Adaptive refinement analysis for the coupled boundary element method-reproducing kernel particle method

Shuai, Ying Yong

2006

Shuai, Y. Y. (2006). Adaptive refinement analysis for the coupled boundary element method-reproducing kernel particle method. Doctoral thesis, Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/12047>

<https://doi.org/10.32657/10356/12047>

Nanyang Technological University

Downloaded on 26 Apr 2023 19:51:21 SGT

**ADAPTIVE REFINEMENT ANALYSIS FOR THE
COUPLED BOUNDARY ELEMENT METHOD -
REPRODUCING KERNEL PARTICLE METHOD**

SHUAI YING YONG

**SCHOOL OF CIVIL AND ENVIRONMENTAL ENGINEERING
NANGYANG TECHNOLOGICAL UNIVERSITY**

2006

Adaptive Refinement Analysis for the Coupled Boundary Element Method - Reproducing Kernel Particle Method

SHUAI YING YONG

School of Civil and Environmental Engineering

**A thesis submitted to Nanyang Technological University
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy**

2006

ACKNOWLEDGEMENTS

I would like to express my gratitude first and foremost to my supervisor, Associate Professor Lee Chi King, for his supervision during the entire period of my research study at Nanyang Technological University. His guidance, diligence, patience and hard work helped me to bring this research work to the final form.

I would also like to thank Associate Professor Lie Seng Tjhen, for his excellent comments on my research proposal.

I am also grateful to Professor Liew Kim Meow for his good suggestion on my research work.

Many thanks are given to Mr. Xu Qiang Xun and others for their beneficial suggestions and warm help in my research. It is my pleasure to have this chance to thank all of you.

The author is thankful to the School of Civil and Environmental Engineering, Nanyang Technological University for the scholarship provided during study.

Special thanks are due to my family for their love and continued support during the course of this work.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS	i
TABLE OF CONTENTS	ii
SUMMARY	vi
LIST OF FIGURES	viii
LIST OF TABLES	xi
LIST OF SYMBOLS	xiii
Chapter 1. INTRODUCTION	1
1.1 Background	1
1.2 Objective	2
1.3 Scope	2
1.4 Organization	3
Chapter 2. LITERATURE REVIEW	4
2.1 A brief review of the boundary element method	4
2.1.1 Background	4
2.1.2 Overview of the BEM	5
2.1.3 <i>A posteriori</i> error estimation and adaptivity of the BEM	6
2.2 A brief review of meshless Methods	8
2.2.1 Background	8
2.2.2 Overview of meshless methods	10
2.2.3 <i>A posteriori</i> error estimation and adaptivity of meshless methods	14
2.3 A brief review of the coupled BEM and meshless methods	17
2.3.1 Background	17
2.3.2 Overview of the coupled BEM and meshless methods	18

2.3.3	Adaptivity of the coupled BEM and meshless methods	19
Chapter 3.	THE MODELLED PROBLEM AND BASIC FORMULATIONS FOR BEM AND RKPM	21
3.1	Two dimensional linear elastostatic problem	21
3.2	The boundary element method	23
3.3	The reproducing kernel particle method	26
3.3.1	Construction of displacement field function	26
3.3.2	Formulations for the RKPM	28
Chapter 4.	THE COUPLED BE-RKPM	30
4.1	Basic formulation for the coupled BE-RKPM	30
4.1.1	Solution by the BEM in Ω^B	32
4.1.2	Solution by the RKPM in Ω^R	33
4.1.3	The coupled BE-RKPM	35
4.1.4	Unit displacement method	36
4.2	<i>A priori</i> error estimation for the BEM, the RKPM and the coupled BE-RKPM	40
4.2.1	Definition of error norm	41
4.2.2	<i>A priori</i> error estimation for the BEM	41
4.2.3	<i>A priori</i> error estimation for the RKPM	42
4.2.4	<i>A priori</i> error estimation for the coupled BE-RKPM	43
4.3	Numerical examples with uniform refinement	45
4.3.1	Timoshenko beam (convex domain, smooth exact solution)	45
4.3.2	Infinite plate with a circular hole (concave domain, smooth exact solution)	48
4.3.3	Domain near a crack tip (convex domain, singular exact solution)	51
4.3.4	L-Shaped domain (concave domain, singular exact solution)	53
4.3.5	A structure on a semi-infinite soil foundation	56
4.4	Closure and discussions	58

Chapter 5. <i>A POSTERIORI</i> ERROR ESTIMATION FOR RKPM AND BEM	61
5.1 <i>A posteriori</i> error estimation for the RKPM	61
5.1.1 Weak form and orthogonality of the RKPM Solution	61
5.1.2 Extraction equation for stress recovery	63
5.1.3 Convergence rate of the recovered stress field	65
5.1.4 The extraction procedure	67
5.1.5 Computation of the recovered stress field $[\tilde{\sigma}(\mathbf{p})]$ and <i>a posteriori</i> error estimation	70
5.1.6 Numerical validation	74
5.1.7 Discussions on obtained numerical results	80
5.2 <i>A posteriori</i> error estimation for the BEM	81
5.2.1 Direct boundary stress based on the BEM solution	81
5.2.2 Recovery boundary stress based on the hyper-singular boundary integration equation	83
5.2.3 <i>A posteriori</i> error estimation based on the residual approach	84
Chapter 6. ADAPTIVE REFINEMENT FOR BEM, RKPM AND COUPLED BE-RKPM	87
6.1 Adaptive refinement for the BEM	87
6.1.1 Global and local refinement indicators for the BEM	87
6.1.2 Automatic refinement based on <i>h</i> -adaptive scheme	88
6.1.3 Numerical examples for the BEM	89
6.1.4 Discussions on obtained BEM numerical results	94
6.2 Adaptive refinement for the RKPM	95
6.2.1 Variable support size determination for an adaptive RKPM mesh	96
6.2.2 Adaptive refinement strategy for the RKPM	98
6.2.3 New point mesh generation and determination of h_i	103
6.2.4 Numerical examples for the RKPM	104
6.2.5 Discussions on obtained RKPM numerical results	111

6.3 Adaptive refinement for the coupled BE-RKPM	111
6.3.1 Displacement compatible condition on interface boundary	111
6.3.2 Automatic refinement strategy for the coupled BE-RKPM	113
6.3.3 Numerical examples for the coupled BE-RKPM	114
6.3.4 Discussions on obtained coupled BE-RKPM numerical results	121
6.4 Closure and discussion	122
Chapter 7. CONCLUSIONS AND RECOMMENDATIONS	126
7.1 Conclusions	126
7.2 Further Investigations	129
REFERENCES	131
APPENDICES	
APPENDIX A Boundary element method based on discontinuous linear element	143
APPENDIX B Integration scheme for RKPM	152
APPENDIX C Singular integration rule	155
APPENDIX D Traction boundary correction based on least error in energy norm	159

SUMMARY

The boundary element method (BEM) and the meshless methods have been applied to solve many engineering problems. It is often desirable and beneficial to combine the BEM and meshless methods in order to exploit their advantages while evading their disadvantages.

In this study, a review is first conducted to provide background on the *a posteriori* error estimation and adaptive refinement for the BEM, the meshless methods and the coupled methods for the BEM and the meshless methods. Based on the formulation of the BEM and the reproducing kernel particle method (RKPM), a new coupled method of the BEM and the RKPM is proposed for two dimensional elastostatic problems. Numerical experiments on four benchmark problems are also carried out to compare the convergence rates of the BEM, the RKPM and the coupled BE-RKPM.

In order to guarantee a certain level of accuracy of the numerical solution, a new *a posteriori* error estimator for the RKPM is constructed by combining the Zienkiewicz and Zhu (Z-Z) error estimation with a new stress recovery procedure. The basic idea of the proposed error estimator is to extract the recovered stress field of the RKPM by means of some extraction functions which are derived in a way similar to the formulation of the BEM. Numerical examples are used again to test the validity of the new *a posteriori* error estimator.

The adaptive refinement technique allows modifications of the mesh, the order of the approximations or other features of numerical algorithms so as to reduce and control numerical error. In this study, automatic adaptive refinement procedures for the BEM, the RKPM and the coupled BE-RKPM are suggested for 2D elastostatic

problems. Numerical studies are then carried out to test the effectiveness of the suggested adaptive strategies by comparing with the results obtained using uniform mesh.

LIST OF FIGURES

Figure 3.1	Two dimensional problem	21
Figure 3.2	Points and elements assignment of the BEM	24
Figure 4.1	The domain of the coupled BE-RKPM	30
Figure 4.2	Unit displacement along the interface boundary	36
Figure 4.3	Cantilever beam with end shear force	46
Figure 4.4	BEM mesh used for the Timoshenko beam	46
Figure 4.5	RKPM mesh used for the Timoshenko beam	47
Figure 4.6	Coupled BE-RKPM mesh used for the Timoshenko beam	47
Figure 4.7	Convergence rates $R(\eta)$ for the Timoshenko beam	48
Figure 4.8	The first quadrant of a plate with a circular hole	49
Figure 4.9	BEM mesh used for the infinite plate with a circular hole	49
Figure 4.10	RKPM mesh used for the infinite plate with a circular hole	49
Figure 4.11	Coupled BE-RKPM mesh used for the infinite plate with a circular hole	50
Figure 4.12	Convergence rates $R(\eta)$ for the infinite plate with a circular hole	50
Figure 4.13	Region near a crack tip	51
Figure 4.14	BEM mesh used for the domain near a crack tip	52
Figure 4.15	RKPM mesh used for the domain near a crack tip	52
Figure 4.16	Coupled BE-RKPM mesh used for the domain near a crack tip	52
Figure 4.17	Convergence rates $R(\eta)$ for the domain near a crack tip	53
Figure 4.18	L-Shaped domain	54
Figure 4.19	BEM mesh used for the L-shaped domain	54
Figure 4.20	RKPM mesh used for the L-shaped domain	55
Figure 4.21	Coupled BE-RKPM mesh used for the L-shaped domain	55

Figure 4.22	Convergence rates $R(\eta)$ for the L-shaped domain	56
Figure 4.23	A structure standing on a semi-infinite soil foundation	57
Figure 4.24	RKPM mesh of structure (region 1) and load cases	57
Figure 4.25	BEM element mesh of soil foundation (region 2)	58
Figure 5.1	The local domain Ω_R in Ω	67
Figure 5.2	Local recovery subdomain Ω_R with circular boundary for interior point $\mathbf{p}$	70
Figure 5.3	Local recovery subdomain for a point close to the domain boundary	70
Figure 5.4	Integration of Eqn. 5.27a when $\Gamma_R = C(\mathbf{p}, r_R)$	71
Figure 5.5	Integration of Eqn. 5.27a when $\Gamma_R = \Gamma_{Ri} \cup \Gamma_{Rb}$	72
Figure 5.6	Recovery stress for boundary point $\mathbf{p}$	73
Figure 5.7	Patch test for the recovery procedure	76
Figure 5.8	Stress components of point $\mathbf{p}$ on the boundary of the BEM domain	81
Figure 6.1	Adaptive BEM mesh used for the Timoshenko beam	89
Figure 6.2	Convergence rates R_{BEM} for the Timoshenko beam	90
Figure 6.3	Adaptive BEM mesh used for the infinite plate with a circular hole	91
Figure 6.4	Convergence rates R_{BEM} for the infinite plate with a circular hole	91
Figure 6.5	Adaptive BEM mesh used for the domain near a crack tip	92
Figure 6.6	Convergence rates R_{BEM} for the domain near a crack tip	93
Figure 6.7	Adaptive BEM mesh used for the L-shaped domain	93
Figure 6.8	Convergence rates R_{BEM} for the L-shaped domain	94
Figure 6.9	The minimum size of domain of influences of points: h_i satisfies the minimum size condition while h_j does not	97
Figure 6.10	Determination of support sizes	98
Figure 6.11	Adaptive RKPM mesh used for the Timoshenko beam	105
Figure 6.12	Convergence rates R_{RKPM} for the Timoshenko beam	106

Figure 6.13	Adaptive RKPM mesh used for the infinite plate with a circular hole	106
Figure 6.14	Convergence rates R_{RKPM} for the infinite plate with a circular hole	107
Figure 6.15	Adaptive RKPM mesh used for the domain near a crack tip	108
Figure 6.16	Convergence rates R_{RKPM} for the domain near a crack tip	109
Figure 6.17	Adaptive RKPM mesh used for the L-shaped domain	109
Figure 6.18	Convergence rates R_{RKPM} for the L-shaped domain	110
Figure 6.19	Displacement compatible condition in adaptive process	112
Figure 6.20	Adjustment of h_i^{new} along interface boundary Γ^{RB}	113
Figure 6.21	Adaptive coupled BE-RKPM mesh used for the Timoshenko beam	116
Figure 6.22	Convergence rates $R_{\text{BE-RKPM}}$ for the Timoshenko beam	117
Figure 6.23	Adaptive coupled BE-RKPM mesh used for the infinite plate with a circular hole	117
Figure 6.24	Convergence rates $R_{\text{BE-RKPM}}$ for the infinite plate with a circular hole	118
Figure 6.25	Adaptive coupled BE-RKPM mesh used for the domain near a crack tip	119
Figure 6.26	Convergence rates $R_{\text{BE-RKPM}}$ for the domain near a crack tip	120
Figure 6.27	Adaptive coupled BE-RKPM mesh used for the L-shape domain	120
Figure 6.28	Convergence rates $R_{\text{BE-RKPM}}$ for the L-shape domain	121
Figure 6.29	Adaptive rates of convergence for the Timoshenko beam	122
Figure 6.30	Adaptive rates of convergence for the infinite plate with a circular hole	122
Figure 6.31	Adaptive rates of convergence for the domain near a crack tip	123
Figure 6.32	Adaptive rates of convergence for the L-shape domain	123

LIST OF TABLES

Table 4.1	Results for the Timoshenko beam	47
Table 4.2	Results for the infinite plate with a circular hole	50
Table 4.3	Results for the domain near a crack tip	53
Table 4.4	Results for the L-shaped domain	55
Table 4.5	Results for vertical displacements along tops of structure of a structure standing on a semi-infinite soil foundation	58
Table 4.6	CPU times for the uniform refinement (mesh 4) using a DELL Precision 450 PC (Xeon 2.8GHz, 1G RAM)	60
Table 5.1	Patch test for recovered stress value $\tilde{\sigma}_{xx}$ along the line $x=1$	77
Table 5.2	Recovery results for the Timoshenko beam	78
Table 5.3	Recovery results for the infinite plate with a circular hole	78
Table 5.4	Recovery results for the domain near a crack tip	79
Table 5.5	Recovery results for the L-shaped domain	79
Table 6.1	Adaptive BEM results of the Timoshenko beam	90
Table 6.2	Adaptive BEM results of the infinite plate with a circular hole	91
Table 6.3	Adaptive BEM results of the domain near a crack tip	92
Table 6.4	Adaptive BEM results of the L-shaped domain	94
Table 6.5	Adaptive RKPM results of the Timoshenko beam	105
Table 6.6	Adaptive RKPM results of the infinite plate with a circular hole	107
Table 6.7	Adaptive RKPM results of the domain near a crack tip	108
Table 6.8	Adaptive RKPM results of the L-shaped domain	110
Table 6.9	Adaptive coupled BE-RKPM results of the Timoshenko beam	116
Table 6.10	Adaptive coupled BE-RKPM results of the infinite plate with a circular hole	118

Table 6.11	Adaptive coupled BE-RKPM results of the domain near a crack tip	119
Table 6.12	Adaptive coupled BE-RKPM results of the L-shape domain	121
Table 6.13	CPU times for the entire automatic adaptive refinement using a DELL Precision 450 PC (Xeon 2.8GHz, 1G RAM)	124
Table 6.14	Ratios of CPU time used in different major processes for adaptive refinement using the BEM and the RKPM	124

LIST OF SYMBOLS

$\{ \}$	Vector
$[\]$	Matrix
$\{ \}^T, [\]^T$	Transposed vector or matrix
$\ \ $	Norm
$\int_{\Omega}$	Domain integration
$\int_{\Gamma}$	Boundary integration
$\sum$	Summation
α	Penalty parameter
δ^i	Dirac delta function
ϵ	Strain vector
η	Relative error
λ	Strength of singularity
μ	Poisson ratio
θ	Angle
$\mathcal{G}$	Effectivity index
σ	Stress vector
τ	Direction of tangential line
Γ	Boundary
Γ_t	Traction boundary
Γ_u	Displacement boundary
Ω	Domain
$\mathbf{b}$	Body force vector
$\mathbf{D}$	Strain-stress matrix

$\mathbf{e}_\sigma$	Pointwise error in stress
$\mathbf{e}_u$	Pointwise error in displacement
E	Young's module
$\mathbf{I}$	Identity matrix
L	Length
$\mathbf{L}$	Strain-displacement matrix
$\mathbf{n}$	Direction of outer normal line
NE	Number of BEM elements
NN	Number of BEM base nodes
NP	Number of RKPM computing points
$\mathbf{N}^B$	Matrix of global interpolation for the BEM
$\mathbf{N}^R$	Matrix of shape function for the RKPM
N_{DoF}	Number of degree of freedom
$\mathbf{Q}$	Surface traction –stress matrix
$\mathbf{R}$	Convergence rate
$\mathbf{t}$	Traction vector
$\bar{\mathbf{t}}$	Prescribed traction
$\hat{\mathbf{t}}_e$	Vector of nodal boundary traction value
$\mathbf{u}$	Displacement vector
$\bar{\mathbf{u}}$	Prescribed displacement
$\hat{\mathbf{u}}_e$	Vector of nodal displacement value
x	x coordinate
y	y coordinate

CHAPTER 1

INTRODUCTION

1.1 Background

It has become clear that in many engineering analyses using finite element method (FEM), mesh generation is a far more time-consuming and expensive task than the assembling of stiffness matrix and solution of linear equation system. Therefore, it has become expedient to explore methods which may be somewhat a bit more expensive from the viewpoint of solution time but involve less time in the preparation of data. In this respect, it should be mentioned that even with powerful mesh generators, three dimensional meshing is still a burdensome task and the conversion of solid models to finite element data is time consuming and often introduces numerous ambiguities.

Recently, considerable research in computational mechanics has been devoted to the development of meshless methods, such as element free Galerkin method (EFGM), reproducing kernel particle method (RKPM), etc., in which the domain of interest is discretized directly by a scattered set of points with any finite elements. The success of meshless methods is due to the development of new shape functions that allow the interpolation of the field variable to be accomplished without the need of a finite element mesh. Therefore, it is ideal for model refinement, adaptivity, fracture problems, and large deformation problems.

In the FEM, two kinds of procedures are widely used for *a posteriori* deriving error estimators: residual-based and post-processing (recovery) based error estimators. Unlike the FEM, the shape function used in meshless methods can be constructed to

be C^1 and thus the stress field is continuous within the whole domain. As a result, the error estimators for meshless methods will be of some form different from the FEM.

The Boundary element method (BEM) has been developed since 1960s. For some specific problems, the BEM is undoubtedly superior to the ‘domain’ type solution such as the FEM and the meshless method. The major advantages of the BEM include reduction of dimension, ease for dealing with problems with infinite domains, and small size of global equations system. As in any other numerical method, the BEM also has disadvantages such as the difficulty in handling nonlinear and discontinuous problems.

When solving the engineering problems, the idea of combining the BEM with other numerical techniques is naturally of great interest in many practical applications. It is often desirable and beneficial to combine ‘boundary’ type methods and ‘domain’ type methods in order to exploit their advantages while evading their disadvantages. In this study, some research work has been done in coupling of the BEM and meshless methods.

1.2 Objective

The main objectives of this study include: (i) To construct a new coupling procedure for the BEM and the RKPM; (ii) To propose a new *a posteriori* error estimation for RKPM; (iii) To develop adaptive schemes for the BEM, the RKPM and the coupled BE-RKPM in solving the linear boundary value problem (BVP).

1.3 Scope

In this study, concentrations will be paid on the following areas:

- 1). The basic formulations of the BEM and the RKPM are reviewed for 2D linear BVP.
- 2). A new coupling procedure for the BEM and the RKPM with numerical experiments are proposed to investigate its validation.

3). The *a posteriori* error estimation scheme for the BEM and the RKPM are constructed. Numerical experiments are done to verify the validity of the proposed *a posteriori* error estimator for the RKPM.

4). Adaptive refinement schemes are developed for the BEM, the RKPM and the coupled BE-RKPM. Numerical experiments are also done to demonstrate their efficiency.

1.4 Organization

Following this introduction, Chapter two presents the literature review on the BEM, the meshless methods and the coupling of BEM and meshless methods. It investigates the related background, new concept and the recent development.

Chapter three reviews the basic formulations of 2D elastostatic problems for the BEM and the RKPM, respectively.

Chapter four describes the implementation details of the suggested coupled BE-RKPM in two dimensional problems. The comparison of benchmark problems for the BEM, the RKPM and the coupled BE-RKPM are also listed in this chapter.

Chapter five concentrates on the *a posteriori* error estimation for the RKPM and the BEM. A new *a posteriori* error estimation based on the extraction function is put forward and numerical experiments on 2D elastostatic problems are then conducted to verify its efficiency for the RKPM.

Chapter six presents the adaptive schemes for the BEM, the RKPM and the coupled BE-RKPM, respectively.

Chapter seven concludes the main findings and results of the research work as well as the limitations of this research.

CHAPTER 2

LITERATURE REVIEW

This chapter reviews the background of the BEM, the meshless methods and the coupled methods. In addition, brief introductions on the *a posteriori* error estimation and adaptivity of the BEM and the meshless methods are presented. The details of the coupled BEM and meshless methods are investigated in Section 2.3.

2.1 A brief review of the boundary element method

2.1.1 Background

Most of the problems in continuum mechanics have more than one mathematical representation. A set of governing differential equations and the associated boundary conditions lead to the differential equation formulations of the problem. The differential equation formulation (the strong formulation) however, is not so amenable to numerical methods. Three kind methods have been applied to transform the differential equation formulation into non-differential equation formulation (the weak formulation).

The first class method is the weighted residual method which can directly handle the differential equation and obtain approximate solutions to problems. The second class method is known as the variational formulation which origins from the energy principles such as the principle of stationary potential energy, the Hamilton's principle in structural dynamics, etc. In this case, the solution of the problem is obtained as the minimum of the functional involved. The third class method is the integral equation formulation, wherein the governing equations are represented in the form of integral equations. The classical Green's function method, for example, can lead to such a formulation.

The BEM is based on the integral equation formulation. For instance, when the elastostatic problem is considered, the resulting integral equations provide the value of the displacement field at any interior point if the boundary values of traction and displacement components are fully known. It makes use of the above integral equations to determine the missing boundary data.

Firstly, the boundary of the domain is discretized into a number of boundary elements. For a three dimensional domain the boundary is a two dimensional surface, and hence two dimensional curved boundary elements such as triangular or quadrilateral elements are used to approximate the actual boundary. For a two dimensional body, the curved boundary line is discretized into straight or curved line elements. Then the boundary displacements and tractions are approximated by piecewise continuous functions over the boundary surface interpolated between boundary points.

Secondly, a limiting form of the integral equation is written at each of these boundary points to arrive at system of boundary integral equations. Numerical integration schemes are then employed to convert these integral equations into a system of algebraic equations whose solution gives the missing boundary displacements and tractions at the boundary points. The same integral equations can also be used to determine the displacements at any interior point.

When the missing boundary data is available, the strain-displacement relations and constitutive equation can then be used to evaluate strains and stresses at pre-selected interior points of interest.

2.1.2 Overview of the BEM

The BEM, in general, can be further classified into two classes, the direct boundary element method and the indirect boundary element method.

1). The indirect boundary element method [1]

The indirect boundary element method uses certain fictitious density functions, which may not bear any direct physical significance, that are assumed to be distributed over the boundary of body under consideration. These density functions

are sought using numerical solution procedures. Nowadays, the indirect formulation is not often used and not so popular.

2). The direct boundary element method [1]

The direct element method makes use of actual physical variable presented in the given problem to form the boundary integral equations. Thus, for example, in elastostatics the boundary integral equations can be written in terms of displacements, tractions and body force terms. Most of the present day researchers seem to favor this method and it is also adopted in this dissertation.

In 1967, Rizzo et al. [2] published an important work in which they used the direct boundary element method to solve two dimensional elastostatic problems. Within a couple of years, Cruse [3, 4], extended Rizzo's work to three dimensional problems of elastostatics. In 1977, Jaswon and Symm [5] presented their work on two dimensional potential problems, in which the numerical procedures were used to solve a set of integral equations and the boundary was discretized into the straight line segments over which the potentials were assumed to be constant. In 1986, Cruse et al. [6] presented their work on elastoplastic analysis of cracked plate, and Cruse [7] also summarized the advancement of the BEM in 1987. Recently, the BEM have been applied by many researchers in fracture mechanics problem [8], probabilistic structural analysis problem [9], three dimensional thermo-elasticity problem [10], and so on. Guiggiani [11] discussed the additional free term, which has been erroneously omitted in the hypersingular boundary integral equations. Mukherjee [12] described the Hadamard finite part and the Cauchy principal value representation of certain singular integrals for linear elasticity problem. More details of the BEM can also be referred to the books by Brebbia [13] and Mohammed [14].

2.1.3 *A posteriori* error estimation and adaptivity of the BEM

Numerical methods can be employed to solve the partial differential equation of a problem when the exact solution is not easy to obtain. During the process of numerical computation, the continuous system of original problem is transformed into a discrete system with finite number of degrees of freedom. As a result,

discretization error will be generated and it can be defined as the difference between the true solution and the numerical approximation. Hence, in practice, it is very important to obtain a reliable *a posteriori* error estimation to guarantee a certain level of accuracy of the numerical solution.

The research in the *a posteriori* error estimation of the BEM began in the early 1980s. Up to date, most of the *a posteriori* error estimators can be classified into three categories:

1). Residual methods

In this method, the value of the residual of the integral equation for each element is used as an indicator for the solution error. The mathematical properties of residual methods were examined by Wedland and Yu [15] for two-dimensional elliptic boundary integral equations. In the computation of residual of the integral equation, it involves the numerical integration along each element, which introduces extra computational cost.

2). Flux projection estimates

This idea was first introduced by Zienkiewicz et al. [16]. In this method, a suitable projection of the numerical solution is used to generate a postprocessed solution with improved accuracy. The numerical solution is then compared against this smoothed solution.

3). Extrapolation error estimates

It was first presented by Rencis and Mullen [17]. It is assumed that the numerical solution obtained by using mesh size h is denoted as σ^h . Then the number of elements is doubled and obtain new numerical solution $\sigma^{h/2}$. A measure of the difference between σ^h and $\sigma^{h/2}$ gives an error estimation.

The aim of the adaptive procedure is to construct an optimal mesh, which corresponds with the adopted *a posteriori* error estimator, and keep the error is equidistributed among the elements. Rank [18] suggested *a posteriori* error estimates and adaptive refinement for the BEM in 1984. Later, Ingber et al. [19]

constructed the element optimization algorithm for the adaptive BEM and Carney [20] presented the mesh redistribution algorithm in adaptive process of the BEM. In 1990, Guiggiani [21] described error indicators, which are based on local reanalysis, in mesh refinement of the BEM. Recently, Liapis [22] and Liang [23] reviewed the error estimation and adaptivity for the BEM. Generally, there are three basic kinds of adaptive techniques.

1). *h*-refinement [24]

It identifies the elements where error is large and refines them into smaller ones.

2). *p*-enrichment [25]

In this method, the number of elements is unchanged and higher order elements are introduced in the regions with large error.

3). *r*-relocation [26]

It keeps the number of elements constant and redistributes elements in an optimal manner.

A combination of above methods can also be applied in practical computations.

2.2 A brief review of meshless methods

2.2.1 Background

The FEM has achieved a great success in engineering. It is based on field function with local approximations, which enable it to deal with complex geometrical domains and be adopted to solve various problems by embedding in a unified formulation. However, the field function provided by the FEM presents a limited regularity and the generating adequate discretization meshes is also not a simple task in particular for complex surface or three dimensional problems.

In some cases such as the simulation of manufacturing and failure process of complex structures, it is necessary to deal with very large deformations of the mesh and to model the propagation of cracks with arbitrary and complex paths. When the FEM is applied to these problems, mesh regeneration is inevitable and this makes the FEM not so efficient when dealing with these problems.

On the contrary, the approximation of the meshless methods is constructed entirely in terms of points in the problem domain. Though meshes may still be needed in parts of the meshless methods, the objective of the meshes is only for integration purpose which can be treated without remeshing with minor costs in accuracy degradation. Thus, it becomes possible to solve engineering problems which are very awkward with mesh-based methods.

The earliest development of the meshless methods is the smoothed particle hydrodynamics (SPH) method. The foundation of the SPH method is the kernel estimate introduced in the basic research works by Lucy [27], Gingold [28] and Monaghan [29]. In SPH, the partial differential equations, such as conservation laws, are transformed into integral equations, and the kernel function then provides the approximation to estimate the values of field variables at discrete points. The first meshless method developed for structural analysis is by Nayroles et al. [30]. They proposed the diffuse element method (DEM) which employs moving least-squares interpolants in conjunction with the Galerkin method to provide a mesh-free computational formulation.

The accuracy of the method is later improved by Belytschko et al. [31], and it is called the element free Galerkin method (EFGM). In the EFGM, the derivatives of interpolants that were omitted in the DEM were introduced and a more accurate numerical integration method is employed to enhance solution accuracy. The EFGM was then applied to practical problems such as dynamic fracture problem [32]. In 1996, Belytschko et al. [33] reviewed the meshless approximations based on moving least-squares, kernels and partitions of unity and proved that the three methods are in most cases identical except for the important fact that partitions of unity enable p -adaptivity to be achieved. Belytschko et al. [34] also examined the completeness of particle methods and derived the reproducing conditions of the approximation or its derivatives.

In 1995, Liu et al. [35] proposed the reproducing kernel particle method (RKPM) to improve the accuracy of SPH method for finite domain problems by introducing a

correction term to the kernel function. The RKPM was then used to solve structure dynamic problem [36] and simulate the large deformation problem [37]. The built-in feature of multiresolution analysis of the RKPM [38] was addressed in 1996 and the corresponding error estimation and convergence properties [39] were presented in 1997. The multiscale analysis was also applied in practical problems, such as computational fluid dynamics [40] and large deformation problems [41]. Later, the application of the RKPM has been extended to other areas, such as analysis of microelectromechanical systems [42] and analysis piezoelectric devices [43]. Aluru presented a point collocation method based on reproducing kernel approximations [44] in 2000 and Aluru et al. [45] also developed the finite cloud method which combines the collocation with a fixed reproducing kernel technique.

Recently, Atluri et al. proposed a new meshless local Petrov-Galerkin method (MLPG) [46] which is based on a local symmetric weak form to solve linear potential problems. In 2000, they also presented the meshless local boundary integral equation method (MLBE) [47] and discussed the difference between MLPG and MLBE. Babuska et al. [48] developed the partition of unity method (PUM), which allows for the construction of conforming Ansatz spaces to local properties determined by the user. Naganshima [49, 50] described a new meshless approach referred to as the Node-By-Node Meshless (NBNM), in which only nodal data is utilized to discretize the governing equations. Oñate et al. [51] put forward the finite point method which is based on a weighted least square interpolation of point data and point collocation for evaluating the approximation integrals. Duarte et al. [52] brought forth the *hp* cloud method which uses the radial basis functions of varying size of supports and with polynomial-reproducing properties of arbitrary order. Yagawa et al. [53] developed the free mesh method in which some temporary triangular elements are set around a computing point of analyzed domain and the contributions from the element matrices of the above temporary elements are assembled to the total stiffness matrix.

2.2.2 Overview of meshless methods

Generally, three essential ingredients are involved in the meshless methods: the first one is to construct an approximation of the field function, the second one is to

get the discrete form of the original partial differential equation and the third one is to enforce the essential boundary condition of the problem.

2.2.2.1 Approximation of the field function

The meshless methods, which based on the global field approximation function which has good local character, can solve many engineering problems without mesh generation and simulate the singular behavior. The objective of the meshless methods is to eliminate the meshing constraints due to the FEM and to construct the approximation function entirely in terms of points. However, not all commonly used functions can be chosen as the field approximation functions. Three most popular methods, by which field functions are constructed, will be described as the following.

1). Kernel function method

The earliest version of this kind function is the SPH kernel function [27] and it was later improved by Liu [36] by introducing a correction function in order to enhance its accuracy near or on the boundaries of the problem domain.

2). Moving-least-square method (MLSM)

This kind of functions is based on the moving least square interpolation functions [54]. The approximations originated in scattered data fitting where it has been used in the DEM [30] and the EFGM [31].

3). Partitions of unity method (PUM)

This kind of functions is developed by Babuska [48] which is motivated by the need of new techniques for the solution of singular or special problems where the classical FEM approaches fail or are prohibitively expensive. The additional feature of the PUM is that finite element spaces can be easily constructed to any desired regularity.

In practical application, the field function is not always continuous in the computing area in practical applications, such as the crack propagation. Rao et al. [55] developed a meshless method to analyze linear elastic cracked structures subject to

single- or mixed-mode loading conditions. Basically, there are three kind criteria to simulate problems with non-continuous field function.

1). Visibility criterion [56]

It is the simplest way to simulate the discontinuities in which the discontinuous interior boundary such as the crack tip is viewed as opaque during the construction of weight function.

2). Diffraction method [57]

The essence of this method is to treat the discontinuous surface (line) as opaque but consider and evaluate those rays that pass around the corner of discontinuity.

3). Transparency method [58]

In this method, the transparency of the surface (line) is endowed with a varying degree. As a result, the shape function and its derivative are continuous around the tip of a discontinuity.

2.2.2.2 Construction of discrete equation

1). Collocation method [44]

The collocation method is based on piecewise polynomial approximation and represents a powerful class of methods for the numerical solution of initial-value problems for functional differential and integral equations arising in a wide spectrum of applications. The key idea in a point collocation approach is to satisfy the governing partial differential equation at each of the points covering the domain of interest. If a Dirichlet or a Neumann boundary condition is imposed on a point that is on the boundary, then an equation that satisfies the boundary condition is developed for the boundary point instead of satisfying the governing partial differential equation.

2). Galerkin method based on weak form [31]

The Galerkin method is a special case of the method of weighted residuals. The convergence rate of numerical results of the Galerkin method is better and more stable than that of the collocation method, but it suffer from extra computational

cost in forming stiffness matrix. Several integration schemes, such as nodal integration [50], cell or mesh integration [59], evaluation point approach [30], have been suggested by many researchers for the meshless methods.

2.2.2.3 Enforcement of essential boundary conditions

In many meshless methods, the approximation of field function is non-interpolation function, i.e., the field function does not have the Kronecker Delta property. Several approaches have been suggested to cope with the problem in the meshless methods are summarized as following:

1). Direct collocation method

Belytschko et al. [32] presented the element-free Galerkin method for dynamic crack propagation in which the essential boundary functions is imposed by using the nodal values of the trial function to enforce exactly only at the prescribed boundary points.

2). Lagrange multiplier method

Belytschko et al. [31] proposed an element-free Galerkin method, in which the Lagrange multiplier is introduced into the weak form of the governing equation.

3). Modified variational principle

Mukherjee et al. [60] discussed the boundary conditions for the EFGM, in which the essential boundary conditions were enforced through a modified variational principle, in which the stiffness matrix is positively definite and banded.

4). Penalty method

Zhu et al. [61] developed this method for the EFGM, in which the essential boundary conditions are imposed by introducing with penalty coefficient. It is simple and yields a symmetric positive stiffness matrix without adding extra unknowns which is a significant advantage in practical application.

5). Coupling with finite element

Hegen [62] provided the EFGM coupled with finite element approaches, in which

the problem domain is divided into the meshless part and the FEM part, which are connected with the help of a Lagrange multiplier. In the coupled meshless/FE approach, elements are placed around the essential boundary of the domain so that the essential boundary conditions are solved by the standard FEM.

6). Admissible approximation approach

Gosz and Liu [63] proposed two admissible meshless approximations, one in which the RKPM correction function equals zero at the boundary, and another in which the domain of the window function is selected such that the approximate vanishes at the boundary. This feature is particularly useful for generating approximations near boundaries that conform to the geometry of the boundary. Additional issues such as degeneration of shape functions from 2D to 1D and moment matrix conditioning are also addressed.

7). D'Alembert's principle

Günther and Liu [64] presented an algorithm based on the D'Alembert's principle that can be used for general constraints both in the meshless methods and the finite elements. In their scheme, n differential equations and m constraints are replaced by $n-m$ equations through choosing $n-m$ generalized variables, in which the constraints is orthogonalized by the Gram-Schmidt algorithm.

Recently, Gavete et al. [65] developed new implementation of essential boundary conditions for the EFGM in which the constraint variational principle with a penalty function is employed to solve 2D potential problems. Zhang et al. [66] proposed a displacement constraint equations method in which the essential boundary conditions are treated as a displacement constraint to the discrete equations obtained from Galerkin methods.

2.2.3 *A posteriori* error estimation and adaptivity of meshless methods

It is well known that the stresses (or derivatives) calculated in the FEM do not possess interelement continuity and have a low accuracy at points and element

boundaries. Many researchers proposed various continuous stress fields defined by shape functions and smoothed nodal parameters to obtain a more accurate distribution of stresses. Basically, the *a posteriori* error estimator of the FEM can be classified into two main groups:

1). Residual type error estimator

The residual type error estimators, first introduced by Babuska and Rheinboldt [67], are based on evaluating equilibrium residuals, inter-element traction jumps and surface traction equilibrium. In practical application, the residuals of the finite element solution can be computed implicitly or explicitly. The implicit residual error estimators are determined by solving local boundary value problems for the error, which use the residuals of the finite element solution as data. The explicit residual error estimations are, on the other hand, expressed directly by the residuals of the discretization solution. Later, Ainsworth and Oden [68] addressed the problem of computing *a posteriori* error estimates for the approximations to elliptic boundary value problems. In 1997, they [69] also presented a summary account of the subject of *a posteriori* error estimation for finite element approximations of problems in mechanics, such as the linear elliptic boundary value problems and nonlinear problems.

2). Recovery type error estimator.

The recovery type error estimators are computed by post-processing technique of the finite element solution such as the recovered gradient values from the original solution. In 1984, Babuska and Miller [70] discussed a method for post-processing a finite element solution to obtain higher accuracy approximations for displacements, stresses, etc. The post-processing technique was also applied to calculate generalized stress intensity factors near corner points [71] and deal with the questions of adaptive mesh selection and *a posteriori* error estimation [72]. In 1987, Zienkiewicz and Zhu [73] presented the famous Zienkiewicz-Zhu (Z-Z) error estimator. In their method, the error criterion is based on a projected stress which is obtained by using fitting approximation. The energy norm of the difference between the projected stress and the computed stress is then used as an estimate of the error.

The major step forward of Z-Z error estimation was in the development of the superconvergent patch recovery (SPR) methods [74].

Unlike the FEM, the meshless methods offer several different characters in the *a posteriori* error estimation and adaptivity:

1). Though it is very easy to add points to existed mesh without needing of any element structure, the insertion procedure is more complicated because simply inserting points between existing points could degrade the quality of the approximation solution. For example, in a one dimensional problem, if two points are simply added in a high gradient region, the reduction of error could be small due to the poor behavior of the approximation [75].

2). The shape function of the meshless methods can be easily constructed to be C^1 and thus the stress field is always continuous through the whole problem domain.

