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CLOSED RANGE AND CYCLICITY OF COMPOSITION OPERATORS ON HILBERT SPACES OF ENTIRE FUNCTIONS

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ABSTRACT. We discover a criterion for having a closed range and completely characterize the cyclic behavior of composition operators on a class of Hilbert spaces of entire functions in the complex plane $\mathbb{C}$.

1. INTRODUCTION

Actions of composition operators on function spaces and their properties have been attractive topics in operator theory. These operators have been analyzed on various spaces of holomorphic and entire functions (see, e.g., [5, 10] and references therein). Some of the interesting properties of such operators are having a closed range and their cyclic behavior.

Let $\mathcal{H}$ be a Hilbert space of holomorphic functions in a domain $G \subseteq \mathbb{C}$, and φ be an analytic map from G into itself. The *composition operator* C_φ acting on $\mathcal{H}$ is a linear operator defined by the rule

$$C_\varphi f(z) = f \circ \varphi(z), \quad z \in G, \quad f \in \mathcal{H}.$$

Denote by $\mathcal{R}(C_\varphi)$ the range of C_φ . Suppose C_φ is bounded, we say that C_φ has a *closed range* if

$$\mathcal{R}(C_\varphi) = \overline{\mathcal{R}(C_\varphi)}.$$

The operator C_φ is said to be (1) *cyclic*, (2) *supercyclic*, or (3) *hypercyclic* if there is a vector $f \in \mathcal{H}$ whose *orbit*

$$\text{Orb}(C_\varphi, f) := \{C_\varphi^n f : n \in \mathbb{N}\}$$

satisfies respectively

- (1) $\overline{\text{span}(\text{Orb}(C_\varphi, f))} = \mathcal{H}$,
- (2) $\overline{\{\lambda y : y \in \text{Orb}(C_\varphi, f), \lambda \in \mathbb{C}\}} = \mathcal{H}$, or
- (3) $\overline{\text{Orb}(C_\varphi, f)} = \mathcal{H}$,

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and f is called a *cyclic* (*supercyclic*, or *hypercyclic* respectively) *vector* for C_φ . It is clear that (3) implies (2), which in turn implies (1). The converse implications are not true in general (see, e.g., [1]).

The study of cyclic behavior for composition operators is especially natural, because $C_\varphi^n = C_{\varphi_n}$, where

$$\varphi_0(z) = z \quad \text{and} \quad \varphi_n = \varphi \circ \varphi \circ \cdots \circ \varphi \quad (n \text{ times}).$$

The reader can find examples from [2, 3], in which Bourdon and Shapiro have done an intensive study of cyclic and hypercyclic linear fractional composition operators on the classical Hardy space $H^2(\mathbb{D})$, where $\mathbb{D}$ is the unit disc of the complex plane.

One of the most popular tools that helps us construct many types of Hilbert spaces of analytic functions is the sequence space ℓ_β^2 . Specifically, to each sequence of positive real numbers $\beta = (\beta_n)_{n=0}^\infty$, we associate the following *weighted sequence space*

$$\ell_\beta^2 = \left\{ \mathbf{a} = (a_n) : \|\mathbf{a}\|_{\ell_\beta^2} := \left(\sum_{n=0}^\infty |a_n|^2 \beta_n^2 \right)^{1/2} < \infty \right\}.$$

This is a Hilbert space with the following inner product,

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\ell_\beta^2} = \sum_{n=0}^\infty a_n \overline{b_n} \beta_n^2, \quad \text{for all } \mathbf{a} = (a_n), \mathbf{b} = (b_n) \in \ell_\beta^2.$$

The space ℓ_β^2 in connection with weighted shift operators on spaces of holomorphic functions was first studied in [11], and is important in the research of many (weighted) function spaces. Composition operators on Hilbert spaces induced by ℓ_β^2 with different weights have also been widely investigated (see, for example, [13]).

Overall, there are still many unsolved problems, and most of current works concentrate on spaces of holomorphic functions in the unit disc or unit ball. We refer the reader to the book [5], the paper [14], and the references therein for more detailed information.

