Design of recursive digital filters using frequency-response masking technique

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Design of Recursive Digital Filters Using Frequency-Response Masking Technique

LIU QINGLAI

School of Electrical and Electronic Engineering

A thesis submitted to the Nanyang Technological University
in partial fulfillment of the requirement for the degree of
Doctor of Philosophy

2020

Statement of Originality

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Authorship Attribution Statement

This thesis contains materials from two papers published in the following peer-reviewed journals, and one paper presented in the following conference in which I am listed as the first author.

Chapter 3 contains materials published in Q. Liu, Y. C. Lim and Z. Lin, “Design of pipelined IIR filters using two-stage frequency-response masking technique,” *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 66, no. 5, pp. 873–877, May 2019.

The contributions of the co-authors are as follows:

- I came up with the idea, did the derivations and also the computer simulations.
- I wrote the drafts of the manuscript. The manuscript was revised by Prof Lim Yong Ching and Prof Lin Zhiping.

Chapter 4 contains materials published in Q. Liu, Y. C. Lim, Z. Lin and X. Lai, “Design of IIR frequency-response masking filters with near linear phase using constrained optimization,” in *Proc. IEEE. Int. Symp. Circuits and Syst.*, Baltimore, MD, USA, May 2017, pp. 1–4.

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- I came up with the idea, did the derivations and also the computer simulations.
- I wrote the drafts of the manuscript. The manuscript was revised by Prof Lim Yong Ching, Prof Lin Zhiping and Prof Lai Xiaoping.

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The contributions of the co-authors are as follows:

- I came up with the idea, did the derivations and also the computer simulations.
- I wrote the drafts of the manuscript. The manuscript was revised by Prof Lim Yong Ching and Prof Lin Zhiping.

5 Feb 2020

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Contents

Acknowledgements	i
Summary	vii
List of Figures	xiv
List of Tables	xvi
List of Abbreviations	xvii
List of Notations	xviii
1 Introduction	1
1.1 Motivation	3
1.2 Objectives	4
1.3 Major Contributions of the Thesis	5
1.4 Organization of the Thesis	6
2 Literature Review on Frequency-Response Masking Technique	8
2.1 Basic Principle	9
2.2 Multistage FRM Technique	16

CONTENTS

2.3	Optimization Methods	17
2.4	Modified Structures	21
2.5	FRM Technique for the Design of IIR Filters	23
2.5.1	Pipelined IIR Filters Based on the FRM Technique	24
2.5.2	FRM-Based IIR Filters With IIR Subfilters	26
2.6	Applications	27
3	Design of Pipelined IIR Filters Using Two-Stage Frequency-Response Masking Technique	29
3.1	Filter Structure	30
3.2	Joint Optimization of Subfilters	33
3.2.1	Problem Formulation	33
3.2.2	Solution Technique	37
3.3	Multiplier Saving	43
3.4	Example	45
3.5	Multi-Stage Masking Design	51
3.6	Conclusion	55
4	Design of Approximately Linear Phase Pipelined IIR Filters Using Frequency-Response Masking Technique	56
4.1	Filter Transfer Functions	57
4.2	Problem Formulation	60
4.2.1	Passband Phase Error and Magnitude Error Constraints	61
4.2.2	Magnitude Constraints in Stopband and Transition Band	64
4.2.3	Optimization Problem	64
4.3	Solution Technique	65

CONTENTS

4.4	Design Examples	70
4.4.1	Example One	70
4.4.2	Example Two	76
4.5	Conclusion	77
5	A Class of IIR Filters Synthesized Using Frequency-Response Masking Technique	80
5.1	Filter Structure	82
5.2	Ripple Analysis	84
5.3	Joint Optimization of IIR Subfilters	96
5.3.1	Problem Formulation	96
5.3.2	Solution Technique	98
5.3.3	Initial Solution Generation	101
5.3.4	Design Example	103
5.4	Relationship Between M and Total Number of Multipliers	105
5.5	Coefficient Sensitivity and Roundoff Noise Analysis	108
5.6	Comparison With FRM-Based Pipelined IIR Filters	124
5.7	Conclusion	126
6	Conclusion and Recommendations	127
6.1	Conclusion	127
6.2	Recommendations for Further Research	129
	Author's Publications	132
	Bibliography	134
	Appendix A Coefficient Values of the Filters Designed in Chapter 3	153

CONTENTS

Appendix B Coefficient Values of the Filters Designed in Chapter 4 158

Appendix C Coefficient Values of the Filters Designed in Chapter 5 166

Summary

The implementation of recursive infinite impulse response (IIR) digital filters suffers from two problems: 1) the feedback loop in the IIR filters limits the opportunities for pipelined implementation, and 2) the coefficient sensitivity and roundoff noise of the IIR filters are very high when their transition bands are very narrow.

This thesis presents new methods for designing high-speed pipelined IIR filters and IIR filters having low coefficient sensitivity and roundoff noise. The proposed methods are based on the frequency-response masking (FRM) technique. To design pipelined IIR filters using the FRM technique, the bandedge shaping filters are IIR filters whose feedback loop contains several delays and the masking filter are finite impulse response (FIR) filters. Both nonlinear phase and approximately linear phase FRM-based pipelined IIR filters are considered in the thesis. For the design of FRM-based pipelined IIR filters with nonlinear phase, we propose a novel two-stage FRM approach where the bandedge shaping filters in the second stage are IIR filters consisting of a parallel connection of two allpass filters. For a given magnitude response specification and a given number of pipeline stages in the feedback loop, our two-stage FRM approach requires fewer multipliers than the one-stage FRM approach if the number of pipeline stages is larger than four. In the FRM-based pipelined IIR filters with approximately linear phase, the bandedge shaping filters are a general IIR

filter and its delay complementary filter. To design this kind of filters, we present a constrained optimization method that independently controls the passband phase error and magnitude error.

A class of FRM-based IIR filters with IIR subfilters is also introduced in the thesis. In the proposed filter class, the bandedge shaping filters are a parallel connection of two allpass filters and the masking filters are approximately linear phase IIR filters composed of a parallel connection of a delay line and an allpass filter. By means of theoretical analysis and computer simulation, we demonstrate that the proposed class of FRM-based IIR filters has much better coefficient sensitivity and roundoff noise performances than conventional IIR filters.

List of Figures

2.1	The structure of a filter synthesized using the FRM technique.	11
2.2	Synthesis of a very sharp filter using subfilters $H_a(z^M)$, $H_{ma}(z)$ and $H_{mc}(z)$ where the transition band of the sharp filter is provided by one of the transition bands of $H_a(z^M)$. This is refereed to as Case A design.	11
2.3	Synthesis of a very sharp filter using subfilters $H_a(z^M)$, $H_{ma}(z)$ and $H_{mc}(z)$ where the transition band of the sharp filter is provided by one of the transition bands of $H_c(z^M)$. This is refereed to as Case B design.	12
2.4	The structure of a filter synthesized using the three-stage FRM technique.	17
2.5	The structure of the pipelined IIR filter proposed in [59]. The filter is synthesized using the FRM technique.	25
3.1	The structure of pipelined IIR filter synthesized using the two-stage FRM technique.	31
3.2	The synthesis of $H_a^{(1)}(z)$ and $H_c^{(1)}(z)$ using the FRM technique.	32
3.3	Magnitude responses of (a) bandedge shaping filters $H_a^{(2)}(z^9)$ (solid line) and $H_c^{(2)}(z^9)$ (dashed line) and (b) masking filters $F_{ma}^{(2)}(z^3)$ (solid line) and $F_{mc}^{(2)}(z^3)$ (dashed line).	48

LIST OF FIGURES

3.4	Magnitude responses of (a) bandedge shaping filters $H_a^{(1)}(z^3)$ (solid line) and $H_c^{(1)}(z^3)$ (dashed line) and (b) masking filters $F_{ma}^{(1)}(z)$ (solid line) and $F_{mc}^{(1)}(z)$ (dashed line).	48
3.5	Magnitude responses of (a) overall filter $H(z)$ and (b) $H(z)$ in the passband for the two-stage design with $M_1 = 3$ and $M_2 = 3$	49
3.6	Magnitude responses of (a) bandedge shaping filters $H_a^{(2)}(z^{15})$ (solid line) and $H_c^{(2)}(z^{15})$ (dashed line) and (b) masking filters $F_{ma}^{(2)}(z^5)$ (solid line) and $F_{mc}^{(2)}(z^5)$ (dashed line).	49
3.7	Magnitude responses of (a) bandedge shaping filters $H_a^{(1)}(z^5)$ (solid line) and $H_c^{(1)}(z^5)$ (dashed line) and (b) masking filters $F_{ma}^{(1)}(z)$ (solid line) and $F_{mc}^{(1)}(z)$ (dashed line).	50
3.8	Magnitude responses of (a) overall filter $H(z)$ and (b) $H(z)$ in the passband for the two-stage design with $M_1 = 5$ and $M_2 = 3$	50
3.9	The structure of pipelined IIR filter synthesized using the three-stage FRM technique.	51
3.10	Magnitude responses of (a) bandedge shaping filters $H_a^{(3)}(z^{45})$ (solid line) and $H_c^{(3)}(z^{45})$ (dashed line) and (b) masking filters $F_{ma}^{(3)}(z^{15})$ (solid line) and $F_{mc}^{(3)}(z^{15})$ (dashed line).	53
3.11	Magnitude responses of (a) bandedge shaping filters $H_a^{(2)}(z^{15})$ (solid line) and $H_c^{(2)}(z^{15})$ (dashed line) and (b) masking filters $F_{ma}^{(2)}(z^5)$ (solid line) and $F_{mc}^{(2)}(z^5)$ (dashed line).	53
3.12	Magnitude responses of (a) bandedge shaping filters $H_a^{(1)}(z^5)$ (solid line) and $H_c^{(1)}(z^5)$ (dashed line) and (b) masking filters $F_{ma}^{(1)}(z)$ (solid line) and $F_{mc}^{(1)}(z)$ (dashed line).	54

LIST OF FIGURES

3.13	Magnitude responses of (a) overall filter $H(z)$ and (b) $H(z)$ in the passband for the three-stage design with $M_1 = 5$, $M_2 = 3$ and $M_3 = 3$.	54
4.1	Structure of the FRM-based pipelined IIR filter in [61]. $H_a(z)$ is an IIR filter. $H_{ma}(z)$ and $H_{mc}(z)$ are linear phase FIR filters. M and d are positive integers.	57
4.2	The complex plane where each point represents a possible value of $\tilde{H}(\omega_i, \mathbf{x})$	62
4.3	(a) Magnitude response, (b) passband ripple, (c) passband phase error and (d) passband group delay of the designed FRM filter in Design I of Example One.	73
4.4	(a) Magnitude response, (b) passband ripple, (c) passband phase error and (d) passband group delay of the designed FRM filter in Design II of Example One.	74
4.5	(a) Magnitude response, (b) passband ripple, (c) passband phase error and (d) passband group delay of the designed FRM filter in Design III of Example One.	75
4.6	(a) Magnitude response, (b) passband ripple, (c) passband phase error and (d) passband group delay of the designed FRM filter in Design I of Example Two.	78
4.7	(a) Magnitude response, (b) passband ripple, (c) passband phase error and (d) passband group delay of the designed FRM filter in Design II of Example Two.	79

LIST OF FIGURES

5.1	The distance from the outermost poles to the unit circle (ρ_{min}) versus transition width plot for an elliptic IIR lowpass filter with passband edge fixed at 0.4π , maximum passband peak-to-peak ripple 0.1 dB and minimum stopband attenuation 60 dB. The discontinuity of the plot is caused by a change in filter order.	81
5.2	The structure of the proposed class of FRM-based IIR filters. $K_m = \max(K_{ma}, K_{mc})$. (a) The overall filter. (b) The masking filter $H_{ma}(z)$. (c) The masking filter $H_{mc}(z)$	83
5.3	Magnitude responses of the subfilters in Case A.	85
5.4	Magnitude responses of the subfilters in Case B.	86
5.5	Magnitude response illustrations for (a) $H_x(z)$ where $x = a, ma, mc$ and (b) $H(z)$	87
5.6	Frequency ranges I and II for the analysis of $\delta(\omega)$ in (a) Case A and (b) Case B.	91
5.7	Illustrations of frequency regions Λ and $[0, \omega_{pmc}] \setminus \Lambda$ in Case A.	92
5.8	Magnitude response of the subfilters in the design example of Section 5.3.4. (a) Bandedge shaping filters $H_a(z^M)$ (solid line) and $H_c(z^M)$ (dashed line). (b) Masking filters $H_{ma}(z)$ (solid line) and $H_{mc}(z)$ (dashed line).	104
5.9	Magnitude response of the FRM-based IIR filter in the design example of Section 5.3.4. (a) Overall response. (b) Passband response.	104
5.10	(a) N_{mi} versus $1/\Delta_{mi}$ for $A_{mi}(z)$ with $\omega_{pmi} = 0.25\pi$. (b) N_{mi} versus ω_{pmi} for $A_{mi}(z)$ with $\Delta_{mi} = 0.2\pi$. For both plots, the peak phase response ripple magnitude of $A_{mi}(z)$ is 0.01 in both $[0, \omega_{pmi}]$ and $[\omega_{smi}, \pi]$	106
5.11	N_{Ha} , $N_{ma} + N_{mc}$ and N versus M	107

LIST OF FIGURES

5.12	Roundoff noise evaluation model for the proposed class of FRM-based IIR filters. z^{-*} represents $z^{-(K_m-K_{mi})}$	112
5.13	Implementation structures for (a) first-order allpass section and (b) second-order allpass section. λ_i is the scaling multiplier. $s_i(n)$ and $m_{il}(n)$ represent the input signals to the allpass sections and the multipliers, respectively.	114
5.14	(a) Passband and (b) stopband magnitude responses of the proposed FRM-based IIR filter with coefficient values quantized to 12 fractional bits (dashed line). The ideal magnitude response is shown in solid line.	120
5.15	(a) Passband and (b) stopband magnitude responses of the Butterworth filter with coefficient values quantized to 12 fractional bits (dashed line). The ideal magnitude response is shown in solid line.	121
5.16	(a) Passband and (b) stopband magnitude responses of the Chebyshev filter with coefficient values quantized to 12 fractional bits (dashed line). The ideal magnitude response is shown in solid line.	121
5.17	(a) Passband and (b) stopband magnitude responses of the inverse Chebyshev filter with coefficient values quantized to 12 fractional bits (dashed line). The ideal magnitude response is shown in solid line.	122
5.18	(a) Passband and (b) stopband magnitude responses of the elliptic filter with coefficient values quantized to 12 fractional bits (dashed line). The ideal magnitude response is shown in solid line.	122
5.19	(a) Passband and (b) stopband magnitude responses of the proposed FRM-based IIR filter with coefficient values quantized to 8 fractional bits (dashed line). The ideal magnitude response is shown in solid line.	123

LIST OF FIGURES

5.20	(a) Passband and (b) stopband magnitude responses of the proposed FRM-based IIR filter with quantized coefficient values (dashed line). The coefficient values of $A_1(z)$ and $A_2(z)$ are quantized to 11 fractional bits and the coefficient values of $A_{ma}(z)$ and $A_{mc}(z)$ are quantized to 10 and 9 fractional bits, respectively. The ideal magnitude response is shown in solid line.	123
5.21	Transposed structures exploiting the coefficient symmetry for implementing (a) odd-length and (b) even-length linear phase FIR filters. L is the filter length. $e_i(n)$ is the roundoff noise.	125
5.22	(a) Passband and (b) stopband magnitude responses of the FRM-based pipelined IIR filter with coefficient values quantized to 8 fractional bits (dashed line). The ideal magnitude response is shown in solid line. . .	126
6.1	The structure of a class of FRM-based IIR filters with approximately linear phase. $K_m = \max(K_{ma}, K_{mc})$	130
6.2	The frequency region $P = P_1 \cup P_2 \cup \dots \cup P_T$ where the phase responses of the masking filters need to be matched. T is the number of transition bands of the bandedge shaping filters within the common passband of the masking filters.	131

List of Tables

3.1	Multiplier Savings in Percentage for Different Number of Pipeline Stages in the Feedback Loop. For the Two-Stage (TS) and One-Stage (OS) Designs, the Filter Orders and Lengths Are Organized in the Form of $(N_1, N_2, L_{2ma}, L_{2mc}, L_{1ma}, L_{1mc})$ and $(N_1, N_2, L_{ma}, L_{mc})$, Respectively.	46
4.1	Performance Comparison of Designs I-III in Example One and the Filter in [59]	72
4.2	Performance Comparison of Designs I and II in Example Two and the Filter in [61]	77
5.1	Coefficient Sensitivity and Roundoff Noise Gain Comparison for the FRM-Based IIR Filter and the Classical IIR Filters	117
5.2	Hardware Cost of the FRM-Based IIR Filter and the Classical IIR Filters	119
A.1	Coefficient Values of $A_1(z)$, $A_2(z)$, $F_{ma}^{(2)}(z)$, $F_{mc}^{(2)}(z)$, $F_{ma}^{(1)}(z)$ and $F_{mc}^{(1)}(z)$ for the Two-Stage Design With $M_1 = 3$ and $M_2 = 3$	154
A.2	Coefficient Values of $A_1(z)$, $A_2(z)$, $F_{ma}^{(2)}(z)$, $F_{mc}^{(2)}(z)$, $F_{ma}^{(1)}(z)$ and $F_{mc}^{(1)}(z)$ for the Two-Stage Design With $M_1 = 5$ and $M_2 = 3$	155

LIST OF TABLES

A.3	Coefficient Values of $A_1(z)$, $A_2(z)$, $F_{ma}^{(3)}(z)$, $F_{mc}^{(3)}(z)$, $F_{ma}^{(2)}(z)$, $F_{mc}^{(2)}(z)$ for the Three-Stage Design With $M_1 = 5$, $M_2 = 3$ and $M_3 = 3$	156
A.4	Coefficient Values of $F_{ma}^{(1)}(z)$ and $F_{mc}^{(1)}(z)$ for the Three-Stage Design With $M_1 = 5$, $M_2 = 3$ and $M_3 = 3$	157
B.1	Coefficient Values of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ in Design I of Example One	159
B.2	Coefficient Values of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ in Design II of Exam- ple One	160
B.3	Coefficient Values of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ in Design III of Ex- ample One	161
B.4	Coefficient Values of $H_a(z)$ in Design I of Example Two	162
B.5	Coefficient Values of $H_{ma}(z)$ and $H_{mc}(z)$ in Design I of Example Two	163
B.6	Coefficient Values of $H_a(z)$ in Design II of Example Two	164
B.7	Coefficient Values of $H_{ma}(z)$ and $H_{mc}(z)$ in Design II of Example Two	165
C.1	Coefficient Values of the FRM-Based IIR Filter in the Design Example of Section 5.3.4	167
C.2	Coefficient Values of the FRM-Based IIR Filter in the Numerical Ex- ample of Section 5.5	168

List of Abbreviations

CCP	Convex-concave procedure
DFP	Davidon-Fletcher-Powell
DSP	Digital signal processing
ECG	Electrocardiogram
EEG	Electroencephalogram
FIR	Finite impulse response
FPGA	Field programmable gate array
FRM	Frequency-response masking
IFIR	Interpolated finite impulse response
IIR	Infinite impulse response
SCP	Sequential convex programming
SDP	Semidefinite programming
SOCP	Second-order cone programming
WLS	Weighted least square

List of Notations

$A \cap B$	Intersection of two sets, $\{x : x \in A \text{ and } x \in B\}$
$A \cup B$	Union of two sets, $\{x : x \in A \text{ or } x \in B\}$
$A \setminus B$	Difference between two sets, $\{x : x \in A \text{ and } x \notin B\}$
$\lfloor x \rfloor$	The largest integer less than or equal to x
$\lceil x \rceil$	The smallest integer greater than or equal to x
$\mathbf{A}^T$	Transpose of a matrix $\mathbf{A}$
∇	Gradient operator
$\frac{\partial f}{\partial x}$	Partial derivative of a function $f(x, y, \dots)$ with respect to the variable x
$E\{x\}$	Expected value of x
$\angle x$	Phase of a complex number x
$\text{Re}\{x\}$	Real part of a complex number x
$\text{Im}\{x\}$	Imaginary part of a complex number x
$\bar{x}$	Conjugate of a complex number x
$ \cdot $	Modulus operator
$\ \cdot\ _p$	L_p norm

Chapter 1

Introduction

Digital filters are key components in digital signal processing (DSP) systems. The applications of digital filters can be found in many areas, such as audio systems, biological signal processing, communication systems, image processing and speech synthesis. Digital filters can be categorized into finite impulse response (FIR) digital filters and infinite impulse response (IIR) digital filters. FIR and IIR filters are often referred to as nonrecursive and recursive filters, respectively. The z -transform transfer function $H(z)$ of a digital filter can be written as

$$H(z) = \frac{\sum_{k=0}^N b_k z^{-k}}{1 + \sum_{k=1}^M a_k z^{-k}} \quad (1.1)$$

where b_k and a_k are the coefficients of the digital filter. For an FIR filter, $a_k = 0$. Although $H(z)$ represents a z -transform transfer function, in the literature on digital filters, the phrase “a filter $H(z)$ ”, or sometimes simply “ $H(z)$ ”, is frequently referred to a filter whose z -transform transfer function is $H(z)$.

In digital filter design, if only the magnitude response specifications are of interest, IIR filters are more attractive than FIR filters because the orders of the IIR filters are much lower than the FIR filters for the same specifications. For applications requiring linear phase responses, both FIR and IIR filters can be used. For FIR filters, exact linear phase responses can be achieved if they possess symmetric or antisymmetric impulse responses [1]. Exact linear phase is not achievable for causal and stable IIR filters [2]. IIR filters can be designed to possess approximately linear phase using sophisticated optimization methods [3–12]. The stability of the IIR filters is an important issue to be considered in the design of IIR filters.

The computational complexity of a digital filter is primarily determined by the number of multipliers required in the implementation and the number of bits used for the multiplier coefficients. The length of an FIR filter is inversely proportional to the width of its transition band for the same passband and stopband ripple specifications [13]. Thus, FIR filters with very narrow transition bands (sharp FIR filters) are very complex. In 1986, Y. C. Lim [17] proposed a technique that significantly reduces the complexity of sharp FIR filters. This technique is called the frequency-response masking (FRM) technique. In the FRM technique, the filter transfer function $H(z)$ is expressed as a function of several subfunctions $H_a(z^M)$, $H_c(z^M)$, $H_{ma}(z)$ and $H_{mc}(z)$ in the form

$$H(z) = H_a(z^M)H_{ma}(z) + H_c(z^M)H_{mc}(z) \quad (1.2)$$

where M is a positive integer. The subfilters $H_a(z^M)$ and $H_c(z^M)$ are called bandedge shaping filters. $H_a(z^M)$ and $H_c(z^M)$ are complementary in some sense. The subfilters $H_{ma}(z)$ and $H_{mc}(z)$ are called masking filters. The subfunction $H_a(z^M)$ ($H_c(z^M)$) is obtained from $H_a(z)$ ($H_c(z)$) by replacing z by z^M . Filters $H_a(z)$ and $H_c(z)$

1.1. MOTIVATION

can be referred to as prototype bandedge shaping filters. Replacing z by z^M in the transfer function corresponds to replacing each delay element of the filter by M delays elements; there is no increase in the number of multipliers.

When the FRM technique is used to design an FIR filter, each subfunction in Eq. (1.2) is the transfer function of an FIR filter. $H_c(z^M)$ can be written as $z^{-P} - H_a(z^M)$ where P is the signal delay in $H_a(z^M)$. Filters $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ have very low complexity. Thus, the complexity of a sharp FIR filter designed using the FRM technique is extremely low compared with that designed without using the FRM technique.

1.1 Motivation

For a given set of design specifications, IIR filters generally require much fewer multipliers than FIR filters. However, the implementation of IIR filters faces two challenges:

1. Unlike FIR filters, the high-speed pipelined implementation of IIR filters are not straightforward as a result of the feedback loop.
2. IIR filters with very narrow transition bands have very high coefficient sensitivity and roundoff noise since their outermost poles are extremely close to the unit circle.

The FRM technique is very well suited to tackle these challenges. The reasons are as follows.

1. In the FRM technique, the transfer functions of the bandedge shaping filters are functions of z^M . If an IIR filter is designed using the FRM technique with IIR bandedge shaping filters and FIR masking filters, the feedback loop of the filter will have M delays. Thus, it is suitable for pipelined implementation.

1.2. OBJECTIVES

2. In the FRM technique, the prototype bandedge shaping filters $H_a(z)$ and $H_c(z)$, and the masking filters $H_{ma}(z)$ and $H_{mc}(z)$ have wider transition widths than that of the filter designed without using the FRM technique. If a sharp IIR filter is designed using the FRM technique with IIR subfilters, the poles of $H_a(z)$, $H_c(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ will move away from the unit circle, thus they have better coefficient sensitivity and roundoff noise performances. It can be expected that the IIR filter designed using the FRM technique with IIR subfilters has lower coefficient sensitivity and roundoff noise compared with the IIR filter designed without using the FRM technique.

The above reasons motivate us to investigate the design of IIR filters using the FRM technique. Although the design of FIR filters using the FRM technique has been intensively studied during the past decades, little effort has been devoted to the design of IIR filters using the FRM technique.

1.2 Objectives

The general objective of this thesis is to investigate the design of IIR filters using the FRM technique. The IIR filters designed using the FRM technique are referred to as FRM-based IIR filters or IIR FRM filters in the thesis. Limitations of previous work on designing FRM-based IIR filters lead to three specific objectives, as will be explained in the following.

- The design of nonlinear phase pipelined IIR filters using the FRM technique was first proposed by Johansson and Wanhammar [59]. A drawback of their approach is that the total number of multipliers becomes very large when the number of pipeline stages is large.

Objective 1: Develop a new approach to designing nonlinear phase FRM-based pipelined IIR filters that require fewer multipliers.

- An optimization method for designing FRM-based pipelined IIR filters with approximately linear phase was proposed by Lu and Hinamoto [61]. A limitation of their method is that it cannot control the passband magnitude error and phase error independently.

Objective 2: Develop an optimization method that independently controls the passband magnitude error and phase error of the approximately linear phase FRM-based pipelined IIR filters.

- The design of nonlinear phase FRM-based IIR filters with IIR subfilters was first considered by Shen and Lim [63]. A limitation of their method is that the design process is often trapped in some undesirable local optima. In their article, the coefficient sensitivity and roundoff noise properties of the FRM-based IIR filters with IIR subfilters were not studied.

Objective 3: Propose a new class of FRM-based IIR filters with IIR subfilters that is easy to optimize and investigate its coefficient sensitivity and roundoff noise properties.

1.3 Major Contributions of the Thesis

The major contributions of this thesis are summarized as follows.

1. A novel two-stage FRM approach for the design of nonlinear phase pipelined IIR filters is proposed. The subfilters in the proposed filter structure are optimized jointly. It is shown that the proposed two-stage masking design requires fewer

1.4. ORGANIZATION OF THE THESIS

multipliers than the one-stage masking design when the number of pipeline stages in the feedback loop is larger than four. The multi-stage FRM approach is also discussed.

2. A constrained optimization method for the design of FRM-based pipelined IIR filters with approximately linear phase is developed. In the proposed method, the passband phase error and magnitude error can be independently controlled. FRM-based pipelined IIR filters with very small phase error can be designed by the proposed method.
3. A new class of FRM-based IIR filters with IIR subfilters is presented. It is easy to generate good initial solutions for the proposed class of FRM-based IIR filters so that very good designs can be obtained by optimizing the IIR subfilters jointly. Theoretical analysis and numerical simulations demonstrate that the proposed class of FRM-based IIR filters has lower coefficient sensitivity and roundoff noise compared with conventional IIR filters.

1.4 Organization of the Thesis

The rest of this thesis is organized as follows.

Chapter 2 presents a comprehensive literature review on the FRM technique. Basic principle of the FRM technique, the multistage FRM technique, optimization methods for the design of FRM-based filters, modifications of FRM technique, FRM technique for the design of IIR filters and the applications of the FRM technique are reviewed.

Chapters 3-5 are organized around the three research contributions of the thesis.

Chapter 3 proposes a novel two-stage FRM approach for the design of pipelined

1.4. ORGANIZATION OF THE THESIS

IIR filters with nonlinear phase. An optimization method that optimizes the subfilters jointly is presented. The number of multipliers required by the proposed two-stage FRM approach is compared with that required by the one-stage FRM approach through theoretical analysis and simulations.

Chapter 4 presents a constrained optimization method for the design of FRM-based pipelined IIR filters with approximately linear phase. The proposed optimization method independently controls the phase error and magnitude error of the passband. The effectiveness of the proposed method is demonstrated by several design examples.

Chapter 5 introduces a new class of FRM-based IIR filters where all the subfilters are IIR filters. An optimization method that optimizes the IIR subfilters jointly is presented. The coefficient sensitivity and roundoff noise performances of the proposed class of FRM-based IIR filters are analyzed and then compared with classical IIR filters by the means of a numerical example.

Chapter 6 concludes the thesis and recommends several directions for further research.

Chapter 2

Literature Review on Frequency-Response Masking Technique

The FRM technique was first proposed by Lim [17] in 1986 for the design of very sharp linear phase FIR filters. The FRM technique significantly reduces the arithmetic complexity of sharp FIR filters compared to the minimax optimum filter meeting the same specifications. Digital filters designed by using the FRM technique are generally known as FRM filters. This chapter provides a literature review on the FRM technique. In Section 2.1, the basic principle of the FRM technique is presented. Extensions of the FRM technique and hardware implementation of FRM filters are also briefly introduced. Section 2.2 reviews the multistage FRM technique. Modifications of FRM technique to further reduce the arithmetic complexity of sharp FIR filters are discussed in 2.4. In Section 2.3, various optimization techniques for the design of FRM filters are presented. Section 2.5 reviews the design of IIR filters using the

2.1. BASIC PRINCIPLE

FRM technique. Section 2.6 gives a brief introduction on the applications of the FRM technique.

