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A note on the weight distribution of some cyclic codes*

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Abstract

Let \mathbb{F}_q be the finite field with q elements and C_n the cyclic group of order n , where n is a positive integer relatively prime to q . Let H, K be subgroups of C_n such that H is a proper subgroup of K . In this note, the weight distribution of the cyclic codes of length n over \mathbb{F}_q with generating idempotents \hat{K} and $e_{H,K} = \hat{H} - \hat{K}$ are explicitly determined, where $\hat{K} = 1/|K| \sum_{g \in K} g$ and $\hat{H} = 1/|H| \sum_{g \in H} g$. Our result naturally gives a new characterization of a theorem by Sharma-Bakshi [18] that determines the weight distribution of all irreducible cyclic codes of length p^m over \mathbb{F}_q , where p is an odd prime and q is a primitive root modulo p^m . Finally, two examples are presented to illustrate our results.

Keywords: Cyclic code, weight distribution, generating idempotent, finite field.

2010 Mathematics Subject Classification: 94B05; 94B15

1 Introduction

Let \mathbb{F}_q be the finite field of order q and n a positive integer relatively prime to q . A linear code C of length n over \mathbb{F}_q is called *cyclic* if it is an ideal of the semisimple group algebra $\mathcal{R} = \mathbb{F}_q C_n$, where $C_n = \langle x \rangle$ is the cyclic group of order n . An element e of \mathcal{R} satisfying $e^2 = e$ is called an *idempotent*. It is well known that each cyclic code of length n over \mathbb{F}_q contains a unique idempotent which generates the code. This idempotent is called the *generating idempotent* of the cyclic code. A cyclic code is said to be *irreducible* if its generating idempotent is primitive (e.g. see [10, Ch 4]). Irreducible cyclic codes are also known as minimal cyclic codes, which have been studied by many authors (e.g. see [1], [2], [9], [12], [15]-[18]).

Every row vector $a = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{F}_q^n$ is identified with an element $\sum_{j=0}^{n-1} a_j x^j \in \mathcal{R}$. The *Hamming weight* of a , denoted by $wt(a)$, is the number of nonzero components of a . For a cyclic code C , the *Hamming weight* of C is defined as the smallest Hamming weight of nonzero codewords of C . Furthermore, the sequence $(A_0 = 1, A_1, \dots, A_n)$ is called the *Hamming weight distribution* of C (weight distribution of C for short), where A_j denotes the number of codewords of Hamming weight j in C . Although the Hamming weight distribution does not completely specify a code, they provide important information on estimating the error correcting capability and the probability of error detection. The weight distribution of cyclic codes has been a hot subject of study. Different methods are employed to determine the specific weight distribution of cyclic codes, including the use of pseudorandom sequence, Gauss sums, Gauss periods, Gröbner basis and so on (e.g. see [4]-[7], [11], [13], [17]-[23]). However, as mentioned in [23], the problem of determining the weight distribution of cyclic codes turns out to be very difficult in general and is only settled for a few special cases in literature.

Sharma and Bakshi in [18] determined the weight distribution of all irreducible cyclic codes of length p^m over \mathbb{F}_q , where p is an odd prime different from the characteristic of the field and q is a primitive root modulo p^m . The proof for [18, Theorem 2] is long, requiring several nontrivial lemmas, and the hardest part of the paper [18]. In this note, using an idea from [14], we extend this theorem to a more general setting. To be more specific, let H, K be subgroups of C_n such that H is a proper subgroup of K , i.e.

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H is a subgroup of K with $H \neq K$. The weight distribution of the cyclic codes of length n over \mathbb{F}_q with generating idempotents

$$\hat{K} = \frac{1}{|K|} \sum_{g \in K} g \quad \text{and} \quad e_{H,K} = \hat{H} - \hat{K} = \frac{1}{|H|} \sum_{g \in H} g - \frac{1}{|K|} \sum_{g \in K} g, \quad (1.1)$$

are explicitly determined, which yields a new characterization of [18, Theorem 2] in the sense that our result appears to be more compact and simple. Our main result is given as follows:

Theorem 1.1. *Let C_n be the cyclic group of order n with subgroups H and K , where n is a positive integer relatively prime to q and H is a proper subgroup of K . Let $\mathcal{R} = \mathbb{F}_q C_n$, $C_K = \mathcal{R}\hat{K}$ and $C_{H,K} = \mathcal{R}e_{H,K}$ be the cyclic codes of length n over \mathbb{F}_q with generating idempotents \hat{K} and $e_{H,K}$, respectively, as in (1.1). Denote by $A_i^{(C_K)}$ (resp. $A_i^{(C_{H,K})}$) the number of codewords with Hamming weight i in C_K (resp. $C_{H,K}$). Suppose $|K| = k$ and $|H| = h$. We then have*

- (1) C_K has parameters $[n, n/k, k]$, and its weight distribution is given by $A_0^{(C_K)} = 1$ and $A_r^{(C_K)} = 0$ if $k \nmid r$; for $0 < i \leq n$ with $i = ki'$, $A_i^{(C_K)} = \binom{n/k}{i'} (q-1)^{i'}$.
- (2) $C_{H,K}$ has parameters $[n, n/h - n/k, 2h]$, and its weight distribution is given by $A_0^{(C_{H,K})} = 1$ and $A_r^{(C_{H,K})} = 0$ if $h \nmid r$; for $0 < i \leq n$ with $i = hi'$,

$$A_{i'h}^{(C_{H,K})} = \sum_{\substack{0 \leq \ell_1, \ell_2, \dots, \ell_{n/k} \leq k/h \\ \ell_1 + \dots + \ell_{n/k} = i'}} |\Omega_{\ell_1}| \cdots |\Omega_{\ell_{n/k}}|$$

where the summation is extended over all n/k -tuples $(\ell_1, \ell_2, \dots, \ell_{n/k})$ of integers with $0 \leq \ell_1, \ell_2, \dots, \ell_{n/k} \leq k/h$ and $\ell_1 + \ell_2 + \dots + \ell_{n/k} = i'$, and where $|\Omega_\ell|$, $0 \leq \ell \leq k/h$, is given explicitly as follows:

$$|\Omega_\ell| = \binom{k/h}{\ell} \frac{(q-1)^\ell + (-1)^\ell (q-1)}{q}.$$

It is a special case for Theorem 1.1 that $n = p^m$, a power of an odd prime p , and that q is a primitive root modulo p^m . In this case, Theorem 1.1 applies to determine the weight distribution of all irreducible cyclic codes of length p^m over \mathbb{F}_q . Note that all the subgroups of $C_{p^m} = \langle x \rangle$ form a chain:

$$C_{p^m} = \langle x^{p^0} \rangle \supset \langle x^p \rangle \supset \cdots \supset \langle x^{p^m} \rangle = \{1\},$$

and that the set of primitive idempotents of $\mathcal{R} = \mathbb{F}_q C_{p^m}$ (see [15, Theorem 3.5] or [9, Corollary 2]) is given by

$$e_0 = \widehat{C_{p^m}} = \widehat{\langle x^{p^0} \rangle} \quad \text{and} \quad e_j = \widehat{\langle x^{p^j} \rangle} - \widehat{\langle x^{p^{j-1}} \rangle}, \quad 1 \leq j \leq m. \quad (1.2)$$

Corollary 1.2. *Let p be an odd prime such that q is a primitive root modulo p^m . Let $C_j = \mathcal{R}e_j$, the irreducible cyclic code of length p^m over \mathbb{F}_q with generating idempotent e_j as in (1.2), for each $0 \leq j \leq m$. Denote by $A_i^{(j)}$ the number of codewords with Hamming weight i in C_j . We have*

(1)

$$A_i^{(0)} = \begin{cases} 1, & \text{if } i = 0; \\ q-1, & \text{if } i = p^m; \\ 0, & \text{otherwise.} \end{cases}$$