In this paper, we mainly consider spaces of entire functions, i.e. when $G = \mathbb{C}$. There have been a few findings regarding cyclicity of some linear operators acting on specific classes of Hilbert spaces of entire functions.

Amongst the few known results, the most notable work [4] deals with hypercyclicity of translation operators on the class of spaces E_γ^2 . Let $\gamma = (\gamma_n)$ be a sequence of positive real numbers such that $(n\gamma_n/\gamma_{n-1})_{n=1}^\infty$ is monotonically decreasing, and

$$E_\gamma^2 = \left\{ f(z) = \sum_{n=0}^\infty a_n z^n \text{ entire: } \|f\| = \left(\sum_{n=0}^\infty |a_n|^2 \gamma_n^{-2} \right)^{1/2} < \infty \right\}.$$

Clearly, if we define $\beta_n = \gamma_n^{-1}$, then E_γ^2 contains power series whose coefficient sequences form a subset of the sequence space ℓ_β^2 induced by $\beta = (\beta_n)$.

Having observed the connection between E_γ^2 and ℓ_β^2 , in the papers [7, 8], we demonstrated the construction of various types of Hilbert spaces from ℓ_β^2 , where E_γ^2 is a particular case of all of them, and studied action of composition operators on them. One of those types of spaces, which we simply denote by $\mathcal{H}(\beta)$ in the sequel, is our main concern for this work. A remarkable property of such a space $\mathcal{H}(\beta)$ is that translation operators are bounded on it.

In the present paper, we study the closed range and cyclic behaviors of composition operators on any space of the aforementioned class $\mathcal{H}(\beta)$. As far as we know, these topics have never been treated before.

The structure of the paper is as follows. The theory of $\mathcal{H}(\beta)$ is briefed in Section 2. We establish a criterion for a composition operator C_φ to have a closed range in Section 3, where Theorem 6 is the main result. In Section 4, we investigate the cyclic properties of C_φ , and provide in Theorem 13 a *complete characterization* for C_φ to be cyclic, supercyclic or hypercyclic on $\mathcal{H}(\beta)$.

2. COMPOSITION OPERATORS ON HILBERT SPACES OF ENTIRE FUNCTIONS

Let $\beta = (\beta_n)$ be a sequence of positive numbers, and ℓ_β^2 be the corresponding weighted sequence space. We introduced in [7, 8] the following inner product space of (formal) power series,

$$(2.1) \quad \mathcal{H}(\beta) := \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n, (a_n) \in \ell_\beta^2 \right\},$$

with the inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta_n^2,$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are in $\mathcal{H}(\beta)$.

Define the following function $\mu_\beta : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$,

$$\mu_\beta(n) = \frac{n\beta_{n-1}}{\beta_n}.$$

By (P) we mean the following set of two conditions

$$(P) \quad (i) \liminf_{n \rightarrow \infty} \frac{\log \beta_n}{n \log n} < +\infty, \quad \text{and} \quad (ii) \sup_n \mu_\beta(n) < \infty.$$

By [8], the following statements about the space $\mathcal{H}(\beta)$ are true.

Remark 1. If the sequence $\beta = (\beta_n)$ satisfies (P) , then the following are true about the space $\mathcal{H}(\beta)$:

- 1) We have $\liminf_{n \rightarrow \infty} \beta_n^{1/n} = +\infty$, and any power series $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\beta)$ is an entire function. Moreover, $\mathcal{H}(\beta)$ is a Hilbert space, with the complete norm defined by the rule

$$\|f\| := \|(a_n)\|_{\ell_\beta^2} = \left(\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \right)^{1/2}.$$

- 2) Condition (P)(ii) implies

$$(2.2) \quad \liminf_{n \rightarrow \infty} \frac{\log \beta_n}{n \log n} \geq 1, .$$

Indeed, there exists $T > 0$ such that for all $n \in \mathbb{Z}^+$, we have $\beta_n^{-1} \beta_{n-1} < n^{-1} T$. Hence,

$$\frac{\beta_0}{\beta_n} = \frac{\beta_0}{\beta_1} \cdot \frac{\beta_1}{\beta_2} \cdots \frac{\beta_{n-1}}{\beta_n} < \frac{T^n}{n!}.$$