2.1 Basic Principle

For a linear phase FIR filter designed using the FRM technique [17], it is synthesized using a system of FIR subfilters $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ in the form

$$H(z) = H_a(z^M)H_{ma}(z) + \left[z^{-M(N_a-1)/2} - H_a(z^M) \right] H_{mc}(z) \quad (2.1)$$

where M is a positive integer and N_a is the length of $H_a(z)$. The value $(N_a - 1)/2$ is the group delay of $H_a(z)$. Filter $H_a(z)$ is referred to as the prototype bandedge shaping filter. Filter $H_c(z) = z^{-(N_a-1)/2} - H_a(z)$ is the complementary filter of $H_a(z)$. The bandedge shaping filter $H_a(z^M)$ is obtained by replacing each delay element of $H_a(z)$ by M delay elements. Filters $H_{ma}(z)$ and $H_{mc}(z)$ are referred to as masking filters. The transfer functions $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ are given by

$$H_a(z) = \sum_{k=0}^{N_a-1} h_a(k)z^{-k}, \quad h_a(N_a - 1 - k) = h_a(k) \quad (2.2a)$$

$$H_{ma}(z) = \sum_{k=0}^{N_{ma}-1} h_{ma}(k)z^{-k}, \quad h_{ma}(N_{ma} - 1 - k) = h_{ma}(k) \quad (2.2b)$$

$$H_{mc}(z) = \sum_{k=0}^{N_{mc}-1} h_{mc}(k)z^{-k}, \quad h_{mc}(N_{mc} - 1 - k) = h_{mc}(k) \quad (2.2c)$$

where N_{ma} and N_{mc} are the lengths of $H_{ma}(z)$ and $H_{mc}(z)$, respectively. N_a is odd. N_{ma} and N_{mc} are either both odd or both even. When $N_{ma} \neq N_{mc}$, i.e., the group delays of $H_{ma}(z)$ and $H_{mc}(z)$ are different, delay elements must be added to either $H_{ma}(z)$ or $H_{mc}(z)$ to equalize their group delays. For $N_{ma} > N_{mc}$, $(N_{ma} - N_{mc})/2$

2.1. BASIC PRINCIPLE

delays are added to $H_{mc}(z)$. For $N_{ma} < N_{mc}$, $(N_{mc} - N_{ma})/2$ delays are added to $H_{ma}(z)$.

The frequency response of the overall filter $H(z)$ can be expressed as

$$H(e^{j\omega}) = H(\omega)e^{-j\frac{M(N_a-1)+\max(N_{ma}-1, N_{mc}-1)}{2}\omega} \quad (2.3)$$

where $H(\omega)$ is the zero-phase frequency response of $H(z)$ and can be written as

$$H(\omega) = H_a(M\omega)H_{ma}(\omega) + [1 - H_a(M\omega)]H_{mc}(\omega) \quad (2.4)$$

where $H_a(\omega)$, $H_{ma}(\omega)$ and $H_{mc}(\omega)$ are the zero-phase frequency responses of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$, respectively. $H_a(\omega)$ is given by.

$$H_a(\omega) = h_a\left(\frac{N_a-1}{2}\right) + \sum_{n=1}^{\frac{N_a-1}{2}} 2h_a\left(\frac{N_a-1}{2} - n\right) \cos(\omega n) \quad (2.5)$$

$H_{ma}(\omega)$ and $H_{mc}(\omega)$ are given by

$$H_{mr}(\omega) = \begin{cases} h_{mr}\left(\frac{N_{mr}-1}{2}\right) + \sum_{n=1}^{\frac{N_{mr}-1}{2}} 2h_{mr}\left(\frac{N_{mr}-1}{2} - n\right) \cos(\omega n), & \text{for } N_{mr} \text{ odd} \\ \sum_{n=1}^{\frac{N_{mr}}{2}} 2h_{mr}\left(\frac{N_{mr}}{2} - n\right) \cos\left(\omega\left(n - \frac{1}{2}\right)\right), & \text{for } N_{mr} \text{ even} \end{cases} \quad (2.6)$$

where $r = a, c$.

The filter structure of a linear phase FIR filter designed using the FRM technique is shown in Fig. 2.1. Note that the delay term $z^{-M(N_a-1)/2}$ can be implemented efficiently by sharing the delay elements of $H_a(z^M)$. Figs. 2.2 and 2.3 illustrate the synthesis of a very sharp filter $H(z)$ using subfilters $H_a(z^M)$, $H_{ma}(z)$ and $H_{mc}(z)$. As can be seen from Figs. 2.2 and 2.3, the transition band of $H(z)$ is provided by

2.1. BASIC PRINCIPLE

Figure 2.1: The structure of a filter synthesized using the FRM technique.

Figure 2.2: Synthesis of a very sharp filter using subfilters $H_a(z^M)$, $H_{ma}(z)$ and $H_{mc}(z)$ where the transition band of the sharp filter is provided by one of the transition bands of $H_a(z^M)$. This is referred to as Case A design.

2.1. BASIC PRINCIPLE

Figure 2.3: Synthesis of a very sharp filter using subfilters $H_a(z^M)$, $H_{ma}(z)$ and $H_{mc}(z)$ where the transition band of the sharp filter is provided by one of the transition bands of $H_c(z^M)$. This is referred to as Case B design.

one of the transition bands of either $H_a(z^M)$ or $H_c(z^M)$ (the complimentary filter of $H_a(z^M)$). These two design cases are referred to as Case A design and Case B design, respectively.

The passband and stopband edges of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ can be found if the passband and stopband edges of $H(z)$ and the value of M are given. Denote the bandedges of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ as $\{\omega_{pa}, \omega_{sa}\}$, $\{\omega_{pma}, \omega_{sma}\}$ and $\{\omega_{pmc}, \omega_{smc}\}$, respectively. Let the bandedges of $H(z)$ be $\{\omega_p, \omega_s\}$. For Case A de-

2.1. BASIC PRINCIPLE

signs,

$$\omega_{pa}^{(A)} = \omega_p M - 2m_A \pi \quad (2.7a)$$

$$\omega_{sa}^{(A)} = \omega_s M - 2m_A \pi \quad (2.7b)$$

where $m_A = \lfloor \omega_p M / (2\pi) \rfloor$ is the largest integer less than or equal to $\omega_p M / (2\pi)$. For Case B designs,

$$\omega_{pa}^{(B)} = 2m_B \pi - \omega_s M \quad (2.8a)$$

$$\omega_{sa}^{(B)} = 2m_B \pi - \omega_p M \quad (2.8b)$$

where $m_B = \lceil \omega_s M / (2\pi) \rceil$ is the smallest integer greater than or equal to $\omega_s M / (2\pi)$. Given M , ω_p and ω_s , the corresponding design case is either Case A or Case B, but cannot be both. If $\omega_{sa}^{(A)} \leq \pi$, Case A design is used. The bandedges of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ are given by

$$\omega_{pa} = \omega_{pa}^{(A)} \quad (2.9a)$$

$$\omega_{sa} = \omega_{sa}^{(A)} \quad (2.9b)$$

$$\omega_{pma} = \omega_p = \frac{2m_A \pi + \omega_{pa}}{M} \quad (2.9c)$$

$$\omega_{sma} = \frac{2(m_A + 1)\pi - \omega_{sa}}{M} \quad (2.9d)$$

$$\omega_{pmc} = \frac{2m_A \pi - \omega_{pa}}{M} \quad (2.9e)$$

$$\omega_{smc} = \omega_s = \frac{2m_A \pi + \omega_{sa}}{M}. \quad (2.9f)$$

If $\omega_{sa}^{(B)} \leq \pi$, Case B design is used. The bandedges of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ are

2.1. BASIC PRINCIPLE

given by

$$\omega_{pa} = \omega_{pa}^{(B)} \quad (2.10a)$$

$$\omega_{sa} = \omega_{sa}^{(B)} \quad (2.10b)$$

$$\omega_{pma} = \frac{2(m_B - 1)\pi + \omega_{sa}}{M} \quad (2.10c)$$

$$\omega_{sma} = \omega_s = \frac{2m_B\pi - \omega_{pa}}{M} \quad (2.10d)$$

$$\omega_{pmc} = \omega_p = \frac{2m_B\pi - \omega_{sa}}{M} \quad (2.10e)$$

$$\omega_{smc} = \frac{2m_B\pi + \omega_{pa}}{M}. \quad (2.10f)$$

Note that if $\omega_{pa} = 0$ or $\omega_{sa} = \pi$, the solution is not feasible and the corresponding M is discarded. From Eqs. (2.9) and (2.10), we have

$$\omega_{sa} - \omega_{pa} = M(\omega_s - \omega_p) \quad (2.11a)$$

$$(\omega_{sma} - \omega_{pma}) + (\omega_{smc} - \omega_{pmc}) = \frac{2\pi}{M}. \quad (2.11b)$$

Thus, the transition width of $H_a(z)$ is M times that of $H(z)$ and the sum of the transition widths of $H_{ma}(z)$ and $H_{mc}(z)$ is $2\pi/M$.

The filter length of $H_a(z)$ is approximately $1/M$ th of that of the minimax optimum FIR filter because the transition width of $H_a(z)$ is M times wider. Since $H_a(z^M)$ is obtained by replacing each delay element of $H_a(z)$ by M delay elements, the arithmetic complexity of $H_a(z^M)$ is the same as $H_a(z)$ although the transition width of $H_a(z^M)$ is a factor of M narrower than that of $H_a(z)$. The filter length of $H_a(z^M)$ is approximately the same as the minimax optimum FIR filter. However, $H_a(z^M)$ has very sparse coefficients; the number of nonzero coefficients is only about $1/M$ of its filter length. Thus, the arithmetic complexity of $H_a(z^M)$ is approximately

2.1. BASIC PRINCIPLE

$1/M$ th of that of the minimax optimum FIR filter. The masking filters $H_{ma}(z)$ and $H_{mc}(z)$ have wide transition widths and their filter lengths are very short compared to the minimax optimum filter. Thus, the filter synthesized using FRM technique consisting of a system of subfilters with very low arithmetic complexities. This is how the FRM technique achieves very narrow transition width with very low arithmetic complexity.

The filter length of $H_a(z^M)$ is $M(N_a - 1) + 1$. The filter lengths of $H_{ma}(z)$ and $H_{mc}(z)$ are small compared to that of $H_a(z^M)$. Thus, the equivalent filter length of the overall filter $H(z)$, $M(N_a - 1) + \max(N_{ma}, N_{mc})$, is primarily contributed by the filter length of $H_a(z^M)$. The equivalent filter length of an FIR filter synthesized using the FRM technique is only slightly longer than the minimax optimum FIR filter meeting the same set of design specifications.

The FRM technique can also be used to synthesize bandpass filter [18] and band-stop filter [20]. When the FRM technique is used to design FIR filter with very narrow passband, the branch $[z^{-M(N_a-1)/2} - H_a(z^M)] H_{mc}(z)$ is absent. The transfer function of the overall filter becomes $H(z) = H_a(z^M)H_{ma}(z)$. This is the transfer function of the interpolated FIR (IFIR) filters [14–16] where $H_{ma}(z)$ is called the interpolator. Thus, IFIR filters can be considered as a special case of the FRM filters.

The extensions of FRM technique to the design of various type of filters have also been reported in the literature. These filters include half-band filters [39, 40], decimators and interpolators [41–43] in multirate systems, FIR differentiators [44], two-dimensional filters [19, 45–47], low delay FIR filters [48–50], filter banks [51–58], IIR filters [59–63], Hilbert transformers [64, 65] and variable bandedge filters [66–68].

The implementation of FRM filters on hardware platforms such as field programmable gate array (FPGA) has been investigated in several articles [69–73]. Sev-

2.2. MULTISTAGE FRM TECHNIQUE

eral techniques are proposed in [69] for reducing memory fetches in FPGA implementation of FRM filters. In [70], a modified FRM filter suitable for FPGA implementation is proposed where the bandedge shaping filter is implemented using two to three cascaded short filters. Architectures for time-multiplexed FRM filters are introduced in [73].

2.2 Multistage FRM Technique

If the filter length of $H_a(z)$ is too long, its arithmetic complexity can be reduced by using FRM technique [17, 19, 21]. This approach can be adopted recursively which leads to multistage FRM design. In a K -stage FRM design, the z -transform transfer function $H(z)$ is expressed as

$$H(z) = H_a^{(0)}(z) \quad (2.12a)$$

$$H_a^{(0)}(z) = H_a^{(1)}(z^{M_1})H_{ma}^{(1)}(z) + \left[z^{-M_1(N_a^{(1)}-1)/2} - H_a^{(1)}(z^{M_1}) \right] H_{mc}^{(1)}(z) \quad (2.12b)$$

$$H_a^{(1)}(z) = H_a^{(2)}(z^{M_2})H_{ma}^{(2)}(z) + \left[z^{-M_2(N_a^{(2)}-1)/2} - H_a^{(2)}(z^{M_2}) \right] H_{mc}^{(2)}(z) \quad (2.12c)$$

$\vdots$

$$H_a^{(i-1)}(z) = H_a^{(i)}(z^{M_i})H_{ma}^{(i)}(z) + \left[z^{-M_i(N_a^{(i)}-1)/2} - H_a^{(i)}(z^{M_i}) \right] H_{mc}^{(i)}(z) \quad (2.12d)$$

$\vdots$

$$H_a^{(K-1)}(z) = H_a^{(K)}(z^{M_K})H_{ma}^{(K)}(z) + \left[z^{-M_K(N_a^{(K)}-1)/2} - H_a^{(K)}(z^{M_K}) \right] H_{mc}^{(K)}(z) \quad (2.12e)$$

where $H_a^{(i)}(z)$, $H_{ma}^{(i)}(z)$ and $H_{mc}^{(i)}(z)$, $i = 1, 2, \dots, K$, are FIR filters with symmetrical impulse responses and $N_a^{(i)}$ is the filter length of $H_a^{(i)}(z)$. Let the filter lengths of $H_{ma}^{(i)}(z)$ and $H_{mc}^{(i)}(z)$ be $N_{ma}^{(i)}$ and $N_{mc}^{(i)}$, respectively. $N_a^{(i)}$ for $i = 1, 2, \dots, K$ and

2.3. OPTIMIZATION METHODS

Figure 2.4: The structure of a filter synthesized using the three-stage FRM technique.

$N_{ma}^{(i)}$ and $N_{mc}^{(i)}$ for $i = 2, \dots, K$ are odd. $N_{ma}^{(1)}$ and $N_{mc}^{(1)}$ are either both odd or both even. If $N_{ma}^{(i)} > N_{mc}^{(i)}$, $(N_{ma}^{(i)} - N_{mc}^{(i)})/2$ delays are added to $H_{mc}^{(i)}(z)$. If $N_{ma}^{(i)} < N_{mc}^{(i)}$, $(N_{mc}^{(i)} - N_{ma}^{(i)})/2$ delays are added to $H_{ma}^{(i)}(z)$. The structure of a filter synthesized using the three-stage FRM technique is shown in Fig. 2.4.

2.3 Optimization Methods

In a filter design problem, the design specifications, in the case of lowpass filter, can be written as

$$1 - \delta_p \leq |H(e^{j\omega})| \leq 1 + \delta_p, \quad \forall \omega \in [0, \omega_p] \quad (2.13a)$$

$$|H(e^{j\omega})| \leq \delta_s, \quad \forall \omega \in [\omega_s, \pi] \quad (2.13b)$$

where δ_p and δ_s are the maximum allowed passband and stopband ripples, respectively. For the design of FIR filters synthesized using one-stage FRM technique, the design problem can be expressed as the following feasibility problem if the coefficient

2.3. OPTIMIZATION METHODS

symmetry is exploited

$$\underset{\mathbf{x}}{\text{minimize}} \quad 0 \quad (2.14a)$$

$$\text{subject to} \quad |H(\omega, \mathbf{x}) - 1| \leq \delta_p, \quad \forall \omega \in [0, \omega_p] \quad (2.14b)$$

$$|H(\omega, \mathbf{x})| \leq \delta_s, \quad \forall \omega \in [\omega_s, \pi] \quad (2.14c)$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_a^T & \mathbf{x}_{ma}^T & \mathbf{x}_{mc}^T \end{bmatrix}^T \quad (2.15a)$$

$$\mathbf{x}_a = \begin{bmatrix} h_a(0) & h_a(1) & \cdots & h_a\left(\frac{N_a-1}{2}\right) \end{bmatrix}^T \quad (2.15b)$$

$$\mathbf{x}_{mr} = \begin{cases} \begin{bmatrix} h_{mr}(0) & h_{mr}(1) & \cdots & h_{mr}\left(\frac{N_{mr}-1}{2}\right) \end{bmatrix}^T, & \text{for } N_{mr} \text{ odd} \\ \begin{bmatrix} h_{mr}(0) & h_{mr}(1) & \cdots & h_{mr}\left(\frac{N_{mr}}{2} - 1\right) \end{bmatrix}^T, & \text{for } N_{mr} \text{ even} \end{cases} \quad (2.15c)$$

$$H(\omega, \mathbf{x}) = H_a(M\omega, \mathbf{x}_a)H_{ma}(\omega, \mathbf{x}_{ma}) + [1 - H_a(M\omega, \mathbf{x}_a)]H_{mc}(\omega, \mathbf{x}_{mc}). \quad (2.15d)$$

Problem (2.14) is a nonconvex and nonlinear optimization problem. Many optimization methods have been proposed in the literature for designing FRM filters, i.e., find the design variable $\mathbf{x}$ such that the frequency response of the filter meets the given design specifications.

In the original FRM technique [17], filters $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ are designed separately in two steps. In the first step, $H_{mr}(z)$, $r = a, c$ are designed such that $H_{mr}(\omega)$ approximates unity for $\omega \in [0, \omega_{pmr}]$ with peak ripple magnitude equal to $0.85\delta_p$ to $0.9\delta_p$ and zero for $\omega \in [\omega_{smr}, \pi]$ with peak ripple magnitude equal to $0.85\delta_s$ to $0.9\delta_s$. In order to reduce the length of $H_{mr}(z)$, the constraint for the ripple of $H_{mr}(\omega)$ can be relaxed in some frequency ranges; in these frequency ranges, the effect

2.3. OPTIMIZATION METHODS

of $H_{mr}(\omega)$ is masked by either the passband or stopband of $H_a(z^M)$. The contribution of the ripples of $H_{ma}(z)$ or $H_{mc}(z)$ to the final ripple value is second order. Thus, a significantly smaller weight on the ripple is adequate. In the second step, $H_a(z)$ is designed by solving the following problem

$$\underset{\mathbf{x}_a}{\text{minimize}} \quad 0 \quad (2.16a)$$

$$\text{subject to} \quad |H(\omega, \mathbf{x}_a) - 1| \leq \delta_p, \quad \forall \omega \in \Omega_p \quad (2.16b)$$

$$|H(\omega, \mathbf{x}_a)| \leq \delta_s, \quad \forall \omega \in \Omega_s \quad (2.16c)$$

where Ω_p and Ω_s are the passband and stopband frequency grids, respectively.

$$H(\omega, \mathbf{x}_a) = H_a(M\omega, \mathbf{x}_a) [H_{ma}(\omega) - H_{mc}(\omega)] + H_{mc}(\omega) \quad (2.17a)$$

$$\Omega_p = \begin{cases} [\omega_{pmc}, \omega_p], & \text{for Case A} \\ [\omega_{pma}, \omega_p], & \text{for Case B} \end{cases} \quad (2.17b)$$

$$\Omega_s = \begin{cases} [\omega_s, \omega_{sma}], & \text{for Case A} \\ [\omega_s, \omega_{smc}], & \text{for Case B.} \end{cases} \quad (2.17c)$$

Problem (2.16) can be solved by linear programming [17] or Remez multiple exchange algorithm [21].

In the design methods proposed in [22–26], the subfilters are optimized alternately. In [22], $\mathbf{x}_a$ is optimized using weighted least square (WLS) approach by fixing $\mathbf{x}_{ma}$ and $\mathbf{x}_{mc}$; $\mathbf{x}_{ma}$ is optimized using WLS approach by fixing $\mathbf{x}_a$ and $\mathbf{x}_{mc}$; $\mathbf{x}_{mc}$ is optimized using WLS approach by fixing $\mathbf{x}_a$ and $\mathbf{x}_{ma}$. These three problems are solved alternately until a satisfactory design is found. In [23], $\mathbf{x}_a$ is optimized using WLS approach by fixing $\mathbf{x}_{ma}$ and $\mathbf{x}_{mc}$; $\mathbf{x}_{ma}$ and $\mathbf{x}_{mc}$ are optimized using WLS approach by

2.3. OPTIMIZATION METHODS

fixing $\mathbf{x}_a$. These two problems are solved alternately. In these two WLS approaches, the minimax design can be achieved by updating the weight function using the technique in [106]. Zhao *et al.* [26] presented an alternating strategy where $\{\mathbf{x}_{ma}, \mathbf{x}_{mc}\}$ and $\{\mathbf{x}_a, \mathbf{x}_{mc}\}$ are optimized alternatively.

The FRM filter designed by the separate optimization method and the alternating optimization method is only a local optimum. To obtain a better design, all the subfilters may be optimized simultaneously, i.e., the coefficient vector $\mathbf{x}$ consisting of the coefficient vectors of the subfilters is optimized. In [27, 28], a two-step approach is proposed. In the first step, a suboptimal solution is obtained by using a simple design method. In the second step, the suboptimal solution obtained in the first step is used as the initial design for further joint optimization using nonlinear optimization algorithm. Yu and Lim [29] proposed a WLS approach for jointly optimizing the subfilters. A quasi-Newton method is employed where the search direction is updated by using the Davidon-Fletcher-Powell (DFP) method. The minimax design is achieved by updating the weight function using the technique in [106]. Lu and Hinamoto [30–32] explored the use of convex optimization [108] for designing FRM filters. The design methods based on semidefinite programming (SDP), second-order cone programming (SOCP) and convex-concave procedure (CCP) have been introduced in their articles [30], [31] and [32], respectively. The joint optimization approaches based on artificial intelligence techniques such as genetic algorithm [33–35], artificial bee colony algorithm [36] and neural network [37] have been reported in the literature.

The FRM filter designed by independently optimizing the subfilters [17] has good finite wordlength properties. The FRM filter designed by using the joint optimization techniques has much smaller ripple compared to the separate optimization method

[17]. For the joint optimization problem, there exists many local optima with similar frequency response performances but their coefficient sensitivities may be significantly different. Thus, the joint optimization techniques may produce filters with undesirable finite word-length performances. To tackle this problem, Lim *et al.* [38] proposed a joint optimization approach that incorporates coefficient sensitivity measure in the objective function of the optimization problem. Several coefficient sensitivity measures for the FRM filter are introduced. The filter designed using the method in [38] has excellent finite wordlength properties and its frequency response performance is similar to the filter designed using joint optimization without incorporating the coefficient sensitivity measure.

2.4 Modified Structures

Many modified FRM structures have been proposed. The arithmetic complexity of these structures is lower than the filter designed using the original FRM technique.

In [74], a modified FRM technique suitable for the design of narrow passband filters is presented. In this method, the masking filters also possess periodic frequency responses like the bandedge shaping filters. The extra unwanted frequency response components produced by the periodic masking filters are eliminated by a filter $E(z)$ with wide transition band. The transfer function of the overall filter is given by

$$H(z) = \left\{ H_a(z^M) H_{ma}(z^N) + \left[z^{-M(N_a-1)/2} - H_a(z^M) \right] H_{mc}(z^N) \right\} E(z). \quad (2.18)$$

Lim and Lian [75] proposed a masking filter factorization approach for reducing the complexity of the masking filters. In their approach, the masking filters $H_{ma}(z)$

2.4. MODIFIED STRUCTURES

and $H_{mc}(z)$ are written as

$$H_{ma}(z) = H'_{ma}(z)H_{mx}(z) \quad (2.19a)$$

$$H_{mc}(z) = H'_{mc}(z)H_{mx}(z) \quad (2.19b)$$

where $H_{mx}(z)$ is a common filter for $H_{ma}(z)$ and $H_{mc}(z)$, and $H'_{ma}(z)$ and $H'_{mc}(z)$ are a pair of equalizers. Three methods have been developed for deriving $H_{mx}(z)$.

In [76], the bandedge shaping filter $H_a(z)$ is designed as a narrow band filter by properly selecting the value of M . Then, $H_a(z)$ is implemented as an IFIR filter. The design example given in [76] shows that a saving of 21% in the number of multipliers is achieved compared to the original FRM technique. The design approach in [76] can be treated as a special case of the two-stage FRM technique where $H_a^{(1)}(z) = H_a^{(2)}(z^{M_2})H_{ma}^{(2)}(z)$. It is shown in [80] that the arithmetic complexity can be further reduced if the interpolator ($H_{ma}^{(2)}(z)$) of the IFIR bandedge shaping filter ($H_a^{(1)}(z)$) is implemented as a cyclotomic polynomial prefilter [107]. A problem of the method [76] is that one of the masking filters has very high arithmetic complexity because it has very narrow transition band. This problem is tackled in [77] and improved design method is proposed. The design example of [77] yields 28% saving in the number of multipliers compared to the approach in [76].

In [78], prefilter-equalizer cascade filters [105] are used to replace the subfilters in the FRM approach, which results in a significant saving in the arithmetic complexity compared to the original FRM technique.

Gustafsson *et al.* [79] introduced a single filter FRM technique where all the subfilters are obtained from the same prototype filter by replacing each delay of the prototype filter by different number of delays. The subfilters are identical except that they have different periods. The use of identical subfilters leads to an area-efficient

time-multiplexed implementation.

A serial-masking scheme is introduced in [81] where the masking task is performed in two stages in series instead of in parallel as in the original FRM technique. Such a design may lower the arithmetic complexity of the masking filter in the first stage. In [82], the serial-masking scheme is adopted for the design of masking filters and the prefilter-equalizer based filter [78] is used for the bandedge shaping filter.

In the conventional FRM technique, the bandedge shaping filter has a periodic frequency response. In [83, 84], the bandedge shaping filter with non-periodic frequency response is considered. The method proposed in [83, 84] is based on two-stage FRM technique where the bandedge shaping filter in the first stage has a non-periodic frequency response and its transfer function can be expressed as

$$H_a^{(1)}(z) = H_a^{(2)}(z^R)H_{ma}^{(2)}(z^P) + \left[z^{-R(N_a^{(2)}-1)/2} - H_a^{(2)}(z^R) \right] H_{mc}^{(2)}(z^Q) \quad (2.20)$$

where R , P and Q are positive integers. Note that in the conventional two-stage FRM design, $P = Q$. R is an integer multiple of P and Q . This condition is not required in [83, 84]. In [83], $H_a^{(2)}(z)$, $H_{ma}^{(2)}(z)$ and $H_{mc}^{(2)}(z)$ are the same filter. The generalized case is discussed in [84] where $H_a^{(2)}(z)$, $H_{ma}^{(2)}(z)$ and $H_{mc}^{(2)}(z)$ are allowed to be different.

2.5 FRM Technique for the Design of IIR Filters

Although there have been many reports on the design of FIR filters using the FRM technique, reports on the design of IIR filters using the FRM technique are significantly less. In this section, the existing methods for the design of IIR FRM filters are reviewed.

2.5.1 Pipelined IIR Filters Based on the FRM Technique

There are two methods available for designing pipelined IIR filters. The first method is the transformation technique [111, 112] which can be generally classified into two categories: clustered look-ahead [114, 117] and scattered look-ahead [115, 116] transformations. In transformation techniques, the original filter transfer function is transformed into a new pipelineable transfer function by adding several cancelling pole-zero pairs. The pole-zero cancellation becomes inexact under finite word-length condition [113]. The second method is based on special filter design technique where the denominator polynomial of the transfer function is a function of z^M . This approach does not involve pole-zero cancellation. The pipelined IIR filter based on the FRM technique proposed by Johansson and Wanhammar [59] belongs to this category. The method in [59] has been successfully extended to the design of IIR interpolators and decimators for interpolation and decimation by a factor of two [41]. The design of narrowband and wideband pipelined IIR filters using the FRM technique is presented in [118–121].

Fig. 2.5 shows the structure of the IIR FRM filter proposed by Johansson and Wanhammar [59]. The masking filters are linear phase FIR filters whose z -transform transfer functions are $F_{ma}(z)$ and $F_{mc}(z)$. The bandedge shaping filters are a pair of power complementary IIR filters realized as a parallel connection of two allpass filters [125] whose z -transform transfer functions are $A_1(z^M)$ and $A_2(z^M)$, where M is a positive integer. The transfer functions of the bandedge shaping filters are given by

$$H_a(z^M) = \frac{A_1(z^M) + A_2(z^M)}{2} \quad (2.21a)$$

$$H_c(z^M) = \frac{A_1(z^M) - A_2(z^M)}{2}. \quad (2.21b)$$

Figure 2.5: The structure of the pipelined IIR filter proposed in [59]. The filter is synthesized using the FRM technique.

When implemented using pipelined arithmetic, the IIR bandedge shaping filters can be pipelined by M stages since the feedback loops of $A_1(z^M)$ and $A_2(z^M)$ contain M delays. A design approach where all subfilters are optimized separately is adopted in [59]. The peak ripple magnitude of the IIR filter $H_a(z)$ [$H_c(z)$] and the FIR filters $F_{ma}(z)$ and $F_{mc}(z)$ are selected such that the ripple of the overall filter meets the given design specifications. The filter obtained by the separate optimization approach is not optimum. Thus, joint optimization of subfilters is also considered in [59] where the filter obtained by the separate optimization is used as the initial solution. In [60], it is shown that $H_a(z)$ can be designed as a minimal Q -factor elliptic filter [122, 123].

For the filter in Fig. 2.5, it can achieve approximately linear phase in the passband if one of the allpass filters is a pure delay line. In this case, the bandedge shaping filter becomes a class of approximately linear phase IIR filter consisting of a parallel connection of a delay line and an allpass filter [124]. Let the allpass filter $A_1(z)$ be a pure delay line. When $H_a(z)$ is a lowpass filter, $A_1(z) = z^{-(N_2-1)}$ where N_2 is the order of $A_2(z)$. Thus, the system delay is given by $M(N_2 - 1) + \max(N_{ma} - 1, N_{mc} - 1)/2$. Another approach to designing FRM-based pipelined IIR filters with approximately linear phase is to use a general form IIR filter $H_a(z)$ and its delay

2.5. FRM TECHNIQUE FOR THE DESIGN OF IIR FILTERS

complementary filter $H_c(z) = z^{-d} - H_a(z)$ as prototype bandedge shaping filters where d is a positive integer. The masking filters are still linear phase FIR filters. This structure is introduced in [61]¹. The system delay of this type of approximately linear phase IIR FRM filter is $Md + \max(N_{ma} - 1, N_{mc} - 1)/2$ which provides some design freedom to choose the system delay (in comparison to the approach in [59]) since d does not rely on the filter order. This is useful when low system delay is desired. In [61], the IIR bandedge shaping filter and FIR masking filters are jointly optimized such that the frequency response of the overall filter approximates $e^{-j\{Md + \max(N_{ma} - 1, N_{mc} - 1)/2\}\omega}$ in the passband and approximates 0 in the stopband. The poles of the IIR bandedge shaping filter is also constrained to be inside the unit circle to ensure the filter stability.

