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REPRESENTING SYSTEMS GENERATED BY REPRODUCING KERNELS

EMMANUEL FRICAIN, LE HAI KHOI, AND PASCAL LEFÈVRE

ABSTRACT. In this paper, we study the representing and absolutely representing systems in the context of reproducing kernel Hilbert spaces. We prove in particular that, in many classical spaces including weighted Hardy and Dirichlet spaces and de Branges–Rovnyak spaces, there cannot exist absolutely representing systems of reproducing kernels.

1. INTRODUCTION

In finite dimensional vector spaces, the notion of basis is classical and completely understood. It corresponds to systems which are on one hand free and on the other hand spanning. In infinite dimensional Banach spaces, things become more complicated. There are several notion of basis (Hamel, Schauder..). In general the notion of freedom corresponds to minimality. A natural analog of spanning systems appears to be representing systems. This notion enables to represent each element of the space as a series of elements of the representing system. The difference with the notion of Schauder basis comes from the fact that the coefficients in the representation are not necessarily unique. The theory of representing systems not only allows to study rather deep functional properties of (holomorphic) functions in different functions spaces, but also has many important applications to functional equations, say representation of solutions in series of simpler functions, like exponential functions or rational functions. Results in this subject were first obtained by A.F. Leont'ev and later on by Yu.F. Korobeinik and his school, who developed systematically this theory in Fréchet–Schwartz or dual Fréchet–Schwartz spaces. See the survey article [5] and references therein for more details.

A specific family of representing systems is absolutely representing systems (see the Definition 2.2 below). According to Vershynin [8], this concept goes back to a Ukrainian edition of Banach's classical book but was mainly studied in the context of locally convex topological spaces. The theory of absolutely representing systems in Banach spaces was not very well

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explored. In [8], some interesting connections are made with ε -nets in uniformly smooth and in B -convex Banach spaces.

In this paper, we mainly focus on absolutely representing systems in the context of reproducing kernel Hilbert spaces. In particular, we are interested in the following question.

Question 1.1. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space on a set X and let $k_x^{\mathcal{H}}$ be the reproducing kernel of $\mathcal{H}$ at a point $x \in X$. Can we characterize the sequence of points $(x_n)_{n \geq 1}$ in X such that $(k_{x_n}^{\mathcal{H}})_{n \geq 1}$ is a representing system (respectively an absolutely representing system) for its closed linear span?*

Reproducing kernels are an important tool in the theory of Hilbert spaces of holomorphic functions, very often explicitly known but not always (e.g. in Dirichlet $\mathfrak{D}_\mu$ type spaces considered in this paper). Sometimes they form a Schauder basic sequence. When it's not the case, it is natural to wonder whether they form an absolutely representing system or merely a representing system.

In Section 2, we give some general preliminaries on representing systems in Banach spaces setting. In Section 3, we prove that there cannot exist absolutely representing systems of reproducing kernels in the Hardy space H^2 and we also give a necessary condition for representing systems. In the last section, we generalize the non-existence of absolutely representing systems of reproducing kernels in various Hilbert spaces, including weighted Hardy spaces, weighted Dirichlet spaces and de Branges–Rovnyak spaces.

2. REPRESENTING SYSTEM AND ABSOLUTELY REPRESENTING SYSTEMS IN BANACH SPACES

2.1. Definition. We recall here some notion of geometry of Banach spaces we need.

Definition 2.1. *Let $\mathfrak{X}$ be a complex and separable Banach space. A sequence of normalized elements $(x_n)_{n \geq 1}$ is said to be*

- (1) complete in $\mathfrak{X}$ if $\text{span}(x_n : n \geq 1) = \mathfrak{X}$;
- (2) minimal if for every $n \geq 1$, $x_n \notin \text{span}(x_k : k \neq n)$;
- (3) uniformly minimal if

$$\inf_{n \geq 1} \text{dist}(x_n, \text{span}(x_k : k \neq n)) > 0;$$

- (4) a Riesz basis if it is complete and there exists two constants $c, C > 0$ such that

$$c \sum_n |a_n|^2 \leq \left\| \sum_n a_n x_n \right\|^2 \leq C \sum_n |a_n|^2.$$

where $\text{span}(\dots)$ denotes the closed linear span generated by $(\dots)$

Definition 2.2. Let $\mathfrak{X}$ be a complex and separable Banach space of infinite dimension. A sequence of non-zero elements $(x_n)_{n \geq 1}$ is said to be a representing system for $\mathfrak{X}$ if, for every element $x \in \mathfrak{X}$, there exists a sequence of complex numbers $(a_n)_{n \geq 1}$ such that

$$(2.1) \quad x = \sum_{n=1}^{\infty} a_n x_n,$$

where the series converges in norm of $\mathfrak{X}$.

It should be noted that in the definition, we do not ask that the sequence $(a_n)_{n \geq 1}$ is unique. If we impose the uniqueness, then it coincides with the notion of Schauder basis. In other words, the sequence $(x_n)_{n \geq 1}$ is a (Schauder) basis if and only if it is a representing system which is minimal.