This implies $\beta_n > T^{-n} \beta_0 n!$, so

$$(2.3) \quad \frac{\log \beta_n}{n \log n} > \frac{\log(n!) + \log \beta_0}{n \log n} - \frac{\log T}{\log n}.$$

Now by Stirling's approximation (see, e.g., [9]), one has

$$(2.4) \quad \liminf_{n \rightarrow \infty} \frac{\log(n!)}{n \log n} = \liminf_{n \rightarrow \infty} \frac{n(\log n - 1) + \log(\sqrt{2\pi n})}{n \log n} = 1.$$

Combining equations (2.3) and (2.4) yields

$$\liminf_{n \rightarrow \infty} \frac{\log \beta_n}{n \log n} \geq \liminf_{n \rightarrow \infty} \frac{\log(n!)}{n \log n} \geq 1.$$

From this, it can be deduced that the entire power series f above has *finite* growth order ρ_f defined by

$$\rho_f := \limsup_{n \rightarrow \infty} \frac{n \log n}{\log \frac{1}{|a_n|}} < +\infty.$$

- 3) Condition (P)(i) guarantees that there exists a sequence $(a_n) \in \ell_\beta^2$ such that the induced power series f has *positive* growth order ρ_f .
 4) Given $u \in \mathbb{C}$, we denote by T_u the translation operator acting on $\mathcal{H}(\beta)$ defined by

$$T_u f(z) = f(z + u), \quad (f \in \mathcal{H}(\beta), z \in \mathbb{C}).$$

Then the operator T_u is bounded on $\mathcal{H}(\beta)$ for any $u \in \mathbb{C}$.

Note that the spaces E_γ^2 in [4] are just special cases of $\mathcal{H}(\beta)$ because of the following result.

Remark 2 ([8, Proposition 2.3 (iv)]). Suppose $\beta = (\beta_n)$ is a sequence of positive numbers such that μ_β is monotonically decreasing. Then β satisfies (P).

We also note the following result from [8, Thms 3.2, 3.5 and Cor 3.9] about composition operators acting on $\mathcal{H}(\beta)$ that is important for later sections.

Theorem 3. *Let $\beta = (\beta_n)$ satisfy condition (P) and φ be an entire function. The following statements are true for the induced composition operator C_φ acting on the Hilbert space $\mathcal{H}(\beta)$.*

- (1) C_φ is bounded if and only if $\varphi(z) = az + b$ ($a, b \in \mathbb{C}$) with $|a| \leq 1$;
- (2) C_φ is compact if and only if $\varphi(z) = az + b$ ($a, b \in \mathbb{C}$) with $|a| < 1$.

Finally, let $a, b \in \mathbb{C}$ and $a \neq 1$. For each $n \in \mathbb{N}$, we define the following polynomial of degree n , which clearly belongs to $\mathcal{H}(\beta)$,

$$\omega_n(z) = \left(z - \frac{b}{1-a} \right)^n.$$

In the sequel, we fix a particular β satisfying (P), and concern composition operators acting on the induced space $\mathcal{H}(\beta)$.

3. CLOSED RANGE OF C_φ

Denote by $\mathcal{N}(C_\varphi)$ the kernel of C_φ . We have the following important remark.

Remark 4. Suppose $\mathcal{N}(C_\varphi) = \{0\}$ (i.e. C_φ is injective), an easy application of open mapping theorem implies that the range $\mathcal{R}(C_\varphi)$ of C_φ is closed if and only if C_φ is bounded from below, i.e. there exists $M > 0$ such that

$$M\|f\|_{\mathcal{H}} \leq \|C_\varphi f\|_{\mathcal{H}}, \quad \text{for all } f \in \mathcal{H}.$$

We first prove the following lemma.