2.5.2 FRM-Based IIR Filters With IIR Subfilters

Compared with FIR filters, the number of multipliers required by IIR filters is much smaller for the same design specifications. This motivates the use of IIR filters as masking filters in addition to the IIR bandedge shaping filters. For the FRM filters where all the subfilters are IIR filters, two designs are available in the literature.

Chen *et al.* [62] presented a class of approximately linear phase IIR FRM filter where the masking filters are approximately linear phase IIR filters realized as a parallel connection of a delay line and an allpass filter, and the bandedge shaping filters are general IIR filter and its delay complementary filter. In the design approach proposed in [62], the approximately linear phase IIR masking filters are designed first using semidefinite programming [126]. Then, a bandedge shaping FIR filter is designed using semidefinite programming to compensate for the ripples of the masking filters. Finally, the bandedge shaping IIR filter is obtained by applying model reduc-

¹The pipeline implementation is not mentioned directly in [61]. In [61], the authors aim to use IIR bandedge shaping filters to reduce the signal delay of the FRM filters.

tion technique to the designed bandedge shaping FIR filter. Since the subfilters are not jointly optimized, the performance of the filter is sub-optimal.

Shen and Lim [63] proposed a class of nonlinear phase IIR FRM filters where all the subfilters are IIR filters realized as a parallel connection of two allpass filters. The allpass filters are realized as wave digital filters [128] and optimized jointly using semi-infinite programming. To initialize the optimization process, the subfilters are initialized as elliptic filters. Since the masking filters are nonlinear phase IIR filters, the initial overall filter has enormous peaks in frequency regions corresponding to the transition bands of the bandedge shaping filters within the common passband of the masking filters (this is because the phase responses of the masking filters are not matched in those frequency regions). This initial design is far away from the optimum. Thus, the optimization process in [63] is often trapped in some bad local optima.

2.6 Applications

The FRM technique has many applications in various fields. Diniz *et al.* [94] applied FRM technique to the design of computational efficient prototype filter in the cosine-modulated transmultiplexer. FIR filters designed using FRM technique for processing electrocardiogram (ECG) and electroencephalogram (EEG) signals are demonstrated in [95] and [103], respectively. Fixed nonuniform filter banks [96, 104] and reconfigurable filter banks [100, 102] based on FRM technique are designed for hearing aids. The use of FRM technique in software radio are presented in [90, 98]. In [97], the FRM technique is applied to active array beamforming. Beampatterns with sharp transition bands and low sidelobes are achieved with fewer sensor elements than conventional approaches. In the speech recognition system proposed in [92], FRM filter

2.6. APPLICATIONS

is used for modulation spectrum filtering. FRM-based filters and filter banks are used in many digital audio applications such as graphic equalizer [85–89] and musical notes recognition [91, 93]. In [99], Lim *et al.* proposed a frequency-domain analog speech scrambler. In the proposed system, the digitized speech signal is processed by an FRM filter bank which has very low computational complexity and sharp transition bands. The outputs of the FRM filter bank are scrambled in the frequency domain and then summed together to generate the scrambled digital speech signal. Smitha and Vinod [101] presented an FRM-based multi-resolution filter bank for spectrum sensing in military radio receiver.

Chapter 3

Design of Pipelined IIR Filters Using Two-Stage Frequency-Response Masking Technique

For the FRM-based pipelined IIR filter proposed by Johansson and Wanhammar [59] as shown in Fig. 2.5, it is a one-stage masking design. For given passband and stopband ripples, the number of multipliers of the masking filters increases when the number of pipeline stages in the feedback loop (M) increases. This is because the transition width of the masking filter decreases when M increases. The number of multipliers of the IIR bandedge shaping filter is not sensitive to the change of M . When M is large, the total number of multipliers of the filter in Fig. 2.5 is very large since the number of multipliers of the masking filters is very large. It is shown in [59] that the number of multipliers of the masking filters can be reduced by using the

3.1. FILTER STRUCTURE

constrained masking filters, i.e., $F_{mc}(z) = z^{-K} - (-1)^K F_{ma}(-z)$ where integer K is the delay of the masking filters. However, this technique is only effective when the filter bandedges are between $(1 - 1/M)\pi/2$ and $(1 + 1/M)\pi/2$. This range is very restrictive when M is very large. Thus, the usage of the constrained masking filters is very limited.

In this chapter, we present a novel two-stage FRM approach for the synthesis of pipelined IIR filters. The proposed two-stage masking design requires much fewer multipliers than the one-stage masking design for the same number of pipeline stages in the feedback loop and for the same set of magnitude response specifications. The proposed technique is independent of the positions of the bandedges. The structure of the proposed two-stage FRM filter is presented in Section 3.1. The two-stage FRM filter is designed by optimizing all the subfilters jointly, which is described in Section 3.2. Section 3.3 shows that our two-stage masking design requires fewer multipliers than the one-stage masking design when the number of pipeline stages in the feedback loop is larger than four. A design example is given in Section 3.4 to demonstrate the effectiveness of our method. The proposed method can be extended to multi-stage masking design. An example of three-stage design is presented in Section 3.5.

Throughout this chapter, we assume that all the FIR subfilters have zero phase. Causality can be restored by inserting the appropriate number of delays into the FIR subfilters.

3.1 Filter Structure

Fig. 3.1 shows the structure of the pipelined IIR filter synthesized using the two-stage FRM technique where M_1 and M_2 are positive integers. The feedback loop can be pipelined by $M_1 M_2$ stages. Let $H(z)$ be the z -transform transfer function of the

3.1. FILTER STRUCTURE

Figure 3.1: The structure of pipelined IIR filter synthesized using the two-stage FRM technique.

overall filter. $H(z)$ is given by

$$H(z) = H_a^{(1)}(z^{M_1})F_{ma}^{(1)}(z) + H_c^{(1)}(z^{M_1})F_{mc}^{(1)}(z) \quad (3.1)$$

where

$$H_a^{(1)}(z) = H_a^{(2)}(z^{M_2})F_{ma}^{(2)}(z) + H_c^{(2)}(z^{M_2})F_{mc}^{(2)}(z) \quad (3.2a)$$

$$H_c^{(1)}(z) = H_a^{(2)}(z^{M_2})(1 - F_{ma}^{(2)}(z)) + H_c^{(2)}(z^{M_2})(1 - F_{mc}^{(2)}(z)). \quad (3.2b)$$

$\{H_a^{(1)}(z), H_c^{(1)}(z)\}$ and $\{H_a^{(2)}(z), H_c^{(2)}(z)\}$ are the prototype bandedge shaping filters in the first and second stage of the two-stage masking design, respectively. $\{F_{ma}^{(1)}(z), F_{mc}^{(1)}(z)\}$ and $\{F_{ma}^{(2)}(z), F_{mc}^{(2)}(z)\}$ are the masking filters in the first and second stage of the two-stage masking design, respectively. They are linear phase FIR filters with symmetrical impulse responses. The synthesis of $H_a^{(1)}(z)$ and $H_c^{(1)}(z)$ is illustrated in Fig. 3.2 where $H_a^{(2)}(e^{jM_2\omega})$, $H_c^{(2)}(e^{jM_2\omega})$, $F_{ma}^{(2)}(\omega)$, $F_{mc}^{(2)}(\omega)$, $H_a^{(1)}(e^{j\omega})$ and $H_c^{(1)}(e^{j\omega})$ are the frequency responses of $H_a^{(2)}(z^{M_2})$, $H_c^{(2)}(z^{M_2})$, $F_{ma}^{(2)}(z)$, $F_{mc}^{(2)}(z)$, $H_a^{(1)}(z)$ and $H_c^{(1)}(z)$, respectively.

The prototype bandedge shaping filters in the second stage, $H_a^{(2)}(z)$ and $H_c^{(2)}(z)$,

Figure 3.2: The synthesis of $H_a^{(1)}(z)$ and $H_c^{(1)}(z)$ using the FRM technique.

3.2. JOINT OPTIMIZATION OF SUBFILTERS

are composed of two allpass filters connected in parallel. Their transfer functions are given by

$$H_a^{(2)}(z) = \frac{A_1(z) + A_2(z)}{2} \quad (3.3a)$$

$$H_c^{(2)}(z) = \frac{A_1(z) - A_2(z)}{2} \quad (3.3b)$$

where $A_x(z)$ ($x = 1, 2$) is an allpass filter whose frequency response magnitude $|A_x(e^{j\omega})|$ equals unity. $H_a^{(2)}(z)$ and $H_c^{(2)}(z)$ are a pair of power complementary filters. From Eqs. (3.2) and (3.3), we have

$$H_a^{(1)}(z) + H_c^{(1)}(z) = A_1(z^{M_2}). \quad (3.4)$$

Since the sum of $H_a^{(1)}(z)$ and $H_c^{(1)}(z)$ is an allpass filter, $H_a^{(1)}(z)$ and $H_c^{(1)}(z)$ are allpass complementary [129].

3.2 Joint Optimization of Subfilters

To obtain an optimum design for $H(z)$, the subfilters are jointly optimized.

3.2.1 Problem Formulation

Let the order of $A_x(z)$ ($x = 1, 2$) be N_x . When $H_a^{(2)}(z)$ and $H_c^{(2)}(z)$ are lowpass and highpass filters, respectively, the orders of $A_1(z)$ and $A_2(z)$ differ by one. In the following discussion, we use $N_2 = N_1 + 1$. The frequency response of $A_x(z)$ can be written as

$$A_x(e^{j\omega}) = e^{-jN_x\omega} \tilde{A}_x(e^{j\omega}) \quad (3.5)$$

3.2. JOINT OPTIMIZATION OF SUBFILTERS

where

$$\tilde{A}_x(e^{j\omega}) = \prod_{i=1}^{\lceil N_x/2 \rceil} \frac{1 + a_{xi}(1)e^{j\omega} + a_{xi}(2)e^{j2\omega}}{1 + a_{xi}(1)e^{-j\omega} + a_{xi}(2)e^{-j2\omega}}. \quad (3.6)$$

Denote the coefficient vector of $A_x(z)$ as

$$\mathbf{a}_x = \begin{bmatrix} a_{x1}(1) & a_{x1}(2) & a_{x2}(1) & a_{x2}(2) & \cdots & a_{x\lceil N_x/2 \rceil}(1) & a_{x\lceil N_x/2 \rceil}(2) \end{bmatrix}^T \quad (3.7)$$

where $\lceil N_x/2 \rceil$ denotes the smallest integer larger than or equal to $N_x/2$. Let the impulse response of $F_{mr}^{(i)}(z)$ ($i = 1, 2$ and $r = a, c$) be $h_{mr}^{(i)}(n)$, $n = 0, \dots, L_{imr} - 1$ where L_{imr} is the length of $F_{mr}^{(i)}(z)$. For $\{F_{ma}^{(1)}(z), F_{mc}^{(1)}(z)\}$, L_{1ma} and L_{1mc} are both odd or both even. For $\{F_{ma}^{(2)}(z), F_{mc}^{(2)}(z)\}$, L_{2ma} and L_{2mc} are odd. The frequency response of $F_{mr}^{(i)}(z)$ can be written as

$$F_{mr}^{(i)}(\omega) = h_{mr}^{(i)}\left(\frac{L_{imr} - 1}{2}\right) + \sum_{n=1}^{\frac{L_{imr}-1}{2}} 2h_{mr}^{(i)}\left(\frac{L_{imr} - 1}{2} - n\right) \cos(\omega n) \quad (3.8)$$

for L_{imr} odd, and written as

$$F_{mr}^{(i)}(\omega) = \sum_{n=1}^{\frac{L_{imr}}{2}} 2h_{mr}^{(i)}\left(\frac{L_{imr}}{2} - n\right) \cos\left(\omega\left(n - \frac{1}{2}\right)\right) \quad (3.9)$$

for L_{imr} even. Denote the coefficient vector of $F_{mr}^{(i)}(z)$ as

$$\mathbf{f}_{imr} = \begin{bmatrix} h_{mr}^{(i)}\left(\frac{L_{imr}-1}{2}\right) & 2h_{mr}^{(i)}\left(\frac{L_{imr}-1}{2} - 1\right) & \cdots & 2h_{mr}^{(i)}(0) \end{bmatrix}^T \quad (3.10)$$

3.2. JOINT OPTIMIZATION OF SUBFILTERS

for L_{imr} odd, and as

$$\mathbf{f}_{imr} = \begin{bmatrix} 2h_{mr}^{(i)}\left(\frac{L_{imr}}{2} - 1\right) & 2h_{mr}^{(i)}\left(\frac{L_{imr}}{2} - 2\right) & \cdots & 2h_{mr}^{(i)}(0) \end{bmatrix}^T \quad (3.11)$$

for L_{imr} even.

Let the coefficient vector of the overall filter be

$$\mathbf{p} = \begin{bmatrix} \mathbf{a}_1^T & \mathbf{a}_2^T & \mathbf{f}_{2ma}^T & \mathbf{f}_{2mc}^T & \mathbf{f}_{1ma}^T & \mathbf{f}_{1mc}^T \end{bmatrix}^T. \quad (3.12)$$

Let $\tilde{H}(e^{j\omega}, \mathbf{p})$ denote the frequency response of the overall filter without the linear phase term $e^{-jM_1M_2N_1\omega}$.

$$\begin{aligned} \tilde{H}(e^{j\omega}, \mathbf{p}) &= \tilde{H}_a^{(1)}(e^{jM_1\omega}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{f}_{2ma}, \mathbf{f}_{2mc})F_{ma}^{(1)}(\omega, \mathbf{f}_{1ma}) \\ &\quad + \tilde{H}_c^{(1)}(e^{jM_1\omega}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{f}_{2ma}, \mathbf{f}_{2mc})F_{mc}^{(1)}(\omega, \mathbf{f}_{1mc}) \end{aligned} \quad (3.13)$$

where

$$\begin{aligned} \tilde{H}_a^{(1)}(e^{jM_1\omega}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{f}_{2ma}, \mathbf{f}_{2mc}) &= \tilde{H}_a^{(2)}(e^{jM_1M_2\omega}, \mathbf{a}_1, \mathbf{a}_2)F_{ma}^{(2)}(M_1\omega, \mathbf{f}_{2ma}) \\ &\quad + \tilde{H}_c^{(2)}(e^{jM_1M_2\omega}, \mathbf{a}_1, \mathbf{a}_2)F_{mc}^{(2)}(M_1\omega, \mathbf{f}_{2mc}) \\ \tilde{H}_c^{(1)}(e^{jM_1\omega}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{f}_{2ma}, \mathbf{f}_{2mc}) &= \tilde{H}_a^{(2)}(e^{jM_1M_2\omega}, \mathbf{a}_1, \mathbf{a}_2)[1 - F_{ma}^{(2)}(M_1\omega, \mathbf{f}_{2ma})] \\ &\quad + \tilde{H}_c^{(2)}(e^{jM_1M_2\omega}, \mathbf{a}_1, \mathbf{a}_2)[1 - F_{mc}^{(2)}(M_1\omega, \mathbf{f}_{2mc})] \\ \tilde{H}_a^{(2)}(e^{jM_1M_2\omega}, \mathbf{a}_1, \mathbf{a}_2) &= \frac{1}{2}\{\tilde{A}_1(e^{jM_1M_2\omega}, \mathbf{a}_1) + e^{-jM_1M_2\omega}\tilde{A}_2(e^{jM_1M_2\omega}, \mathbf{a}_2)\} \\ \tilde{H}_c^{(2)}(e^{jM_1M_2\omega}, \mathbf{a}_1, \mathbf{a}_2) &= \frac{1}{2}\{\tilde{A}_1(e^{jM_1M_2\omega}, \mathbf{a}_1) - e^{-jM_1M_2\omega}\tilde{A}_2(e^{jM_1M_2\omega}, \mathbf{a}_2)\}. \end{aligned}$$

The joint optimization of subfilters can be formulated as the following optimization

3.2. JOINT OPTIMIZATION OF SUBFILTERS

problem

$$\underset{\delta, \mathbf{p}}{\text{minimize}} \quad \delta \quad (3.15a)$$

$$\text{subject to} \quad W(\omega)|\tilde{H}(e^{j\omega}, \mathbf{p})|^2 - 1| \leq \delta, \quad \forall \omega \in \Omega_p \quad (3.15b)$$

$$W(\omega)|\tilde{H}(e^{j\omega}, \mathbf{p})| \leq \delta, \quad \forall \omega \in \Omega_s \quad (3.15c)$$

$$\mathbf{Q}_x \mathbf{a}_x < \mathbf{v}_x, \quad x = 1, 2 \quad (3.15d)$$

where

$$\mathbf{Q}_x = \begin{bmatrix} \mathbf{D} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D} \end{bmatrix} \quad (3.16)$$

$$\mathbf{D} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 0 & 1 \end{bmatrix} \quad (3.17)$$

$$\mathbf{v}_x = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T \quad (3.18)$$

$W(\omega)$ is a prescribed weight function, and Ω_p and Ω_s are the sets of passband and stopband grid points, respectively. The dimension of $\mathbf{Q}_x$ is $3 \lceil N_x/2 \rceil \times 2 \lceil N_x/2 \rceil$ and the dimension of $\mathbf{v}_x$ is $3 \lceil N_x/2 \rceil \times 1$. Constraints (3.15d) ensure the stability of the allpass filters $A_1(z)$ and $A_2(z)$. Note that we use $W(\omega)|\tilde{H}(e^{j\omega}, \mathbf{p})|^2 - 1| \leq \delta$ instead of $W(\omega)|\tilde{H}(e^{j\omega}, \mathbf{p})| - 1| \leq \delta$ as the constraint for the passband magnitude response because $|\tilde{H}(e^{j\omega}, \mathbf{p})|$ may not be differentiable at certain points.

3.2.2 Solution Technique

Problem (3.15) can be rewritten in the form

$$\underset{\delta, \mathbf{p}}{\text{minimize}} \quad \delta \quad (3.19a)$$

$$\text{subject to} \quad \left\| \begin{bmatrix} W(\omega_1) \{ |\tilde{H}(e^{j\omega_1}, \mathbf{p})|^2 - 1 \} \\ W(\omega_2) \{ |\tilde{H}(e^{j\omega_2}, \mathbf{p})|^2 - 1 \} \\ \vdots \\ W(\omega_{L_p}) \{ |\tilde{H}(e^{j\omega_{L_p}}, \mathbf{p})|^2 - 1 \} \end{bmatrix} \right\|_{\infty} \leq \delta, \quad \omega_i \in \Omega_p \quad (3.19b)$$

$$\left\| \begin{bmatrix} W(\omega_1) \tilde{H}(e^{j\omega_1}, \mathbf{p}) \\ W(\omega_2) \tilde{H}(e^{j\omega_2}, \mathbf{p}) \\ \vdots \\ W(\omega_{L_s}) \tilde{H}(e^{j\omega_{L_s}}, \mathbf{p}) \end{bmatrix} \right\|_{\infty} \leq \delta, \quad \omega_i \in \Omega_s \quad (3.19c)$$

$$\mathbf{Q}_x \mathbf{a}_x < \mathbf{v}_x, \quad x = 1, 2 \quad (3.19d)$$

where L_p and L_s are the number of passband and stopband grid points, respectively.

Problem (3.19) can be solved using sequential convex programming (SCP) [110]. The SCP is an iterative method where a convex approximation of the original problem is solved in each iteration. The solution may be a local optimum. The convex approximation of problem (3.19) can be obtained by using linear approximations of $|\tilde{H}(e^{j\omega_i}, \mathbf{p})|^2$ in the passband constraint (3.19b) and $\tilde{H}(e^{j\omega_i}, \mathbf{p})$ in the stopband constraint (3.19c).

Let $\mathbf{p}(k)$ be the coefficient vector at the start of the k th iteration. In the vicinity of $\mathbf{p}(k)$, $|\tilde{H}(e^{j\omega}, \mathbf{p})|^2$ and $\tilde{H}(e^{j\omega}, \mathbf{p})$ can be approximated using first-order Taylor

3.2. JOINT OPTIMIZATION OF SUBFILTERS

expansion

$$|\tilde{H}(e^{j\omega}, \mathbf{p})|^2 \approx |\tilde{H}(e^{j\omega}, \mathbf{p}(k))|^2 + \nabla^T \{|\tilde{H}(e^{j\omega}, \mathbf{p}(k))|^2\} \cdot (\mathbf{p} - \mathbf{p}(k)) \quad (3.20a)$$

$$\tilde{H}(e^{j\omega}, \mathbf{p}) \approx \tilde{H}(e^{j\omega}, \mathbf{p}(k)) + \nabla^T \tilde{H}(e^{j\omega}, \mathbf{p}(k)) \cdot (\mathbf{p} - \mathbf{p}(k)). \quad (3.20b)$$

Since $|\tilde{H}(e^{j\omega}, \mathbf{p})|^2 = \text{Re}^2[\tilde{H}(e^{j\omega}, \mathbf{p})] + \text{Im}^2[\tilde{H}(e^{j\omega}, \mathbf{p})]$, $\nabla \{|\tilde{H}(e^{j\omega}, \mathbf{p})|^2\}$ can be computed from $\tilde{H}(e^{j\omega}, \mathbf{p})$ and $\nabla \tilde{H}(e^{j\omega}, \mathbf{p})$ using Eq. (3.21)

$$\nabla \{|\tilde{H}(e^{j\omega}, \mathbf{p})|^2\} = 2 \text{Re}[\tilde{H}(e^{j\omega}, \mathbf{p})] \text{Re}[\nabla \tilde{H}(e^{j\omega}, \mathbf{p})] + 2 \text{Im}[\tilde{H}(e^{j\omega}, \mathbf{p})] \text{Im}[\nabla \tilde{H}(e^{j\omega}, \mathbf{p})]. \quad (3.21)$$

In Eq. (3.21), $\nabla \tilde{H}(e^{j\omega}, \mathbf{p})$ is given by

$$\nabla \tilde{H}(e^{j\omega}, \mathbf{p}) = \left[\frac{\partial \tilde{H}(e^{j\omega}, \mathbf{p})}{\partial \mathbf{a}_1^T} \quad \frac{\partial \tilde{H}(e^{j\omega}, \mathbf{p})}{\partial \mathbf{a}_2^T} \quad \frac{\partial \tilde{H}(e^{j\omega}, \mathbf{p})}{\partial \mathbf{f}_{2ma}^T} \quad \frac{\partial \tilde{H}(e^{j\omega}, \mathbf{p})}{\partial \mathbf{f}_{2mc}^T} \quad \frac{\partial \tilde{H}(e^{j\omega}, \mathbf{p})}{\partial \mathbf{f}_{1ma}^T} \quad \frac{\partial \tilde{H}(e^{j\omega}, \mathbf{p})}{\partial \mathbf{f}_{1mc}^T} \right]^T \quad (3.22)$$

3.2. JOINT OPTIMIZATION OF SUBFILTERS

where

$$\begin{aligned} \frac{\partial \tilde{H}(e^{j\omega}, \mathbf{p})}{\partial \mathbf{a}_1} &= \frac{1}{2} \left\{ [F_{ma}^{(2)}(M_1\omega, \mathbf{f}_{2ma}) + F_{mc}^{(2)}(M_1\omega, \mathbf{f}_{2mc})] \right. \\ &\quad \times [F_{ma}^{(1)}(M_1\omega, \mathbf{f}_{1ma}) - F_{mc}^{(1)}(M_1\omega, \mathbf{f}_{1mc})] + 2F_{mc}^{(1)}(M_1\omega, \mathbf{f}_{1mc}) \Big\} \\ &\quad \times \nabla \tilde{A}_1(e^{jM_1M_2\omega}, \mathbf{a}_1) \end{aligned} \quad (3.23a)$$

$$\begin{aligned} \frac{\partial \tilde{H}(e^{j\omega}, \mathbf{p})}{\partial \mathbf{a}_2} &= \frac{1}{2} \left\{ [F_{ma}^{(2)}(M_1\omega, \mathbf{f}_{2ma}) - F_{mc}^{(2)}(M_1\omega, \mathbf{f}_{2mc})] \right. \\ &\quad \times [F_{ma}^{(1)}(M_1\omega, \mathbf{f}_{1ma}) - F_{mc}^{(1)}(M_1\omega, \mathbf{f}_{1mc})] + 2F_{mc}^{(1)}(M_1\omega, \mathbf{f}_{1mc}) \Big\} \\ &\quad \times e^{-jM_1M_2\omega} \nabla \tilde{A}_2(e^{jM_1M_2\omega}, \mathbf{a}_2) \end{aligned} \quad (3.23b)$$

$$\frac{\partial \tilde{H}(e^{j\omega}, \mathbf{p})}{\partial \mathbf{f}_{2ma}} = \tilde{H}_a^{(2)}(e^{jM_1M_2\omega}, \mathbf{a}_1, \mathbf{a}_2) [F_{ma}^{(1)}(M_1\omega, \mathbf{f}_{1ma}) - F_{mc}^{(1)}(M_1\omega, \mathbf{f}_{1mc})] \mathbf{c}_{2ma} \quad (3.23c)$$

$$\frac{\partial \tilde{H}(e^{j\omega}, \mathbf{p})}{\partial \mathbf{f}_{2mc}} = \tilde{H}_c^{(2)}(e^{jM_1M_2\omega}, \mathbf{a}_1, \mathbf{a}_2) [F_{ma}^{(1)}(M_1\omega, \mathbf{f}_{1ma}) - F_{mc}^{(1)}(M_1\omega, \mathbf{f}_{1mc})] \mathbf{c}_{2mc} \quad (3.23d)$$

$$\frac{\partial \tilde{H}(e^{j\omega}, \mathbf{p})}{\partial \mathbf{f}_{1ma}} = \tilde{H}_a^{(1)}(e^{jM_1\omega}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{f}_{2ma}, \mathbf{f}_{2mc}) \mathbf{c}_{1ma} \quad (3.23e)$$

$$\frac{\partial \tilde{H}(e^{j\omega}, \mathbf{p})}{\partial \mathbf{f}_{1mc}} = \tilde{H}_c^{(1)}(e^{jM_1\omega}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{f}_{2ma}, \mathbf{f}_{2mc}) \mathbf{c}_{1mc} \quad (3.23f)$$

$$\mathbf{c}_{imr}(\omega) = \begin{cases} \left[\begin{matrix} 1 & \cos(\omega) & \cdots & \cos\left(\frac{L_{imr}-1}{2}\omega\right) \end{matrix} \right]^T, & \text{for } L_{imr} \text{ odd} \\ \left[\begin{matrix} \cos\left(\frac{1}{2}\omega\right) & \cos\left(\frac{3}{2}\omega\right) & \cdots & \cos\left(\frac{L_{imr}-1}{2}\omega\right) \end{matrix} \right]^T, & \text{for } L_{imr} \text{ even.} \end{cases} \quad (3.23g)$$

3.2. JOINT OPTIMIZATION OF SUBFILTERS

The expression for $\nabla \tilde{A}_x(e^{j\omega}, \mathbf{a}_x)$ in Eqs. (3.23a) and (3.23b) is given by

$$\nabla \tilde{A}_x(e^{j\omega}, \mathbf{a}_x) = \left[\frac{\partial \tilde{A}_x(e^{j\omega}, \mathbf{a}_x)}{\partial a_{x1}(1)} \quad \frac{\partial \tilde{A}_x(e^{j\omega}, \mathbf{a}_x)}{\partial a_{x1}(2)} \quad \dots \right. \\ \left. \frac{\partial \tilde{A}_x(e^{j\omega}, \mathbf{a}_x)}{\partial a_{xi}(1)} \quad \frac{\partial \tilde{A}_x(e^{j\omega}, \mathbf{a}_x)}{\partial a_{xi}(2)} \quad \dots \quad \frac{\partial \tilde{A}_x(e^{j\omega}, \mathbf{a}_x)}{\partial a_{x\lceil N_x/2 \rceil}(1)} \quad \frac{\partial \tilde{A}_x(e^{j\omega}, \mathbf{a}_x)}{\partial a_{x\lceil N_x/2 \rceil}(2)} \right]^T \quad (3.24a)$$

$$\frac{\partial \tilde{A}_x(e^{j\omega}, \mathbf{a}_x)}{\partial a_{xi}(1)} = \frac{\tilde{A}_x(e^{j\omega}, \mathbf{a}_x) \times 2j \sin(\omega) (1 - a_{xi}(2))}{(1 + a_{xi}(1)e^{j\omega} + a_{xi}(2)e^{j2\omega})(1 + a_{xi}(1)e^{-j\omega} + a_{xi}(2)e^{-j2\omega})} \quad (3.24b)$$

$$\frac{\partial \tilde{A}_x(e^{j\omega}, \mathbf{a}_x)}{\partial a_{xi}(2)} = \frac{\tilde{A}_x(e^{j\omega}, \mathbf{a}_x) \times 2j \sin(\omega) (2 \cos(\omega) + a_{xi}(1))}{(1 + a_{xi}(1)e^{j\omega} + a_{xi}(2)e^{j2\omega})(1 + a_{xi}(1)e^{-j\omega} + a_{xi}(2)e^{-j2\omega})}, \\ \text{for } a_{xi}(2) \neq 0. \quad (3.24c)$$

Note that (3.20) is a good approximation only when $\mathbf{p}$ is close to $\mathbf{p}(k)$. Thus, we may impose constraint $\|\mathbf{p} - \mathbf{p}(k)\|_\infty \leq \beta$ on $\mathbf{p}$ where β is a prescribed small positive number.