Definition 2.3. We say that a sequence $\mathfrak{X} = (x_n)_{n \geq 1}$ is an absolutely representing system for $\mathfrak{X}$ if we have (2.1) and moreover

$$\sum_{n \geq 1} |a_n| \|x_n\| < \infty.$$

2.2. General results on representing and absolutely representing systems. It should be noted that every separable Banach space admits an absolutely representing system. That follows directly from Banach–Mazur theorem (see for instance [1]). Nevertheless we give the proof for completeness.

Proposition 2.4. Let $\mathfrak{X}$ be a separable Banach space. Then there exists a sequence $(x_n)_{n \geq 1}$ which forms an absolutely representing system for $\mathfrak{X}$.

Proof. Since $\mathfrak{X}$ is separable, we can find a sequence of vectors $(x_n)_{n \geq 1}$ dense in the unit ball of $\mathfrak{X}$. We shall prove that $(x_n)_{n \geq 1}$ is an absolutely representing system for $\mathfrak{X}$, which means that, for every $x \in \mathfrak{X}$, there is a sequence $(a_n)_{n \geq 1} \in \ell^1$ such that

$$x = \sum_{n=1}^{\infty} a_n x_n.$$

Let $x \in \mathfrak{X}$. Without loss of generality, we can of course assume that $\|x\| = 1$. Then we can choose $n_1 \geq 1$ so that

$$\|x - x_{n_1}\| \leq \frac{1}{2}.$$

Assume that we have chosen $n_1 < n_2 < \dots < n_j$ satisfying

$$(2.2) \quad \left\| x - x_{n_1} - \frac{1}{2}x_{n_2} - \dots - \frac{1}{2^{j-1}}x_{n_j} \right\| \leq \frac{1}{2^j}.$$

Since $2^j(x - x_{n_1} - \cdots - \frac{1}{2^{j-1}x_{n_j}})$ lies in the unit ball of $\mathfrak{X}$, we can choose $n_{j+1} > n_j$ so that

$$\left\| 2^j(x - x_{n_1} - \cdots - \frac{1}{2^{j-1}x_{n_j}}) - x_{n_{j+1}} \right\| < \frac{1}{2}.$$

Hence

$$\left\| x - x_{n_1} - \frac{1}{2}x_{n_2} - \cdots - \frac{1}{2^j}x_{n_{j+1}} \right\| \leq \frac{1}{2^{j+1}},$$

and by induction we obtain (2.2) for all $j \geq 1$. Take now

$$a_n = \begin{cases} \frac{1}{2^{j-1}} & \text{if } n = n_j, j \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, of course $(a_n)_{n \geq 1} \in \ell^1$ and letting $j \rightarrow \infty$ in (2.2) gives

$$x = \sum_{n=1}^{\infty} a_n x_n.$$

□

The following characterization of absolutely representing systems will be useful for us. As usual, given a Banach space $\mathfrak{X}$, we denote by $\mathfrak{X}^*$ the dual space of $\mathfrak{X}$.

Lemma 2.5. *Let $\mathfrak{X}$ be a Banach space, $(x_n)_{n \geq 1}$ be a sequence of vectors in $\mathfrak{X}$, $\|x_n\| = 1$, $n \geq 1$. The following are equivalent:*

- (i) *the sequence $(x_n)_{n \geq 1}$ is an absolutely representing system for $\mathfrak{X}$;*
- (ii) *there exists $C > 0$ such that for every $x^* \in \mathfrak{X}^*$, $\|x^*\| = 1$, there is an integer $n \geq 1$ such that $|x^*(x_n)| \geq C$.*

Proof. First note that the map T defined, for $a = (a_n)_{n \geq 1} \in \ell^1$, by

$$(2.3) \quad T(a) = \sum_{n=1}^{\infty} a_n x_n$$

is linear and bounded from ℓ^1 into $\mathfrak{X}$. Moreover, an easy computation shows that its adjoint $T^* : \mathfrak{X}^* \rightarrow \ell^\infty$ is given by

$$T^*(x^*) = (x^*(x_n))_{n \geq 1}, \quad x^* \in \mathfrak{X}^*.$$

It remains to note that $(i) \iff T \text{ onto} \iff T^* \text{ bounded below} \iff (ii)$. □

As already mentioned, a Schauder basis is a representing system which is minimal. For a class of Banach spaces, including Hilbert spaces, the situation for absolutely representing systems changes dramatically.

Proposition 2.6. *Let $\mathfrak{X}$ be a Banach space which is not isomorphic to ℓ^1 and let $(x_n)_{n \geq 1}$ be a sequence of unit vectors in $\mathfrak{X}$ which is a basis for $\mathfrak{X}$. Then $(x_n)_{n \geq 1}$ cannot be an absolutely representing system for $\mathfrak{X}$.*

Proof. Let $T : \ell^1 \rightarrow \mathfrak{X}$ be the map defined by (2.3). It is clear that T is a linear and bounded map from ℓ^1 into $\mathfrak{X}$. Since $(x_n)_{n \geq 1}$ is a basis, T is also one-to-one. Assume that $(x_n)_{n \geq 1}$ is an absolutely representing system for $\mathfrak{X}$. Then the operator T is also onto and thus an isomorphism, which contradicts the fact that $\mathfrak{X}$ is not isomorphic to ℓ^1 . $\square$

Note that Proposition 2.6 applies in particular to Hilbert spaces.

