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COMPLEX SYMMETRIC WEIGHTED COMPOSITION OPERATORS ON $\mathcal{H}_\gamma(\mathbb{D})$

RENON LIM & LE HAI KHOI

ABSTRACT. We study the complex symmetric structure of weighted composition operators of the form $W_{\psi,\varphi}$ on the Hilbert space $\mathcal{H}_\gamma(\mathbb{D})$ of holomorphic functions over the open unit disk $\mathbb{D}$ with reproducing kernels $K_w^{(\gamma)} = (1 - \bar{w}z)^{-\gamma}$, where $\gamma \in \mathbb{N}$. First, we consider conjugations on $\mathcal{H}_\gamma(\mathbb{D})$ of the form $\mathcal{A}_{u,v}f = u \cdot \overline{f \circ v}$ (such conjugations are also known as weighted composition conjugations) and characterize them into two classes, denoted by $\mathcal{C}_1$ and $\mathcal{C}_2$. Then, we obtain explicit conditions for $W_{\psi,\varphi}$ when it is $\mathcal{C}_1$ -symmetric and $\mathcal{C}_2$ -symmetric respectively.

1. INTRODUCTION

The general study of complex symmetry has origins that can be traced back to both operator theory and complex analysis, and was initiated by Garcia and Putinar in [4, 5]. Thereafter, a number of papers were devoted to the topic and contributed significantly to the study of complex symmetry (see [3] and the references therein). In the past decade, complex symmetric operators have become particularly important in both theoretical aspects and applications. Renewed interest in non-Hermitian quantum mechanics and spectral analysis of certain complex symmetric operators has brought forth more research into complex symmetry. Present results show that bounded complex symmetric operators are very diverse, which include the likes of Volterra integration operator, normal operator, compressed Toeplitz operators, etc. Recently, in [11], some interpolation properties of complex symmetric operators on Hilbert spaces, with application to complex symmetric Toeplitz operators, have been studied.

1.1. Basic definitions and notation. Let $\mathcal{H}$ denote a separable complex Hilbert space. A mapping $\mathcal{A}$ acting on $\mathcal{H}$ is said to be *anti-linear* (or *conjugate-linear*) if for all $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{C}$,

$$\mathcal{A}(\alpha x + y) = \bar{\alpha}\mathcal{A}x + \mathcal{A}y.$$

The *boundedness* of an anti-linear operator is defined in the same manner as a linear operator.

For a bounded anti-linear operator $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}$, its *adjoint* $\mathcal{A}^*$ satisfies

$$\langle \mathcal{A}x, y \rangle = \overline{\langle x, \mathcal{A}^*y \rangle} = \langle \mathcal{A}^*y, x \rangle$$

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for all $x, y \in \mathcal{H}$. We say that $\mathcal{A}$ is *unitary* if $\mathcal{A}\mathcal{A}^* = \mathcal{A}^*\mathcal{A} = I$, where $I : \mathcal{H} \rightarrow \mathcal{H}$ is the identity operator. Also, $\mathcal{A}$ is *self-adjoint* if $\mathcal{A} = \mathcal{A}^*$, i.e. for all $x, y \in \mathcal{H}$, $\langle \mathcal{A}x, y \rangle = \langle \mathcal{A}y, x \rangle$.

An anti-linear operator $\mathcal{C} : \mathcal{H} \rightarrow \mathcal{H}$ is a **conjugation** if $\mathcal{C}$ is **(a) involutive**, i.e. $\mathcal{C}^2 = I$ and **(b) isometric**, i.e. $\|\mathcal{C}x\| = \|x\|$ for all $x \in \mathcal{H}$. Note that a conjugation is necessarily bounded as it is isometric. A bounded linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is said to be **complex symmetric** if there exists a conjugation $\mathcal{C} : \mathcal{H} \rightarrow \mathcal{H}$ such that $T = \mathcal{C}T^*\mathcal{C}$ (equivalent to either $T\mathcal{C} = \mathcal{C}T^*$ or $T^* = \mathcal{C}T\mathcal{C}$). We also say that T is a $\mathcal{C}$ -symmetric operator.

A *reproducing kernel Hilbert space* (RKHS) on a set X is a Hilbert space on X for which the linear evaluation functional on X is bounded. By a simple application of the Riesz representation theorem, there is a unique vector in the RKHS, say K_w , such that for all f in the RKHS,

$$f(w) = \langle f, K_w \rangle$$

for all $w \in X$. We call K_w the *reproducing kernel*. It is well known that a reproducing kernel uniquely defines a RKHS. We refer the interested reader to [12] for more details on RKHS.

Let $\mathbb{D}$ be an open unit disk on the complex plane. For all $w, z \in \mathbb{D}$ and $\gamma \in \mathbb{N}$, consider the reproducing kernel

$$(1.1) \quad K_w^{(\gamma)}(z) = (1 - \bar{w}z)^{-\gamma}.$$

We denote by $\mathcal{H}_\gamma(\mathbb{D})$, the Hilbert space of holomorphic functions on $\mathbb{D}$ with reproducing kernel $K_w^{(\gamma)}(z)$. It is worth noting that when $\gamma = 1$, we get the classical Hardy space, while when $\gamma = 2$, we obtain the classical Bergman space. In fact, such a space was studied in [10] for several complex variables and $\gamma > 0$. However, we will restrict our study to just $\gamma \in \mathbb{N}$. This restriction will be justified later on.

For any $w, z \in \mathbb{D}$, the norm of $K_w^{(\gamma)}(z)$ is given by

$$\|K_w^{(\gamma)}\| = (1 - |w|^2)^{-\frac{\gamma}{2}}.$$

Hence we can write the normalized reproducing kernel as

$$(1.2) \quad k_w^{(\gamma)}(z) = \frac{K_w^{(\gamma)}(z)}{\|K_w^{(\gamma)}\|} = \frac{(1 - |w|^2)^{\frac{\gamma}{2}}}{(1 - \bar{w}z)^\gamma}.$$

Now, let $\psi : \mathbb{D} \rightarrow \mathbb{C}$ and $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions. For all $f \in \mathcal{H}_\gamma(\mathbb{D})$ and $z \in \mathbb{D}$, the functions ψ and φ induce a linear *weighted composition operator* $W_{\psi, \varphi}$ on $\mathcal{H}_\gamma(\mathbb{D})$ defined by

$$W_{\psi, \varphi}f(z) = \psi(z) \cdot f(\varphi(z)).$$

It is well-known that the operator $W_{\psi, \varphi}$ is bounded on $\mathcal{H}_\gamma(\mathbb{D})$ if and only if $W_{\psi, \varphi}$ acts from $\mathcal{H}_\gamma(\mathbb{D})$ into itself. This is because boundedness and invariance of a weighted composition operator are equivalent on a RKHS, which is due to a simple application of the Closed Graph Theorem.

1.2. Problems for investigation. Weighted composition operators on various function spaces has been studied extensively during the past few decades (see [1, 13] and references therein). Traditional choice of topics studied with respect to weighted composition operators includes: boundedness, compactness, compact difference, essential norm, closed range, etc. With the recent advances in the study of complex symmetry, it makes sense to ask the following question: are there any complex symmetric weighted composition operators?

Within the last decade, the above question was studied by Jung and colleagues (see [9]) as well as by Garcia and Hammond (see [2]). Independently, they obtained a criterion for the complex symmetric structure of $W_{\psi,\varphi}$ on Hardy spaces with respect to the what we now know as the standard conjugation, i.e. those of the form

$$\mathcal{J}f(z) = \overline{f(\bar{z})},$$

where f belongs to the Hardy space. Also, a criterion for the complex symmetric structure of $W_{\psi,\varphi}$ was discovered on Fock space $\mathcal{F}^2(\mathbb{C})$ (see [6]) and $\mathcal{F}^2(\mathbb{C}^n)$ (see [7]) with respect to a more general conjugation of the form

$$\mathcal{A}_{u,v}f(z) = u(z) \cdot \overline{f(\overline{v(z)})},$$

where u and v are entire functions and f belongs to the Fock space. Inspired by these recent results, we attempt to study the complex symmetric structure of $W_{\psi,\varphi}$ on $\mathcal{H}_\gamma(\mathbb{D})$ with respect to conjugations of the form

$$\mathcal{A}_{u,v}f(z) = u(z) \cdot \overline{f(\overline{v(z)})},$$

where $u : \mathbb{D} \rightarrow \mathbb{C}$ and $v : \mathbb{D} \rightarrow \mathbb{D}$ are holomorphic functions and $f \in \mathcal{H}_\gamma(\mathbb{D})$. Note that both the operators $W_{\psi,\varphi}$ and $\mathcal{A}_{u,v}$ are not necessarily invariant on $\mathcal{H}_\gamma(\mathbb{D})$.