3.2. JOINT OPTIMIZATION OF SUBFILTERS

Define two matrices $\mathbf{G}_{pk}$ and $\mathbf{G}_{sk}$ and two column vectors $\mathbf{e}_{pk}$ and $\mathbf{e}_{sk}$ as

$$\mathbf{G}_{pk} = \begin{bmatrix} W(\omega_1) \nabla^T \{ |\tilde{H}(e^{j\omega_1}, \mathbf{p}(k))|^2 \} \\ \vdots \\ W(\omega_{L_p}) \nabla^T \{ |\tilde{H}(e^{j\omega_{L_p}}, \mathbf{p}(k))|^2 \} \end{bmatrix}, \quad \omega_i \in \Omega_p \quad (3.25a)$$

$$\mathbf{e}_{pk} = \begin{bmatrix} W(\omega_1) \{ |\tilde{H}(e^{j\omega_1}, \mathbf{p}(k))|^2 - 1 \} \\ \vdots \\ W(\omega_{L_p}) \{ |\tilde{H}(e^{j\omega_{L_p}}, \mathbf{p}(k))|^2 - 1 \} \end{bmatrix}, \quad \omega_i \in \Omega_p \quad (3.25b)$$

$$\mathbf{G}_{sk} = \begin{bmatrix} W(\omega_1) \nabla^T \tilde{H}(e^{j\omega_1}, \mathbf{p}(k)) \\ \vdots \\ W(\omega_{L_s}) \nabla^T \tilde{H}(e^{j\omega_{L_s}}, \mathbf{p}(k)) \end{bmatrix}, \quad \omega_i \in \Omega_s \quad (3.25c)$$

$$\mathbf{e}_{sk} = \begin{bmatrix} W(\omega_1) \tilde{H}(e^{j\omega_1}, \mathbf{p}(k)) \\ \vdots \\ W(\omega_{L_s}) \tilde{H}(e^{j\omega_{L_s}}, \mathbf{p}(k)) \end{bmatrix}, \quad \omega_i \in \Omega_s \quad (3.25d)$$

The optimization problem to be solved in the k th iteration can be written as

$$\underset{\delta, \mathbf{p}}{\text{minimize}} \quad \delta \quad (3.26a)$$

$$\text{subject to} \quad \|\mathbf{G}_{pk} \mathbf{p} + \mathbf{d}_{pk}\|_\infty \leq \delta \quad (3.26b)$$

$$\|\mathbf{G}_{sk} \mathbf{p} + \mathbf{d}_{sk}\|_\infty \leq \delta \quad (3.26c)$$

$$\mathbf{Q}_x \mathbf{a}_x < \mathbf{v}_x, \quad x = 1, 2 \quad (3.26d)$$

$$\|\mathbf{p} - \mathbf{p}(k)\|_\infty \leq \beta \quad (3.26e)$$

3.2. JOINT OPTIMIZATION OF SUBFILTERS

where

$$\mathbf{d}_{pk} = -\mathbf{G}_{pk}\mathbf{p}(k) + \mathbf{e}_{pk} \quad (3.27a)$$

$$\mathbf{d}_{sk} = -\mathbf{G}_{sk}\mathbf{p}(k) + \mathbf{e}_{sk} \quad (3.27b)$$

If the relative difference of the objective function values in two consecutive iterations is less than a prescribed tolerance ε , the optimization process is terminated. The optimization process may also be terminated when a prescribed number of iterations, K_{max} , have been carried out. Let δ_0 be the objective function value at the start of the k th iteration. The algorithm for designing pipelined IIR filters using two-stage FRM technique is summarized as follows.

Step 1: Find an initial solution $\mathbf{p}(1)$. Let $k = 1$.

Step 2: Compute $\mathbf{G}_{pk}$, $\mathbf{G}_{sk}$, $\mathbf{d}_{pk}$ and $\mathbf{d}_{sk}$.

Step 3: Solve problem (3.26) for δ and $\mathbf{p}$.

Step 4: When $k \geq 2$, if $|\delta - \delta_0|/\delta_0 \leq \varepsilon$, return $\mathbf{p}$. If $k = K_{max}$, return $\mathbf{p}$.

Step 5: $k = k + 1$, $\mathbf{p}(k) = \mathbf{p}$ and $\delta_0 = \delta$. Go to Step 2.

It is important to find a good initial solution such that the optimization procedure converges to a much better local optimum or hopefully the global optimum. Given M_1 , M_2 , N_1 , N_2 , L_{2ma} , L_{2mc} , L_{1ma} , L_{1mc} and the bandedges of the overall filter, a good initialization technique is as follows.

Step 1: Compute the bandedges of $H_a^{(2)}(z)[H_c^{(2)}(z)]$, $F_{ma}^{(2)}(z)$, $F_{mc}^{(2)}(z)$, $F_{ma}^{(1)}(z)$ and $F_{mc}^{(1)}(z)$ using the formulas in [17].

3.3. MULTIPLIER SAVING

Step 2: Initialize $F_{ma}^{(2)}(z)$, $F_{mc}^{(2)}(z)$, $F_{ma}^{(1)}(z)$ and $F_{mc}^{(1)}(z)$ as linear phase FIR filters with appropriate bandedges.

Step 3: Initialize $H_a^{(2)}(z)$ or $H_c^{(2)}(z)$ as an odd-order digital elliptic filter. Then, $A_1(z)$ and $A_2(z)$ are obtained based on the poles of the elliptic filter.

3.3 Multiplier Saving

For a given magnitude response specification and a given number of pipeline stages in the feedback loop, the number of multipliers of the IIR bandedge shaping filter will not have a significant difference in the one-stage and two-stage masking designs. However, the number of multipliers of the FIR masking filters may be significantly different.

The length L of a lowpass linear phase FIR filter with transition width $\Delta \cdot f_s$, where f_s is the sampling frequency, passband ripple magnitude δ_1 and stopband ripple magnitude δ_2 , is given by [13]

$$L \approx \frac{\Phi_1(\delta_1, \delta_2)}{\Delta} - \Phi_2(\delta_1, \delta_2)\Delta + 1 \quad (3.28)$$

where

$$\begin{aligned} \Phi_1(\delta_1, \delta_2) = & [0.005309(\log_{10} \delta_1)^2 + 0.07114 \log_{10} \delta_1 - 0.4761] \\ & \times \log_{10} \delta_2 - 0.00266(\log_{10} \delta_1)^2 - 0.5941 \log_{10} \delta_1 - 0.4278 \end{aligned} \quad (3.29a)$$

$$\Phi_2(\delta_1, \delta_2) = 11.01217 + 0.51244(\log_{10} \delta_1 - \log_{10} \delta_2). \quad (3.29b)$$

3.3. MULTIPLIER SAVING

The first term in (3.28) is the dominant term when $\Delta \leq 0.2$. Thus, we have

$$L \approx \frac{\Phi_1(\delta_1, \delta_2)}{\Delta}, \text{ for } \Delta \leq 0.2. \quad (3.30)$$

Let the maximum allowed passband and stopband ripple magnitudes of $H(z)$ be δ_p and δ_s , respectively. When estimating the lengths of the masking filters, the passband and stopband ripple magnitudes of $F_{mr}^{(i)}(z)$ ($i = 1, 2$ and $r = a, c$) are chosen as δ_p and δ_s , respectively. The sum of the normalized transition widths of $F_{ma}^{(i)}(z)$ and $F_{mc}^{(i)}(z)$ is $1/M_i$. It has been shown in [19] that $L_{ima} + L_{imc}$ is minimum when $\Delta_{ima} = \Delta_{imc} = 1/(2M_i)$ where Δ_{ima} and Δ_{imc} are the normalized transition widths of $F_{ma}^{(i)}(z)$ and $F_{mc}^{(i)}(z)$, respectively. Thus, for the two-stage masking design, the total number of coefficients of the masking filters is given by

$$L_{T2S} \approx 4\Phi_1(\delta_p, \delta_s)(M_1 + M_2). \quad (3.31)$$

If the number of pipeline stages in the feedback loop of the one-stage masking design is the same as that of the two-stage masking design, i.e., $M = M_1M_2$, the total number of coefficients of the masking filters for one-stage masking design is given by

$$L_{T1S} \approx 4\Phi_1(\delta_p, \delta_s)M_1M_2. \quad (3.32)$$

In the FRM technique, the value of M is larger than one. From (3.31) and (3.32), we have $L_{T2S} \leq L_{T1S}$. The equality holds when $M_1 = M_2 = 2$. Thus, the two-stage masking design requires fewer multipliers than the one-stage masking design when the number of pipeline stages in the feedback loop is larger than four.

There may exist a few possible values of (M_1, M_2) satisfying $M_1M_2 = M_{des}$ where

3.4. EXAMPLE

M_{des} is the desired number of pipeline stages in the feedback loop. The number of multipliers is minimum when $M_1 = M_2 = \sqrt{M_{des}}$. Thus, M_1 and M_2 should be chosen such that $M_1 \approx M_2$.

3.4 Example

We choose the design specifications given in Example 1 of [59] as an example to illustrate our method. The design specifications are: passband edge = 0.495π , stopband edge = 0.5π , maximum passband peak-to-peak ripple = 0.2 dB, minimum stopband attenuation = 50 dB. Our design algorithm is implemented using MATLAB and CVX [109]. The weight function $W(\omega)$ is determined as follows. In the passband of $H(z)$, we have $||H(e^{j\omega})|^2 - 1| \approx 2||H(e^{j\omega})| - 1|$. If we choose $W(\omega) = 1$ for the passband, $W(\omega)$ for the stopband is determined as

$$W(\omega) = 2 \times \frac{1 - 10^{-0.1/20}}{10^{-50/20}} \approx 7.24, \text{ for } \omega \in \Omega_s.$$

The values of β and ε are 5×10^{-3} and 1×10^{-5} , respectively.

For the purpose of comparison, one-stage FRM filter [59] is also designed using the joint optimization method similar to that described in Section 3.2. Filters with different number of pipeline stages in the feedback loop are designed. The actual lengths of the FIR masking filters are often shorter than the estimated lengths obtained using (3.28). The total number of multipliers required by the two-stage and one-stage masking designs is listed in Table 3.1. As can be seen from Table 3.1, our two-stage masking design produces significant savings in the total number of multipliers compared to the one-stage design. The magnitude responses of subfilters and overall filter in the two-stage design with $M_1 = 3$ and $M_2 = 3$ are shown in Figs. 3.3-

3.4. EXAMPLE

Table 3.1: Multiplier Savings in Percentage for Different Number of Pipeline Stages in the Feedback Loop. For the Two-Stage (TS) and One-Stage (OS) Designs, the Filter Orders and Lengths Are Organized in the Form of $(N_1, N_2, L_{2ma}, L_{2mc}, L_{1ma}, L_{1mc})$ and $(N_1, N_2, L_{ma}, L_{mc})$, Respectively.

Pipeline stages	M	Orders and lengths	Multipliers	Saving (%)
9	TS: $M_1 = 3, M_2 = 3$	(4, 5, 11, 13, 11, 13)	35	27.1
	OS: $M = 9$	(4, 5, 38, 40)	48	
15	TS: $M_1 = 5, M_2 = 3$	(3, 4, 13, 11, 20, 20)	40	42.9
	OS: $M = 15$	(3, 4, 68, 58)	70	
21	TS: $M_1 = 7, M_2 = 3$	(3, 4, 9, 13, 28, 28)	47	50.0
	OS: $M = 21$	(3, 4, 78, 96)	94	
25	TS: $M_1 = 5, M_2 = 5$	(3, 4, 17, 23, 20, 18)	47	57.3
	OS: $M = 25$	(3, 4, 89, 115)	110	
27	TS: $M_1 = 9, M_2 = 3$	(3, 4, 13, 9, 32, 34)	52	54.8
	OS: $M = 27$	(3, 4, 122, 94)	115	
35	TS: $M_1 = 7, M_2 = 5$	(3, 4, 25, 17, 24, 22)	52	63.6
	OS: $M = 35$	(3, 4, 162, 110)	143	
45	TS: $M_1 = 9, M_2 = 5$	(2, 3, 15, 25, 34, 34)	60	67.0
	OS: $M = 45$	(2, 3, 136, 218)	182	

3.5. The coefficient values are shown in Table A.1 (in Appendix A). The magnitude responses of subfilters and overall filter in the two-stage design with $M_1 = 5$ and $M_2 = 3$ are shown in Figs. 3.6-3.8. The coefficient values are shown in Table A.2 (in Appendix A). For these two designs ($M_1 = 3$ and $M_2 = 3$; $M_1 = 5$ and $M_2 = 3$), the numbers of grid points are both 2500 in the passband and stopband. The grid points are evenly distributed.

For the two-stage masking design, the number of multipliers increases with $M_1 +$

3.4. EXAMPLE

M_2 as shown in Section 3.3. In general, for the two-stage design, the number of multipliers increases when the number of pipeline stages increases, i.e., M_1M_2 increases. For the cases where M_1M_2 increases but $M_1 + M_2$ remains unchanged, the number of multipliers may be unchanged or approximately the same. This can be seen in Table I when M_1M_2 increases from 21 to 25 and when M_1M_2 increases from 27 to 35.

Compared to the one-stage design, the two-stage design requires two extra adders for obtaining $1 - F_{ma}^{(2)}(z^{M_1})$ and $1 - F_{mc}^{(2)}(z^{M_1})$ and requires two extra adders for summing the outputs of $F_{ma}^{(2)}(z^{M_1})$ and $F_{mc}^{(2)}(z^{M_1})$ and the outputs of $1 - F_{ma}^{(2)}(z^{M_1})$ and $1 - F_{mc}^{(2)}(z^{M_1})$. However, the total number of adders required by the FIR masking filters in the two-stage design is significantly less than that in the one-stage design. Thus, the two-stage design requires much fewer adders than the one-stage design. For the example where the number of pipeline stages is 15, 80 adders is saved by using the two-stage design.

3.4. EXAMPLE

Figure 3.3: Magnitude responses of (a) bandedge shaping filters $H_a^{(2)}(z^9)$ (solid line) and $H_c^{(2)}(z^9)$ (dashed line) and (b) masking filters $F_{ma}^{(2)}(z^3)$ (solid line) and $F_{mc}^{(2)}(z^3)$ (dashed line).

Figure 3.4: Magnitude responses of (a) bandedge shaping filters $H_a^{(1)}(z^3)$ (solid line) and $H_c^{(1)}(z^3)$ (dashed line) and (b) masking filters $F_{ma}^{(1)}(z)$ (solid line) and $F_{mc}^{(1)}(z)$ (dashed line).

3.4. EXAMPLE

Figure 3.5: Magnitude responses of (a) overall filter $H(z)$ and (b) $H(z)$ in the pass-band for the two-stage design with $M_1 = 3$ and $M_2 = 3$.

Figure 3.6: Magnitude responses of (a) bandedge shaping filters $H_a^{(2)}(z^{15})$ (solid line) and $H_c^{(2)}(z^{15})$ (dashed line) and (b) masking filters $F_{ma}^{(2)}(z^5)$ (solid line) and $F_{mc}^{(2)}(z^5)$ (dashed line).

3.4. EXAMPLE

Figure 3.7: Magnitude responses of (a) bandedge shaping filters $H_a^{(1)}(z^5)$ (solid line) and $H_c^{(1)}(z^5)$ (dashed line) and (b) masking filters $F_{ma}^{(1)}(z)$ (solid line) and $F_{mc}^{(1)}(z)$ (dashed line).

Figure 3.8: Magnitude responses of (a) overall filter $H(z)$ and (b) $H(z)$ in the pass-band for the two-stage design with $M_1 = 5$ and $M_2 = 3$.

3.5. MULTI-STAGE MASKING DESIGN

Figure 3.9: The structure of pipelined IIR filter synthesized using the three-stage FRM technique.

3.5 Multi-Stage Masking Design

Our proposed technique may be extended to multi-stage masking design. We illustrate this by using an example of three-stage masking design. Fig. 3.9 shows the structure of the pipelined IIR filter synthesized using the three-stage FRM technique. The feedback loop is pipelined by $M_1M_2M_3$ stages. The z -transform transfer function of the overall filter $H(z)$ is given by

$$H(z) = H_a^{(1)}(z^{M_1})F_{ma}^{(1)}(z) + H_c^{(1)}(z^{M_1})F_{mc}^{(1)}(z) \quad (3.33a)$$

$$H_a^{(1)}(z) = H_a^{(2)}(z^{M_2})F_{ma}^{(2)}(z) + H_c^{(2)}(z^{M_2})F_{mc}^{(2)}(z) \quad (3.33b)$$

$$H_c^{(1)}(z) = H_a^{(2)}(z^{M_2})(1 - F_{ma}^{(2)}(z)) + H_c^{(2)}(z^{M_2})(1 - F_{mc}^{(2)}(z)) \quad (3.33c)$$

$$H_a^{(2)}(z) = H_a^{(3)}(z^{M_3})F_{ma}^{(3)}(z) + H_c^{(3)}(z^{M_3})F_{mc}^{(3)}(z) \quad (3.33d)$$

$$H_c^{(2)}(z) = H_a^{(3)}(z^{M_3})(1 - F_{ma}^{(3)}(z)) + H_c^{(3)}(z^{M_3})(1 - F_{mc}^{(3)}(z)) \quad (3.33e)$$

$$H_a^{(3)}(z) = \frac{A_1(z) + A_2(z)}{2} \quad (3.33f)$$

$$H_c^{(3)}(z) = \frac{A_1(z) - A_2(z)}{2}. \quad (3.33g)$$

Let the filter lengths of $F_{ma}^{(3)}$ and $F_{mc}^{(3)}$ be L_{3ma} and L_{3mc} , respectively. Given M_1 ,

3.5. MULTI-STAGE MASKING DESIGN

M_2 , N_1 , N_2 , L_{3ma} , L_{3mc} , L_{2ma} , L_{2mc} , L_{1ma} , L_{1mc} and the bandedges of the overall filter, a good initialization technique for a three-stage masking design is as follows.

Step 1: Compute the bandedges of $H_a^{(3)}(z)[H_c^{(3)}(z)]$, $F_{ma}^{(3)}(z)$, $F_{mc}^{(3)}(z)$, $F_{ma}^{(2)}(z)$, $F_{mc}^{(2)}(z)$, $F_{ma}^{(1)}(z)$ and $F_{mc}^{(1)}(z)$ using the formulas in [17].

Step 2: Initialize $F_{ma}^{(1)}(z)$ and $F_{mc}^{(1)}(z)$ as linear phase FIR filters with appropriate bandedges.

Step 3: The initial designs for $A_1(z)$, $A_2(z)$, $F_{ma}^{(3)}(z)$, $F_{mc}^{(3)}(z)$, $F_{ma}^{(2)}(z)$ and $F_{mc}^{(2)}(z)$ are obtained by designing $H_a^{(1)}(z)$ using the joint optimization approach presented in Section 3.2.

Consider the design in Table 3.1 whose number of pipeline stages is 45. If a three-stage design is used, we may choose $M_1 = 5$, $M_2 = 3$ and $M_3 = 3$ and the total number of multipliers is 53. Compared to the two-stage design, the three-stage design produces a 11.7% saving in total number of multipliers. The magnitude responses of subfilters and overall filter in the three-stage design are shown in Figs. 3.10-3.13. The coefficient values are shown in Tables A.3-A.4 (in Appendix A).

3.5. MULTI-STAGE MASKING DESIGN

Figure 3.10: Magnitude responses of (a) bandedge shaping filters $H_a^{(3)}(z^{45})$ (solid line) and $H_c^{(3)}(z^{45})$ (dashed line) and (b) masking filters $F_{ma}^{(3)}(z^{15})$ (solid line) and $F_{mc}^{(3)}(z^{15})$ (dashed line).

Figure 3.11: Magnitude responses of (a) bandedge shaping filters $H_a^{(2)}(z^{15})$ (solid line) and $H_c^{(2)}(z^{15})$ (dashed line) and (b) masking filters $F_{ma}^{(2)}(z^5)$ (solid line) and $F_{mc}^{(2)}(z^5)$ (dashed line).

3.5. MULTI-STAGE MASKING DESIGN

Figure 3.12: Magnitude responses of (a) bandedge shaping filters $H_a^{(1)}(z^5)$ (solid line) and $H_c^{(1)}(z^5)$ (dashed line) and (b) masking filters $F_{ma}^{(1)}(z)$ (solid line) and $F_{mc}^{(1)}(z)$ (dashed line).

Figure 3.13: Magnitude responses of (a) overall filter $H(z)$ and (b) $H(z)$ in the passband for the three-stage design with $M_1 = 5$, $M_2 = 3$ and $M_3 = 3$.

3.6 Conclusion

In this chapter, we have presented a two-stage FRM technique for designing pipelined IIR filter. In our design, all the subfilters are optimized jointly. It is shown that, when the number of pipeline stages in the feedback loop is larger than four, our two-stage masking design requires fewer multipliers than the one-stage masking design [59] meeting the same magnitude response specifications. The superiority of the two-stage design over the one-stage design is also demonstrated by an example. The proposed method can be used to design any piecewise constant frequency response such as a bandpass filter.

Chapter 4

Design of Approximately Linear Phase Pipelined IIR Filters Using Frequency-Response Masking Technique

This chapter presents the design of approximately linear phase pipelined IIR filters using the FRM technique. The filter structure we considered is the one proposed in [61] where the bandedge shaping filters are an IIR filter and its delay complementary filter, and the masking filters are linear phase FIR filters, as shown in Fig. 4.1. The optimization method in [61] only constrains the frequency-response error magnitude but not its phase. In this chapter, we present a constrained optimization method where the phase error and magnitude error are controlled independently by imposing constraints on phase error and frequency-response error whose real and imaginary parts are bounded by an ellipse. Our proposed method is an extension of the IIR

Figure 4.1: Structure of the FRM-based pipelined IIR filter in [61]. $H_a(z)$ is an IIR filter. $H_{ma}(z)$ and $H_{mc}(z)$ are linear phase FIR filters. M and d are positive integers.

filter design technique presented in [12] to the more complicated IIR FRM filters.

This Chapter is organized as follows. The z -transform transfer functions of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ are formulated in Section 4.1. The optimization problem is formulated in 4.2. The solution technique is presented in 4.3. Design examples are given in Section 4.4.

4.1 Filter Transfer Functions

The z -transform transfer function of the IIR prototype bandedge shaping filter $H_a(z)$ with numerator order N_a and denominator order L_a can be expressed as

$$H_a(z, \mathbf{b}, \mathbf{a}) = \frac{\phi_n^T(z) \mathbf{b}}{\prod_{i=1}^{\lceil \frac{L_a}{2} \rceil} (1 + \phi_d^T(z) \mathbf{a}_i)} \quad (4.1)$$

4.1. FILTER TRANSFER FUNCTIONS

where

$$\mathbf{b} = \begin{bmatrix} b_0 & b_1 & \cdots & b_{N_a} \end{bmatrix}^T \quad (4.2a)$$

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_1^T & \mathbf{a}_2^T & \cdots & \mathbf{a}_{\lceil \frac{L_a}{2} \rceil}^T \end{bmatrix}^T \quad (4.2b)$$

$$\mathbf{a}_i = \begin{bmatrix} a_{i1} & a_{i2} \end{bmatrix}^T, \quad i = 1, 2, \dots, \left\lceil \frac{L_a}{2} \right\rceil \quad (4.2c)$$

$$\phi_n(z) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-N_a} \end{bmatrix}^T \quad (4.2d)$$

$$\phi_d(z) = \begin{bmatrix} z^{-1} & z^{-2} \end{bmatrix}^T. \quad (4.2e)$$

Vectors $\mathbf{b}$, $\mathbf{a}$ and $\mathbf{a}_i$ are the coefficient vectors associated with the numerator, denominator and the i th denominator factor. If L_a is odd, $\mathbf{a}_1 = [a_{11} \ 0]^T$. The denominator of $H_a(z)$ is represented as a cascade of second-order sections such that necessary and sufficient stability condition [4] can be applied. If all the poles are required to be inside a circle of radius r in the complex plane, the condition can be expressed in matrix form as

$$\mathbf{Q}\mathbf{a} \leq \gamma \quad (4.3)$$

4.1. FILTER TRANSFER FUNCTIONS

where

$$\mathbf{Q} = \begin{bmatrix} \mathbf{\Gamma} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{\Gamma} \end{bmatrix} \quad (4.4a)$$

$$\mathbf{\Gamma} = \begin{bmatrix} r & -1 \\ -r & -1 \\ 0 & 1 \end{bmatrix} \quad (4.4b)$$

$$\boldsymbol{\gamma} = \begin{bmatrix} r^2 & r^2 & \cdots & r^2 \end{bmatrix}^T. \quad (4.4c)$$

The transfer functions of the FIR masking filters are given by

$$H_{ma}(z, \mathbf{h}_{ma}) = \boldsymbol{\phi}_{ma}^T(z) \mathbf{h}_{ma} \quad (4.5a)$$

$$H_{mc}(z, \mathbf{h}_{mc}) = \boldsymbol{\phi}_{mc}^T(z) \mathbf{h}_{mc} \quad (4.5b)$$

where

$$\mathbf{h}_{ma} = \begin{bmatrix} h_{ma}(0) & h_{ma}(1) & \cdots & h_{ma}(N_{ma} - 1) \end{bmatrix}^T \quad (4.6a)$$

$$\mathbf{h}_{mc} = \begin{bmatrix} h_{mc}(0) & h_{mc}(1) & \cdots & h_{mc}(N_{mc} - 1) \end{bmatrix}^T \quad (4.6b)$$

$$\boldsymbol{\phi}_{ma}(z) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-(N_{ma}-1)} \end{bmatrix}^T \quad (4.6c)$$

$$\boldsymbol{\phi}_{mc}(z) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-(N_{mc}-1)} \end{bmatrix}^T. \quad (4.6d)$$

Vectors $\mathbf{h}_{ma}$ and $\mathbf{h}_{mc}$ are coefficient vectors of $H_{ma}(z)$ and $H_{mc}(z)$, respectively, and

4.2. PROBLEM FORMULATION

they are symmetrical, i.e.,

$$h_{ma}(N_{ma} - 1 - k) = h_{ma}(k), \quad i = 0, 1, \dots, N_{ma} - 1 \quad (4.7a)$$

$$h_{mc}(N_{mc} - 1 - k) = h_{mc}(k), \quad i = 0, 1, \dots, N_{mc} - 1. \quad (4.7b)$$

N_{ma} and N_{mc} are filter lengths of $H_{ma}(z)$ and $H_{mc}(z)$, respectively, and they are either both even or both odd.

4.2 Problem Formulation

In this section, we formulate the optimization problem for designing FRM-based pipelined IIR filter with approximately linear phase.

Let the transformed frequency response of $H_a(z)$ be $\tilde{H}_a(\omega, \mathbf{b}, \mathbf{a})$ where

$$\tilde{H}_a(\omega, \mathbf{b}, \mathbf{a}) = H_a(e^{j\omega}, \mathbf{b}, \mathbf{a})e^{jd\omega}. \quad (4.8)$$

The zero-phase frequency responses of the FIR masking filters are given by

$$H_{mr}(\omega, \mathbf{f}_{mr}) = \begin{cases} h_{mr}\left(\frac{N_{mr}-1}{2}\right) + \sum_{n=1}^{\frac{N_{mr}-1}{2}} 2h_{mr}\left(\frac{N_{mr}-1}{2} - n\right) \cos(\omega n), & \text{for } N_{mr} \text{ odd} \\ \sum_{n=1}^{\frac{N_{mr}}{2}} 2h_{mr}\left(\frac{N_{mr}}{2} - n\right) \cos\left(\omega\left(n - \frac{1}{2}\right)\right), & \text{for } N_{mr} \text{ even} \end{cases} \quad (4.9)$$

where $r = a, c$ and

$$\mathbf{f}_{mr} = \begin{cases} \left[h_{mr}\left(\frac{N_{mr}-1}{2}\right), 2h_{mr}\left(\frac{N_{mr}-1}{2} - 1\right), \dots, 2h_{mr}(0) \right]^T, & \text{for } N_{mr} \text{ odd} \\ \left[2h_{mr}\left(\frac{N_{mr}}{2} - 1\right), 2h_{mr}\left(\frac{N_{mr}}{2} - 2\right), \dots, 2h_{mr}(0) \right]^T, & \text{for } N_{mr} \text{ even.} \end{cases} \quad (4.10)$$

4.2. PROBLEM FORMULATION

The transformed frequency response of the overall filter can be written as

$$\tilde{H}(\omega, \mathbf{x}) = \tilde{H}_a(\omega, \mathbf{b}, \mathbf{a})[H_{ma}(\omega, \mathbf{f}_{ma}) - H_{mc}(\omega, \mathbf{f}_{mc})] + H_{mc}(\omega, \mathbf{f}_{mc}) \quad (4.11)$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{b}^T & \mathbf{a}^T & \mathbf{f}_{ma}^T & \mathbf{f}_{mc}^T \end{bmatrix}^T. \quad (4.12)$$

$\tilde{H}(\omega, \mathbf{x})$ represents the frequency response of the overall filter $H(e^{j\omega}, \mathbf{x})$ without the linear phase term $e^{-j\omega\{Md + \max\{(N_{ma}-1)/2, (N_{mc}-1)/2\}\}}$, i.e.,

$$\tilde{H}(\omega, \mathbf{x}) = H(e^{j\omega}, \mathbf{x})e^{j\omega\{Md + \max\{(N_{ma}-1)/2, (N_{mc}-1)/2\}\}} \quad (4.13a)$$

$$\angle \tilde{H}(\omega, \mathbf{x}) = \angle H(e^{j\omega}, \mathbf{x}) + \omega\{Md + \max\{(N_{ma}-1)/2, (N_{mc}-1)/2\}\}. \quad (4.13b)$$

For the design of lowpass filters, the desired $\tilde{H}(\omega, \mathbf{x})$ is given by

$$\tilde{H}_{des}(\omega) = \begin{cases} 1 & 0 \leq \omega \leq \omega_p \\ 0 & \omega_s \leq \omega \leq \pi \end{cases} \quad (4.14)$$

where ω_p and ω_s denote the passband and stopband edges, respectively.

4.2.1 Passband Phase Error and Magnitude Error Constraints

In order to achieve a design with near linear phase and at the same time, independently control the phase error and magnitude error, the following constraints for the passband

Figure 4.2: The complex plane where each point represents a possible value of $\tilde{H}(\omega_i, \mathbf{x})$.

4.2. PROBLEM FORMULATION

should be incorporated

$$W_\varphi \left| \angle \tilde{H}(\omega, \mathbf{x}) \right| \leq \eta, \quad \forall \omega \in \Omega_p \quad (4.15a)$$

$$\left| \left| \tilde{H}(\omega, \mathbf{x}) \right| - 1 \right| \leq \eta, \quad \forall \omega \in \Omega_p \quad (4.15b)$$

where W_φ is the weight function and $\Omega_p = \{\omega_i, 1 \leq i \leq N_p\} \subset [0, \omega_p]$ is a set of N_p frequency sample points in the passband. The phase error constraint (4.15a) and the magnitude error constraint (4.15b) jointly define the feasible region $ABCD$ in Fig. 4.2. The constraint (4.15b) may be approximated by

$$\left| \tilde{H}(\omega, \mathbf{x}) - 1 \right| \leq \eta, \quad \forall \omega \in \Omega_p \quad (4.16)$$

which is the region inside the circle centered at $(1,0)$ as shown in Fig. 4.2. Nevertheless, substituting (4.15b) by (4.16) will exclude feasible regions such as EBF . A better substitution is to replace (4.15b) by the ellipse with aspect ratio $\lambda > 1$

$$\left| \text{Re}[\tilde{H}(\omega, \mathbf{x}) - 1] + j\lambda^{-1} \text{Im}[\tilde{H}(\omega, \mathbf{x}) - 1] \right| \leq \eta, \quad \forall \omega \in \Omega_p. \quad (4.17)$$

It is shown in Fig. 4.2 that the feasible region defined by (4.15a) and (4.17) is much closer to $ABCD$ than that defined by (4.15a) and (4.16). Thus, we use (4.15a) and (4.17) to constrain the passband phase error and magnitude error. Note that the method in [61] only constrains the frequency-response error in the passband using (4.16).