(2) *Let j be an integer with $1 \leq j \leq m$. We have $A_0^{(j)} = 1$ and $A_r^{(j)} = 0$ if $p^{m-j} \nmid r$. Assume that $0 < i \leq p^m$ with $i = p^{m-j}i'$. We then have*

$$A_{i'p^{m-j}}^{(j)} = \sum_{\substack{0 \leq \ell_1, \ell_2, \dots, \ell_{p^j-1} \leq p \\ \ell_1 + \dots + \ell_{p^j-1} = i'}} |\Omega_{\ell_1}| \cdots |\Omega_{\ell_{p^j-1}}|$$

where the summation is extended over all p^{j-1} -tuples $(\ell_1, \ell_2, \dots, \ell_{p^j-1})$ of integers with $0 \leq \ell_1, \ell_2, \dots, \ell_{p^j-1} \leq p$ and $\ell_1 + \ell_2 + \dots + \ell_{p^j-1} = i'$, and where $|\Omega_\ell|$, $0 \leq \ell \leq p$, is given explicitly as follows:

$$|\Omega_\ell| = \binom{p}{\ell} \frac{(q-1)^\ell + (-1)^\ell (q-1)}{q}.$$

2 Proof of Theorem 1.1

We need the following lemma.

Lemma 2.1. *Let m be a positive integer and ℓ be an integer with $0 \leq \ell \leq m$. Let*

$$\Omega_\ell = \left\{ (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{F}_q^m \mid \sum_{t=1}^m \lambda_t = 0 \text{ and } wt((\lambda_1, \lambda_2, \dots, \lambda_m)) = \ell \right\}.$$

We then have

$$|\Omega_\ell| = \binom{m}{\ell} \frac{(q-1)^\ell + (-1)^\ell (q-1)}{q}.$$

Proof. It is clear that $|\Omega_0| = 1$. Let $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Suppose hereafter that $1 \leq \ell \leq m$ and

$$S_\ell = \left\{ (\mu_1, \mu_2, \dots, \mu_\ell) \mid \mu_j \in \mathbb{F}_q^* \text{ for all } 1 \leq j \leq \ell \text{ and } \sum_{j=1}^{\ell} \mu_j = 0 \right\}.$$

Let $t_\ell = |\Omega_\ell|$ and $s_\ell = |S_\ell|$. It is clear that $t_\ell = \binom{m}{\ell} s_\ell$, which means that the problem of finding t_ℓ can be entirely reduced to those of determining s_ℓ . At this point, we claim that s_ℓ satisfies the recurrence relation $s_{\ell+1} = (q-1)^\ell - s_\ell$ for $\ell \geq 1$ and with $s_1 = 0$.

To see this, let

$$S'_\ell = \left\{ (\mu_1, \mu_2, \dots, \mu_\ell) \mid \mu_j \in \mathbb{F}_q^* \text{ for all } 1 \leq j \leq \ell \text{ and } \sum_{j=1}^{\ell} \mu_j \neq 0 \right\}.$$

It is easy to see that $s'_\ell = |S'_\ell| = (q-1)^\ell - s_\ell$. Note, for any $(\mu_1, \mu_2, \dots, \mu_\ell, \mu_{\ell+1}) \in S_{\ell+1}$, that the map $f : S_{\ell+1} \rightarrow S'_\ell$, defined by $f((\mu_1, \mu_2, \dots, \mu_\ell, \mu_{\ell+1})) = (\mu_1, \mu_2, \dots, \mu_\ell)$, is a bijection. This gives $s_{\ell+1} = |S'_\ell| = (q-1)^\ell - s_\ell$, as claimed. We then see that

$$s_\ell = (q-1)^{\ell-1} - (q-1)^{\ell-2} + \dots + (-1)^{\ell-2} (q-1). \quad (2.1)$$

If $q = 2$ and $\ell \geq 1$ is odd, it is straightforward to check that $s_\ell = 0$; if $q = 2$ and $\ell \geq 1$ is even, then $s_\ell = 1$. We can assume, therefore, that $q > 2$: After reducing, Eq. (2.1) becomes

$$s_\ell = \frac{(q-1)^\ell + (-1)^\ell (q-1)}{q}. \quad (2.2)$$

It is a trivality that Eq. (2.2) also holds true when $\ell = 0$ or $q = 2$ and $\ell \geq 1$. In conclusion, we have

$$s_\ell = \frac{(q-1)^\ell + (-1)^\ell (q-1)}{q} \quad (2.3)$$

for any prime power q and positive integer ℓ with $0 \leq \ell \leq m$. This completes the proof. \square

We are now in a position to give a proof for our main result.