Lemma 5. *Let C_φ be a bounded composition operator on $\mathcal{H}(\beta)$, i.e. $\varphi(z) = az + b$ with $a, b \in \mathbb{C}$, $|a| \leq 1$. Then C_φ is injective if and only if $a \neq 0$.*

Proof. • **Necessity:** Assume that $a = 0$, that is $\varphi(z) = b$. For the nonzero function $f(z) = z - b \in \mathcal{H}(\beta)$, we have $C_\varphi f(z) = 0$, for all $z \in \mathbb{C}$. This shows that the kernel of C_φ is not trivial, and hence it is not injective: a contradiction.

• **Sufficiency:** Suppose $a \neq 0$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{N}(C_\varphi)$, that is,

$$f(az + b) = \sum_{n=0}^{\infty} a_n (az + b)^n = 0.$$

Since $T_{-b/a}$ is bounded on $\mathcal{H}(\beta)$ by point 4) of Remark 1, we have

$$0 = (T_{-b/a} C_\varphi) f(z) = \sum_{n=0}^{\infty} a_n a^n z^n, \quad \text{for all } z \in \mathbb{C}.$$

Since $a \neq 0$, the above equality happens if and only if $a_n = 0$ for all n , so $\mathcal{N}(C_\varphi) = \{0\}$. $\square$

With Lemma 5, we prove the following criterion for C_φ to have a closed range, which is the main result of this section.

Theorem 6. *Let C_φ be a bounded composition operator on $\mathcal{H}(\beta)$, i.e. $\varphi(z) = az + b$ with $a, b \in \mathbb{C}$, $|a| \leq 1$. Then the range of C_φ is closed if and only if exactly one of the following conditions holds:*

- (1) $|a| = 1$,
- (2) $a = 0$, i.e. φ is a constant function.

Furthermore, $\mathcal{R}(C_\varphi) = \mathcal{H}(\beta)$ in case (1), and $\mathcal{R}(C_\varphi) = \mathbb{C}$ in case (2).

Proof. By Theorem 3, we have $\varphi(z) = az + b$ with $|a| \leq 1$.

• **Necessity** Assume on the contrary that $0 < |a| < 1$. Note that

$$(3.1) \quad C_\varphi \omega_n(z) = \left(az + b - \frac{b}{1-a} \right)^n = a^n \left(z - \frac{b}{1-a} \right)^n = a^n \omega_n(z),$$

which implies that

$$\|C_\varphi \omega_n\| = |a|^n \|\omega_n\|.$$

Since $0 < |a| < 1$, C_φ is injective by Lemma 5. In addition, we have

$$\lim_{n \rightarrow \infty} |a|^n = \lim_{n \rightarrow \infty} \|\omega_n\|^{-1} \|C_\varphi \omega_n\| = 0.$$

Hence C_φ is not bounded from below. By Remark 4, the range of C_φ is not closed, which is a contradiction.

Thus $|a| = 1$ or $a = 0$.

• **Sufficiency:** We consider each of the two cases for a .

- *Case 1:* $a = 0$.

In this case, we have $\varphi(z) = b$, so $\mathcal{R}(C_\varphi) \subseteq \mathbb{C}$. Take any $z_0 \in \mathbb{C}$. Consider the constant power series $f(z) \equiv z_0 \in \mathcal{H}(\beta)$. Clearly, $C_\varphi f(z) = f(b) = z_0$ and so $z_0 \in \mathcal{R}(C_\varphi)$. This implies $\mathbb{C} \subseteq \mathcal{R}(C_\varphi) \subseteq \overline{\mathcal{R}(C_\varphi)} \subseteq \mathbb{C}$. Hence, $\mathcal{R}(C_\varphi) = \overline{\mathcal{R}(C_\varphi)} = \mathbb{C}$.

- *Case 2:* $|a| = 1$.

We claim in this case that C_φ is surjective, i.e. $\mathcal{R}(C_\varphi) = \mathcal{H}(\beta)$. The closedness of $\mathcal{R}(C_\varphi)$ is then shown because $\mathcal{H}(\beta)$ is complete.