Chung and Belytschko [76] developed local and global error estimates for the Element-free Galerkin (EFG) method. The essence of this error estimator is to use the difference between the values of projected stress, which is obtained by using an approximation with a different domain of influence, and those obtained directly from the EFG solution.

Duarte and Oden [77] derived an explicit *a posteriori* error estimator by calculating the interior and boundary residuals for the *hp* cloud method. It makes use of the high regularity of the *hp* cloud approximations to avoid the calculation of flux jumps in the domain.

Liu and Jun [36] proposed an approximate measure to properly detect the high-scale response using the built-in feature of the RKPM multiresolution analysis. It utilized the reproducing kernel as the low pass filter for estimation of high gradient location.

Other approaches for the error estimation of meshless method which are based on local Taylor series expansion were suggested by Häussler-Combe and Korn [78]

and Gavete et al. [79]. Recently, Lee and Zhou [80, 81] carried out a comprehensive study on the theories and natures of some commonly used stress recovery schemes for the EFGM and concluded that these schemes are, in fact, particular cases of a general recovery scheme.

The adaptive computational methods allow modifications of the mesh, the order of the approximations or other features of numerical algorithms so as to reduce and control numerical error. Duarte and Oden [77] developed a new meshless method called *hp* clouds in which allows *h*, *p* and *hp* adaptivity in the *hp* clouds context with the same remarkable features of *hp* finite element methods but without the burden of a mesh. Liu and Jun [36] developed an adaptive algorithm in which the shape function is decomposed into a low scale (scaling function) component and high scale (wave function) component.

2.3 A brief review of coupled BEM and meshless methods

2.3.1 Background

Over years two kinds of approximate numerical techniques have been developed. The first kind method is the so-called domain type methods such as the finite difference method, the FEM and the meshless methods. The second one is the so-called boundary type methods such as the BEM. The coupling of different numerical methods has drawn attention due to the potential advantages associated with each of the methods being considered in the coupling approach. Belytschko et al. [82] developed procedure for coupled EFGM with FEM, in which the continuity and consistency are preserved on the interface elements. Krongauz et al. [83] also proposed a coupling technique for the EFGM in which the combined shape functions are constructed so that the functions are continuous and maintain the ability to reproduce the polynomial basis of the moving least-square approximant. Liu et al. [84] presented a multiple field procedure to enrich the FEM with the RKPM and the RKPM with analytical functions.

The domain type methods are capable of solving problems of any complexity with reasonable accuracy in a finite domain. The boundary type methods are capable of

solving simple problems in an infinite or semi-infinite domain. For some specific problems, the BEM is undoubtedly superior to the domain type methods. Many efforts have been concentrated on the coupled of BEM and FEM. Li et al [85] proposed a local finite element coupling approach for the coupling of sub-regions of elastic solids discretized by means of finite elements and boundary elements, respectively. Zeng et al. [86] developed a mixed variational principle for coupling finite element and boundary integral methods in interface problems, using the generalized Poisson's equation as a prototype situation. Belytschko et al. [87] presented a variationally coupled finite element-boundary element for transient problems. Zeng et al. [88] also provided a stable domain and boundary integral method for the solution of scattering and radiation problems of elastic structures submerged in an acoustic medium. Pullan [89] described a high-order (cubic Hermite) coupled finite element/boundary element procedure for solving electrocardiographic potential problems to be ultimately used for solving forward and inverse problems on an anatomically accurate human torso. Oysu et al. [90] introduced a coupled finite element and boundary element method for the analysis of two-dimensional elastic contact problems without friction, in which the Lagrange multiplier method is used to apply the contact constraints. Song et al. [91] developed a coupled finite element and boundary element method to predict the magnetic vector and scalar potential distributions in the droplets levitated in an alternating magnetic or electrostatic field. Yu et al. [92] presented a linear θ method to improve the stability of the time domain BEM/FEM coupling formulation.

2.3.2 Overview of coupled BEM and meshless methods

With the advent of the meshless methods, it is desirable and beneficial to combine the BEM with the meshless methods together for certain problems, such as large deformation and growth of cracks with arbitrary paths. Moreover, since the numerical integration of the meshless methods involved in the calculating the stiffness matrix is computationally very expensive especially for problems with infinite or semi-infinite domains, the computation cost of the meshless methods can be reduced by coupling the meshless methods with the BEM, i.e., the domain size of the meshless methods is decreased. For instance, the meshless methods can be

used only in sub-domains, such as the areas with crack growth, and the BEM is used in the remaining non-singular part of the domain, such as the infinite or semi-infinite areas.

Since the shape functions of the meshless methods lack the Kronecker Delta function property, the major difficulty of coupling of the BEM and the meshless methods is how to satisfy the displacement and traction compatibility condition along the interface between the domains of the two methods.

In 2001, Gu and Liu [93] presented the coupled EFG/BE methods in which an interface element is introduced to keep the displacement compatible between the EFG domain and the BEM domain. In 2000, Liu and Gu [94] developed the coupled EFG/HBE (hybrid displacement boundary element) for continuum mechanics problems in which the compatibility condition on the interface boundary is introduced into the variational formulations of the EFG and the HBE using Lagrange multipliers. In 2003, Li, Paulino and Aluru [95] proposed the coupling of the mesh-free cloud method with the boundary element method in which the finite cloud method (FCM) and the BEM share the same set of points at common boundaries.

2.3.3 Adaptivity of coupled BEM and meshless methods

The adaptivity is also very important for the coupled method to achieve a better numerical solution. Liu et al. [84] proposed the wavelet based adaptivity approach to refine the grid in which the RKPM p -enrichment and hp -enrichment is introduced to the FEM. In their scheme, the coupling between the FEM and the RKPM are performed through multiple field formulation as well as by imposing the completeness or reproducing conditions and the multiresolution concept is used to establish criteria in determining critical regions for local refinement.

Karutz et al. [96] presented an adaptive algorithm for the coupling of the finite element and element free Galerkin discretization. In their method, the problem domain is first discretized by the FEM and then detected the sensible zone for a

meshfree discretization; the adaption of meshfree EFG-discretization zones is finally included into the FE-discretized structure.

Carstensen et al. [97] developed the adaptive h -versions of the symmetric FEM/BEM coupling for linear and nonlinear interface problems, in which an adaptive feedback algorithm is used for the mesh refinement of the coupling procedure. Their algorithm is based on an *a posteriori* error estimate which gives a computable error estimate up to a multiplicative constant.

CHAPTER 3

THE MODELLED PROBLEM AND BASIC FORMULATIONS FOR BEM AND RKPM

In the first section of this chapter, the two dimensional linear elastostatic problem is introduced as the modelled problem. The basic formulations of the BEM and the RKPM for the modelled problem are then discussed in Sections 3.2 and 3.3, respectively. It should be highlighted here that the modelled problem and all the formulations presented in this chapter will also be adopted in the rest of this study.

3.1 Two dimensional linear elastostatic problem

Figure 3.1 Two dimensional problem

Considering the two dimensional linear elastostatic stress analysis problem in the domain Ω bounded by $\Gamma = \Gamma_u \cup \Gamma_t$ (as shown in Figure 3.1), the corresponding governing differential equation is represented as

$$\mathbf{L}^T \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \text{ in } \Omega \quad (3.1a)$$

$$\boldsymbol{\sigma} = \mathbf{D} \boldsymbol{\varepsilon} \quad (3.1b)$$

$$\boldsymbol{\varepsilon} = \mathbf{L} \mathbf{u} \quad (3.1c)$$

where,

$$\mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \quad (3.2a)$$

$$\mathbf{D} = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \text{ (for plane stress problems)} \quad (3.2b)$$

In Eqn. 3.1, $\boldsymbol{\sigma} = \{\sigma_x, \sigma_y, \tau_{xy}\}^T = \{\sigma_{11}, \sigma_{22}, \sigma_{12}\}^T$ is the stress tensor; $\mathbf{b} = \{b_x, b_y\}^T = \{b_1, b_2\}^T$ is the body force vector; $\boldsymbol{\varepsilon}$ is the strain vector; $\mathbf{u} = \{u_x, u_y\}^T = \{u_1, u_2\}^T$ is the displacement vector; $\mathbf{D}$ is the strain-stress matrix, E is the Young's module and μ is the Poisson ratio.

On the boundary Γ , the relation between the surface traction $\mathbf{t}$ and the stress $\boldsymbol{\sigma}$ can be written by as

$$\mathbf{t} = \boldsymbol{\sigma} \mathbf{n} = \mathbf{Q} \boldsymbol{\sigma} \quad \text{on } \Gamma \quad (3.2c)$$

in Eqn. 3.2c, $\mathbf{Q}$ is a linear operator which relates $\boldsymbol{\sigma}$ and $\mathbf{t}$, which is expressed as

$$\mathbf{Q} = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \quad (3.2d)$$

where, $\mathbf{n}$ is the unit outward normal to the boundary of the domain Ω . θ is the angle between the outer normal line $\mathbf{n}$ and the x axis.

The corresponding boundary conditions are given as following:

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u \text{ (essential boundary)} \quad (3.3a)$$

$$\boldsymbol{\sigma} \mathbf{n} = \mathbf{Q} \boldsymbol{\sigma} = \mathbf{Q} \mathbf{D} \mathbf{L} \mathbf{u} = \bar{\mathbf{t}} \quad \text{on } \Gamma_t \text{ (traction boundary)} \quad (3.3b)$$

in Eqn. 3.3, $\mathbf{u}$ is the global displacement (a vector function) in the problem domain Ω . $\bar{\mathbf{u}}$ and $\bar{\mathbf{t}}$ are the prescribed boundary displacement and prescribed boundary traction along the boundaries Γ_u and Γ_t respectively such that $\Gamma = \Gamma_u \cup \Gamma_t$ and $\Gamma_u \cap \Gamma_t = \emptyset$. In this moment, it is assumed that $\bar{\mathbf{u}}$ and $\bar{\mathbf{t}}$ are sufficiently smooth functions.

3.2 The boundary element method

Brebbia and Dominguez [13] presented their book on the BEM, in which the formulations of BEM for 2D and 3D elastostatic problem is discussed with details. In their book, the basic formulations for the modelled problem are developed and adopted in this study. The related details are summarized as following:

The principle of virtual displacements for the linear elastic material of the modelled problem can be written as

$$\int_{\Omega} \mathbf{u}^* (\mathbf{L}^T \boldsymbol{\sigma} + \mathbf{b}) d\Omega = \int_{\Gamma_u} \mathbf{t}^* (\mathbf{u} - \bar{\mathbf{u}}) d\Gamma - \int_{\Gamma_t} \mathbf{u}^* (\mathbf{t} - \bar{\mathbf{t}}) d\Gamma \quad (3.4)$$

where, $\mathbf{u}^*$ is the virtual displacement and $\mathbf{t}^*$ is the virtual surface traction corresponding to $\mathbf{u}^*$. The first term of left side in Eqn. 3.4 can then be integrated by parts and the following can be arrived as

$$\int_{\Omega} \mathbf{u}^* \mathbf{b} d\Omega + \int_{\Omega} (\mathbf{L}^T \boldsymbol{\sigma}^*) \mathbf{u} d\Omega = \int_{\Gamma_u} (\mathbf{u}^* \mathbf{t} - \mathbf{t}^* \bar{\mathbf{u}}) d\Gamma - \int_{\Gamma_t} (\mathbf{t}^* \mathbf{u} - \mathbf{u}^* \bar{\mathbf{t}}) d\Gamma \quad (3.5)$$

in Eqn. 3.5, $\boldsymbol{\sigma}^*$ can be taken as the fundamental solution of the following equation.

$$\mathbf{L} \boldsymbol{\sigma}^* + \delta^i = \mathbf{0} \quad (3.6)$$

where, δ^i is the Dirac Delta function.

Then the following boundary integral equation (BIE) can be obtained by substituting Eqn. 3.6 into Eqn. 3.5

$$\mathbf{c}^i \mathbf{u}^i + \int_{\Gamma} \mathbf{t}^* \mathbf{u} d\Gamma = \int_{\Gamma} \mathbf{u}^* \mathbf{t} d\Gamma + \int_{\Omega} \mathbf{u}^* \mathbf{b} d\Omega \quad (3.7)$$

where, $\mathbf{c}^i$ is the coefficient and its detailed expression can be found in Appendix A.

In the numerical computation process, the boundary Γ needs to be discretized into NN base points and NE elements, i.e., $\Gamma = \sum_{Q=1}^{NE} \Gamma_Q$, Γ_Q is the boundary on which the element Q is located (shown in Figure 3.2). Within the elements, the values of $\mathbf{u}$ and $\mathbf{t}$ are interpolated as

$$\hat{\mathbf{u}} = (\mathbf{N}^B)^T \hat{\mathbf{u}}_e \quad (3.8a)$$

$$\hat{\mathbf{t}} = (\mathbf{N}^B)^T \hat{\mathbf{t}}_e \quad (3.8b)$$

where, $\mathbf{N}^B$ is the global interpolation function for $\hat{\mathbf{u}}$ and $\hat{\mathbf{t}}$ which can be calculated from the element interpolation functions φ_1 and φ_2 defined in Appendix A. $\hat{\mathbf{u}}_e$ and $\hat{\mathbf{t}}_e$ are the vectors of the nodal values of $\hat{\mathbf{u}}$ and $\hat{\mathbf{t}}$ at the base points, respectively.

Figure 3.2 Points and elements arrangement of the BEM

Substituting Eqn.3.8 into Eqn. 3.7, the discrete boundary integration equation can be obtained as following:

$$\mathbf{H} \hat{\mathbf{u}}_e = \mathbf{G} \hat{\mathbf{t}}_e + \mathbf{F} \quad (3.9)$$

where,

$$\hat{\mathbf{u}}_e = \{\hat{u}_1^1, \hat{u}_1^2, \hat{u}_2^1, \hat{u}_2^2, \dots, \hat{u}_{NN}^1, \hat{u}_{NN}^2\}^T, \quad \hat{\mathbf{t}}_e = \{\hat{t}_1^1, \hat{t}_1^2, \hat{t}_2^1, \hat{t}_2^2, \dots, \hat{t}_{NN}^1, \hat{t}_{NN}^2\}^T \quad (3.10)$$

and

$$\mathbf{H} = \mathbf{c} + \int_{\Gamma} (\mathbf{N}^B)^T \mathbf{t}^* d\Gamma = [H_{lk}^{ij}] \quad (3.11a)$$

$$H_{lk}^{ij} = c_{lk}^i + \sum_t h_{lk}^{i,t,m} \quad (3.11b)$$

where, $\mathbf{c}$ is the matrix associated with coefficient $\mathbf{c}^i$, $(l, k, m) = 1, 2$ and $(i, j) = 1, 2, \dots, NN$.

When $i = j$, the term H_{lk}^{ij} will be singular and it can be computed by the rigid displacement method [98] as

$$H_{lk}^{ii} = -\sum_{j=1}^{NN} H_{lk}^{ij} \quad i \neq j, \text{ if the boundary is a finite boundary} \quad (3.11c)$$

$$H_{lk}^{ii} = \mathbf{I} - \sum_{j=1}^{NN} H_{lk}^{ij} \quad i \neq j, \text{ if the boundary is an infinite boundary} \quad (3.11d)$$

The matrix $\mathbf{G}$ in Eqn. 3.9 is defined as

$$\mathbf{G} = \int_{\Gamma} (\mathbf{N}^B)^T \mathbf{u}^* d\Gamma = [G_{lk}^{ij}] \quad (3.12a)$$

$$G_{lk}^{ij} = \sum_t g_{lk}^{i,t,m} \quad (3.12b)$$

where, $(l, k, m) = 1, 2$ and $(i, j) = 1, 2, \dots, NN$.

The term $\mathbf{F}$ in Eqn. 3.9 is the contribution due to the body force and is given by

$$\mathbf{F} = \int_{\Omega} \mathbf{u}^* \mathbf{b} d\Omega = \{F_1^1, F_2^1, F_1^2, F_2^2, \dots, F_1^{NN}, F_2^{NN}\}^T \quad (3.13a)$$

$$\begin{Bmatrix} F_1^i \\ F_2^i \end{Bmatrix} = \begin{Bmatrix} \sum_{Q=1}^{NE} \int_{\Gamma_Q} f_1^* d\Gamma \\ \sum_{Q=1}^{NE} \int_{\Gamma_Q} f_2^* d\Gamma \end{Bmatrix} \quad (3.13b)$$

In the above equations, it is noted that the symbol $\sum_t$ implies the summation

extend to all the elements, $\mathbf{I}$ is the identity matrix, $h_{lk}^{i,t,m}$ and $g_{lk}^{i,t,m}$ are the element matrixes, f_1^* and f_2^* are the constant term of the i th point associated to body force.

For the detailed expressions of $h_{lk}^{i,t,m}$, $g_{lk}^{i,t,m}$, f_1^* and f_2^* , please kindly refer to Appendix A.

3.3 The reproducing kernel particle method

The RKPM is one of the most important meshless methods, which was first proposed by Liu et al. [35] in 1995. Aluru [44] also developed a point collocation method based on reproducing kernel approximations, in which the RKPM formulations for 1D and 2D problem are presented and the corresponding convergence rate is also addressed. In this study, the basic formulations used for solving the modelled problem are summarized as below.

3.3.1 Construction of displacement field function

The RKPM is similar to the SPH method in the sense that it begins the formulation from the kernel estimate of the exact solution. However, the RKPM modifies the kernel function by introducing a correction function in order to enhance its accuracy near the boundaries of the problem domain. Due to the correction function, the kernel function of the RKPM attains the consistency conditions over all the domain of the problem.

In two dimensions, the corrected kernel approximation to field displacement function $\mathbf{u}(x,y)$ can be written as

$$\hat{\mathbf{u}}(x,y) = \int_{\Omega} \bar{w}_d(x-s,y-t) \mathbf{u}(s,t) dsdt \quad (3.14)$$

where, $\bar{w}_d(x-s,y-t)$ is the corrected kernel function which is given by

$$\bar{w}_d(x-s,y-t) = C(x,y,s,t) w_d(x-s,y-t) \quad (3.15)$$

in Eqn. 3.15, $C(x,y,s,t)$ is the correction function [35]. If $C(x,y,s,t)$ is taken to be unity, i.e., $C(x,y,s,t) = 1$, the approximation function of the RKPM will reduce to that of the classical SPH method [28]. In this study, the correction function is typically expressed as a linear combination of polynomial basis function up to second order [44]. $w_d(x-s,y-t)$ is the kernel function and the two-dimensional form of it can be expressed as

$$w_d(x-s,y-t) = \frac{1}{d_x} w\left(\frac{x-s}{d_x}\right) \frac{1}{d_y} w\left(\frac{y-t}{d_y}\right) \quad (3.16)$$

in Eqn. 3.16, d_x , d_y are the dilation parameters along the x and y directions, respectively. The kernel function can be taken as the wavelet functions or the spline functions and the following cubic B -Spline function is adopted in this study

$$w(z) = \begin{cases} 0, & z < -1 \\ \frac{4}{3} + 4z + 4z^2 + \frac{4}{3}z^3, & -1 \leq z < -\frac{1}{2} \\ \frac{2}{3} - 4z^2 - 4z^3, & -\frac{1}{2} \leq z < 0 \\ \frac{2}{3} - 4z^2 + 4z, & 0 \leq z < \frac{1}{2} \\ \frac{4}{3} - 4z + 4z^2 - \frac{4}{3}z^3, & \frac{1}{2} \leq z \leq 1 \\ 0, & z > 1 \end{cases} \quad (3.17)$$

In practical application, the approximation function needs to be discretized in the problem domain and the corresponding discrete form for 2D problems can be written as

$$\begin{aligned} \hat{\mathbf{u}}(x,y) &= \sum_{i=1}^{NP} \bar{w}_d(x-x_i, y-y_i) \hat{\mathbf{u}}(x_i, y_i) \Delta V_i \\ &= \sum_{i=1}^{NP} N_i^R(x,y) \hat{\mathbf{u}}_i = (\mathbf{N}^R)^T \hat{\mathbf{u}}_e \end{aligned} \quad (3.18)$$

where, $\{x_i, y_i\}$ are the coordinates of the i th point in the problem domain, NP is the number of computing points, $\bar{w}_d(x-x_i, y-y_i) = C(x,y,x_i,y_i) w_d(x-x_i, y-y_i)$ is the discrete form of corrected kernel function associated with the i th point, $\hat{\mathbf{u}}(x_i, y_i)$ and ΔV_i are the nodal value and the volume (or area in 2D problems) associated with the i th point, respectively. $N_i^R(x,y)$ is the shape function for the i th point, which can be expressed as

$$N_i^R(x,y) = C(x,y,x_i,y_i) w_d(x-x_i, y-y_i) \Delta V_i \quad (3.19)$$

$\mathbf{N}^R$ is the matrix of the shape function $N_i^R(x,y)$ and $\hat{\mathbf{u}}_e$ is the vector of nodal displacement value.

3.3.2 Formulations for RKPM

For the 2D linear elastostatic problem defined in Eqn. 3.1, the essential boundary condition involved in Eqn. 3.3 can then be included by introducing the penalty multiplier method into the weak form based on the amended variational principles [61], i.e., to find $\mathbf{u} \in (H^1(\Omega))$ such that

$$\Pi = \frac{1}{2} \int_{\Omega} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\Omega - \int_{\Omega} \mathbf{u}^T \mathbf{b} d\Omega - \int_{\Gamma_t} \mathbf{u}^T \bar{\mathbf{t}} d\Gamma - \frac{\alpha}{2} \int_{\Gamma_u} (\mathbf{u} - \bar{\mathbf{u}})^T (\mathbf{u} - \bar{\mathbf{u}}) d\Gamma \quad (3.20)$$

is stationary.

where, $H^i(\Omega)$ denotes the Sobolev space of order i , the scalar $\alpha \gg 1$ [61] is the penalty parameter which is used to enforce the essential boundary condition and is set as $\alpha = (10^3 \sim 10^9)E$ [61] in this study.

Setting $\mathbf{u} \approx \hat{\mathbf{u}}$ in Eqn. 3.20 and using the stationary condition $\partial \Pi = 0$, the final matrix form of governing equation can be arrived as

$$\mathbf{K} \hat{\mathbf{u}}_e = \mathbf{f} \quad (3.21)$$

where,

$$\begin{aligned} \mathbf{K}_{ij} &= \int_{\Omega} \mathbf{B}_i^T \mathbf{D} \mathbf{B}_j d\Omega + \alpha \int_{\Gamma_u} N_i^R \mathbf{S} N_j^R d\Gamma \\ &= \int_{\Omega} \frac{E}{1-\mu^2} \begin{bmatrix} N_{i,x}^R & 0 & N_{i,y}^R \\ 0 & N_{i,y}^R & N_{i,x}^R \end{bmatrix} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{bmatrix} N_{j,x}^R & 0 \\ 0 & N_{j,y}^R \\ N_{j,y}^R & N_{j,x}^R \end{bmatrix} d\Omega + \\ &\quad \alpha \int_{\Gamma_u} \begin{bmatrix} N_i^R & 0 \\ 0 & N_i^R \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} N_j^R & 0 \\ 0 & N_j^R \end{bmatrix} d\Gamma \end{aligned} \quad (3.22a)$$

and

$$\begin{aligned} \mathbf{f}_i &= \int_{\Omega} N_i^R \mathbf{b} d\Omega + \int_{\Gamma_t} N_i^R \bar{\mathbf{t}} d\Gamma + \alpha \int_{\Gamma_u} N_i^R \mathbf{S} \bar{\mathbf{u}} d\Gamma \\ &= \int_{\Omega} \begin{bmatrix} N_i^R & 0 \\ 0 & N_i^R \end{bmatrix} \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} d\Omega + \int_{\Gamma_t} \begin{bmatrix} N_i^R & 0 \\ 0 & N_i^R \end{bmatrix} \begin{Bmatrix} \bar{t}_x \\ \bar{t}_y \end{Bmatrix} d\Gamma + \\ &\quad \alpha \int_{\Gamma_u} \begin{bmatrix} N_i^R & 0 \\ 0 & N_i^R \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{Bmatrix} \bar{u}_x \\ \bar{u}_y \end{Bmatrix} d\Gamma \end{aligned} \quad (3.22b)$$

in Eqn. 3.22, $(i, j) = 1, 2, \dots, NP$, and

$$\mathbf{B}_i = \begin{bmatrix} N_{i,x}^R & 0 \\ 0 & N_{i,y}^R \\ N_{i,y}^R & N_{i,x}^R \end{bmatrix} \quad (3.23a)$$

$$\mathbf{S} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \text{ and } S_k = \begin{cases} 1 & \text{if } u_i \text{ is prescribed on } \Gamma_u, k = 1, 2. \\ 0 & \text{if } u_i \text{ is not prescribed on } \Gamma_u, k = 1, 2. \end{cases} \quad (3.23b)$$

In order to obtain the integrals in Eqns. 3.22 and 3.23, the Gaussian integration rule based on the background mesh is used in this study. For the details of the integration scheme, please kindly refer to Appendix B.

When the nodal displacement values $\hat{\mathbf{u}}_e$ are available, the approximated displacements $\hat{\mathbf{u}} = \{\hat{u}_x, \hat{u}_y\}^T = \{\hat{u}_1, \hat{u}_2\}^T$ and the approximated stresses $\hat{\boldsymbol{\sigma}} = \{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\tau}_{xy}\}^T = \{\hat{\sigma}_{11}, \hat{\sigma}_{22}, \hat{\sigma}_{12}\}^T$ of any point $\{x, y\}$ in the problem domain Ω can then be easily computed as

$$\hat{\mathbf{u}} = (\mathbf{N}^R)^T \hat{\mathbf{u}}_e = \sum_{i=1}^{NP} \begin{bmatrix} N_i^R & 0 \\ 0 & N_i^R \end{bmatrix} \begin{Bmatrix} \hat{u}_e^i(x) \\ \hat{u}_e^i(y) \end{Bmatrix} \quad (3.24a)$$

$$\hat{\boldsymbol{\sigma}} = \mathbf{D} \mathbf{L} \hat{\mathbf{u}} = \mathbf{D} \mathbf{B} \hat{\mathbf{u}}_e = \sum_{i=1}^{NP} \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{bmatrix} N_{i,x}^R & 0 \\ 0 & N_{i,y}^R \\ N_{i,y}^R & N_{i,x}^R \end{bmatrix} \begin{Bmatrix} \hat{u}_e^i(x) \\ \hat{u}_e^i(y) \end{Bmatrix} \quad (3.24b)$$

where, $\hat{u}_e^i(x)$ and $\hat{u}_e^i(y)$ are the nodal values along x and y direction of the RKPM computing i th point, respectively.

CHAPTER 4

THE COUPLED BE-RKPM

The main objective of this Chapter is to suggest a new procedure for the direct coupling of the BEM and RKPM for the 2D elastostatic problem. A penalty method is used here to enforce the displacement compatibility condition along the interface boundary between the BEM and RKPM domains. By using the present method, it is neither necessary to introduce any interface elements nor to modify the formulations of the BEM and the RKPM. The *a priori* error estimation for the suggested coupled BE-RKPM is also discussed in this chapter. In the final part of this chapter, numerical experiments are done to verify the efficiency of the coupled BE-RKPM.

4.1 Basic formulation for the coupled BE-RKPM

Figure 4.1 The domain of the coupled BE-RKPM

Considering the modelled problem defined in Eqns 3.1-3.3 (Section 3.1) over domain $\Omega = \Omega^R \cup \Omega^B$ (as shown in Figure 4.1), The BEM and the RKPM are employed in Ω^B and Ω^R , respectively. Γ^{BR} (Γ^{RB}) is the interface boundary between Ω^B and Ω^R . $\mathbf{u}^{RB}$ ($\mathbf{t}^{RB}$) and $\mathbf{u}^{BR}$ ($\mathbf{t}^{BR}$) are the displacements (tractions) along the interface with respect to Ω^R and Ω^B , respectively.

The boundary conditions are given by

For the BEM in Ω^B , $\Gamma^B = \Gamma_u^B \cup \Gamma_t^B \cup \Gamma^{BR}$:

$$\text{Essential boundary} \quad \mathbf{u} = \bar{\mathbf{u}}^B \quad \text{on } \Gamma_u^B \quad (4.1a)$$

$$\text{Natural boundary} \quad \mathbf{t} = \bar{\mathbf{t}}^B \quad \text{on } \Gamma_t^B \quad (4.1b)$$

The boundary conditions on the interface boundary Γ^{BR} for Ω^B are listed as

$$\text{Interface boundary} \quad \mathbf{u} = \mathbf{u}^{BR} = \mathbf{u}^{RB} \quad \text{on } \Gamma^{BR} \quad (4.1c)$$

$$\text{Interface boundary} \quad \mathbf{t} = \mathbf{t}^{BR} = -\mathbf{t}^{RB} \quad \text{on } \Gamma^{BR} \quad (4.1d)$$

For the RKPM in Ω^R , $\Gamma^R = \Gamma_u^R \cup \Gamma_t^R \cup \Gamma^{RB}$:

$$\text{Essential boundary} \quad \mathbf{u} = \bar{\mathbf{u}}^R \quad \text{on } \Gamma_u^R \quad (4.2a)$$

$$\text{Natural boundary} \quad \mathbf{t} = \bar{\mathbf{t}}^R \quad \text{on } \Gamma_t^R \quad (4.2b)$$

The boundary condition on the interface boundary Γ^{RB} for Ω^R is expressed as

$$\text{Interface boundary} \quad \mathbf{u} = \mathbf{u}^{RB} \quad \text{on } \Gamma^{RB} \quad (4.2c)$$

$$\text{Interface boundary} \quad \mathbf{t} = \mathbf{t}^{RB} \quad \text{on } \Gamma^{RB} \quad (4.2d)$$

4.1.1 Solution by the BEM in Ω^B

Following the BEM formulation in Section 3.2, the boundary integral equation for Ω^B can be written as

$$\mathbf{c}^i \mathbf{u}^i + \int_{\Gamma^B + \Gamma^{BR}} \mathbf{t}^* \mathbf{u} d\Gamma = \int_{\Gamma^B + \Gamma^{BR}} \mathbf{u}^* \mathbf{t} d\Gamma + \int_{\Omega^B} \mathbf{u}^* \mathbf{b} d\Omega, \text{ sum over } i, i=1,2. \quad (4.3)$$

Substituting the boundary condition Eqn. 4.1 into Eqn. 4.3, the following equation can be arrived as

$$\begin{aligned} & \mathbf{c}^i \mathbf{u}^i + \int_{\Gamma_u^B} \mathbf{t}^* \bar{\mathbf{u}}^B d\Gamma + \int_{\Gamma_t^B} \mathbf{t}^* \mathbf{u}_{\Gamma_t^B}^B d\Gamma + \int_{\Gamma^{BR}} \mathbf{t}^* \mathbf{u}^{RB} d\Gamma \\ &= \int_{\Gamma_u^B} \mathbf{u}^* \mathbf{t}_{\Gamma_u^B}^B d\Gamma + \int_{\Gamma_t^B} \mathbf{u}^* \bar{\mathbf{t}}^B d\Gamma + \int_{\Gamma^{BR}} \mathbf{u}^* (-\mathbf{t}^{RB}) d\Gamma + \int_{\Omega^B} \mathbf{u}^* \mathbf{b} d\Gamma \end{aligned} \quad (4.4)$$

in Eqn. 4.4, $\mathbf{u}_{\Gamma_t^B}^B$ is the unknown displacement on the boundary Γ_t^B and $\mathbf{t}_{\Gamma_u^B}^B$ is the unknown traction on the boundary Γ_u^B , summing over $i, i=1,2$.

In practical computation, it is assumed that $\bar{\mathbf{u}}^B$, $\bar{\mathbf{t}}^B$, $\mathbf{u}_{\Gamma_t^B}^B$, $\mathbf{t}_{\Gamma_u^B}^B$, $\mathbf{u}^{RB}$ and $\mathbf{t}^{RB}$ are interpolated in the similar manner as (see Appendix A for details)

$$\bar{\mathbf{u}}^B = (\mathbf{N}^B)^T \bar{\mathbf{u}}_e^B \quad (4.5a)$$

$$\bar{\mathbf{t}}^B = (\mathbf{N}^B)^T \bar{\mathbf{t}}_e^B \quad (4.5b)$$

$$\mathbf{u}_{\Gamma_t^B}^B = (\mathbf{N}^B)^T \hat{\mathbf{u}}_{\Gamma_t^B(e)}^B \quad (4.5c)$$

$$\mathbf{t}_{\Gamma_u^B}^B = (\mathbf{N}^B)^T \hat{\mathbf{t}}_{\Gamma_u^B(e)}^B \quad (4.5d)$$

$$\mathbf{u}^{RB} = (\mathbf{N}^B)^T \hat{\mathbf{u}}_e^{RB} \quad (4.5e)$$

$$\mathbf{t}^{RB} = (\mathbf{N}^B)^T \hat{\mathbf{t}}_e^{RB} \quad (4.5f)$$

in Eqn. 4.5, $\mathbf{N}^B$ is the element interpolation function for the displacement and traction of BEM, $\bar{\mathbf{u}}_e^B$ is the known nodal displacement vector on Γ_u^B , $\bar{\mathbf{t}}_e^B$ is the

known nodal traction vector on Γ_t^B , $\hat{\mathbf{u}}_{\Gamma_t^B(e)}^B$ is the unknown nodal displacement vector Γ_t^B , $\hat{\mathbf{t}}_{\Gamma_u^B(e)}^B$ is the unknown nodal traction vector on Γ_u^B , $\hat{\mathbf{u}}_e^{RB}$ and $\hat{\mathbf{t}}_e^{RB}$ are the unknown nodal displacement vector and nodal traction vector on Γ^{RB} (Γ^{RB}), respectively.

Substituting Eqn.4.5 into Eqn. 4.4, the linear equations system for Ω^B can be arrived as

$$\mathbf{H}_{\Gamma_u^B}^B \bar{\mathbf{u}}_e^B + \mathbf{H}_{\Gamma_t^B}^B \hat{\mathbf{u}}_{\Gamma_t^B(e)}^B + \mathbf{H}^{RB} \hat{\mathbf{u}}_e^{RB} = \mathbf{G}_{\Gamma_u^B}^B \hat{\mathbf{t}}_{\Gamma_u^B(e)}^B + \mathbf{G}_{\Gamma_t^B}^B \bar{\mathbf{t}}_e^B - \mathbf{G}^{RB} \hat{\mathbf{t}}_e^{RB} + \mathbf{F}^B \quad (4.6)$$

where,

$$\mathbf{H}_{\Gamma_u^B}^B = \mathbf{c} + \int_{\Gamma_u^B} \mathbf{t}^* (\mathbf{N}^B)^T d\Gamma \quad (4.7a)$$

$$\mathbf{H}_{\Gamma_t^B}^B = \mathbf{c} + \int_{\Gamma_t^B} \mathbf{t}^* (\mathbf{N}^B)^T d\Gamma \quad (4.7b)$$

$$\mathbf{H}^{RB} = \mathbf{c} + \int_{\Gamma^{RB}} \mathbf{t}^* (\mathbf{N}^B)^T d\Gamma \quad (4.7c)$$

$$\mathbf{G}_{\Gamma_u^B}^B = \int_{\Gamma_u^B} \mathbf{u}^* (\mathbf{N}^B)^T d\Gamma \quad (4.7d)$$

$$\mathbf{G}_{\Gamma_t^B}^B = \int_{\Gamma_t^B} \mathbf{u}^* (\mathbf{N}^B)^T d\Gamma \quad (4.7e)$$

$$\mathbf{G}^{RB} = \int_{\Gamma^{RB}} \mathbf{u}^* (\mathbf{N}^B)^T d\Gamma \quad (4.7f)$$

$$\mathbf{F}^B = \int_{\Omega^B} \mathbf{u}^* \mathbf{b} d\Omega \quad (4.7g)$$

4.1.2 Solution by the RKPM in Ω^R

Similar to the RKPM process described in Section 3.3, the approximation function $\hat{\mathbf{u}}^R(x,y)$ in Ω^R can be introduced as

$$\hat{\mathbf{u}}^R(x,y) = \sum_{i=1}^{NP} N_i^R(x,y) \hat{\mathbf{u}}_i^R = (\mathbf{N}^R)^T \hat{\mathbf{u}}_e^R \quad (4.8)$$

Then the displacement $\mathbf{u}^{\text{RB}}$ along boundary Γ^{RB} can also be expressed in the general RKPM form as

$$\mathbf{u}^{\text{RB}} = (\mathbf{N}^{\text{R}})^T \hat{\mathbf{u}}_e^{\text{R}} \quad (4.9)$$

From Eqn. 4.9, it can be seen that $\mathbf{u}^{\text{RB}}$ will depend not only on the RKPM points on the interface boundary Γ^{RB} but also on a set of RKPM points near the interface boundary.

Recalling Eqns. 4.5e and 4.9 for the approximation of the displacement $\mathbf{u}^{\text{RB}}$, two different interpolation formulations are employed, i.e., the displacement along the interface boundary Γ^{RB} is not fully compatible between Ω^{B} and Ω^{R} . The interface boundary Γ^{RB} is assumed to be an essential boundary for RKPM and then a penalty formulation is employed here to enforce the displacement compatible condition $\mathbf{u}^{\text{BR}} = \mathbf{u}^{\text{RB}}$, which is introduced as

$$\frac{\alpha}{2} \int_{\Gamma^{\text{RB}}} (\hat{\mathbf{u}}^{\text{R}} - (\mathbf{N}^{\text{B}})^T \hat{\mathbf{u}}_e^{\text{RB}})^T (\hat{\mathbf{u}}^{\text{R}} - (\mathbf{N}^{\text{B}})^T \hat{\mathbf{u}}_e^{\text{RB}}) d\Gamma \quad (4.10)$$

It can be further expect that as the computation model is refined, the integral defined in Eqn. 4.10 will converge to zero.