4.2. PROBLEM FORMULATION

4.2.2 Magnitude Constraints in Stopband and Transition Band

The stopband magnitude constraint is given by

$$W_s \left| \tilde{H}(\omega, \mathbf{x}) \right| \leq \eta, \quad \forall \omega \in \Omega_s \quad (4.18)$$

where $W_s > 0$ is a prescribed weight, and $\Omega_s = \{\omega_i, 1 \leq i \leq N_s\} \subset [\omega_s, \pi]$ is a set of N_s frequency sample points in the stopband.

We impose constraints on the magnitude response in the transition band to prevent undesirable peak

$$\left| \tilde{H}(\omega, \mathbf{x}) \right| \leq m_t, \quad \forall \omega \in \Omega_t \quad (4.19)$$

where $m_t > 0$ is a preset upper bound, and $\Omega_t = \{\omega_i, 1 \leq i \leq N_t\} \subset (\omega_p, \omega_s)$ is a set of N_t frequency sample points in the transition band.

4.2.3 Optimization Problem

Based on (4.15a), (4.17), (4.18), (4.19) and (4.3), the filter design problem at hand can be formulated as the following optimization problem.

$$\underset{\eta, \mathbf{x}}{\text{minimize}} \quad \eta \quad (4.20a)$$

$$\text{subject to} \quad W_\varphi \left| \angle \tilde{H}(\omega, \mathbf{x}) \right| \leq \eta, \quad \forall \omega \in \Omega_p \quad (4.20b)$$

$$\left| \text{Re}[\tilde{H}(\omega, \mathbf{x}) - 1] + j\lambda^{-1} \text{Im}[\tilde{H}(\omega, \mathbf{x}) - 1] \right| \leq \eta, \quad \forall \omega \in \Omega_p \quad (4.20c)$$

$$W_s \left| \tilde{H}(\omega, \mathbf{x}) \right| \leq \eta, \quad \forall \omega \in \Omega_s \quad (4.20d)$$

$$\left| \tilde{H}(\omega, \mathbf{x}) \right| \leq m_t, \quad \forall \omega \in \Omega_t \quad (4.20e)$$

$$\mathbf{Q}\mathbf{a} \leq \gamma. \quad (4.20f)$$

4.3 Solution Technique

The sequential convex programming (SCP) [110] is used to solve problem (4.20). Let the value of $\mathbf{x}$ at the start of the k th iteration be

$$\mathbf{x}_k = \begin{bmatrix} \mathbf{b}_k^T & \mathbf{a}_k^T & \mathbf{f}_{mak}^T & \mathbf{f}_{mck}^T \end{bmatrix}^T. \quad (4.21)$$

Let

$$\tilde{\Lambda}(\omega, \mathbf{x}) = \text{Re}[\tilde{H}(\omega, \mathbf{x}) - 1] + j\lambda^{-1} \text{Im}[\tilde{H}(\omega, \mathbf{x}) - 1]. \quad (4.22)$$

$\angle \tilde{H}(\omega, \mathbf{x})$ and $\tilde{\Lambda}(\omega, \mathbf{x})$ at the vicinity of $\mathbf{x}_k$ can be approximated using first-order Taylor expansion

$$\angle \tilde{H}(\omega, \mathbf{x}_k + \boldsymbol{\delta}) \approx \angle \tilde{H}(\omega, \mathbf{x}_k) + \mathbf{g}_\varphi^T(\omega, \mathbf{x}_k) \boldsymbol{\delta} \quad (4.23a)$$

$$\tilde{\Lambda}(\omega, \mathbf{x}_k + \boldsymbol{\delta}) \approx \tilde{\Lambda}(\omega, \mathbf{x}_k) + \mathbf{g}_\lambda^T(\omega, \mathbf{x}_k) \boldsymbol{\delta} \quad (4.23b)$$

where $\boldsymbol{\delta}$ is a vector of small displacements given by

$$\boldsymbol{\delta} = \begin{bmatrix} \boldsymbol{\delta}_b^T & \boldsymbol{\delta}_a^T & \boldsymbol{\delta}_{fma}^T & \boldsymbol{\delta}_{fmc}^T \end{bmatrix}^T. \quad (4.24)$$

The gradient vectors $\mathbf{g}_\varphi(\omega, \mathbf{x})$ (the derivation of $\mathbf{g}_\varphi(\omega, \mathbf{x})$ can be found in the Appendix of [12]) and $\mathbf{g}_\lambda(\omega, \mathbf{x})$ are given by

$$\mathbf{g}_\varphi(\omega, \mathbf{x}) = \text{Im} \left\{ \frac{\overline{\tilde{H}(\omega, \mathbf{x})} \mathbf{g}(\omega, \mathbf{x})}{|\tilde{H}(\omega, \mathbf{x})|^2} \right\}. \quad (4.25a)$$

$$\mathbf{g}_\lambda(\omega, \mathbf{x}) = \text{Re}[\mathbf{g}(\omega, \mathbf{x})] + j\lambda^{-1} \text{Im}[\mathbf{g}(\omega, \mathbf{x})] \quad (4.25b)$$

4.3. SOLUTION TECHNIQUE

where $\mathbf{g}(\omega, \mathbf{x})$ is the gradient of $\tilde{H}(\omega, \mathbf{x})$. $\overline{\tilde{H}}(\omega, \mathbf{x})$ denotes the complex conjugate of $\tilde{H}(\omega, \mathbf{x})$. $\mathbf{g}(\omega, \mathbf{x})$ can be expressed as

$$\mathbf{g}(\omega, \mathbf{x}) = \begin{bmatrix} [H_{ma}(\omega, \mathbf{f}_{ma}) - H_{mc}(\omega, \mathbf{f}_{mc})] \frac{\partial \tilde{H}_a(M\omega, \mathbf{b}, \mathbf{a})}{\partial \mathbf{b}} \\ [H_{ma}(\omega, \mathbf{f}_{ma}) - H_{mc}(\omega, \mathbf{f}_{mc})] \frac{\partial \tilde{H}_a(M\omega, \mathbf{b}, \mathbf{a})}{\partial \mathbf{a}} \\ \tilde{H}_a(M\omega, \mathbf{b}, \mathbf{a}) \mathbf{c}_{ma}(\omega) \\ [1 - \tilde{H}_a(M\omega, \mathbf{b}, \mathbf{a})] \mathbf{c}_{mc}(\omega) \end{bmatrix} \quad (4.26)$$

where

$$\frac{\partial \tilde{H}_a(M\omega, \mathbf{b}, \mathbf{a})}{\partial \mathbf{b}} = \frac{e^{jM\omega} \phi_n(e^{jM\omega})}{\prod_{i=1}^{\frac{L_a}{2}} [1 + \phi_d^T(e^{jM\omega}) \mathbf{a}_i]} \quad (4.27a)$$

$$\frac{\partial \tilde{H}_a(M\omega, \mathbf{b}, \mathbf{a})}{\partial \mathbf{a}} = \begin{bmatrix} \frac{\partial \tilde{H}_a(M\omega, \mathbf{b}, \mathbf{a})}{\partial a_{11}} & \frac{\partial \tilde{H}_a(M\omega, \mathbf{b}, \mathbf{a})}{\partial a_{12}} & \dots \\ \frac{\partial \tilde{H}_a(M\omega, \mathbf{b}, \mathbf{a})}{\partial a_{i1}} & \frac{\partial \tilde{H}_a(M\omega, \mathbf{b}, \mathbf{a})}{\partial a_{i2}} & \dots & \frac{\partial \tilde{H}_a(M\omega, \mathbf{b}, \mathbf{a})}{\partial a_{\lceil L_a/2 \rceil 1}} & \frac{\partial \tilde{H}_a(M\omega, \mathbf{b}, \mathbf{a})}{\partial a_{\lceil L_a/2 \rceil 2}} \end{bmatrix} \quad (4.27b)$$

$$\frac{\partial \tilde{H}_a(M\omega, \mathbf{b}, \mathbf{a})}{\partial a_{i1}} = -\frac{\tilde{H}_a(M\omega, \mathbf{b}, \mathbf{a}) e^{-jM\omega}}{1 + \phi_d^T(e^{jM\omega}) \mathbf{a}_i} \quad (4.27c)$$

$$\frac{\partial \tilde{H}_a(M\omega, \mathbf{b}, \mathbf{a})}{\partial a_{i2}} = -\frac{\tilde{H}_a(M\omega, \mathbf{b}, \mathbf{a}) e^{-j2M\omega}}{1 + \phi_d^T(e^{jM\omega}) \mathbf{a}_i}, \quad \text{for } a_{i2} \neq 0 \quad (4.27d)$$

$$\mathbf{c}_{mr}(\omega) = \begin{cases} \left[1 \quad \cos(\omega) \quad \dots \quad \cos\left(\frac{N_{mr}-1}{2}\omega\right) \right]^T, & \text{for } N_{mr} \text{ odd} \\ \left[\cos\left(\frac{1}{2}\omega\right) \quad \cos\left(\frac{3}{2}\omega\right) \quad \dots \quad \cos\left(\frac{N_{mr}-1}{2}\omega\right) \right]^T, & \text{for } N_{mr} \text{ even} \end{cases} \quad (4.27e)$$

4.3. SOLUTION TECHNIQUE

Substituting (4.23a) and (4.23b) into (4.20b) and (4.20c), respectively, we have

$$W_\varphi \|\mathbf{G}_\varphi(\mathbf{x}_k)\boldsymbol{\delta} + \mathbf{p}_\varphi(\mathbf{x}_k)\|_\infty \leq \eta \quad (4.28a)$$

$$\|\mathbf{G}_\lambda(\mathbf{x}_k)\boldsymbol{\delta} + \mathbf{p}_\lambda(\mathbf{x}_k)\|_\infty \leq \eta \quad (4.28b)$$

where

$$\mathbf{G}_\varphi(\mathbf{x}_k) = \begin{bmatrix} \mathbf{g}_\varphi(\omega_1, \mathbf{x}_k)^T \\ \vdots \\ \mathbf{g}_\varphi(\omega_{N_p}, \mathbf{x}_k)^T \end{bmatrix}, \quad \omega_i \in \Omega_p \quad (4.29a)$$

$$\mathbf{p}_\varphi(\mathbf{x}_k) = \begin{bmatrix} \angle \tilde{H}(\omega_1, \mathbf{x}_k) & \cdots & \angle \tilde{H}(\omega_{N_p}, \mathbf{x}_k) \end{bmatrix}^T, \quad \omega_i \in \Omega_p \quad (4.29b)$$

$$\mathbf{G}_\lambda(\mathbf{x}_k) = \begin{bmatrix} \mathbf{g}_\lambda(\omega_1, \mathbf{x}_k)^T \\ \vdots \\ \mathbf{g}_\lambda(\omega_{N_p}, \mathbf{x}_k)^T \end{bmatrix}, \quad \omega_i \in \Omega_p \quad (4.29c)$$

$$\mathbf{p}_\lambda(\mathbf{x}_k) = \begin{bmatrix} \tilde{\Lambda}(\omega_1, \mathbf{x}_k) & \cdots & \tilde{\Lambda}(\omega_{N_p}, \mathbf{x}_k) \end{bmatrix}^T, \quad \omega_i \in \Omega_p. \quad (4.29d)$$

Constraints (4.18) and (4.19) at the vicinity of $\mathbf{x}_k$ can be approximated as

$$W_s \|\mathbf{G}_s(\mathbf{x}_k)\boldsymbol{\delta} + \mathbf{s}(\mathbf{x}_k)\|_\infty \leq \eta \quad (4.30a)$$

$$\|\mathbf{G}_t(\mathbf{x}_k)\boldsymbol{\delta} + \mathbf{t}(\mathbf{x}_k)\|_\infty \leq m_t \quad (4.30b)$$

4.3. SOLUTION TECHNIQUE

where

$$\mathbf{G}_s(\mathbf{x}_k) = \begin{bmatrix} \mathbf{g}(\omega_1, \mathbf{x}_k)^T \\ \vdots \\ \mathbf{g}(\omega_{N_s}, \mathbf{x}_k)^T \end{bmatrix}, \quad \omega_i \in \Omega_s \quad (4.31a)$$

$$\mathbf{s}(\mathbf{x}_k) = \begin{bmatrix} \tilde{H}(\omega_1, \mathbf{x}_k) & \cdots & \tilde{H}(\omega_{N_s}, \mathbf{x}_k) \end{bmatrix}^T, \quad \omega_i \in \Omega_s \quad (4.31b)$$

$$\mathbf{G}_t(\mathbf{x}_k) = \begin{bmatrix} \mathbf{g}(\omega_1, \mathbf{x}_k)^T \\ \vdots \\ \mathbf{g}(\omega_{N_t}, \mathbf{x}_k)^T \end{bmatrix}, \quad \omega_i \in \Omega_t \quad (4.31c)$$

$$\mathbf{t}(\mathbf{x}_k) = \begin{bmatrix} \tilde{H}(\omega_1, \mathbf{x}_k) & \cdots & \tilde{H}(\omega_{N_t}, \mathbf{x}_k) \end{bmatrix}^T, \quad \omega_i \in \Omega_t. \quad (4.31d)$$

The optimization problem to be solved in the k th iteration is the minimization of η subject to constraints (4.28a), (4.28b), (4.30a), (4.30b), the pole radius constraint and an additional step size constraint, as summarized in (4.32)

$$\underset{\eta, \boldsymbol{\delta}}{\text{minimize}} \quad \eta \quad (4.32a)$$

$$\text{subject to} \quad W_\varphi \|\mathbf{G}_\varphi(\mathbf{x}_k)\boldsymbol{\delta} + \mathbf{p}_\varphi(\mathbf{x}_k)\|_\infty \leq \eta \quad (4.32b)$$

$$\|\mathbf{G}_\lambda(\mathbf{x}_k)\boldsymbol{\delta} + \mathbf{p}_\lambda(\mathbf{x}_k)\|_\infty \leq \eta \quad (4.32c)$$

$$W_s \|\mathbf{G}_s(\mathbf{x}_k)\boldsymbol{\delta} + \mathbf{s}(\mathbf{x}_k)\|_\infty \leq \eta \quad (4.32d)$$

$$\|\mathbf{G}_t(\mathbf{x}_k)\boldsymbol{\delta} + \mathbf{t}(\mathbf{x}_k)\|_\infty \leq m_t \quad (4.32e)$$

$$\mathbf{Q}\boldsymbol{\delta}_a \leq \boldsymbol{\gamma} - \mathbf{Q}\mathbf{a}_k \quad (4.32f)$$

$$\|\boldsymbol{\delta}\|_\infty \leq \beta. \quad (4.32g)$$

The value of β is chosen sufficiently small such that the linear approximation is valid.

4.3. SOLUTION TECHNIQUE

Problem (4.32) can be solved efficiently using convex optimization software [109]. Let the convergence tolerance and the maximum allowed number of iterations be ε ($\varepsilon > 0$) and K_{max} , respectively. The filter design is carried out by the following algorithm.

Step 1: Find an initial solution $\mathbf{x}_1$. Let $k = 1$.

Step 2: Compute $\mathbf{G}_\lambda(\mathbf{x}_k)$, $\mathbf{p}_\lambda(\mathbf{x}_k)$, $\mathbf{G}_\varphi(\mathbf{x}_k)$, $\mathbf{p}_\varphi(\mathbf{x}_k)$, $\mathbf{G}_s(\mathbf{x}_k)$, $\mathbf{s}(\mathbf{x}_k)$, $\mathbf{G}_t(\mathbf{x}_k)$ and $\mathbf{t}(\mathbf{x}_k)$.

Step 3: Solve problem (4.32) for η and δ . $\mathbf{x} = \mathbf{x}_k + \delta$.

Step 4: When $k \geq 2$, if $|\eta - \eta_0|/\eta_0 \leq \varepsilon$, return $\mathbf{x}$. If $k = K_{max}$, return $\mathbf{x}$.

Step 5: $\mathbf{x}_{k+1} = \mathbf{x}$, $k = k + 1$ and $\eta_0 = \eta$. Go to Step 2.

A good initial point $\mathbf{x}_0$ can be obtained using the following procedure.

Step 1: Compute the bandedges of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ using the formulas in [17].

Step 2: Initialize $H_{ma}(z)$ and $H_{mc}(z)$ using Remez exchange algorithm.

Step 3: For the IIR filter $H_a(z)$, the denominator $\mathbf{a}$ is initialized as a zero vector.

The numerator $\mathbf{b}$ is initialized by solving the following convex optimization problem

$$\underset{\mathbf{b}}{\text{minimize}} \quad \left\| \tilde{H}_a(\omega, \mathbf{b}) - H_{ades}(\omega) \right\|_\infty, \quad \omega \in \Omega_{pa} \cup \Omega_{sa} \quad (4.33)$$

where Ω_{pa} and Ω_{sa} are passband and stopband frequency sample points, respectively, and $H_{ades}(\omega)$ equals unity in the passband of $H_a(z)$ and equals zero in the stopband of $H_a(z)$.

4.4 Design Examples

Two design examples are given in this section to demonstrate our method.

4.4.1 Example One

In this example, we choose the design specifications in Example 2 of [59] where an FRM-based pipelined IIR filter with approximately linear phase is designed. The bandedge shaping filters in [59] consist of a parallel connection of a pure delay line and an allpass filter. The design specifications are: $\omega_p = 0.35\pi$, $\omega_s = 0.4\pi$, maximum passband peak-to-peak ripple = 0.2 dB, minimum stopband attenuation = 40 dB and maximum passband phase error = 0.01 rad. When the number of pipeline stages in the feedback loop is 4 ($M = 4$), by using simultaneous optimization, a pipelined IIR filter with 0.2 dB maximum passband peak-to-peak ripple, 49.4 dB minimum stopband attenuation and 0.01 rad maximum passband phase error is obtained in [59]. The order of the IIR bandedge shaping filter is 15 and the lengths of FIR masking filters are 18 for both. Since the IIR bandedge shaping filter is a parallel connection of a pure delay line and an allpass filter, it requires 8 multipliers. The total number of multipliers is 26. The system delay of this filter is 36.5 ($7 \times 4 + 8.5$). If a linear phase FIR filter is designed to meet the same specifications, the filter length is 91. The number of multipliers is 46 and the group delay is 45.

We present three designs to compare with [59]. In all the designs, we choose $M = 4$ (the same number of pipeline stages as [59]), $m_t = 1$, $r = 0.9$, $\lambda = 10$, $\beta = 0.01$ and $\varepsilon = 1 \times 10^{-6}$. Based on the ripples of the design in [59], W_s is determined as

$$W_s = \frac{1 - 10^{-0.1/20}}{10^{-49.4/20}} \approx 3.3782.$$

4.4. DESIGN EXAMPLES

For the filter in [59], the ratio of the maximum passband ripple to the maximum passband phase error is given by

$$W_{\varphi 0} = \frac{1 - 10^{-0.1/20}}{0.01} \approx 1.1447.$$

In this example, W_{φ} is chosen as a multiple of $W_{\varphi 0}$. We use 875, 10 and 1500 frequency sample points in the passband, transition band and stopband, respectively, which are evenly distributed in the respective frequency region.

Design I: In this design, we choose $d = 7$, $N_a = 9$, $L_a = 4$, $N_{ma} = 10$, $N_{mc} = 14$ and $W_{\varphi} = 1.2 \times W_{\varphi 0}$. After 107 iterations, our design algorithm converges to a filter with 0.179 dB maximum passband peak-to-peak ripple, 50.33 dB minimum stopband attenuation and 0.00748 rad maximum passband phase error. The magnitude responses, passband phase error and passband group delay of the designed filter are plotted in Fig. 4.3. The filter coefficients are shown in Table B.1 (in Appendix B).

Design II: In this design, we choose $d = 7$, $N_a = 13$, $L_a = 4$, $N_{ma} = 10$, $N_{mc} = 14$ and $W_{\varphi} = 10 \times W_{\varphi 0}$. After 164 iterations, our design algorithm produces a filter with 0.184 dB maximum passband peak-to-peak ripple, 49.99 dB minimum stopband attenuation and 0.00095 rad maximum passband phase error. It can be seen that by using a larger weight for the phase error, the filter with very small phase error is obtained. The magnitude responses, passband phase error and passband group delay of the designed filter are plotted in Fig. 4.4. The filter coefficients are shown in Table B.2 (in Appendix B).

Design III: The filter with reduced delay is considered in this design. We choose $d = 5$, $N_a = 10$, $L_a = 4$, $N_{ma} = 10$, $N_{mc} = 14$, and $W_{\varphi} = W_{\varphi 0}$. After 119 iterations, our design algorithm produces a filter with 0.181 dB maximum passband peak-to-peak ripple, 50.13 dB minimum stopband attenuation and 0.00908 rad maximum passband

4.4. DESIGN EXAMPLES

Table 4.1: Performance Comparison of Designs I-III in Example One and the Filter in [59]

	T	δ_{pp} (dB)	δ_s (dB)	δ_φ (rad)	D	$r_{\max}$
Design I	26	0.179	50.33	0.00748	34.5	0.877
Design II	30	0.184	49.99	0.00095	34.5	0.9
Design III	27	0.181	50.13	0.00908	26.5	0.9
[59]	26	0.2	49.4	0.01	36.5	N/A

phase error. The magnitude responses, passband phase error and passband group delay of the designed filter are plotted in Fig. 4.5. The filter coefficients are shown in Table B.3 (in Appendix B).

The comparisons of Designs I-III in Example One and the filter in [59] are summarized in Table 4.1 where T , δ_{pp} , δ_s and δ_φ , D and $r_{\max}$ represents the total number of multipliers, maximum passband peak-to-peak ripple, minimum stopband attenuation, maximum passband phase error, system delay and maximum pole radius, respectively. It can be seen from Table 4.1 that all our three designs have smaller values of δ_{pp} , δ_s and δ_φ than [59]. The number of multipliers of Design I is the same as [59]. The maximum phase error of Design II is only 9.5% of that of [59] at the cost of 4 additional multipliers. The system delay of Design III is 72.6% of that of [59] at the cost of 1 additional multiplier.

4.4. DESIGN EXAMPLES

Figure 4.3: (a) Magnitude response, (b) passband ripple, (c) passband phase error and (d) passband group delay of the designed FRM filter in Design I of Example One.

4.4. DESIGN EXAMPLES

Figure 4.4: (a) Magnitude response, (b) passband ripple, (c) passband phase error and (d) passband group delay of the designed FRM filter in Design II of Example One.

4.4. DESIGN EXAMPLES

Figure 4.5: (a) Magnitude response, (b) passband ripple, (c) passband phase error and (d) passband group delay of the designed FRM filter in Design III of Example One.

4.4.2 Example Two

In this example, we consider the following design specifications: $\omega_p = 0.6\pi$, $\omega_s = 0.61\pi$, $M = 9$, maximum passband ripple = 0.01 (0.0864 dB), minimum stopband attenuation = 40 dB and maximum passband phase error = 0.01 rad. We present two designs and they are compared with the design presented in Section V-E of [61] where the design of IIR FRM filter with the same bandedges are considered. In the example of [61], $M = 9$, $d = 9$, $N_a = 14$, $L_a = 10$, $N_{ma} = 41$ and $N_{mc} = 33$. The designed filter in [61] has a maximum passband ripple of 0.0775 dB and a minimum stopband attenuation of 40.8921 dB. The maximum pole radius of $H_a(z)$ is 0.9487. In our designs, we choose $M = 9$, $d = 9$, $W_s = 1$, $m_t = 1$, $r = 0.9487$, $\lambda = 10$, $\beta = 5 \times 10^{-3}$ and $\varepsilon = 1 \times 10^{-6}$. We use 1500, 4 and 1000 frequency sample points in the passband, transition band and stopband, respectively, which are evenly distributed in the respective frequency region.

Design I: In this design, we choose $N_a = 16$, $L_a = 8$, $N_{ma} = 37$, $N_{mc} = 25$ and $W_\varphi = 1$. After 269 iterations, our design algorithm produces a filter with 0.0737 dB maximum passband ripple, 41.1030 dB minimum stopband attenuation and 0.00846 rad maximum passband phase error. The magnitude responses, passband phase error and passband group delay of the designed filter are plotted in Fig. 4.6. The filter coefficients are shown in Tables B.4 and B.5 (in Appendix B).

Design II: In this design, we choose $N_a = 22$, $L_a = 8$, $N_{ma} = 41$, $N_{mc} = 25$ and $W_\varphi = 2$. After 265 iterations, our design algorithm produces a filter with 0.0764 dB maximum passband ripple, 41.4143 dB minimum stopband attenuation and 0.00434 rad maximum passband phase error. The magnitude responses, passband phase error and passband group delay of the designed filter are plotted in Fig. 4.7. The filter coefficients are shown in Tables B.6 and B.7 (in Appendix B).

4.5. CONCLUSION

Table 4.2: Performance Comparison of Designs I and II in Example Two and the Filter in [61]

	T	δ_p (dB)	δ_s (dB)	δ_φ (rad)	D	$r_{\max}$
Design I	57	0.0737	41.1030	0.00846	99	0.9487
Design II	65	0.0764	41.4143	0.00434	101	0.9487
[61]	63	0.0775	40.8921	N/A	101	0.9487

The comparisons of Designs I and II in Example Two and the filter in [61] are summarized in Table 4.2 where δ_p is the maximum passband ripple. As can be seen from Table 4.2, our designs have smaller passband ripple and larger stopband attenuation compared with [61]. Design I requires fewer multipliers and has shorter system delay than [61]. The maximum phase error of Design II is 51.3% of that of Design I. The smaller phase error is achieved at the cost of 8 extra multipliers.

If a linear phase FIR filter is designed to meet the same specifications of Example Two, the filter length is 389. The number of multipliers required is 195 and the group delay is 194. If FIR FRM technique is used, the number of multipliers can be reduced drastically, however, the group delay will be larger than 194. Thus, compared to FIR FRM filters, the approximately linear phase IIR FRM filters designed in Example Two can achieve at least 46.4% reduction in system delay.

4.5 Conclusion

In this chapter, we have presented a constrained optimization method for the design of FRM-based pipelined IIR filters with approximately linear phase, which independently controls the phase error and magnitude error. The proposed method produces FRM-based pipelined IIR filters with very small phase error.

4.5. CONCLUSION

Figure 4.6: (a) Magnitude response, (b) passband ripple, (c) passband phase error and (d) passband group delay of the designed FRM filter in Design I of Example Two.

4.5. CONCLUSION

Figure 4.7: (a) Magnitude response, (b) passband ripple, (c) passband phase error and (d) passband group delay of the designed FRM filter in Design II of Example Two.

Chapter 5

A Class of IIR Filters Synthesized Using Frequency-Response Masking Technique

Shen and Lim [63] proposed a FRM-based IIR filter where all the subfilters are IIR filters consisting of a parallel connection of two allpass filters. The allpass filters are optimized jointly. However, the optimization process is often trapped in undesired local optimum solutions. In this chapter, we present a new class of FRM-based IIR filters with IIR subfilters. In the proposed filter class, the bandedge shaping filters are a parallel connection of two allpass filters and the masking filters are approximately linear phase IIR filters composed of a parallel connection of a delay line and an allpass filter. This type of approximately linear phase IIR filters was originally introduced in [124]. It has been used as masking filters for the design of low delay FRM filter [62]. Using approximately linear phase IIR filters as masking filters instead of using nonlinear phase IIR filters as masking filters like [63] enables better initial solutions

Figure 5.1: The distance from the outermost poles to the unit circle ($\rho_{\min}$) versus transition width plot for an elliptic IIR lowpass filter with passband edge fixed at 0.4π , maximum passband peak-to-peak ripple 0.1 dB and minimum stopband attenuation 60 dB. The discontinuity of the plot is caused by a change in filter order.

for joint optimization because the bounds on the magnitude response for the entire passband and stopband are available from ripple analysis. This leads to much better designs than [63] where it is hard to choose a good initial point.

For a very sharp IIR filter, the distance from its outermost poles to the unit circle is roughly proportional to its transition width, as illustrated in Fig. 5.1. Thus, the outermost poles of very sharp IIR filters are extremely close to the unit circle which results in very high coefficient sensitivity and roundoff noise magnification [1]. We will show that the coefficient sensitivity and roundoff noise of very sharp IIR filters can be significantly reduced if they are designed using the FRM technique with IIR subfilters. A numerical example demonstrates that the new class of FRM-based IIR filters has

5.1. FILTER STRUCTURE

much better coefficient sensitivity and roundoff noise performances compared with classical IIR filters.

The remaining of this chapter is organized as follows. Section 5.1 presents the structure of the proposed FRM-based IIR filter. The magnitude response ripple of the proposed filter is analyzed in Section 5.2. Section 5.3 introduces a joint optimization approach for designing the filter. In the FRM technique, each delay of the prototype bandedge shaping filter is replaced by M delays. The relationship between M and the total number of multipliers is discussed in Section 5.4. The coefficient sensitivity and roundoff noise performance are analyzed in Section 5.5. The comparison of the proposed class of FRM-based IIR filters and the FRM-based pipelined IIR filters is presented in Section 5.6.

5.1 Filter Structure

The structure of the proposed class of FRM-based IIR filters is shown in Fig. 5.2. The z -transform transfer functions of the prototype bandedge shaping filters $\{H_a(z), H_c(z)\}$ and the masking filters $\{H_{ma}(z), H_{mc}(z)\}$ are given by

$$H_a(z) = \frac{A_1(z) + A_2(z)}{2} \quad (5.1a)$$

$$H_c(z) = \frac{A_1(z) - A_2(z)}{2} \quad (5.1b)$$

$$H_{mi}(z) = \frac{z^{-K_{mi}} + A_{mi}(z)}{2}, \quad i = a, c \quad (5.1c)$$

where $A_x(z)$ ($x = 1, 2, ma, mc$) is the transfer function of an allpass filter whose frequency response magnitude $|A_x(e^{j\omega})| = 1$ and K_{mi} is a positive integer. $H_a(z)$ and $H_c(z)$ are a pair of power complementary filters. The phase response of the passband of $H_{mi}(z)$ is approximately $-K_{mi}\omega$. Let $K_m = \max(K_{ma}, K_{mc})$. The

5.1. FILTER STRUCTURE

Figure 5.2: The structure of the proposed class of FRM-based IIR filters. $K_m = \max(K_{ma}, K_{mc})$. (a) The overall filter. (b) The masking filter $H_{ma}(z)$. (c) The masking filter $H_{mc}(z)$.

phase responses of the masking filters can be made approximately equal by inserting $K_m - K_{mi}$ delays to $H_{mi}(z)$.