Our solution to this problem can thought of as a two-step approach. First, we must find all such $\mathcal{A}_{u,v}$ that are conjugations, i.e. find holomorphic functions $u : \mathbb{D} \rightarrow \mathbb{C}$ and $v : \mathbb{D} \rightarrow \mathbb{D}$ such that $\mathcal{A}_{u,v}$ is **(a)** invariant on $\mathcal{H}_\gamma(\mathbb{D})$, **(b)** involutive and **(c)** isometric. Second, we find all such $W_{\psi,\varphi}$ that are complex symmetric with respect to $\mathcal{A}_{u,v}$, i.e. find holomorphic functions $\psi : \mathbb{D} \rightarrow \mathbb{C}$ and $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ such that **(a)** $W_{\psi,\varphi}$ is invariant on $\mathcal{H}_\gamma(\mathbb{D})$ and **(b)** $W_{\psi,\varphi} = \mathcal{A}_{u,v}W_{\psi,\varphi}^*\mathcal{A}_{u,v}$.

The following result, mentioned in the context of the Fock space in [7], holds for any separable complex Hilbert space $\mathcal{H}$.

Proposition 1.1. *$\mathcal{A}$ is a conjugation on $\mathcal{H}$ if and only if $\mathcal{A}$ is self-adjoint and unitary on $\mathcal{H}$.*

The proof is a straightforward checking of definitions. Thus it is clear that the standard conjugation $\mathcal{J}$ is self-adjoint and unitary.

The structure of the paper is as follows. In Section 2, we find explicit conditions for $\mathcal{A}_{u,v}$ to be self-adjoint and unitary respectively. We then combine these two sets of expression to obtain a criterion for $\mathcal{A}_{u,v}$ to be a conjugation. We will see that $\mathcal{A}_{u,v}$ can be either of the forms (2.11) or (2.12). We denote the conjugations by $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. In Section 3, we find explicit conditions for $W_{\psi,\varphi}$ that is complex symmetric with respect

to $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. In Section 4, we give some concluding remarks and raise some open questions pertaining to our research.

2. WEIGHTED COMPOSITION CONJUGATIONS

In this section, we characterize anti-linear weighted composition operators on $\mathcal{H}_\gamma(\mathbb{D})$ that are conjugations, which we name as *weighted composition conjugations*.

2.1. Preliminary results for the $\mathcal{J}$ -symmetry of $W_{\psi,\varphi}$ on $\mathcal{H}_\gamma(\mathbb{D})$. Let $\gamma \in \mathbb{N}$. If $W_{\psi,\varphi}$ is bounded on $\mathcal{H}_\gamma(\mathbb{D})$, then it is clear for all $w \in \mathbb{D}$, we have

$$(2.1) \quad W_{\psi,\varphi}^* K_w^{(\gamma)} = \overline{\psi(w)} K_{\varphi(w)}^{(\gamma)}.$$

This formula (2.1) will be useful in many of the proofs provided in this paper.

The following proposition is similar to those in [2, 9] for the (weighted) Hardy spaces. It can be proved in the same manner, using in particular formula (2.1).

Proposition 2.1. *Let $\gamma \in \mathbb{N}$. Let $\psi : \mathbb{D} \rightarrow \mathbb{C}$ and $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions such that $W_{\psi,\varphi} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$. If $W_{\psi,\varphi}$ is complex symmetric, then $\psi(z)$ is either identically zero or non-zero for all $z \in \mathbb{D}$.*

Obviously, if ψ is identically zero, $W_{\psi,\varphi}$ must also be identically zero. Such a case is trivial and we shall assume from now on, ψ is not identically zero. Thus Proposition 2.1 tells us $\psi(z)$ is never zero at any $z \in \mathbb{D}$.

The next result is an immediate corollary of the fact that the linear span of reproducing kernels is dense in any RKHS.

Proposition 2.2. *Let $\gamma \in \mathbb{N}$, $W_{\psi,\varphi} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$, and $\mathcal{C}$ a conjugation on $\mathcal{H}_\gamma(\mathbb{D})$. Then $W_{\psi,\varphi}$ is $\mathcal{C}$ -symmetric if and only if for all $w \in \mathbb{D}$,*

$$W_{\psi,\varphi} \mathcal{C} K_w^{(\gamma)} = \mathcal{C} W_{\psi,\varphi}^* K_w^{(\gamma)}.$$

Now we find a criterion for the $\mathcal{J}$ -symmetry of $W_{\psi,\varphi}$ on $\mathcal{H}_\gamma(\mathbb{D})$, which allows us to study properties of weighted composition conjugations.

The following proposition is identical to Proposition 2.9 of [2] for the weighted Hardy spaces. We note that the necessity part can be proved in the same manner as in [2]. Nevertheless, we give its proof by another, “differentiation” method, which gives us the way to obtain an explicit necessary condition for weighted composition operators to be complex symmetric in Section 3. We also would like to emphasize the need for $\gamma \in \mathbb{N}$ in the proof of the sufficiency part. It should be noted that the requirement $\gamma \in \mathbb{N}$ is essential, as in general, for any $a, b \in \mathbb{C}$ and $\gamma > 0$, $(ab)^\gamma \neq a^\gamma b^\gamma$, while the equality holds when $\gamma \in \mathbb{N}$.

Proposition 2.3. *Let $\gamma \in \mathbb{N}$. Let $\psi : \mathbb{D} \rightarrow \mathbb{C}$ and $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions such that $W_{\psi,\varphi} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$. Then $W_{\psi,\varphi}$ is $\mathcal{J}$ -symmetric if and only if for all $z \in \mathbb{D}$,*

$$\psi(z) = \frac{b}{(1 - a_0 z)^\gamma} \quad \text{and} \quad \varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z},$$

where $a_0, a_1 \in \mathbb{D}$ and $b \in \mathbb{C}$.

Proof. Necessity. Suppose that $W_{\psi,\varphi}$ is $\mathcal{J}$ -symmetric. By Proposition 2.2, we can write

$$(2.2) \quad W_{\psi,\varphi} \mathcal{J} K_w^{(\gamma)}(z) = \mathcal{J} W_{\psi,\varphi}^* K_w^{(\gamma)}(z)$$

for all $w, z \in \mathbb{D}$.

We simplify the left hand side of (2.2) to get

$$\begin{aligned} W_{\psi,\varphi} \mathcal{J} K_w^{(\gamma)}(z) &= \psi(z) \overline{K_w^{(\gamma)}(\overline{\varphi(z)})} \\ &= \frac{\psi(z)}{(1 - w\varphi(z))^\gamma} \end{aligned} \quad \text{by (1.1)}$$

and simplify the right hand side of (2.2) to get

$$\begin{aligned} \mathcal{J} W_{\psi,\varphi}^* K_w^{(\gamma)}(z) &= \mathcal{J} \left(\overline{\psi(w)} K_{\varphi(w)}(z) \right) \quad \text{by (2.1)} \\ &= \psi(w) \overline{K_{\varphi(w)}(\overline{z})} \\ &= \frac{\psi(w)}{(1 - z\varphi(w))^\gamma} \quad \text{by (1.1)} \end{aligned}$$

Hence (2.2) reduces to

$$(2.3) \quad \frac{\psi(w)}{(1 - z\varphi(w))^\gamma} = \frac{\psi(z)}{(1 - w\varphi(z))^\gamma}$$

for all $w, z \in \mathbb{D}$.