Based on the above assumption, the amended variational equation for Ω^{R} can be given by

$$\begin{aligned} \Pi = & \frac{1}{2} \int_{\Omega^{\text{R}}} (\hat{\boldsymbol{\varepsilon}}^{\text{R}})^T \hat{\boldsymbol{\sigma}}^{\text{R}} d\Omega - \int_{\Omega^{\text{R}}} (\hat{\mathbf{u}}^{\text{R}})^T \mathbf{b} d\Omega - \int_{\Gamma_{\text{t}}^{\text{R}}} (\hat{\mathbf{u}}^{\text{R}})^T \bar{\mathbf{t}}^{\text{R}} d\Gamma \\ & + \frac{\alpha}{2} \int_{\Gamma_{\text{u}}^{\text{R}}} (\hat{\mathbf{u}}^{\text{R}} - \bar{\mathbf{u}}^{\text{R}})^T (\hat{\mathbf{u}}^{\text{R}} - \bar{\mathbf{u}}^{\text{R}}) d\Gamma + \frac{\alpha}{2} \int_{\Gamma^{\text{RB}}} (\hat{\mathbf{u}}^{\text{R}} - (\mathbf{N}^{\text{B}})^T \hat{\mathbf{u}}_e^{\text{RB}})^T (\hat{\mathbf{u}}^{\text{R}} - (\mathbf{N}^{\text{B}})^T \hat{\mathbf{u}}_e^{\text{RB}}) d\Gamma \end{aligned} \quad (4.11)$$

Using the stationary condition in Eqn. 4.11, the final linear equations system for Ω^{R} can be obtained as

$$\mathbf{K}^{\text{R}} \hat{\mathbf{u}}_e^{\text{R}} = \mathbf{f}^{\text{R}} + \mathbf{f}^{\text{RB}} \quad (4.12)$$

where,

$$\mathbf{K}_{ij}^R = \int_{\Omega^R} \mathbf{B}_i^T \mathbf{D} \mathbf{B}_j d\Omega + \alpha \int_{\Gamma_u^R + \Gamma^{RB}} N_i^R \mathbf{S} N_j^R d\Gamma \quad (4.13a)$$

$$\mathbf{f}_i^R = \int_{\Omega^R} N_i^R \mathbf{b} d\Omega + \int_{\Gamma_t^R} N_i^R \bar{\mathbf{t}}^R d\Gamma + \alpha \int_{\Gamma_u^R} N_i^R \mathbf{S} \bar{\mathbf{u}}^R d\Gamma \quad (4.13b)$$

$$\mathbf{f}_i^{RB} = \alpha \int_{\Gamma^{RB}} N_i^R \mathbf{S} (\mathbf{N}^B)^T \hat{\mathbf{u}}_e^{RB} d\Gamma \quad (4.13c)$$

4.1.3 The coupled BE-RKPM

Based on the analyses of the BEM part and the RKPM part, the final linear equations system for the coupled BE-RKPM is summarized as

In Ω^B ,

$$\mathbf{H}_{\Gamma_t^B}^B \hat{\mathbf{u}}_{\Gamma_t^B(e)}^B + \mathbf{H}_e^{RB} \hat{\mathbf{u}}_e^{RB} - \mathbf{G}_{\Gamma_u^B}^B \hat{\mathbf{t}}_{\Gamma_u^B(e)}^B + \mathbf{G}_e^{RB} \hat{\mathbf{t}}_e^{RB} = \bar{\mathbf{F}}^B \quad (4.14)$$

in Eqn. 4.14,

$$\bar{\mathbf{F}}^B = -\mathbf{H}_{\Gamma_u^B}^B \bar{\mathbf{u}}_e^B + \mathbf{G}_{\Gamma_t^B}^B \bar{\mathbf{t}}_e^B + \mathbf{F}^B \quad (4.15)$$

In Ω^R ,

$$\mathbf{K}^R \hat{\mathbf{u}}_e^R + \mathbf{K}^{RB} \hat{\mathbf{u}}_e^{RB} = \mathbf{f}^R \quad (4.16)$$

in Eqn. 4.16,

$$\mathbf{K}^{RB} = -\alpha \int_{\Gamma^{RB}} \mathbf{N}^R \mathbf{S} (\mathbf{N}^B)^T d\Gamma \quad (4.17)$$

On the interface boundary Γ^{RB} , the nodal traction $\hat{\mathbf{t}}_e^{RB}$ can be computed from the stress $\hat{\boldsymbol{\sigma}}^R$ of Ω^R as (see Eqn. 3.2c for details)

$$\hat{\mathbf{t}}_e^{RB} = \mathbf{Q} \hat{\boldsymbol{\sigma}}^R = \mathbf{Q} \mathbf{D} \mathbf{B} \hat{\mathbf{u}}_e^{RB} \quad (4.18)$$

Combining the Eqns 4.14-4.18 together, the final matrix equation for the coupled BE-RKPM is rearranged as

$$\begin{bmatrix} \mathbf{K}^R & \mathbf{K}^{RB} & \mathbf{0} & \mathbf{0} \\ \mathbf{G}^{RB} & \mathbf{QDB} & \mathbf{H}^{RB} & \mathbf{H}_{\Gamma_t^B}^B - \mathbf{G}_{\Gamma_u^B}^B \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}}_e^R \\ \hat{\mathbf{u}}_e^{RB} \\ \hat{\mathbf{u}}_{\Gamma_t^B(e)}^B \\ \hat{\mathbf{t}}_{\Gamma_u^B(e)}^B \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^R \\ \overline{\mathbf{F}}^B \end{Bmatrix} \quad (4.19)$$

Though the coupled linear system defined in Eqn. 4.19 is typically non-symmetric, it can be conveniently solved by means of sparse solver [95, 100].

4.1.4 Unit displacement method

Since Ω^B and Ω^R are all contained the essential boundary condition, the *unit displacement method* can be adopted here to avoid the non-symmetric structure of the coupled linear system. The key idea of the *unit displacement method* is first to find out the relationship between $\hat{\mathbf{u}}_e^{RB}$ and $\hat{\mathbf{t}}_e^{RB}$ and then to substitute the relationship into the coupled system (Eqn. 4.19) and finally to get the results for the problem.

Figure 4.2 Unit displacement along the interface boundary

In respect that the displacement compatible condition $\mathbf{u}^{BR} = \mathbf{u}^{RB}$ is weakly imposed by the penalty method (Eqn. 4.10), the BEM base point mesh need not be coincident with RKPM point mesh along the interface boundary Γ^{BR} . In practical computation, the displacement $\mathbf{u}^{BR} = (\mathbf{N}^B)^T \hat{\mathbf{u}}_e^{RB}$ on the interface boundary Γ^{BR} can be viewed as the essential boundary condition for the RKPM part. In respect that the nodal value $\hat{\mathbf{u}}_e^{RB}$ is unknown, the unit base vectors are used here.

It is assumed here that there are m BEM base points on Γ^{RB} (shown in Figure 4.2).

For the i th point along Γ^{BR} , the unit base vectors, $\mathbf{e}_{2i-1}$, $\mathbf{e}_{2i}$, in the x and y directions respectively are defined as

$$\mathbf{e}_{2i-1} = \underbrace{\{0, 0, \dots, 1, 0, \dots, 0\}^T}_{2i-1 \text{ terms}} \quad (4.20a)$$

$$\mathbf{e}_{2i} = \underbrace{\{0, 0, \dots, 1, 0, \dots, 0\}^T}_{2i \text{ terms}} \quad (4.20b)$$

The nodal displacement vector $\hat{\mathbf{u}}_e^{\text{RB}}$ can then be written in component form as

$$\hat{\mathbf{u}}_e^{\text{RB}} = \{\hat{u}_1^{\text{RB}}, \hat{u}_2^{\text{RB}}, \dots, \hat{u}_{2m}^{\text{RB}}\}^T = [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{2m}] \begin{Bmatrix} \hat{u}_1^{\text{RB}} \\ \hat{u}_2^{\text{RB}} \\ \vdots \\ \hat{u}_{2m}^{\text{RB}} \end{Bmatrix} \quad (4.21)$$

in Eqn. 4.21, the terms $\hat{u}_{2i-1}^{\text{RB}}$ and $\hat{u}_{2i}^{\text{RB}}$ are the displacement components along x and y directions at the i th point on Γ^{RB} respectively.

The displacement vector $\hat{\mathbf{u}}_e^{\text{R}}$ in Eqn. 4.12 can be decomposed into two parts, $\hat{\mathbf{u}}_e^{\text{R(1)}}$ and $\hat{\mathbf{u}}_e^{\text{R(2)}}$ such that

$$\hat{\mathbf{u}}_e^{\text{R}} = \hat{\mathbf{u}}_e^{\text{R(1)}} + \hat{\mathbf{u}}_e^{\text{R(2)}} \quad (4.22a)$$

$$\mathbf{K}^{\text{R}} \hat{\mathbf{u}}_e^{\text{R(1)}} = \mathbf{f}^{\text{R}} \quad (4.22b)$$

$$\mathbf{K}^{\text{R}} \hat{\mathbf{u}}_e^{\text{R(2)}} = \mathbf{f}^{\text{RB}} \quad (4.22c)$$

in Eqn. 4.22, the first component $\hat{\mathbf{u}}_e^{\text{R(1)}}$ is the nodal values caused by internal body forces and non-interface boundary conditions. It can be solved as $\hat{\mathbf{u}}_e^{\text{R(1)}} = (\mathbf{K}^{\text{R}})^{-1} \mathbf{f}^{\text{R}}$ without the consideration of Ω^{B} . The second component $\hat{\mathbf{u}}_e^{\text{R(2)}}$ is the nodal values caused by the interface boundary conditions.

From Eqns. 4.13c and 4.21, $\mathbf{f}^{\text{RB}}$ can be expressed as

$$\begin{aligned}
 \mathbf{f}_i^{\text{RB}} &= \alpha \int_{\Gamma^{\text{RB}}} N_i^{\text{R}} \mathbf{S} (\mathbf{N}^{\text{B}})^T \hat{\mathbf{u}}_e^{\text{RB}} d\Gamma \\
 &= \left[\alpha \int_{\Gamma^{\text{RB}}} N_i^{\text{R}} \mathbf{S} (\mathbf{N}^{\text{B}})^T [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{2m}] d\Gamma \right] \left\{ \begin{matrix} \hat{u}_1^{\text{RB}} \\ \hat{u}_2^{\text{RB}} \\ \vdots \\ \hat{u}_{2m}^{\text{RB}} \end{matrix} \right\} \\
 &= \left[\mathbf{f}_{\mathbf{e}_1}^{\text{RB}}, \mathbf{f}_{\mathbf{e}_2}^{\text{RB}}, \dots, \mathbf{f}_{\mathbf{e}_{2m}}^{\text{RB}} \right] \left\{ \begin{matrix} \hat{u}_1^{\text{RB}} \\ \hat{u}_2^{\text{RB}} \\ \vdots \\ \hat{u}_{2m}^{\text{RB}} \end{matrix} \right\} = \mathbf{F}^{\text{RB}} \hat{\mathbf{u}}_e^{\text{RB}}
 \end{aligned} \tag{4.23}$$

where,

$$\mathbf{f}_{\mathbf{e}_i}^{\text{RB}} = \alpha \int_{\Gamma^{\text{RB}}} N_i^{\text{R}} \mathbf{S} (\mathbf{N}^{\text{B}})^T \mathbf{e}_i d\Gamma \tag{4.24a}$$

$$\mathbf{F}^{\text{RB}} = [\mathbf{f}_{\mathbf{e}_1}^{\text{RB}}, \mathbf{f}_{\mathbf{e}_2}^{\text{RB}}, \dots, \mathbf{f}_{\mathbf{e}_{2m}}^{\text{RB}}] \tag{4.24b}$$

Let $\hat{\mathbf{u}}_{\mathbf{e}_i}^{\text{R}(2)}$ be the nodal value cause by the unit base vector $\mathbf{e}_i$ such that

$$\mathbf{K}^{\text{R}} \hat{\mathbf{u}}_{\mathbf{e}_i}^{\text{R}(2)} = \mathbf{f}_{\mathbf{e}_i}^{\text{RB}} \text{ or } \hat{\mathbf{u}}_{\mathbf{e}_i}^{\text{R}(2)} = (\mathbf{K}^{\text{R}})^{-1} \mathbf{f}_{\mathbf{e}_i}^{\text{RB}} \tag{4.25}$$

Then by Eqns. 4.22, 4.23 and 4.24, the vector $\hat{\mathbf{u}}_e^{\text{R}(2)}$ can be expressed as

$$\begin{aligned}
 \hat{\mathbf{u}}_e^{\text{R}(2)} &= \sum_{i=1}^{2m} \hat{\mathbf{u}}_{\mathbf{e}_i}^{\text{R}(2)} \hat{\mathbf{u}}_i^{\text{RB}} = \sum_{i=1}^{2m} (\mathbf{K}^{\text{R}})^{-1} \mathbf{f}_{\mathbf{e}_i}^{\text{RB}} \hat{\mathbf{u}}_i^{\text{RB}} \\
 &= (\mathbf{K}^{\text{R}})^{-1} \sum_{i=1}^{2m} \mathbf{f}_{\mathbf{e}_i}^{\text{RB}} \hat{\mathbf{u}}_i^{\text{RB}} = (\mathbf{K}^{\text{R}})^{-1} \mathbf{F}^{\text{RB}} \hat{\mathbf{u}}_e^{\text{RB}}
 \end{aligned} \tag{4.26}$$

Hence, by combining Eqns. 4.8, 4.22, 4.23, 4.24 and 4.26, the displacement $\hat{\mathbf{u}}^{\text{R}}$ in Ω^{R} and along Γ^{RB} can be expressed as

$$\begin{aligned}
 \hat{\mathbf{u}}^{\text{R}} &= (\mathbf{N}^{\text{R}})^T \hat{\mathbf{u}}_e^{\text{R}} \\
 &= (\mathbf{N}^{\text{R}})^T (\hat{\mathbf{u}}_e^{\text{R}(1)} + \hat{\mathbf{u}}_e^{\text{R}(2)}) \\
 &= (\mathbf{N}^{\text{R}})^T (\mathbf{K}^{\text{R}})^{-1} (\mathbf{f}^{\text{R}} + \mathbf{F}^{\text{RB}} \hat{\mathbf{u}}_e^{\text{RB}})
 \end{aligned} \tag{4.27}$$

Thus, from the constitute relation defined in Eqns.3.1b and 3.1c (defined in Section 3.1), the stress $\hat{\boldsymbol{\sigma}}^R$ in Ω^R is defined by

$$\hat{\boldsymbol{\sigma}}^R = \mathbf{D} \mathbf{L} (\mathbf{N}^R)^T (\mathbf{K}^R)^{-1} (\mathbf{f}^R + \mathbf{F}^{RB} \hat{\mathbf{u}}_e^{RB}) \quad (4.28)$$

Along Γ^{RB} , the relationship between the stress $\hat{\boldsymbol{\sigma}}^R$ and the traction $\mathbf{t}^{RB}$ can be expressed as

$$\mathbf{t}^{RB} = \mathbf{Q} \hat{\boldsymbol{\sigma}}^R \quad (4.29)$$

The nodal traction vector $\hat{\mathbf{t}}_e^{RB}$ can also be written in component form as

$$\hat{\mathbf{t}}_e^{RB} = \{\hat{t}_1^{RB}, \hat{t}_2^{RB}, \dots, \hat{t}_{2m}^{RB}\}^T \quad (4.30)$$

where, $\hat{t}_{2i-1}^{RB}$ and $\hat{t}_{2i}^{RB}$ are nodal traction components along x and y direction associated to the i th point on Γ^{RB} .

Then using Eqns. 4.28, 4.29 and 4.30, at the i th point on Γ^{RB} , the following equation can be arrived as

$$\begin{aligned} \begin{Bmatrix} \hat{t}_{2i-1}^{RB} \\ \hat{t}_{2i}^{RB} \end{Bmatrix} &= (\mathbf{Q} \mathbf{D} \mathbf{L} (\mathbf{N}^R)^T) \Big|_{\mathbf{p}_i} (\mathbf{K}^R)^{-1} (\mathbf{f}^R + \mathbf{F}^{RB} \Big|_{\mathbf{p}_i} \hat{\mathbf{u}}_e^{RB}) \\ &= \begin{Bmatrix} \hat{t}_{2i-1}^{RB(1)} \\ \hat{t}_{2i}^{RB(1)} \end{Bmatrix} + \begin{bmatrix} \mathbf{W}_{2i-1}^{RB} \\ \mathbf{W}_{2i}^{RB} \end{bmatrix} \hat{\mathbf{u}}_e^{RB} \end{aligned} \quad (4.31)$$

in Eqn. 4.31, $\mathbf{p}_i = \{x_i, y_i\}$ are the coordinates of the i th point on Γ^{RB} . The terms

$\{\hat{t}_{2i-1}^{RB(1)}, \hat{t}_{2i}^{RB(1)}\}^T$ and the $2 \times (2m)$ matrix $\begin{bmatrix} \mathbf{W}_{2i-1}^{RB} \\ \mathbf{W}_{2i}^{RB} \end{bmatrix}$ are given by

$$\begin{Bmatrix} \hat{t}_{2i-1}^{RB(1)} \\ \hat{t}_{2i}^{RB(1)} \end{Bmatrix} = (\mathbf{Q} \mathbf{D} \mathbf{L} (\mathbf{N}^R)^T) \Big|_{\mathbf{p}_i} (\mathbf{K}^R)^{-1} \mathbf{f}^R \quad (4.32a)$$

$$\begin{bmatrix} \mathbf{W}_{2i-1}^{RB} \\ \mathbf{W}_{2i}^{RB} \end{bmatrix} = (\mathbf{Q} \mathbf{D} \mathbf{L} (\mathbf{N}^R)^T) \Big|_{\mathbf{p}_i} (\mathbf{K}^R)^{-1} \mathbf{F}^{RB} \Big|_{\mathbf{p}_i} \quad (4.32b)$$

It is noted here that the term $(\mathbf{K}^R)^{-1}$ will never be computed explicitly in actual computation. It is only required to assembly and factorize $\mathbf{K}^R$ once and then back substitute $2m+1$ times for the vector $\mathbf{f}^R$ and the $(2NP) \times (2m)$ matrix $\mathbf{F}^{RB}$.

Thus, by repeating Eqns. 4.32 for all m points on Γ^{RB} , the final relationship between $\hat{\mathbf{t}}_e^{RB}$ and $\hat{\mathbf{u}}_e^R$ can be written as

$$\hat{\mathbf{t}}_e^{RB} = \mathbf{W}^{RB} \hat{\mathbf{u}}_e^{RB} + \hat{\mathbf{t}}_e^{RB(1)} \quad (4.33)$$

such that

$$\mathbf{W}^{RB} = [\mathbf{W}_1^{RB}, \mathbf{W}_2^{RB}, \dots, \mathbf{W}_{2m}^{RB}]^T \quad (4.34a)$$

$$\hat{\mathbf{t}}_e^{RB(1)} = \{\hat{t}_1^{RB(1)}, \hat{t}_2^{RB(1)}, \dots, \hat{t}_{2m}^{RB(1)}\}^T \quad (4.34b)$$

Once the relationship between $\hat{\mathbf{t}}_e^{RB}$ and $\hat{\mathbf{u}}_e^R$ is known, the coupled system defined in Eqn. 4.19 can be solved independently and the details are as follow:

In Ω^B , substituting Eqn. 4.33 into Eqn. 4.14 produces

$$\left[(\mathbf{H}^{RB} + \mathbf{G}^{RB} \mathbf{W}^{RB}) \mathbf{H}_{\Gamma_t^B}^B, -\mathbf{G}_{\Gamma_u^B}^B \right] \begin{Bmatrix} \hat{\mathbf{u}}_e^{RB} \\ \hat{\mathbf{u}}_{\Gamma_t^B(e)}^B \\ \hat{\mathbf{t}}_{\Gamma_u^B(e)}^B \end{Bmatrix} = \bar{\bar{\mathbf{F}}}^B \quad (4.35)$$

where, $\bar{\bar{\mathbf{F}}}^B = -\mathbf{G}^{RB} \hat{\mathbf{t}}_e^{RB(1)} + \bar{\mathbf{F}}^B$.

Thus $\hat{\mathbf{u}}_e^{RB}$, $\hat{\mathbf{u}}_{\Gamma_t^B(e)}^B$ and $\hat{\mathbf{t}}_{\Gamma_u^B(e)}^B$ in Ω^B can be solved from the Eqn. 4.35 without considering Ω^R , and by substituting $\hat{\mathbf{u}}_e^{RB}$ into Eqn. 4.27, the displacement $\hat{\mathbf{u}}_e^R$ in Ω^R can be calculated immediately.

4.2 *A priori* error estimation for the BEM, the RKPM and the coupled BE-RKPM

A priori error estimation can provide an estimation of exact energy norm and a qualitative description the rate of the numerical solution. It can be used to predict the convergence rate of the numerical methods. In this part, the *a priori* error

estimations for the BEM and the RKPM are introduced and then the convergence rate of the coupled BE-RKPM is discussed with proof.

4.2.1 Definition of error norm

The errors in displacement $\mathbf{e}_u$ and stress $\mathbf{e}_\sigma$ are defined as the differences between the exact solution and approximated solutions

$$\mathbf{e}_u = \mathbf{u} - \hat{\mathbf{u}} \quad (4.36a)$$

$$\mathbf{e}_\sigma = \boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}} \quad (4.36b)$$

in Eqn. 4.36, $\mathbf{u}$ is the exact displacement solution and $\boldsymbol{\sigma}$ is the exact stress solution. $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\sigma}}$, are, respectively, the approximated displacement and stress obtained by the BEM, the RKPM and the coupled BE-RKPM.

In general, such a pointwise error measures are difficult to use in practice and the energy norm of the error is often used instead. For the 2D elastostatic problem defined by Eqns. 3.1-3.3, the energy norm of the exact solution $\|\mathbf{u}\|$ and the energy norm of the error $\|\mathbf{e}_\sigma\|$ of the approximated solution are defined as [101]

$$\|\mathbf{u}\|^2 = \int_{\Omega} (\mathbf{L} \mathbf{u})^T \mathbf{D} (\mathbf{L} \mathbf{u}) d\Omega = \int_{\Omega} \boldsymbol{\sigma}^T \mathbf{D}^{-1} \boldsymbol{\sigma} d\Omega \quad (4.37a)$$

$$\|\mathbf{e}_\sigma\|^2 = \int_{\Omega} (\mathbf{L} \mathbf{e}_u)^T \mathbf{D} (\mathbf{L} \mathbf{e}_u) d\Omega = \int_{\Omega} \mathbf{e}_\sigma^T \mathbf{D}^{-1} \mathbf{e}_\sigma d\Omega \quad (4.37b)$$

The absolute value of the energy norm varies from problem to problem and is not particularly useful by itself. Hence the relative error in energy norm, η , defined as

$$\eta = \frac{\|\mathbf{e}_\sigma\|}{\|\mathbf{u}\|} \quad (4.38)$$

is usually adopted as a measure of the accuracy of the approximated solution.

4.2.2 *A priori* error estimation for the BEM

For the collocation BEM, if the model is under uniform h -refinement, the *a priori* error estimation can be written in the form [102]

$$\|\mathbf{e}_\sigma\|_B \leq C^B (h^B)^{\min(P^B, \lambda)} \quad (4.39)$$

where, h^B is the size of the boundary element used, $P^B \geq 1$ is the order of the interpolation functions $\mathbf{N}^B$ used in the BEM. C^B is a constant independent of both h^B and P^B . $\lambda < 1$ is the strength of singularity of the problem. For the BEM, since only the boundary of the problem domain is discretized, h^B is only inversely proportional to N_{DOF}^B , the number of degree of freedoms of the BEM. Thus, in terms of N_{DOF}^B , Eqn. 4.39 can also be written in the form as

$$\|\mathbf{e}_\sigma\|_B \leq C^B (N_{\text{DOF}}^B)^{-\min(P^B, \lambda)} \quad (4.40)$$

4.2.3 *A priori error estimation for the RKPM*

For the RKPM, the penalty method is used in enforcing the essential boundary conditions, which can be a source of error. However, it can be expected that as the computation model is refined, the penalty term $\frac{\alpha}{2} \int_{\Gamma_u^R} (\hat{\mathbf{u}}^R - \bar{\mathbf{u}}^R)^T (\hat{\mathbf{u}}^R - \bar{\mathbf{u}}^R) d\Gamma$ defined in Eqn. 4.11 will converge to zero, so the error can be ignored in the *a priori* error estimations for the RKPM and coupled BE-RKPM. Then the *a priori* error estimation under uniform h -refinement can be written as [103]

$$\|\mathbf{e}_\sigma\|_R \leq C^R (h^R)^{\min(P^R, \lambda)} \quad (4.41)$$

in Eqn. 4.41, h^R is the spacing of nodal points in the domain. The term P^R is the order of the RKPM approximation such that $P^R = q - 1$ and $q \geq 2$ is the completed order of polynomial used in the correction function (Eqn. 3.15) of the RKPM formulation. C^R is a constant independent of both h^R and P^R . For 2D problems, h^R is inversely proportional to $(N_{\text{DOF}}^R)^{1/2}$ where N_{DOF}^R is the number of degree of freedoms of the RKPM. Thus, in terms of N_{DOF}^R , the *a priori* error estimation can be rewritten as

$$\|\mathbf{e}_\sigma\|_R \leq C^R (N_{\text{DOF}}^R)^{-\min(P^R, \lambda)/2} \quad (4.42)$$

4.2.4 *A priori* error estimation for the coupled BE-RKPM

For the coupled BE-RKPM, the problem domain is divided into two subdomains, Ω^B and Ω^R , the following equations can be established as

$$\|\mathbf{e}_\sigma\|^2 = \|\mathbf{e}_\sigma\|_B^2 + \|\mathbf{e}_\sigma\|_R^2 \quad (4.43a)$$

$$N_{\text{DOF}} = N_{\text{DOF}}^B + N_{\text{DOF}}^R \quad (4.43b)$$

in Eqn. 4.43, N_{DOF} and $\|\mathbf{e}_\sigma\|$ are the total degree of the freedom and error norm for the whole problem domain, respectively. $\|\mathbf{e}_\sigma\|_B$ is the energy norm of error in Ω^B , $\|\mathbf{e}_\sigma\|_R$ is the energy norm of error in Ω^R . N_{DOF}^B is the degree of freedom for Ω^B and N_{DOF}^R is the degree of freedom for Ω^R .

Now assume that under uniform h -refinement, the *a priori* error estimation for the whole problem domain can be written as

$$\|\mathbf{e}_\sigma\| = C (N_{\text{DOF}})^{-\gamma} \quad (4.44)$$

where, C is a constant independent of the overall convergence rate γ and N_{DOF} .

Let $\gamma^B = \min(P^B, \lambda)$, $\gamma^R = \min(P^R, \lambda)/2$ be the convergence rates of the BEM and RKPM respectively. Substituting Eqns. 4.40, 4.42, and 4.44 into Eqn. 4.43a, the following equation can be obtained as

$$\begin{aligned} \frac{C^2}{(N_{\text{DOF}})^{2\gamma}} &= \frac{(C^B)^2}{(N_{\text{DOF}}^B)^{2\gamma^B}} + \frac{(C^R)^2}{(N_{\text{DOF}}^R)^{2\gamma^R}} \\ &\geq \frac{(C^B)^2}{(N_{\text{DOF}})^{2\gamma^B}} + \frac{(C^R)^2}{(N_{\text{DOF}})^{2\gamma^R}} \\ &\geq \frac{(C^B)^2}{(N_{\text{DOF}})^{2\max(\gamma^B, \gamma^R)}} + \frac{(C^R)^2}{(N_{\text{DOF}})^{2\max(\gamma^B, \gamma^R)}} \\ &= \frac{(C^B)^2 + (C^R)^2}{(N_{\text{DOF}})^{2\max(\gamma^B, \gamma^R)}} \end{aligned} \quad (4.45)$$

As a result, when $N_{\text{DOF}} \rightarrow \infty$, it can be concluded that $\gamma \leq \max(\gamma^B, \gamma^R)$.

Furthermore, Eqn. 4.43b can be rewritten as

$$N_{\text{DOF}}^{\text{B}} = \xi N_{\text{DOF}} \quad \text{and} \quad N_{\text{DOF}}^{\text{R}} = (1 - \xi) N_{\text{DOF}} \quad \text{for some } 0 < \xi < 1 \quad (4.46)$$

Hence,

$$\begin{aligned} \frac{C^2}{(N_{\text{DOF}})^{2\gamma}} &= \frac{(C^{\text{B}})^2}{(N_{\text{DOF}}^{\text{B}})^{2\gamma^{\text{B}}}} + \frac{(C^{\text{R}})^2}{(N_{\text{DOF}}^{\text{R}})^{2\gamma^{\text{R}}}} \\ &\leq \frac{(C^{\text{B}})^2}{(N_{\text{DOF}}^{\text{B}})^{2\min(\gamma^{\text{B}}, \gamma^{\text{R}})}} + \frac{(C^{\text{R}})^2}{(N_{\text{DOF}}^{\text{R}})^{2\min(\gamma^{\text{B}}, \gamma^{\text{R}})}} \\ &= \frac{(C^{\text{B}})^2}{(\xi N_{\text{DOF}})^{2\min(\gamma^{\text{B}}, \gamma^{\text{R}})}} + \frac{(C^{\text{R}})^2}{((1 - \xi) N_{\text{DOF}})^{2\min(\gamma^{\text{B}}, \gamma^{\text{R}})}} \\ &= \frac{(C^{\text{B}})^2}{\xi^{2\min(\gamma^{\text{B}}, \gamma^{\text{R}})}} \frac{1}{(N_{\text{DOF}})^{2\min(\gamma^{\text{B}}, \gamma^{\text{R}})}} + \frac{(C^{\text{R}})^2}{(1 - \xi)^{2\min(\gamma^{\text{B}}, \gamma^{\text{R}})}} \frac{1}{(N_{\text{DOF}})^{2\min(\gamma^{\text{B}}, \gamma^{\text{R}})}} \\ &\leq \max \left(\frac{(C^{\text{B}})^2}{\xi^{2\min(\gamma^{\text{B}}, \gamma^{\text{R}})}}, \frac{(C^{\text{R}})^2}{(1 - \xi)^{2\min(\gamma^{\text{B}}, \gamma^{\text{R}})}} \right) \frac{1}{(N_{\text{DOF}})^{2\min(\gamma^{\text{B}}, \gamma^{\text{R}})}} \end{aligned} \quad (4.47)$$

Thus as $N_{\text{DOF}} \rightarrow \infty$, $\gamma \geq \min(\gamma^{\text{B}}, \gamma^{\text{R}})$.

Finally, combining Eqns. 4.45 and 4.47, the following equation can be arrived as

$$\min(\gamma^{\text{B}}, \gamma^{\text{R}}) \leq \gamma \leq \max(\gamma^{\text{B}}, \gamma^{\text{R}}) \quad (4.48)$$

Thus, the convergence rate of the coupled BE-RKPM will always lie between the convergence rates of the BEM and the RKPM. In practical computation, the actual convergence rate of the coupled BE-RKPM will also relate to the partition of the BEM subdomain and the RKPM subdomain.

In case that no singularity is present and the approximation order of the BEM (P^{B}) and the RKPM (P^{R}) used are at the same orders, from Eqns. 4.40, 4.42 and 4.48, γ will be in the range

$$P^{\text{R}} \leq \gamma \leq P^{\text{B}} \quad (4.49)$$

In the thesis, the approximation order of the RKPM P^{R} is always set 1.0 (see Section 3.3.1). When the constant boundary elements or linear boundary elements

are used, the convergence rate of the coupled BE-RKPM using uniform element–point mesh will lie between 0.5 and 1.0 and when the quadratic boundary elements are used, the convergence rate of the coupled BE-RKPM using uniform element–point mesh will lie between 0.5 and 2.0.

In case that singularity is present, since $\min(P^B, P^R) > \lambda$, from Eqns. 4.40, 4.42 and 4.48, γ

$$\lambda/2 \leq \gamma \leq \lambda \quad (4.50)$$

The selection of different element types for the BEM subdomain, such as the constant boundary element, linear boundary element or quadratic boundary element, will only have little effect on the convergence rate of the coupled BE-RKPM using adaptive element-point mesh.

4.3 Numerical examples with uniform refinement

In this part, numerical studies on four 2D benchmark linear BVP problems are done to investigate the efficiency of different solutions obtained by the BEM, the RKPM and the coupled BE-RKPM based on four different uniform point/element mesh density, in which the following results will be computed:

- 1). The relative error of three methods, η (Eqn. 4.38).
- 2). The convergence rate of three methods, $R(\eta) = \frac{\log(\eta_i / \eta_{i+1})}{\log(N_{\text{DOF},i+1} / N_{\text{DOF},i})}$, where, η_i are $N_{\text{DOF},i}$ the relative error and the number of degree of freedoms of the i th refinement analysis respectively.

In addition, the coupled BE-RKPM is applied on a structure on a semi-infinite soil foundation problem and compared its results with the results of the coupled BE-FEM and the coupled BE-EFGM.

4.3.1 Timoshenko beam (convex domain, smooth exact solution)

The first example is a cantilever beam loaded under end shear force as shown as Figure 4.3. The exact stress solution of this problem is given by Timoshenko and Goodier[104]

$$\sigma_x(x,y) = -\frac{Py}{I}(L-x) \quad (4.51a)$$

$$\sigma_y(x,y) = 0.0 \quad (4.51b)$$

$$\tau_{xy}(x,y) = -\frac{P}{2I}(h^2/4 - y^2) \quad (4.51c)$$

where, $I = h^3/12$.

The meshes used in the BEM, the RKPM and the coupled BE-RKPM are shown in Figures 4.4, 4.5 and 4.6 respectively. The results obtained are summarized in Table 4.1 while the convergence rates in energy norm are plotted in Figure 4.7, respectively.

Figure 4.3 Cantilever beam subjected to a distributed shear force at the right-end

Figure 4.4 BEM mesh used for the Timoshenko beam

Mesh 1) 27 Points

Mesh 2) 85 Points

Mesh 3) 126 Points

Mesh 4) 297 Points

Figure 4.5 RKPM mesh used for the Timoshenko beam

Mesh 1) 8 Elements and 15 Points

Mesh 2) 14 Elements and 45 Points

Mesh 3) 18 Elements and 66 Points

Mesh 4) 32 Elements and 153 Points

Figure 4.6 Coupled BE-RKPM mesh used for the Timoshenko beam

Mesh	BEM			RKPM			coupled BE-RKPM		
	N_{DOF}	η (%)	$R(\eta)$	N_{DOF}	η (%)	$R(\eta)$	N_{DOF}	η (%)	$R(\eta)$
1	48	16.2		54	14.5		54	15.8	
2	72	10.4	1.09	170	6.9	0.65	130	8.3	0.73
3	104	7.1	1.03	252	5.1	0.76	184	6.5	0.70
4	144	4.5	1.40	594	2.9	0.66	402	3.8	0.69

Table 4.1 Results for the Timoshenko beam

Figure 4.7 Convergence rates $R(\eta)$ for the Timoshenko beam

4.3.2 Infinite plate with a circular hole (concave domain, smooth exact solution)

In the second example, an infinite plate with a circular hole as shown Figure 4.8 is considered. The exact solution of this problem is given by [101]

$$\sigma_x(x,y) = 1 - \frac{a^2}{r^2} \left(\frac{3}{2} \cos 2\theta + \cos 4\theta \right) + \frac{3a^4}{2r^4} \cos 4\theta \quad (4.52a)$$

$$\sigma_y(x,y) = -\frac{a^2}{r^2} \left(\frac{1}{2} \cos 2\theta - \cos 4\theta \right) - \frac{3a^4}{2r^4} \cos 4\theta \quad (4.52b)$$

$$\tau_{xy}(x,y) = -\frac{a^2}{r^2} \left(\frac{1}{2} \sin 2\theta + \sin 4\theta \right) + \frac{3a^4}{2r^4} \sin 4\theta \quad (4.52c)$$

For this example, the meshes used in different analyses are shown in Figure 4.9, 4.10 and 4.11. The results obtained are summarized in Table 4.2 while the convergence rates in energy norm are plotted in Figure 4.12, respectively.

Figure 4.8 The first quadrant of a plate with a circular hole

Figure 4.9 BEM mesh used for the infinite plate with a circular hole

Figure 4.10 RKPM mesh used for the infinite plate with a circular hole

Figure 4.11 Coupled BE-RKPM mesh used for the infinite plate with a circular hole

Mesh	BEM			RKPM			coupled BE-RKPM		
	N_{DOF}	η (%)	$R(\eta)$	N_{DOF}	η (%)	$R(\eta)$	N_{DOF}	η (%)	$R(\eta)$
1	40	7.6		72	7.8		50	7.7	
2	72	4.1	1.14	162	4.5	0.68	88	5.1	0.73
3	116	2.4	1.12	240	3.5	0.64	106	4.5	0.67
4	144	1.8	1.33	564	2.1	0.60	248	2.4	0.72

Table 4.2 Results for the infinite plate with a circular hole

Figure 4.12 Convergence rates $R(\eta)$ for the infinite plate with a circular hole

4.3.3 Domain near a crack tip (convex domain, singular exact solution)

In the third problem, the problem domain is a region near the crack tip as shown in Figure 4.13. It is loaded by tractions given by an exact solution corresponding to the first symmetric mode of the stress intensity factor solution. The stress field in the domain is given by [105]

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) \quad (4.53a)$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) \quad (4.53b)$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \left(\sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}\right) \quad (4.53c)$$

Exploiting symmetry, only half of the domain is modelled. In this study, K_I is set to be $\sqrt{2\pi}$ for convenience and a singularity exists at $(0, 0)$ with λ equals to 0.5.

Figure 4.13 Region near a crack tip

For this problem, the meshes used in the BEM, the RKPM and the coupled BE-RKPM analyses are shown in Figures 4.14, 4.15 and 4.16 respectively. The results obtained are summarized in Table 4.3 while the convergence rates in energy norm are plotted in Figure 4.17, respectively.

Mesh 1) 12 Elements

Mesh 2) 24 Elements

Mesh 3) 30 Elements

Mesh 4) 48 Elements

Figure 4.14 BEM mesh used for the domain near a crack tip

Mesh 1) 15 Points

Mesh 2) 45 Points

Mesh 3) 66 Points

Mesh 4) 153 Points

Figure 4.15 RKPM mesh used for the domain near a crack tip

Mesh 1) 18 Elements and 9 Points

Mesh 2) 34 Elements and 25 Points

Mesh 3) 43 Elements and 36 Points

Mesh 4) 68 Elements and 81 Points

Figure 4.16 Coupled BE-RKPM mesh used for the domain near a crack tip

Mesh	BEM			RKPM			coupled BE-RKPM		
	N_{DOF}	η (%)	$R(\eta)$	N_{DOF}	η (%)	$R(\eta)$	N_{DOF}	η (%)	$R(\eta)$
1	48	24.4		30	29.7		66	23.1	
2	96	18.4	0.41	90	20.9	0.32	138	18.5	0.30
3	120	16.8	0.41	132	18.8	0.28	184	17.0	0.29
4	192	14.2	0.36	306	15.2	0.25	338	14.3	0.28

Table 4.3 Results for the domain near a crack tip

Figure 4.17 Convergence rates $R(\eta)$ for the domain near a crack tip

4.3.4 L-shaped domain (concave domain, singular exact solution)

In the fourth example, an L-shaped plane as shown in Figure 4.18 is considered. It is loaded by tractions corresponding to the following exact stress solution [101]

$$\sigma_x = \lambda r^{(\lambda-1)} \{ (2 - Q(\lambda+1)) \cos((\lambda-1)\theta) - (\lambda-1) \cos((\lambda-3)\theta) \} \quad (4.54a)$$

$$\sigma_y = \lambda r^{(\lambda-1)} \{ (2 + Q(\lambda+1)) \cos((\lambda-1)\theta) + (\lambda-1) \cos((\lambda-3)\theta) \} \quad (4.54b)$$

$$\tau_{xy} = \lambda r^{(\lambda-1)} \{ (\lambda-1) \sin((\lambda-3)\theta) + Q(\lambda+1) \sin((\lambda-1)\theta) \} \quad (4.54c)$$

in Eqns. 4.54, $Q=0.543075579$ and $\lambda = 0.544483737$.

The traction conditions satisfy equilibrium and hence only rigid-body constraints have to be applied at point **3** and **6**. In this example, similar to example 3, a singular point exists at point **3**.

Figure 4.18 L-shaped domain

The meshes used for the BEM, the RKPM and the coupled BE-RKPM are shown in Figures 4.19, 4.20 and 4.21, respectively. The results obtained are summarized in Table 4.4 and the convergence rates in energy norm are plotted in Figure 4.22, respectively.