For any $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\beta)$, consider

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{a^n} (z - b)^n.$$

We show that $g \in \mathcal{H}(\beta)$. Indeed, let $h(z) = \sum_{n=0}^{\infty} a_n a^{-n} z^n$, then $\|h\| = \|f\|$. The translation operator $T_{-b} : f(z) \mapsto f(z - b)$ is bounded on $\mathcal{H}(\beta)$ and hence

$$\|g\| = \|T_{-b} h\| \leq \|T_{-b}\| \|h\| = \|T_{-b}\| \|f\| < \infty.$$

Furthermore, it is easy to check that $C_\varphi g = f$, so C_φ is surjective. $\square$

4. CYCLIC BEHAVIOR OF C_φ

For a bounded composition operator C_φ on $\mathcal{H}(\beta)$ with $\varphi(z) = az + b$, $|a| \leq 1$, we note the following simple fact:

$$(4.1) \quad \varphi_n(z) = \begin{cases} z + nb, & a = 1; \\ a^n z + \frac{b(1-a^n)}{1-a}, & a \neq 1. \end{cases}$$

We list *all possibilities* for a and b .

- (I) $a = 0$;
- (II) $a = 1$ and $b = 0$, i.e. $C_\varphi = id$, the identity operator on $\mathcal{H}(\beta)$;
- (III) $a = 1$ and $b \neq 0$, i.e. C_φ is a (nontrivial) translation T_b ;
- (IV) $a \neq 1$ but $a^k = 1$ for some $k \in \mathbb{Z}^+$, i.e. a is a k th root of unity;
- (V) $a \neq 0$ and $a^k \neq 1$ for all $k \in \mathbb{Z}^+$.

Note these five conditions are pairwise exclusive.

The following result stated and proved in [6] helps describe the cyclic behavior of C_φ .

Lemma 7. *Suppose Λ is a bounded linear operator on a Hilbert space $\mathcal{H}$. If there exists a spanning set of eigenvectors for Λ corresponding to pairwise distinct eigenvalues, then Λ is cyclic and its set of cyclic vectors is dense in $\mathcal{H}$.*

We also need the following result from [4] for the hypercyclicity of C_φ .

Lemma 8. *Fix some $u \neq 0$. Suppose H is a Hilbert space of entire functions that satisfies the following:*

- (i) *The set of polynomials is dense in H .*
- (ii) *The topology of H is stronger than the topology of uniform convergence on compact subsets of $\mathbb{C}$.*
- (iii) *T_u is bounded on H .*
- (iv) *There exists $\gamma = (\gamma_n)$, for which μ_γ is monotonically decreasing, such that the Hilbert space $\mathcal{H}(\gamma)$ defined in the same way as (2.1) is a subspace of H .*

Then T_u is hypercyclic on H .

Note that with the sequence γ (if it exists) in (iv) of Lemma 8, $\mathcal{H}(\gamma)$ can be constructed according to (2.1) because of Remark 2.

Finally, we include the following lemma about supercyclicity of composition operator.

Lemma 9 ([1]). *Suppose Λ is a bounded supercyclic linear operator on a Banach space X , and the set $\{\Lambda^n, n \in \mathbb{N}\}$ is uniformly bounded. Then for each $x \in X$,*

$$\lim_{n \rightarrow \infty} \Lambda^n x = 0.$$

Now, we are able to prove the following sufficient conditions for the characterization of cyclic behavior of C_φ on $\mathcal{H}(\beta)$.

Proposition 10. *Let C_φ be a bounded composition operator on $\mathcal{H}(\beta)$, i.e. $\varphi(z) = az + b$ with $a, b \in \mathbb{C}$, $|a| \leq 1$. The following statements are true.*

- (1) *If φ satisfies one of the conditions (I), (II) or (IV), then C_φ is not cyclic.*
- (2) *If φ satisfies condition (V), then C_φ is cyclic.*
- (3) *If φ satisfies condition (III), then C_φ is hypercyclic.*

Proof. (1) Clearly, with conditions (I) or (II), C_φ is not cyclic, because for any vector $f \in \mathcal{H}(\beta)$, $\text{span}(\text{Orb}(C_\varphi, f))$ is of dimension at most 1, while $\mathcal{H}(\beta)$ is of infinite-dimension.