Since (2.3) holds for all $w \in \mathbb{D}$, we set $w = 0$ to get

$$(2.4) \quad \psi(z) = \frac{b}{(1 - a_0 z)^\gamma}$$

for all $z \in \mathbb{D}$, where $a_0 = \varphi(0)$ and $b = \psi(0)$.

Now we substitute (2.4) back into (2.3) to get

$$(1 - a_0 w)^\gamma (1 - z\varphi(w))^\gamma = (1 - a_0 z)^\gamma (1 - w\varphi(z))^\gamma$$

for all $w, z \in \mathbb{D}$. Since φ is holomorphic, we differentiate the above equation with respect to w to get

$$\begin{aligned} (1 - a_0 w)^\gamma \gamma (1 - z\varphi(w))^{\gamma-1} (-z\varphi'(w)) + (1 - z\varphi(w))^\gamma \gamma (1 - a_0 w)^{\gamma-1} (-a_0) \\ = (1 - a_0 z)^\gamma \gamma (1 - w\varphi(z))^{\gamma-1} (-\varphi(z)). \end{aligned}$$

Finally, we set $w = 0$ again to get

$$\begin{aligned} \gamma(1 - a_0 z)^{\gamma-1} (-a_1 z) + (1 - a_0 z)^\gamma (-\gamma a_0) &= (1 - a_0 z)^\gamma (-\gamma \varphi(z)) \\ \iff \varphi(z) &= a_0 + \frac{a_1 z}{1 - a_0 z} \end{aligned}$$

for all $z \in \mathbb{D}$, where $a_1 = \varphi'(0)$.

Sufficiency. Suppose that for all $z \in \mathbb{D}$

$$\psi(z) = \frac{b}{(1 - a_0 z)^\gamma} \quad \text{and} \quad \varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z},$$

where $a_0, a_1 \in \mathbb{D}$ and $b \in \mathbb{C}$.

Since $W_{\psi,\varphi}$ is bounded on $\mathcal{H}_\gamma(\mathbb{D})$, it suffices to prove that (2.2) holds for all $w, z \in \mathbb{D}$.

On one hand, we have

$$\begin{aligned}
W_{\psi,\varphi} \mathcal{J} K_w^{(\gamma)}(z) &= \frac{\psi(z)}{(1 - w\varphi(z))^\gamma} \\
&= \frac{b}{(1 - a_0 z)^\gamma} \cdot \left(1 - w \left(a_0 + \frac{a_1 z}{1 - a_0 z} \right) \right)^{-\gamma} \\
&= b((1 - a_0 z) - a_0 w(1 - a_0 z) - a_1 w z)^{-\gamma} \\
&= b(1 - a_0(w + z) + wz(a_0^2 - a_1))^{-\gamma}.
\end{aligned}$$

Here it is essential that the third equality holds, due to $\gamma \in \mathbb{N}$.

On the other hand, we have

$$\begin{aligned}
\mathcal{J} W_{\psi,\varphi}^* K_w^{(\gamma)}(z) &= \frac{\psi(w)}{(1 - z\varphi(w))^\gamma} \\
&= b(1 - a_0(w + z) + wz(a_0^2 - a_1))^{-\gamma}.
\end{aligned}$$

Thus (2.2) holds for all $w, z \in \mathbb{D}$. $\square$

In the rest of the paper, we shall no longer mention the fact $\gamma \in \mathbb{N}$ in other proofs to keep the exposition short.

We remark that Proposition 2.3 contains the results of [9] as a particular case. Simply set $\gamma = 1$.

2.2. Weighted composition conjugations. Let $\gamma \in \mathbb{N}$. Let $u : \mathbb{D} \rightarrow \mathbb{C}$ and $v : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions. For all $f \in \mathcal{H}_\gamma(\mathbb{D})$ and $z \in \mathbb{D}$, the functions u and v induce an anti-linear operator on $\mathcal{H}_\gamma(\mathbb{D})$ defined by

$$\mathcal{A}_{u,v} f(z) = u(z) \cdot \overline{f(v(z))}.$$

$\mathcal{A}_{u,v}$ is known as an anti-linear weighted composition operator on $\mathcal{H}_\gamma(\mathbb{D})$. The standard conjugation $\mathcal{J} f(z) = \overline{f(\bar{z})}$ is a specific case of $\mathcal{A}_{u,v}$. Simply set $u(z) = 1$ and $v(z) = z$.

From this point on, we always assume that $\mathcal{A}_{u,v} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$. It is easy to see that $W_{u,v} = \mathcal{A}_{u,v} \mathcal{J}$, thus $W_{u,v}$ is invariant (hence bounded) on $\mathcal{H}_\gamma(\mathbb{D})$. Equivalently, we have

$$(2.5) \quad \mathcal{A}_{u,v} = W_{u,v} \mathcal{J},$$

which is a composition of two bounded operators, thus $\mathcal{A}_{u,v}$ is bounded too.

We first note the following property which relates the adjoint of composite operators to the composition of adjoints.

Proposition 2.4. *Let $S, T : \mathcal{H} \rightarrow \mathcal{H}$ be bounded operators that are either linear or anti-linear. Then $(ST)^* = T^* S^*$.*

The proof of this proposition is straightforward, as it suffices to consider three cases: both S and T are linear, both S and T are anti-linear, and without loss of generality, S is linear whilst T is anti-linear.

We now establish the relationship between $\mathcal{A}_{u,v}$ and $W_{u,v}$ when $\mathcal{A}_{u,v}$ is either self-adjoint or unitary.

Proposition 2.5. *Let $u : \mathbb{D} \rightarrow \mathbb{C}$ and $v : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions such that $W_{u,v} : \mathcal{H} \rightarrow \mathcal{H}$. Then $\mathcal{A}_{u,v}$ is self-adjoint if and only if $W_{u,v}$ is $\mathcal{J}$ -symmetric.*

Proof. By (2.5), we have

$$\mathcal{A}_{u,v} = \mathcal{A}_{u,v}^* \iff W_{u,v}\mathcal{J} = (W_{u,v}\mathcal{J})^*.$$

By Proposition 2.4 and self-adjointness of $\mathcal{J}$ respectively, we get

$$(W_{u,v}\mathcal{J})^* = \mathcal{J}^*W_{u,v}^* = \mathcal{J}W_{u,v}^*.$$

Now, the involutivity of $\mathcal{J}$ gives us

$$W_{u,v}\mathcal{J} = \mathcal{J}W_{u,v}^* \iff W_{u,v} = \mathcal{J}W_{u,v}^*\mathcal{J}.$$

□

Proposition 2.6. *Let $u : \mathbb{D} \rightarrow \mathbb{C}$ and $v : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions such that $W_{u,v} : \mathcal{H} \rightarrow \mathcal{H}$. Then $\mathcal{A}_{u,v}$ is unitary if and only if $W_{u,v}$ is unitary.*

Proof. By (2.5), we have

$$\begin{aligned} \mathcal{A}_{u,v}\mathcal{A}_{u,v}^* &= \mathcal{A}_{u,v}^*\mathcal{A}_{u,v} = I \\ \iff (W_{u,v}\mathcal{J})(W_{u,v}\mathcal{J})^* &= (W_{u,v}\mathcal{J})^*(W_{u,v}\mathcal{J}) = I. \end{aligned}$$

By Proposition 2.4, we get

$$W_{u,v}\mathcal{J}\mathcal{J}^*W_{u,v}^* = I \text{ and } \mathcal{J}^*W_{u,v}^*W_{u,v}\mathcal{J} = I$$

respectively. Since $\mathcal{J}$ is self-adjoint and $\mathcal{J}^2 = I$, we have

$$W_{u,v}W_{u,v}^* = I \text{ and } W_{u,v}^*W_{u,v} = I.$$

□

As a result of Propositions 2.5 and 2.6, we are now able to study self-adjoint $\mathcal{A}_{u,v}$ and unitary $\mathcal{A}_{u,v}$ in terms of $W_{u,v}$, which was already investigated in [10] and Proposition 2.3. An automorphism on $\mathbb{D}$ is a holomorphic function $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ that is also bijective. The set of all automorphisms of $\mathbb{D}$ is

$$\text{Aut}(\mathbb{D}) = \left\{ \beta \cdot \frac{\alpha - z}{1 - \bar{\alpha}z} : \alpha \in \mathbb{D}, \beta \in \partial\mathbb{D} \right\}.$$

Corollary 3.6 of [10] gives equivalent assertions for bounded (or equivalently, invariant) $W_{\psi,\varphi}$. We restate the corollary in the context of our work.

