R}{\lambda}\right)(\theta_z, \phi_x, \gamma, \eta) + \underbrace{[(\mathbf{n}(M))^T, \dots, (\mathbf{n}(M))^T]^T}_{=\boldsymbol{\nu}}, \quad (35)$$

where $\mathbf{s}$ is an $M \times 1$ vector with entries that are complex-valued zero-mean Gaussian, with a covariance matrix of $\mathbf{\Gamma}_s = \sigma_s^2 \mathbf{I}_M$, and statistically independent of each other. Here, $\mathbf{I}_M$ denotes an $M \times M$ identity matrix, and $\otimes$ symbolizes the Kronecker product. Moreover, $\boldsymbol{\nu}$ represents a $3M \times 1$ noise vector with a spatio-temporal covariance matrix of $\mathbf{\Gamma}_n = \mathbf{I}_M \otimes \mathbf{\Gamma}_0$.

Define $\boldsymbol{\psi} := [\theta_z, \phi_x, \frac{R}{\lambda}, \sigma_n^2]^T$. As the direction-finding formulas in Section IV-1 requires no prior information on $\frac{R}{\lambda}$ and σ_n^2 , the Fisher information matrix (FIM) is to be 4×4 in

Fig. 5: How the proposed polarization estimators ($\hat{\gamma}$ and $\hat{\eta}$) would be degraded by an imperfect fore-knowledge of the incident source's direction-of-arrival (θ_z, ϕ_x).

size:

$$\mathbf{J} = \begin{bmatrix} J_{\theta_z, \theta_z} & J_{\theta_z, \phi_x} & J_{\theta_z, \frac{R}{\lambda}} & J_{\theta_z, \sigma_n^2} \\ J_{\phi_x, \theta_z} & J_{\phi_x, \phi_x} & J_{\phi_x, \frac{R}{\lambda}} & J_{\phi_x, \sigma_n^2} \\ J_{\frac{R}{\lambda}, \theta_z} & J_{\frac{R}{\lambda}, \phi_x} & J_{\frac{R}{\lambda}, \frac{R}{\lambda}} & J_{\frac{R}{\lambda}, \sigma_n^2} \\ J_{\sigma_n^2, \theta_z} & J_{\sigma_n^2, \phi_x} & J_{\sigma_n^2, \frac{R}{\lambda}} & J_{\sigma_n^2, \sigma_n^2} \end{bmatrix},$$

whose (i, j) th entry equals (see equation (8.34) of [10])¹¹

$$[\mathbf{J}]_{i,j} = \text{Tr} \left(\mathbf{\Gamma}^{-1} \frac{\partial \mathbf{\Gamma}}{\partial [\psi]_i} \mathbf{\Gamma}^{-1} \frac{\partial \mathbf{\Gamma}}{\partial [\psi]_j} \right),$$

where $\text{Tr}(\cdot)$ represents the trace operator.¹²

¹¹That equation (8.34) may be used, because the collected data vector, ζ , has entries here that are Gaussian, zero-mean, and with a covariance matrix of $\mathbf{\Gamma} = \mathbf{I}_M \otimes \sigma_s^2 \mathbf{a} \mathbf{a}^H + \mathbf{I}_M \otimes \mathbf{\Gamma}_0$.

¹²The above involves partial derivatives of $\mathbf{a}(\frac{R}{\lambda})(\theta_z, \phi_x, \gamma, \eta)$, whose entries depend on $J_n(\xi)$, where $J_n(\xi)$ represents Bessel function of the first kind and order n , as before. From [3] (Appendix IV, page 936): $\frac{\partial J_1(\xi)}{\partial \xi} = J_0(\xi) - \frac{1}{\xi} J_1(\xi)$.

The inverse of the above Fisher information matrix gives

$$\text{CRB} = \left[\begin{bmatrix} J_{\theta_z, \theta_z} & J_{\theta_z, \phi_x} \\ J_{\phi_x, \theta_z} & J_{\phi_x, \phi_x} \end{bmatrix} - \begin{bmatrix} J_{\theta_z, \frac{R}{\lambda}} & J_{\theta_z, \sigma_n^2} \\ J_{\phi_x, \frac{R}{\lambda}} & J_{\phi_x, \sigma_n^2} \end{bmatrix} \begin{bmatrix} J_{\frac{R}{\lambda}, \frac{R}{\lambda}} & J_{\frac{R}{\lambda}, \sigma_n^2} \\ J_{\sigma_n^2, \frac{R}{\lambda}} & J_{\sigma_n^2, \sigma_n^2} \end{bmatrix}^{-1} \begin{bmatrix} J_{\frac{R}{\lambda}, \theta_z} & J_{\frac{R}{\lambda}, \phi_x} \\ J_{\sigma_n^2, \theta_z} & J_{\sigma_n^2, \phi_x} \end{bmatrix} \right]^{-1},$$