Figure 4.19 BEM mesh used for the L-shaped domain

Figure 4.20 RKPM mesh used for the L-shaped domain

Figure 4.21 Coupled BE-RKPM mesh used for the L-shaped domain

Mesh	BEM			RKPM			coupled BE-RKPM		
	N_{DOF}	η (%)	$R(\eta)$	N_{DOF}	η (%)	$R(\eta)$	N_{DOF}	η (%)	$R(\eta)$
1	64	20.5		66	23.5		78	20.9	
2	96	16.6	0.52	134	18.5	0.34	178	16.1	0.32
3	128	14.8	0.40	226	15.9	0.29	450	11.5	0.36
4	160	13.5	0.41	342	14.3	0.26	622	10.3	0.34

Table 4.4 Results for the L-shaped domain

Figure 4.22 Convergence rates $R(\eta)$ for the L-shaped domain

4.3.5 A structure on a semi-infinite soil foundation

In the last example, the suggested coupled BE-RKPM is applied for the solution of a soil-structure interaction problem as shown in Figure 4.23, which has been solved by Brebbia and Georgion [106] using the BEM and the coupled BE-FEM and by Liu and Gu [93] using the coupled BE-EFGM. As shown in Figure 4.24, the infinite soil foundation is treated by truncating the soil region at a finite distance from the structure [106]. Note that such truncation treatment is used here as it is convenient to compare the suggested coupled BE-RKPM with other researchers' solutions. As shown in Figures 4.24 and 4.25, the RKPM and the BEM are used in regions 1 and 2 respectively. Two loading cases are applied (Figure 4.24): Case 1 considers five concentrated vertical loads along the top and Case 2 considers and additional load acting at a corner.

The results obtained by using the FEM [106], the BE-FEM [106], the coupled BE-EFGM [93] and the suggested coupled BE-RKPM are shown in Table 4.5. From the results, it can be seen that the coupled BE-RKPM is in excellent agreement with the results obtained by other methods. It should be noted that only in the coupled BE-RKPM, only 40 BE elements and 35 RKPM points are used.

Figure 4.23 A structure standing on a semi-infinite soil foundation

Figure 4.24 RKPM mesh of structure (region 1) and load cases

Figure 4.25 Boundary element mesh of soil foundation (region 2)

Points	Displacements ($\times 10^{-4}$)			
	FE*	BE-FEM*	BE-EFGM#	coupled BE-RKPM
Load Case 1				
1	1.41	1.40	1.42	1.42
2	1.34	1.33	1.33	1.35
3	1.32	1.32	1.32	1.33
4	1.34	1.33	1.33	1.35
5	1.41	1.40	1.42	1.42
Load Case 2				
1	-3.39	-3.55	-3.58	-3.59
2	-0.97	-1.05	-1.04	-1.06
3	1.35	1.35	1.34	1.32
4	3.61	3.70	3.68	3.72
5	6.00	6.17	6.13	6.18

*Results obtained from Reference [106] #Results obtained from Reference [93]

Table 4.5 Results for vertical displacements along top of structure of a structure standing on a semi-infinite soil foundation

4.4 Closure and discussions

In this Chapter, the coupled BE-RKPM is proposed by direct coupling of the BEM and the RKPM. The *a priori* error estimation for the coupled BE-RKPM is also discussed with details. Numerical experiments on four benchmark problems are given to compare the convergence rates of the BEM, the RKPM and the coupled

BE-RKPM. A semi-infinite domain problem is also used to verify the efficiency of the coupled BE-RKPM by comparing it with other coupled methods. From the numerical results obtained, it can be concluded that

1). Regarding the accuracy of the solutions obtained, from Tables 4.1-4.4, it can be seen that in examples 4.3.1-4.3.4, the BEM solutions are always the most accurate solutions while the coupled BE-RKPM solutions are more accurate than the RKPM solutions. This can be explained by the fact that the BEM is a semi-analytical numerical method which makes use of the fundamental solution in its formulation and thus leads to more accurate solutions. Hence, in practical applications of the coupled BE-RKPM, it should be preferred to limit the size of RKPM subdomain only to where rapid refinement/remodeling/change in domain geometry occurs.

2). Regarding to the convergence rates of the solutions, from Figures 4.7 and 4.12, when no singularity is present in the problem domain, for both the RKPM and the BEM, the theoretical convergence rates of 0.5 and 1.0 are achieved. Furthermore, as predicted by Eqn. 4.49, the convergence rate of the coupled BE-RKPM solution is in the predicted range of (0.5,1.0).

When singularity is present in the problem domain, as seen from Figures 4.17 and 4.22, the convergence rates of all methods are affected as predicted by Eqn. 4.50. Moreover, the convergence rate of the coupled BE-RKPM lies between the range $(\lambda/2 \leq \gamma \leq \lambda)$. Thus, in examples 1-4, the actual convergence rate obtained from the coupled BE-RKPM is always higher than the convergence rate obtained from the RKPM.

Hence, it can be concluded that the suggested coupled procedure does not impair the convergence rate of the solution despite the compatibility conditions along interface boundary is only imposed in a weak sense.

3). Regarding the computational cost of the solutions, detailed timings (as shown in Table 4.6) indicated that for the same point/element mesh density, the BEM is approximately ten times faster than the RKPM. Furthermore, additional CPU time

needed for the coupled solution is negligible when comparing with the CPU time used in the RKPM. Hence, by coupled the BEM with RKPM, a significant saving of computational cost can be achieved, especially for problems involving infinite/semi-infinite domain.

Problem	BEM (minutes)	RKPM (minutes)	coupled BE-RKPM (minutes)
Example 1	2	120	50
Example 2	2	120	25
Example 3	3	90	40
Example 4	3	100	110
Structure-soil problem			9

Table 4.6 CPU times for the uniform refinement (mesh 4)
using a DELL Precision 450 PC (Xeon 2.8GHz, 1G RAM)

CHAPTER 5

A POSTERIORI ERROR ESTIMATIONS FOR RKPM AND BEM

The *a posteriori* error estimations for the RKPM and the BEM are discussed in this Chapter. In Section 5.1, a new *a posteriori* error estimation based on the well-known Zienkiewicz and Zhu (Z-Z) error estimation approach is proposed for the RKPM. Numerical experiments on 2D elastostatic problems are done to verify the efficiency of the new *a posteriori* error estimator. In Section 5.2, the *a posteriori* error estimator based on the residual of boundary stress components between the conventional BEM solution and the hyper-singular boundary integral equation (HBIE) is introduced for the error estimation of the BEM.

5.1 *A posteriori* error estimation for the RKPM

The new *a posteriori* error estimator is different from many recovery type error estimators suggested previously in such a way that, rather than using the least squares fitting approach, the recovery stress field is constructed by an extraction function approach.

5.1.1 Weak form and orthogonality of the RKPM solution

It is well known that the strong form of the boundary value problem (BVP) (defined in Eqn. 3.1) is equivalent to the following weak (virtual work) defined as [107]

$$\int_{\Omega} (\mathbf{L} \mathbf{v}')^T \mathbf{D} (\mathbf{L} \mathbf{u}) d\Omega - \int_{\Omega} (\mathbf{v}')^T \mathbf{b} d\Omega - \int_{\Gamma_t} (\mathbf{v}')^T \bar{\mathbf{t}} d\Gamma = 0, \quad \forall \mathbf{v}' \in \mathbf{V}(\mathbf{0}) \quad (5.1)$$

where, $\mathbf{u} \in \mathbf{V}(\bar{\mathbf{u}})$ is the exact solution and $\mathbf{v}'$ is the test function. The normed vector spaces $\mathbf{V}(\mathbf{g})$ is defined as

$$\mathbf{V}(\mathbf{g}) = \{\mathbf{w} \in \mathbf{H}^1(\Omega) : \mathbf{w} = \mathbf{g} \text{ on } \Gamma_u\} \quad (5.2)$$

where $\mathbf{H}^k(\Omega)$ is the k th order Sobolev space defined over Ω .

The first integral term in Eqn. 5.1 is a bounded coercive bilinear functional and is commonly denoted as $A(\mathbf{v}', \mathbf{u})$ such that

$$A(\mathbf{v}', \mathbf{u}) = \int_{\Omega} (\mathbf{L} \mathbf{v}')^T \mathbf{D} (\mathbf{L} \mathbf{u}) d\Omega \quad (5.3)$$

The bilinear functional in Eqn. 5.3 is also known as the energy inner product of the problem and can be expressed in terms of the stress tensor

$$A(\mathbf{v}', \mathbf{u}) = \int_{\Omega} (\boldsymbol{\sigma}')^T \mathbf{D}^{-1} \boldsymbol{\sigma} d\Omega = \langle \boldsymbol{\sigma}', \boldsymbol{\sigma} \rangle, \text{ where } \boldsymbol{\sigma}' = \mathbf{D} \mathbf{L} (\mathbf{v}') \quad (5.4)$$

In actual numerical computation, the use of the RKPM approximation (Eqn. 3.18) simply means that the infinite dimension space $\mathbf{V}(\mathbf{0})$ and $\mathbf{V}(\bar{\mathbf{u}})$ are replaced by their finite dimension counterparts $\hat{\mathbf{V}}(\mathbf{0})$ and $\hat{\mathbf{V}}(\bar{\mathbf{u}})$ such that Eqn. 5.1 becomes

$$A(\hat{\mathbf{v}}', \hat{\mathbf{u}}) = \int_{\Omega} (\hat{\mathbf{v}}')^T \mathbf{b} d\Omega + \int_{\Gamma_t} (\hat{\mathbf{v}}')^T \bar{\mathbf{t}} d\Gamma, \quad \forall \hat{\mathbf{v}}' \in \hat{\mathbf{V}}(\mathbf{0}) \text{ and } \hat{\mathbf{u}} \in \hat{\mathbf{V}}(\bar{\mathbf{u}}) \quad (5.5)$$

in Eqn. 5.5, $\hat{\mathbf{v}}' \in \hat{\mathbf{V}}(\mathbf{0})$ is the test function while $\hat{\mathbf{u}} \in \hat{\mathbf{V}}(\bar{\mathbf{u}})$ is the RKPM approximation (trial function). It is well known that the error of the solution is orthogonal to any function in the test space such that

$$A(\hat{\mathbf{v}}', \mathbf{u} - \hat{\mathbf{u}}) = \langle \hat{\boldsymbol{\sigma}}', \boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}} \rangle = 0, \text{ where } \hat{\boldsymbol{\sigma}}' = \mathbf{D} \mathbf{L} \hat{\mathbf{v}}', \forall \hat{\mathbf{v}}' \in \hat{\mathbf{V}}(\mathbf{0}) \quad (5.6)$$

It is noted that since $\forall \hat{\mathbf{v}}' \in \hat{\mathbf{V}}(\mathbf{0}) = \mathbf{H}^1(\Omega)$ and from Eqn. 5.6, $\hat{\boldsymbol{\sigma}}' \in \mathbf{H}^0(\Omega)$. Furthermore, the RKPM stress field $\hat{\boldsymbol{\sigma}}$ is continuous and belongs to an essentially dense subspace $\hat{\mathbf{E}} \in \mathbf{H}^0(\Omega)$.

Since Eqn. 5.6 holds for any $\boldsymbol{\sigma}' \in \mathbf{H}(\Omega)$, the term $\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}$ can then be spanned into an error subspace [107] and there exists a self-adjoint orthogonal projection operator $\hat{\mathbf{P}} : \mathbf{H}^0(\Omega) \rightarrow \hat{\mathbf{E}}$ such that

$$\hat{\mathbf{P}} = \hat{\mathbf{P}}^*, \quad \|\hat{\mathbf{P}}\| = 1 \text{ and } \hat{\mathbf{P}}(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) = \mathbf{0} \Rightarrow \hat{\mathbf{P}} \boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}} \quad (5.7)$$

in Eqn. 5.7, where $\hat{\mathbf{P}}^*$ is the adjoint of $\hat{\mathbf{P}}$. Since the subspace $\hat{\mathbf{E}}$ is essentially dense, Eqn. 5.7 also implies that

$$\hat{\mathbf{P}} \boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}} \rightarrow \boldsymbol{\sigma}, \text{ as } N_{\text{DOF}} \rightarrow \infty \quad (5.8)$$

where, N_{DOF} is the degree of freedom of the RKPM solution. Obviously, Eqn. 5.8 is simply an alternative statement for the convergence of the RKPM solution when the numerical model is properly refined.

5.1.2 Extraction equation for stress recovery

The modelled problem (defined in Section. 3.1) can also be expressed abstractly in an extraction operator form as [70]:

$$\boldsymbol{\sigma} = \boldsymbol{\Psi} \boldsymbol{\sigma} + \boldsymbol{\Phi} \quad (5.9a)$$

where, $\boldsymbol{\Psi} : \mathbf{H}^0(\Omega) \rightarrow \mathbf{H}^0(\Omega)$ is a compact extraction operator such that $\boldsymbol{\Psi}^{-1}$ exists and assure that Eqn. 5.9 has a unique solution. The term $\boldsymbol{\Phi} \in \mathbf{H}^0(\Omega)$, in general, is the forcing term [70] which may be computed using only the problem's input data (the body force $\mathbf{b}$ in Eqn. 3.1). Since the BVP has only a unique solution $\mathbf{u}$ (and hence, $\boldsymbol{\sigma}$), when the exact solution $\boldsymbol{\sigma}$ is replaced by the approximated RKPM solution $\hat{\boldsymbol{\sigma}} = \mathbf{D} \mathbf{L} \hat{\mathbf{u}}$, an extracted (or recovered) stress field $\tilde{\boldsymbol{\sigma}} \approx \boldsymbol{\sigma}$ can be obtained as

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\Psi} \hat{\boldsymbol{\sigma}} + \boldsymbol{\Phi} \quad (5.9b)$$

The pointwise error of the RKPM stress field is defined as the difference between exact $\boldsymbol{\sigma}$ and the RKPM solution $\hat{\boldsymbol{\sigma}}$

$$\mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}} \quad (5.10a)$$

and the error in energy norm is calculated as

$$\|\mathbf{e}_{\boldsymbol{\sigma}}\| = \left\{ \int_{\Omega} (\mathbf{L}(\mathbf{u} - \hat{\mathbf{u}}))^T \mathbf{D} (\mathbf{L}(\mathbf{u} - \hat{\mathbf{u}})) d\Omega \right\}^{1/2} = \left\{ \int_{\Omega} (\mathbf{e}_{\boldsymbol{\sigma}})^T \mathbf{D}^{-1} (\mathbf{e}_{\boldsymbol{\sigma}}) d\Omega \right\}^{1/2} \quad (5.10b)$$

The pointwise error, $\tilde{\mathbf{e}}_{\boldsymbol{\sigma}}$ and the error norm, $\|\tilde{\mathbf{e}}_{\boldsymbol{\sigma}}\|$ of the recovered stress field can then be defined as

$$\tilde{\mathbf{e}}_{\sigma} = \boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}} \quad (5.11a)$$

$$\|\tilde{\mathbf{e}}_{\sigma}\| = \left\{ \int_{\Omega} (\tilde{\mathbf{e}}_{\sigma})^T \mathbf{D}^{-1} (\tilde{\mathbf{e}}_{\sigma}) d\Omega \right\}^{1/2} \quad (5.11b)$$

Obviously, if the recovered stress is more accurate than the RKPM stress, it could be used to estimate the error of the RKPM solution and the estimated error norm [16], the post-processing error $\bar{\mathbf{e}}_{\sigma}$ between the recovered stress $\tilde{\boldsymbol{\sigma}}$ and the approximated solution $\hat{\boldsymbol{\sigma}}$, and the corresponding error norm $\|\bar{\mathbf{e}}_{\sigma}\|$ can be written as

$$\bar{\mathbf{e}}_{\sigma} = \tilde{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}} \quad (5.12a)$$

$$\|\bar{\mathbf{e}}_{\sigma}\| = \left\{ \int_{\Omega} (\bar{\mathbf{e}}_{\sigma})^T \mathbf{D}^{-1} (\bar{\mathbf{e}}_{\sigma}) d\Omega \right\}^{1/2} \quad (5.12b)$$

From Eqns. 5.10, 5.11 and 5.12, the three pointwise errors, $\mathbf{e}_{\sigma}$, $\tilde{\mathbf{e}}_{\sigma}$ and $\bar{\mathbf{e}}_{\sigma}$ are related by

$$\mathbf{e}_{\sigma} = \tilde{\mathbf{e}}_{\sigma} + \bar{\mathbf{e}}_{\sigma} \quad (5.13)$$

By using the triangle inequality to the norms of the terms on both sides of Eqn. 5.14, it can be shown that

$$\left| 1 - \frac{\|\tilde{\mathbf{e}}_{\sigma}\|}{\|\mathbf{e}_{\sigma}\|} \right| \leq \frac{\|\bar{\mathbf{e}}_{\sigma}\|}{\|\mathbf{e}_{\sigma}\|} \leq \left| 1 + \frac{\|\tilde{\mathbf{e}}_{\sigma}\|}{\|\mathbf{e}_{\sigma}\|} \right| \quad (5.14a)$$

Hence, the estimated error norm will be asymptotically exact if the convergence rate of the recovered stress field $\tilde{\boldsymbol{\sigma}}$ is higher than that of $\hat{\boldsymbol{\sigma}}$ as N_{DOF} approaches

infinity. In this case, the effectivity index, $\mathcal{J} = \frac{\|\bar{\mathbf{e}}_{\sigma}\|}{\|\mathbf{e}_{\sigma}\|}$, of the error estimator will

converge to unity [108] That is,

$$\lim_{N_{\text{DOF}} \rightarrow \infty} \frac{\|\tilde{\mathbf{e}}_{\sigma}\|}{\|\mathbf{e}_{\sigma}\|} = 0 \Rightarrow \lim_{N_{\text{DOF}} \rightarrow \infty} \frac{\|\bar{\mathbf{e}}_{\sigma}\|}{\|\mathbf{e}_{\sigma}\|} = \lim_{N_{\text{DOF}} \rightarrow \infty} \mathcal{J} = 1 \quad (5.14b)$$

5.1.3 Convergence rate of the recovered stress field

From Eqn. 5.14, it can be seen that if the extraction equation (Eqn. 5.9b) is employed to generate a recovered stress field for the *a posteriori* error estimation of the RKPM solution, the convergence rate of the recovered solution will significantly affect the performance and reliability of the *a posteriori* error estimator. The recovered stress field should, ideally, converge at a higher rate than the RKPM stress field as this will ensure the asymptotically exactness of the effectivity index. Towards this end, it could be proved that if the extraction operator Ψ (Eqn. 5.9) is compact and $\hat{\sigma} = \mathbf{D} \mathbf{L} \hat{\mathbf{u}}$ is the RKPM solution (from the Galerkin method), then the recovered stress field $\tilde{\sigma}$ generated will converge at a higher rate than σ . Details of the convergence properties of the extracted stress field could be found in reference [109-112] and a brief outline of the proof will be given here.

From Eqns. 5.9-5.11, the relationship between $\tilde{\mathbf{e}}_{\sigma}$ and $\mathbf{e}_{\sigma}$ can also be written as

$$\tilde{\mathbf{e}}_{\sigma} = \Psi \mathbf{e}_{\sigma} \quad (5.15a)$$

and from Eqn. 5.8

$$\hat{\mathbf{P}}(\mathbf{e}_{\sigma}) = \mathbf{0} \quad (5.15b)$$

Hence, combining Eqn. 5.15a with 5.15b, one can write

$$\tilde{\mathbf{e}}_{\sigma} = \Psi (\mathbf{I} - \hat{\mathbf{P}}) \mathbf{e}_{\sigma} \quad (5.15c)$$

Furthermore, from Eqn. 5.8, it can be seen that the projection operator $\hat{\mathbf{P}}$ is strong convergent but not uniformly convergent to the identity operator itself. That is

$$\lim_{N_{\text{DOF}} \rightarrow \infty} \|\Psi (\mathbf{I} - \hat{\mathbf{P}}) \sigma\| = 0, \quad \forall \sigma \in \mathbf{H}^0(\Omega) \quad (5.16)$$

but $\lim_{N_{\text{DOF}} \rightarrow \infty} \|\Psi (\mathbf{I} - \hat{\mathbf{P}})\| = 0$ is not assured. Hence, in order to establish the

convergence property of the term $\frac{\|\tilde{\mathbf{e}}_{\sigma}\|}{\|\mathbf{e}_{\sigma}\|}$, further analysis on the convergence of the

operator $\Psi (\mathbf{I} - \hat{\mathbf{P}})$ is needed [109].

Now consider the definition of the term $\|\Psi(\mathbf{I} - \hat{\mathbf{P}})\|$

$$\|\Psi(\mathbf{I} - \hat{\mathbf{P}})\| = \|\Psi^* - \hat{\mathbf{P}}\Psi^*\| = \sup_{\mathbf{z} \in \mathbf{H}^0(\Omega), \|\mathbf{z}\|=1} \|\Psi^*\mathbf{z} - \hat{\mathbf{P}}\Psi^*\mathbf{z}\| \quad (5.17a)$$

where Ψ^* is the adjoint of Ψ .

Assuming that $\lim_{N_{\text{DOF}} \rightarrow \infty} \|\Psi(\mathbf{I} - \hat{\mathbf{P}})\| \neq 0$, then there exists $\varepsilon > 0$ and a sequence $\{\mathbf{z}_{N_{\text{DOF}}}\} \subset \mathbf{H}^0(\Omega)$ such that

$$\|\Psi^*\mathbf{z}_{N_{\text{DOF}}} - \hat{\mathbf{P}}\Psi^*\mathbf{z}_{N_{\text{DOF}}}\| > \varepsilon, \quad \|\mathbf{z}_{N_{\text{DOF}}}\| = 1, \quad \forall \mathbf{z}_{N_{\text{DOF}}} \in \mathbf{H}^0(\Omega) \quad (5.17b)$$

Since Ψ is compact, the adjoint Ψ^* will also be compact. Hence, the sequences and $\{\mathbf{z}_{N_{\text{DOF}}}\}$ can always be selected in such a way that the expression $\Psi^*\mathbf{z}_{N_{\text{DOF}}}$ will converge to an element $\mathbf{z} \in \mathbf{H}^0(\Omega)$ as the RKPM model is refined. That is

$$\lim_{N_{\text{DOF}} \rightarrow \infty} \Psi^*\mathbf{z}_{N_{\text{DOF}}} = \mathbf{z} \quad (5.17c)$$

Now by using the identity

$$\begin{aligned} \Psi^*\mathbf{z}_{N_{\text{DOF}}} - \hat{\mathbf{P}}\Psi^*\mathbf{z}_{N_{\text{DOF}}} &\equiv \Psi^*\mathbf{z}_{N_{\text{DOF}}} - \hat{\mathbf{P}}\Psi^*\mathbf{z}_{N_{\text{DOF}}} - \mathbf{z} + \hat{\mathbf{P}}\mathbf{z}_{N_{\text{DOF}}} + \mathbf{z} - \hat{\mathbf{P}}\mathbf{z}_{N_{\text{DOF}}} \\ &\equiv (\mathbf{I} - \hat{\mathbf{P}})(\Psi^*\mathbf{z}_{N_{\text{DOF}}} - \mathbf{z}) + (\mathbf{z} - \hat{\mathbf{P}}\mathbf{z}_{N_{\text{DOF}}}) \end{aligned} \quad (5.18a)$$

One can write

$$\begin{aligned} \|\Psi^*\mathbf{z}_{N_{\text{DOF}}} - \hat{\mathbf{P}}\Psi^*\mathbf{z}_{N_{\text{DOF}}}\| &= \|(\mathbf{I} - \hat{\mathbf{P}})(\Psi^*\mathbf{z}_{N_{\text{DOF}}} - \mathbf{z}) + (\mathbf{z} - \hat{\mathbf{P}}\mathbf{z}_{N_{\text{DOF}}})\| \\ &\leq \|(\mathbf{I} - \hat{\mathbf{P}})(\Psi^*\mathbf{z}_{N_{\text{DOF}}} - \mathbf{z})\| + \|\mathbf{z} - \hat{\mathbf{P}}\mathbf{z}_{N_{\text{DOF}}}\| \end{aligned} \quad (5.18b)$$

However, as $N_{\text{DOF}} \rightarrow \infty$, $\Psi^*\mathbf{z}_{N_{\text{DOF}}} - \mathbf{z} \rightarrow \mathbf{0}$ and $\hat{\mathbf{P}}\mathbf{z}_{N_{\text{DOF}}} \rightarrow \mathbf{z}$, the two RHS term of Eqn. 5.18b will converge to zero which results in a contradiction of the assumption of Eqn. 5.17b. hence, it can be concluded that

$$\lim_{N_{\text{DOF}} \rightarrow \infty} \|\Psi(\mathbf{I} - \hat{\mathbf{P}})\| = 0 \quad (5.19)$$

From Eqn. 5.15c, it can be seen that

$$\|\tilde{\mathbf{e}}_\sigma\| = \|\Psi(\mathbf{I} - \hat{\mathbf{P}})\mathbf{e}_\sigma\| \leq \|\Psi(\mathbf{I} - \hat{\mathbf{P}})\| \|\mathbf{e}_\sigma\| \quad (5.20a)$$

in Eqn. 5.20a, it implies that

$$\frac{\|\tilde{\mathbf{e}}_\sigma\|}{\|\mathbf{e}_\sigma\|} \leq \|\Psi(\mathbf{I} - \hat{\mathbf{P}})\| \Rightarrow \lim_{N_{\text{DOF}} \rightarrow \infty} \frac{\|\tilde{\mathbf{e}}_\sigma\|}{\|\mathbf{e}_\sigma\|} \leq \lim_{N_{\text{DOF}} \rightarrow \infty} \|\Psi(\mathbf{I} - \hat{\mathbf{P}})\| = 0 \quad (5.20b)$$

Hence, from Eqn. 5.14b, one can conclude that if the recovered stress field is constructed by Eqn. 5.9b, the *a posteriori* estimator will be asymptotically exact in the sense that

$$\lim_{N_{\text{DOF}} \rightarrow \infty} \frac{\|\bar{\mathbf{e}}_\sigma\|}{\|\mathbf{e}_\sigma\|} \leq \lim_{N_{\text{DOF}} \rightarrow \infty} \mathcal{J} = 1 \quad (5.21)$$

However, it should be note that Eqn. 5.20 only indicates that the convergence rate of the recovered stress is higher than, but is not always *one* order higher than (i.e., superconvergence) the RKPM stress. Numerical examples given in Section 5.1.6 will show that the difference between the convergence rates of the two stress fields will, depend on the smoothness of the solution and point mesh pattern used.

5.1.4 The extraction procedure

Figure 5.1 The local domain Ω_R in Ω

The idea of recovered stress based on the fundamental solution has been applied on the BEM [113] and the FEM [114]. In the present study, the extraction operator Ψ is a boundary integral operator while the load term is a domain integral operator and Eqn. 5.9b can be written in the form

$$[\tilde{\sigma}(\mathbf{p})] = \int_{\Gamma_R} \Psi \, d\Gamma + \int_{\Omega_R} \Phi \, d\Omega \quad (5.22a)$$

where, $[\tilde{\sigma}(\mathbf{p})]$ is the recovered stress at a given point $\mathbf{p} = (x_1, x_2)$ inside the problem domain (Figure 5.1) with the stress components arranged in a matrix form such that

$$[\tilde{\sigma}(\mathbf{p})] = \begin{bmatrix} \tilde{\sigma}_{xx} & \tilde{\tau}_{xy} \\ \tilde{\tau}_{xy} & \tilde{\sigma}_{yy} \end{bmatrix} = \begin{bmatrix} \tilde{\sigma}_{11} & \tilde{\sigma}_{12} \\ \tilde{\sigma}_{12} & \tilde{\sigma}_{22} \end{bmatrix} \quad (5.22b)$$

The terms Ψ and Φ are 2 by 2 matrices and are functions of the RKPM solution $\hat{\mathbf{u}}$ (Eqn. 3.18) and the body force vector, $\mathbf{b}$ (Eqn. 3.1) such that

$$\Psi = \begin{bmatrix} \psi_{11}(\hat{\mathbf{u}}) & \psi_{12}(\hat{\mathbf{u}}) \\ \psi_{12}(\hat{\mathbf{u}}) & \psi_{22}(\hat{\mathbf{u}}) \end{bmatrix} \quad \text{and} \quad \Phi = \begin{bmatrix} \phi_{11}(\mathbf{b}) & \phi_{12}(\mathbf{b}) \\ \phi_{12}(\mathbf{b}) & \phi_{22}(\mathbf{b}) \end{bmatrix} \quad (5.22c)$$

In Eqn. 5.22a, Ω_R is a local subdomain enclosing the point $\mathbf{p}$ with boundary Γ_R (Figure 5.1.).

In order to obtain the expressions for the operators Ψ and Φ , one can rewrite the weak form of the governing equation (Eqn. 5.1) as

$$\int_{\Omega} \sigma^* (\mathbf{L}^T \sigma + \mathbf{b}) \, d\Omega = 0 \quad (5.23)$$

In Eqn. 5.23, σ^* is the test function used. Suppose that σ^* is selected in such a way that it satisfies the fundamental equation. That is

$$\mathbf{L}^T \sigma^* + \delta^i = 0 \quad (5.24)$$

Where δ^i is the Dirac delta function, then after integration by part for Eqn. 5.23 twice over the whole problem domain, the following integral equation can be obtained for the displacement at point $\mathbf{p}$

$$c_{lk}^p u_k(\mathbf{p}) + \int_{\Gamma} t_{lk}^* u_k \, d\Gamma = \int_{\Gamma} u_{lk}^* t_k \, d\Gamma + \int_{\Omega} u_{lk}^* b_k \, d\Omega, \quad \text{sum over } k; \quad (l, k) = 1, 2 \quad (5.25)$$

In Eqn. 5.25, u_{lk}^* and t_{lk}^* are, respectively, the 2D fundamental displacement and surface tractions solution to Eqn. 5.24, c_{lk}^p is the coefficient associated with point $\mathbf{p}$. Details of the terms in Eqn. 5.25 can be found in Eqns. A.4 and A.5 (Appendix A). By using the constitutive equation in Eqns. 3.1 and Eqn. 5.25, the stress at the point $\mathbf{p}$ can be written as

$$\boldsymbol{\sigma}_{ij}(\mathbf{p}) = \int_{\Gamma} (-s_{kij}^* u_k + d_{kij}^* t_k) d\Gamma + \int_{\Omega} d_{kij}^* b_k d\Omega, \quad \text{sum over } k; (i,j,k) = 1,2 \quad (5.26)$$

where the terms s_{kij}^* and d_{kij}^* can be found in Eqns. A.6 and A.7 (Appendix A).

In practice, the exact displacement and tractions are in general unknown. Hence, the recovered stress at point $\mathbf{p}$, $\tilde{\boldsymbol{\sigma}}_{ij}(\mathbf{p})$ can be obtained by replacing the exact displacement term u_k and the exact traction term t_k by their corresponding RKPM solutions. Furthermore, the domain of integration can be limited to a local subdomain Ω_R (with boundary Γ_R) enclosing point $\mathbf{p}$. Thus, the extracted stress $\tilde{\boldsymbol{\sigma}}_{ij}(\mathbf{p})$ could be obtained by using the expression (c.f. Eqn. 5.26)

$$\tilde{\sigma}_{ij}(\mathbf{p}) = \int_{\Gamma_R} (-s_{kij}^* \hat{u}_k + d_{kij}^* \hat{t}_k) d\Gamma + \int_{\Omega_R} d_{kij}^* b_k d\Omega \quad (5.27a)$$

Eqn. 5.27a can be written in vector form as

$$\tilde{\sigma}_{ij}(\mathbf{p}) = \int_{\Gamma_R} (-\mathbf{s}_{ij}^{*T} \hat{\mathbf{u}} + \mathbf{d}_{ij}^{*T} \hat{\mathbf{t}}) d\Gamma + \int_{\Omega_R} \mathbf{d}_{ij}^{*T} \mathbf{b} d\Omega \quad (5.27b)$$

where,

$$\mathbf{s}_{ij}^* = [s_{1ij}^*, s_{2ij}^*]^T \quad \text{and} \quad \mathbf{d}_{ij}^* = [d_{1ij}^*, d_{2ij}^*]^T \quad (5.27c)$$

By using Eqns. 3.2c and 3.24b which relate $\hat{\mathbf{t}}$ and $\hat{\mathbf{u}}$, Eqn. 5.27b can be rewritten as

$$\begin{aligned} \tilde{\sigma}_{ij}(\mathbf{p}) &= \int_{\Gamma_R} (-\mathbf{s}_{ij}^{*T} \hat{\mathbf{u}} + \mathbf{d}_{ij}^{*T} (\mathbf{Q} \mathbf{D} \mathbf{L} \hat{\mathbf{u}})) d\Gamma + \int_{\Omega_R} \mathbf{d}_{ij}^{*T} \mathbf{b} d\Omega \\ &= \int_{\Gamma_R} \mathbf{d}_{ij}^{*T} (\mathbf{Q} \mathbf{D} \mathbf{L} \hat{\mathbf{u}}) - \mathbf{s}_{ij}^{*T} \hat{\mathbf{u}} d\Gamma + \int_{\Omega_R} \mathbf{d}_{ij}^{*T} \mathbf{b} d\Omega \\ &= \int_{\Gamma_R} (\mathbf{d}_{ij}^{*T} (\mathbf{Q} \mathbf{D} \mathbf{L}) - \mathbf{s}_{ij}^{*T}) \hat{\mathbf{u}} d\Gamma + \int_{\Omega_R} \mathbf{d}_{ij}^{*T} \mathbf{b} d\Omega \end{aligned} \quad (5.27d)$$

Hence, by comparing Eqn. 5.22c with Eqn. 5.27d, it can be concluded that

$$\psi_{ij}(\hat{\mathbf{u}}) = \left((\mathbf{d}_{ij}^*)^T (\mathbf{Q} \mathbf{D} \mathbf{L}) - (\mathbf{s}_{ij}^*)^T \right) \hat{\mathbf{u}} \quad \text{and} \quad \varphi_{ij}(\mathbf{b}) = (\mathbf{d}_{ij}^*)^T \mathbf{b} \quad (5.28)$$

5.1.5 Computation of the recovered stress field $[\tilde{\sigma}(\mathbf{p})]$ and *a posteriori* error estimation

Figure 5.2 Local recovery subdomain Ω_R with circular boundary for interior point $\mathbf{p}$

Figure 5.3 Local recovery subdomain for a point close to the domain boundary

It should be noted that the recovery equation (Eqn. 5.27a) holds irrespective of the size and shape of the local recovery subdomain Ω_R . In this study, in order to simplify the computation task, for a given point $\mathbf{p}$ under consideration, Ω_R is

selected as a circular disc with radius equal to r_R (Figure 5.2). Hence, if the minimum distance between $\mathbf{p}$ and the domain boundary is greater than or equal r_R , the boundary of the recovery subdomain, Γ_R is simply a circle $C(\mathbf{p}, r_R)$ with centre at $\mathbf{p}$ and radius equal to r_R (Figure 5.2). In case that the minimum distance between $\mathbf{p}$ and the domain boundary is less than r_R , Γ_R will consist of two parts such that $\Gamma_R = \Gamma_{Ri} \cup \Gamma_{Rb}$ where Γ_{Ri} is the circular part inside the domain Ω and Γ_{Rb} is the intersection of $C(\mathbf{p}, r_R)$ with Γ (Figure 5.3).

In both cases, Eqn. 5.27a could be evaluated by applying a suitable numerical integration scheme along Γ_R .

For the first boundary integral on the RHS of Eqn. 27a, two possible cases are required to be considered.

- 1). The minimum distance between $\mathbf{p}$ and the domain boundary is greater than or equal to r_R

Figure 5.4 Integration of Eqn. 5.27a when $\Gamma_R = C(\mathbf{p}, r_R)$

In this case, the boundary of the recovery subdomain, Γ_R is complete a circle $C(\mathbf{p}, r_R)$ (Figure 5.4). Since in this case the terms $1/r$ and $1/r^2$ in d_{kij}^* and s_{kij}^* are constant along $C(\mathbf{p}, r_R)$, the first RHS integral of Eqn. 5.27a could be computed by first divides the circle $C(\mathbf{p}, r_R)$ into a number of equal size arc segments that subtends an angle θ at $\mathbf{p}$. Standard Gaussian integration scheme is then applied to

each arc segment. It is found that the accuracy of the integration result is only sensitive to the subtended angle θ but not the order of Gaussian rule used for each segment (see the details in Appendix C). Further numerical investigations found that optimal results could be obtained by setting $\theta = 4^\circ$ and using the single point rule for each arc segment.

2). The minimum distance between $\mathbf{p}$ and the domain boundary is less than r_R

Figure 5.5 Integration of Eqn. 5.27a when $\Gamma_R = \Gamma_{Ri} \cup \Gamma_{Rb}$

In this case, Γ_R will consist of two parts such that $\Gamma_R = \Gamma_{Ri} \cup \Gamma_{Rb}$ where Γ_{Rb} is the intersection of $C(\mathbf{p}, r_R)$ with Γ (Figure 5.5). Obviously, contribution from the first part Γ_{Ri} can be computed in the same way as in the first case. For the second part, since the terms $1/r$ and $1/r^2$ in d_{kij}^* and s_{kij}^* are no longer constant along the line Γ_{Rb} with length equal to L (Figure 5.5), a slightly more complicated integration scheme is needed (see the details in Appendix C). Firstly, Γ_{Rb} will be divided into a number of segments with length ΔL such that

$$\Delta L = \frac{L}{N_{\text{bseg}}} \leq \frac{D}{6}, \quad N_{\text{bseg}} = \text{Int}\left(\frac{6L}{D}\right) + 1 \quad (5.29)$$

In Eqn. 5.29, D is the shortest distance between $\mathbf{p}$ and Γ and $\text{Int}(R)$ is the integer part for the real number R . Secondly, the contribution from Γ_{Rb} is then computed by applying 6-point Gaussian rule along each segment.

For the second domain integral on the RHS of Eqn. 5.27a, if the body force term $\mathbf{b}$ is zero or constant, this integral can be easily computed analytically. In case that $\mathbf{b}$ is not constant within the subdomain, the special Gaussian integration rules described in reference [115] could be used.

By using Eqns. 5.26 to 5.28, the components of the extraction operators Ψ and Φ can be obtained for any point inside Ω . However, when the point $\mathbf{p}$ is located on the domain boundary, some terms defined by Eqns. 5.26b and 5.26c that contain expression like $1/r$ or $1/r^2$ will be singular. In this case, in order to overcome such singular problem, a simple procedure involving the creation of two “offsetting” points and linear interpolation as described below is applied (Figure 5.6).

Figure 5.6 Recovery stress for boundary point $\mathbf{p}$

Assume that the point $\mathbf{p}$ under consideration locates along the domain boundary, two additional points $\mathbf{p}_1$ and $\mathbf{p}_2$ which are lying on the inward normal at $\mathbf{p}$ will be established. The distance between the points $\mathbf{p}$ and $\mathbf{p}_1$, $\mathbf{p}_1$ and $\mathbf{p}_2$ are set equal to a small distance Δp . The recovered stresses at these two points $\tilde{\sigma}(\mathbf{p}_1)$ and $\tilde{\sigma}(\mathbf{p}_2)$ are first computed using Eqn. 5.27a. Finally, the recovered stress the boundary point $\mathbf{p}$, $\tilde{\sigma}(\mathbf{p})$ will be interpolated as

$$\tilde{\sigma}(\mathbf{p}) = 2\tilde{\sigma}(\mathbf{p}_1) - \tilde{\sigma}(\mathbf{p}_2) \quad (5.30)$$

It should be mentioned that since linear relation is assumed in Eqn. 5.30, in order to obtain accurate recovered stress at $\mathbf{p}$, the offsetting distance Δp should be a small value when compared with h_p , the size of domain of influence associates with point

$\mathbf{p}$ for the RKPM (Section 3.3). In the current implementation, Δp is set equal to $0.05 h_p$. Furthermore, if the point $\mathbf{p}$ is located on the traction boundary, the corresponding components of stress recovery solution vector $\tilde{\boldsymbol{\sigma}}(\mathbf{p})$ can be further corrected by the traction boundary condition (see the details in Appendix D).