Proposition 2.7 ([10]). *Let $\gamma \in \mathbb{N}$. Let $\psi : \mathbb{D} \rightarrow \mathbb{C}$ and $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions such that $W_{\psi,\varphi} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$. Then $W_{\psi,\varphi}$ is unitary if and only if for all $z \in \mathbb{D}$,*

$$\varphi \in \text{Aut}(\mathbb{D}) \quad \text{and} \quad \psi = \lambda k_{\varphi^{-1}(0)}^{(\gamma)},$$

where $\lambda \in \partial\mathbb{D}$.

The next result gives an explicit condition for φ when it is both of the form in Proposition 2.3 and $\varphi \in \text{Aut}(\mathbb{D})$.

Proposition 2.8. *Let $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic for all $z \in \mathbb{D}$ and*

$$\varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z},$$

where $a_0, a_1 \in \mathbb{D}$. If $\varphi \in \text{Aut}(\mathbb{D})$, then we have either of the following:

(1) *there exists $\alpha \in \mathbb{D} \setminus \{0\}$ and $\beta \in \partial\mathbb{D}$ such that*

$$(2.6) \quad \varphi(z) = \frac{\bar{\alpha}}{\alpha} \cdot \frac{\alpha - z}{1 - \bar{\alpha}z},$$

$$\text{where } a_0 = \beta\alpha \text{ and } a_1 = \frac{\beta^2\alpha(|\alpha|^2 - 1)}{\bar{\alpha}}.$$

(2) *there exist $\beta \in \partial\mathbb{D}$ such that*

$$(2.7) \quad \varphi(z) = \beta z,$$

$$\text{where } a_0 = 0 \text{ and } a_1 = \beta.$$

Proof. By our initial assumption,

$$\varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z} = \frac{a_0 + (a_1 - a_0^2)z}{1 - a_0 z}$$

for all $z \in \mathbb{D}$. Now suppose that $\varphi \in \text{Aut}(\mathbb{D})$. So there exist $\alpha \in \mathbb{D}$ and $\beta \in \partial\mathbb{D}$ such that for all $z \in \mathbb{D}$,

$$\varphi(z) = \frac{\beta\alpha - \beta z}{1 - \bar{\alpha}z}.$$

Equating both expressions for φ give us

$$\begin{aligned} \frac{a_0 + (a_1 - a_0^2)z}{1 - a_0 z} &= \frac{\beta\alpha - \beta z}{1 - \bar{\alpha}z} \\ \iff (1 - \bar{\alpha}z)(a_0 + (a_1 - a_0^2)z) &= (1 - a_0 z)(\beta\alpha - \beta z) \\ \iff a_0 + (a_1 - a_0^2 - \bar{\alpha}a_0)z + \bar{\alpha}(a_0^2 - a_1)z^2 &= \beta\alpha - \beta(a_0\alpha + 1)z + a_0\beta z^2. \end{aligned}$$

Comparing the constants and the coefficients of z and z^2 respectively,

$$(2.8) \quad a_0 = \beta\alpha,$$

$$(2.9) \quad a_1 - a_0^2 - \bar{\alpha}a_0 = -\beta(a_0\alpha + 1),$$

$$(2.10) \quad \bar{\alpha}(a_0^2 - a_1) = a_0\beta.$$

Case 1. If $\alpha = 0$, then $a_0 = 0$ by (2.8) and $a_1 = -\beta$ by (2.9). Hence $\varphi(z) = -\beta z$. This proves the case of (2.7) since $-\beta \in \partial\mathbb{D}$ too.

Case 2. Now suppose that $\alpha \in \mathbb{D} \setminus \{0\}$. We substitute $a_0 = \beta\alpha$ into (2.10) to get

$$a_1 = \frac{\beta^2\alpha(|\alpha|^2 - 1)}{\bar{\alpha}}$$

and further substitute a_0, a_1 into (2.9) to get

$$\begin{aligned} \frac{\beta^2\alpha(|\alpha|^2 - 1)}{\bar{\alpha}} - (\beta\alpha)^2 - \bar{\alpha}\beta\alpha &= -\beta(\beta\alpha^2 + 1) \\ \iff \beta^2\alpha(|\alpha|^2 - 1) - \beta^2\alpha|\alpha|^2 - \beta\bar{\alpha}|\alpha|^2 &= -\alpha\beta^2|\alpha|^2 - \beta\bar{\alpha} \\ \iff \beta^2\alpha(|\alpha|^2 - 1) &= \beta\bar{\alpha}(|\alpha|^2 - 1). \end{aligned}$$

Since $\beta \neq 0$ and $|\alpha|^2 - 1 \neq 0$, we divide both sides by $\beta(|\alpha|^2 - 1)$ to get $\beta = \frac{\bar{\alpha}}{\alpha}$. Hence we have

$$\begin{aligned}\varphi(z) &= \frac{a_0 + (a_1 - a_0^2)z}{1 - a_0z} = \beta \cdot \frac{\alpha + \left(\frac{\beta\alpha(|\alpha|^2 - 1)}{\bar{\alpha}} - \beta\alpha^2\right)z}{1 - \beta\alpha z} \\ &= \frac{\bar{\alpha}}{\alpha} \cdot \frac{\alpha + (|\alpha|^2 - 1)z - \alpha\bar{\alpha}z}{1 - \bar{\alpha}z} \\ &= \frac{\bar{\alpha}}{\alpha} \cdot \frac{\alpha - z}{1 - \bar{\alpha}z}.\end{aligned}$$

This proves the case of (2.6). $\square$

2.3. Criterion for $\mathcal{A}_{u,v}$ to be a conjugation. We first find an explicit condition for $\mathcal{A}_{u,v}$ to be self-adjoint.

Theorem 2.9. *Let $\gamma \in \mathbb{N}$. Let $u : \mathbb{D} \rightarrow \mathbb{C}$ and $v : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions. Then $\mathcal{A}_{u,v} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$ is self-adjoint if and only if for all $z \in \mathbb{D}$,*

$$u(z) = \frac{b}{(1 - a_0z)^\gamma} \quad \text{and} \quad v(z) = a_0 + \frac{a_1z}{1 - a_0z},$$

where $a_0, a_1 \in \mathbb{D}$ and $b \in \mathbb{C}$.

Proof. We simply combine Propositions 2.3 and 2.5. $\square$

Similarly, we find an explicit condition for $\mathcal{A}_{u,v}$ to be unitary.

Theorem 2.10. *Let $\gamma \in \mathbb{N}$. Let $u : \mathbb{D} \rightarrow \mathbb{C}$ and $v : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions. Then $\mathcal{A}_{u,v} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$ is unitary if and only if for all $z \in \mathbb{D}$,*

$$u = \lambda k_{v^{-1}(0)}^{(\gamma)} \quad \text{and} \quad v \in \text{Aut}(\mathbb{D}),$$

where $\lambda \in \partial\mathbb{D}$.

Proof. We simply combine Propositions 2.6 and 2.7. $\square$

We shall now use the characterization of a conjugation, Theorems 2.9 and 2.10 to characterize all weighted composition conjugations.

Theorem 2.11 (Criterion for conjugation). *Let $\gamma \in \mathbb{N}$. Let $u : \mathbb{D} \rightarrow \mathbb{C}$ and $v : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions. Then $\mathcal{A}_{u,v} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$ is a conjugation if and only if either of the following holds:*

(1) *there exist $\alpha, \lambda \in \partial\mathbb{D}$ such that for all $z \in \mathbb{D}$,*

$$(2.11) \quad u(z) = \lambda \quad \text{and} \quad v(z) = \alpha z.$$

(2) *there exist $\alpha \in \mathbb{D} \setminus \{0\}$ and $\lambda \in \partial\mathbb{D}$ such that for all $z \in \mathbb{D}$,*

$$(2.12) \quad u(z) = \lambda k_\alpha^{(\gamma)}(z) \quad \text{and} \quad v(z) = \frac{\bar{\alpha}}{\alpha} \cdot \frac{\alpha - z}{1 - \bar{\alpha}z}.$$

Proof. Necessity. Suppose that $\mathcal{A}_{u,v}$ is a conjugation. Then $\mathcal{A}_{u,v}$ is both self-adjoint and unitary. By Theorems 2.9 and 2.10 respectively, we have

$$v(z) = a_0 + \frac{a_1z}{1 - a_0z} \quad \text{and} \quad v \in \text{Aut}(\mathbb{D}).$$

By Proposition 2.7, v is either of the form (2.6) or (2.7).