If the order of $A_x(z)$ is N_x , the structure in Fig. 5.2 requires $N_1 + N_2 + N_{ma} + N_{mc}$ multiplications to compute each output sample if the structure implementing the subfilters are canonic in the number of multipliers. In the case of synthesizing lowpass filters, $H_a(z)$ and $H_c(z)$ are lowpass and highpass filters, respectively, and $H_{ma}(z)$ and $H_{mc}(z)$ are lowpass filters. Thus, N_1 and N_2 differ by one (we use $N_2 = N_1 + 1$) [125] and $N_{mi} = K_{mi} + 1$ ($i = a, c$) [124]. The allpass filters can be implemented using any implementation structure such as the low coefficient sensitivity wave digital filter structure or the lattice structure or in a straightforward manner using the cascade of first- and second-order sections.

5.2. RIPPLE ANALYSIS

In cascade form, the transfer function $A_x(z)$ may be written as

$$A_x(z) = z^{-N_x} \prod_{i=1}^{\lceil N_x/2 \rceil} \frac{1 + a_{i1}^{(x)} z + a_{i2}^{(x)} z^2}{1 + a_{i1}^{(x)} z^{-1} + a_{i2}^{(x)} z^{-2}} \quad (5.2)$$

where $\lceil N_x/2 \rceil$ is the smallest integer greater than or equal to $N_x/2$. When N_x is odd, $a_{12}^{(x)} = 0$. $A_x(z)$ is stable if and only if $a_{i1}^{(x)}$ and $a_{i2}^{(x)}$ satisfy the following constraints [4], i.e.,

$$a_{i1}^{(x)} - a_{i2}^{(x)} < 1 \quad (5.3a)$$

$$-a_{i1}^{(x)} - a_{i2}^{(x)} < 1 \quad (5.3b)$$

$$a_{i2}^{(x)} < 1. \quad (5.3c)$$

5.2 Ripple Analysis

In this section, we analyze the magnitude response ripple of the proposed FRM-based IIR filter. The results are important for initializing the joint optimization algorithm, as will be discussed later in Section 5.3.3.

The frequency response of the overall filter, $H(e^{j\omega})$, can be written as

$$H(e^{j\omega}) = H_a(e^{jM\omega})H_{ma}(e^{j\omega})e^{-j(K_m-K_{ma})\omega} + H_c(e^{jM\omega})H_{mc}(e^{j\omega})e^{-j(K_m-K_{mc})\omega}. \quad (5.4)$$

$H_a(e^{jM\omega})$, $H_c(e^{jM\omega})$, $H_{ma}(e^{j\omega})$ and $H_{mc}(e^{j\omega})$ can be expressed as

$$H_i(e^{jM\omega}) = |H_i(e^{jM\omega})|e^{j\phi_i(M\omega)}, \quad i = a, c \quad (5.5a)$$

$$H_{mi}(e^{j\omega}) = |H_{mi}(e^{j\omega})|e^{j\phi_{mi}(\omega)}, \quad i = a, c. \quad (5.5b)$$

Figure 5.3: Magnitude responses of the subfilters in Case A.

where $|H_a(e^{jM\omega})|$, $|H_c(e^{jM\omega})|$, $|H_{ma}(e^{j\omega})|$ and $|H_{mc}(e^{j\omega})|$ are the magnitudes of $H_a(e^{jM\omega})$, $H_c(e^{jM\omega})$, $H_{ma}(e^{j\omega})$ and $H_{mc}(e^{j\omega})$, respectively. They are bounded by unity. $\phi_a(M\omega)$, $\phi_c(M\omega)$, $\phi_{ma}(\omega)$ and $\phi_{mc}(\omega)$ are the phases of $H_a(e^{jM\omega})$, $H_c(e^{jM\omega})$, $H_{ma}(e^{j\omega})$ and $H_{mc}(e^{j\omega})$, respectively.

The transition band of $H(z)$ may be formed by one of the transition bands of $H_a(z^M)$ or $H_c(z^M)$. We refer to these two cases as Case A and Case B, respectively.

Figure 5.4: Magnitude responses of the subfilters in Case B.

The magnitude responses of the subfilters in Case A and Case B may be that shown in Fig. 5.3 and Fig. 5.4, respectively. Given M and the bandedges $\{\omega_p, \omega_s\}$ of $H(z)$, the bandedges $\{\omega_{px}, \omega_{sx}\}$ of $H_x(z)$ ($x = a, c, ma, mc$) can be computed [17]. Let the maximum passband and stopband ripple magnitudes of $H_x(z)$ and $H(z)$ be $\{\delta_{px}, \delta_{sx}\}$ and $\{\delta_p, \delta_s\}$, respectively. Note that even though the magnitude responses of subfilters are bounded by unity, the magnitude response of the overall filter may not be bounded

Figure 5.5: Magnitude response illustrations for (a) $H_x(z)$ where $x = a, ma, mc$ and (b) $H(z)$.

by unity. The magnitude responses of $H_x(z)$ and $H(z)$ are illustrated in Fig. 5.5.

The phase responses of the power complementary filters $H_a(z^M)$ and $H_c(z^M)$ satisfy

$$\phi_a(M\omega) - \phi_c(M\omega) = -\frac{\pi}{2}. \quad (5.6)$$

5.2. RIPPLE ANALYSIS

Thus, the squared magnitude response $|H(e^{j\omega})|^2$ can be written as

$$|H(e^{j\omega})|^2 = U(\omega) + V(\omega) \quad (5.7a)$$

$$U(\omega) = |H_a(e^{jM\omega})|^2 |H_{ma}(e^{j\omega})|^2 + |H_c(e^{jM\omega})|^2 |H_{mc}(e^{j\omega})|^2 \quad (5.7b)$$

$$\begin{aligned} V(\omega) = & 2|H_a(e^{jM\omega})||H_{ma}(e^{j\omega})||H_c(e^{jM\omega})||H_{mc}(e^{j\omega})| \\ & \times \sin(\phi_{ma}(\omega) + K_{ma}\omega - \phi_{mc}(\omega) - K_{mc}\omega). \end{aligned} \quad (5.7c)$$

The upper and lower bounds on $U(\omega)$ are given by

$$\min(|H_{ma}(e^{j\omega})|^2, |H_{mc}(e^{j\omega})|^2) \leq U(\omega) \leq \max(|H_{ma}(e^{j\omega})|^2, |H_{mc}(e^{j\omega})|^2) \leq 1. \quad (5.8)$$

Since $|V(\omega)| \leq 2|H_a(e^{jM\omega})||H_{ma}(e^{j\omega})||H_c(e^{jM\omega})||H_{mc}(e^{j\omega})|$, we have

$$|H(e^{j\omega})| \geq ||H_a(e^{jM\omega})||H_{ma}(e^{j\omega})| - |H_c(e^{jM\omega})||H_{mc}(e^{j\omega})|| \quad (5.9a)$$

$$\begin{aligned} |H(e^{j\omega})| & \leq |H_a(e^{jM\omega})||H_{ma}(e^{j\omega})| + |H_c(e^{jM\omega})||H_{mc}(e^{j\omega})| \\ & \leq |H_a(e^{jM\omega})| + \sqrt{1 - |H_a(e^{jM\omega})|^2} \\ & \leq \sqrt{2} \end{aligned} \quad (5.9b)$$

Let $\delta_a(M\omega)$, $\delta_c(M\omega)$, $\delta_{ma}(\omega)$, $\delta_{mc}(\omega)$ and $\delta(\omega)$ be the ripple functions defined by the

5.2. RIPPLE ANALYSIS

following equalities

$$|H_a(e^{jM\omega})| = 1 - \delta_a(M\omega), \text{ in passbands} \quad (5.10a)$$

$$|H_a(e^{jM\omega})| = \delta_a(M\omega), \text{ in stopbands} \quad (5.10b)$$

$$|H_c(e^{jM\omega})| = 1 - \delta_c(M\omega), \text{ in passbands} \quad (5.10c)$$

$$|H_c(e^{jM\omega})| = \delta_c(M\omega), \text{ in stopbands} \quad (5.10d)$$

$$|H_{ma}(e^{j\omega})| = 1 - \delta_{ma}(\omega), \omega \in [0, \omega_{pma}] \quad (5.10e)$$

$$|H_{ma}(e^{j\omega})| = \delta_{ma}(\omega), \omega \in [\omega_{sma}, \pi] \quad (5.10f)$$

$$|H_{mc}(e^{j\omega})| = 1 - \delta_{mc}(\omega), \omega \in [0, \omega_{pmc}] \quad (5.10g)$$

$$|H_{mc}(e^{j\omega})| = \delta_{mc}(\omega), \omega \in [\omega_{smc}, \pi] \quad (5.10h)$$

$$|H(e^{j\omega})| = 1 + \delta(\omega), \omega \in [0, \omega_p] \quad (5.10i)$$

$$|H(e^{j\omega})| = \delta(\omega), \omega \in [\omega_s, \pi]. \quad (5.10j)$$

Note that $\delta_a(M\omega)$, $\delta_c(M\omega)$, $\delta_{ma}(\omega)$ and $\delta_{mc}(\omega)$ are larger than or equal to zero.

Let the phase responses of allpass filters $A_{ma}(z)$ and $A_{mc}(z)$ be $\varphi_{ma}(\omega)$ and $\varphi_{mc}(\omega)$, respectively. Then, we have

$$\phi_{mi}(\omega) = \frac{\varphi_{mi}(\omega) - K_{mi}\omega}{2}, \quad i = a, c \quad (5.11a)$$

$$|H_{mi}(e^{j\omega})| = \left| \cos\left(\frac{\varphi_{mi}(\omega) + K_{mi}\omega}{2}\right) \right|, \quad i = a, c. \quad (5.11b)$$

Let the ripples of $\varphi_{ma}(\omega)$ and $\varphi_{mc}(\omega)$ be $\delta_{\varphi_{ma}}(\omega)$ and $\delta_{\varphi_{mc}}(\omega)$, respectively. The

5.2. RIPPLE ANALYSIS

phase responses of $A_{ma}(z)$ and $A_{mc}(z)$ can be written as

$$\varphi_{mi}(\omega) = -K_{mi}\omega + \delta_{\varphi mi}(\omega), \quad \omega \in [0, \omega_{pmi}] \quad (5.12a)$$

$$\varphi_{mi}(\omega) = -K_{mi}\omega - \pi + \delta_{\varphi mi}(\omega), \quad \omega \in [\omega_{smi}, \pi]. \quad (5.12b)$$

For $\omega \in [0, \omega_{pmi}]$,

$$|H_{mi}(e^{j\omega})| = \left| \cos\left(\frac{\delta_{\varphi mi}(\omega)}{2}\right) \right| \approx 1 - \frac{\delta_{\varphi mi}^2(\omega)}{8}. \quad (5.13)$$

For $\omega \in [\omega_{smi}, \pi]$,

$$|H_{mi}(e^{j\omega})| = \left| \cos\left(\frac{\delta_{\varphi mi}(\omega) - \pi}{2}\right) \right| \approx \left| \sin\left(\frac{\delta_{\varphi mi}(\omega)}{2}\right) \right| \approx \frac{|\delta_{\varphi mi}(\omega)|}{2}. \quad (5.14)$$

Thus, we have

$$\delta_{mi}(\omega) \approx \frac{\delta_{\varphi mi}^2(\omega)}{8}, \quad \omega \in [0, \omega_{pmi}] \quad (5.15a)$$

$$\delta_{mi}(\omega) \approx \frac{|\delta_{\varphi mi}(\omega)|}{2}, \quad \omega \in [\omega_{smi}, \pi]. \quad (5.15b)$$

We analyze the ripple function of the overall filter, $\delta(\omega)$, in two frequency regions as shown in Fig. 5.6 (for Case A, $\omega_{pma} = \omega_p$ and $\omega_{smc} = \omega_s$; for Case B, $\omega_{pmc} = \omega_p$ and $\omega_{sma} = \omega_s$). The analysis for Case A is carried out first.

Frequency region I: $[0, \omega_{pmc}] \cup [\omega_{sma}, \pi]$. In this frequency region,

$$\begin{aligned} \sin(\phi_{ma}(\omega) + K_{ma}\omega - \phi_{mc}(\omega) - K_{mc}\omega) &= \sin\left(\frac{\delta_{\varphi ma}(\omega) - \delta_{\varphi mc}(\omega)}{2}\right) \\ &\approx \frac{\delta_{\varphi ma}(\omega) - \delta_{\varphi mc}(\omega)}{2}. \end{aligned} \quad (5.16)$$

Denote the frequency region consisting of all the transition bands of $H_a(z^M)$ as Λ . The

Figure 5.6: Frequency ranges I and II for the analysis of $\delta(\omega)$ in (a) Case A and (b) Case B.

frequency region $[0, \omega_{pmc}]$ can be divided into two parts, $[0, \omega_{pmc}] \setminus \Lambda$ and $\Lambda \cap [0, \omega_{pmc}]$. The frequency regions Λ and $[0, \omega_{pmc}] \setminus \Lambda$ are illustrated in Fig. 5.7. For $\omega \in [0, \omega_{pmc}]$,

$$\begin{aligned}
 V(\omega) &\approx |H_a(e^{jM\omega})| |H_c(e^{jM\omega})| \{1 - \delta_{ma}(\omega)\} \{1 - \delta_{mc}(\omega)\} \{\delta_{\varphi ma}(\omega) - \delta_{\varphi mc}(\omega)\} \\
 &\approx |H_a(e^{jM\omega})| |H_c(e^{jM\omega})| \{\delta_{\varphi ma}(\omega) - \delta_{\varphi mc}(\omega)\}.
 \end{aligned} \tag{5.17}$$

 Figure 5.7: Illustrations of frequency regions Λ and $[0, \omega_{pmc}] \setminus \Lambda$ in Case A.

For $\omega \in [0, \omega_{pmc}] \setminus \Lambda$,

$$|H(e^{j\omega})|^2 \leq 1 + V(\omega) \quad (5.18a)$$

$$|H(e^{j\omega})|^2 \geq \min(|H_{ma}(e^{j\omega})|^2, |H_{mc}(e^{j\omega})|^2) + V(\omega) \quad (5.18b)$$

The value of $V(\omega)$ is very small in this frequency region and it is insignificant compared to unity and $\min(|H_{ma}(e^{j\omega})|^2, |H_{mc}(e^{j\omega})|^2)$. Thus, we have

$$\min(|H_{ma}(e^{j\omega})|, |H_{mc}(e^{j\omega})|) \leq |H(e^{j\omega})| \leq 1 + \frac{V(\omega)}{2} \quad (5.19)$$

i.e.,

$$-\max\left(\frac{\delta_{\varphi ma}^2(\omega)}{8}, \frac{\delta_{\varphi mc}^2(\omega)}{8}\right) \leq \delta(\omega) \leq \frac{V(\omega)}{2} \quad (5.20)$$

5.2. RIPPLE ANALYSIS

For $\omega \in \Lambda \cap [0, \omega_{pmc}]$,

$$|H(e^{j\omega})|^2 \leq 1 + |H_a(e^{jM\omega})||H_c(e^{jM\omega})|(|\delta_{\varphi ma}(\omega)| + |\delta_{\varphi mc}(\omega)|) \quad (5.21a)$$

$$\begin{aligned} |H(e^{j\omega})|^2 &\geq \min(|H_{ma}(e^{j\omega})|^2, |H_{mc}(e^{j\omega})|^2) \\ &\quad - |H_a(e^{jM\omega})||H_c(e^{jM\omega})|(|\delta_{\varphi ma}(\omega)| + |\delta_{\varphi mc}(\omega)|) \\ &\approx 1 - \max\left(\frac{\delta_{\varphi ma}^2(\omega)}{8}, \frac{\delta_{\varphi mc}^2(\omega)}{8}\right) \\ &\quad - |H_a(e^{jM\omega})||H_c(e^{jM\omega})|(|\delta_{\varphi ma}(\omega)| + |\delta_{\varphi mc}(\omega)|) \\ &\approx 1 - |H_a(e^{jM\omega})||H_c(e^{jM\omega})|(|\delta_{\varphi ma}(\omega)| + |\delta_{\varphi mc}(\omega)|) \end{aligned} \quad (5.21b)$$

From $|H_a(e^{jM\omega})|^2 + |H_c(e^{jM\omega})|^2 = 1$, we can obtain

$$|H_a(e^{jM\omega})||H_c(e^{jM\omega})| \leq \frac{1}{2}. \quad (5.22)$$

Thus, we have

$$1 - \frac{1}{4}(|\delta_{\varphi ma}(\omega)| + |\delta_{\varphi mc}(\omega)|) \leq |H(e^{j\omega})| \leq 1 + \frac{1}{4}(|\delta_{\varphi ma}(\omega)| + |\delta_{\varphi mc}(\omega)|). \quad (5.23)$$

i.e.,

$$|\delta(\omega)| \leq \frac{1}{4}(|\delta_{\varphi ma}(\omega)| + |\delta_{\varphi mc}(\omega)|). \quad (5.24)$$

For $\omega \in [\omega_{sma}, \pi]$,

$$V(\omega) \approx |H_a(e^{jM\omega})||H_c(e^{jM\omega})|\delta_{ma}(\omega)\delta_{mc}(\omega)\{\delta_{\varphi ma}(\omega) - \delta_{\varphi mc}(\omega)\}. \quad (5.25)$$

5.2. RIPPLE ANALYSIS

The upper bound on $|H(e^{j\omega})|^2$ is given by

$$|H(e^{j\omega})|^2 \leq \max(|H_{ma}(e^{j\omega})|^2, |H_{mc}(e^{j\omega})|^2) + V(\omega). \quad (5.26)$$

The value of $V(\omega)$ is insignificant compared to $\max(|H_{ma}(e^{j\omega})|^2, |H_{mc}(e^{j\omega})|^2)$. Thus,

$$|H(e^{j\omega})| \leq \max(|H_{ma}(e^{j\omega})|, |H_{mc}(e^{j\omega})|) \quad (5.27)$$

i.e.,

$$\delta(\omega) \leq \max\left(\frac{|\delta_{\varphi ma}(\omega)|}{2}, \frac{|\delta_{\varphi mc}(\omega)|}{2}\right). \quad (5.28)$$

Frequency region II: $[\omega_{pmc}, \omega_p] \cup [\omega_s, \omega_{sma}]$.

For $\omega \in [\omega_{pmc}, \omega_p]$, (5.9) leads to

$$|H(e^{j\omega})| \leq 1 + |H_c(M\omega)| = 1 + \delta_c(M\omega) \quad (5.29a)$$

$$\begin{aligned} |H(e^{j\omega})| &\geq |H_a(e^{jM\omega})||H_{ma}(e^{j\omega})| - |H_c(e^{jM\omega})| \\ &= \{1 - \delta_a(M\omega)\}\{1 - \delta_{ma}(\omega)\} - \delta_c(M\omega) \\ &\approx 1 - \delta_a(M\omega) - \delta_{ma}(\omega) - \delta_c(M\omega) \end{aligned} \quad (5.29b)$$

In this frequency region, $\delta_a(M\omega)$ and $\delta_c(M\omega)$ are related by

$$\{1 - \delta_a(M\omega)\}^2 + \delta_c^2(M\omega) = 1. \quad (5.30)$$

Thus, we have

$$\delta_a(M\omega) \approx \frac{\delta_c^2(M\omega)}{2}. \quad (5.31)$$

5.2. RIPPLE ANALYSIS

Taking (5.31) into consideration, the bounds on $|H(e^{j\omega})|$ is given by

$$1 - \delta_{ma}(\omega) - \delta_c(M\omega) \leq |H(e^{j\omega})| \leq 1 + \delta_c(M\omega) \quad (5.32)$$

i.e.,

$$-\frac{\delta_{\varphi ma}^2(\omega)}{8} - \delta_c(M\omega) \leq \delta(\omega) \leq \delta_c(M\omega). \quad (5.33)$$

For $\omega \in [\omega_s, \omega_{sma}]$, (5.9) leads to

$$|H(e^{j\omega})| \leq |H_a(M\omega)| + |H_{mc}(\omega)| \quad (5.34)$$

i.e.,

$$\delta(\omega) \leq \delta_a(M\omega) + \frac{|\delta_{\varphi mc}(\omega)|}{2}. \quad (5.35)$$

Using the same approach, the bounds on $\delta(\omega)$ can be derived for Case B. The results are summarized as follows.

For $\omega \in [0, \omega_{pma}] \setminus \Lambda$,

$$-\max\left(\frac{\delta_{\varphi ma}^2(\omega)}{8}, \frac{\delta_{\varphi mc}^2(\omega)}{8}\right) \leq \delta(\omega) \leq \frac{V(\omega)}{2} \quad (5.36)$$

For $\omega \in \Lambda \cap [0, \omega_{pma}]$,

$$|\delta(\omega)| \leq \frac{1}{4}(|\delta_{\varphi ma}(\omega)| + |\delta_{\varphi mc}(\omega)|). \quad (5.37)$$

5.3. JOINT OPTIMIZATION OF IIR SUBFILTERS

For $\omega \in [\omega_{smc}, \pi]$,

$$\delta(\omega) \leq \max\left(\frac{|\delta_{\varphi ma}(\omega)|}{2}, \frac{|\delta_{\varphi mc}(\omega)|}{2}\right). \quad (5.38)$$

For $\omega \in [\omega_{pma}, \omega_p]$,

$$-\frac{\delta_{\varphi mc}^2(\omega)}{8} - \delta_a(M\omega) \leq \delta(\omega) \leq \delta_a(M\omega). \quad (5.39)$$

For $\omega \in [\omega_s, \omega_{smc}]$,

$$\delta(\omega) \leq \delta_c(M\omega) + \frac{|\delta_{\varphi ma}(\omega)|}{2}. \quad (5.40)$$

5.3 Joint Optimization of IIR Subfilters

5.3.1 Problem Formulation

The frequency response of $A_x(z)$ ($x = 1, 2, ma, mc$) can be written as $A_x(e^{j\omega}) = e^{-jN_x\omega} \tilde{A}_x(e^{j\omega})$ where

$$\tilde{A}_x(e^{j\omega}) = \prod_{i=1}^{\lceil N_x/2 \rceil} \frac{1 + a_{i1}^{(x)} e^{j\omega} + a_{i2}^{(x)} e^{j2\omega}}{1 + a_{i1}^{(x)} e^{-j\omega} + a_{i2}^{(x)} e^{-j2\omega}}. \quad (5.41)$$

We formulate $\tilde{A}_x(e^{j\omega})$ in terms of the coefficients of the second order sections because the necessary and sufficient stability constraints [4] can be easily incorporated. Denote the coefficient vector of $A_x(z)$ as $\mathbf{c}_x$ where

$$\mathbf{c}_x = \begin{bmatrix} a_{11}^{(x)} & a_{12}^{(x)} & \cdots & a_{i1}^{(x)} & a_{i2}^{(x)} & \cdots & a_{\lceil N_x/2 \rceil 1}^{(x)} & a_{\lceil N_x/2 \rceil 2}^{(x)} \end{bmatrix}^T. \quad (5.42)$$

5.3. JOINT OPTIMIZATION OF IIR SUBFILTERS

Let the coefficient vector of the overall filter be

$$\mathbf{c} = \begin{bmatrix} \mathbf{c}_1^T & \mathbf{c}_2^T & \mathbf{c}_{ma}^T & \mathbf{c}_{mc}^T \end{bmatrix}^T. \quad (5.43)$$

The frequency response of $H(z)$ can be written as

$$H(e^{j\omega}, \mathbf{c}) = e^{-j(MN_1 + K_m)\omega} \tilde{H}(e^{j\omega}, \mathbf{c}) \quad (5.44)$$

where

$$\tilde{H}(e^{j\omega}, \mathbf{c}) = \tilde{H}_a(e^{jM\omega}, \mathbf{c}_1, \mathbf{c}_2) \tilde{H}_{ma}(e^{j\omega}, \mathbf{c}_{ma}) + \tilde{H}_c(e^{jM\omega}, \mathbf{c}_1, \mathbf{c}_2) \tilde{H}_{mc}(e^{j\omega}, \mathbf{c}_{mc}) \quad (5.45a)$$

$$\tilde{H}_a(e^{jM\omega}, \mathbf{c}_1, \mathbf{c}_2) = \frac{1}{2} \{ \tilde{A}_1(e^{jM\omega}, \mathbf{c}_1) + e^{-jM\omega} \tilde{A}_2(e^{jM\omega}, \mathbf{c}_2) \} \quad (5.45b)$$

$$\tilde{H}_c(e^{jM\omega}, \mathbf{c}_1, \mathbf{c}_2) = \frac{1}{2} \{ \tilde{A}_1(e^{jM\omega}, \mathbf{c}_1) - e^{-jM\omega} \tilde{A}_2(e^{jM\omega}, \mathbf{c}_2) \} \quad (5.45c)$$

$$\tilde{H}_{ma}(e^{j\omega}, \mathbf{c}_{ma}) = \frac{1}{2} \{ 1 + e^{-j\omega} \tilde{A}_{ma}(e^{j\omega}, \mathbf{c}_{ma}) \} \quad (5.45d)$$

$$\tilde{H}_{mc}(e^{j\omega}, \mathbf{c}_{mc}) = \frac{1}{2} \{ 1 + e^{-j\omega} \tilde{A}_{mc}(e^{j\omega}, \mathbf{c}_{mc}) \}. \quad (5.45e)$$

Let Ω_p and Ω_s be the sets of grid points placed in the passband and stopband, respectively. The joint optimization of IIR subfilters can be formulated as

$$\underset{\delta, \mathbf{c}}{\text{minimize}} \quad \delta \quad (5.46a)$$

$$\text{subject to} \quad W(\omega) ||\tilde{H}(e^{j\omega}, \mathbf{c})|^2 - 1| \leq \delta, \quad \forall \omega \in \Omega_p \quad (5.46b)$$

$$W(\omega) |\tilde{H}(e^{j\omega}, \mathbf{c})| \leq \delta, \quad \forall \omega \in \Omega_s \quad (5.46c)$$

$$\mathbf{Q}_x \mathbf{c}_x < \mathbf{v}_x, \quad x = 1, 2, ma, mc \quad (5.46d)$$

where

$$\mathbf{Q}_x = \begin{bmatrix} \mathbf{D} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D} \end{bmatrix} \quad (5.47)$$

$$\mathbf{D} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 0 & 1 \end{bmatrix} \quad (5.48)$$

$$\mathbf{v}_x = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T \quad (5.49)$$

$W(\omega)$ is a prescribed weight function. The dimension of $\mathbf{Q}_x$ is $3 \lceil N_x/2 \rceil \times 2 \lceil N_x/2 \rceil$ and the dimension of $\mathbf{v}_x$ is $3 \lceil N_x/2 \rceil \times 1$. Constraints (5.46d) ensure the stability of the allpass filter $A_x(z)$.

5.3.2 Solution Technique

We adopt the sequential convex programming (SCP) approach [110] to find a solution to problem (5.46). The SCP is an iterative method where a convex approximation of the original problem is solved in each iteration.

Let $\mathbf{c}(k)$ be the coefficient vector at the start of the k th iteration. For small values of $|c - c(k)|$, $|\tilde{H}(e^{j\omega}, \mathbf{c})|$ and $\tilde{H}(e^{j\omega}, \mathbf{c})$ can be approximated using first-order Taylor expansion

$$|\tilde{H}(e^{j\omega}, \mathbf{c})|^2 \approx |\tilde{H}(e^{j\omega}, \mathbf{c}(k))|^2 + \nabla^T \{|\tilde{H}(e^{j\omega}, \mathbf{c}(k))|^2\} \cdot (\mathbf{c} - \mathbf{c}(k)) \quad (5.50a)$$

$$\tilde{H}(e^{j\omega}, \mathbf{c}) \approx \tilde{H}(e^{j\omega}, \mathbf{c}(k)) + \nabla^T \tilde{H}(e^{j\omega}, \mathbf{c}(k)) \cdot (\mathbf{c} - \mathbf{c}(k)). \quad (5.50b)$$

5.3. JOINT OPTIMIZATION OF IIR SUBFILTERS

$\nabla\{|\tilde{H}(e^{j\omega}, \mathbf{c})|^2\}$ can be computed from $\tilde{H}(e^{j\omega}, \mathbf{c})$ and $\nabla\tilde{H}(e^{j\omega}, \mathbf{c})$ (see Eq. (3.21)).

$\nabla\tilde{H}(e^{j\omega}, \mathbf{c})$ is given by

$$\nabla\tilde{H}(e^{j\omega}, \mathbf{c}) = \begin{bmatrix} \frac{\partial\tilde{H}(e^{j\omega}, \mathbf{c})}{\partial\mathbf{c}_1^T} & \frac{\partial\tilde{H}(e^{j\omega}, \mathbf{c})}{\partial\mathbf{c}_2^T} & \frac{\partial\tilde{H}(e^{j\omega}, \mathbf{c})}{\partial\mathbf{c}_{ma}^T} & \frac{\partial\tilde{H}(e^{j\omega}, \mathbf{c})}{\partial\mathbf{c}_{mc}^T} \end{bmatrix}^T \quad (5.51a)$$

$$\frac{\partial\tilde{H}(e^{j\omega}, \mathbf{c})}{\partial\mathbf{c}_1} = \frac{1}{2}\{\tilde{H}_{ma}(e^{j\omega}, \mathbf{c}_{ma}) + \tilde{H}_{mc}(e^{j\omega}, \mathbf{c}_{mc})\}\nabla\tilde{A}_1(e^{jM\omega}, \mathbf{c}_1) \quad (5.51b)$$

$$\frac{\partial\tilde{H}(e^{j\omega}, \mathbf{c})}{\partial\mathbf{c}_2} = \frac{e^{-jM\omega}}{2}\{\tilde{H}_{ma}(e^{j\omega}, \mathbf{c}_{ma}) - \tilde{H}_{mc}(e^{j\omega}, \mathbf{c}_{mc})\}\nabla\tilde{A}_2(e^{jM\omega}, \mathbf{c}_2) \quad (5.51c)$$

$$\frac{\partial\tilde{H}(e^{j\omega}, \mathbf{c})}{\partial\mathbf{c}_{ma}} = \frac{e^{-j\omega}}{2}\tilde{H}_a(e^{jM\omega}, \mathbf{c}_1, \mathbf{c}_2)\nabla\tilde{A}_{ma}(e^{j\omega}, \mathbf{c}_{ma}) \quad (5.51d)$$

$$\frac{\partial\tilde{H}(e^{j\omega}, \mathbf{c})}{\partial\mathbf{c}_{mc}} = \frac{e^{-j\omega}}{2}\tilde{H}_c(e^{jM\omega}, \mathbf{c}_1, \mathbf{c}_2)\nabla\tilde{A}_{mc}(e^{j\omega}, \mathbf{c}_{mc}) \quad (5.51e)$$

The expression for $\nabla\tilde{A}_x(e^{j\omega}, \mathbf{c}_x)$ can be found in Eq. (3.24).