By using Eqns. 5.9b, 5.22, 5.27 and 5.30, it is possible to compute the recovered stress field $\tilde{\boldsymbol{\sigma}}$ at any location inside the problem domain. Hence, Eqns. 5.12 could be used to compute the estimated pointwise error ($\bar{\mathbf{e}}_\sigma$) and the error norm ($\|\bar{\mathbf{e}}_\sigma\|$) of the RKPM solution. Note that since the integral in Eqn. 5.12 is referred to the whole problem domain, the error norm computed is a global *a posteriori* error estimator. In practice, since the recovered stress is available at any location inside the problem domain, it is also possible to compute the estimated error norm for any subdomain Ω_s as (c.f. Eqn. 5.12)

$$\|\bar{\mathbf{e}}_\sigma\|_{\Omega_s} = \left\{ \int_{\Omega_s} (\bar{\mathbf{e}}_\sigma)^T \mathbf{D}^{-1} (\bar{\mathbf{e}}_\sigma) d\Omega \right\}^{1/2} \quad (5.31)$$

5.1.6 Numerical validation

5.1.6.1 Scope of study

In this section, numerical studies will be carried out to investigate the accuracy and the convergence rate of the recovered stress field. In order to thoroughly test the performance of the recovery procedure, numerical verifications will be carried out by applying the RKPM for the solution of four classic benchmark problems with different domain geometry (convex and concave) and with different smoothness of the exact solution (smooth and singular). For each benchmark problem, linear RKPM is applied and four uniform meshes will be employed to obtain the RKPM solutions. Since the exact solutions for these problems are known, it is possible to compute the relative error of the RKPM solution, η_{RKPM} , and the relative error of the recovered stress, η_{Rv} , as

$$\eta_{\text{RKPM}} = \frac{\|\mathbf{e}_\sigma\|}{\|\mathbf{u}\|} \quad \text{and} \quad \eta_{\text{Rv}} = \frac{\|\tilde{\mathbf{e}}_\sigma\|}{\|\mathbf{u}\|} \quad (5.32a)$$

where $\|\mathbf{u}\|$ is the exact energy norm of the solution

$$\|\mathbf{u}\| = \left\{ \int_{\Omega} \boldsymbol{\sigma}^T \mathbf{D}^{-1} \boldsymbol{\sigma} d\Omega \right\}^{1/2} \quad (5.32b)$$

Furthermore, since linear RKPM is used, when uniform refinement is carried out, it is expected that R_{RKPM} , the convergence rate of η_{RKPM} , defined as

$$R_{\text{RKPM}} = \frac{\log(\eta_{\text{RKPM},i} / \eta_{\text{RKPM},i+1})}{\log(N_{\text{DOF},i+1} / N_{\text{DOF},i})} \quad (5.33)$$

will be given by

$$R_{\text{RKPM}} = \min(p^R / 2, \lambda / 2) \quad (5.34)$$

In Eqn. 5.34, $p^R = 1$ is the order of approximation of the RKPM and $\lambda < 1$ is the strength of singularity of the problem (see the details in Section 4.23). In Eqn. 5.33, $\eta_{\text{RKPM},i}$ and $N_{\text{DOF},i}$ are, respectively, the relative error and the degrees of freedom for the i th RKPM refinement analysis.

Similarly, the convergence rate of the recovered solution, R_{Rv} can be computed as

$$R_{\text{Rv}} = \frac{\log(\eta_{\text{Rv},i} / \eta_{\text{Rv},i+1})}{\log(N_{\text{DOF},i+1} / N_{\text{DOF},i})} \quad (5.35)$$

with $\eta_{\text{Rv},i}$ is the relative error for the recovered solution obtained from the i th refinement analysis.

From Eqn. 5.14b, it can be seen that the effectivity index of the *a posteriori* error estimator will be asymptotically exact if $R_{\text{Rv}} > R_{\text{RKPM}}$.

In the numerical study, for each RKPM solution obtained, three different recovered stress fields will be computed by using three different values of r_R (Figure 5.2) such that

$$r_R = k h, \quad k = 1, 2 \text{ and } 3 \quad (5.36)$$

In Eqn. 5.36, h is the size of the domain of influence used in the RKPM. Since here uniform meshes are employed, h will be a constant for all the nodes in the mesh. For the different values of r_R tested, values of η_{Rv} , R_{Rv} and the corresponding values of effectivity index $\mathcal{G}$ (Eqn. 5.14b) are computed.

Note that the accuracy of the recovery stress could, in general, be improved [70] by adopting a bigger subdomain Ω_R in the recovered procedure. However, it should be mentioned that a larger Ω_R will lead to a higher computational cost for the integration of Eqn. 5.27 and thus in practice, an optimal value of r_R should be used.

5.1.6.2 The patch test

Figure 5.7 Patch test for the recovery procedure

One necessary condition for the asymptotically exactness of the *a posteriori* error estimator is that the recovered stress field obtained from Eqn. 5.27 must be able to reproduce the exact stress field if the solution of the RKPM solution is exact. Since linear RKPM is employed for the solution in the numerical study, a simple patch test for the recovered procedure can be carried out by considering the model problem shown in Figure 5.7, in which, a rectangular plate under plane stress conditions is loaded by uniform traction force on one side. Due to the boundary conditions imposed, a constant stress state of $\sigma_{xx}=1.0$ will be achieved and can be solved exactly by the RKPM solution. The recovered stress $\tilde{\sigma}_{xx}$ along the line $x=1.0$ obtained by using different values of the penalty term α in the RKPM formulation (Eqn. 3.20) are listed in Table 5.1. It could be seen from Table 5.1 that for all the penalty values used, the recovered stress obtained are all very close to the exact value of unity.

Node coordinates		(1.0, 0)	(1.0, 0.25)	(1.0, 0.5)	(1.0, 0.75)	(1.0, 1.0)
Penalty coefficient α	$10^1 E$	1.0027	0.99978	0.99937	0.99994	1.0031
	$10^3 E$	1.0027	0.99979	0.99937	0.99995	1.0031
	$10^6 E$	1.0027	0.99979	0.99937	0.99995	1.0031
	$10^9 E$	1.0027	0.99982	0.99938	0.99993	1.0031

Table 5.1 Patch test for recovered stress value $\tilde{\sigma}_{xx}$ along the line $x=1$

5.1.6.3 Numerical examples

Example 1: Timoshenko beam (convex domain, smooth exact solution)

The first example is the Timoshenko beam problem as shown in Section 4.3.1. The stress recovery results for the whole domain under different point mesh (shown in Figure 4.5) are shown in Table 5.2.

Example 2: Infinite plate with a circular hole (concave domain, smooth exact solution)

In the second example, an infinite plate with a circular hole example as shown in Section 4.3.2 is considered. The stress recovery results for the whole domain under different point mesh (shown in Figure 4.10) are shown in Table 5.3.

Example 3: Domain near a crack tip (convex domain, singular exact solution)

In the third example, the problem domain is a region near the crack tip as shown in Section 4.3.3. The stress recovery results for the whole domain under different point mesh (shown in Figure 4.15) are shown in Table 5.4.

Example 4: L-shaped domain (concave domain, singular exact solution)

The last example is an L-shaped domain problem as shown in Section 4.3.4. The stress recovery results for the whole domain under different point mesh (shown in Figure 4.20) are shown in Table 5.5.

Mesh	1	2	3	4
N_{DOF}	54	170	252	594
h	5	2.5	2.0	1.25
η_{RKPM} (%)	14.53	6.58	5.10	2.89
$\eta_{\text{Rv}}(r_R = h)$ (%)	7.73	2.92	2.1	1.02
$\eta_{\text{Rv}}(r_R = 2h)$ (%)	7.03	2.62	1.87	0.91
$\eta_{\text{Rv}}(r_R = 3h)$ (%)	6.73	2.51	1.8	0.88
R_{RKPM}		0.69	0.76	0.66
$R_{\text{Rv}}(r_R = h)$		0.85	0.84	0.83
$R_{\text{Rv}}(r_R = 2h)$		0.86	0.85	0.84
$R_{\text{Rv}}(r_R = 3h)$		0.86	0.85	0.84
$\mathcal{G}(r_R = h)$	0.59	0.67	0.73	0.82
$\mathcal{G}(r_R = 2h)$	0.63	0.71	0.78	0.83
$\mathcal{G}(r_R = 3h)$	0.67	0.73	0.8	0.85

Table 5.2 Recovery results for the Timoshenko beam

Mesh	1	2	3	4
N_{DOF}	72	164	240	564
h	1.0	0.625	0.5	0.3125
η_{RKPM} (%)	7.78	4.48	3.52	2.13
$\eta_{\text{Rv}}(r_R = h)$ (%)	4.63	2.26	1.63	0.81
$\eta_{\text{Rv}}(r_R = 2h)$ (%)	4.25	2.06	1.49	0.73
$\eta_{\text{Rv}}(r_R = 3h)$ (%)	4.09	1.98	1.42	0.70
R_{RKPM}		0.67	0.63	0.59
$R_{\text{Rv}}(r_R = h)$		0.87	0.85	0.82
$R_{\text{Rv}}(r_R = 2h)$		0.88	0.85	0.83
$R_{\text{Rv}}(r_R = 3h)$		0.88	0.87	0.83
$\mathcal{G}(r_R = h)$	0.62	0.66	0.72	0.82
$\mathcal{G}(r_R = 2h)$	0.65	0.70	0.75	0.84
$\mathcal{G}(r_R = 3h)$	0.68	0.73	0.78	0.86

Table 5.3 Recovery results for the infinite plate with a circular hole

Mesh	1	2	3	4
N_{DOF}	30	90	132	306
h	0.5	0.25	0.2	0.125
η_{RKPM} (%)	29.7	20.9	18.8	15.2
$\eta_{\text{RV}}(r_R = h)$ (%)	19.12	12.45	10.93	8.42
$\eta_{\text{RV}}(r_R = 2h)$ (%)	17.97	11.71	10.24	7.82
$\eta_{\text{RV}}(r_R = 3h)$ (%)	17.21	11.09	9.7	7.41
R_{RKPM}		0.32	0.28	0.25
$R_{\text{RV}}(r_R = h)$		0.39	0.34	0.31
$R_{\text{RV}}(r_R = 2h)$		0.39	0.35	0.32
$R_{\text{RV}}(r_R = 3h)$		0.40	0.35	0.32
$\mathcal{G}(r_R = h)$	0.4	0.48	0.5	0.53
$\mathcal{G}(r_R = 2h)$	0.46	0.54	0.57	0.61
$\mathcal{G}(r_R = 3h)$	0.49	0.56	0.6	0.63

Table 5.4 Recovery results for the domain near a crack tip

Mesh	1	2	3	4
N_{DOF}	66	134	226	342
h	0.3535	0.2357	0.17675	0.1414
η_{RKPM} (%)	23.5	18.5	15.9	14.3
$\eta_{\text{RV}}(r_R = h)$ (%)	15.02	11.4	9.49	8.31
$\eta_{\text{RV}}(r_R = 2h)$ (%)	14.12	10.71	8.87	7.74
$\eta_{\text{RV}}(r_R = 3h)$ (%)	13.52	10.18	8.43	7.35
R_{RKPM}		0.34	0.29	0.26
$R_{\text{RV}}(r_R = h)$		0.39	0.35	0.32
$R_{\text{RV}}(r_R = 2h)$		0.39	0.36	0.33
$R_{\text{RV}}(r_R = 3h)$		0.4	0.36	0.33
$\mathcal{G}(r_R = h)$	0.42	0.47	0.54	0.58
$\mathcal{G}(r_R = 2h)$	0.46	0.55	0.59	0.63
$\mathcal{G}(r_R = 3h)$	0.5	0.58	0.63	0.66

Table 5.5 Recovery results for the L-shaped domain

5.1.7 Discussions on obtained numerical results

From the numerical results listed in Tables 5.2 to 5.5, it can be observed that

1). Regarding the accuracies and convergence rates of the RKPM solutions, it can be seen that in Examples 1 and 2 where the exact solution is smooth, the convergence rate of the RKPM solutions is approximately equal to the theoretical rate of $p^R/2=0.5$. However, such normal convergence rate is reduced to $\lambda/2$ in Examples 3 and 4 when singularity exists in the problem domain.

2). Regarding the accuracy of the recovered stress field, it can be seen from Tables 5.2 to 5.5 that in all four benchmark problems and for all the meshes used, the recovered stress fields are more accurate than the corresponding RKPM solutions. Furthermore, the larger the size of the recovery radius r_R used, the more accurate the recovered stress obtained. Tables 5.2 to 5.4 also show that for the four benchmark examples given, the relative errors of the recovered stress fields obtained in all the cases are roughly equal to one half of the corresponding RKPM solution and these agree with the prediction of Eqn. 5.20a.

3). Regarding the convergence rates of the recovered solutions, it can be seen that, again, they are all greater than their corresponding RKPM counterparts. In all the examples, the convergence rates of the recovered solutions are improved by increasing the recovery radius from h to $2h$. However, no further improvement could be found by changing r_R from $2h$ to $3h$. Hence it appears that the optimal value of r_R should be equal to $2h$. Note that in Examples 1 and 2, the recovered stresses are nearly superconvergent with convergence rates > 0.8 . However, in Examples 3 and 4, despite that the convergence rates of the recovered solutions are higher than the RKPM solutions, they are also affected by the present of singularity and are still less than the normal convergence rate of the RKPM solution.

4). Regarding the effectivity indices of the *a posteriori* error estimator, it can be seen from Tables 5.2 to 5.5 that, in general, a higher value of r_R leads to better values of $\mathcal{J}$. In particular, in Examples 1 and 2 when the recovered stress fields are

almost superconvergent, the values of $\mathcal{J}$ are clearly converging to unity even the final meshes used in these two examples are relatively coarse with $N_{\text{DOF}} < 600$. For Examples 3 and 4, since the convergence rates of the recovered stress fields are affected by the singularity, the values of $\mathcal{J}$ obtained are lower than the results obtained in Examples 1 and 2. However, even in these cases, it can still be seen from Tables 5.4 and 5.5 that values of $\mathcal{J}$ are also increased as the meshes are refined.

5.2 *A posteriori* error estimation for the BEM

A posteriori error estimation and the consequent adaptive strategy are two crucially important factors in the adaptive procedure for the BEM. The basic idea of the *a posteriori* error estimation is to introduce a reliable error indicator for each element. The adaptive process is then driven by the error indicator and the accuracy of the boundary element numerical solution is evaluated at the same time. In this section, the *a posteriori* error estimator based on the residual approach [113] for the modelled problem defined in Section 3.1 is introduced and the details are as given below.

5.2.1 Direct boundary stress based on the BEM solution

Figure 5.8 Stress components of point p on the boundary of the BEM domain

When the nodal value $\hat{\mathbf{u}}_e$ and $\hat{\mathbf{t}}_e$ (in Eqn. 3.9) of base points on the boundary element are all available, the direct stress $\hat{\boldsymbol{\sigma}} = \{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\tau}_{xy}\}^T = \{\hat{\sigma}_{11}, \hat{\sigma}_{22}, \hat{\sigma}_{12}\}^T$ based on the BEM solution of any point $\mathbf{p}$ on the boundary of the BEM domain Ω can be computed by using boundary condition and the numerical tangential derivatives of the boundary displacement. The details are listed as following:

The direct stress components $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\tau}_{xy}$ at point $\mathbf{p}$ on the boundary of domain Ω are shown in Figure 5.8. n - τ is the local coordinate system and x - y is the global coordinates system, θ is the angle between the outer-normal line of the boundary and x axis at the point $\mathbf{p}$.

1). Stress components on local n - τ coordinate system

From the equilibrium condition, the normal stress $\hat{\sigma}_n$ along outer normal line n and the shear stress $\hat{\tau}_{n\tau}$ along the tangential line τ can be computed directly as

$$\hat{\sigma}_n = \hat{t}_1 \cos \theta + \hat{t}_2 \sin \theta \quad (5.37a)$$

$$\hat{\tau}_{n\tau} = -\hat{t}_1 \sin \theta + \hat{t}_2 \cos \theta \quad (5.37b)$$

The normal stress $\hat{\sigma}_\tau$ along the tangential line τ can not be solved directly and the tensor form of the constitutional law (defined in Eqn. 3.1b and 3.1c) is introduced here as

$$\sigma_{ij} = \bar{\lambda} u_{k,k} \delta_{ij} + \bar{G} (u_{i,j} + u_{j,i}), \text{ sum over } k; (i, j, k) = 1, 2 \quad (5.38)$$

where, $\bar{\lambda} = \frac{E\mu}{(1+\mu)(1-2\mu)}$ and $\bar{G} = \frac{E}{2(1+\mu)}$.

By using Eqn. 5.38, $\hat{\sigma}_n$ and $\hat{\sigma}_\tau$ can then be written as

$$\hat{\sigma}_n = \bar{\lambda} (\hat{\epsilon}_n + \hat{\epsilon}_\tau) + 2\bar{G} \hat{\epsilon}_n \quad (5.39a)$$

$$\hat{\sigma}_\tau = \bar{\lambda} (\hat{\epsilon}_n + \hat{\epsilon}_\tau) + 2\bar{G} \hat{\epsilon}_\tau \quad (5.39b)$$

in Eqn. 5.39, $\hat{\epsilon}_n$ is the strain along the outer normal line n and $\hat{\epsilon}_\tau$ is the strain along the tangential line τ .

By eliminating the term $\hat{\varepsilon}_n$ in Eqn. 5.39, the normal stress $\hat{\sigma}_\tau$ can be calculated as

$$\hat{\sigma}_\tau = (2\bar{G} \hat{\varepsilon}_\tau + \mu \hat{\sigma}_n) / (1 - \mu) \quad (5.40)$$

Moreover, the strain $\hat{\varepsilon}_\tau$ can be computed from the numerical tangential derivative of the displacement as

$$\hat{\varepsilon}_\tau = \frac{\partial \hat{u}_\tau}{\partial \tau} = \frac{\partial(-\hat{u}_1 \sin \theta + \hat{u}_2 \cos \theta)}{\partial \tau} = -\frac{\partial \hat{u}_1}{\partial \tau} \sin \theta + \frac{\partial \hat{u}_2}{\partial \tau} \cos \theta \quad (5.41)$$

Eventually, the normal stress $\hat{\sigma}_\tau$ can be obtained by substituting Eqn. 5.41 into Eqn. 5.40 as

$$\hat{\sigma}_\tau = [2\bar{G} (-\frac{\partial \hat{u}_1}{\partial \tau} \sin \theta + \frac{\partial \hat{u}_2}{\partial \tau} \cos \theta) + \mu (\hat{t}_1 \cos \theta + \hat{t}_2 \sin \theta)] / (1 - \mu) \quad (5.42)$$

2). Stress components on global x - y coordinate system

When the stress components $\hat{\sigma}_n$, $\hat{\sigma}_\tau$ and $\hat{t}_{n\tau}$ are available, the stress components $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{t}_{xy}$ can then be obtained by a transformation of coordinates between the global x - y coordinate system and the n - τ coordinates system

$$\hat{\sigma}_x = \hat{\sigma}_n \cos^2 \theta - 2\hat{t}_{n\tau} \sin \theta \cos \theta + \hat{\sigma}_\tau \sin^2 \theta \quad (5.43a)$$

$$\hat{\sigma}_y = \hat{\sigma}_n \sin^2 \theta + 2\hat{t}_{n\tau} \sin \theta \cos \theta + \hat{\sigma}_\tau \cos^2 \theta \quad (5.43b)$$

$$\hat{t}_{xy} = (\hat{\sigma}_n - \hat{\sigma}_\tau) \sin \theta \cos \theta + \hat{t}_{n\tau} (\cos^2 \theta - \sin^2 \theta) \quad (5.43c)$$

5.2.2 Recovery boundary stress based on the hyper-singular boundary integral equation

From Eqn. A.6b defined in Appendix A, the hyper-singular boundary integral equation for the exact stress $\boldsymbol{\sigma} = \{\sigma_x, \sigma_y, \tau_{xy}\}^T = \{\sigma_{11}, \sigma_{22}, \sigma_{12}\}^T$ at the point $\mathbf{p}$ on the boundary of the BEM domain can also be written as

$$\sigma_{ij} = -\int_{\Gamma} s_{ijk}^* u_k d\Gamma + \int_{\Gamma} d_{ijk}^* t_k d\Gamma + \int_{\Gamma} F_{ij}^* d\Gamma, \text{ sum over } k; (i, j, k) = 1, 2 \quad (5.44)$$

In Eqn. 5.44, if the exact solution u_k and t_k are replaced by the approximated solution $\hat{u}_k$ and $\hat{t}_k$ (from the BEM), respectively, the recovery boundary stress $\tilde{\sigma} = \{\tilde{\sigma}_x, \tilde{\sigma}_y, \tilde{\tau}_{xy}\}^T = \{\tilde{\sigma}_{11}, \tilde{\sigma}_{22}, \tilde{\sigma}_{12}\}^T$ based on the hyper-singular BIE (Eqn. 5.44) can be arrived as

$$\tilde{\sigma}_{ij} = -\int_{\Gamma} s_{ijk}^* \hat{u}_k d\Gamma + \int_{\Gamma} d_{ijk}^* \hat{t}_k d\Gamma + \int_{\Gamma} F_{ij}^* d\Gamma, \text{ sum over } k; (i, j, k) = 1, 2 \quad (5.45)$$

5.2.3 *A posteriori* error estimation based on the residual approach

In the residual type *a posteriori* error estimator for the BEM, the value of the residual on the residual of boundary stress components between the BEM solution and the hyper-singular boundary integral equation is used as the indicator for the solution error [113]. The details are described as following:

Generally, the value of $\tilde{\sigma}_{ij}$ will not be the same as that of $\hat{\sigma}_{ij}$ in the elements of boundary of the BEM domain. Then the difference $\bar{\mathbf{e}}_R = \{\bar{e}_{R_{11}}, \bar{e}_{R_{22}}, \bar{e}_{R_{12}}\}^T$ between $\tilde{\sigma}_{ij}$ and $\hat{\sigma}_{ij}$ can be represented as

$$\bar{\mathbf{e}}_{R_{ij}} = \tilde{\sigma}_{ij} - \hat{\sigma}_{ij} = \int_{\Gamma} (-s_{ijk}^* \hat{u}_k + d_{ijk}^* \hat{t}_k) d\Gamma + \int_{\Gamma} F_{ij}^* d\Gamma - \hat{\sigma}_{ij} \quad (5.46)$$

Let e_{ij} be the true error between σ_{ij} and $\hat{\sigma}_{ij}$, i.e.,

$$e_{ij} = \sigma_{ij} - \hat{\sigma}_{ij} \quad (5.47)$$

Combining Eqns. 5.44, 5.46 and 5.47 together, the residual $\bar{\mathbf{e}}_{R_{ij}}$ can then be written as

$$\bar{\mathbf{e}}_{R_{ij}} = e_{ij} - \int_{\Gamma} [-s_{ijk}^* (u_k - \hat{u}_k) + d_{ijk}^* (t_k - \hat{t}_k)] d\Gamma, \text{ sum over } k; (i, j, k) = 1, 2 \quad (5.48)$$

and the matrix form of Eqn. 5.48 can be expressed as

$$\bar{\mathbf{e}}_{R_{ij}} = e_{ij} - \mathbf{v}_{ijk}^u (u_k - \hat{u}_k) - \tilde{\mathbf{v}}_{ijk}^t (t_k - \hat{t}_k) \quad (5.49)$$

where,

$$v_{ijk}^u(u_k - \hat{u}_k) = \int_{\Gamma} -s_{ijk}^*(u_k - \hat{u}_k) d\Gamma \quad (5.50a)$$

$$\tilde{v}_{ijk}^t(t_k - \hat{t}_k) = \int_{\Gamma} d_{ijk}^*(t_k - \hat{t}_k) d\Gamma \quad (5.50b)$$

From the boundary condition defined in Eqn.3.3b, $t_k - \hat{t}_k$ can be represented by e_{ij} , then Eqn. 5.49 can be rewritten as

$$\bar{\mathbf{e}}_{R_{ij}} = -v_{ijk}^u(u_k - \hat{u}_k) + (I_{ijk} - v_{ijk}^t) e_{ij}, \text{ sum over } k; (i, j, k) = 1, 2 \quad (5.51)$$

where,

$$v_{ijk}^t e_{ij} = \int_{\Gamma} d_{ijk}^*(t_k - \hat{t}_k) d\Gamma \quad (5.52)$$

1). For Dirichlet problem [113]

In this case, u_i is prescribed and t_i is unknown, i.e., $u_k - \hat{u}_k = 0$. If the considered boundary is sufficiently smooth (C^1 continuous), and v_{ijk}^t is compact and $I_{ijk} - v_{ijk}^t$ is invertible. Thus the error norm of e_{ij} is bounded by the residual norm of $\bar{\mathbf{e}}_{R_{ij}}$ as $c_1 \|\bar{\mathbf{e}}_R\| \leq \|\mathbf{e}\| \leq c_2 \|\bar{\mathbf{e}}_R\|$, $0 < c_1 < c_2$, c_1 and c_2 are two positive constants. $\|\cdot\|$ is the commonly used energy norm

$$\|\bar{\mathbf{e}}_R\| = \left\{ \int_{\Gamma} \bar{\mathbf{e}}_R^T \mathbf{D}^{-1} \bar{\mathbf{e}}_R d\Gamma \right\}^{1/2}, \quad \|\mathbf{e}\| = \left\{ \int_{\Gamma} \mathbf{e}^T \mathbf{D}^{-1} \mathbf{e} d\Gamma \right\}^{1/2} \quad (5.53)$$

2). For Neumann problem [113]

In this case, t_i is prescribed and u_i is unknown, i.e., $t_k - \hat{t}_k = 0$. Since v_{ijk}^u is a hyper-singular integral operator, it magnifies the high frequency terms of $u_k - \hat{u}_k$, therefore, $\bar{\mathbf{e}}_{R_{ij}}$ can be used as an *a posteriori* error indicator with sufficient smooth boundary condition.

3). For mixed boundary value problem [113]

For most engineering problems, the mixed boundary value problems with non-smooth boundary are often met. The mathematical analysis of error estimation for

these problems is quite difficult, especially for the collocation BEM. However, due to the pseudo-locality of the boundary integral operator, the hyper-singular residual equation can still be used for the error estimation for a certain BEM solution of a mixed boundary value problem.

In practical computation, the boundary will be discretized into NE linear element, and then residual $\bar{\mathbf{e}}_{R_{ij}}$ defined in Eqn. 5.46 can be calculated as

$$\bar{\mathbf{e}}_{R_{ij}} = \sum_{k=1}^{NE} \int_{\Gamma_i} (-s_{ijk}^* \hat{u}_k + d_{ijk}^* \hat{t}_k + F_{ij}^*) d\Gamma - \hat{\sigma}_{ij}, \text{ sum over } k; (i, j, k) = 1, 2 \quad (5.54)$$

When the point residual $\bar{\mathbf{e}}_{R_{ij}}$ is available, the local residual error norm $\|\bar{\mathbf{e}}_R\|_i$ ($i = 1, 2, \dots, NE$), associated with the i th element, can be defined as

$$\|\bar{\mathbf{e}}_R\|_i = \frac{1}{L_i} \left\{ \int_{\Gamma_i} \bar{\mathbf{e}}_R^T \mathbf{D}^{-1} \bar{\mathbf{e}}_R d\Gamma \right\}^{1/2}, \quad i = 1, 2, \dots, NE \quad (5.55a)$$

where, L_i is the length of element Γ_i .

Then the global residual error norm $\|\bar{\mathbf{e}}_R\|$ is defined as

$$\|\bar{\mathbf{e}}_R\| = \frac{1}{L} \left\{ \int_{\Gamma} \bar{\mathbf{e}}_R^T \mathbf{D}^{-1} \bar{\mathbf{e}}_R d\Gamma \right\}^{1/2} \quad (5.55b)$$

where, $L = \sum_{i=1}^{NE} L_i$ is the length of the whole elements Γ .

The results of numerical experiments on benchmark problem for adaptive refinement process will be discussed in Chapter 6. Further details of the numerical experiments on the residual type *a posteriori* error estimator for the BEM can be referred the works by Chen and Yu [113].

CHAPTER 6

ADAPTIVE REFINEMENT FOR BEM, RKPM AND COUPLED BE-RKPM

Based on the *a posteriori* error estimators described in Chapter 5, the automatic adaptive refinement procedure for the BEM, the RKPM and the coupled BE-RKPM are developed in this Chapter. Numerical experiments on the 2D elastostatic problems are also conducted to test the efficiency of the adaptive schemes for the BEM, the RKPM and the coupled BE-RKPM, respectively.

6.1 Adaptive refinement for the BEM

Error estimation and the consequent adaptive mesh refinement are crucially important in numerical engineering analysis. Research on the adaptive refinement strategy for the BEM began in the early 1980s and it has been applied to many engineering problems [22-26]. As a general rule, the accuracy of the boundary solution depends critically on the discretization of the boundary by elements. The distribution of the elements is one of the most important decisions in the adaptive refinement analysis. In this section, the *h*-adaptive refinement scheme based on the residual methods [113] described in Chapter 5 is adopted to guide the adaptive process for the BEM.

6.1.1 Global and local refinement indicators of the BEM

When the initial mesh or the mesh in any iteration step (with NE element) is used for a standard BEM analysis, the recovery boundary stress $\tilde{\sigma}$ can be calculated from Eqn. 5.45. Then the local boundary energy norm $\|\mathbf{u}\|_{\Gamma_i}$ and global boundary energy norm $\|\mathbf{u}\|_{\Gamma}$ can be defined as

$$\|\mathbf{u}\|_{\Gamma_i} = \frac{1}{L_i} \left\{ \int_{\Gamma_i} \tilde{\boldsymbol{\sigma}}^T \mathbf{D}^{-1} \tilde{\boldsymbol{\sigma}} d\Gamma \right\}^{1/2}, \quad i = 1, 2, \dots, NE \quad (6.1a)$$

$$\|\mathbf{u}\|_{\Gamma} = \frac{1}{L} \left\{ \int_{\Gamma} \tilde{\boldsymbol{\sigma}}^T \mathbf{D}^{-1} \tilde{\boldsymbol{\sigma}} d\Gamma \right\}^{1/2} \quad (6.1b)$$

By using the reference stress, the local error indicator $\bar{\eta}_{R_i}$, which estimates the discretization error in the i th element, and the global error indicator $\bar{\eta}_R$, which estimates the global discretization error, can then be defined as

$$\bar{\eta}_{R_i} = \frac{\|\bar{\mathbf{e}}_R\|_i}{\|\mathbf{u}\|_{\Gamma_i}} \quad (i = 1, NE), \quad \bar{\eta}_R = \frac{\|\bar{\mathbf{e}}_R\|}{\|\mathbf{u}\|_{\Gamma}} \quad (6.2)$$

where, $\|\bar{\mathbf{e}}_R\|_i$ and $\|\bar{\mathbf{e}}_R\|$ (Eqn. 5.55) are the local and global residual error norm of the BEM.

6.1.2 Automatic refinement based on h -adaptive scheme

An h -adaptive scheme is adopted in this study and the details of implementation are summarized as following [113]:

- i). Read the initial element mesh data and solve the BIE equation.
- ii). Compute the global indicator $\bar{\eta}_R$ and the local indicators $\bar{\eta}_{R_i}$.
- iii). Stop the mesh refinement and export the data interested, if one of the following conditions are satisfied; otherwise go to step iv:
 - a). $\bar{\eta}_R < \eta_{\text{target}}^G$, where η_{target}^G is the global prescribed target error for the BEM.
 - b). $\max\{\bar{\eta}_{R_i}\} < \eta_{\text{target}}^L$ ($i = 1, 2, \dots, NE$), where η_{target}^L is the local prescribed target error for the BEM.
 - c). The iteration number is equal to a setting number (3 iterations).
- iv). Store the processed data of the current iteration and subdivide those elements which satisfy $\bar{\eta}_R > \eta_{\text{target}}^G$ into two equal size (length) elements.
- v). Subdivide the longer element of two neighbor ones, if the length ratio between them is outside the range (0.25, 4.0). Generate new mesh data and return to Step ii.

6.1.3 Numerical examples for the BEM

6.1.3.1 Scope of study

In this section, the four 2D benchmark problems described in Chapter 4 are reused to test the efficiency of the adopted h -adaptive scheme. $\eta_{\text{target}}^G = 5\%$ and $\eta_{\text{target}}^L = 5\%$ [113] are used for the problems with smooth exact solutions and $\eta_{\text{target}}^G = 15\%$ and $\eta_{\text{target}}^L = 15\%$ [113] are used for the problems with singular exact solutions. For all the adaptive analysis, the following results are computed:

- i). The global error indicator, $\bar{\eta}_R$ (Eqn. 6.2).
- ii). The exact relative error of the adaptive BEM solution, η_{BEM} (Eqn. 4.38).
- iii). The convergence rate of the adaptive BEM solution, R_{BEM} , defined as

$$R_{\text{BEM}} = \frac{\log(\eta_{\text{BEM}, i} / \eta_{\text{BEM}, i+1})}{\log(N_{\text{DOF}, i+1} / N_{\text{DOF}, i})} \quad (6.3)$$

where, $\eta_{\text{BEM}, i}$ are $N_{\text{DOF}, i}$ the relative error and the number of degree of freedoms of the i th refinement analysis of the adaptive BEM respectively.

6.1.3.2 Timoshenko beam (convex domain, smooth exact solution)

The first example is the Timoshenko beam problem as shown in Section 4.3.1. The meshes used for the adaptive BEM are shown in Figure 6.1.

Figure 6.1 Adaptive BEM mesh used for the Timoshenko beam

The results obtained are summarized in Table 6.1 while the rates of convergence for both uniform and adaptive refinements are plotted in Figure 6.2, respectively.

Mesh	Initial	1 st	2 st	3 st
N_{DOF}	80	136	200	280
$\bar{\eta}_R$ (%)	14.7	10.2	8.6	6.5
η_{BEM} (%)	9.3	4.6	2.7	1.7
R_{BEM}		1.33	1.38	1.37

Table 6.1 Adaptive BEM results for the Timoshenko beam

Figure 6.2 Convergence rates R_{BEM} for the Timoshenko beam

6.1.3.3 Infinite plate with a circular hole (concave domain, smooth exact solution)

In the second example, an infinite plate with a circular hole example as shown in Section 4.3.2 is considered. The meshes used for the adaptive BEM are shown in Figure 6.3. The results obtained are summarized in Table 6.2 while the rates of convergence for both uniform and adaptive refinement are shown in Figure 6.4, respectively.

Figure 6.3 Adaptive BEM mesh used for the infinite plate with a circular hole

Mesh	Initial	1 st	2 st	3 st
N_{DOF}	40	80	120	184
$\bar{\eta}_R$ (%)	17.8	13.2	10.6	6.7
η_{BEM} (%)	7.6	3.4	2.0	1.1
R_{BEM}		1.16	1.31	1.40

Table 6.2 Adaptive BEM results for the infinite plate with a circular hole

Figure 6.4 Convergence rates R_{BEM} for the infinite plate with a circular hole

6.1.3.4 Domain near a crack tip (convex domain, singular exact solution)

In the third example, the problem domain is a region near the crack tip as shown in Section 4.3.3. The meshes used for the adaptive BEM are shown in Figure 6.5. The results obtained are summarized in Table 6.3 while the rates of convergence for both uniform and adaptive refinements are plotted in Figure 6.6, respectively.

6.1.3.5 L-shape domain (concave domain, singular exact solution)

The last example is an L-shaped domain problem as shown in Section 4.3.4. The meshes used for the adaptive BEM are shown in Figure 6.7 while the results obtained are summarized in Table 6.4, respectively. The rates of convergence for both uniform and adaptive refinements are plotted in Figure 6.8.

Figure 6.5 Adaptive BEM mesh used for the domain near a crack tip

Mesh	Initial	1 st	2 st	3 st
N_{DOF}	96	128	180	224
$\bar{\eta}_R$ (%)	45.8	32.5	28.2	17.5
η_{BEM} (%)	17.9	15.7	13.3	11.9
R_{BEM}		0.46	0.49	0.51

Table 6.3 Adaptive BEM results for the domain near a crack tip

Figure 6.6 Convergence rates R_{BEM} for the domain near a crack tip

Figure 6.7 Adaptive BEM mesh used for the L-shaped domain

Mesh	Initial	1 st	2 st	3 st
N_{DOF}	80	120	180	244
$\bar{\eta}_R$ (%)	30.9	27.6	20.5	17.1
η_{BEM} (%)	18.3	14.7	11.8	10.0
R_{BEM}		0.54	0.54	0.55

Table 6.4 Adaptive BEM results for the L-shaped domain

Figure 6.8 Convergence rates R_{BEM} for the L-shaped domain

6.1.4 Discussions on obtained BEM numerical results

From the numerical results listed in Tables 6.1-6.4, it can be observed that the convergence rates of adaptive element mesh are greater than that of the uniform mesh (Table 4.1-4.4). For the smooth problems, the element meshes are denser around the place where high stress gradients are located and for the singular problems, the elements meshes are denser around the place where the singular points are located. So from the numerical results, it can be concluded that the adopted h -type mesh refinement can well track the numerical error efficiently.

6.2 Adaptive refinement for the RKPM

After nearly decade's research, the development of an automatic adaptive refinement procedure for the FEM can be considered as a mature area in computational mechanics. This fact is confirmed by the large number of references quoted in the bibliographies and special issues related to adaptive analysis and meshing [116-119]. When compared with the FEM, the development of an automatic adaptive refinement procedure for the meshless methods is a relatively new research area and only received attentions in recently years. As a result, fewer papers related to this topic had been published to date [36, 77-81]. One of the possible explanations for such observation is that many meshless methods like the EFGM [31] and the RKPM [35] are relatively new numerical methods that only received major attention since the last decade. As a result, a reliable *a posteriori* error estimator for the meshless methods, which is an essential ingredient and prerequisite for any adaptive refinement procedure, is only available in the past few years [78-81].

However, it should be pointed out that once a reliable *a posteriori* error estimator is available, development and implementation of an automatic adaptive refinement procedure for the meshless methods is a relatively easy task when compared with the case of the FEM. The main reason is that when compared with the FEM, the meshless methods enjoy much more flexibility in model generation and no complicated algorithm is needed for the generation of an adaptive finite point meshes even for surfaces [120, 121] and 3D [122] problem domains.