where

$$\begin{aligned} J_{\theta_z, \theta_z} &= M \left(\frac{\sigma_s}{\sigma_n} \right)^4 \text{Tr} \left([\mathbf{M}_1^{-1} \mathbf{M}_2]^2 \right), \\ J_{\theta_z, \phi_x} &= J_{\phi_x, \theta_z}^* \\ &= M \left(\frac{\sigma_s}{\sigma_n} \right)^4 \text{Tr} \left(\mathbf{M}_1^{-1} \mathbf{M}_2 \mathbf{M}_1^{-1} \mathbf{M}_3 \right), \\ J_{\theta_z, \frac{R}{\lambda}} &= J_{\frac{R}{\lambda}, \theta_z}^* \\ &= M \left(\frac{\sigma_s}{\sigma_n} \right)^4 \text{Tr} \left(\mathbf{M}_1^{-1} \mathbf{M}_2 \mathbf{M}_1^{-1} \mathbf{M}_4 \right), \\ J_{\theta_z, \sigma_n^2} &= J_{\sigma_n^2, \theta_z}^* \\ &= M \left(\frac{\sigma_s}{\sigma_n} \right)^2 \frac{1}{\sigma_n^2} \text{Tr} \left(\mathbf{M}_1^{-1} \mathbf{M}_2 \mathbf{M}_1^{-1} \right), \\ J_{\phi_x, \phi_x} &= M \left(\frac{\sigma_s}{\sigma_n} \right)^4 \text{Tr} \left([\mathbf{M}_1^{-1} \mathbf{M}_3]^2 \right), \\ J_{\phi_x, \frac{R}{\lambda}} &= J_{\frac{R}{\lambda}, \phi_x}^* \\ &= M \left(\frac{\sigma_s}{\sigma_n} \right)^4 \text{Tr} \left(\mathbf{M}_1^{-1} \mathbf{M}_3 \mathbf{M}_1^{-1} \mathbf{M}_4 \right), \\ J_{\phi_x, \sigma_n^2} &= J_{\sigma_n^2, \phi_x}^* \\ &= M \left(\frac{\sigma_s}{\sigma_n} \right)^2 \frac{1}{\sigma_n^2} \text{Tr} \left(\mathbf{M}_1^{-1} \mathbf{M}_3 \mathbf{M}_1^{-1} \right), \\ J_{\frac{R}{\lambda}, \frac{R}{\lambda}} &= M \left(\frac{\sigma_s}{\sigma_n} \right)^4 \text{Tr} \left([\mathbf{M}_1^{-1} \mathbf{M}_4]^2 \right), \\ J_{\frac{R}{\lambda}, \sigma_n^2} &= J_{\sigma_n^2, \frac{R}{\lambda}}^* \\ &= M \left(\frac{\sigma_s}{\sigma_n} \right)^2 \frac{1}{\sigma_n^2} \text{Tr} \left(\mathbf{M}_1^{-1} \mathbf{M}_4 \mathbf{M}_1^{-1} \right), \\ J_{\sigma_n^2, \sigma_n^2} &= M \frac{1}{\sigma_n^4} \text{Tr} \left(\mathbf{M}_1^{-2} \right). \end{aligned}$$

In the above, the superscript * denotes complex conjugation, and

$$\begin{aligned} \mathbf{M}_1 &:= \mathbf{I}_3 + \left(\frac{\sigma_s}{\sigma_n} \right)^2 \mathbf{a} \left(\frac{R}{\lambda} \right) \mathbf{a} \left(\frac{R}{\lambda} \right)^H, \\ \mathbf{M}_2 &:= \frac{\partial}{\partial \theta_z} \left(\mathbf{a} \left(\frac{R}{\lambda} \right) \mathbf{a} \left(\frac{R}{\lambda} \right)^H \right), \\ \mathbf{M}_3 &:= \frac{\partial}{\partial \phi_x} \left(\mathbf{a} \left(\frac{R}{\lambda} \right) \mathbf{a} \left(\frac{R}{\lambda} \right)^H \right), \\ \mathbf{M}_4 &:= \frac{\partial}{\partial \frac{R}{\lambda}} \left(\mathbf{a} \left(\frac{R}{\lambda} \right) \mathbf{a} \left(\frac{R}{\lambda} \right)^H \right). \end{aligned}$$

APPENDIX B: THE CRAMÉR-RAO BOUND FOR POLARIZATION

The algorithm newly proposed in Section IV-2 requires no prior knowledge of $\frac{R}{\lambda}$ or σ_n^2 . Hence, these are considered here as nuisance parameters.

Equation (35) in Appendix A remains valid here.