Case 1. There exist $\beta \in \partial\mathbb{D}$ such that for all $z \in \mathbb{D}$, v is of the form (2.6). Then

$$v(0) = 0 \iff 0 = v^{-1}(0)$$

and hence Theorem 2.10 tells us $u(z) = \lambda k_0^{(\gamma)}(z) = \lambda$, which is also consistent with Theorem 2.9. This proves (2.11).

Case 2. There exist $\alpha \in \mathbb{D} \setminus \{0\}$ such that for all $z \in \mathbb{D}$, v is of the form (2.7). Then

$$v(\alpha) = 0 \iff \alpha = v^{-1}(0)$$

and hence Proposition 2.3 tells us $u(z) = \lambda k_\alpha^{(\gamma)}(z)$, which is also consistent with Theorem 2.9. This proves (2.12).

Sufficiency. Now suppose that we have either (2.11) or (2.12).

Case 1. There exist $\alpha, \lambda \in \partial\mathbb{D}$ such that for all $z \in \mathbb{D}$, we have (2.11). Clearly, $v \in \text{Aut}(\mathbb{D})$ and $v^{-1}(0) = 0$. We can write

$$u(z) = \lambda k_0^{(\gamma)}(z) = \lambda k_{v^{-1}(0)}^{(\gamma)}(z)$$

and according to Theorem 2.10, $\mathcal{A}_{u,v}$ is unitary. Moreover, we can write u and v into the forms given in Theorem 2.9 by setting $b = \lambda, a_0 = 0$ and $a_1 = \alpha$. Hence $\mathcal{A}_{u,v}$ is self-adjoint.

Case 2. There exist $\alpha \in \mathbb{D} \setminus \{0\}$ and $\lambda \in \partial\mathbb{D}$ such that for all $z \in \mathbb{D}$, we have (2.12). Clearly, $v \in \text{Aut}(\mathbb{D})$ and $v^{-1}(0) = \alpha$. We can write

$$u(z) = \lambda k_\alpha^{(\gamma)}(z) = \lambda k_{v^{-1}(0)}^{(\gamma)}(z)$$

and according to Theorem 2.10, $\mathcal{A}_{u,v}$ is unitary. Moreover, we can write u and v into the forms given in Theorem 2.9 by setting $b = \lambda, a_0 = \bar{\alpha}$ and $a_1 = \frac{\bar{\alpha}}{\alpha}(|\alpha|^2 - 1)$. Hence $\mathcal{A}_{u,v}$ is self-adjoint.

Therefore, $\mathcal{A}_{u,v}$ is a conjugation in both cases. $\square$

We denote $\mathcal{A}_{u,v}$ with u and v of the form (2.11) by $\mathcal{C}_1$ and $\mathcal{A}_{u,v}$ with u and v of the form (2.12) by $\mathcal{C}_2$.

3. COMPLEX SYMMETRIC WEIGHTED COMPOSITION OPERATORS

As a direct consequence of Theorem 2.11, there are only two classes of conjugations $\mathcal{A}_{u,v}$ on $\mathcal{H}_\gamma(\mathbb{D})$, either $\mathcal{C}_1$ or $\mathcal{C}_2$. To obtain explicit conditions for $W_{\psi,\varphi}$ to be complex symmetric, it suffices to find holomorphic functions $\psi : \mathbb{D} \rightarrow \mathbb{C}$ and $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ such that $W_{\psi,\varphi}$ is either **(a)** $\mathcal{C}_1$ -symmetric or **(b)** $\mathcal{C}_2$ -symmetric. We also note that the main results that we are about to present are a generalization of the main results of [8].

Theorem 3.1 ($\mathcal{C}_1$ -symmetry of $W_{\psi,\varphi}$). *Let $\gamma \in \mathbb{N}$. Let $\psi : \mathbb{D} \rightarrow \mathbb{C}$ and $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions such that $W_{\psi,\varphi} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$. Suppose that $\mathcal{C}_1 := \mathcal{A}_{u,v}$ is a conjugation with u and v of the form (2.11), i.e. there exist $\alpha, \lambda \in \partial\mathbb{D}$ such that*

$$u(z) = \lambda \quad \text{and} \quad v(z) = \alpha z.$$

Then $W_{\psi,\varphi}$ is $\mathcal{C}_1$ -symmetric if and only if for all $z \in \mathbb{D}$

$$\psi(z) = \frac{d}{(1 - \alpha c_0 z)^\gamma} \quad \text{and} \quad \varphi(z) = c_0 + \frac{c_1 z}{1 - \alpha c_0 z},$$

where $c_0, c_1 \in \mathbb{D}$ and $d \in \mathbb{C}$.

Proof. Necessity. Suppose that $W_{\psi, \varphi}$ is $\mathcal{C}_1$ -symmetric. By Proposition 2.2, we can write

$$(3.1) \quad W_{\psi, \varphi} \mathcal{C}_1 K_w^{(\gamma)}(z) = \mathcal{C}_1 W_{\psi, \varphi}^* K_w^{(\gamma)}(z)$$

for all $w, z \in \mathbb{D}$.

We simplify the left hand side of (3.1) to get

$$\begin{aligned} W_{\psi, \varphi} \mathcal{C}_1 K_w^{(\gamma)}(z) &= \lambda \psi(z) \overline{K_w^{(\gamma)}(\overline{\alpha \varphi(z)})} \\ &= \frac{\lambda \psi(z)}{(1 - \alpha w \varphi(z))^\gamma} \end{aligned} \quad \text{by (1.1)}$$

and simplify the right hand side of (3.1) to get

$$\begin{aligned} \mathcal{C}_1 W_{\psi, \varphi}^* K_w^{(\gamma)}(z) &= \mathcal{C}_1 \left(\overline{\psi(w)} K_{\varphi(w)}(z) \right) \quad \text{by (2.1)} \\ &= \lambda \psi(w) \overline{K_{\varphi(w)}(\overline{\alpha z})} \\ &= \frac{\lambda \psi(w)}{(1 - \alpha z \varphi(w))^\gamma} \quad \text{by (1.1)} \end{aligned}$$

Hence (3.1) becomes

$$\frac{\lambda \psi(z)}{(1 - \alpha w \varphi(z))^\gamma} = \frac{\lambda \psi(w)}{(1 - \alpha z \varphi(w))^\gamma}$$

and thus

$$(3.2) \quad \psi(z) = \psi(w) \left(\frac{1 - \alpha w \varphi(z)}{1 - \alpha z \varphi(w)} \right)^\gamma$$

for all $w, z \in \mathbb{D}$. Since (3.2) holds for all $w \in \mathbb{D}$, we set $w = 0$ to get

$$(3.3) \quad \psi(z) = \frac{d}{(1 - \alpha c_0 z)^\gamma}$$

for all $z \in \mathbb{D}$, where $c_0 = \varphi(0)$ and $d = \psi(0)$.