Suppose now (IV) occurs. Take any $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\beta)$. Since $a^k = 1$ for some $k \geq 1$, by the formula (4.1) of the n th iterate of φ , we have $C_{\varphi^k} f = f$. Hence, $\text{Orb}(C_\varphi, f) = \{f, C_\varphi f, \dots, C_{\varphi^{k-1}} f\}$, whose span is also finite-dimensional. This fact shows that C_φ is not cyclic.

(2) The functions ω_k 's belong to $\mathcal{H}(\beta)$. Moreover, each ω_k is an eigenvector corresponding to eigenvalue a^k of C_φ , as shown in (3.1). By assumption (V), a^k 's are pairwise distinct. In addition, ω_k 's form a spanning set for $\mathcal{H}(\beta)$, since any $f \in \mathcal{H}(\beta)$ can be uniquely written under the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} c_n \left(z - \frac{b}{1-a} \right)^n = \sum_{n=0}^{\infty} c_n \omega_n(z).$$

Therefore, by Lemma 7, C_φ is cyclic.

(3) The hypercyclicity of C_φ is due to Lemma 8. We need to carefully check all the conditions (i) – (iv) of the lemma for $H = \mathcal{H}(\beta)$.

- By 4) in Remark 1, $C_\varphi = T_b$ when (III) holds, and any translation operator is bounded on $\mathcal{H}(\beta)$, so (iii) holds.

- To see why (i) holds, for any $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\beta)$, consider the sequence of partial sums $(p_{f,N})_{N=0}^{\infty}$, where

$$p_{f,N}(z) = \sum_{n=0}^N a_n z^n \in \mathcal{H}(\beta),$$

we have $\lim_{N \rightarrow \infty} \|p_{f,N} - f\| = 0$, which proves the assertion.

- Let X be the set of all functions in $\mathcal{H}(\beta)$, with the topology of uniform convergence on compact sets, i.e. the topology generated by the seminorms $\|f\|_K = \sup_{z \in K} |f(z)|$ ($f \in X$), where K ranges over compact subsets of $\mathbb{C}$. Note that we still denote by $\|f\|$ the norm of f in $\mathcal{H}(\beta)$. We want to show that the identity map $\iota : \mathcal{H}(\beta) \rightarrow X$ is continuous, so that open sets in X are also $\mathcal{H}(\beta)$ -open.

Associate to each $z \in \mathbb{C}$ the function $M(z) = \left(\sum_{n=0}^{\infty} \beta_n^{-2} |z|^{2n} \right)^{1/2}$. By Remark 1 (1), we have $\beta_n^{1/n} \rightarrow \infty$ as $n \rightarrow \infty$, so $M(z)$ is convergent.

By the Cauchy–Schwarz inequality, for any $z \in \mathbb{C}$, we have

$$(4.2) \quad \begin{aligned} |f(z)| &= \left| \sum_{n=0}^{\infty} a_n z^n \right| \\ &\leq \left(\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} \frac{|z|^{2n}}{\beta_n^2} \right)^{1/2} = M_z \|f\|. \end{aligned}$$

For any compact subset K , there exists a number $r_K > 0$ such that K is contained in the ball $B(0, r_K)$ centered at the origin with radius r_K , thus (4.2) implies

$$\|f\|_K = \sup_{z \in K} |f(z)| \leq M_{r_K} \|f\|.$$

The inequality above shows that ι is continuous (cf. [12, Thm 1 pg 42]). Therefore (ii) holds.