3. REPRESENTING SYSTEMS IN HARDY SPACES

Let H^2 be the Hardy space on the open unit disc $\mathbb{D}$. It is a reproducing kernel Hilbert space with reproducing kernel given by $k_\lambda(z) = (1 - \bar{\lambda}z)^{-1}$, where $\lambda, z \in \mathbb{D}$. This means that

$$f(\lambda) = \langle f, k_\lambda \rangle_2, \quad \text{where } f \in H^2.$$

We denote by $\hat{k}_\lambda$ the normalized reproducing kernel, $\hat{k}_\lambda = k_\lambda / \|k_\lambda\|_2$. Let us recall that if $(\lambda_n)_{n \geq 1}$ is a sequence of distinct points in the unit disc, then we have the following dichotomy: if $(\lambda_n)_{n \geq 1}$ satisfies the Blaschke condition, that is if

$$\sum_{n \geq 1} (1 - |\lambda_n|) < \infty,$$

then the sequence $(\hat{k}_{\lambda_n})_{n \geq 1}$ is minimal and

$$\text{span}(\hat{k}_{\lambda_n} : n \geq 1) = K_B,$$

where B is the associated Blaschke product $B = \prod_{n \geq 1} b_{\lambda_n}$, b_{λ_n} is the single Blaschke factor defined by $b_{\lambda_n}(z) = \frac{|\lambda_n|}{\lambda_n}(\lambda_n - z)(1 - \bar{\lambda}_n z)^{-1}$, and K_B is the associated model subspace defined by $K_B = (BH^2)^\perp$. On the contrary, if $(\lambda_n)_{n \geq 1}$ does not satisfy the Blaschke condition, then the sequence $(\hat{k}_{\lambda_n})_{n \geq 1}$ is not minimal and

$$\text{span}(\hat{k}_{\lambda_n} : n \geq 1) = H^2.$$

See for instance [3, Lemma 14.16].

In the case when the sequence $(\lambda_n)_{n \geq 1}$ is a Blaschke sequence, we can easily characterize the representing systems formed by reproducing kernels.

Proposition 3.1. *Let $(\lambda_n)_{n \geq 1}$ be a Blaschke sequence of distinct points in $\mathbb{D}$ and B the associated Blaschke product. The following are equivalent:*

- (i) *the sequence $(\hat{k}_{\lambda_n})_{n \geq 1}$ is a representing system for K_B ;*
- (ii) *the sequence $(\hat{k}_{\lambda_n})_{n \geq 1}$ is a Riesz basis for K_B ;*

- (iii) the sequence $(\hat{k}_{\lambda_n})_{n \geq 1}$ is uniformly minimal;
(iv) the sequence $(\lambda_n)_{n \geq 1}$ satisfies the Carleson condition, that is

$$\inf_{n \geq 1} |B_n(\lambda_n)| > 0,$$

where $B_n = B/b_{\lambda_n}$.

Proof. The equivalence between (ii), (iii) and (iv) is well-known (see [3, 6]). The implication (ii) $\implies$ (i) is obvious. It remains to prove (i) $\implies$ (iii). Since $(\lambda_n)_{n \geq 1}$ is a Blaschke sequence, we know that the sequence $(\hat{k}_{\lambda_n})_{n \geq 1}$ is minimal. If it is furthermore assumed to be a representing system, then it should be a (Schauder) basis. Hence, in particular, it should be uniformly minimal. $\square$

It is quite easy to prove that there does not exist absolutely representing systems formed by reproducing kernels of H^2 .

Theorem 3.2. *Let $(\lambda_n)_{n \geq 1}$ be a sequence of distinct points in $\mathbb{D}$. Then the sequence $(\hat{k}_{\lambda_n})_{n \geq 1}$ cannot be an absolutely representing system for its closed linear span in H^2 .*

Proof. Assume that the sequence $(\hat{k}_{\lambda_n})_{n \geq 1}$ is an absolutely representing system for its closed linear span. We shall distinguish two cases.

First case: $(\lambda_n)_{n \geq 1}$ is a Blaschke sequence. Let B be the associated Blaschke product. Then we know that the sequence $(\hat{k}_{\lambda_n})_{n \geq 1}$ generates the space K_B and according to Proposition 3.1, it should be a Riesz basis for K_B . Then apply Proposition 2.6 to get a contradiction.