In this section, an automatic adaptive refinement procedure for the RKPM will be suggested. The proposed refinement procedure will employ the new stress extraction scheme which has been described in Chapter 5 for the construction of an extracted (recovered) stress field. The extracted stress field will then be used with Zienkiewicz and Zhu (Z-Z) error estimation approach for the *a posteriori* error estimation of the RKPM solution. An adaptive refinement strategy will be suggested to control the discretization error of the RKPM solution. New point spacing required to reduce the error of the RKPM solution will be determined by

making use of the *a priori* convergence rate and the quantitative error distribution information provided by the *a posteriori* error estimator. The refined point mesh is then generated by a simple point mesh generation scheme [121].

6.2.1 Variable support size determination for an adaptive RKPM mesh

In an adaptive refinement analysis, unlike the case of uniform refinement in which the spacings for all the points are roughly the same, the spacings of points in an adaptive point mesh could vary greatly in the problem domain. Hence, a constant support size for all the points in the mesh will not be able to reflect the local point spacing of an adaptive mesh. If support sizes for all the points are set equal to the smallest point spacing in the mesh, it is possible that the problem domain may not be covered completely. Furthermore, the shape functions for some points located at where the point spacing is large may fail to be constructed. However, setting the support size based on the maximum point spacing will increase the computational cost and reduce the accuracy of the RKPM solution [123]. As a result, during the adaptive refinement analysis, a variable support size scheme to define the domains of influence for points should be adopted. In the current implementation, the support size for points in the mesh will be determined by imposing the follow two essential conditions:

1). Minimum support size condition [124]

For all the points in the problem domain, its domain of influences must contain at least six points (Figure 6.9).

2). Minimum cover condition

The union of the domains of influences for all the points in the mesh must completely cover the whole problem domain. The minimum cover condition simply means that the problem domain must be completely covered by the union of the domain of influences of all the points. It is a necessary condition to fulfill the completeness of partition of unity of the RKPM shape functions. i.e.,

$$\sum_{i=1}^{NP} N_i^R \Omega_i = \Omega \quad (6.4)$$

where Ω_i is the domain of influences associates with the i th computing point.

Figure 6.9 The minimum size of domain of influences of points: h_i satisfies the minimum size condition while h_j does not

Note that the minimum size condition can only define the lower bound of the support size of points in the mesh. Fulfillment of this condition obviously does not imply that the minimum cover condition is satisfied automatically. However, in general, it is always preferred to use the minimum support size whenever is possible as it will reduce the computational cost and increase the accuracy of the RKPM solution. In this study, the support sizes of the points in the point mesh will be determined by the following steps.

- i). Scan over all the points within the point mesh, compute h_i , the minimum support sizes for each point i ($i = 1, 2, \dots, NP$) such that they will satisfy the minimum support size condition. In addition, $d_{\min}$, the minimum distance between any two points of the mesh will be computed.
- ii). A virtual grid which contains square cells with size equal to $d_{\min}$ will be generated to cover the entire problem domain. For a typical grid cell j , the centre of the cell is denoted as $\mathbf{O}_j$.
- iii). Scan over all the cells that overlap with the problem domain, Ω . For a typical

cell j (Figure 6.10), find out six nearest points that are visible to $\mathbf{O}_j$. The distances between these six points with $\mathbf{O}_j$, d_k , $k = 1, 2, \dots, 6$ are computed.

iv). Check whether the original minimum support sizes of these six points satisfy the condition

$$h_k \geq d_k + \sqrt{1/2} d_{\min} \quad \text{for } k = 1, 2, \dots, 6 \quad (6.5)$$

For the k th point, the minimum support size, h_k will be kept if Eqn. 6.5 is satisfied.

Otherwise, the value of h_k will be increased to $h_k = d_k + \sqrt{1/2} d_{\min}$.

The above procedure will ensure that any point of the problem domain will be inside the domain of influence associates to at least one point of the mesh and the RKPM shape function can always be constructed at any point inside the problem domain.

Figure 6.10: Determination of support sizes

6.2.2 Adaptive refinement strategy for the RKPM

6.2.2.1 Acceptable RKPM solution

In any adaptive refinement procedure, the ultimate objective is to achieve an RKPM solution with relative error, η_{RKPM} , less than some user prescribed target value,

$$\eta_{\text{target}}$$

$$\eta_{\text{RKPM}} \leq \eta_{\text{target}} \quad (6.6)$$

Since during the adaptive refinement analysis, the exact energy norm of the problem (Eqn. 4.37a) is not known in advance, it is approximated by $\|\tilde{\mathbf{u}}\|$, the energy norm of the RKPM solution (c.f. Eqn. 4.37a)

$$\|\mathbf{u}\| \approx \|\tilde{\mathbf{u}}\| = \left\{ \int_{\Omega} \tilde{\boldsymbol{\sigma}}^T \mathbf{D}^{-1} \tilde{\boldsymbol{\sigma}} d\Omega \right\}^{1/2} \quad (6.7)$$

and the estimated relative error of the RKPM solution, $\bar{\eta}$ can be computed as

$$\eta_{\text{RKPM}} \approx \bar{\eta} = \frac{\|\bar{\mathbf{e}}_{\sigma}\|}{\|\tilde{\mathbf{u}}\|} \quad (6.8)$$

The adaptive refinement RKPM procedure will be terminated if the following accuracy condition is reached (c.f. Eqn. 6.6)

$$\bar{\eta} \leq \eta_{\text{target}} \quad \text{or} \quad \|\bar{\mathbf{e}}_{\sigma}\| \leq \eta_{\text{target}} \|\tilde{\mathbf{u}}\| \quad (6.9)$$

6.2.2.2 Global and local refinement indicators and new point spacing calculation

From the experience obtained from adaptive FE analyses [108, 125-129], an adaptive refinement procedure will be optimal only when both the global and local errors of the solution are controlled properly and distributed equally among all the degree of freedoms of the numerical model. Since the recovered stress field could be evaluated at any point inside Ω , the notion of estimated error density, $\bar{\mathbf{e}}_d$ can be introduced as

$$\bar{\mathbf{e}}_d = \gamma^2 (\bar{\mathbf{e}}_{\sigma})^T \mathbf{D}^{-1} (\bar{\mathbf{e}}_{\sigma}) = \gamma^2 (\tilde{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}})^T \mathbf{D}^{-1} (\tilde{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}) \quad (6.10)$$

where γ is a compensational factor >1 employed to improve the adaptive efficiency and accuracy of the error indicators. A value of $\gamma=1.2$ is used in this study.

Using Eqn. 6.10, the global error in energy norm for the whole problem domain can then be re-defined as

$$\|\bar{\mathbf{e}}_{\sigma}\| = \gamma \left\{ \int_{\Omega} \bar{\mathbf{e}}_d d\Omega \right\}^{1/2} \quad (6.11)$$

Hence, using Eqn. 6.9, a global refinement indicator, ζ_g , can be defined as

$$\zeta_g = \frac{\|\bar{\mathbf{e}}_\sigma\|}{\eta_{\text{target}} \|\tilde{\mathbf{u}}\|} \quad (6.12)$$

Obviously, a value of $\zeta_g \leq 1$ implies that the global target accuracy criterion is satisfied while $\zeta_g > 1$ indicates further refinement is needed. Note that the global error indicator can only indicate whether further refinement is needed or not. It alone cannot indicate where and how to refine the point mesh. Thus, in order to construct an optimal point mesh that will able to eliminate the effect of singularity, a local error refinement indicator is required. Ideally, in an optimal mesh [130], the error contribution should be equally distributed among all the degree of freedoms of the model. In case of finite element analysis, it implies that the error norm should be equally distributed among all the elements of the mesh. In the meshless methods, since the domains of influence of points in the point mesh, in general, overlap each another, the notion of error norm contribution from individual point of the mesh should be different from that of adaptive FE analysis. Recall that the RKPM shape functions and the domains of influence of points will satisfy the partition of unity condition (Eqn. 6.4), the global error norm of the RKPM solution (Eqn. 5.10b) can be re-written in the form

$$\|\mathbf{e}_\sigma\| = \left\{ \int_{\Omega} (\mathbf{e}_\sigma)^T \mathbf{D}^{-1}(\mathbf{e}_\sigma) d\Omega \right\}^{1/2} = \left\{ \int_{\Omega} \sum_{i=1}^{NP} N_i^R \left((\mathbf{e}_\sigma)^T \mathbf{D}^{-1}(\mathbf{e}_\sigma) \right) d\Omega \right\}^{1/2} \quad (6.13a)$$

By changing the order of integration of Eqn. 6.13a, one has

$$\|\mathbf{e}_\sigma\| = \left\{ \sum_{i=1}^{NP} \left(\int_{\Omega_i} N_i^R \left((\mathbf{e}_\sigma)^T \mathbf{D}^{-1}(\mathbf{e}_\sigma) \right) d\Omega \right) \right\}^{1/2} = \left\{ \sum_{i=1}^{NP} \left(\|\mathbf{e}_\sigma\|_i \right)^2 \right\}^{1/2} \quad (6.13b)$$

where $\|\mathbf{e}_\sigma\|_i$ is the local error norm contribution associates with the i th point such that

$$\|\mathbf{e}_\sigma\|_i = \left\{ \int_{\Omega_i} N_i^R (\mathbf{e}_\sigma)^T \mathbf{D}^{-1}(\mathbf{e}_\sigma) d\Omega \right\}^{1/2}, \quad i = 1, 2, \dots, NP \quad (6.14)$$

The equal error norm contribution condition implies that in an optimal point mesh, for any two points i and j in the problem domain, the following conditions should be valid

$$\|\mathbf{e}_\sigma\|_i = \|\mathbf{e}_\sigma\|_j \quad \text{for } i, j = 1, 2, \dots, NP \quad (6.15a)$$

In Eqn. 6.15a, $\|\mathbf{e}_\sigma\|_i$ and $\|\mathbf{e}_\sigma\|_j$ are, respectively, the error norm contributions associate with the i th point and the j th point. In case that the above equal distribution of error norm condition is achieved, one can write

$$\|\mathbf{e}_\sigma\|^2 = \sum_{i=1}^{NP} (\|\mathbf{e}_\sigma\|_i)^2 = NP (\|\mathbf{e}_\sigma\|_i)^2 \Rightarrow \|\mathbf{e}_\sigma\|_i = \frac{\|\mathbf{e}_\sigma\|}{\sqrt{NP}} \quad \text{for } i = 1, 2, \dots, NP \quad (6.15b)$$

During the adaptive refinement process, the estimated error norm contribution associates with i th point of the point mesh can be defined in a similar manner (c.f. Eqn. 6.14)

$$\|\bar{\mathbf{e}}_\sigma\|_i = \gamma^2 \left\{ \int_{\Omega_i} N_i^R (\bar{\mathbf{e}}_\sigma)^T \mathbf{D}^{-1} (\bar{\mathbf{e}}_\sigma) d\Omega \right\}^{1/2} = \gamma \left\{ \int_{\Omega_i} N_i^R \bar{\mathbf{e}}_d d\Omega \right\}^{1/2} \quad (6.16a)$$

and it is hoped that during the adaptive refinement, an equilibration of local estimated error norm contribution can be achieved. That is (c.f. Eqn. 6.15)

$$\|\bar{\mathbf{e}}_\sigma\|_i = \|\bar{\mathbf{e}}_\sigma\|_j \quad \text{and} \quad \|\bar{\mathbf{e}}_\sigma\|_i = \frac{\|\bar{\mathbf{e}}_\sigma\|}{\sqrt{NP}} \quad \text{for } i, j = 1, 2, \dots, NP \quad (6.16b)$$

With the above results, the local error indicator for the i th point, ζ_i , can be defined as

$$\zeta_i = \frac{\|\bar{\mathbf{e}}_\sigma\|_i}{\left(\|\bar{\mathbf{e}}_\sigma\| / \sqrt{NP} \right)} \quad (6.17a)$$

Using Eqn. 6.9, Eqn. 6.17a can be more conveniently written in the form

$$\zeta_i = \frac{\|\bar{\mathbf{e}}_\sigma\|_i}{\|\mathbf{e}_\sigma\|_{\text{allow}}} = \frac{\sqrt{NP} \|\bar{\mathbf{e}}_\sigma\|_i}{\eta_{\text{target}} \|\tilde{\mathbf{u}}\|} \quad (6.17b)$$

in Eqn. 6.17b, the term $\|\mathbf{e}_\sigma\|_{\text{allow}}$ is the allowable error norm contribution and is given by

$$\|\mathbf{e}_\sigma\|_{\text{allow}} = \frac{\eta_{\text{target}} \|\tilde{\mathbf{u}}\|}{\sqrt{NP}} \quad (6.17c)$$

Obviously, a value of $\zeta_i > 1$ clearly means that the point mesh requires local refinement at the i th point.

It should be stressed that from Eqn. 6.14, the error norm contribution for the i th point, $\|\mathbf{e}_\sigma\|_i$, is not equal to the error norm defined over Ω_i , the domain of influence associates with the i th point. That is

$$\|\mathbf{e}_\sigma\|_i \neq \|\mathbf{e}_\sigma\|_{\Omega_i} = \left\{ \int_{\Omega_i} (\mathbf{e}_\sigma)^T \mathbf{D}^{-1} (\mathbf{e}_\sigma) d\Omega \right\}^{1/2} \quad (6.18a)$$

In fact, since the domains of influence of points of the point mesh in general will overlap each another, the following inequality will be held (c.f. Eqn. 6.15b)

$$\|\mathbf{e}_\sigma\|^2 \leq \sum_{i=1}^{NP} (\|\mathbf{e}_\sigma\|_{\Omega_i})^2 \quad (6.18b)$$

However, the *a priori* convergence rate of Eqn. 4.41 is also applicable to $\|\mathbf{e}_\sigma\|_i$ [130]. i.e.,

$$\|\mathbf{e}_\sigma\|_i \leq C (h_i)^{\beta_i} \quad (6.19)$$

where h_i is the local point spacing (support size) defined for the i th point. β_i is the local convergence rate for the i th point and is given by

$$\beta_i = \begin{cases} p^R, & \text{if } \Omega_i \text{ contains no singular point} \\ \lambda, & \text{if } \Omega_i \text{ contains a singular point with strength} = \lambda \end{cases} \quad (6.20)$$

With the definitions of the global ζ_g and the local ζ_i error indicators, the new point spacing of the point mesh can be computed by the following steps.

- i). Retrieve the existing point spacing information, h_i^{old} from the existing mesh.

ii). Compute the global error indicator, ζ_g and the local error indicators ζ_i , $i = 1, 2, \dots, NP$.

iii). Check for convergence conditions

The adaptive refinement procedure will be stopped if it found that the condition $\zeta_g \leq 1$ is true. Otherwise, the new point spacing will be computed by using step iv.

iv). Define the new point spacing, h_i^{new} .

By using Eqn. 6.19, if h_i^{new} is the required new point spacing for the i th point, one has

$$\|\mathbf{e}_\sigma\|_i \approx C \left(h_i^{\text{old}}\right)^{\beta_i} \text{ and } \|\mathbf{e}_\sigma\|_{\text{allow}} \approx C \left(h_i^{\text{new}}\right)^{\beta_i} \quad (6.21a)$$

Hence, using Eqn. 6.17b, the relationship between h_i^{old} and h_i^{new} can be determined as

$$\zeta_i = \frac{\|\mathbf{e}_\sigma\|_i}{\|\mathbf{e}_\sigma\|_{\text{allow}}} = \left(\frac{h_i^{\text{old}}}{h_i^{\text{new}}}\right)^{\beta_i} \quad (6.21b)$$

Therefore, the new point spacing for the i th point, h_i^{new} can be computed as

$$h_i^{\text{new}} = \frac{h_i^{\text{old}}}{(\zeta_i)^{1/\beta_i}} \quad (6.21c)$$

6.2.3 New point mesh generation and determination of h_i

The new point mesh spacing information given by Eqn. 6.21c will be defined at the nodal point of the current mesh. It can be directly converted to a set of metric specification [131] for the generation of the new adaptive finite point mesh. In this study, the surface point mesh generator used is the one developed by Lee [121] that employs the boundary offsetting technique to generate points and the metric specification to control the point spacing of the mesh. After the new point mesh is generated, the support sizes for the points of the mesh will be determined by using the procedure described in Section 6.2.1 so that the minimum support size and minimum cover conditions will be satisfied.

6.2.4 Numerical examples for the RKPM

6.2.4.1 Scope of study

In this section, the four 2D benchmark problems described in Chapter 4 are used to investigate the performance of the proposed adaptive RKPM refinement procedure. In order to validate the effectiveness of the adaptive refinement procedure, both uniform and adaptive refinement will be carried out. The point meshes employed during the uniform refinements are the same as those used in Part I of the current study.

In all examples, a coarse uniform mesh will be used as the initial mesh for the adaptive analysis. The target relative error η_{target} is set to be 2.5% for problems with smooth exact solution and is set to be 5% for problems with singular solution. The adaptive procedure will be continued until Eqn. 6.6 is satisfied or when totally three refinements (four analyses) are carried out. For all the uniform and adaptive analyses done, the following results will be computed:

- i). The exact relative error of the RKPM solution, η_{RKPM} (Eqn. 5.32a).
- ii). The relative error of the recovered solution, η_{Rv} (Eqn. 5.32a).
- iii). The estimated relative error, $\bar{\eta}$ (Eqn. 6.8).
- iv). The convergence rate of the RKPM solution, R_{RKPM} (Eqn. 5.33).
- v). The convergence rate of the recovered stress, R_{Rv} (Eqn. 5.35).
- vi). The effectivity index of the *a posteriori* error estimator, $\mathcal{J}$ (Eqn. 5.14b).

Since the quadratic RKPM (Section 3.3) is applied for the solution of the problem, it is expected that the optimal convergence rate (with respect to N_{DOF}) of the RKPM solution should equal to $p^{\text{R}}/2 = 0.5$ (Eqn. 4.41). During the RKPM analysis, the penalty parameter α used will be set equal to $\alpha = 5 \times 10^6 \text{ E}$ (Eqn. 3.20). For the stress recovery procedure, based on the numerical results obtained in Chapter 4, the radius of subdomain for stress extraction, r_{R} (Figure 5.2) is set equal to $2h$ where h is the support size of the point under consideration.

6.2.4.2 Timoshenko beam (convex domain, smooth exact solution)

The first example is the Timoshenko beam problem as shown in Section 4.3.1. The adaptive point meshes generated are shown in Figure 6.11. All the results obtained are summarized in Table 6.5 while the rates of convergence for both uniform and adaptive refinements are plotted in Figure 6.12, respectively.

Figure 6.11 Adaptive RKPM mesh used for the Timoshenko beam

Mesh	Initial	1 st	2 st	3 st
N_{DOF}	30	168	434	912
$\eta_{\text{RKPM}} (\%)$	21.54	6.45	3.41	2.1
$\eta_{\text{Rv}} (\%)$	11.67	2.61	1.16	0.62
$\bar{\eta} (\%)$	16.37	5.55	3.31	2.18
R_{RKPM}		0.70	0.67	0.65
R_{Rv}		0.87	0.85	0.84
$\mathcal{Q}$	0.76	0.86	0.97	1.04

Table 6.5 Adaptive RKPM results for the Timoshenko beam

From Table 6.5 and Figure 6.12, it can be seen that the optimal convergence rate is achieved by both uniform and adaptive refinements with adaptive refinement slightly outperformed the uniform refinement. In fact, it can be seen from Figure. 6.11 that even the exact solution for this problem is smooth, spacings of points are reduced near the support end of the beam in which high stress gradients are located.

Regarding the effectivity indices obtained, from Table 6.5, it can be seen that they converged to a value close to the ideal value of unity as the meshes are refined and the recovered stress fields are more accurate and converged at a higher rate than the RKPM solution.

Figure 6.12 Convergence rates R_{RKPM} for the Timoshenko beam

6.2.4.3 Infinite plate with a circular hole (concave domain, smooth exact solution)

In the second example, an infinite plate with a circular hole example as shown in Section 4.3.2 is considered. The adaptive meshes generated are shown in Figure 6.13. The results obtained for adaptive refinement are shown in Table 6.6 while the convergence rates are shown in Figures 6.14, respectively.

Figure 6.13 Adaptive RKPM mesh used for the infinite plate with a circular hole

From the numerical results obtained, it can also be concluded that adaptive refinement is again slightly outperformed the uniform refinement and this can be

explained by the fact that denser points were generated near the opening of the plate where high stress gradients are located. Similar to the first example, the recovered stress field is more accurate and converged at a higher rate than the RKPM solution. As a result, the effectivity indices converged to the ideal value of unity as the meshes are adaptively refined.

Mesh	Initial	1 st	2 st	3 st
N_{DOF}	54	98	224	482
η_{RKPM} (%)	9.41	6.24	3.59	2.22
η_{Rv} (%)	4.97	2.94	1.44	0.76
$\bar{\eta}$ (%)	7.25	5.05	3.27	2.15
R_{RKPM}		0.69	0.67	0.63
R_{Rv}		0.88	0.86	0.83
$\mathcal{G}$	0.77	0.81	0.91	0.97

Table 6.6 Adaptive RKPM results for the infinite plate with a circular hole

Figure 6.14 Convergence rates R_{RKPM} for the infinite plate with a circular hole

6.2.4.4 Region near a crack tip (convex domain, singular exact solution)

In the third example, the problem domain is a region near the crack tip as shown in Section 4.3.3. The adaptive meshes generated are shown in Figure 6.15. The results obtained are listed in Table 6.7 while the convergence history of the adaptive refinement is plotted in Figure 6.16, respectively.

Figure 6.15 Adaptive RKPM mesh used for the domain near a crack tip

Mesh	Initial	1 st	2 st	3 st
N_{DOF}	30	298	938	2300
$\eta_{\text{RKPM}} (\%)$	29.7	12.13	7.76	5.52
$\eta_{\text{Rv}} (\%)$	17.97	6.70	4.14	2.89
$\bar{\eta} (\%)$	16.33	8.85	6.05	4.75
R_{RKPM}		0.39	0.39	0.38
R_{Rv}		0.43	0.42	0.40
$\mathcal{G}$	0.55	0.73	0.78	0.86

Table 6.7 Adaptive RKPM results for the domain near a crack tip

From Table 6.7 and Figure 6.16, it can be seen that significant improvement in the rate of convergence is obtained (0.25 for uniform refinement, ≈ 0.40 for adaptive refinement). During adaptive refinement, rapid reduction of point spacing was carried out near the singular point and the effect of singularity is largely eliminated. Even though in this problem a singular point exists inside the problem domain, both the accuracy and convergence rates of the recovered solutions are higher than the RKPM solutions. As a result, convergence of the effectivity indices is observed. Note that the relatively poor value of the effectivity index for the first refinement

mesh (0.73) may be due to the coarse initial mesh used ($N_{\text{DOF}}=30$). From this example, it can be concluded that the adaptive refinement scheme suggested and the point mesh generation procedure successfully lead to the generation of nearly optimal point meshes.

Figure 6.16 Convergence rates R_{RKPM} for the domain near a crack tip

6.2.4.5 L-shape domain (concave domain, singular exact solution)

The last example is an L-shaped domain problem as shown in Section 4.3.4. The adaptive point meshes generated are shown in Figure 6.17. The results obtained from adaptive refinement are listed in Table 6.8 while the convergence history is plotted in Figure 6.18, respectively.

Figure 6.17 Adaptive RKPM mesh used for the L-shaped domain

Mesh	Initial	1 st	2 st	3 st
N_{DOF}	66	242	758	2110
η_{RKPM} (%)	23.5	13.79	8.73	5.86
η_{Rv} (%)	14.12	7.97	4.88	3.17
$\bar{\eta}$ (%)	12.93	9.93	6.90	4.92
R_{RKPM}		0.41	0.40	0.39
R_{Rv}		0.44	0.43	0.42
$\mathcal{I}$	0.55	0.72	0.79	0.84

Table 6.8 Adaptive RKPM results for the L-shaped domain

Figure 6.18 Convergence rates R_{RKPM} for the L-shaped domain

Similar to Example 3, significant improvement in the convergence rate was obtained during adaptive refinement (from 0.27 for uniform refinement to 0.4 for adaptive refinement). Extensive reduction in point spacing can be observed near the re-entrance corner and both the accuracy and the convergences rate of the recovered solutions are higher than the RKPM solution. Regarding the performance of the *a posteriori* error estimator, it can be concluded that asymptotic convergence trend of the effectivity index can be observed. Again, the relatively poor results for the initial ($\mathcal{I}=0.55$) and first refinement ($\mathcal{I}=0.72$) meshes were due to the coarse initial

mesh ($N_{\text{DOF}}=66$) used. In this example, the adaptive refinement scheme and the point mesh generator again eliminated most of the effects of the singular point.

6.2.5 Discussions on obtained RKPM numerical results

From the numerical studies above, it is found that by using the proposed adaptive refinement procedure, the user prescribed target accuracy can be reached in a few refinement steps. For problems with smooth exact solutions, even though there is no significant improvement in the convergence rate is obtained, the adaptive scheme can lead to solutions with higher accuracy. For problems with singular solutions, it is found that the proposed refinement process can lead to the generation of nearly optimal point meshes. As a result, when compared with uniform refinement, the effect of singularity is largely eliminated and nearly optimal convergence rates are achieved. The numerical tests also re-confirm that the recovered stress fields generated by the recovery procedure described in Chapter 5 converge at a higher rate than the RKPM solutions.

6.3 Adaptive refinement for the coupled BE-RKPM

When the adaptive schemes for the BEM and the RKPM are available, the adaptive refinement procedure of the coupled BE-RKPM can be implemented. The only difficulty for the coupled BE-RKPM is how to keep the displacement condition compatible in each adaptive process. In this section, the adaptive refinement strategy for the coupled BE-RKPM is proposed and its efficiency is also investigated through numerical experiments on four 2D benchmark problems.

6.3.1 Displacement compatible condition on interface boundary

Considering the interface boundary Γ^{RB} between the RKPM domain Ω^{R} and the BEM domain Ω^{B} (shown in Figure 6.19), $\mathbf{p}$ is a point located on Γ^{RB} . $\Omega_{\text{sup}}(\mathbf{p})$ is the support domain associated with $\mathbf{p}$ in Ω^{R} with support size as $h_{\mathbf{p}}$. Q is the element associated with $\mathbf{p}$ in Ω^{B} with element length as L_Q .

Figure 6.19 Displacement compatible condition in adaptive process

In general, the displacement function $\mathbf{u}(s, t)$ in the neighbor subdomain of any point $\{x, y\}$ in the domain $\Omega = \Omega^R \cup \Omega^B$ (shown in Figure 6.19), can be expanded by a Taylor series as

$$\begin{aligned} \mathbf{u}(s, t) = & \mathbf{u}(x, y) - (x - s) \frac{\partial \mathbf{u}}{\partial x} - (y - t) \frac{\partial \mathbf{u}}{\partial y} + \frac{(x - s)^2}{2!} \frac{\partial^2 \mathbf{u}}{\partial x^2} + \\ & \frac{(y - t)^2}{2!} \frac{\partial^2 \mathbf{u}}{\partial y^2} + (x - s)(y - t) \frac{\partial^2 \mathbf{u}}{\partial x \partial y} + \dots \end{aligned} \quad (6.22)$$

where, $\{s, t\}$ is a point in the neighbor subdomain of point $\{x, y\}$.

For the point $\mathbf{p} = \{x, y\}$ located on the interface boundary Γ^{RB} , the displacement can be respectively approximated in Ω^B and Ω^R as:

In the element Q (Ω^B part), since the linear element (Section 3.2) is used in this study, so only the linear items are kept in the Taylor series, i.e.,

$$\mathbf{u}^B(s, t) = \mathbf{u}(x, y) - (x - s) \frac{\partial \mathbf{u}}{\partial x} - (y - t) \frac{\partial \mathbf{u}}{\partial y} \quad (6.23a)$$

In $\Omega_{\text{sup}}(\mathbf{p})$ (Ω^R part), since the correction function is a linear combination of polynomial basis function up to second order, so from the reproducing condition [44], only the linear and quadratic items are kept in the Taylor series, i.e.,

$$\begin{aligned} \mathbf{u}^R(s, t) = & \mathbf{u}(x, y) - (x-s) \frac{\partial \mathbf{u}}{\partial x} - (y-t) \frac{\partial \mathbf{u}}{\partial y} + \frac{(x-s)^2}{2!} \frac{\partial^2 \mathbf{u}}{\partial x^2} + \\ & \frac{(y-t)^2}{2!} \frac{\partial^2 \mathbf{u}}{\partial y^2} + (x-s)(y-t) \frac{\partial^2 \mathbf{u}}{\partial x \partial y} \end{aligned} \quad (6.23b)$$

From the displacement compatible condition (Eqn. 4.10), it required that

$$\frac{\alpha}{2} \int_{\Gamma^{\text{RB}}} (\mathbf{u}^R(s, t) - \mathbf{u}^B(s, t))^T (\mathbf{u}^R(s, t) - \mathbf{u}^B(s, t)) d\Gamma \rightarrow 0 \quad (6.24)$$

In practical computation, the neighbor domain L_Q of $\mathbf{u}^B(s, t)$ will not overlap entirely with the neighbor domain of $\mathbf{u}^R(s, t)$. In order to satisfy the displacement compatible condition, it is better to keep the values of L_Q and h_p suitably to ensure Eqn. 6.24 converge to zero. In practical computation, it can be simply let $L_Q = h_p$ for convenience.

6.3.2 Automatic refinement strategy for the coupled BE-RKPM

Based on the adaptive strategy for the BEM and the RKPM, the adaptive procedure for the coupled BE-RKPM is summarized as following:

Figure 6.20 Adjustment of h_i^{new} along interface boundary Γ^{RB}

- i). Read the initial element and point mesh data and solve the coupled linear system.
- ii). Compute the adaptive parameters ($\bar{\eta}_R$ (Eqn. 6.2), $\bar{\eta}_{R_i}$ (Eqn. 6.2) for the BEM

- ξ_g (Eqn. 6.12), ξ_i (Eqn. 6.17) for the RKPM) needs for the subdomains Ω^B and Ω^R , respectively.
- iii). Determine the adaptive element mesh for Ω^B and adaptive point mesh for Ω^R , respectively (c.f. Section 6.1.2 and 6.2.2).
 - iv). Adjust h_i^{new} (Eqn. 6.21c) of the RKPM point mesh on the interface boundary Γ^{RB} according to the following steps.
 - a). In Ω^B , find all the end points of the BEM elements which is located on the interface boundary Γ^{RB} , compute the lengths (L_i and L_{i+1}) of neighbor elements associated with the boundary element end point $\mathbf{p}_i$. (shown in Figure 6.20)
 - b). In Ω^R , for each RKPM point $\mathbf{p}_j$, scan over the element end point and find the nearest element end point $\mathbf{p}_i$. If the element end point $\mathbf{p}_i$ is in the support domain of RKPM point $\mathbf{p}_j$, adjust the value of h_i^{new} such that $h_i^{\text{new}} \rightarrow (L_i + L_{i+1})/2$ (from the displacement compatible condition described in section 6.3.1).
 - c). In Ω^R , after adjustment of h_i^{new} , generate the new point mesh for the RKPM.
 - v). Stop the mesh refinement and export the data interested, if one of the following conditions are satisfied; otherwise go to step ii.
 - a). The termination condition for the adaptive refinement of the BEM and the RKPM is satisfied (c.f. section 6.1.2 and 6.2.2).
 - b). The iteration number is equal to a setting number (3 iterations).

6.3.3 Numerical examples for the coupled BE-RKPM

6.3.3.1 Scope of study

In this section, four 2D benchmark problems described in Chapter 4 are re-used to test the efficiency of the proposed adaptive scheme strategy for the coupled BE-RKPM. In the BEM subdomain Ω^B , $\eta_{\text{target}}^G = 5\%$ and $\eta_{\text{target}}^L = 5\%$. In the RKPM

subdomain Ω^R , The target relative error η_{target} is set to be 2.5% for problems with smooth exact solution and is set to be 5% for problems with singularity solution. For all the adaptive analysis, the following results are computed:

- i). The exact relative error of the adaptive coupled BE-RKPM solution, $\eta_{\text{BE-RKPM}}$ (Eqn. 4.38).
- ii). The convergence rate of the adaptive coupled BE-RKPM solution, $R_{\text{BE-RKPM}}$, defined as

$$R_{\text{BE-RKPM}} = \frac{\log(\eta_{\text{BE-RKPM},i} / \eta_{\text{BE-RKPM},i+1})}{\log(N_{\text{DOF},i+1} / N_{\text{DOF},i})} \quad (6.25)$$

where, $\eta_{\text{BE-RKPM},i}$ are $N_{\text{DOF},i}$ the relative error and the number of degree of freedoms of the i th refinement analysis of the adaptive coupled BE-RKPM respectively.

6.3.3.2 Timoshenko beam (convex domain, smooth exact solution)

The first example is the Timoshenko beam problem as shown in Section 4.3.1. The meshes used for the adaptive coupled BE-RKPM are shown in Figure 6.21. The results obtained are summarized in Table 6.9 while the rates of convergence for both uniform and adaptive refinements are plotted in Figure 6.22, respectively.

6.3.3.3 Infinite plate with a circular hole (concave domain, smooth exact solution)

In the second example, an infinite plate with a circular hole example as shown in Section 4.3.2 is considered. The meshes used for the adaptive coupled BE-RKPM are shown in Figure 6.23. The results obtained are summarized in Table 6.10 while the rates of convergence for both uniform and adaptive refinements are plotted in Figure 6.24, respectively.

Initial mesh with 12 elements and 15 points

1st mesh with 24 elements and 45 points

2st mesh with 48 elements and 174 points

3st mesh with 72 elements and 264 points

Figure 6.21 Adaptive coupled BE-RKPM mesh used for the Timoshenko beam

Mesh	Initial	1 st	2 st	3 st
N_{DOF}	70	170	508	752
$\eta_{\text{BE-RKPM}}$ (%)	15.7	8.1	3.6	2.7
$R_{\text{BE-RKPM}}$		0.75	0.74	0.73

Table 6.9 Adaptive coupled BE-RKPM results for the Timoshenko beam

Figure 6.22 Convergence rates $R_{BE-RKPM}$ for the Timoshenko beam

Figure 6.23 Adaptive coupled BE-RKPM mesh used for the infinite plate with a circular hole

Mesh	Initial	1 st	2 st	3 st
N_{DOF}	88	158	312	624
$\eta_{\text{BE-RKPM}}$ (%)	5.2	3.4	2.0	1.2
$R_{\text{BE-RKPM}}$		0.72	0.75	0.74

Table 6.10 Adaptive coupled BE-RKPM results for the infinite plate with a circular hole

Figure 6.24 Convergence rates $R_{\text{BE-RKPM}}$ for the infinite plate with a circular hole

6.3.3.4 Domain near a crack tip (convex domain, singular exact solution)

In the third example, the problem domain is a region near the crack tip as shown in Section 4.3.3. The meshes used for the adaptive coupled BE-RKPM are shown in Figure 6.25. The results obtained are summarized in Table 6.11 while the rates of convergence for both uniform and adaptive refinements are plotted in Figure 6.26, respectively.

6.3.3.5 L-shape domain (concave domain, singular exact solution)

The last example is an L-shaped domain problem as shown in Section 4.3.4. The meshes used for the adaptive coupled BE-RKPM are shown in Figure 6.27. The results obtained are summarized in Table 6.12 while the rates of convergence for both uniform and adaptive refinements are plotted in Figure 6.28, respectively.

Figure 6.25 Adaptive coupled BE-RKPM mesh used for the domain
near a crack tip

Mesh	Initial	1 st	2 st	3 st
N_{DOF}	66	206	574	902
$\eta_{\text{BE-RKPM}}$ (%)	23.2	15.1	9.9	8.3
$R_{\text{BE-RKPM}}$		0.38	0.41	0.39

Table 6.11 Adaptive coupled BE-RKPM results for the domain near a crack tip

Figure 6.26 Convergence rates $R_{BE-RKPM}$ for the domain near a crack tip

Figure 6.27 Adaptive coupled BE-RKPM mesh used for the L-shape domain

Mesh	Initial	1 st	2 st	3 st
N_{DOF}	186	280	992	1290
$\eta_{\text{BE-RKPM}}$ (%)	15.9	13.3	7.5	6.7
$R_{\text{BE-RKPM}}$		0.44	0.45	0.43

Table 6.12 Adaptive coupled BE-RKPM results for the L-shape domain

Figure 6.28 Convergence rates $R_{\text{BE-RKPM}}$ for the L-shape domain

6.3.4 Discussions on obtained coupled BE-RKPM numerical results

From the adaptive point/element mesh shown in Figures 6.21, 6.23, 6.25 and 6.27, it can be seen that the proposed adaptive refinement strategy for the coupled BE-RKPM can satisfy the displacement compatible condition and identify the place where, high stress gradients or singularity are located. The ratio between the density of the RKPM point mesh and the BEM element length is suitable along the interface boundary Γ^{RB} between the RKPM domain Ω^{R} and BEM domain Ω^{B} .

From the obtained numerical results, it can be observed that for the problems with smooth exact solution, the convergence rates of adaptive element-point mesh are a bit greater than that of the uniform point mesh (Section 4.3); for the problems with singular exact solution, the convergence rates of adaptive element-point mesh are

much greater than that of the uniform point mesh (Section 4.3) and the effects of singular point are largely eliminated.

6.4 Closure and discussion

Based on the numerical results of the adaptive refinement of the BEM (Section 6.1, the RKPM (Section 6.2) and the coupled BE-RKPM (Section 6.3), the comparison of rates of convergence in energy norm for the adaptive results of three methods on four 2D benchmark problems are plotted in Figures 6.29-6.32, respectively.

Figure 6.29 Adaptive rates of convergence for the Timoshenko beam

Figure 6.30 Adaptive rates of convergence for the infinite plate with a circular hole

Figure 6.31 Adaptive rates of convergence for the domain near a crack tip

Figure 6.32 Adaptive rates of convergence for the L-shape domain

From the comparisons in Figures 6.29-6.32, it can be conclude that

- 1). Regarding to the convergence rates of the solutions, the rate of convergence for the adaptive BEM is the highest one, the rate of convergence for the adaptive RKPM is the lowest one and the rate of convergence for the adaptive coupled BE-RKPM is between that of the adaptive BEM and that of the adaptive RKPM. The

accuracy of the results for the adaptive coupled BE-RKPM is mainly controlled by that of the adaptive RKPM.

2). Regarding the computing cost on automatic refinement procedure for the BEM, the RKPM and the coupled BE-RKPM, since there is singular integration involved in adaptive process of the BEM (Eqn. 5.45) and the RKPM (Eqn. 5.26), the time cost (as shown in Table 6.13) for adaptive refinement of the BEM and the RKPM is at the same order for the same point/element mesh density. Especially when the number of freedom of point/element mesh is big, the time cost of the automatic refinement process of the coupled BE-RKPM will become comparatively high.

Problem	BEM (hours)	RKPM (hours)	coupled BE-RKPM (hours)
Example 1	3	48	30
Example 2	2	35	40
Example 3	2.5	65	45
Example 4	2.5	65	50

Table 6.13 CPU times for the entire automatic adaptive refinement using a DELL Precision 450 PC (Xeon 2.8GHz, 1G RAM)

Major process	BEM	RKPM
Analysis	2%	90%
Error estimation	97%	9%
Refinement	<1%	<1%

Table 6.14 Ratios of CPU time used in different major processes for adaptive refinement using the BEM and the RKPM

In addition, the ratios of CPU time used in different major processes of adaptive refinements for the BEM and the RKPM are shown in Table 6.14. From Table 6.14, it can be seen that the total CPU time needed in the adaptive refinement are longer than the corresponding uniform refinement. For the BEM, since there is singular integration involved in the calculation of the point residual $\bar{\mathbf{e}}_{R_{ij}}$ (Eqn. 5.54), the time

cost for the local residual error norm $\|\bar{\mathbf{e}}_R\|_i$ (Eqn. 5.55a) and the global residual error norm $\|\bar{\mathbf{e}}_R\|$ (Eqn. 5.55b) is comparatively expensive in the process of BEM error estimation (See Table 6.14).