Now we substitute (3.3) back into (3.2) to get

$$\begin{aligned} \frac{d}{(1 - \alpha c_0 z)^\gamma} &= \frac{d}{(1 - \alpha c_0 w)^\gamma} \left(\frac{1 - \alpha w \varphi(z)}{1 - \alpha z \varphi(w)} \right)^\gamma \\ \iff (1 - \alpha c_0 w)^\gamma (1 - \alpha z \varphi(w))^\gamma &= (1 - \alpha c_0 z)^\gamma (1 - \alpha w \varphi(z))^\gamma \end{aligned}$$

for all $w, z \in \mathbb{D}$. Since φ is holomorphic, we differentiate the above equation with respect to w to get

$$\begin{aligned} &(1 - \alpha c_0 w)^\gamma \gamma (1 - \alpha z \varphi(w))^{\gamma-1} (-\alpha z \varphi'(w)) \\ &+ (1 - \alpha z \varphi(w))^\gamma \gamma (1 - \alpha c_0 w)^{\gamma-1} (-\alpha c_0) \\ &= (1 - \alpha c_0 z)^\gamma \gamma (1 - \alpha w \varphi(z))^{\gamma-1} (-\alpha \varphi(z)). \end{aligned}$$

Now we set $w = 0$ again to get

$$\begin{aligned} z \varphi'(0) (1 - \alpha c_0 z)^{\gamma-1} + c_0 (1 - \alpha c_0 z)^\gamma &= \varphi(z) (1 - \alpha c_0 z)^\gamma \\ \iff \varphi(z) &= c_0 + \frac{c_1 z}{1 - \alpha c_0 z} \end{aligned}$$

for all $z \in \mathbb{D}$, where $c_1 = \varphi'(0)$.

Sufficiency. Suppose that for all $z \in \mathbb{D}$,

$$\psi(z) = \frac{d}{(1 - \alpha c_0 z)^\gamma} \quad \text{and} \quad \varphi(z) = c_0 + \frac{c_1 z}{1 - \alpha c_0 z},$$

where $c_0, c_1 \in \mathbb{D}$ and $d \in \mathbb{C}$.

By our assumption, $W_{\psi, \varphi}$ is bounded on $\mathcal{H}_\gamma(\mathbb{D})$. Hence it suffices to prove that (3.1) holds for all $w, z \in \mathbb{D}$. Indeed we have

$$\begin{aligned} W_{\psi, \varphi} \mathcal{C}_1 K_w^{(\gamma)}(z) &= \psi(z) \cdot \frac{\lambda}{(1 - w \alpha \varphi(z))^\gamma} \\ &= \frac{d\lambda}{(1 - \alpha c_0 z)^\gamma} \left(1 - w \alpha \left(c_0 + \frac{c_1 z}{1 - \alpha c_0 z} \right) \right)^{-\gamma} \\ &= \frac{d\lambda}{(1 - \alpha c_0 z - w \alpha (c_0(1 - \alpha c_0 z) + c_1 z))^\gamma} \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_1 W_{\psi, \varphi}^* K_w^{(\gamma)}(z) &= \psi(w) \cdot \frac{\lambda}{(1 - \varphi(w) \alpha z)^\gamma} \\ &= \frac{d\lambda}{(1 - \alpha c_0 w)^\gamma} \left(1 - \left(c_0 + \frac{c_1 w}{1 - \alpha c_0 w} \right) \alpha z \right)^{-\gamma} \\ &= \frac{d\lambda}{(1 - \alpha c_0 w - z \alpha (c_0(1 - \alpha c_0 w) + c_1 w))^\gamma} \\ &= \frac{d\lambda}{(1 - \alpha c_0 z - w \alpha (c_0(1 - \alpha c_0 z) + c_1 z))^\gamma}. \end{aligned}$$

Thus (3.1) holds for all $w, z \in \mathbb{D}$. $\square$

Note that Theorem 3.1 contains Proposition 2.3 as a particular case, which in turn contains the results of [2, 9]. Simply set $\alpha = \lambda = 1$ and we have $A_{u, v} = \mathcal{J}$. The proof of the next theorem is similar to that of Theorem 3.1, but contains lengthier algebraic manipulations.

Theorem 3.2 ($\mathcal{C}_2$ -symmetry of $W_{\psi, \varphi}$). *Let $\gamma \in \mathbb{N}$. Let $\psi : \mathbb{D} \rightarrow \mathbb{C}$ and $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic functions such that $W_{\psi, \varphi} : \mathcal{H}_\gamma(\mathbb{D}) \rightarrow \mathcal{H}_\gamma(\mathbb{D})$. Suppose that $\mathcal{C}_2 := \mathcal{A}_{u, v}$ is a conjugation with u and v of the form (2.12), i.e. there exist $\alpha \in \mathbb{D} \setminus \{0\}$ and $\lambda \in \mathbb{D}$ such that*

$$u(z) = \lambda k_\alpha^{(\gamma)}(z) \quad \text{and} \quad v(z) = \frac{\bar{\alpha}}{\alpha} \cdot \frac{\alpha - z}{1 - \bar{\alpha} z}.$$

Then $W_{\psi, \varphi}$ is $\mathcal{C}_2$ -symmetric if and only if for all $z \in \mathbb{D}$

$$\begin{aligned} \psi(z) &= \frac{d(c_0^2 - \alpha c_1)^\gamma}{(c_0^2 - \alpha c_1 - (c_1 - c_2)z)^\gamma} \quad \text{and} \\ \varphi(z) &= \frac{\alpha(\bar{\alpha}c_0^2 - c_1) - (|\alpha|^2 c_1 - c_2)z}{\bar{\alpha}(c_0^2 - \alpha c_1) - \bar{\alpha}(c_1 - c_2)z}, \end{aligned}$$

where $c_0, c_1, c_2, d \in \mathbb{C}$.

Proof. Necessity. Suppose that $W_{\psi, \varphi}$ is $\mathcal{C}_2$ -symmetric. By Proposition 2.2, we can write

$$(3.4) \quad W_{\psi, \varphi} \mathcal{C}_2 K_w^{(\gamma)}(z) = \mathcal{C}_2 W_{\psi, \varphi}^* K_w^{(\gamma)}(z)$$

for all $w, z \in \mathbb{D}$.

We simplify the left hand side of (3.4) to get

$$\begin{aligned}
W_{\psi, \varphi} \mathcal{C}_2 K_w^{(\gamma)}(z) &= \frac{\psi(z)u(\varphi(z))}{(1 - w \cdot v(\varphi(z)))^\gamma} && \text{by (1.1)} \\
&= \frac{\lambda \left(1 - |\alpha|^2\right)^{\frac{\gamma}{2}}}{(1 - \bar{\alpha}\varphi(z))^\gamma} \cdot \frac{\psi(z)}{\left(1 - w \left(\frac{\bar{\alpha}}{\alpha} \cdot \frac{\alpha - \varphi(z)}{1 - \bar{\alpha}\varphi(z)}\right)\right)^\gamma} && \text{by (1.2)} \\
&= \frac{\alpha^\gamma \lambda \psi(z) \left(1 - |\alpha|^2\right)^{\frac{\gamma}{2}}}{(\alpha(1 - \bar{\alpha}\varphi(z)) - \bar{\alpha}w(\alpha - \varphi(z)))^\gamma}
\end{aligned}$$

and simplify the right hand side of (3.4) to get

$$\begin{aligned}
\mathcal{C}_2 W_{\psi, \varphi}^* K_w^{(\gamma)}(z) &= \mathcal{C}_2 \left(\overline{\psi(w)} K_{\varphi(w)}(z) \right) && \text{by (2.1)} \\
&= \frac{u(z)\psi(w)}{(1 - \varphi(w)v(z))^\gamma} && \text{by (1.1)} \\
&= \frac{\lambda \left(1 - |\alpha|^2\right)^{\frac{\gamma}{2}}}{(1 - \bar{\alpha}z)^\gamma} \cdot \frac{\psi(w)}{\left(1 - \varphi(w) \left(\frac{\bar{\alpha}}{\alpha} \cdot \frac{\alpha - z}{1 - \bar{\alpha}z}\right)\right)^\gamma} && \text{by (1.2)} \\
&= \frac{\alpha^\gamma \lambda \psi(w) \left(1 - |\alpha|^2\right)^{\frac{\gamma}{2}}}{(\alpha(1 - \bar{\alpha}\varphi(w)) - \bar{\alpha}z(\alpha - \varphi(w)))^\gamma}.
\end{aligned}$$

In order to simplify the exposition, we introduce for any $w, z \in \mathbb{D}$, the following functions

$$p_z(w) := \alpha(1 - \bar{\alpha}\varphi(z)) - \bar{\alpha}w(\alpha - \varphi(z))$$

and

$$q_z(w) := \alpha(1 - \bar{\alpha}\varphi(w)) - \bar{\alpha}z(\alpha - \varphi(w)).$$

We note that by the above notation, we have $p_z(w) = q_w(z)$ for any $w, z \in \mathbb{D}$.