Second case: $(\lambda_n)_{n \geq 1}$ is not a Blaschke sequence. In that case, we know that

$$\text{span}(\hat{k}_{\lambda_n} : n \geq 1) = H^2.$$

According to Lemma 2.5, there is a constant $C > 0$ such that for any $h \in H^2$, $\|h\|_2 = 1$, we can find $n \geq 1$ such that $|h(\lambda_n)|(1 - |\lambda_n|^2)^{1/2} \geq C$. Choose N sufficiently large so that $2 \leq (N + 1)C^2$. Consider $\varphi(t) = (1 - t)t^N$ and point out that

$$\sup_{t \in [0,1]} \varphi(t) \leq \frac{1}{N + 1}.$$

For $h(z) = z^N$, there is an integer n such that

$$C^2 \leq |h(\lambda_n)|^2(1 - |\lambda_n|^2) = |\lambda_n|^{2N}(1 - |\lambda_n|^2) = \varphi(|\lambda_n|^2) \leq \frac{1}{N + 1} \leq \frac{C^2}{2},$$

which gives the desired contradiction in that second case. $\square$

We give now a necessary condition so that a sequence of reproducing kernels forms a representing system for H^2 .

Theorem 3.3. *Let $(\lambda_n)_{n \geq 1}$ be a sequence in $\mathbb{D}$ which is not a Blaschke sequence. Assume that*

$$(3.1) \quad \sum_n |\lambda_n - \lambda_{n+1}| < \infty.$$

Then the sequence $(k_{\lambda_n})_{n \geq 1}$ is not a representing system for H^2 .

Proof. Condition (3.1) obviously implies that $(\lambda_n)_{n \geq 1}$ is convergent to some point in $\overline{\mathbb{D}}$. In particular, there exists $\zeta_0 \in \mathbb{T} = \partial\mathbb{D}$ such that

$$(3.2) \quad \inf_{n \geq 1} |\lambda_n - \zeta_0| > 0.$$

Assume now that the sequence $(k_{\lambda_n})_{n \geq 1}$ is a representing system for H^2 . Then there exists a sequence of complex numbers $(a_n)_{n \geq 1}$ such that

$$\frac{1}{(1 - \overline{\zeta_0}z)^{1/4}} = \sum_{n=1}^{\infty} a_n \frac{1}{1 - \overline{\lambda_n}z},$$

where the series converges in the H^2 -norm. In particular, the series is also pointwise convergent and taking $z = 0$, we get that the series $\sum_n a_n$ is convergent. Moreover, using (3.2), we have

$$\left| \frac{1}{1 - \overline{\lambda_n}\zeta_0} - \frac{1}{1 - \overline{\lambda_{n+1}}\zeta_0} \right| = \frac{|\lambda_n - \lambda_{n+1}|}{|1 - \overline{\lambda_n}\zeta_0||1 - \overline{\lambda_{n+1}}\zeta_0|} \lesssim |\lambda_n - \lambda_{n+1}|,$$

where the symbol $A \lesssim B$ means that there is an absolute constant $C > 0$ such that $A \leq CB$.

Using now (3.1), we can apply Abel's theorem to get that the series

$$\sum_n a_n \frac{1}{1 - \overline{\lambda_n}\zeta_0}$$

is convergent and

$$\lim_{z \xrightarrow[\triangleleft]{} \zeta_0} \sum_{n=1}^{\infty} a_n \frac{1}{1 - \overline{\lambda_n}z} = \sum_{n=1}^{\infty} a_n \frac{1}{1 - \overline{\lambda_n}\zeta_0},$$

where $\lim_{z \xrightarrow[\triangleleft]{} \zeta_0}$ denotes the non-tangential limit at ζ_0 . Hence $(1 - \overline{\zeta_0}z)^{-1/4}$ has a non-tangential limit at ζ_0 which gives a contradiction. $\square$

It would be interesting to answer the following open question.

Question 3.4. *Can we construct a sequence of points $(\lambda_n)_n$ in $\mathbb{D}$ which is not a Blaschke sequence and so that $(k_{\lambda_n})_n$ forms a representing system for H^2 ?*

4. ABSOLUTELY REPRESENTING SYSTEMS IN VARIOUS REPRODUCING KERNEL HILBERT SPACES

We can generalize Theorem 3.2 for a large class of reproducing kernel Hilbert spaces. Let $\mathcal{H}$ be a reproducing kernel Hilbert space of analytic functions on the unit disc $\mathbb{D}$. We say that $\mathcal{H}$ is *invariant with respect to the difference quotients* if, for every $\lambda \in \mathbb{D}$, we have $f \in \mathcal{H} \implies Q_\lambda f \in \mathcal{H}$, where

$$(Q_\lambda f)(z) = \frac{f(z) - f(\lambda)}{z - \lambda} = \sum_{k=0}^{\infty} \frac{f^{(k+1)}(\lambda)}{(k+1)!} (z - \lambda)^k, \quad z \in \mathbb{D}.$$

Many classical spaces are invariant with respect to the difference quotients, e.g. the Hardy space H^2 , the Dirichlet space $\mathcal{D}$, the Bergman space $\mathcal{B}$,...