The Fisher information matrix here is also 4×4 in size, as the estimation formulas in Section IV-2 requires no prior information on $\frac{R}{\lambda}$, σ_n^2 . Hence,

$$\mathbf{J} = \begin{bmatrix} J_{\gamma,\gamma} & J_{\gamma,\eta} & J_{\gamma,\frac{R}{\lambda}} & J_{\gamma,\sigma_n^2} \\ J_{\eta,\gamma} & J_{\eta,\eta} & J_{\eta,\frac{R}{\lambda}} & J_{\eta,\sigma_n^2} \\ J_{\frac{R}{\lambda},\gamma} & J_{\frac{R}{\lambda},\eta} & J_{\frac{R}{\lambda},\frac{R}{\lambda}} & J_{\frac{R}{\lambda},\sigma_n^2} \\ J_{\sigma_n^2,\gamma} & J_{\sigma_n^2,\eta} & J_{\sigma_n^2,\frac{R}{\lambda}} & J_{\sigma_n^2,\sigma_n^2} \end{bmatrix},$$

has its (i, j) th entry as given by (36). Therefore,

$$\text{CRB} = \left[\begin{bmatrix} J_{\gamma,\gamma} & J_{\gamma,\eta} \\ J_{\eta,\gamma} & J_{\eta,\eta} \end{bmatrix} - \begin{bmatrix} J_{\gamma,\frac{R}{\lambda}} & J_{\gamma,\sigma_n^2} \\ J_{\eta,\frac{R}{\lambda}} & J_{\eta,\sigma_n^2} \end{bmatrix} \right. \\ \left. \begin{bmatrix} J_{\frac{R}{\lambda},\frac{R}{\lambda}} & J_{\frac{R}{\lambda},\sigma_n^2} \\ J_{\sigma_n^2,\frac{R}{\lambda}} & J_{\sigma_n^2,\sigma_n^2} \end{bmatrix}^{-1} \begin{bmatrix} J_{\frac{R}{\lambda},\gamma} & J_{\frac{R}{\lambda},\eta} \\ J_{\sigma_n^2,\gamma} & J_{\sigma_n^2,\eta} \end{bmatrix} \right]^{-1},$$

where

$$\begin{aligned} J_{\gamma,\gamma} &= M \left(\frac{\sigma_s}{\sigma_n} \right)^4 \text{Tr} \left([\mathbf{M}_1^{-1} \mathbf{M}_5]^2 \right), \\ J_{\eta,\eta} &= J_{\eta,\gamma}^* \\ &= M \left(\frac{\sigma_s}{\sigma_n} \right)^4 \text{Tr} \left(\mathbf{M}_1^{-1} \mathbf{M}_5 \mathbf{M}_1^{-1} \mathbf{M}_6 \right), \\ J_{\gamma,\frac{R}{\lambda}} &= J_{\frac{R}{\lambda},\gamma}^* \\ &= M \left(\frac{\sigma_s}{\sigma_n} \right)^4 \text{Tr} \left(\mathbf{M}_1^{-1} \mathbf{M}_5 \mathbf{M}_1^{-1} \mathbf{M}_4 \right), \\ J_{\gamma,\sigma_n^2} &= J_{\sigma_n^2,\gamma}^* \\ &= M \left(\frac{\sigma_s}{\sigma_n} \right)^2 \frac{1}{\sigma_n^2} \text{Tr} \left(\mathbf{M}_1^{-1} \mathbf{M}_5 \mathbf{M}_1^{-1} \right), \\ J_{\eta,\eta} &= M \left(\frac{\sigma_s}{\sigma_n} \right)^4 \text{Tr} \left([\mathbf{M}_1^{-1} \mathbf{M}_6]^2 \right), \\ J_{\eta,\frac{R}{\lambda}} &= J_{\frac{R}{\lambda},\eta}^* \\ &= M \left(\frac{\sigma_s}{\sigma_n} \right)^4 \text{Tr} \left(\mathbf{M}_1^{-1} \mathbf{M}_6 \mathbf{M}_1^{-1} \mathbf{M}_4 \right), \\ J_{\eta,\sigma_n^2} &= J_{\sigma_n^2,\eta}^* \\ &= M \left(\frac{\sigma_s}{\sigma_n} \right)^2 \frac{1}{\sigma_n^2} \text{Tr} \left(\mathbf{M}_1^{-1} \mathbf{M}_6 \mathbf{M}_1^{-1} \right), \end{aligned}$$

where

$$\begin{aligned} \mathbf{M}_5 &:= \frac{\partial}{\partial \gamma} \left(\mathbf{a} \left(\frac{R}{\lambda} \right) \mathbf{a} \left(\frac{R}{\lambda} \right)^H \right), \\ \mathbf{M}_6 &:= \frac{\partial}{\partial \eta} \left(\mathbf{a} \left(\frac{R}{\lambda} \right) \mathbf{a} \left(\frac{R}{\lambda} \right)^H \right). \end{aligned}$$

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