• The most difficult and interesting part is to prove (iv). We have to show that there exists a weight $\gamma = (\gamma_n)$ with $\mu_\gamma(n) = n\gamma_{n-1}/\gamma_n$ monotonically decreasing, such that the space $\mathcal{H}(\gamma)$ is contained in $\mathcal{H}(\beta)$, where $\mathcal{H}(\gamma)$ is defined as

$$\mathcal{H}(\gamma) = \left\{ g(z) = \sum_{n=0}^{\infty} b_n z^n : \|g\|_{\mathcal{H}(\gamma)}^2 = \sum_{n=0}^{\infty} |b_n|^2 \gamma_n^2 < \infty \right\}.$$

We will construct a γ satisfying, in addition, that $\gamma_n \geq \beta_n$ for all n . In this case, we have $\mathcal{H}(\gamma) \subseteq \mathcal{H}(\beta)$, because for any $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $\mathcal{H}(\gamma)$,

$$\|g\|^2 = \|g\|_{\mathcal{H}(\beta)}^2 = \sum_{n=0}^{\infty} |b_n|^2 \beta_n^2 \leq \sum_{n=0}^{\infty} |b_n|^2 \gamma_n^2 = \|g\|_{\mathcal{H}(\gamma)}^2 < \infty,$$

which implies $g \in \mathcal{H}(\beta)$. Note that if such γ is built, γ automatically satisfies (P), by Remark 2.

We inductively construct γ as follow. By (2.2), there is some $L \in [1, +\infty)$ such that

$$\liminf_{n \rightarrow \infty} \frac{\log \beta_n}{n \log n} = L.$$

Hence, put $c := L/2$, there exists $N \in \mathbb{N}$ such that

$$\frac{\log \beta_n}{n \log n} \geq c, \quad \text{for all } n \geq N.$$

Without the loss of generality, we can let $N = 2$. Hence,

$$(4.3) \quad \beta_n \geq n^{cn}, \text{ for all } n \geq 2.$$

- Put $\gamma_0 = \beta_0$ and $\gamma_1 = \beta_1$.

- Since (n^{cn}) is monotonically increasing (as $c > 0$), we can choose $n_2 > n_1 = 1$, such that

$$n_2^{cn_2} > \max \left\{ \frac{2\gamma_1^2}{\gamma_0}, \beta_2 \right\},$$

and put $\gamma_2 = \beta_{n_2}$. By (4.3), $\gamma_2 \geq n_2^{cn_2}$. We then have

$$\gamma_2 > \beta_2 \quad \text{and} \quad \frac{\gamma_0}{\gamma_1} > \frac{2\gamma_1}{\gamma_2}.$$

- For $j \geq 3$, suppose we have found positive integers $n_{j-1} > n_{j-2} > \dots > n_2 > n_1 = 1$ and a set of positive real numbers $\{\gamma_0, \gamma_1, \dots, \gamma_{j-1}\}$ satisfying

$$\frac{\gamma_0}{\gamma_1} > \frac{2\gamma_1}{\gamma_2} > \dots > \frac{(j-1)\gamma_{j-2}}{\gamma_{j-1}} \quad \text{and} \quad \gamma_s \geq \beta_s, \quad s = 0, 1, \dots, j-1.$$

Similarly to the argument in the case $j = 2$, we take $n_j > n_{j-1}$, so that

$$n_j^{cn_j} > \max \left\{ \frac{j\gamma_{j-1}^2}{(j-1)\gamma_{j-2}}, \beta_j \right\}$$

and put $\gamma_j = \beta_{n_j}$. Note that in this case, we have

$$\gamma_j > \beta_j \quad \text{and} \quad \frac{(j-1)\gamma_{j-2}}{\gamma_{j-1}} > \frac{j\gamma_{j-1}}{\gamma_j}.$$

Thus we have constructed a sequence (γ_n) that satisfies both conditions $\gamma_n \geq \beta_n$ for all $n \in \mathbb{N}$ and $\mu_\gamma(n)$ is monotonically decreasing. Hence, (iv) is satisfied.

The proof is complete. $\square$

Now we study the supercyclicity of the composition operator C_φ . By condition (P)(ii), let $B = \sup_n \mu_\beta(n)$, then $0 < B < \infty$.