Lemma 4.1. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space which is invariant with respect to the difference quotients. Let $(k_{\lambda_n}^{\mathcal{H}})_{n \geq 1}$ be a sequence of reproducing kernels of $\mathcal{H}$. Then we have the following dichotomy: either $(k_{\lambda_n}^{\mathcal{H}})_{n \geq 1}$ is minimal, or $(k_{\lambda_n}^{\mathcal{H}})_{n \geq 1}$ is complete in $\mathcal{H}$.*

Proof. Assume that $(k_{\lambda_n}^{\mathcal{H}})_{n \geq 1}$ is not complete in $\mathcal{H}$ and let us prove that the sequence is minimal. There exists $f \in \mathcal{H}$, $f \not\equiv 0$, such that $f(\lambda_n) = 0$, $n \geq 1$. Fix an integer $N \geq 1$ and denote by ℓ the smallest integer such that $f^{(\ell)}(\lambda_N) \neq 0$. Then define $g = Q_{\lambda_N}^\ell f$ where $Q_{\lambda_N}^\ell = Q_{\lambda_N} \circ Q_{\lambda_N} \circ \dots \circ Q_{\lambda_N}$ (ℓ -times). By hypothesis, the function g belongs to $\mathcal{H}$. Moreover, since

$$(Q_\lambda^s f)(z) = \sum_{k=0}^{\infty} \frac{f^{(k+s)}(\lambda)}{(k+s)!} (z - \lambda)^k,$$

we have $g(\lambda_N) \neq 0$ and, by induction, it is easy to check that $g(\lambda_n) = 0$, $n \geq 1$, $n \neq N$. Hence

$$k_{\lambda_N}^{\mathcal{H}} \notin \text{span} (k_{\lambda_n}^{\mathcal{H}} : n \neq N),$$

which proves the minimality of $(k_{\lambda_n}^{\mathcal{H}})_{n \geq 1}$. □

4.1. Weighted Hilbert spaces of analytic functions and absolutely representing systems. Given a positive integrable function ω on $[0, 1]$ which is not zero almost everywhere on a neighborhood of 1, we extend it by $\omega(z) = \omega(|z|)$, $z \in \mathbb{D}$, and call such ω a weight function. We denote by $\mathfrak{D}_\omega$ the space consisting of analytic functions f on $\mathbb{D}$ such that

$$\|f'\|_\omega^2 = \int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z) < \infty,$$

where $dA(z) = dxdy/\pi$ is normalized area measure on $\mathbb{D}$. The Hilbert space $\mathfrak{D}_\omega$ is endowed with the norm

$$\|f\|_{\mathfrak{D}_\omega}^2 = |f(0)|^2 + \|f'\|_\omega^2.$$

A simple computation shows that an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to $\mathfrak{D}_\omega$ if and only if

$$\|f\|_{\mathfrak{D}_\omega}^2 = \sum_{n=0}^{\infty} |a_n|^2 \omega_n < \infty,$$

where $\omega_0 = 1$ and

$$\omega_n = 2n^2 \int_0^1 r^{2n-1} \omega(r) dr, \quad n \geq 1.$$

The following result is probably known but we couldn't find reference so we include a proof.

Lemma 4.2. *Let ω be a weight function. Then $\mathfrak{D}_\omega$ is invariant with respect to the difference quotients and it is a reproducing kernel Hilbert space whose kernel is given by*

$$k_\lambda^\omega(z) = \sum_{k=0}^{\infty} \frac{\bar{\lambda}^k}{\omega_k} z^k, \quad (z, \lambda \in \mathbb{D}).$$

Proof. To prove that $\mathfrak{D}_\omega$ is invariant with respect to the difference quotients, let $f \in \mathfrak{D}_\omega$ and $\lambda \in \mathbb{D}$. Fix $\delta > 0$ such that the closed disc $K = \overline{D}(\lambda, \delta)$ is contained in $\mathbb{D}$. If $g = Q_\lambda f$, then write

$$\int_{\mathbb{D}} |g'(z)|^2 \omega(z) dA(z) = \int_K |g'(z)|^2 \omega(z) dA(z) + \int_{\mathbb{D} \setminus K} |g'(z)|^2 \omega(z) dA(z).$$

On one hand, note that

$$\int_K |g'(z)|^2 \omega(z) dA(z) \leq \sup_{z \in K} |g'(z)|^2 \int_{\mathbb{D}} \omega(z) dA(z) \leq 2\|\omega\|_{L^1[0,1]} \sup_{z \in K} |g'(z)|^2 < \infty.$$

On the other hand, we have

$$\begin{aligned} \int_{\mathbb{D} \setminus K} |g'(z)|^2 \omega(z) dA(z) &= \int_{\mathbb{D} \setminus K} \frac{|f(z) - f(\lambda) - (z - \lambda)f'(\lambda)|^2}{|z - \lambda|^4} \omega(z) dA(z) \\ &\lesssim \int_{\mathbb{D}} (|f(z)|^2 + |f(\lambda)|^2 + |f'(\lambda)|^2) \omega(z) dA(z). \end{aligned}$$

Since $f \in \mathfrak{D}_\omega$, we have

$$\int_{\mathbb{D}} (|f(\lambda)|^2 + |f'(\lambda)|^2) \omega(z) dA(z) \leq 2\|\omega\|_{L^1[0,1]} |f(\lambda)|^2 + \|f'\|_\omega^2 < \infty.$$

Writing $f(z) = \sum_{n \geq 0} a_n z^n$, it remains to note that

$$\begin{aligned} \int_{\mathbb{D}} |f(z)|^2 \omega(z) dA(z) &= 2 \sum_{n=0}^{\infty} |a_n|^2 \int_0^1 r^{2n+1} \omega(r) dr \\ &\lesssim \sum_{n=1}^{\infty} |a_n|^2 n^2 \int_0^1 r^{2n-1} \omega(r) dr + |a_0|^2 \\ &= \sum_{n=0}^{\infty} |a_n|^2 \omega_n = \|f\|_{\mathfrak{D}_\omega}^2 < \infty. \end{aligned}$$

Hence

$$\int_{\mathbb{D}} |g'(z)|^2 \omega(z) dA(z) < \infty,$$

and $g = Q_\lambda f \in \mathfrak{D}_\omega$. We have thus proved that $\mathfrak{D}_\omega$ is invariant with respect to the difference quotients.