The approximations in (5.50) are good approximations only when $\mathbf{c}$ is close to $\mathbf{c}(k)$. Thus, we may impose constraint $\|\mathbf{c} - \mathbf{c}(k)\|_\infty \leq \beta$ on $\mathbf{c}$ where β is a prescribed small number.

5.3. JOINT OPTIMIZATION OF IIR SUBFILTERS

Define two matrices $\mathbf{G}_{pk}$ and $\mathbf{G}_{sk}$ and two column vectors $\mathbf{e}_{pk}$ and $\mathbf{e}_{sk}$ as

$$\mathbf{G}_{pk} = \begin{bmatrix} W(\omega_1) \nabla^T \{ |\tilde{H}(e^{j\omega_1}, \mathbf{c}(k))|^2 \} \\ \vdots \\ W(\omega_{L_p}) \nabla^T \{ |\tilde{H}(e^{j\omega_{L_p}}, \mathbf{c}(k))|^2 \} \end{bmatrix}, \omega_i \in \Omega_p \quad (5.52a)$$

$$\mathbf{e}_{pk} = \begin{bmatrix} W(\omega_1) \{ |\tilde{H}(e^{j\omega_1}, \mathbf{c}(k))|^2 - 1 \} \\ \vdots \\ W(\omega_{L_p}) \{ |\tilde{H}(e^{j\omega_{L_p}}, \mathbf{c}(k))|^2 - 1 \} \end{bmatrix}, \omega_i \in \Omega_p \quad (5.52b)$$

$$\mathbf{G}_{sk} = \begin{bmatrix} W(\omega_1) \nabla^T \tilde{H}(e^{j\omega_1}, \mathbf{c}(k)) \\ \vdots \\ W(\omega_{L_s}) \nabla^T \tilde{H}(e^{j\omega_{L_s}}, \mathbf{c}(k)) \end{bmatrix}, \omega_i \in \Omega_s \quad (5.52c)$$

$$\mathbf{e}_{sk} = \begin{bmatrix} W(\omega_1) \tilde{H}(e^{j\omega_1}, \mathbf{c}(k)) \\ \vdots \\ W(\omega_{L_s}) \tilde{H}(e^{j\omega_{L_s}}, \mathbf{c}(k)) \end{bmatrix}, \omega_i \in \Omega_s \quad (5.52d)$$

where L_p and L_s are the number of passband and stopband grid points, respectively.

The optimization problem to be solved in the k th iteration can be written as

$$\underset{\delta, \mathbf{c}}{\text{minimize}} \quad \delta \quad (5.53a)$$

$$\text{subject to} \quad \|\mathbf{G}_{pk} \mathbf{c} + \mathbf{d}_{pk}\|_\infty \leq W(\omega) \delta \quad (5.53b)$$

$$\|\mathbf{G}_{sk} \mathbf{c} + \mathbf{d}_{sk}\|_\infty \leq W(\omega) \delta \quad (5.53c)$$

$$\mathbf{Q}_x \mathbf{c}_x < \mathbf{v}_x, \quad x = 1, 2, ma, mc \quad (5.53d)$$

$$\|\mathbf{c} - \mathbf{c}(k)\|_\infty \leq \beta \quad (5.53e)$$

where

$$\mathbf{d}_{pk} = -\mathbf{G}_{pk}\mathbf{c}(k) + \mathbf{e}_{pk} \quad (5.54a)$$

$$\mathbf{d}_{sk} = -\mathbf{G}_{sk}\mathbf{c}(k) + \mathbf{e}_{sk} \quad (5.54b)$$

If the relative difference of the objective function values in two consecutive iterations is less than a prescribed tolerance ε , the optimization process is terminated. The optimization process may also be terminated when a prescribed number of iterations, K_{max} , have been carried out. Let δ_0 be the objective function value at the start of the k th iteration. The algorithm for designing the proposed class of FRM-based IIR filters is summarized as follows.

Step 1: Find an initial solution $\mathbf{c}(1)$. Let $k = 1$.

Step 2: Compute $\mathbf{G}_{pk}$, $\mathbf{G}_{sk}$, $\mathbf{d}_{pk}$ and $\mathbf{d}_{sk}$.

Step 3: Solve problem (5.53) for δ and $\mathbf{c}$.

Step 4: When $k \geq 2$, if $|\delta - \delta_0|/\delta_0 \leq \varepsilon$, return $\mathbf{c}$. If $k = K_{max}$, return $\mathbf{c}$.

Step 5: $k = k + 1$, $\mathbf{c}(k) = \mathbf{c}$ and $\delta_0 = \delta$. Go to Step 2.

The solution obtained by our design algorithm may be a local optimum. Thus, it is important to generate a good initial solution. A good initialization technique will be introduced in the following section.

5.3.3 Initial Solution Generation

Consider the design of a lowpass filter with passband edge ω_p , stopband edge ω_s , maximum passband ripple δ_p and maximum stopband ripple δ_s . After the bandedges

5.3. JOINT OPTIMIZATION OF IIR SUBFILTERS

of $H_a(z)[H_c(z)]$, $H_{ma}(z)$ and $H_{mc}(z)$ are computed, $A_{mi}(z)$ ($i = a, c$) is initialized such that its phase response approximates $-K_{mi}\omega$ and $-K_{mi}\omega - \pi$ for $\omega \in [0, \omega_{pmi}]$ and $\omega \in [\omega_{smi}, \pi]$, respectively. We use the method in [127] to design $A_{mi}(z)$. As shown in Section 5.2, the ripple of $H(z)$ is bounded by $\max(\delta_{\varphi_{ma}}^2(\omega)/8, \delta_{\varphi_{mc}}^2(\omega)/8)$ and $\max(|\delta_{\varphi_{ma}}(\omega)|/2, |\delta_{\varphi_{mc}}(\omega)|/2)$ in most part of its passband and stopband, respectively. Based on this, a good estimation for N_{mi} can be obtained by the following procedure.

Step 1: Set $N_{miu} = 1$ and $N_{mil} = 1$.

Step 2: Solve the following feasibility problem with $N_{mi} = N_{miu}$

$$\text{minimize} \quad 0 \tag{5.55}$$

$$\text{subject to} \quad |\delta_{\varphi_{mi}}(\omega)| \leq 2\sqrt{2\delta_p}, \quad \forall \omega \in [0, \omega_{pmi}] \tag{5.56}$$

$$|\delta_{\varphi_{mi}}(\omega)| \leq 2\delta_s, \quad \forall \omega \in [\omega_{smi}, \pi]. \tag{5.57}$$

Step 3: If problem (5.55) is feasible, go to Step 4. Otherwise, let $N_{mil} = N_{miu}$ and $N_{miu} = 2 \times N_{miu}$. Go to Step 2.

Step 4: Perform a bisection search between N_{miu} and N_{mil} to find the minimum N_{mi} such that problem (5.55) is feasible. Return N_{mi} .

The above procedure finds the lowest-order $A_{mi}(z)$ that makes the problem in (5.55) feasible.

The prototype bandedge shaping filter can be initialized as an odd-order elliptic filter. We choose the elliptic filter with bandedges $\{\omega_{pa}, \omega_{sa}\}$ and stopband ripple magnitude δ_s to initialize $H_a(z)$ and $H_c(z)$ for Case A and Case B, respectively. Coefficients of $A_1(z)$ and $A_2(z)$ are obtained based on the poles of the elliptic filter.

5.3. JOINT OPTIMIZATION OF IIR SUBFILTERS

Note that the initial solution of [63] often has enormous peaks in the frequency regions corresponding to the transition bands of the bandedge shaping filters within the common passband of the masking filters ($\omega \in \Lambda \cap [0, \min(\omega_{pma}, \omega_{pmc})]$). The proposed filter class does not have this problem because the ripple of $H(z)$ is bounded by $(|\delta_{\varphi ma}(\omega)| + |\delta_{\varphi mc}(\omega)|)/4$ in those frequency regions.

5.3.4 Design Example

In this section, we present an example to demonstrate the design of the proposed class of FRM-based IIR filters.

The design specifications used in the example of [63] are: $\omega_p = 0.4\pi$, $\omega_s = 0.42\pi$, maximum passband peak-to-peak ripple = 0.4 dB and minimum stopband attenuation = 50 dB. The filter designed in [63] has a maximum passband peak-to-peak ripple of 0.256 dB and a minimum stopband attenuation of 53.6 dB. The value of M is 6. The orders of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ are 15, 9 and 9, respectively. The total number of multipliers is 33. Those filter orders are very high because the design in [63] is only a local optimum. In our design, we use the same value of M . The orders of $H_a(z)$, $A_{ma}(z)$ and $A_{mc}(z)$ are determined as 7, 6 and 8, respectively. We use 2000 and 2900 evenly spaced grid points in the passband and stopband, respectively (the bandedges are included in the grid points). Our optimization algorithm converges in 30 iterations. The magnitude responses of the subfilters and overall filter in our design are shown in Figs. 5.8 and 5.9, respectively. The coefficient values of the second-order sections are shown in Table C.1 (in Appendix C). The maximum passband peak-to-peak ripple is 0.128 dB and the minimum stopband attenuation is 57.49 dB. The total number of multipliers of our filter is 21. Our design achieves a 36.4% saving in the total number of multipliers compared to that of [63].

Figure 5.8: Magnitude response of the subfilters in the design example of Section 5.3.4. (a) Bandedge shaping filters $H_a(z^M)$ (solid line) and $H_c(z^M)$ (dashed line). (b) Masking filters $H_{ma}(z)$ (solid line) and $H_{mc}(z)$ (dashed line).

Figure 5.9: Magnitude response of the FRM-based IIR filter in the design example of Section 5.3.4. (a) Overall response. (b) Passband response.

5.4 Relationship Between M and Total Number of Multipliers

The bandedges of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ are determined by M for a given set of the bandedges of the overall filter. Thus, M will affect the number of multipliers of these filters. For the proposed class of FRM-based IIR filters, the total number of multipliers N is $N_1 + N_2 + N_{ma} + N_{mc}$ where N_1 , N_2 , N_{ma} and N_{mc} are the orders of $A_1(z)$, $A_2(z)$, $A_{ma}(z)$, $A_{mc}(z)$, respectively. Let the order of $H_a(z)$ be N_{Ha} , i.e., $N_{Ha} = N_1 + N_2$. N_{Ha} is insensitive to changes in the value of M . Thus, N is affected mainly by $N_{ma} + N_{mc}$ for different values of M . There is no estimation formula available in the literature for determining N_{mi} , $i = a, c$. In the following, we use simulation examples to illustrate the relationship between N_{mi} and bandedges.

Consider the design of $A_{mi}(z)$ whose peak phase response ripple magnitude is 0.01 in both $[0, \omega_{pmi}]$ and $[\omega_{smi}, \pi]$. We calculate the minimum value of N_{mi} using the procedure in Section 5.3.3 for two cases. In the first case, ω_{pmi} is fixed at 0.25π and the transition width, $\Delta_{mi} = \omega_{smi} - \omega_{pmi}$, is varying from 0.01π to 0.5π . In the second case, Δ_{mi} is fixed at 0.2π and ω_{pmi} is varying from 0.1π to 0.7π . A number of 1000 designs are carried out for each case. Figs. 5.10(a) and 5.10(b) show the N_{mi} versus $1/\Delta_{mi}$ plot and the N_{mi} versus ω_{pmi} plot, respectively. It can be observed from Fig. 5.10 that N_{mi} is roughly inversely proportional to Δ_{mi} and N_{mi} is relatively independent of the location of the transition band.

The sum of the transition widths of $H_{ma}(z)$ and $H_{mc}(z)$, $\Delta_{ma} + \Delta_{mc}$, is $2\pi/M$. When M increases, $\Delta_{ma} + \Delta_{mc}$ decreases. Thus, $N_{ma} + N_{mc}$ increases, to the first degree of approximation, proportionately with M . At some values of M , one of Δ_{ma} and Δ_{mc} is very small and the other is very large. For those cases, $N_{ma} + N_{mc}$ may be very large. Thus, $N_{ma} + N_{mc}$ does not increase monotonically. Since N_{Ha} is very

5.4. RELATIONSHIP BETWEEN M AND TOTAL NUMBER OF MULTIPLIERS

Figure 5.10: (a) N_{mi} versus $1/\Delta_{mi}$ for $A_{mi}(z)$ with $\omega_{pmi} = 0.25\pi$. (b) N_{mi} versus ω_{pmi} for $A_{mi}(z)$ with $\Delta_{mi} = 0.2\pi$. For both plots, the peak phase response ripple magnitude of $A_{mi}(z)$ is 0.01 in both $[0, \omega_{pmi}]$ and $[\omega_{smi}, \pi]$.

5.4. RELATIONSHIP BETWEEN M AND TOTAL NUMBER OF MULTIPLIERS

Figure 5.11: N_{Ha} , $N_{ma} + N_{mc}$ and N versus M .

slightly affected as M changes, the total number of multipliers N roughly increases with increasing M . Note that N does not increase monotonically with increasing M .

For the purpose of illustration, we choose the design of a FRM-based IIR filter with $\omega_p = 0.4\pi$, $\omega_s = 0.402\pi$, maximum passband peak-to-peak ripple = 0.2 dB and minimum stopband attenuation = 60 dB. We design 10 filters with M ranging from 3 to 14 (the designs with $M = 5, 10$ are not feasible). The plots of N_{Ha} , $N_{ma} + N_{mc}$ and N versus M are shown in Fig. 5.11. As can be seen from Fig. 5.11, N roughly increases with increasing M .

5.5 Coefficient Sensitivity and Roundoff Noise Analysis

For an IIR filter synthesized using a system of IIR subfilters based on the FRM technique, the poles of $H_a(z)$, $H_c(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ are further away from the unit circle compared with the conventional IIR filter designed without using the FRM technique because $H_a(z)$, $H_c(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ have much wider transition widths. Thus, the coefficient sensitivity and roundoff noise performances of $H_a(z)$, $H_c(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ are much better than those of the conventional IIR filters. $H_a(z^M)[H_c(z^M)]$ is obtained from $H_a(z)[H_c(z)]$ by replacing one delay by M delays. If $H_a(z)[H_c(z)]$ has a pole p close to the unit circle and its distance to the unit circle is ρ , $H_a(z^M)[H_c(z^M)]$ will have M poles with a distance of $1 - (1 - \rho)^{1/M} \approx \rho/M$ to the unit circle. When the filter coefficients are quantized, only the deviation of one of those M poles, denoted as λ_k , will significantly affect the overall magnitude response. The magnitude response changes of $H_a(z^M)$ and $H_c(z^M)$ due to the deviations of the other $M - 1$ poles occur either in the passbands of the masking filters which has a marginal effect on the overall magnitude response (for the power complementary bandedge shaping filters, $(|H_a(e^{jM\omega})| + \Delta|H_a(e^{jM\omega})|)^2 + (|H_c(e^{jM\omega})| + \Delta|H_c(e^{jM\omega})|)^2 = 1$), or in the stopbands of the masking filters which are suppressed by the masking filters. The pole λ_k is less sensitive to coefficient quantization than the pole p because when the distance of p to the unit circle changes by $\Delta\rho$, the distance of λ_k to the unit circle changes by only $\Delta\rho/M$. It can be seen that the FRM-based IIR filter has lower coefficient sensitivity than the conventional IIR filter. The roundoff noise gain of the FRM-based IIR filter can be written as a weighted sum of the roundoff noise gains of $H_a(z^M)$, $H_c(z^M)$, $H_{ma}(z)$ and $H_{mc}(z)$. The roundoff noise gain of $H_a(z^M)[H_c(z^M)]$ is the same as that of $H_a(z)[H_c(z)]$ [118] which is much smaller than that of the conventional IIR filter. The roundoff noise gains of $H_{ma}(z)$ and $H_{mc}(z)$ are also

much smaller than the conventional IIR filter. Thus, the FRM-based IIR filter has smaller roundoff noise gain.

In the following, we present a numerical example to compare the coefficient sensitivity and roundoff noise of the proposed class of FRM-based IIR filters with the classical IIR filters, i.e., Butterworth, Chebyshev, inverse Chebyshev, and elliptic filters. Note that the magnitude response of the Chebyshev (inverse Chebyshev) filter is equiripple (monotonically decreasing) in the passband and monotonically decreasing (equiripple) in the stopband.

Numerical Example:

Consider the design of a lowpass filter with $\omega_p = 0.8\pi$, $\omega_s = 0.801\pi$, maximum passband peak-to-peak ripple = 0.2 dB and minimum stopband attenuation = 50 dB. The classical IIR filters are designed as a parallel connection of two allpass filters¹. The orders of Butterworth, Chebyshev, inverse Chebyshev and elliptic filters are 1361, 79, 79 and 13, respectively. For the FRM-based IIR filter with $M = 7$, the orders of $H_a(z)$, $A_{ma}(z)$ and $A_{mc}(z)$ are 13, 5 and 8, respectively. The coefficient values of the second-order sections are shown in Table C.2 (in Appendix C).

The coefficient sensitivity of a digital filter can be measured using a statistical approach [38, 130, 131]. Let $c[n]$ ($n = 1, 2, \dots, N$) be the n th coefficient value of filter $H(z)$. When $c[n]$ is quantized to discrete values, $c[n]$ changes to $c[n] + \Delta c[n]$ and the magnitude response $|H(e^{j\omega})|$ changes to $|H(e^{j\omega})| + \Delta|H(e^{j\omega})|$. If the coefficient quantization step size is Q_c ,

$$E\{(\Delta c[n])^2\} = \frac{Q_c^2}{12} \quad (5.58)$$

¹The Chebyshev, inverse Chebyshev and elliptic filters are designed using Tapio Saramäki's classical IIR filter design program.

5.5. COEFFICIENT SENSITIVITY AND ROUND-OFF NOISE ANALYSIS

where $E\{x\}$ denotes the expected value of x . Assume that for $i \neq j$,

$$E\{\Delta c[i]\Delta c[j]\} = 0. \quad (5.59)$$

The magnitude response deviation due to coefficient quantization can be measured by $E\{(\Delta|H(e^{j\omega})|)^2\}$. In the passband of $H(z)$ ($|H(e^{j\omega})| \approx 1$),

$$(\Delta|H(e^{j\omega})|)^2 \approx \frac{1}{4}\{\Delta(|H(e^{j\omega})|^2)\}^2 \quad (5.60)$$

because

$$\begin{aligned} \Delta(|H(e^{j\omega})|^2) &\equiv \{|H(e^{j\omega})| + \Delta|H(e^{j\omega})|\}^2 - |H(e^{j\omega})|^2 \\ &= 2|H(e^{j\omega})|\Delta|H(e^{j\omega})| + (\Delta|H(e^{j\omega})|)^2 \\ &\approx 2\Delta|H(e^{j\omega})|. \end{aligned} \quad (5.61)$$

In the stopband of $H(z)$ ($|H(e^{j\omega})|$ is very small),

$$(\Delta|H(e^{j\omega})|)^2 \approx |\Delta H(e^{j\omega})|^2. \quad (5.62)$$

because

$$\begin{aligned} |\Delta H(e^{j\omega})| &\equiv |\{|H(e^{j\omega})| + \Delta|H(e^{j\omega})|\}e^{\Delta\phi(\omega)} - |H(e^{j\omega})|\}| \\ &\approx |\{|H(e^{j\omega})| + \Delta|H(e^{j\omega})|\}\{\cos[\Delta\phi(\omega)] + j\sin[\Delta\phi(\omega)]\} - |H(e^{j\omega})|\}| \\ &\approx |\{|H(e^{j\omega})| + \Delta|H(e^{j\omega})|\}(1 + j\Delta\phi(\omega)) - |H(e^{j\omega})|\}| \\ &\approx |\Delta|H(e^{j\omega})| + j|H(e^{j\omega})|\Delta\phi(\omega)| \\ &\approx |\Delta|H(e^{j\omega})| \end{aligned} \quad (5.63)$$

5.5. COEFFICIENT SENSITIVITY AND ROUND-OFF NOISE ANALYSIS

where $\Delta\phi(\omega)$ denotes the phase response deviation. The value of $\Delta(|H(e^{j\omega})|^2)$ can be approximated by

$$\Delta(|H(e^{j\omega})|^2) \approx \sum_{n=1}^N \frac{\partial\{|H(e^{j\omega})|^2\}}{\partial c[n]} \Delta c[n]. \quad (5.64)$$

The value of $\Delta H(e^{j\omega})$ can be approximated by

$$\Delta H(e^{j\omega}) \approx \sum_{n=1}^N \frac{\partial H(e^{j\omega})}{\partial c[n]} \Delta c[n]. \quad (5.65)$$

Thus,

$$\mathbb{E}\{(\Delta|H(e^{j\omega})|)^2\} \approx \Gamma(\omega) \frac{Q_c^2}{12} \quad (5.66)$$

where

$$\Gamma(\omega) = \begin{cases} \frac{1}{4} \sum_{n=1}^N \left\{ \frac{\partial(|H(e^{j\omega})|^2)}{\partial c[n]} \right\}^2, & \text{for } \omega \in [0, \omega_p] \\ \sum_{n=1}^N \left| \frac{\partial H(e^{j\omega})}{\partial c[n]} \right|^2, & \text{for } \omega \in [\omega_s, \pi]. \end{cases} \quad (5.67)$$

We use

$$\sqrt{\frac{1}{\omega_p} \int_0^{\omega_p} \Gamma(\omega) d\omega} \quad (5.68)$$

and

$$\sqrt{\frac{1}{\pi - \omega_s} \int_{\omega_s}^{\pi} \Gamma(\omega) d\omega} \quad (5.69)$$

to measure the passband and stopband sensitivity, respectively.

Figure 5.12: Roundoff noise evaluation model for the proposed class of FRM-based IIR filters. z^{-*} represents $z^{-(K_m - K_{mi})}$.

The roundoff noise performance is evaluated assuming that wordlength truncation occurs only at the output of each multiplier. If the wordlength is truncated by rounding, a roundoff noise with power $Q^2/12$ is injected at the output of each multiplier, where Q is the signal quantization step size. The roundoff noise evaluation model for the proposed class of FRM-based IIR filters is shown in Fig. 5.12. From Fig. 5.12, the output noise $e(z)$ is given by

$$\begin{aligned}
 e(z) &= \{e_1(z) + e_2(z)\}H_{ma}(z)z^{-(K_m - K_{ma})} + \{e_1(z) - e_2(z)\}H_{mc}(z)z^{-(K_m - K_{mc})} \\
 &\quad + e_{ma}(z)z^{-(K_m - K_{ma})} + e_{mc}(z)z^{-(K_m - K_{mc})} \\
 &= e_1(z)\left\{H_{ma}(z)z^{-(K_m - K_{ma})} + H_{mc}(z)z^{-(K_m - K_{mc})}\right\} \\
 &\quad + e_2(z)\left\{H_{ma}(z)z^{-(K_m - K_{ma})} - H_{mc}(z)z^{-(K_m - K_{mc})}\right\} \\
 &\quad + e_{ma}(z)z^{-(K_m - K_{ma})} + e_{mc}(z)z^{-(K_m - K_{mc})}
 \end{aligned} \tag{5.70}$$

Let the noise power for $e_x(z)$ be σ_x^2 ($x = 1, 2, ma, mc$). the total noise power at the

output is given by

$$\sigma^2 = \sigma_1^2 \|\tilde{H}_{ma}(e^{j\omega}) + \tilde{H}_{mc}(e^{j\omega})\|_2^2 + \sigma_2^2 \|\tilde{H}_{ma}(e^{j\omega}) - \tilde{H}_{mc}(e^{j\omega})\|_2^2 + \sigma_{ma}^2 + \sigma_{mc}^2 \quad (5.71)$$

where

$$\|\tilde{H}_{ma}(e^{j\omega}) + \tilde{H}_{mc}(e^{j\omega})\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{H}_{ma}(e^{j\omega}) + \tilde{H}_{mc}(e^{j\omega})|^2 d\omega \quad (5.72a)$$

$$\|\tilde{H}_{ma}(e^{j\omega}) - \tilde{H}_{mc}(e^{j\omega})\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{H}_{ma}(e^{j\omega}) - \tilde{H}_{mc}(e^{j\omega})|^2 d\omega. \quad (5.72b)$$

We implement the allpass filters in the FRM-based IIR filter and the classical IIR filters as a cascade of first- and second-order allpass sections. Our choice of implementation structures for the allpass sections is those shown in Fig. 5.13 because they are canonic in both multipliers and delays. The structures in Fig. 5.13 are from Mitra and Hirano [134]. For the implementation of $A_1(z^M)$ and $A_2(z^M)$, z^{-1} is replaced by z^{-M} . When one's- or two's-complement arithmetic is used, overflow constraints are required at the inputs to the multipliers to avoid internal signal overflow [133] (in one's- or two's-complement numbering system, partial sum overflow needs not to be considered; the correct final sum will be obtained as long as the final sum does not overflow), i.e., for a first-order allpass section, $|m_{i1}(n)| \leq 1$ and for a second-order allpass section, $|m_{i1}(n)| \leq 1$ and $|m_{i2}(n)| \leq 1$. We use a scaling multiplier λ_i to scale the signal levels. Assume that the input signal to the filter is a sinusoid with unit magnitude. Its Fourier transform $X(e^{j\omega})$ satisfies

$$\|X(e^{j\omega})\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})| d\omega = 1. \quad (5.73)$$

For the purpose of analysis, λ_i is selected as the largest value such that the overflow

Figure 5.13: Implementation structures for (a) first-order allpass section and (b) second-order allpass section. λ_i is the scaling multiplier. $s_i(n)$ and $m_{il}(n)$ represent the input signals to the allpass sections and the multipliers, respectively.

constraints are satisfied. Let $T_{i1}(z)$ and $T_{i2}(z)$ be the unscaled z -transform transfer functions from $s_i(n)$ to $m_{i1}(n)$ and $m_{i2}(n)$, respectively. According to Hölder's

inequality,

$$\begin{aligned}
 |m_{il}(n)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \lambda_i T_{il}(e^{j\omega}) S_i(e^{j\omega}) e^{j\omega n} d\omega \right| \\
 &\leq \frac{\lambda_i}{2\pi} \int_{-\pi}^{\pi} |T_{il}(e^{j\omega}) S_i(e^{j\omega})| d\omega \\
 &\leq \lambda_i \|T_{il}(e^{j\omega})\|_{\infty} \|S_i(e^{j\omega})\|_1 \\
 &\leq \frac{\lambda_i}{2} \|T_{il}(e^{j\omega})\|_{\infty}
 \end{aligned} \tag{5.74}$$

where $S_i(e^{j\omega})$ is the Fourier transform of $S_i(n)$. Thus, for a second-order allpass section,

$$\lambda_i = 2 \min\{\|T_{i1}(e^{j\omega})\|_{\infty}^{-1}, \|T_{i2}(e^{j\omega})\|_{\infty}^{-1}\}. \tag{5.75}$$

where $T_{i1}(e^{j\omega})$ and $T_{i2}(e^{j\omega})$ are given by

$$T_{i1}(e^{j\omega}) = \frac{(1 - a_{i2})e^{-j\omega}}{1 + a_{i1}e^{-j\omega} + a_{i2}e^{-2j\omega}} \tag{5.76a}$$

$$T_{i2}(e^{j\omega}) = \frac{1 + a_{i1}e^{-j\omega} + e^{-2j\omega}}{1 + a_{i1}e^{-j\omega} + a_{i2}e^{-2j\omega}} \tag{5.76b}$$

and for a first-order allpass section,

$$\lambda_i = 2\|T_{i1}(e^{j\omega})\|_{\infty}^{-1} \tag{5.77}$$

where $T_{i1}(e^{j\omega})$ is given by

$$T_{i1}(e^{j\omega}) = \frac{1 - e^{-j\omega}}{1 + a_{i1}e^{-j\omega}}. \tag{5.78}$$

Eqs. (5.76) and (5.78) can be obtained using the method in [132]. Let $Q_{il}(z)$ be the

5.5. COEFFICIENT SENSITIVITY AND ROUND-OFF NOISE ANALYSIS

z -transform transfer function from the output of the multiplier a_{il} to the output of the allpass section. For the second-order allpass section,

$$Q_{i1}(z) = Q_{i2}(z) = \frac{1 - z^{-2}}{1 + a_{i1}z^{-1} + a_{i2}z^{-2}}. \quad (5.79)$$

For the first-order allpass section,

$$Q_{i1}(z) = \frac{1 + z^{-1}}{1 + a_{i1}z^{-1}}. \quad (5.80)$$

It is shown in [134] that for the second-order allpass section,

$$\|Q_{i1}(e^{j\omega})\|_2^2 = \|Q_{i2}(e^{j\omega})\|_2^2 = \frac{2}{1 - a_{i2}} \quad (5.81)$$

and for the first-order allpass section,

$$\|Q_{i1}(e^{j\omega})\|_2^2 = \frac{2}{1 + a_{i1}}. \quad (5.82)$$

Thus, σ_x^2 is given by

$$\sigma_x^2 = \frac{Q^2}{12} \sum_{i=1}^{\lceil N_x/2 \rceil} \frac{\tau_i^2}{\lambda_i^2} \quad (5.83)$$

where

$$\tau_i^2 = \begin{cases} \frac{4}{1 - a_{i2}}, & \text{for a second-order allpass section} \\ \frac{2}{1 + a_{i1}}, & \text{for a first-order allpass section.} \end{cases} \quad (5.84)$$

The coefficient sensitivity measures and roundoff noise gains for the FRM-based IIR filter and the classical IIR filters are calculated and shown in Table 5.1. As can be

5.5. COEFFICIENT SENSITIVITY AND ROUND-OFF NOISE ANALYSIS

Table 5.1: Coefficient Sensitivity and Roundoff Noise Gain Comparison for the FRM-Based IIR Filter and the Classical IIR Filters

	Butter.	Cheb.	Inv. Cheb.	Ellip.	FRM
Passband sensitivity	1.36	8.16	0.77	2.82	0.37
Stopband sensitivity	120.51	24.56	172.27	21.48	3.17
$\sigma^2/(Q^2/12)$ (dB)	42.6	40.0	43.8	34.4	23.2

seen from Table 5.1, the FRM filter has smaller sensitivity measures than the classical IIR filters. The roundoff noise gain of the FRM filter is 20 dB, 17 dB, 21 dB and 11 dB better than those of the Butterworth, Chebyshev, inverse Chebyshev and elliptic filters, respectively.