For the RKPM, the size of the cell of the background integration mesh (see the integration scheme in Appendix B) is judged by the minimum spacing of the RKPM computing points. When the RKPM computing point mesh becomes dense, the corresponding background integration mesh will also become dense and the time cost on the calculation of the stiff matrix components $\mathbf{K}_{ij}$ of is comparatively high in the process of RKPM analysis (See Table 6.14). This is why the time cost of the adaptive RKPM are much expensive than that of the uniform RKPM. In the process of RKPM error estimation, though there is singular integration involved in calculating the recovered stress field (see Section 5.1.5), the time cost on the process of the error estimation is acceptable.

For the coupled BE-RKPM, the total CPU time cost of the BEM subdomain is mainly controlled by the process of BEM error estimation, and the total CPU time cost of the RKPM subdomain is mainly controlled by the process of RKPM analysis. Whereupon, ratio of CPU time used in different major processes of the entire automatic adaptive refinement can not provided for different element-point mesh.

CHAPTER 7

CONCLUSIONS AND FURTHER INVESTIGATIONS

7.1 Conclusions

A comprehensive study on the convergence rates, the *a posteriori* error estimation and automatic refinement procedures for the boundary element method (BEM), the reproducing kernel particle method (RKPM) and the coupled BE-RKPM for 2D linear elastostatic problem has been carried out in this research. The contribution on originality of the study mainly focuses on four areas related to the coupled method of the BEM and the RKPM, the *a posteriori* error estimation for the RKPM and the development of automatic adaptive refinement procedures for the RKPM and the coupled BE-RKPM.

1). Coupled method of the BEM and the RKPM

A new coupled method of the BEM and the RKPM is described in Chapter 4. Based on the basic formulations of the BEM and the RKPM, which are described in Chapter 3, the coupled BE-RKPM is developed to take the advantages of the BEM and RKPM, in which a penalty method is used to preserve the continuity and compatibility of the displacement along the interface boundary between the BEM and RKPM domains. In order to avoid the non-symmetric structure of the final coupled linear system, the unit displacement method is developed for the coupled BE-RKPM without causing extra computational cost. *A priori* error estimation for the coupled BE-RKPM is also discussed and it has found that the convergence rate of the coupled BE-RKPM always lie between the convergence rates of the BEM and the RKPM. Numerical experiments on four 2D benchmark boundary value problems have been carried out to test the validity of the proposed coupled BE-

RKPM by comparing its convergence rate with that of the BEM and the RKPM. In addition, the suggested coupled BE-RKPM is also applied for the solution of a soil-structure interaction problem and compared its result with that of the coupled BE-FEM and the coupled BE-EFGM. From the numerical results, it has found that when compared with the RKPM, a significant saving of computational cost can be achieved in the coupled BE-RKPM, especially for problems involving infinite/semi-infinite domain and it prefers to limit the size of the RKPM subdomain only to where rapid refinement/remodeling/change in domain geometry occurs.

2). The *a posteriori* error estimation for the RKPM

A new *a posteriori* error estimation based on the stress recovery procedure for the RKPM is proposed in Chapter 5. The basic principle of the stress recovery procedure suggested is to employ some local and compact extraction functions which are similar to the boundary integral technique for the generation of the recovery stress field. The recovery stress field is then compared with the RKPM solution for the *a posteriori* error estimation. In order to validate the accuracy and convergence rate of the recovered stress field obtained and the reliability of the *a posteriori* error estimator, four numerical examples involving 2D benchmark boundary value problems that with different domain geometry and smoothness of the exact solution are reused. Uniform refinements are carried out and recovery subdomains with different sizes are constructed to investigate the effect of subdomain size on the performance of the recovery procedure. The numerical results obtained indicate that the recovered stress field is more accurate and converges at a higher rate than the original RKPM stress field. Furthermore, it is also shown that, in general, both the accuracy and convergence rate of the recovered stress field can be improved by increasing the size of the recovery subdomain. For the performance of the *a posteriori* error estimation scheme, since the recovery stress field converges at a higher rate than the RKPM solution, asymptotic convergence of the effectivity indices is observed. However, as the convergence rate of the recovered stress is affected by the present of singularity, the performance of the error estimator is affected for such problems and lower values of effectivity

indices are obtained when compared with the cases that the exact solutions are smooth.

3). Automatic adaptive refinement procedures for the RKPM

A new automatic adaptive refinement procedure for the RKPM is suggested in Chapter 6. The suggested new adaptive refinement strategy is based on the partition of unity property of the RKPM shape functions for the definition of the local refinement indicators. The local refinement indicators are then used for the definition of new node spacings during the adaptive point mesh generation. When the new point mesh spacing information is available, the new point mesh can be generated by using the surface point mesh generator. A simple but effective variable support size definition scheme for the new point mesh has also been suggested for adaptive meshes to ensure that the RKPM shape functions can always be constructed in any point inside the problem domain. From the numerical results obtained for the four benchmark problems, it has found that the convergence rate of the adaptive point mesh for the RKPM is higher than that for the uniform mesh for the RKPM, and the proposed refinement procedure can guide the process for the RKPM effectively.

4). Automatic adaptive refinement procedures for the coupled BE-RKPM

A new automatic adaptive refinement procedure for the coupled BE-RKPM is also suggested in Chapter 6. In the adaptive refinement scheme for the coupled BE-RKPM, the h -adaptive scheme based on the residual method is adopted to guide the adaptive process for the BEM part, and the adaptive refinement procedure previously proposed for the RKPM is adopted for the RKPM part, respectively. During the coupled adaptive refinement process, the displacement compatible condition on the interface boundary are enforced by directly keeping the support domains size of RKPM points in line with the length of BEM elements in each adaptive steps. From the numerical results of four benchmark problems, it has found that the convergence rates of adaptive element-point mesh are greater than that of the uniform point mesh and the displacement compatible condition are satisfied along the interface boundary between the BEM part and the RKPM part.

7.2 Further investigations

With regard to the further work of this study, several issues remained to be investigated in the future are as the following:

1). Further improvement on practical implementation

Currently, the study is only concentrated in the development of theory rather than to produce an optimized commercial program. The current computer program is not yet optimized in speed performance and further improvement in CPU time is still possible. For the problem on singular integration, i.e., when the kernels of the boundary integrals in the extract equation involve or hyper-singular functions, such as the functions of order $(1/r^2)$ and $(1/r^3)$ for 2D or 3D problems, the integrals are unbounded to an extent that the usual Riemann integration is not possible and the integrals only exist in the sense of their Cauchy principal values. In such a situation, the stress recovery scheme based on the regularized hyper-singular boundary integration equation can be developed to avoid the numerical integration with hyper-singular kernel. The basic idea of the regularized method is based on the singularity subtraction technique, in which, the singular kernel is subtracted out in such a way that it leaves the remaining integrand regular and the subtracted singular part can be generally be integrated analytically.

For the problem on background integration mesh, further improvement in CPU time can only be obtained by increasing the speed on solving a 6×6 linear equations set to get the shape function $N_i^R(x,y)$ at each integration point $\mathbf{p}(x,y)$ (see Section 3.3.1) or by using the parallel computing algorithm directly to speed up the computation to what is required. It is believed that if all the above improvements on various processes of the adaptive analysis procedure are implemented the CPU time need shall be reduced by a factor of 5-10 times.

2). Stress recovery scheme for other meshless methods

In this study, though the suggested stress recovery scheme is discussed only for the RKPM, the same idea of stress recovery can also be applied to other Galerkin based meshless methods without changing the basic stress recovery formulation. The

validity of the proposed *a posteriori* error estimation can then be verified through its application on the other Galerkin based meshless methods, such as the element free Galerkin method (EFGM).

3). 3D stress recovery scheme for meshless methods

Though the suggested stress recovery scheme is only used on 2D elastostatic problem, it can also be extended to 3D elastostatic problem theoretically. However, the order of the singular kernel function of fundamental solution for 3D elastostatic problem is at $(1/r^2)$ in which the numerical integration scheme will be more complex than the 2D case. Suitable size of support domain of the stress recovery scheme for 3D elastostatic problem also needs to be decided without increasing much more extra computation cost.

4). Coupled method of symmetric Galerkin BEM and meshless methods

The traditional collocation boundary element method has some certain unpleasant features. The most pertinent is the lack of symmetry for some coefficient matrices, which makes it less efficient to couple BEM with other symmetric schemes. The idea of coupling of the symmetric Galerkin BEM and the meshless methods can be developed to avoid the non-symmetric structure of the final coupled linear system, which is very important for practical engineering problems involving with large degree of freedoms.

5). Computation of stress intensity factor (SIF)

Recently, there are some publications [132,133] reported that some researchers begin to use the meshless methods to compute the stress intensive factor. In the coupled BE-RKPM, the RKPM can be used to simulate the crack propagation just by inserting computing points near crack zone without modifying the corresponding shape function. However, just like that in the FEM, the initial selection of the route for the crack propagation will have great influence on the value of SIF. Moreover, the rule of inserting RKPM computing point is also needed to be constructed at the same time.

REFERENCES

1. Constanda C. (1999), Direct and Indirect Boundary Integral Equation Methods, Chapman & Hall / CRC.
2. Rizzo F. J. (1967), "An Integral Equation Approach to Boundary Value Problems of Classical Elastostatics", Quaterly of Applied Mathematics, Vol. 25, pp. 83-95.
3. Cruse T. A. (1969), "Numerical Solutions in Three-Dimensional Elastostatics", International Journal of Solids& Structures, Vol. 5, pp. 1259-1274.
4. Cruse T. A. (1973), "Application of the Boundary-Integral Equation Method to Three Dimensional Stress Analysis", Computers & Structures, Vol. 3, pp. 509-527.
5. Jaswon M. A., Symm G. T. (1977), Integral Equation Methods in Potential Theory and Elastostatics, Academic Press, London.
6. Cruse T. A., Polch E. Z. (1986), "Elastoplastic BIE Analysis of Cracked Plates and Related Problems, Part 1: Formulation". International Journal for Numerical Methods in Engineering, Vol. 23, pp. 429-437.
7. Cruse T. A. (1987), "Recent Advances in Boundary Element Analysis Methods", Computer Methods in Applied Mechanics and Engineering, Vol. 62, pp. 227-244.
8. Cruse T. A. (1988), "Boundary Element Analysis in Computational Fracture Mechanics", Kluwer Academic Publishers, Dordrecht.
9. Cruse T. A., Unruh J. F., Wu Y. T., Harren S. V. (1990), "Probabilistic Structural Analysis for Advanced Space Propulsion Systems", Journal of Engineering for Gas and Turbines Power, Vol. 112, pp. 251-260.
10. Rizzo F. J., Shippy D. J. (1997), "An Advanced Boundary Integral Equation Method for Three-Dimensional Thermo-Elasticity", International Journal for Numerical Methods in Engineering, Vol. 11, pp.1259-1274.
11. Guiggiani M. (1995), "Hypersingular Boundary Integral Equations Have an Additional Free Term", Computational Mechanics, Vol. 16, pp. 245-248.

12. Mukherjee S. (2000), "CPV and HFP Integrals and Their Applications in the Boundary Element Methods", International Journal of Solids and Structures, Vol. 37, pp. 6623-6634.
13. Brebbia C. A., Dominguez J. (1992), Boundary Elements: an Introductory Course (Second Edition), Computational Mechanics Publications, Southampton Boston.
14. Mohammed A. (2001), Boundary Element Analysis Theory and Programming, CRC Press, New York.
15. Wedland W. L., Yu D. (1988), "Adaptive Boundary Element Methods for Strong Elliptic Integral Equations", Numerische Mathematik, Vol. 53, pp. 539-558.
16. Zienkiewicz O. C., Zhu J. Z. (1987), "A Simple Error Estimator and Adaptive Procedure for Practical Engineering Analysis", International Journal for Numerical Methods in Engineering, Vol. 24, pp. 337-357.
17. Rencis J. J., Mullen R. L. (1986), "Solution of Elasticity Problems by a Self-Adaptive Mesh Refinement Technique for Boundary Element Computation", International Journal for Numerical Methods in Engineering, Vol. 23, pp. 1509-1527.
18. Rank E. (1984), "A *Posteriori* Error Estimates and Adaptive Refinement for Some Boundary Integral Element Methods", Babuska I. et al. (eds), Proc. AREFEC Conference, Lisbon.
19. Ingber M. S., Mitra A. K. (1986), "Grid Optimization for the Boundary Element Method", International Journal for Numerical Methods in Engineering, Vol. 23, pp. 2121-2136.
20. Carney G. F., Kennon S. (1987), "Adaptive Mesh Redistribution for a Boundary Element (Panel) Method", International Journal for Numerical Methods in Engineering, Vol. 24, pp. 2315-2325.
21. Guiggiani M. (1990), "Error Indicators for Adaptive Mesh Refinement in the Boundary Element Method Using Local Reanalysis", International Journal for Numerical Methods in Engineering, Vol. 29, pp. 1247-1269.
22. Liapis S. (1994), "A Review of Error Estimation and Adaptivity in the Boundary Element Method", Engineering Analysis with Boundary Elements, Vol. 14, pp. 315-323.
23. Liang M. T., Chen J. T., Yang S. S. (1999), "Error Estimation for Boundary Element Method", Engineering Analysis with Boundary Elements, Vol. 23, pp. 257-265.

24. Charafi A., Neves A. C., Wrobel L. C. (1995), " h -Hierarchical Adaptive Boundary Element Method Using Local Reanalysis", International Journal for Numerical Methods in Engineering, Vol. 38, pp. 2185-2207.
25. Alarcon E., Reverter A. (1986), " p -Adaptive Boundary Elements", International Journal for Numerical Methods in Engineering, Vol. 23, pp. 801-829.
26. Abe. K. (1996), " r -Adaptive Boundary Element Method for Unsteady Free-Surface Flow Analysis", International Journal for Numerical Methods in Engineering, Vol. 39, pp. 2769-2787.
27. Lucy L. B. (1977), "A Numerical Approach to the Testing of the Fission Hypothesis", Astronomical Journal, Vol. 82, pp. 1013-1024.
28. Gingold R. A., Monaghan J. J. (1977), "Smoothed Particle Hydrodynamics: Theory and Application to Non-spherical Stars", Monthly Notices of the Royal Astronomical Society, Vol. 181, pp.375-389.
29. Monaghan J. J. (1992), "Smooth Particle Hydrodynamics", Annual Review of Astronomy and Astrophysics, Vol. 30, pp. 543-574.
30. Nayroles B., Touzot G., Villon P. (1992), "Generalizing the FEM: Diffuse Approximation and Diffuse Elements", Computational Mechanics, Vol. 10, pp. 307-318.
31. Belytschko T., Lu Y. Y., Gu L. (1994), "Element-Free Galerkin Methods", International Journal for Numerical Methods in Engineering, Vol. 37, pp. 229-256.
32. Belytschko T., Tabbara M. (1996), "Dynamic Fracture using Element-Free Galerkin Method", International Journal for Numerical Methods in Engineering, Vol. 39, pp. 923-938.
33. Belytschko T., Krongauz Y., Organ D., Fleming M., Krysl P. (1996), "Meshless Methods: An Overview and Recent Developments", Computer methods in Applied Mechanics and Engineering, Vol. 139, pp. 3-47.
34. Belytschko T., Krongauz Y., Dolbow J., Garlach C. (1998), "On the Completeness of Meshfree Particle Methods", International Journal for Numerical Methods in Engineering, Vol. 43, pp. 785-819.
35. Liu W. K., Sukky J., Zhang Y. F. (1995), "Reproducing Kernel Particle Methods", International Journal for Numerical Methods in Fluids, Vol. 20, pp. 1081- 1106.

36. Liu W. K., Jun S., Li S., Adee J., Belytschko T. (1995), "Reproducing Kernel Particle Methods for structural dynamics", International Journal for Numerical Methods in Engineering, Vol. 38, pp. 1655-1679.
37. Jun S., Liu W. K., Belytschko T. (1996), "Explicit Reproducing Kernel Particle Methods for Large Deformation Problems", International Journal of Numerical Methods for Engineering, Vol. 41, pp. 137-166.
38. Liu W. K., Chen Y., Chang C. T., Belytschko T. (1996), "Advances in Multiple Scale Kernel Particle Methods", Computational Mechanics, Vol. 18, pp. 73-111.
39. Liu W. K., Sukky J., Zhang Y. F. (1997), "Multiresolution Reproducing Kernel Particle Methods", Computational Mechanics, Vol. 20, pp. 295-309.
40. Liu W. K., Sukky J., Dirk T. S., Chen Y. J., Wei H. (1997), "Multiresolution Reproducing Kernel Particle Method for Computational Fluid Dynamics", International Journal for Numerical Methods in Fluids, Vol. 24, pp. 1391-1415.
41. Liu W. K., Sukky J. (1998), "Multiple-Scale Reproducing Kernel Particle Methods for Large Deformation Problems", International Journal for Numerical Methods in Engineering, Vol. 41, pp. 1339-1362.
42. Aluru N. R. (1999), "A Reproducing Kernel Particle Method for Meshless Analysis of Microelectromechanical Systems, Computational mechanics, Vol. 23, pp. 324-338.
43. Ohs R. R., Aluru N. R. (2000), "Meshless Analysis of Piezoelectric Devices", Computational Mechanics, Vol. 27, pp. 23-36.
44. Aluru N. R. (2000), "A Point Collocation Method Based on Reproducing Kernel Approximations", International Journal for Numerical Methods in Engineering, Vol. 47, pp. 1083-1121.
45. Aluru N. R., Li G. (2001), "Finite Cloud Method: A True Meshless Technique Based on a Fixed Reproducing Kernel Approximation", International Journal for Numerical Methods in Engineering, Vol. 50, pp. 2373-2410.
46. Atluri S. N., Zhu T. (1998), "A New Meshless Local Petrov-Galerkin (MLPG) Approach in Computational Mechanics", Computational Mechanics, Vol. 22, pp. 117-127.
47. Atluri S. N., Zhu T. (2000), "New Concepts in Meshless Methods", International Journal for Numerical Methods in Engineering, Vol. 47, pp. 537-556.

48. Babuska I., Melenk J. M. (1997), "The Partition of Unity Method", International Journal for Numerical Methods in Engineering, Vol. 40, pp. 727-758.
49. Nagashima T. (1996), "Development of Meshless Method based on Node-by-Node Procedures (Formulation and Application to Two-Dimensional Elastic Problems)", Transactions of the Japan Society of Mechanics Engineering, Vol. 62(603), pp. 2474-2480.
50. Nagashima T. (1999), "Node-by-Node Meshless Approach and its Applications to Structural Analysis", International Journal for Numerical Methods in Engineering, Vol. 46, pp. 341-385.
51. Onate E., Idelsohn S., Zienkiewicz O. C., Taylor R. L. (1996), "A Finite Point Method in Computational Mechanics: Application to Convective Transport and Fluid Flow", International Journal for Numerical Methods in Engineering, Vol. 39, pp. 3839-3866.
52. Duarte C. A., Oden J. T. (1996), "HP Clouds-an HP Meshless Method", Numerical Method for Partial Differential Equations, Vol. 12, pp. 673-705.
53. Yagawa G., Yamada T. (1996), "Free Mesh Method: A New Meshless Method", Computational Mechanics, Vol. 18, pp. 383-386.
54. Lancaster P., Salkauskas K. (1981), "Surface Generated by Moving Least Square Methods", Mathematics of Computation, Vol. 37(155), pp.141-158.
55. Rao B. N., Rahman S. (2000), "A Efficient Meshless Method for Fracture Analysis of Cracks", Computational Mechanics, Vol. 26, pp.398-408.
56. Kryl P., Belytschko T. (1997), "Element Free Galerkin Method: Convergence of the Continuous and Discontinuous Shape Functions", Computer Methods in Application Mechanics and Engineering, Vol. 148, pp. 257-277.
57. Krongauz Y., Belytschko T. (1998), "EFG Approximation with Discontinuous Derivatives", International Journal for Numerical Methods in Engineering, Vol. 41, pp. 1215-1233.
58. Organ D., Fleming M., Tery T., Belytschko T. (1996), "Continuous Meshless Approximations for Nonconvex Bodies by Diffraction and Transparency", Computational Mechanics, Vol. 18, pp. 225-235.
59. Dolbow J., Belytschko T. (1999), "Numerical Integration of the Galerkin Weak Form in Meshless Methods", Computational Mechanics, Vol. 23, pp. 219-230.

60. Mukherjee Y. X., Mukherjee S. (1997), "On Boundary Conditions in the Element Free Galerkin Methods", Computational Mechanics, Vol. 19, pp. 267-270.
61. Zhu T., Atluri S. N. (1998), "A Modified Collocation and a Penalty Formulation For Enforcing the Essential Boundary Conditions in the Element Free Galerkin Method", Computational Mechanics, Vol. 21, pp. 211-222.
62. Hegen D. (1996), "Element-Free Galerkin Methods in Combination with Finite Element Approaches", Computer Methods in Applied Mechanics and Engineering, Vol. 135, pp. 143-166.
63. Gosz J., Liu W. K. (1996), "Admissible Approximations for Essential Boundary Conditions in the Reproducing Kernel Particle Method", Computational Mechanics, Vol. 19, pp. 120-135.
64. Günther F. C., Liu W. K. (1998), "Implementation of Boundary Conditions for Meshless Methods", Computer Methods in Applied Mechanics and Engineering, Vol. 163, pp. 205-230.
65. Gavete L., Benito J. J., Falcón S., Ruiz A. (2000), "Implementation of Essential Boundary Conditions in a Meshless Method", Communications in Numerical Methods in Engineering, Vol. 16, pp. 409-421.
66. Zhang X., Liu X., Lu M. W., Chen Y. (2001), "Imposition of Essential Boundary Conditions by Displacement Constraint Equations in Meshless Methods", Communications in Numerical Methods in Engineering, Vol. 17, pp.165-178.
67. Babuska I., Rheinboldt W. C. (1978), "Error Estimations for Adaptive Finite Element Method", International Journal for Numerical Methods in Engineering, Vol. 12, pp. 345-405.
68. Ainsworth M., Oden J. T. (1993), "A Unified Approach to *a Posteriori* Error Estimation Using Element Residual Method", Numerische Mathematik, Vol. 65, pp. 23-50.
69. Ainsworth M., Oden J. T., (1997) "A *Posteriori* Estimation in Finite Element Analysis", Computer Methods in Applied Mechanics and Engineering, Vol. 142, pp. 1-88.
70. Babuska B., Miller A., (1984) "The Post-Processing Approach in the Finite Element Method – Part 1: Calculation of Displacements, Stress and other Higher Derivatives of the Displacements", International Journal for Numerical Methods in Engineering, Vol. 20, pp. 1085-1109.
71. Babuska B., Miller A., (1984) "The Post-Processing Approach in the Finite Element Method – Part 2: The Calculation of Stress Intensity Factors",

- International Journal for Numerical Methods in Engineering, Vol. 20, pp. 1111-1129.
72. Babuska B., Miller A., (1984) "The Post-Processing Approach in the Finite Element Method – Part 3: A *Posteriori* Error Estimation and Adaptive Mesh Selection", International Journal for Numerical Methods in Engineering, Vol. 20, pp. 2311-2324.
 73. Zienkiewicz O. C., Zhu J. Z. (1987), "A Simple Error Estimator and Adaptive Procedure for Practical Engineering Analysis", International Journal for Numerical Methods in Engineering, Vol. 24, pp. 337-357.
 74. Zhu J. Z., Zienkiewicz O. C. (1990), "Superconvergence Recovery Technique and a *Posteriori* Error Estimators", International Journal for Numerical Methods in Engineering, Vol. 30, pp. 1321-1339.
 75. Belytschko T., Liu W. K., Sinnger M. (1998), "On Adaptivity and Error Criteria for Meshfree Methods", Advances in Adaptive Computational Methods in Mechanics, Eds. Ladevèze P. and Oden J. T., pp. 217-228.
 76. Chung H. J., Belytschko T. (1998), "An Error Estimate in the EFG Method", Computational Mechanics, Vol. 21, pp. 91-100.
 77. Duarte C. A. M., Oden J. T. (1996), "A *hp* Adaptive Method Using Clouds", Computer Methods in Applied Mechanics and Engineering, Vol. 139, pp. 237-262.
 78. Häussler-Combe U., Korn C. (1998), "An Adaptive Approach with the Element-Free Galerkin Method", Computer Methods in Applied Mechanics and Engineering, Vol. 162, pp. 203-222.
 79. Gavete L., Cuesta J. L., Ruiz A. (2002), "A Procedure for Approximation of the Error in the EFG Method", International Journal for Numerical Methods in Engineering, Vol. 53, pp. 677-690.
 80. Lee C. K., Zhou C. E. (2004), "On Error Estimation and Adaptive Refinement for Element Free Galerkin Method. Part I: Stress Recovery and A *Posteriori* Error Estimation", Computers & Structures, Vol. 82, pp. 413-428.
 81. Lee C. K., Zhou C. E. (2004), "On Error Estimation and Adaptive Refinement for Element Free Galerkin Method. Part II: Adaptive Refinement", Computers & Structures, Vol. 82, pp. 429-443.
 82. Belytschko T., Organ D., Krongauz Y. (1995), "A Coupled Finite Element-Element-Free Galerkin Method", Computational Mechanics, Vol. 17, pp. 186-195.

83. Krongauz Y., Belytschko T. (1996), "Enforcement of Essential Boundary Conditions in Meshless Approximations Using Finite Elements", *Computer Methods in Applied Mechanics and Engineering*, Vol. 131, pp. 133–145.
84. Liu W. K., Uras R. A., Chen Y. (1997), "Enrichment of the Finite Element Method with the Reproducing Kernel Particle Method", *Journal of Applied Mechanics*, Vol. 64, pp. 861-870.
85. Li H., Han G., Mang H. A., Torzicky P. (1986), "A New Method for the Coupling of Finite Element and Boundary Element Discretized Subdomains of Elastic Bodies", *Computer Methods in Applied Mechanics and Engineering*, Vol. 54, pp. 161–185.
86. Zeng X., Kallivokas L. F., Bielak J. (1993), "A Symmetric Variational Finite Element-Boundary Integral Equation Coupling Method", *Computers & Structures*, Vol. 46, pp. 995-1000.
87. Belytschko T., Lu Y. Y. (1994), "A Variational Coupled FE-BE Method for Transient Problems", *International Journal for Numerical Methods in Engineering*, Vol. 37, pp. 91-105.
88. Zeng X., Bielak J. (1995), "Stable Symmetric Finite Element-Boundary Integral Coupling Methods for Fluid-Structure Interface Problems", *Engineering Analysis with Boundary Elements*, Vol. 15, pp. 79-91.
89. Pullan A. (1996), "High-Order Coupled Finite Element / Boundary Element Torso Model", *IEEE Transactions on Biomedical Engineering*, Vol. 43, pp. 292-298.
90. Oysu C., Fenner R. T. (1998), "Two-Dimensional Elastic Contact Analysis by Coupling Finite Element and Boundary Element Methods", *Journal of Strain Analysis for Engineering Design*, Vol. 33, pp. 459-468.
91. Song S. P., Li B. Q. (1999), "Coupled Boundary / Finite Element Method for the Computation of Magnetically and Electrostatically Levitated Droplet Shapes", *International Journal for Numerical Methods in Engineering*, Vol. 44, pp. 1055-1077.
92. Yu G., Mansur W. J., Carrer J. A. M., Lie S. T. (2001), "A More Stable Scheme for BEM / FEM Coupling Applied to Two-Dimensional Elastodynamics", *Computers & Structures*, Vol. 79, pp. 811-823.
93. Gu Y. T., Liu G. R. (2001), "A Coupled Element Free Galerkin / Boundary Element Method for Stress Analysis of Two-Dimensional Solids", *Computer methods in Applied Mechanics and Engineering*, Vol. 190, pp. 4405-4419.

94. Liu G. R., Gu Y. T. (2000), "Coupling of Element Free Galerkin and Hybrid Boundary Element Methods using Modified Variational Formulation", *Computational Mechanics*, Vol. 26, pp.166-173.
95. Li G., Paulino G. H., Aluru N. R. (2003), "Coupling of the Mesh-Free Finite Cloud Method with the Boundary Element Method: a Collocation Approach", *Computer Methods in Applied Mechanics and Engineering*, Vol. 192, pp. 2355-2375.
96. Karutz H., Chudoba R., Krätzig W. B. (2002), "Automatic Adaptive Generation of a Coupled Finite Element / Element-Free Galerkin Discretization", *Finite Elements in Analysis and Design*, Vol. 38, pp. 1075-1091.
97. Carstensen C., Funken S. A., Stephan E. P. (1997), "On the Adaptive Coupling of FEM and BEM in 2-D-Elasticity", *Numerische Mathematik*, Vol. 77, pp. 187-221.
98. Wrobel L. C. (2002), "The Boundary Element Method", Volume 1: *Applications in Thermo-Fluids and Acoustics*, Wiley & Sons, Inc.
99. Stroud, A.H. and Secrest, D. (1975), *Gaussian Quadrature Formulas*, Prentice-Hall, N.Y.
100. Heath M. T. (1997), *Scientific Computing: An Introductory Survey*, McGraw-Hill.
101. Szabo B. A., Babuska I. (1991), *Finite Element Analysis*, Wiley & Sons, Inc.
102. Chen G., Zhou J. (1992), *Boundary Element Methods*, Computational Mathematics and Applications.
103. Weimin H., Meng X. (2001), "Error Analysis of the Reproducing Kernel Particle Method", *Computer Methods in Applied Mechanics and Engineering*, Vol. 190, pp. 6157-6181.
104. Timoshenko S. P., Goodier J. N. (1970), *Theory of Elasticity*, third edition, McGraw Hill, New York.
105. Kobayshi A. S. (1986), *Computational Methods in the Mechanics of Fracture*, North-Holland, Amsterdam.
106. Brebbia C. A., Georgiou P. (1979), "Combination of Boundary and Finite Elements in Elastostatics", *Applied Mathematical Modelling*, Vol. 3, pp. 212-220.
107. Reddy B. D. (1986), "Functional Analysis and Boundary-Value Problems: an Introductory Treatment", John Wiley & Sons Inc.

108. Zienkiewicz O. C., Zhu J. Z. (1992), "The Superconvergent Patch Recovery and *a Posteriori* Error Estimates. Part 1: The Recovery Technique", International Journal for Numerical Methods in Engineering, Vol. 33, pp. 1331-1364.
109. Sloan I. H. (1976), "Improvement by Iteration for Compact Operator Equation", Mathematics of Computation, Vol. 30, pp. 758-764.
110. Atkinson K. A. (1990), "A survey of Numerical Methods for Solving Nonlinear Integral Equations", Journal of Integral Equation and Applications, Vol. 3, pp. 15-46.
111. Kaneko H., Xu Y. (1996), "Superconvergence of the Iterated Galerkin Methods for Hammerstein Equations, SIAM Journal on Numerical Analysis, Vol. 33, pp. 1048-1064.
112. Gerg P., Oden J. T., Demkowicz (1996), "Numerical Solution and *a Posteriori* Error Estimation of Exterior Acoustics Problems by a Boundary Element Method at High Wave Numbers", Journal of the Acoustics Society of America, Vol. 100, pp. 335-345.
113. Chen H. B., Yu D. H., Schnack E. (2003), "A Simple *A Posteriori* Error Estimation for Adaptive BEM in Elasticity", Computational Mechanics, Vol.30, pp. 343-354.
114. Grätsch T., Hartmann F. (2003), "Finite Element Recovery Techniques for Local Quantities of Linear Problems Using Fundamental Solutions", Computational Mechanics, Vol. 33, pp. 15-21.
115. Wrobel L. C. (2002), "The Boundary Element Method", Volume 1: Applications in Thermo-Fluids and Acoustics, Wiley & Sons, Inc.
116. Zienkiewicz O. C., Zhu J. Z. (1990), "The Three R's of Engineering Analysis and Error Estimation and Adaptivity", Computer Methods in Applied Mechanics and Engineering, Vol. 82, pp. 95-113.
117. Mackerle, J. (2001), "Error Estimates and Adaptive Finite Element Methods: A Bibliography (1990-2000)", Engineering Computations, Vol. 18, pp. 802-914.
118. Mackerle, J. (2001), "2D and 3D Finite Element Meshing and Remeshing: A Bibliography (1990-2001)", Engineering Computations, Vol. 18, pp. 1108-1197.
119. Lo S. H. (2002), "Finite Element Mesh Generation and Adaptive Meshing", Progress in Structural Engineering and Materials, Vol. 4, pp. 381-300.

120. Piegl L., Tiller W. (1987), "Curve and Surface Constructions using Rational B-Splines", Computer-Aided Design, Vol. 19, pp.485-498.
121. Lee C. K. (2000), "A New Finite Point Generation Scheme using Metric Specifications", International Journal of Numerical Methods for Engineering, Vol. 48, pp. 1423-1444.
122. Lohner R. and Onate E. (1998), "An Advancing Front Point Generation Technique", Communications in Numerical Methods in Engineering, Vol. 14, pp. 1097-1108.
123. Sukumar N., Moran B., Belytschko T. (1998), "The Natural Element Method in Solid Mechanics", International Journal for Numerical Methods in Engineering, Vol. 43, pp. 839-887.
124. Arash Y., Ali K., Shahram S., Hosein A. R. B. (2001), "Topological Aspects of Meshless Methods and Nodal Ordering for Meshless Discretizations", International Journal of Numerical Methods for Engineering, Vol. 52, pp. 921-938.
125. Zienkiewicz O. C., Zhu J. Z. (1992), "The Superconvergent Patch Recovery and *a Posteriori* Error Estimates. Part 2: Error Estimates and Adaptivity", International Journal for Numerical Methods in Engineering, Vol. 33, pp. 1365-1382.
126. Lee C. K., Lo S. H. (1992), "An Automatic Adaptive Refinement Finite Element Procedure for 2D Elastostatic Analysis", International Journal for Numerical Methods in Engineering, Vol. 35, pp. 1967-1989.
127. Lee C. K., Hobbs R. E. (1997), "Automatic Adaptive Refinement for Shell Analysis using Nine-Node Assumed Strain Element", International Journal for Numerical Methods in Engineering, Vol. 40, pp. 3601-3638.
128. Lee C. K., Hobbs R.E. (1998), "On Solving Nearly Incompressible 2D Problems using an Adaptive Refinement Procedure", Communications in Numerical Methods in Engineering, Vol. 14, pp. 409-418.
129. Lee C. K., Hobbs R. E. (1998), "Automatic Adaptive Refinement for Plate Bending Problems using Reissner-Mindlin Plate Bending Elements", International Journal for Numerical Methods in Engineering, Vol. 41, pp. 1-63.
130. Oñate E., Bugeda G. (1993), "A Study of Mesh Optimality Criteria in Adaptive Finite Element Analysis", Engineering Computations, Vol. 10, pp. 307-321.
131. Lee C. K. (1999), "Automatic Adaptive Mesh Generation using Metric Advancing Front Approach", Engineering Computation, Vol. 16. pp. 230-263.

132. Rao B. N., Rahman S. (2003), “An Enriched Meshless Method for Non-linear Fracture Mechanics”, *International Journal for Numerical Methods in Engineering*, Vol. 59, pp. 197–223.

133. Marc Duflot, Hung Nguyen-Dang (2004), “A Meshless Method with Enriched Weight Functions for Fatigue Crack Growth”, *International Journal for Numerical Methods in Engineering*, Vol. 59, pp. 1945 – 1961.

APPENDIX A

BOUNDARY ELEMENT METHOD BASED ON DISCONTINUOUS LINEAR ELEMENT

A.1 Discontinuous linear element

In this study, the discontinuous element model [13] is used for the boundary element method (BEM) (shown in Figure A.1). The points **1** and **2** are the base points of element Q , which are shifted from the ends of element Q with offset as a and b respectively. If $a = b = 0$, the element model will be reduced to the conventional BEM linear element model.

Figure A.1 Discontinuous element model

The displacements $\hat{\mathbf{u}} = \{\hat{u}_1^Q, \hat{u}_2^Q\}^T$ and tractions $\hat{\mathbf{t}} = \{\hat{t}_1^Q, \hat{t}_2^Q\}^T$ of any point $(-1 \leq \zeta \leq 1)$ on the element Q can then be linearly interpolated by the nodal values of base points **1** ($\hat{u}_1^{Q,1}$, $\hat{u}_2^{Q,1}$, $\hat{t}_1^{Q,1}$ and $\hat{t}_2^{Q,1}$) and **2** ($\hat{u}_1^{Q,2}$, $\hat{u}_2^{Q,2}$, $\hat{t}_1^{Q,2}$ and $\hat{t}_2^{Q,2}$) as

$$\begin{Bmatrix} \hat{u}_1^Q \\ \hat{u}_2^Q \end{Bmatrix} = \begin{bmatrix} \varphi_1 & 0 & \varphi_2 & 0 \\ 0 & \varphi_1 & 0 & \varphi_2 \end{bmatrix} \begin{Bmatrix} \hat{u}_1^{Q,1} \\ \hat{u}_2^{Q,1} \\ \hat{u}_1^{Q,2} \\ \hat{u}_2^{Q,2} \end{Bmatrix} \quad (\text{A.1a})$$

$$\begin{Bmatrix} \hat{t}_1^Q \\ \hat{t}_2^Q \end{Bmatrix} = \begin{bmatrix} \varphi_1 & 0 & \varphi_2 & 0 \\ 0 & \varphi_1 & 0 & \varphi_2 \end{bmatrix} \begin{Bmatrix} \hat{t}_1^{Q,1} \\ \hat{t}_2^{Q,1} \\ \hat{t}_1^{Q,2} \\ \hat{t}_2^{Q,2} \end{Bmatrix} \quad (\text{A.1b})$$

where, φ_1 and φ_2 are linear element interpolation functions and can be given by

$$\varphi_1 = \left[\frac{1}{2}(1-\xi)(L-b) - \frac{1}{2}(1+\xi)b \right] / (L-a-b) \quad (\text{A.2a})$$

$$\varphi_2 = \left[-\frac{1}{2}(1-\xi)a + \frac{1}{2}(1+\xi)(L-a) \right] / (L-a-b) \quad (\text{A.2b})$$

If $a = b$, the interpolation functions can be further simplified as

$$\varphi_1 = \frac{1}{2} - \frac{1}{2} \frac{L}{(L-2a)} \xi \quad (\text{A.2c})$$

$$\varphi_2 = \frac{1}{2} + \frac{1}{2} \frac{L}{(L-2a)} \xi \quad (\text{A.2d})$$

In this study, the Eqns. A.2c and A.2d is adopted for simplicity.