Using a duality argument and Hadamard's formula, we see that $\mathfrak{D}_\omega$ is a reproducing kernel Hilbert space (that is it has bounded evaluation at every point $\lambda \in \mathbb{D}$) if and only if

$$(4.1) \quad \liminf_{n \rightarrow \infty} \omega_n^{1/n} \geq 1.$$

By hypothesis, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$, we have

$$C_\varepsilon := \int_{1-\varepsilon}^1 \omega(r) dr > 0,$$

which gives

$$\omega_n \geq 2n^2 \int_{1-\varepsilon}^1 r^{2n-1} \omega(r) dr \geq 2n^2 C_\varepsilon (1-\varepsilon)^{2n-1}$$

Hence we get

$$\liminf_{n \rightarrow \infty} \omega_n^{1/n} \geq (1-\varepsilon)^2.$$

Letting ε go to zero, we obtain (4.1) and $\mathfrak{D}_\omega$ is a reproducing kernel Hilbert space.

Finally, let $\lambda \in \mathbb{D}$ and $f(z) = \sum_{n \geq 0} a_n z^n \in \mathfrak{D}_\omega$. By (4.1), the function

$$k_\lambda^\omega(z) = \sum_{k=0}^{\infty} \frac{\bar{\lambda}^k}{\omega_k} z^k.$$

is an analytic function on $\mathbb{D}$ which belongs to $\mathfrak{D}_\omega$. Moreover, we have

$$\langle f, k_\lambda^\omega \rangle_{\mathfrak{D}_\omega} = \sum_{n=0}^{\infty} a_n \frac{\lambda^n}{\omega_n} \omega_n = \sum_{n=0}^{\infty} a_n \lambda^n = f(\lambda),$$

which proves that k_λ^ω is the reproducing kernel for $\mathfrak{D}_\omega$. □

Lemma 4.3. *Let ω be a weight function and*

$$S_N = \sum_{k=0}^N \frac{1}{\omega_k}.$$

Then

$$\limsup_{N \rightarrow \infty} \omega_N S_N = \infty.$$

Proof. Assume that $\limsup_{N \rightarrow \infty} \omega_N S_N < \infty$. Then there exists a constant C such that for every $n \geq 1$, we have $1 \leq \omega_n S_n < C$. Writing

$$S_n \leq \frac{C}{\omega_n} = C(S_n - S_{n-1}),$$

we get $qS_{n-1} \leq S_n$, where $q = C/(C-1) > 1$. Hence $S_n \geq q^n S_0 = q^n$ (recall that $\omega_0 = 1$), which gives

$$\omega_n^{1/n} \leq C^{1/n} q^{-1},$$

and that contradicts (4.1). $\square$

We have

$$\|k_\lambda^\omega\|_{\mathfrak{D}_\omega}^2 = \sum_{k=0}^{\infty} \frac{|\lambda|^{2k}}{\omega_k}.$$

and we denote by $\widehat{k_\lambda^\omega}$ the normalized reproducing kernel.

Theorem 4.4. *Let ω be a weight function and $(\lambda_n)_{n \geq 1}$ be a sequence of distinct points in $\mathbb{D}$. Then the sequence $(\widehat{k_{\lambda_n}^\omega})_{n \geq 1}$ cannot be an absolutely representing system for its closed linear span in $\mathfrak{D}_\omega$.*

Proof. Assume that the sequence $(\widehat{k_{\lambda_n}^\omega})_{n \geq 1}$ is an absolutely representing system for its closed linear span. Since $\mathfrak{D}_\omega$ is an Hilbert space, arguing as in the proof of Theorem 3.2, we focus directly on the case when the sequence $(\widehat{k_{\lambda_n}^\omega})_{n \geq 1}$ is not minimal. Then, according to Lemma 4.2 and Lemma 4.1, we have

$$\text{span}(\widehat{k_{\lambda_n}^\omega} : n \geq 1) = \mathfrak{D}_\omega.$$

It follows from Lemma 2.5 that there is a constant $C > 0$ such that for every $h \in \mathfrak{D}_\omega$, we can find $n \geq 1$ such that

$$|h(\lambda_n)| \geq C \|k_{\lambda_n}^\omega\|_{\mathfrak{D}_\omega} \|h\|_{\mathfrak{D}_\omega}.$$