Now, we can write (3.4) as

$$\frac{\alpha^\gamma \lambda \psi(z) \left(1 - |\alpha|^2\right)^{\frac{\gamma}{2}}}{p_z(w)^\gamma} = \frac{\alpha^\gamma \lambda \psi(w) \left(1 - |\alpha|^2\right)^{\frac{\gamma}{2}}}{q_z(w)^\gamma}$$

and thus

$$(3.5) \quad \frac{\psi(z)}{p_z(w)^\gamma} = \frac{\psi(w)}{q_z(w)^\gamma}$$

for all $w, z \in \mathbb{D}$. Since (3.5) holds for all $w \in \mathbb{D}$, we set $w = 0$ and thus

$$(3.6) \quad \psi(z) = \frac{\psi(0)p_z(0)^\gamma}{q_z(0)^\gamma}$$

for all $z \in \mathbb{D}$.

Now we substitute (3.6) back into (3.5) to get

$$\frac{\psi(0)p_z(0)^\gamma}{p_z(w)^\gamma q_z(0)^\gamma} = \frac{\psi(0)p_w(0)^\gamma}{q_z(w)^\gamma q_w(0)^\gamma},$$

which can also be written as

$$\frac{\psi(0)p_z(0)^\gamma}{p_z(w)^\gamma q_z(0)^\gamma} = \frac{\psi(0)q_0(w)^\gamma}{q_z(w)^\gamma p_0(w)^\gamma},$$

which can be further reduced to

$$\left(\frac{p_z(0)}{q_z(0)}\right)^\gamma (q_z(w)p_0(w))^\gamma = (q_0(w)p_z(w))^\gamma$$

for all $w, z \in \mathbb{D}$. Since φ is holomorphic, we are able to differentiate $p_z(w)$ and $q_z(w)$, thus we can differentiate the above equation with respect to w to get

$$\begin{aligned} & \left(\frac{p_z(0)}{q_z(0)}\right)^\gamma \gamma (q_z(w)p_0(w))^{\gamma-1} (q_z(w)p'_0(w) + q'_z(w)p_0(w)) \\ &= \gamma (q_0(w)p_z(w))^{\gamma-1} (q_0(w)p'_z(w) + q'_0(w)p_z(w)) \end{aligned}$$

for all $w, z \in \mathbb{D}$. We further let $w = 0$ to get

$$\begin{aligned} & \left(\frac{p_z(0)}{q_z(0)}\right)^\gamma (q_z(0)p_0(0))^{\gamma-1} (q_z(0)p'_0(0) + q'_z(0)p_0(0)) \\ &= (q_0(0)p_z(0))^{\gamma-1} (q_0(0)p'_z(0) + q'_0(0)p_z(0)) \end{aligned}$$

which reduces to

$$(3.7) \quad p_z(0) (q_z(0)p'_0(0) + q'_z(0)p_0(0)) = q_z(0) (q_0(0)p'_z(0) + q'_0(0)p_z(0))$$

for all $z \in \mathbb{D}$.

Denote $c_0 = \alpha(1 - \bar{\alpha}\varphi(0))$ and $\beta = \bar{\alpha}(\alpha - \varphi(0))$. Then we have

$$\begin{aligned} p_z(0) &= \alpha(1 - \bar{\alpha}\varphi(z)), \\ q_z(0) &= c_0 - \beta z, \\ p_0(0) &= q_0(0) = c_0, \\ p'_0(0) &= -\beta, \\ q'_z(0) &= -\bar{\alpha}\varphi'(0)(\alpha - z), \\ p'_z(0) &= -\bar{\alpha}(\alpha - \varphi(z)), \\ q'_0(0) &= -|\alpha|^2 \varphi'(0). \end{aligned}$$

We substitute the above expressions into (3.7) to get

$$\begin{aligned}
& \alpha(1 - \bar{\alpha}\varphi(z)) (-\beta(c_0 - \beta z) - \bar{\alpha}c_0\varphi'(0)(\alpha - z)) \\
&= (c_0 - \beta z) \left(-\bar{\alpha}c_0(\alpha - \varphi(z)) - \alpha|\alpha|^2\varphi'(0)(1 - \bar{\alpha}\varphi(z)) \right) \\
\iff & \alpha(1 - \bar{\alpha}\varphi(z)) (-\beta(c_0 - \beta z) - \bar{\alpha}c_0\varphi'(0)(\alpha - z)) \\
&= -\bar{\alpha}c_0(c_0 - \beta z)(\alpha - \varphi(z)) \\
&\quad - \alpha(1 - \bar{\alpha}\varphi(z)) \left(|\alpha|^2 c_0\varphi'(0) - |\alpha|^2 \beta\varphi'(0)z \right) \\
\iff & \alpha(1 - \bar{\alpha}\varphi(z)) \left(-\beta c_0 + \beta^2 z + \bar{\alpha}c_0\varphi'(0)z - |\alpha|^2 \beta\varphi'(0)z \right) \\
&= -|\alpha|^2 c_0^2 + |\alpha|^2 \beta c_0 z + \bar{\alpha}\varphi(z)(c_0^2 - \beta c_0 z) \\
\iff & \alpha \left(-\beta c_0 + \beta^2 z + \bar{\alpha}c_0\varphi'(0)z - |\alpha|^2 \beta\varphi'(0)z \right) \\
&\quad - \bar{\alpha}\varphi(z) \cdot \alpha \left(-\beta c_0 + \beta^2 z + \bar{\alpha}c_0\varphi'(0)z - |\alpha|^2 \beta\varphi'(0)z \right) \\
&= -|\alpha|^2 c_0^2 + |\alpha|^2 \beta c_0 z + \bar{\alpha}\varphi(z)(c_0^2 - \beta c_0 z) \\
\iff & |\alpha|^2 c_0^2 - |\alpha|^2 \beta c_0 z - \alpha\beta c_0 + \alpha\beta^2 z + |\alpha|^2 c_0\varphi'(0)z - \alpha|\alpha|^2 \beta\varphi'(0)z \\
&= \bar{\alpha}\varphi(z) \left(c_0^2 - \beta c_0 z - \alpha\beta c_0 + \alpha\beta^2 z + |\alpha|^2 c_0\varphi'(0)z - \alpha|\alpha|^2 \beta\varphi'(0)z \right)
\end{aligned}$$

for all $z \in \mathbb{D}$.

Denote $c_1 = \beta c_0$ and $c_2 = |\alpha|^2 c_0\varphi'(0) - \alpha|\alpha|^2 \beta\varphi'(0) + \alpha\beta^2$. Then we get

$$|\alpha|^2 c_0^2 - \alpha c_1 - \left(|\alpha|^2 c_1 - c_2 \right) z = \bar{\alpha}\varphi(z) \left(c_0^2 - \alpha c_1 - (c_1 - c_2)z \right)$$

for all $z \in \mathbb{D}$. Dividing both sides by $\bar{\alpha}$, we get

$$(3.8) \quad \varphi(z) = \frac{\alpha(\bar{\alpha}c_0^2 - c_1) - \left(|\alpha|^2 c_1 - c_2 \right) z}{\bar{\alpha}(c_0^2 - \alpha c_1) - \bar{\alpha}(c_1 - c_2)z}$$

for all $z \in \mathbb{D}$. Let $d = \psi(0)$. Finally, we substitute (3.8) into (3.6) to get

$$\begin{aligned}
\psi(z) &= \frac{d\alpha^\gamma}{(c_0 - \beta z)^\gamma} \left(1 - \frac{\alpha(\bar{\alpha}c_0^2 - c_1) - \left(|\alpha|^2 c_1 - c_2 \right) z}{c_0^2 - \alpha c_1 - (c_1 - c_2)z} \right)^\gamma \\
&= \frac{d\alpha^\gamma}{(c_0 - \beta z)^\gamma} \left(\frac{c_0^2(1 - |\alpha|^2) - c_1(1 - |\alpha|^2)z}{c_0^2 - \alpha c_1 - (c_1 - c_2)z} \right)^\gamma \\
&= \frac{d}{(c_0 - \beta z)^\gamma} \left(\alpha c_0(1 - |\alpha|^2) \cdot \frac{c_0 - \beta z}{c_0^2 - \alpha c_1 - (c_1 - c_2)z} \right)^\gamma \\
&= \frac{d(c_0^2 - \alpha c_1)^\gamma}{(c_0^2 - \alpha c_1 - (c_1 - c_2)z)^\gamma}
\end{aligned}$$

for all $z \in \mathbb{D}$.