A.2 BEM formulation based on discontinuous element

A.2.1 Boundary integration equation

Figure A.2 Points and elements assignment of the BEM

The two dimensional elastostatic problem defined in Eqn. 3.1 is introduced as the model problem, in which the problem domain Ω shown in Figure A.2 is surrounded by the boundary Γ , $\mathbf{p}$ is the source point and $\mathbf{q}$ is the field point. $\mathbf{n}$ is the outer normal line of the boundary. The corresponding boundary integration equation (BIE) of BEM is introduced as following [13]:

$$\mathbf{c}^i \mathbf{u}^i + \int_{\Gamma} \mathbf{t}^* \mathbf{u} d\Gamma = \int_{\Gamma} \mathbf{u}^* \mathbf{t} d\Gamma + \int_{\Omega} \mathbf{u}^* \mathbf{b} d\Omega, \text{ sum over } i; i = 1, 2. \quad (\text{A.3})$$

It is noted that the integration direction is anticlockwise along the outer boundary and clockwise along the inner boundary.

In practical computation, the boundary Γ is first discretized into NN points and NE elements (shown in Figure A.2), i.e., $\Gamma = \sum_{Q=1}^{NE} \Gamma_Q$, Γ_Q is the boundary on which the element Q is located. Then the discrete matrix form of BIE can be obtained by substituting Eqns. A.1 and A.2 into Eqn. A.3 as

$$\begin{bmatrix} \mathbf{c}_{11}^{\mathbf{p}} & \mathbf{c}_{12}^{\mathbf{p}} \\ \mathbf{c}_{21}^{\mathbf{p}} & \mathbf{c}_{22}^{\mathbf{p}} \end{bmatrix} \begin{Bmatrix} u_1(\mathbf{p}) \\ u_2(\mathbf{p}) \end{Bmatrix} + \sum_{Q=1}^{NE} \int_{\Gamma_Q} \begin{bmatrix} t_{11}^* \phi_1 & t_{12}^* \phi_1 & t_{11}^* \phi_2 & t_{12}^* \phi_2 \\ t_{21}^* \phi_1 & t_{22}^* \phi_1 & t_{21}^* \phi_2 & t_{22}^* \phi_2 \end{bmatrix} \begin{Bmatrix} \hat{u}_1^{Q,1} \\ \hat{u}_2^{Q,1} \\ \hat{u}_1^{Q,2} \\ \hat{u}_2^{Q,2} \end{Bmatrix} d\Gamma = \quad (\text{A.4})$$

$$\sum_{Q=1}^{NE} \int_{\Gamma_Q} \begin{bmatrix} u_{11}^* \phi_1 & u_{12}^* \phi_1 & u_{11}^* \phi_2 & u_{12}^* \phi_2 \\ u_{21}^* \phi_1 & u_{22}^* \phi_1 & u_{21}^* \phi_2 & u_{22}^* \phi_2 \end{bmatrix} \begin{Bmatrix} \hat{t}_1^{Q,1} \\ \hat{t}_2^{Q,1} \\ \hat{t}_1^{Q,2} \\ \hat{t}_2^{Q,2} \end{Bmatrix} d\Gamma + \sum_{Q=1}^{NE} \int_{\Gamma_Q} \begin{Bmatrix} f_1^* \\ f_2^* \end{Bmatrix} d\Gamma$$

where, $\mathbf{u}(\mathbf{p}) = \{u_1(\mathbf{p}), u_2(\mathbf{p})\}^T$ is the displacement associated with point $\mathbf{p}$, $\mathbf{c}_{lk}^{\mathbf{p}}$ is a coefficient related with the position of point $\mathbf{p}$ and is showed as

$$\mathbf{c}_{lk}^{\mathbf{p}} = \begin{cases} \delta_{lk} & \mathbf{p} \in \Omega \\ \frac{1}{2} \delta_{lk} & \mathbf{p} \in \Gamma \text{ (smooth boudary point)} \\ \delta_{lk} + \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_{\varepsilon}} t_{lk}^* d\Gamma & \mathbf{p} \in \Gamma \text{ (non - smooth boundary point)} \end{cases} \quad (\text{A.5a})$$

u_{lk}^* and t_{lk}^* are the k th components of displacements and tractions due to a unit nodal load in the l direction. In 2D problems, u_{lk}^* and t_{lk}^* can be expressed as

$$u_{lk}^* = \frac{1}{8\pi\bar{G}(1-\mu)} \left\{ \left[(3-4\mu) \ln \frac{1}{r} - \frac{7-8\mu}{2} \right] \delta_{lk} + \frac{\partial r}{\partial x_l} \frac{\partial r}{\partial x_k} \right\} \quad (\text{A.5b})$$

$$t_{lk}^* = \frac{-1}{4\pi(1-\mu)r} \left\{ \frac{\partial r}{\partial n} [(1-2\mu) \delta_{lk} + 2 \frac{\partial r}{\partial x_l} \frac{\partial r}{\partial x_k}] - (1-2\mu) \left(\frac{\partial r}{\partial x_l} n_k - \frac{\partial r}{\partial x_k} n_l \right) \right\} \quad (\text{A.5c})$$

in Eqns. A.5a, A.5b and A.5c, $(l, i, j, k) = 1, 2$, $\bar{G} = \frac{E}{2(1+\mu)}$, E is the Young's module and μ is the Poisson ratio. n_l and n_k are the direction cosines of the normal with respect to x_l , $\frac{\partial r}{\partial \mathbf{n}}$ is the directional derivative along the outer normal line $\mathbf{n}$ and x_k . $r = [(x_1^q - x_1^p)^2 + (x_2^q - x_2^p)^2]^{1/2}$.

$$f_l^* = \frac{r}{8\pi\bar{G}} (2 \ln \frac{1}{r} - 1) (b_l n_j \frac{\partial r}{\partial x_j} - \frac{1}{2(1-\mu)} b_j \frac{\partial r}{\partial x_j} n_l) \text{ sum over } j; j = 1, 2. \quad (\text{A.5d})$$

in Eqn. A.5d, b_l is the component of body force and is set to be constant in this part.

When the nodal displacements and tractions of the base points on each element are available, the displacement and stress of any inner points in the domain Ω can then be given by

$$u_l = - \int_{\Gamma} t_{lk}^* \hat{u}_k d\Gamma + \int_{\Gamma} u_{lk}^* \hat{t}_k d\Gamma + \int_{\Gamma} f_l^* d\Gamma \quad (\text{A.6a})$$

$$\sigma_{ij} = - \int_{\Gamma} s_{ijk}^* \hat{u}_k d\Gamma + \int_{\Gamma} d_{ijk}^* \hat{t}_k d\Gamma + \int_{\Gamma} F_{ij}^* d\Gamma \quad (\text{A.6b})$$

where,

$$s_{ijk}^* = \frac{\bar{G}}{2\pi(1-\mu)r^2} \{ 2r_m n_m [(1-2\mu) r_k \delta_{ij} + \mu (r_j \delta_{ik} + r_i \delta_{jk}) - 4r_i r_j r_k] + \quad (\text{A.7a})$$

$$2\mu (n_i r_j + n_j r_i) r_k + (1-2\mu) (2n_k r_i r_j + n_j \delta_{ik} + n_i \delta_{jk}) - (1-4\mu) n_k \delta_{ij} \}$$

$$d_{ijk}^* = \frac{1}{4\pi(1-\mu)r} [(1-2\mu)(r_i\delta_{jk} + r_j\delta_{ik} - r_k\delta_{ij}) + 2r_i r_j r_k] \quad (\text{A.7b})$$

$$F_{ij}^* = \frac{1}{8\pi} \left\{ 2n_m r_m (b_i r_j + b_j r_i) + \frac{1}{1-\mu} \left[\mu \delta_{ij} (2n_m r_m b_s r_s + (1-2\ln \frac{1}{r} b_m r_m) - b_m r_m (n_i r_j + n_j r_i) + \frac{1-2\mu}{2} (1-2\ln \frac{1}{r}) (b_i n_j + b_j n_i) \right] \right\} \quad (\text{A.7c})$$

where, $r_i = \frac{\partial r}{\partial x_i}$. sum over k, m ; $(i, j, k, m) = 1, 2$.

A.2.2 Element matrix of boundary integration equation

Conveniently, the sign matrix $\mathbf{h}^Q$ and $\mathbf{g}^Q$ are first introduced as

$$\mathbf{h}^Q = \int_{\Gamma_Q} \begin{bmatrix} t_{11}^* \varphi_1 & t_{12}^* \varphi_1 & t_{11}^* \varphi_2 & t_{12}^* \varphi_2 \\ t_{21}^* \varphi_1 & t_{22}^* \varphi_1 & t_{21}^* \varphi_2 & t_{22}^* \varphi_2 \end{bmatrix} d\Gamma = \begin{bmatrix} h_{11}^{\mathbf{p},Q,1} & h_{12}^{\mathbf{p},Q,1} & h_{11}^{\mathbf{p},Q,2} & h_{12}^{\mathbf{p},Q,2} \\ h_{21}^{\mathbf{p},Q,1} & h_{22}^{\mathbf{p},Q,1} & h_{21}^{\mathbf{p},Q,2} & h_{22}^{\mathbf{p},Q,2} \end{bmatrix} \quad (\text{A.8a})$$

$$\mathbf{g}^Q = \int_{\Gamma_Q} \begin{bmatrix} u_{11}^* \varphi_1 & u_{12}^* \varphi_1 & u_{11}^* \varphi_2 & u_{12}^* \varphi_2 \\ u_{21}^* \varphi_1 & u_{22}^* \varphi_1 & u_{21}^* \varphi_2 & u_{22}^* \varphi_2 \end{bmatrix} d\Gamma = \begin{bmatrix} g_{11}^{\mathbf{p},Q,1} & g_{12}^{\mathbf{p},Q,1} & g_{11}^{\mathbf{p},Q,2} & g_{12}^{\mathbf{p},Q,2} \\ g_{21}^{\mathbf{p},Q,1} & g_{22}^{\mathbf{p},Q,1} & g_{21}^{\mathbf{p},Q,2} & g_{22}^{\mathbf{p},Q,2} \end{bmatrix} \quad (\text{A.8b})$$

in which,

$$h_{lk}^{\mathbf{p},Q,m} = \int_{\Gamma_Q} t_{lk}^* \varphi_m d\Gamma, \quad g_{lk}^{\mathbf{p},Q,m} = \int_{\Gamma_Q} u_{lk}^* \varphi_m d\Gamma, \quad (l, k, m) = 1, 2 \quad (\text{A.8c})$$

The integration terms in Eqn. A.4 are computed as following:

A.2.2.1 Point $\mathbf{p}$ is not located on the element Q

In this case, the values of $\mathbf{h}^Q$ and $\mathbf{g}^Q$ can be computed by the standard Gaussian Quadrature method without any problem. However, since $h_{lk}^{\mathbf{p},Q,m}$ and $g_{lk}^{\mathbf{p},Q,m}$ contain items $\frac{1}{r}$ and $\ln(\frac{1}{r})$ respectively, the value of $h_{lk}^{\mathbf{p},Q,m}$ and $g_{lk}^{\mathbf{p},Q,m}$ will become very large when r is small (for example, when the inner point approaches the

boundary). In order to keep the integration precision, different number of Gaussian integration points need to be adopted on this situation.

A.2.2.2 Point $\mathbf{p}$ is repeated with the base points of element Q

In Figure A.3, the discontinuous element Q is shown with length as L . The points **1** and **2** are the base points with the offset a from the ends of element Q , respectively. $\mathbf{n}$ is the outer normal line of element Q . When the point $\mathbf{p}$ repeats with the base points **1** or **2** of the element Q , the corresponding integration of $\mathbf{h}^Q$ and $\mathbf{g}^Q$ of element Q are calculated as following:

Figure A.3 Point $\mathbf{p}$ on element Q

1). Point $\mathbf{p}$ repeating with the point **1** of element Q , i.e., $\mathbf{p} = (Q, \mathbf{1})$

The computation of $g_{lk}^{\mathbf{p},Q,m}$ is as follows:

$$\begin{aligned}
 g_{11}^{\mathbf{p},Q,1} = & -\frac{3-4\mu}{8\pi \bar{G} (1-\mu)} \frac{L}{2} \left\{ \ln \frac{L-2a}{2} + \frac{(L-a)(L-3a)}{L(L-2a)} \ln \frac{2L-2a}{L-2a} + \right. \\
 & \left. \frac{a(2L-3a)}{L(L-2a)} \ln \frac{2a}{L-2a} - 1.5 \right\} - \\
 & \frac{1}{8\pi \bar{G} (1-\mu)} \frac{(7-8\mu)L}{4} + \frac{1}{8\pi \bar{G} (1-\mu)} \frac{L}{2} \frac{(x_1^{(2)} - x_1^{(1)})^2}{(L-2a)^2}
 \end{aligned} \tag{A.9a}$$

$$\begin{aligned}
 g_{22}^{\mathbf{p},Q,1} = & -\frac{3-4\mu}{8\pi \overline{G} (1-\mu)} \frac{L}{2} \ln \frac{L-2a}{2} + \left\{ \frac{(L-a)(L-3a)}{L(L-2a)} \ln \frac{2L-2a}{L-2a} + \right. \\
 & \left. \frac{a(2L-3a)}{L(L-2a)} \ln \frac{2a}{L-2a} - 1.5 \right\} - \\
 & \frac{1}{8\pi \overline{G} (1-\mu)} \frac{(7-8\mu)L}{4} + \frac{1}{8\pi \overline{G} (1-\mu)} \frac{L}{2} \frac{(x_2^{(2)} - x_2^{(1)})^2}{(L-2a)^2}
 \end{aligned} \tag{A.9b}$$

$$\begin{aligned}
 g_{11}^{\mathbf{p},Q,2} = & -\frac{3-4\mu}{8\pi \overline{G} (1-\mu)} \frac{L}{2} \left\{ \ln \frac{L-2a}{2} + \frac{(L-a)^2}{L(L-2a)} \ln \frac{2L-2a}{L-2a} - \right. \\
 & \left. \frac{a^2}{L(L-2a)} \ln \frac{2a}{L-2a} - 0.5 \right\} - \\
 & \frac{1}{8\pi \overline{G} (1-\mu)} \frac{(7-8\mu)L}{4} + \frac{1}{8\pi \overline{G} (1-\mu)} \frac{L}{2} \frac{(x_1^{(2)} - x_1^{(1)})^2}{(L-2a)^2}
 \end{aligned} \tag{A.9c}$$

$$\begin{aligned}
 g_{22}^{\mathbf{p},Q,2} = & -\frac{3-4\mu}{8\pi \overline{G} (1-\mu)} \frac{L}{2} \left\{ \ln \frac{L-2a}{2} + \frac{(L-a)^2}{L(L-2a)} \ln \frac{2L-2a}{L-2a} - \right. \\
 & \left. \frac{a^2}{L(L-2a)} \ln \frac{2a}{L-2a} - 0.5 \right\} - \\
 & \frac{1}{8\pi \overline{G} (1-\mu)} \frac{(7-8\mu)L}{4} + \frac{1}{8\pi \overline{G} (1-\mu)} \frac{L}{2} \frac{(x_2^{(2)} - x_2^{(1)})^2}{(L-2a)^2}
 \end{aligned} \tag{A.9d}$$

$$g_{12}^{\mathbf{p},Q,1} = g_{21}^{\mathbf{p},Q,1} = g_{12}^{\mathbf{p},Q,2} = g_{21}^{\mathbf{p},Q,2} = \frac{1}{8\pi \overline{G} (1-\mu)} \frac{L}{2} \frac{(x_1^{(2)} - x_1^{(1)})(x_2^{(2)} - x_2^{(1)})}{(L-2a)^2} \tag{A.9e}$$

The computation of $h_{lk}^{\mathbf{p},Q,m}$ is as follows:

$$h_{11}^{\mathbf{p},Q,2} = \int_{\Gamma_Q} t_{11}^* \varphi_2 d\Gamma = 0 \tag{A.9f}$$

$$h_{12}^{\mathbf{p},Q,2} = \int_{\Gamma_Q} t_{12}^* \varphi_2 d\Gamma = -\frac{1-2\mu}{4\pi (1-\mu)} \frac{L}{L-2a} \tag{A.9g}$$

$$h_{21}^{\mathbf{p},Q,2} = \int_{\Gamma_Q} t_{21}^* \varphi_2 d\Gamma = \frac{1-2\mu}{4\pi (1-\mu)} \frac{L}{L-2a} \tag{A.9h}$$

$$h_{22}^{p,Q,2} = \int_{\Gamma_Q} t_{22}^* \varphi_2 d\Gamma = 0 \quad (\text{A.9i})$$

2). Point **p** repeating with the point **2** of element Q , i.e., $\mathbf{p} = (Q, 2)$

The values of $g_{11}^{p,Q,1}$, $g_{11}^{p,Q,2}$, $g_{22}^{p,Q,1}$ and $g_{22}^{p,Q,2}$ of $\mathbf{p} = (Q, 2)$, will equal the values of $g_{11}^{p,Q,2}$, $g_{11}^{p,Q,1}$, $g_{22}^{p,Q,2}$ and $g_{22}^{p,Q,1}$ of $\mathbf{p} = (Q, 1)$, respectively. The remained values are calculated as following:

$$g_{12}^{p,Q,1} = g_{21}^{p,Q,1} = g_{12}^{p,Q,2} = g_{21}^{p,Q,2} = \frac{1}{8\pi \bar{G} (1-\mu)} \frac{L}{2} \frac{(x_1^{(2)} - x_1^{(1)})(x_2^{(2)} - x_2^{(1)})}{(L-2a)^2} \quad (\text{A.10a})$$

The value of $h_{lk}^{p,Q,m}$ is as following:

$$h_{11}^{p,Q,1} = \int_{\Gamma_Q} t_{11}^* \varphi_2 d\Gamma = 0 \quad (\text{A.10b})$$

$$h_{12}^{p,Q,1} = \int_{\Gamma_Q} t_{12}^* \varphi_2 d\Gamma = -\frac{1-2\mu}{4\pi (1-\mu)} \frac{L}{L-2a} \quad (\text{A.10c})$$

$$h_{21}^{p,Q,1} = \int_{\Gamma_Q} t_{21}^* \varphi_2 d\Gamma = \frac{1-2\mu}{4\pi (1-\mu)} \frac{L}{L-2a} \quad (\text{A.10d})$$

$$h_{22}^{p,Q,1} = \int_{\Gamma_Q} t_{22}^* \varphi_2 d\Gamma = 0 \quad (\text{A.10e})$$

A.2.3 Forcing term of boundary integration equation

When the point **p** is on the element Q , the integration of $\int_{\Gamma_Q} f_1^* d\Gamma$ and $\int_{\Gamma_Q} f_2^* d\Gamma$ will be singular. The corresponding exact integration results are given by following formulation as

$$\begin{aligned} \int_{\Gamma_Q} f_i^* d\Gamma = & -\frac{b_i n_j \frac{\partial r}{\partial x_j} - \frac{1}{2(1-\mu)} b_j \frac{\partial r}{\partial x_j} n_i}{8\pi \bar{G}} \{[(L-a)^2 + a^2] \ln \frac{L-2a}{2} + \\ & (L-a)^2 \ln \frac{2L-2a}{L-2a} + a^2 \ln \frac{2a}{L-2a}\} \end{aligned} \quad (\text{A.11})$$

It is noted that all the formulae above are only for the plain strain problems. As for the plain stress problems, a transformation for the material parameters μ is needed and it is expressed as

$$\mu: \mu \rightarrow \mu/(1 + \mu) \quad (\text{A.12})$$

APPENDIX B

INTEGRATION SCHEME FOR RKPM

B.1 Gaussian integration rule

The basic idea of the Gaussian quadrature integration rule is to provide the most accurate possible formulae by choosing the quadrature point x_i and weights w_i to give exact results for polynomial of highest degree possible. In this part, the related formulations for one, two and three-dimensional domains are introduced as following:

1). One-dimension Quadrature [99]

$$I = \int_{-1}^1 f(x) dx \cong \sum_{i=1}^n w_i f(x_i) \quad (\text{B.1})$$

where x_i is the coordinate of the i th integration point, w_i is the associated weighting factor, and n is the total number of integration points; they are listed in Table B.1. The associated error is $E_n = 0$ ($d^{2n} f / dx^{2n}$).

2). Two- and Three-dimensional Quadrature for Rectangles [99]

The formulations for 2D and 3D problems are obtained by combining expression (Eqn. B.1) in the form as

$$\int_{-1}^1 \int_{-1}^1 f(x,y) dx dy \cong \sum_{j=1}^n \sum_{i=1}^n w_i w_j f(x_i, y_j) \quad (\text{B.2})$$

$$\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 f(x,y,z) dx dy dz \cong \sum_{k=1}^n \sum_{j=1}^n \sum_{i=1}^n w_i w_j w_k f(x_i, y_j, z_k) \quad (\text{B.3})$$

	$\pm x_i$	w_i
$n=1$	0.0	2.0
$n=2$	0.577350269189626	1.000000000000000
$n=4$	0.339981043584856 0.861136311594053	0.652145154862546 0.347854845137454
$n=6$	0.238619186083197 0.661209386466265 0.932469514203152	0.467913934572691 0.360761573048139 0.171324492379170
$n=8$	0.183434642495650 0.525532409916329 0.796666477413627 0.960289856497536	0.362683783378362 0.313706645877887 0.222381034453374 0.101228536290376
$n=9$	0.000000000000000 0.324253423403809 0.613371432700590 0.836031107326636 0.968160239507626	0.330239355001260 0.312347077040004 0.260610696402935 0.180648160694857 0.081274388361574
$n=10$	0.148874338981631 0.433395394129247 0.679409568299024 0.865063366688985 0.973906528517172	0.295524224714753 0.269266719309996 0.219086362515982 0.149451349150581 0.066671344308688
$n=12$	0.125233408511469 0.467831498998180 0.587317954286617 0.769902674194305 0.904117256370475 0.981560634246719	0.249147045813403 0.233492536538355 0.203167426723066 0.160078328543346 0.106939325995318 0.047175336386512

Table B.1 The Gaussian integration coefficient

B.2 Integration scheme based on the background mesh

It is assumed here that the support domain of 2D kernel function of RKPM is a rectangle area. In Figure B.1, the 2D problem domain for RKPM is Ω , $\mathbf{p}_i$ and $\mathbf{p}_j$ are two computing points of RKPM and their intersection area is as $\tilde{\Omega}$ and r_x and r_y are the length of interesting rectangle along x direction and y direction, respectively. A background mesh structure [31] which is independent of the RKPM computing points is shown as the broken line.

Figure B.1 Integration scheme based on the background mesh

In each cell of background mesh, the Gaussian quadrature rule $n_Q \times n_Q$ is used. The number of the quadrature points depends on the number of RKPM computing points in each cell, i.e., $n_Q = \sqrt{m} + 2$, m is the number of the RKPM computing points in a cell. Then the integration over the whole RKPM domain Ω can be computed only in the intersecting shaded region $\tilde{\Omega}$.

APPENDIX C

SINGULAR INTEGRAITON RULE

C.1 Singular integration along circle part

In this part, the numerical experiments for singular integration with integral kernel as $1/r$ and $1/r^2$ along the circle are conducted to test the efficiency of the Gaussian integration rules.

As shown in Figure C.1, **p** is the center of the circle and It is assumed that the circle is first divided into many arcs with angle as 2θ , and then the Gaussian integration is applied in the chord of each arc.

Figure C.1 Singular integration along circle part

C.1.1 Integration kernel as $1/r$

$$I = \int_c \frac{1}{r} d\Gamma = 2\pi = 6.28319 \quad (C.1)$$

The corresponding results of numerical integration experiment for different Gaussian integration rules are listed in Table C.1.

$I(1/r)$		1	2	4	6	8
θ	0.1°	6.28319	6.28319	6.28319	6.28319	6.28319
	0.2°	6.28321	6.2832	6.2832	6.2832	6.2832
	0.4°	6.28329	6.28324	6.28324	6.28324	6.28324
	0.6°	6.28341	6.2833	6.2833	6.2833	6.2833
	0.8°	6.28359	6.28339	6.28339	6.28339	6.28339
	1.0°	6.28382	6.2835	6.2835	6.2835	6.2835
	2.0°	6.28574	6.28446	6.28446	6.28446	6.28446
	4.0°	6.29341	6.28829	6.2883	6.2883	6.2883

Table C.1 Numerical experiments for singular integration with integral kernel as $1/r$ along circle

C.1.2 Integration kernel as $1/r^2$

$$I = \int_c \frac{1}{r^2} d\Gamma = 2\pi r = 6.28319 / r \quad (C.2)$$

The corresponding results of numerical integration experiment for different Gaussian integration rules are listed in Table C.2.

$I(1/r^2)$		1	2	4	6	8
θ	0.1°	6.2832/r	6.28319/r	6.28319/r	6.28319/r	6.28319/r
	0.2°	6.28325/r	6.28322/r	6.28322/r	6.28322/r	6.28322/r
	0.4°	6.28344/r	6.28334/r	6.28334/r	6.28334/r	6.28334/r
	0.6°	6.28376/r	6.28353/r	6.28353/r	6.28353/r	6.28353/r
	0.8°	6.28421/r	6.2838/r	6.2838/r	6.2838/r	6.2838/r
	1.0°	6.28478/r	6.28414/r	6.28414/r	6.28414/r	6.28414/r
	2.0°	6.28957/r	6.28701/r	6.28702/r	6.28702/r	6.28702/r
	4.0°	6.30878/r	6.29851/r	6.29853/r	6.29853/r	6.29853/r

Table C.2 Numerical experiments for singular integration with integral kernel as $1/r^2$ along circle

C.2 Singular integration along domain boundary part

In this part, the numerical experiments for singular integration with integral kernel as $1/r$ and $1/r^2$ along the boundary part are conducted to test the efficiency of the Gaussian integration rules.

Figure B.2 Singular integration along boundary part

As shown in Figure C.2, $\mathbf{p}$ is the center point of an arc. Q is element of the boundary part within the arc and the length of the element Q is L . The distance between the point $\mathbf{p}$ and the element Q is D . The singular integration results for different ratios of D/L and different Gaussian integration rules are listed as following:

C.2.1 Integration kernel as $1/r$

$$I = \int_{-L/2}^{L/2} \frac{1}{r} dx = \int_{-L/2}^{L/2} \frac{1}{\sqrt{x^2 + D^2}} dx = \ln \frac{\sqrt{1/4 + D^2/L^2} + 1/2}{\sqrt{1/4 + D^2/L^2} - 1/2} \quad (\text{C.3})$$

The corresponding results of numerical integration experiment for different ratios of D/L and different Gaussian integration rules are listed in Table C.3.

D/L			0.5	1	3.5	6
$I(1/r)$	Exact		1.762747	0.962424	0.284751	0.166474
	Numerical	2	1.732051	0.960769	0.284747	0.166474
		4	1.762054	0.962420	0.284751	0.166474
		6	1.762730	0.962424	0.284751	0.166474
		8	1.762747	0.962424	0.284751	0.166474

Table C.3 Numerical experiments for singular integration with integral kernel as $1/r$ along boundary

C.2.2 Integration kernel as $1/r^2$

$$I = \int_{-L/2}^{L/2} \frac{1}{r^2} dx = \int_{-L/2}^{L/2} \frac{1}{x^2 + D^2} dx = \frac{1}{D} \arctan\left(\frac{L/2}{D}\right) - \frac{1}{D} \arctan\left(-\frac{L/2}{D}\right) \quad (\text{C.4})$$

The corresponding results of numerical integration experiment for different ratios of D/L and different Gaussian integration rules are listed in Table C.4.

D/L			0.5	1	3.5	6
$I(1/r^2)$	Exact		3.14159	0.927295	0.081084	0.0277137
	Numerical	2	3	0.923077	0.0810811	0.0277136
		4	3.13725	0.927282	0.081084	0.0277137
		6	3.14146	0.927295	0.081084	0.0277137
		8	3.14159	0.927295	0.081084	0.0277137

Table C.4 Numerical experiments for singular integration with integral kernel as $1/r^2$ along boundary

APPENDIX D

TRACTION BOUNDARY CORRECTION BASED ON LEAST ERROR IN ENERGY NORM

D.1 Traction boundary condition

Figure D.1 Transformation between nodal traction and nodal stress

The boundary traction condition shown in Figure D.1 can be written as

$$\sigma_x \cos \theta + \tau_{xy} \sin \theta = \bar{t}_x \quad (\text{D.1a})$$

$$\sigma_y \sin \theta + \tau_{xy} \cos \theta = \bar{t}_y \quad (\text{D.1b})$$

where, θ is the angle between the outer normal line of the boundary and x axis.

The Eqns. D.1a and D.1b can also be written in matrix form as

$$\begin{Bmatrix} \bar{t}_x \\ \bar{t}_y \end{Bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad (\text{D.2})$$

If the exact stresses σ_x , σ_y and τ_{xy} in Eqn. D.1 is replaced by the approximated stresses σ_x^h , σ_y^h and τ_{xy}^h , the following formulae can then be arrived as

$$\sigma_x^h \cos \theta + \tau_{xy}^h \sin \theta = t_x^h \quad (\text{D.3a})$$

$$\sigma_y^h \sin \theta + \tau_{xy}^h \cos \theta = t_y^h \quad (\text{D.3b})$$

Subtracting Eqn. D.3 from Eqn. D.1, the error terms between the exact stresses and the approximated stresses can be obtained as

$$e_{\sigma_x} \cos \theta + e_{\tau_{xy}} \sin \theta = e_{t_x} \quad (\text{D.4a})$$

$$e_{\sigma_y} \sin \theta + e_{\tau_{xy}} \cos \theta = e_{t_y} \quad (\text{D.4b})$$

where,

$$e_{\sigma_x} = \sigma_x - \sigma_x^h, \quad e_{\sigma_y} = \sigma_y - \sigma_y^h, \quad e_{\tau_{xy}} = \tau_{xy} - \tau_{xy}^h \quad (\text{D.5a})$$

$$e_{t_x} = t_x - t_x^h, \quad e_{t_y} = t_y - t_y^h \quad (\text{D.5b})$$

D.2 Boundary correction based on least error energy norm

D.2.1 $\bar{t}_x$ and $\bar{t}_y$ are all known

It is obviously that the correction terms e_{σ_x} , e_{σ_y} and $e_{\tau_{xy}}$ can not be computed only from Eqn. D.4 and an additional condition based on the least error energy norm is introduced and the related details are shown as following:

Firstly, the error energy function $F(e_{\sigma_x}, e_{\sigma_y}, e_{\tau_{xy}})$ is introduced as

$$F(e_{\sigma_x}, e_{\sigma_y}, e_{\tau_{xy}}) = \mathbf{e}_{\sigma}^T \mathbf{D}^{-1} \mathbf{e}_{\sigma} = e_{\sigma_x}^2 + e_{\sigma_y}^2 - 2\mu e_{\sigma_x} e_{\sigma_y} + 2(1 + \mu) e_{\tau_{xy}}^2 \quad (\text{D.6})$$

where,

$$\mathbf{D} = \frac{E}{1 - \mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1 - \mu}{2} \end{bmatrix} \quad (\text{for plane stress problems}) \quad (\text{D.7a})$$

$$\mathbf{e}_\sigma = \{e_{\sigma_x}, e_{\sigma_y}, e_{\tau_{xy}}\}^T \quad (\text{D.7b})$$

In Eqn. D.7a, $\mathbf{D}$ is the strain-stress matrix, E is the Young's module and μ is the Poisson ratio.

Secondly, it can be assumed that when $F(e_{\sigma_x}, e_{\sigma_y}, e_{\tau_{xy}})$ approaches the least value, the corresponding error terms e_{σ_x} , e_{σ_y} and $e_{\tau_{xy}}$ are the needed solution. From Eqn.

D.4, e_{σ_x} and e_{σ_y} can be represented by $e_{\tau_{xy}}$ as

$$e_{\sigma_x} = \frac{(e_{t_x} - e_{\tau_{xy}} \sin \theta)}{\cos \theta} \quad (\text{D.8a})$$

$$e_{\sigma_y} = \frac{(e_{t_y} - e_{\tau_{xy}} \cos \theta)}{\sin \theta} \quad (\text{D.8b})$$

Substituting Eqn. D.8 into Eqn. D.6, the error energy function can be computed as

$$\begin{aligned} F(e_{\sigma_x}, e_{\sigma_y}, e_{\tau_{xy}}) &= \frac{(e_{t_x} - e_{\tau_{xy}} \sin \theta)^2}{\cos^2 \theta} + \frac{(e_{t_y} - e_{\tau_{xy}} \cos \theta)^2}{\sin^2 \theta} \\ &- 2\mu \frac{(e_{t_x} - e_{\tau_{xy}} \sin \theta)(e_{t_y} - e_{\tau_{xy}} \cos \theta)}{\sin \theta \cos \theta} + 2(1 + \mu) e_{\tau_{xy}}^2 \end{aligned} \quad (\text{D.9})$$

If $F(e_{\sigma_x}, e_{\sigma_y}, e_{\tau_{xy}})$ approaches to the least value, it must satisfy the following condition as

$$\begin{aligned} \frac{\partial F(e_{\sigma_x}, e_{\sigma_y}, e_{\tau_{xy}})}{\partial e_{\tau_{xy}}} &= \frac{2(e_{t_x} - e_{\tau_{xy}} \sin \theta)(-\sin \theta)}{\cos^2 \theta} + \frac{2(e_{t_y} - e_{\tau_{xy}} \cos \theta)(-\cos \theta)}{\sin^2 \theta} \\ &- 2\mu \frac{(e_{t_x} - e_{\tau_{xy}} \sin \theta) \cos \theta + (e_{t_y} - e_{\tau_{xy}} \cos \theta) \sin \theta}{\sin \theta \cos \theta} + 4(1 + \mu) e_{\tau_{xy}} = 0 \end{aligned} \quad (\text{D.10})$$

Finally, by combining Eqns. D.4a, D.4b and D.10, e_{σ_x} , e_{σ_y} and $e_{\tau_{xy}}$ can be solved as

$$e_{\tau_{xy}} = e_{t_x} (\sin^3 \theta - \mu \sin \theta \cos^2 \theta) + e_{t_y} (\cos^3 \theta - \mu \cos \theta \sin^2 \theta) \quad (\text{D.11a})$$

$$e_{\sigma_x} = e_{t_x} (\cos \theta + (1 + \mu) \sin^2 \theta \cos \theta) + e_{t_y} (\cos^2 \theta \sin \theta - \mu \sin^3 \theta) \quad (\text{D.11b})$$

$$e_{\sigma_y} = e_{t_y} (\sin \theta + (1 + \mu) \cos^2 \theta \sin \theta) + e_{t_x} (\sin^2 \theta \cos \theta - \mu \cos^3 \theta) \quad (\text{D.11c})$$

D.2.2 Only $\bar{t}_x$ is known

Similarly, when only $\sigma_x \cos \theta + \tau_{xy} \sin \theta = \bar{t}_x$ is known, the error energy function is represented as

$$F(e_{\sigma_x}, e_{\sigma_y}, e_{\tau_{xy}}) = \frac{(e_{t_x} - e_{\tau_{xy}} \sin \theta)^2}{\cos^2 \theta} + e_{\sigma_y}^2 - 2\mu e_{\sigma_y} \frac{(e_{t_y} - e_{\tau_{xy}} \cos \theta)}{\cos \theta} + 2(1 + \mu) e_{\tau_{xy}}^2 \quad (\text{D.12})$$

Then the corresponding least error energy norm conditions are computed as

$$\frac{\partial F(e_{\sigma_x}, e_{\sigma_y}, e_{\tau_{xy}})}{\partial e_{\tau_{xy}}} = \frac{2(e_{t_x} - e_{\tau_{xy}} \sin \theta)(-\sin \theta)}{\cos^2 \theta} + 2\mu e_{\sigma_y} \frac{\sin \theta}{\cos \theta} + 4(1 + \mu) e_{\tau_{xy}} = 0 \quad (\text{D.13a})$$

$$\frac{\partial F(e_{\sigma_x}, e_{\sigma_y}, e_{\tau_{xy}})}{\partial e_{\sigma_y}} = 2e_{\sigma_y} - 2\mu \frac{(e_{t_x} - e_{\tau_{xy}} \sin \theta)}{\cos \theta} = 0 \quad (\text{D.13b})$$

Finally, by combining Eqns. D.4a and D.13, e_{σ_x} , e_{σ_y} and $e_{\tau_{xy}}$ are solved as

$$e_{\tau_{xy}} = \frac{e_{t_x} (1 - \mu) \sin \theta}{1 + \cos^2 \theta - \mu \sin^2 \theta} \quad (\text{D.14a})$$

$$e_{\sigma_x} = \frac{2e_{t_x} \cos \theta}{1 + \cos^2 \theta - \mu \sin^2 \theta} \quad (\text{D.14b})$$

$$e_{\sigma_y} = \frac{2\mu e_{t_x} \cos \theta}{1 + \cos^2 \theta - \mu \sin^2 \theta} \quad (\text{D.14c})$$

D.2.3 Only $\bar{t}_y$ is known

Similarly, when only $\sigma_y \sin \theta + \tau_{xy} \cos \theta = \bar{t}_y$ is known, the error energy function is represented as

$$F(e_{\sigma_x}, e_{\sigma_y}, e_{\tau_{xy}}) = e_{\sigma_x}^2 + \frac{(e_{t_y} - e_{\tau_{xy}} \cos \theta)^2}{\sin^2 \theta} - 2\mu e_{\sigma_x} \frac{(e_{t_y} - e_{\tau_{xy}} \cos \theta)}{\sin \theta} + 2(1 + \mu) e_{\tau_{xy}}^2 \quad (\text{D.15})$$

Then the corresponding least error energy norm conditions are computed as

$$\frac{\partial F(e_{\sigma_x}, e_{\sigma_y}, e_{\tau_{xy}})}{\partial e_{\tau_{xy}}} = \frac{2(e_{t_y} - e_{\tau_{xy}} \cos \theta)(-\cos \theta)}{\sin^2 \theta} + 2\mu e_{\sigma_x} \frac{\cos \theta}{\sin \theta} + 4(1 + \mu) e_{\tau_{xy}} = 0 \quad (\text{D.16a})$$

$$\frac{\partial F(e_{\sigma_x}, e_{\sigma_y}, e_{\tau_{xy}})}{\partial e_{\sigma_x}} = 2e_{\sigma_x} - 2\mu \frac{(e_{t_y} - e_{\tau_{xy}} \cos \theta)}{\sin \theta} = 0 \quad (\text{D.16b})$$

Finally, by combining Eqns. D.4b and D.16, e_{σ_x} , e_{σ_y} and $e_{\tau_{xy}}$ are solved as

$$e_{\tau_{xy}} = \frac{e_{t_y} (1 - \mu) \cos \theta}{1 + \sin^2 \theta - \mu \cos^2 \theta} \quad (\text{D.17a})$$

$$e_{\sigma_x} = \frac{2\mu e_{t_y} \sin \theta}{1 + \sin^2 \theta - \mu \cos^2 \theta} \quad (\text{D.17b})$$

$$e_{\sigma_y} = \frac{2e_{t_y} \sin \theta}{1 + \sin^2 \theta - \mu \cos^2 \theta} \quad (\text{D.17c})$$

D.3 Corrected solution after traction boundary correction

When the error terms e_{σ_x} , e_{σ_y} and $e_{\tau_{xy}}$ are all available, the corrected stress terms can be computed as

$$\tilde{\sigma}_x = e_{\sigma_x} + \sigma_x^h \quad (\text{D.18a})$$

$$\tilde{\sigma}_y = e_{\sigma_y} + \sigma_y^h \quad (\text{D.18b})$$

$$\tilde{\tau}_{xy} = e_{\tau_{xy}} + \tau_{xy}^h \quad (\text{D.18c})$$