Applying this to $h(z) = z^N$ for $N \in \mathbb{N}$, there is $n \geq 1$ such that

$$|\lambda_n|^{2N} \geq C^2 \|k_{\lambda_n}^\omega\|_{\mathfrak{D}_\omega}^2 \|z^N\|_{\mathfrak{D}_\omega}^2.$$

Note that

$$\|z^N\|_{\mathfrak{D}_\omega}^2 = \omega_N,$$

and

$$\|k_{\lambda_n}^\omega\|_{\mathfrak{D}_\omega}^2 \geq \sum_{k=0}^N \frac{|\lambda_n|^{2k}}{\omega_k} \geq |\lambda_n|^{2N} S_N.$$

Hence we obtain $|\lambda_n|^{2N} \geq C^2 |\lambda_n|^{2N} S_N \omega_N$, that is

$$1 \geq C^2 S_N \omega_N,$$

for every $N \in \mathbb{N}$. According to Lemma 4.3, letting N tend to ∞ in the last estimate, we get the desired contradiction in that second case. $\square$

Example 4.5. *Let $\omega(r) = (1 - r^2)^\alpha$ with $\alpha > -1$. Then ω is a weight function and we can apply Theorem 4.4 to $\mathfrak{D}_\omega$.*

For $\alpha = 0$, we get $\omega_n = n$ and $\mathfrak{D}_\omega$ is the classical Dirichlet space $\mathfrak{D}$. Therefore we can apply Theorem 4.4 to the Dirichlet space.

For $\alpha = 1$, we get $\omega_n = \frac{n}{n+1}$, $n \geq 0$, and since $\omega_n \sim 1$, then $\mathfrak{D}_\omega = H^2$. Thus Theorem 4.4 is a generalization of Theorem 3.2.

For $\alpha = 2$, we get $\omega_n = \frac{2n}{(n+1)(n+2)}$, $n \geq 1$, and since $\omega_n \sim \frac{2}{n}$, we get $\mathfrak{D}_\omega$ is the classical Bergman space $\mathfrak{B}$ and we can apply Theorem 4.4 to the Bergman space.

4.2. Dirichlet type spaces and absolutely representing systems. We now study an example when the weight ω is not (necessarily) radial but corresponds to the Poisson transform of a positive finite Borel measure μ on $\mathbb{T}$. Let

$$\omega(z) = P[\mu](z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|z - \zeta|^2} d\mu(\zeta)$$

and let $\mathfrak{D}_\mu = \mathfrak{D}_\omega$ be the space consisting of analytic functions f on $\mathbb{D}$ satisfying

$$\mathcal{D}_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P[\mu](z) dA(z) < \infty.$$

It is known (see [2, Theorem 7.1.2]) that $\mathfrak{D}_\mu \subset H^2$ and $\mathfrak{D}_\mu$ is a reproducing kernel Hilbert space whose norm is given by

$$\|f\|_\mu^2 = \|f\|_2^2 + \mathcal{D}_\mu(f), \quad f \in \mathfrak{D}_\mu.$$

It is also known that $\mathfrak{D}_\mu$ is invariant with respect to the difference quotients. Denote by $\widehat{k_\lambda^\mu}$ the normalized reproducing kernel for $\mathfrak{D}_\mu$ at point $\lambda \in \mathbb{D}$.

Theorem 4.6. *Let μ be a positive finite Borel measure on $\mathbb{T}$ and $(\lambda_n)_{n \geq 1}$ a sequence of distinct points in $\mathbb{D}$. Then the sequence $(\widehat{k_{\lambda_n}^\mu})_{n \geq 1}$ cannot be an absolutely representing system for its closed linear span in $\mathfrak{D}_\mu$.*

Proof. The proof is along the same lines as the proof of Theorem 4.4. First note that a simple computation shows that for every integer $N \geq 0$, we have

$$(4.2) \quad \|z^N\|_\mu^2 = 1 + N\|\mu\|.$$

Assume that the sequence $(\widehat{k_{\lambda_n}^\mu})_{n \geq 1}$ is an absolutely representing system for its closed linear span. Since $\mathfrak{D}_\mu$ is an Hilbert space, arguing as in the proof of Theorem 3.2, we focus directly on the case when the sequence $(\widehat{k_{\lambda_n}^\mu})_{n \geq 1}$ is not minimal. Then, since $\mathfrak{D}_\mu$ is invariant with respect to the difference quotients, we can apply Lemma 4.1, and we get

$$\text{span}(\widehat{k_{\lambda_n}^\mu} : n \geq 1) = \mathfrak{D}_\mu.$$

It follows from Lemma 2.5 that there is a constant $C > 0$ such that for every $h \in \mathfrak{D}_\mu$, we can find $n \geq 1$ such that

$$|h(\lambda_n)| \geq C \|k_{\lambda_n}^\mu\|_\mu \|h\|_\mu.$$

Applying this to $h(z) = z^N$ for $N \in \mathbb{N}$, there is $n \geq 1$ such that

$$|\lambda_n|^{2N} \geq C^2 \|k_{\lambda_n}^\mu\|_\mu^2 \|z^N\|_\mu^2.$$

Since $1 \in \mathfrak{D}_\mu$ and $\|1\|_\mu = 1$, note that

$$1 = \langle 1, k_{\lambda_n}^\mu \rangle_\mu \leq \|k_{\lambda_n}^\mu\|_\mu,$$

and we deduce

$$1 \geq |\lambda_n|^{2N} \geq C^2 \|z^N\|_\mu^2.$$

If we let N tend to ∞ we obtain a contradiction with (4.2) in that second case. $\square$