Let D denote the differential operator on $\mathcal{H}(\beta)$, i.e.

$$D : f \mapsto f', \quad (f \in \mathcal{H}(\beta)).$$

Note that D is a bounded operator (see, e.g., [8, Prop 2.5]). Then for any $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\beta)$, we have

$$\begin{aligned} \|Df\|^2 &= \left\| \sum_{n=1}^{\infty} n a_n z^{n-1} \right\|^2 = \sum_{n=1}^{\infty} n^2 |a_n|^2 \beta_{n-1}^2 \\ &= \sum_{n=1}^{\infty} \mu_\beta(n)^2 |a_n|^2 \beta_n^2 \leq B^2 \|f\|^2, \end{aligned}$$

which implies that $\|D\| \leq B$.

Lemma 11. *Suppose C_φ be a bounded composition operator on $\mathcal{H}(\beta)$, i.e. $\varphi(z) = az + b$ with $|a| \leq 1$. Then $\|C_\varphi\| \leq e^{|b|B}$.*

Proof. Denote by C_ψ the composition operator induced by $\psi(z) = az$. Clearly, C_ψ is bounded on $\mathcal{H}(\beta)$ and $\|C_\psi\| \leq 1$ since $|a| \leq 1$.

For any $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\beta)$ with $\|f\| = 1$ and $w \in \mathbb{C}$, we have the following identity (see, e.g. [8]),

$$f(z+w) = \sum_{n=0}^{\infty} \frac{w^n}{n!} f^{(n)}(z) = \sum_{n=0}^{\infty} \frac{w^n}{n!} D^n f(z).$$

Therefore,

$$\begin{aligned} \|C_\varphi f\| &= \|(C_\psi T_b)f\| \leq \|C_\psi\| \|T_b f\| \leq \|f(z+b)\| \\ &= \left\| \sum_{k=0}^{\infty} \frac{b^k}{k!} D^k f(z) \right\| \leq \sum_{k=0}^{\infty} \frac{|b|^k}{k!} \|D^k f\| \leq e^{|b|B} \|f\|. \quad \square \end{aligned}$$

Proposition 12. *Let $\varphi(z) = az + b$ with $|a| \leq 1$ and C_φ be the composition operator on $\mathcal{H}(\beta)$ induced by φ . If $a \neq 1$, then C_φ is not supercyclic.*

Proof. Assume C_φ is supercyclic. For any $n \in \mathbb{N}$, by Lemma 11,

$$\|C_\varphi^n\| = \|C_{\varphi^n}\| = \left\| C_{a^n z + \frac{b(1-a^n)}{1-a}} \right\| \leq e^{\left| \frac{b(1-a^n)}{1-a} \right| B} \leq e^{\frac{2B|b|}{|1-a|}},$$

so $\{C_\varphi^n, n \in \mathbb{N}\}$ is uniformly bounded. However, for $\omega_0(z) \equiv 1 \in \mathcal{H}(\beta)$,

$$C_\varphi^n \omega_0 = 1 \not\rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which contradicts Lemma 9. $\square$

Summarizing the results in Propositions 10 and 12, we obtain a *complete characterization* of the cyclicity of C_φ on $\mathcal{H}(\beta)$.

Theorem 13. *Let C_φ be the bounded composition operator on $\mathcal{H}(\beta)$, i.e. $\varphi(z) = az + b$, with $a, b \in \mathbb{C}$, $|a| \leq 1$. Then*

- (1) C_φ is hypercyclic if and only if C_φ is supercyclic, if and only if $a = 1$ and $b \neq 0$;
- (2) C_φ is cyclic but not supercyclic if and only if $a \neq 0$ and $a^k \neq 1$ for all $k \in \mathbb{Z}^+$.
- (3) C_φ is not cyclic if and only if one of the following conditions holds
 - $a = 0$, i.e. φ is a constant function;
 - $a = 1$ and $b = 0$, i.e. φ is the identity map;
 - $a \neq 1$ but $a^k = 1$ for some $k \in \mathbb{Z}^+$.

As a concluding remark, we have the following result about the cyclicity of compact composition operators, which can be deduced from Theorem 13(3) and then Theorem 13(2).

Corollary 14. *Let C_φ be a compact composition operator on $\mathcal{H}(\beta)$ induced by a non-constant entire function φ . Then C_φ is cyclic but not supercyclic and hence, not hypercyclic on $\mathcal{H}(\beta)$.*

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