Sufficiency. Suppose that for all $z \in \mathbb{D}$,

$$\psi(z) = \frac{d(c_0^2 - \alpha c_1)^\gamma}{(c_0^2 - \alpha c_1 - (c_1 - c_2)z)^\gamma} \quad \text{and}$$

$$\varphi(z) = \frac{\alpha(\bar{\alpha}c_0^2 - c_1) - (|\alpha|^2 c_1 - c_2)z}{\bar{\alpha}(c_0^2 - \alpha c_1) - \bar{\alpha}(c_1 - c_2)z},$$

where $c_0, c_1, c_2, d \in \mathbb{C}$.

By our assumption, $W_{\psi, \varphi}$ is bounded on $\mathcal{H}_\gamma(\mathbb{D})$. Hence it suffices to prove that (3.4) holds for all $w, z \in \mathbb{D}$. Clearly we have

$$W_{\psi, \varphi} \mathcal{C}_2 K_w^{(\gamma)}(z) = \frac{d(c_0^2 - \alpha c_1)^\gamma}{(c_0^2 - \alpha c_1 - (c_1 - c_2)z)^\gamma} \cdot \frac{\alpha^\gamma \lambda \left(1 - |\alpha|^2\right)^{\frac{\gamma}{2}}}{(\alpha(1 - \bar{\alpha}\varphi(z)) - \bar{\alpha}w(\alpha - \varphi(z)))^\gamma}$$

and

$$\mathcal{C}_2 W_{\psi, \varphi}^* K_w^{(\gamma)}(z) = \frac{d(c_0^2 - \alpha c_1)^\gamma}{(c_0^2 - \alpha c_1 - (c_1 - c_2)w)^\gamma} \cdot \frac{\alpha^\gamma \lambda \left(1 - |\alpha|^2\right)^{\frac{\gamma}{2}}}{(\alpha(1 - \bar{\alpha}\varphi(w)) - \bar{\alpha}z(\alpha - \varphi(w)))^\gamma}.$$

Note that the numerators of $W_{\psi, \varphi} \mathcal{C}_2 K_w^{(\gamma)}(z)$ and $\mathcal{C}_2 W_{\psi, \varphi}^* K_w^{(\gamma)}(z)$ are the same thus it suffices to check if their denominators are the same. We denote

$$D := (c_0^2 - \alpha c_1 - (c_1 - c_2)z)(\alpha(1 - \bar{\alpha}\varphi(z)) - \bar{\alpha}w(\alpha - \varphi(z)))$$

and

$$D_* := (c_0^2 - \alpha c_1 - (c_1 - c_2)w)(\alpha(1 - \bar{\alpha}\varphi(w)) - \bar{\alpha}z(\alpha - \varphi(w))).$$

Substituting φ gives us

$$\begin{aligned} D &= (c_0^2 - \alpha c_1 - (c_1 - c_2)z) \left(\alpha \left(1 - |\alpha|^2\right) \cdot \frac{c_0^2 - c_1 z}{c_0^2 - \alpha c_1 - (c_1 - c_2)z} \right. \\ &\quad \left. - \alpha \bar{\alpha} w + \bar{\alpha} w \frac{\alpha(\bar{\alpha}c_0^2 - c_1) - (|\alpha|^2 c_1 - c_2)z}{\bar{\alpha}(c_0^2 - \alpha c_1) - \bar{\alpha}(c_1 - c_2)z} \right) \\ &= \alpha \left(1 - |\alpha|^2\right) (c_0^2 - c_1 z) - |\alpha|^2 w (c_0^2 - \alpha c_1 - (c_1 - c_2)z) \\ &\quad + w \left(\alpha(\bar{\alpha}c_0^2 - c_1) - (|\alpha|^2 c_1 - c_2)z \right) \\ &= \alpha c_0^2 \left(1 - |\alpha|^2\right) - \alpha c_1 \left(1 - |\alpha|^2\right) z - |\alpha|^2 (c_0^2 - \alpha c_1) w \\ &\quad + |\alpha|^2 (c_1 - c_2) w z + \alpha(\bar{\alpha}c_0^2 - c_1) w - (|\alpha|^2 c_1 - c_2) w z \\ &= \alpha c_0^2 \left(1 - |\alpha|^2\right) - \alpha c_1 \left(1 - |\alpha|^2\right) (w + z) \\ &\quad - \bar{\alpha} \left(|\alpha|^2 c_1 - c_2 - \alpha c_1 + \alpha c_2 \right) w z \end{aligned}$$

Likewise, we have

$$\begin{aligned} D_* &= \alpha c_0^2 \left(1 - |\alpha|^2\right) - \alpha c_1 \left(1 - |\alpha|^2\right) (z + w) \\ &\quad - \bar{\alpha} \left(|\alpha|^2 c_1 - c_2 - \alpha c_1 + \alpha c_2 \right) z w \end{aligned}$$

by interchanging the variables w and z . So we indeed have $D = D_*$ for all $w, z \in \mathbb{D}$. $\square$

By setting $\gamma = 1$ in Theorems 3.1 and 3.2, we obtain the main results of [8]. For $\gamma = 2$, we obtain the results for the case of Bergman space which, to the best of our knowledge, has not been published prior to this paper.

4. CONCLUDING REMARKS AND OPEN QUESTIONS

In this paper, we have managed to characterize weighted composition conjugations into two classes, namely $\mathcal{C}_1$ and $\mathcal{C}_2$ (see Theorem 2.11). We then proceeded to characterize $W_{\psi,\varphi}$ on $\mathcal{H}_\gamma(\mathbb{D})$ that are $\mathcal{C}_1$ -symmetric and $\mathcal{C}_2$ -symmetric (see Theorems 3.1 and 3.2), thereby completely characterizing $W_{\psi,\varphi}$ on $\mathcal{H}_\gamma(\mathbb{D})$ which are complex symmetric with respect to all weighted composition conjugations.

Let $G \subset \mathbb{C}$ be a domain and $\beta = (\beta_j)$ be a sequence of positive real numbers. Consider following space

$$\mathcal{H}(G, \beta) := \left\{ f(z) = \sum_{j=0}^{\infty} a_j z^j \in \mathcal{O}(G) : \|f\|^2 = \sum_{j=0}^{\infty} |a_j|^2 \beta_j^2 < \infty \right\}.$$

If $G = \mathbb{C}$, we have the Hilbert space of entire functions. Further, if $\beta_j = \sqrt{j!}$, we have the Fock space, which was studied in [6]. If $G = \mathbb{D}$, then we have the weighted Hardy spaces. Further, if $\beta_j = 1$, we have the Hardy space; if $\beta_j = \frac{1}{\sqrt{j+1}}$, we have the Bergman space; if $\beta_j = \sqrt{j+1}$, we have the Dirichlet space.

Question 4.1. *What is the complex symmetric structure of $W_{\psi,\varphi}$ with respect to weighted composition conjugations on $\mathcal{H}(G, \beta)$?*

In the statement of Theorems 3.1 and 3.2, we note that φ is independent of $\gamma \in \mathbb{N}$. This leads us to suspect that even if we have $\gamma > 0$, we will arrive at the same formula for φ . However as mentioned earlier, our proof requires $\gamma \in \mathbb{N}$ due to the multiplication of complex numbers raised to power of γ . Is there a way to arrive at the formulae of φ and ψ without considering such multiplication?

Question 4.2. *Do Theorems 3.1 and 3.2 hold true for real positive γ ?*

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