4.3. De Branges–Rovnyak spaces and absolutely representing systems. In this section we study the case of de Branges–Rovnyak spaces. We denote by $\mathcal{B}_{H^\infty}$ the unit ball of H^∞ . Recall that, given a function $b \in \mathcal{B}_{H^\infty}$, the de Branges–Rovnyak space $\mathcal{H}(b)$ is the reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$ whose kernel is

$$k_\lambda^b(z) := \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z}, \quad z, \lambda \in \mathbb{D}.$$

Note that

$$\|k_\lambda^b\|_b^2 = \frac{1 - |b(\lambda)|^2}{1 - |\lambda|^2}.$$

We also denote by $\widehat{k}_\lambda^b$ the normalized reproducing kernel of $\mathcal{H}(b)$ at point λ .

The space $\mathcal{H}(b)$ can also be defined as the range space $(I - T_b T_{\bar{b}})^{1/2} H^2$ equipped with the norm which makes $(I - T_b T_{\bar{b}})^{1/2}$ a partial isometry. Here T_φ is the Toeplitz operator on H^2 with symbol $\varphi \in L^\infty(\mathbb{T})$ defined by

$$T_\varphi f = P_+(\varphi f), \quad f \in H^2,$$

where P_+ is the orthogonal projection from $L^2(\mathbb{T})$ onto H^2 . The book [7] is the classic reference for $\mathcal{H}(b)$ spaces (see also [4]).

For our result, we focus on the case when b is a non-extreme point of $\mathcal{B}_{H^\infty}$, equivalently, $\log(1 - |b|) \in L^1(\mathbb{T}, m)$ where m is normalized Lebesgue measure on $\mathbb{T}$. Then there exists a unique outer function $a \in \mathcal{B}_{H^\infty}$, called the *Pythagorean mate* for b , such that $a(0) > 0$ and $|a|^2 + |b|^2 = 1$ almost everywhere on $\mathbb{T}$.

When b is non-extreme and a is its Pythagorean mate, an important vector space of functions is

$$\mathcal{M}(a) := T_a H^2 = a H^2.$$

We also equip this space with the range norm which makes T_a an isometry from H^2 onto $\mathcal{M}(a)$. Then $\mathcal{M}(a)$ is a Hilbert space which is included into $\mathcal{H}(b)$. Moreover, the inclusion $\mathcal{M}(a) \subset \mathcal{H}(b)$ is contractive (see [7, p. 24]). Finally, we recall that $\mathcal{H}(b)$ is invariant with respect to the difference quotients.

Theorem 4.7. *Let b be a non-extreme point of $\mathcal{B}_{H^\infty}$ and $(\lambda_n)_{n \geq 1}$ be a sequence of distinct points in $\mathbb{D}$. Then the sequence $(\widehat{k_{\lambda_n}^b})_{n \geq 1}$ cannot be an absolutely representing system for its closed linear span in $\mathcal{H}(b)$.*

Proof. Assume that the sequence $(\widehat{k_{\lambda_n}^b})_{n \geq 1}$ is an absolutely representing system for its closed linear span. Since $\mathcal{H}(b)$ is an Hilbert space, arguing as in the proof of Theorem 3.2, we focus directly on the case when the sequence $(\widehat{k_{\lambda_n}^b})_{n \geq 1}$ is not minimal. Then, according to Lemma 4.1, we have

$$\text{span}(\widehat{k_{\lambda_n}^b} : n \geq 1) = \mathcal{H}(b).$$

It follows from Lemma 2.5 that there is a constant $C > 0$ such that for every $h \in \mathcal{H}(b)$, we can find $n \geq 1$ such that

$$|h(\lambda_n)| \geq C \|k_{\lambda_n}^b\|_b \|h\|_b.$$

As in the proof for the H^2 case, choose N sufficiently large so that $2 \leq (N+1)C^2$. Consider $h(z) = z^N a(z)$. Since $h \in \mathcal{M}(a) \subset \mathcal{H}(b)$ and

$$1 = \|z^N\|_2 = \|h\|_{\mathcal{M}(a)} \leq \|h\|_b,$$

there is an integer n such that

$$C^2(1 - |b(\lambda_n)|^2) \leq |a(\lambda_n)|^2 |\lambda_n|^{2N} (1 - |\lambda_n|^2).$$

Remind now that $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. Hence a standard argument based on Poisson integral gives that $|a(z)|^2 + |b(z)|^2 \leq 1$ for every $z \in \mathbb{D}$, which gives

$$C^2 \leq |\lambda_n|^{2N} (1 - |\lambda_n|^2) \leq \frac{1}{N+1} \leq \frac{C^2}{2}.$$

Thus we get the desired contradiction in that second case. □

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