Proof of Theorem 1.1 Let T (resp. S) be an ordered transversal of H in K (resp. K in C_n). This gives $|T| = k/h > 1$ and $|S| = n/k$. We then have an ordered transversal of H in C_n ,

$$ST = \{st \mid s \in S, t \in T\}.$$

(1) Clearly, $\{s\hat{K} \mid s \in S\}$ is an \mathbb{F}_q -basis of C_K , which gives that C_K has dimension n/k . Note, for any distinct $s_1, s_2 \in S$, that $\{s_1 g \mid g \in K\}$ and $\{s_2 g \mid g \in K\}$ are disjoint. Therefore, $A_r^{(C_K)} = 0$ if $k \nmid r$, and that for $0 < i \leq n$ with $i = ki'$, $A_i^{(C_K)} = \binom{n/k}{i'} (q-1)^{i'}$.

(2) It is easy to see that $\{st\hat{H} \mid s \in S, t \in T\}$ is an \mathbb{F}_q -basis of $\mathcal{R}\hat{H}$. Note that

$$e_{H,K}\hat{H} = (\hat{H} - \hat{K})\hat{H} = \hat{H} - \hat{H}\hat{K} = \hat{H} - \hat{K} = e_{H,K},$$

where the third equality holds because $H \subseteq K$ and $g\hat{K} = \hat{K}$ for any $g \in H$. We then have

$$C_{H,K} = \mathcal{R}e_{H,K} \subseteq \mathcal{R}\hat{H}.$$

Given any $\alpha = \sum_{s \in S, t \in T} \lambda_{s,t} st\hat{H} \in \mathcal{R}\hat{H}$ with $\lambda_{s,t} \in \mathbb{F}_q$ for all $s \in S$ and $t \in T$,

$$\alpha e_{H,K} = \left(\sum_{s \in S, t \in T} \lambda_{s,t} st\hat{H} \right) (\hat{H} - \hat{K}) = \alpha - \sum_{s \in S, t \in T} \lambda_{s,t} s\hat{K}.$$

Using the fact that $\alpha \in \mathcal{R}e_{H,K}$ if and only if $\alpha e_{H,K} = \alpha$, it follows that $\alpha \in \mathcal{R}e_{H,K}$ precisely when

$$\sum_{s \in S, t \in T} \lambda_{s,t} s\hat{K} = \sum_{s \in S} \left(\sum_{t \in T} \lambda_{s,t} \right) s\hat{K} = 0.$$

We then have that $\sum_{s \in S} \left(\sum_{t \in T} \lambda_{s,t} \right) s\hat{K} = 0$ if and only if $\sum_{t \in T} \lambda_{s,t} = 0$ for any $s \in S$. This means that the sum of the elements in each row of the $|S| \times |T|$ matrix $[\lambda_{s,t}]_{s \in S, t \in T}$ is equal to 0.

It is clear that $A_0^{(C_{H,K})} = 1$, and that $A_r^{(C_{H,K})} = 0$ if $h \nmid r$ as $\mathcal{R}e_{H,K} \subseteq \mathcal{R}\hat{H}$. Let i , $0 < i \leq n$, be an integer with $h \mid i$, say $i = hi'$. For any n/k -tuple $(\ell_1, \ell_2, \dots, \ell_{n/k})$ of integers with $0 \leq \ell_1, \ell_2, \dots, \ell_{n/k} \leq k/h$ and $\ell_1 + \ell_2 + \dots + \ell_{n/k} = i'$, define

$$\Omega_{(\ell_1, \ell_2, \dots, \ell_{n/k})} = \left\{ [\lambda_{s,t}] \in \mathbb{F}_q^{(n/k) \times (k/h)} \mid \sum_{t \in T} \lambda_{s,t} = 0 \text{ and } wt((\lambda_{s,t})) = \ell_s \text{ for every } 1 \leq s \leq n/k \right\},$$

where $(\lambda_{s,t}) \in \mathbb{F}_q^{k/h}$ denotes the s th row of the matrix $[\lambda_{s,t}] \in \mathbb{F}_q^{(n/k) \times (k/h)}$. It follows that

$$A_{i'}^{(C_{H,K})} = \sum_{\substack{0 \leq \ell_1, \ell_2, \dots, \ell_{n/k} \leq k/h \\ \ell_1 + \dots + \ell_{n/k} = i'}} |\Omega_{(\ell_1, \ell_2, \dots, \ell_{n/k})}| = \sum_{\substack{0 \leq \ell_1, \ell_2, \dots, \ell_{n/k} \leq k/h \\ \ell_1 + \dots + \ell_{n/k} = i'}} |\Omega_{\ell_1}| \cdots |\Omega_{\ell_{n/k}}|.$$

Lemma 2.1 then applies to give the desired result.

Finally, it follows from $|\Omega_0| = 1$ and $|\Omega_1| = 0$ that $A_h^{(C_{H,K})} = 0$ and $A_{2h}^{(C_{H,K})} = \frac{n}{k} |\Omega_2|$, which implies that the Hamming weight of $C_{H,K}$ is equal to $2h$ as $|\Omega_2| \neq 0$. We are done. \square

3 Examples

In this section, two examples are presented to illustrate our results.

Example 3.1. *With the notation of Theorem 1.1, take $n = 2^3 \times 3 \times 5 = 120$ and $q = 7$. Let H, K be subgroups of $C_{120} = \langle x \rangle$ with order $h = 20$ and $k = 60$, respectively. Thus $H = \langle x^6 \rangle$ and $K = \langle x^2 \rangle$, which gives that $\hat{H} = \frac{1}{20} \sum_{j=0}^{19} x^{6j}$ and $\hat{K} = \frac{1}{60} \sum_{j=0}^{59} x^{2j}$ are idempotents of $\mathcal{R} = \mathbb{F}_7 C_{120}$.*

It follows from Theorem 1.1 that C_K is a cyclic code over \mathbb{F}_7 with parameters $[120, 2, 60]$, and its weight distribution is given by (We just list the A_i which are nonzero.):

$$A_i^{(C_K)} = \begin{cases} 1, & \text{if } i = 0; \\ 12, & \text{if } i = 60; \\ 36, & \text{if } i = 120. \end{cases}$$

Using Theorem 1.1 again, $C_{H,K}$ is a cyclic code over \mathbb{F}_7 has parameters $[120, 4, 40]$, and its weight distribution is given by Table 1.

Example 3.2. *With the notation of Theorem 1.1, take $n = 3 \times 5^2 = 75$ and $q = 2$. Let H, K be subgroups of $C_{75} = \langle x \rangle$ with order $h = 3$ and $k = 75$, respectively. Thus $H = \langle x^{25} \rangle$ and $K = \langle x \rangle$, which gives that $\hat{H} = 1 + x^{25} + x^{50}$ and $\hat{K} = \sum_{j=0}^{74} x^j$ are idempotents of $\mathcal{R} = \mathbb{F}_2 C_{75}$.*

Hamming weight	Frequency	Hamming weight	Frequency
0	1	80	324
40	36	100	1080
60	60	120	900

Table 1: Weight distribution of $C_{H,K}$ of length 120 over \mathbb{F}_7

We then know from Theorem 1.1 that C_K is a cyclic code over \mathbb{F}_2 having parameters $[75, 1, 75]$, and its weight distribution is given by (We just list the A_i which are nonzero.):

$$A_i^{(C_K)} = \begin{cases} 1, & \text{if } i = 0; \\ 1, & \text{if } i = 75. \end{cases}$$

Using Theorem 1.1 again, $C_{H,K}$ is a cyclic code over \mathbb{F}_2 having parameters $[75, 24, 6]$, and its weight distribution is given by Table 2.

Hamming weight	Frequency	Hamming weight	Frequency
0	1	42	4457400
6	300	48	2042975
12	12650	54	480700
18	177100	60	53130
24	1081575	66	2300
30	3268760	72	25
36	5200300		

Table 2: Weight distribution of $C_{H,K}$ of length 75 over \mathbb{F}_2

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