We have conducted several simulations to study the magnitude responses of the FRM-based IIR filter and the classical IIR filters with quantized coefficient values. When the coefficient value is quantized to b fractional bits, it is multiplied by 2^b , rounded to the nearest integer and then divided by 2^b . This means that more bits are used for coefficients whose magnitudes are larger than unity.

Simulation I: In this simulation, the coefficient values are quantized to 12 fractional bits. The magnitude responses of the FRM filter, Butterworth filter, Chebyshev filter, inverse Chebyshev filter and elliptic filter with coefficient values quantized to 12 fractional bits are shown in Figs. 5.14, 5.15, 5.16, 5.17 and 5.18, respectively. As can be seen in Fig. 5.14, the magnitude response of the FRM filter with quantized coefficients nearly overlaps with the ideal magnitude response (the magnitude response with infinite precision coefficient values). For the FRM filter, the maximum passband peak-to-peak ripple increases by 0.0037 dB and the minimum stopband attenuation decreases by 3.2 dB compared to the ideal magnitude response. For the Butterworth filter, the maximum passband peak-to-peak ripple increases by 0.12 dB and the minimum stopband attenuation decreases by 21.3 dB. For the Chebyshev

5.5. COEFFICIENT SENSITIVITY AND ROUND-OFF NOISE ANALYSIS

filter, the maximum passband peak-to-peak ripple increases by 0.11 dB and the minimum stopband attenuation decreases by 18.1 dB. For the inverse Chebyshev filter, the maximum passband peak-to-peak ripple decreases by 0.028 dB and the minimum stopband attenuation decreases by 16.8 dB. For the elliptic filter, the maximum passband peak-to-peak ripple increases by 0.069 dB and the minimum stopband attenuation decreases by 25.4 dB.

Simulation II: In this simulation, the coefficient values are quantized to 8 fractional bits. In this case, the Butterworth, Chebyshev, inverse Chebyshev and elliptic filters are unstable whereas the FRM filter is still stable. Note that the Chebyshev filter starts to become unstable when the coefficient values are quantized to 9 fractional bits. The magnitude response of the FRM filter with coefficient values quantized to 8 fractional bits is shown in Fig. 5.19. Its maximum passband peak-to-peak ripple increases by 0.11 dB and its minimum stopband attenuation decreases by 5.7 dB compared to the ideal magnitude response.

Simulation III: If it is required that the passband peak-to-peak ripple must be less than 0.2 dB and the stopband attenuation must be larger than 50 dB, the coefficient values of $A_1(z)$ and $A_2(z)$ in the FRM filter can be quantized to as low as 11 fractional bits and the coefficient values of $A_{ma}(z)$ and $A_{mc}(z)$ can be quantized to as low as 10 and 9 fractional bits, respectively. The magnitude response of the FRM filter with quantized coefficients is shown in Fig. 5.20. The coefficient values of the Butterworth, Chebyshev, inverse Chebyshev and elliptic filters require at least 18, 15, 19 and 17 fractional bits, respectively.

The above simulations demonstrate that the coefficient sensitivity of the FRM-based IIR filter is much lower than those of the classical IIR filters.

It can be seen from the Numerical Example that the coefficient sensitivity and

5.5. COEFFICIENT SENSITIVITY AND ROUND-OFF NOISE ANALYSIS

Table 5.2: Hardware Cost of the FRM-Based IIR Filter and the Classical IIR Filters

	Butter.	Cheb.	Inv. Cheb.	Ellip.	FRM
Multipliers	1361	79	79	13	26
Adders	2724	160	160	28	59
Delays	1361	79	79	13	118

roundoff noise performances of the FRM-based IIR filter are significantly better than those of the classical IIR filters.

The hardware cost of the FRM-based IIR filter and the classical IIR filters are compared in Table 5.2. The FRM-based IIR filter requires more multipliers, adders and delay elements than the elliptic filter, requires more delay elements but fewer multipliers and adders than the Chebyshev and inverse Chebyshev filters, and requires much fewer multipliers, adders and delay elements than the Butterworth filter. Note that each filter type has its own advantages and disadvantages comparing to another filter type. The preferred filter type to be used in an implementation depends on the specific application and the implementation requirement.

The above hardware cost comparison does not consider the case where the coefficient wordlength is fixed a priori. For an elliptic filter, if its bandedges remain the same and its order is increased, its ripples will decrease. Thus, a higher order elliptic filter with overdesigned passband and stopband ripples may be used to meet the required specifications when the coefficient wordlength is fixed a priori. However, the coefficient sensitivity and roundoff noise performances of the overdesigned elliptic filter will be worse since its poles will move towards the unit circle. If the coefficient wordlength is fixed a priori as 12 fractional bits and the frequency response requirement is that the passband peak-to-peak ripple must be less than 0.2 dB and the stopband attenuation must be larger than 50 dB, the FRM-based IIR filter with

Figure 5.14: (a) Passband and (b) stopband magnitude responses of the proposed FRM-based IIR filter with coefficient values quantized to 12 fractional bits (dashed line). The ideal magnitude response is shown in solid line.

coefficient values shown in Table C.2 (in Appendix C) can meet the requirements. The requirements can also be met by an overdesigned elliptic filter with 6.4×10^{-4} dB passband peak-to-peak ripple and 154.87 dB stopband attenuation. This elliptic filter requires 35 multipliers which is 34.6% larger than the FRM-based IIR filter. The roundoff noise gain of the elliptic filter is 42.3 dB which is 19 dB worse than the FRM-based IIR filter.

Figure 5.15: (a) Passband and (b) stopband magnitude responses of the Butterworth filter with coefficient values quantized to 12 fractional bits (dashed line). The ideal magnitude response is shown in solid line.

Figure 5.16: (a) Passband and (b) stopband magnitude responses of the Chebyshev filter with coefficient values quantized to 12 fractional bits (dashed line). The ideal magnitude response is shown in solid line.

Figure 5.17: (a) Passband and (b) stopband magnitude responses of the inverse Chebyshev filter with coefficient values quantized to 12 fractional bits (dashed line). The ideal magnitude response is shown in solid line.

Figure 5.18: (a) Passband and (b) stopband magnitude responses of the elliptic filter with coefficient values quantized to 12 fractional bits (dashed line). The ideal magnitude response is shown in solid line.

Figure 5.19: (a) Passband and (b) stopband magnitude responses of the proposed FRM-based IIR filter with coefficient values quantized to 8 fractional bits (dashed line). The ideal magnitude response is shown in solid line.

Figure 5.20: (a) Passband and (b) stopband magnitude responses of the proposed FRM-based IIR filter with quantized coefficient values (dashed line). The coefficient values of $A_1(z)$ and $A_2(z)$ are quantized to 11 fractional bits and the coefficient values of $A_{ma}(z)$ and $A_{mc}(z)$ are quantized to 10 and 9 fractional bits, respectively. The ideal magnitude response is shown in solid line.

5.6 Comparison With FRM-Based Pipelined IIR Filters

In this section, we compare our design with the FRM-based pipelined IIR filter in [59] whose structure is shown in Fig. 2.5. The FRM-based pipelined IIR filter requires 43 multipliers ($M = 7$) to meet the same set of specifications if it is designed by joint optimization (the order of the IIR bandedge shaping filter is 13 and the filter lengths of the linear phase FIR masking filters $F_{ma}(z)$ and $F_{mc}(z)$ are 26 and 34, respectively). The allpass filters are also implemented using the allpass sections in Fig. 5.13. The FIR filters are implemented using the transposed direct-form structure exploiting the coefficient symmetry as shown in Fig. 5.21 where $e_i(n)$ represents the roundoff noise. The passband and stopband sensitivity measures of the FRM-based pipelined IIR filter are 5.34 and 5.81, respectively. To achieve a passband peak-to-peak ripple less than 0.2 dB and a stopband attenuation larger than 50 dB, the coefficient values of the IIR bandedge shaping filters require at least 13 fractional bits and the coefficient values of the FIR masking filters $F_{ma}(z)$ and $F_{mc}(z)$ require at least 12 and 13 fractional bits, respectively. When the coefficient values are quantized to 8 fractional bits, the maximum passband peak-to-peak ripple increases by 0.25 dB and the minimum stopband attenuation decreases by 24.7 dB compared to the ideal magnitude response. The magnitude response of the FRM-based pipelined filter with coefficient values quantized to 8 fractional bits is shown in Fig. 5.22. The total noise power at the output of the FRM-based pipelined IIR filter is given by

$$\sigma^2 = \sigma_1^2 \|F_{ma}(\omega) + F_{mc}(\omega)\|_2^2 + \sigma_2^2 \|F_{ma}(\omega) - F_{mc}(\omega)\|_2^2 + (L_{ma} + L_{mc}) \frac{Q^2}{12} \quad (5.85)$$

where $F_{ma}(\omega)$ and $F_{mc}(\omega)$ are the zero-phase frequency responses of the masking filters, and L_{ma} and L_{mc} are the filter lengths. The roundoff noise gain is calculated

Figure 5.21: Transposed structures exploiting the coefficient symmetry for implementing (a) odd-length and (b) even-length linear phase FIR filters. L is the filter length. $e_i(n)$ is the roundoff noise.

to be 25.5 dB. As can be seen, our design with IIR subfilters requires fewer multipliers and has lower coefficient sensitivity and roundoff noise gain. Note that unlike the FRM-based pipelined IIR filters, the proposed class of FRM-based IIR filters with IIR subfilters is not naturally pipelined due to the feedback loops of the IIR masking filters.

Figure 5.22: (a) Passband and (b) stopband magnitude responses of the FRM-based pipelined IIR filter with coefficient values quantized to 8 fractional bits (dashed line). The ideal magnitude response is shown in solid line.

5.7 Conclusion

A class of IIR filters synthesized using the FRM technique has been presented in this chapter. The subfilters are IIR filters and they are jointly optimized. The coefficient sensitivity and roundoff noise performances of the new class of IIR filters are much superior to the conventional IIR filters.

Chapter 6

Conclusion and Recommendations

In this final chapter of the thesis, the conclusion is given in Section 6.1 and some suggestions for further research are made in Section 6.2.

6.1 Conclusion

The three objectives posed in Section. 1.2 have been fulfilled by the research work presented in Chapters 3-5.

- **Objective 1:** Develop a new approach to designing nonlinear phase FRM-based pipelined IIR filters that require fewer multipliers.

For objective 1, a novel two-stage FRM approach has been proposed in Chapter. 3. In the first stage, the bandedge shaping filters are FRM-based pipelined IIR filters and the masking filters are linear phase FIR filters. In the second stage, the bandedge shaping filters are a parallel connection of two allpass filters and the masking filters are linear phase FIR filters. It has been shown that when the number of pipeline stages in the feedback loop is larger than four, the two-stage masking design requires

6.1. CONCLUSION

fewer multipliers than the one-stage masking design meeting the same magnitude response specifications. The proposed two-stage FRM approach can be extended to the multi-stage FRM approach.

- **Objective 2:** Develop an optimization method that independently controls the passband magnitude error and phase error of the approximately linear phase FRM-based pipelined IIR filters.

For objective 2, a constrained optimization method which independently controls the passband phase error and magnitude error has been presented in Chapter. 4. The independent control of phase error and magnitude error is achieved by imposing two constraints for the passband. The first constraint is on the phase error and the second constraint is on the frequency-response error whose real and imaginary parts are bounded by an ellipse. Simulation examples have shown that the proposed constrained optimization method produces FRM-based pipelined IIR filters with very small phase error.

- **Objective 3:** Propose a new class of FRM-based IIR filters with IIR subfilters that is easy to optimize and investigate its coefficient sensitivity and roundoff noise properties.

For objective 3, A class of FRM-based IIR filters with IIR subfilters has been introduced in Chapter. 5. In the proposed class of FRM-based IIR filters, the bandedge shaping filters are a parallel connection of two allpass filters and the masking filters are approximately linear phase IIR filters composed of a parallel connection of a delay line and an allpass filter. The magnitude response ripple of the proposed class of FRM-based IIR filters has been analyzed in different frequency regions. Based on the ripple analysis, a method for generating good initial solutions for further joint optimization

has been described. It has been shown by both theoretical analysis and numerical example that the proposed class of FRM-based IIR filters has much better coefficient sensitivity and roundoff noise performances compared with the conventional IIR filters. The proposed class of FRM-based IIR filters requires more multipliers than an elliptic filter. However, for a given coefficient word-length, the FRM-based IIR filter may require fewer multipliers.

The FRM-based IIR filters considered in this thesis are all designed by optimizing the subfilters jointly. These joint optimization problems are nonconvex optimization problems. The sequential convex programming (SCP) has been successfully used to find satisfactory solutions. The stability of all the FRM-based IIR filters presented in the thesis are ensured because the necessary and sufficient stability constraints have been incorporated in the formulation of the joint optimization problems.

6.2 Recommendations for Further Research

Further research may be carried out in the following directions.

In Chapter 5, we have used a bisection search to determine the minimum orders of $A_{ma}(z)$ and $A_{mc}(z)$ to meet a set of phase response specifications. It will be interesting to develop an empirical formula for estimating the filter order of an allpass filter given the specifications on its phase response.

The FRM-based IIR filter introduced in Chapter 5 has nonlinear phase response. If the allpass filter $A_1(z^M)$ is a delay line z^{-MN_1} , the phase response of the overall filter can be constrained to be approximately linear in the passband. This leads to a class of FRM-based IIR filters with approximately linear phase. The structure of this class of approximately linear phase IIR filters is shown in Fig. 6.1. The design method which optimizes $A_2(z)$, $A_{ma}(z)$ and $A_{mc}(z)$ jointly can be developed. The

Figure 6.1: The structure of a class of FRM-based IIR filters with approximately linear phase. $K_m = \max(K_{ma}, K_{mc})$.

relationship between the value of M and the total number of multipliers may be investigated. The filter in Fig. 6.1 and the approximately linear phase IIR filter consisting of a delay line and an allpass filter can be compared in terms of the total number of multipliers, coefficient sensitivity and roundoff noise.

When designing FRM-based filter with IIR subfilters, it is important to constrain the phase responses of the two IIR masking filters to be approximately equal. For the class of nonlinear phase FRM-based IIR filters in Chapter 5, this is achieved by selecting the masking filters as approximately linear phase IIR filters composed of a parallel connection of a delay line and an allpass filter, and cascading an appropriate number of delay elements to one of the masking filters. However, it is not necessary to match the phase responses of the masking filters in the entire common passband of the masking filters. The phase responses of the masking filters only need to be matched in frequency region P corresponding to the transition bands of the bandedge shaping filters within the common passband of the masking filters, as shown in Fig. 6.2. Thus, the IIR masking filters may possess nonlinear phase if their phase responses can be designed to be matched accurately for $\omega \in P$. One way to do this is to cascade an allpass filter with one of the masking filters to compensate for the phase difference of

Figure 6.2: The frequency region $P = P_1 \cup P_2 \cup \dots \cup P_T$ where the phase responses of the masking filters need to be matched. T is the number of transition bands of the bandedge shaping filters within the common passband of the masking filters.

the two masking filters in frequency region P . A systematic design approach may be developed.

In recent years, the design of sparse filters [135–139] becomes an emerging topic in filter design community due to the advances in sparse representation [140]. Sparse filters are filters with many zero-valued coefficients. The bandedge shaping filters of the filters designed using the FRM technique are sparse filters because they have many zero-valued coefficients. A sparse filter design problem can be generally formulated as minimizing the l_0 -norm of the coefficient vector subject to some constraints on filter performance. It would be beneficial to incorporate the sparse filter design techniques into the FRM technique to further reduce the complexity of the filter.

Author's Publications

The work presented in this thesis has been accompanied by the following publications.

Journal Papers

1. Q. Liu, Y. C. Lim and Z. Lin, "Design of pipelined IIR filters using two-stage frequency-response masking technique," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 66, no. 5, pp. 873–877, May 2019.
2. Q. Liu, Y. C. Lim and Z. Lin, "A class of IIR filters synthesized using frequency-response masking technique," *IEEE Signal Process. Lett.*, vol. 26, no. 11, pp. 1693–1697, Nov. 2019.

Conference Papers

1. Q. Liu, Y. C. Lim, Z. Lin and X. Lai, "Design of IIR frequency-response masking filters with near linear phase using constrained optimization," in *Proc. IEEE. Int. Symp. Circuits and Syst.*, Baltimore, MD, USA, May 2017, pp. 1–4.
2. Q. Liu, Y. C. Lim and Z. Lin, "Design of sparse frequency-response masking filters with arbitrary delay using l_1 -minimization," in *Proc. IEEE Int. Conf. Digit. Signal Process.*, London, UK, Aug. 2017, pp. 1–4.
3. Q. Liu, Y. C. Lim and Z. Lin, "Design of recursive filters using frequency-response

masking technique," in *Proc. IEEE Int. Conf. Digit. Signal Process.*, Shanghai, China, Nov. 2018, pp. 1–5.

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Appendix A

Coefficient Values of the Filters Designed in Chapter 3

Table A.1: Coefficient Values of $A_1(z)$, $A_2(z)$, $F_{ma}^{(2)}(z)$, $F_{mc}^{(2)}(z)$, $F_{ma}^{(1)}(z)$ and $F_{mc}^{(1)}(z)$ for the Two-Stage Design With $M_1 = 3$ and $M_2 = 3$

$a_{11}(1) = -0.422913923301446$	$a_{21}(1) = -0.331722538592126$
$a_{11}(2) = 0.239026223499850$	$a_{21}(2) = 0.000000000000000$
$a_{12}(1) = -0.285236909862866$	$a_{22}(1) = -0.345406781171487$
$a_{12}(2) = 0.794063679675224$	$a_{22}(2) = 0.483719394395487$
	$a_{23}(1) = -0.219162854501889$
	$a_{23}(2) = 0.945637116500811$
$h_{ma}^{(2)}(0) = -0.023994923609302$	$h_{mc}^{(2)}(0) = 0.025048142180963$
$h_{ma}^{(2)}(1) = -0.078954791618166$	$h_{mc}^{(2)}(1) = -0.064460633745396$
$h_{ma}^{(2)}(2) = 0.005048094991784$	$h_{mc}^{(2)}(2) = 0.062479427779326$
$h_{ma}^{(2)}(3) = 0.189615180926258$	$h_{mc}^{(2)}(3) = 0.050252455803790$
$h_{ma}^{(2)}(4) = 0.303144379725393$	$h_{mc}^{(2)}(4) = -0.244814591793955$
$h_{ma}^{(2)}(5) = 0.621505631993002$	$h_{mc}^{(2)}(5) = 0.254613015346426$
	$h_{mc}^{(2)}(6) = 1.285604299086454$
$h_{ma}^{(1)}(0) = 0.006092003929236$	$h_{mc}^{(1)}(0) = -0.025211553758036$
$h_{ma}^{(1)}(1) = -0.006828903572602$	$h_{mc}^{(1)}(1) = 0.001387910170039$
$h_{ma}^{(1)}(2) = -0.064143097216619$	$h_{mc}^{(1)}(2) = 0.053968882813118$
$h_{ma}^{(1)}(3) = 0.004146314483984$	$h_{mc}^{(1)}(3) = -0.066457481014556$
$h_{ma}^{(1)}(4) = 0.295718916805567$	$h_{mc}^{(1)}(4) = -0.119411175529703$
$h_{ma}^{(1)}(5) = 0.490076012883382$	$h_{mc}^{(1)}(5) = 0.289692950154134$
	$h_{mc}^{(1)}(6) = 0.618541938367870$

Table A.2: Coefficient Values of $A_1(z)$, $A_2(z)$, $F_{ma}^{(2)}(z)$, $F_{mc}^{(2)}(z)$, $F_{ma}^{(1)}(z)$ and $F_{mc}^{(1)}(z)$ for the Two-Stage Design With $M_1 = 5$ and $M_2 = 3$

$a_{11}(1) = 0.392178919257735$	$a_{21}(1) = 0.526338754929022$
$a_{11}(2) = 0.000000000000000$	$a_{21}(2) = 0.344477735714563$
$a_{12}(1) = 0.403321799782520$	$a_{22}(1) = 0.367715987627043$
$a_{12}(2) = 0.622962482782130$	$a_{22}(2) = 0.897874990280713$
$h_{ma}^{(2)}(0) = -0.023971821827727$	$h_{mc}^{(2)}(0) = -0.009750471728404$
$h_{ma}^{(2)}(1) = -0.053383192172249$	$h_{mc}^{(2)}(1) = 0.054317748630854$
$h_{ma}^{(2)}(2) = -0.053593374608275$	$h_{mc}^{(2)}(2) = -0.002088312263685$
$h_{ma}^{(2)}(3) = 0.044741642245504$	$h_{mc}^{(2)}(3) = -0.118512917293624$
$h_{ma}^{(2)}(4) = 0.195738377622483$	$h_{mc}^{(2)}(4) = 0.241692140679120$
$h_{ma}^{(2)}(5) = 0.190838915003943$	$h_{mc}^{(2)}(5) = 0.763263289408957$
$h_{ma}^{(2)}(6) = 0.261304368450267$	
$h_{ma}^{(1)}(0) = -0.006174845786805$	$h_{mc}^{(1)}(0) = 0.008928644269163$
$h_{ma}^{(1)}(1) = 0.004774264078534$	$h_{mc}^{(1)}(1) = -0.006976160125584$
$h_{ma}^{(1)}(2) = 0.012157356331036$	$h_{mc}^{(1)}(2) = -0.029564738978457$
$h_{ma}^{(1)}(3) = -0.016360153407917$	$h_{mc}^{(1)}(3) = 0.001778538625737$
$h_{ma}^{(1)}(4) = -0.012369129145799$	$h_{mc}^{(1)}(4) = 0.043799863590131$
$h_{ma}^{(1)}(5) = 0.051563500442353$	$h_{mc}^{(1)}(5) = 0.004010001455358$
$h_{ma}^{(1)}(6) = -0.001801809142000$	$h_{mc}^{(1)}(6) = -0.087819828790875$
$h_{ma}^{(1)}(7) = -0.114365872463247$	$h_{mc}^{(1)}(7) = -0.051913118330990$
$h_{ma}^{(1)}(8) = 0.086890263327473$	$h_{mc}^{(1)}(8) = 0.181032342211771$
$h_{ma}^{(1)}(9) = 0.494460856174286$	$h_{mc}^{(1)}(9) = 0.404739512763396$

Table A.3: Coefficient Values of $A_1(z)$, $A_2(z)$, $F_{ma}^{(3)}(z)$, $F_{mc}^{(3)}(z)$, $F_{ma}^{(2)}(z)$, $F_{mc}^{(2)}(z)$ for the Three-Stage Design With $M_1 = 5$, $M_2 = 3$ and $M_3 = 3$

$a_{11}(1) = -0.847176576048662$	$a_{21}(1) = -0.473832394502086$
$a_{11}(2) = 0.441549557186340$	$a_{21}(2) = 0.000000000000000$
	$a_{22}(1) = -0.803388816162434$
	$a_{22}(2) = 0.803716849528456$
$h_{ma}^{(3)}(0) = -0.069079623321699$	$h_{mc}^{(3)}(0) = 0.015846822960580$
$h_{ma}^{(3)}(1) = 0.004890375845306$	$h_{mc}^{(3)}(1) = 0.002696953492536$
$h_{ma}^{(3)}(2) = 0.126439971472424$	$h_{mc}^{(3)}(2) = -0.018766179971889$
$h_{ma}^{(3)}(3) = 0.215495508703917$	$h_{mc}^{(3)}(3) = 0.061263151281036$
$h_{ma}^{(3)}(4) = 0.277425763208318$	$h_{mc}^{(3)}(4) = -0.007254138899188$
	$h_{mc}^{(3)}(5) = -0.093414296807919$
	$h_{mc}^{(3)}(6) = 0.195930839694304$
	$h_{mc}^{(3)}(7) = 0.679442237618583$
$h_{ma}^{(2)}(0) = -0.024710722145988$	$h_{mc}^{(2)}(0) = 0.058138455066569$
$h_{ma}^{(2)}(1) = -0.047896172705750$	$h_{mc}^{(2)}(1) = 0.058881791939212$
$h_{ma}^{(2)}(2) = -0.005385356221826$	$h_{mc}^{(2)}(2) = -0.079395966229584$
$h_{ma}^{(2)}(3) = 0.125980911477046$	$h_{mc}^{(2)}(3) = -0.032337728852242$
$h_{ma}^{(2)}(4) = 0.259709186359779$	$h_{mc}^{(2)}(4) = 0.351914683488471$
$h_{ma}^{(2)}(5) = 0.321821254600198$	$h_{mc}^{(2)}(5) = 0.591738814074494$

Table A.4: Coefficient Values of $F_{ma}^{(1)}(z)$ and $F_{mc}^{(1)}(z)$ for the Three-Stage Design With $M_1 = 5$, $M_2 = 3$ and $M_3 = 3$

$h_{ma}^{(1)}(0) = -0.000605762914197$	$h_{mc}^{(1)}(0) = 0.003086407054565$
$h_{ma}^{(1)}(1) = -0.011084952546864$	$h_{mc}^{(1)}(1) = 0.000723636954899$
$h_{ma}^{(1)}(2) = 0.002182239286358$	$h_{mc}^{(1)}(2) = -0.008810642986543$
$h_{ma}^{(1)}(3) = 0.017414130461862$	$h_{mc}^{(1)}(3) = -0.014722781384188$
$h_{ma}^{(1)}(4) = -0.016184881550048$	$h_{mc}^{(1)}(4) = 0.001784360049684$
$h_{ma}^{(1)}(5) = -0.024232622145381$	$h_{mc}^{(1)}(5) = 0.027734527931045$
$h_{ma}^{(1)}(6) = 0.051736158199953$	$h_{mc}^{(1)}(6) = 0.021060863878641$
$h_{ma}^{(1)}(7) = 0.011776227638019$	$h_{mc}^{(1)}(7) = -0.035036873032185$
$h_{ma}^{(1)}(8) = -0.117573899629706$	$h_{mc}^{(1)}(8) = -0.074800591102176$
$h_{ma}^{(1)}(9) = 0.076066044216098$	$h_{mc}^{(1)}(9) = 0.002498923638708$
$h_{ma}^{(1)}(10) = 0.505030942552917$	$h_{mc}^{(1)}(10) = 0.196945544187321$
	$h_{mc}^{(1)}(11) = 0.370458301267620$

Appendix B

Coefficient Values of the Filters Designed in Chapter 4

Table B.1: Coefficient Values of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ in Design I of Example One

$b_0 = -0.019545605597870$	$a_{11} = -0.284662507228169$
$b_1 = 0.001915187881285$	$a_{12} = 0.074880078993927$
$b_2 = 0.024836685054927$	$a_{21} = 0.054542392613613$
$b_3 = -0.006538523982126$	$a_{22} = 0.769309082194525$
$b_4 = -0.078509083930200$	
$b_5 = 0.019250286587166$	
$b_6 = 0.264664765365045$	
$b_7 = 0.855208509247925$	
$b_8 = 0.586886446497942$	
$b_9 = 0.551958799741761$	
$h_{ma}(0) = -0.012607888279527$	$h_{mc}(0) = 0.012551876107864$
$h_{ma}(1) = -0.020905524956565$	$h_{mc}(1) = 0.036759035503392$
$h_{ma}(2) = 0.035229096583707$	$h_{mc}(2) = 0.009932435013174$
$h_{ma}(3) = 0.185934800752786$	$h_{mc}(3) = -0.071078552126251$
$h_{ma}(4) = 0.330324386572190$	$h_{mc}(4) = -0.057214975283322$
	$h_{mc}(5) = 0.173363214949099$
	$h_{mc}(6) = 0.442405350935870$

Table B.2: Coefficient Values of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ in Design II of Example One

$b_0 = -0.005990560564487$	$a_{11} = 0.072460270411784$
$b_1 = -0.006397509308157$	$a_{12} = 0.809999999975069$
$b_2 = 0.005713525047172$	$a_{21} = 0.592833427792146$
$b_3 = 0.003237159691050$	$a_{22} = 0.509456656629142$
$b_4 = -0.018652627095511$	
$b_5 = -0.015489308130416$	
$b_6 = 0.074156312752058$	
$b_7 = 1.047168706043916$	
$b_8 = 0.960249951528256$	
$b_9 = 1.511900806467344$	
$b_{10} = 0.697848347156875$	
$b_{11} = 0.440818132029406$	
$b_{12} = 0.017136162907826$	
$b_{13} = -0.005445919479788$	
$h_{ma}(0) = -0.012762920613152$	$h_{mc}(0) = 0.044635995607073$
$h_{ma}(1) = -0.024191202601870$	$h_{mc}(1) = 0.126288662422099$
$h_{ma}(2) = 0.030570854106516$	$h_{mc}(2) = 0.061883358637544$
$h_{ma}(3) = 0.186708522898158$	$h_{mc}(3) = -0.180204622601497$
$h_{ma}(4) = 0.336087730955295$	$h_{mc}(4) = -0.254098501064584$
	$h_{mc}(5) = 0.146932787161724$
	$h_{mc}(6) = 0.684890063519926$

Table B.3: Coefficient Values of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ in Design III of Example One

$b_0 =$	0.028207848830201	$a_{11} =$	-0.220814395786388
$b_1 =$	-0.017015400300889	$a_{12} =$	0.809999999974431
$b_2 =$	-0.025962611028823	$a_{21} =$	-0.198185152955859
$b_3 =$	0.032175732775983	$a_{22} =$	0.673019101385034
$b_4 =$	0.177273181386211		
$b_5 =$	0.849889365256655		
$b_6 =$	0.448374859887363		
$b_7 =$	1.161728285440149		
$b_8 =$	0.310830420456016		
$b_9 =$	0.474708218888126		
$b_{10} =$	0.068283643410351		
$h_{ma}(0) =$	-0.012081609233919	$h_{mc}(0) =$	0.018003264197924
$h_{ma}(1) =$	-0.020168074963742	$h_{mc}(1) =$	0.044744726627827
$h_{ma}(2) =$	0.037556905152543	$h_{mc}(2) =$	0.009096674167589
$h_{ma}(3) =$	0.185677689473821	$h_{mc}(3) =$	-0.085691918485431
$h_{ma}(4) =$	0.326864714716930	$h_{mc}(4) =$	-0.072093999238849
		$h_{mc}(5) =$	0.174810147197723
		$h_{mc}(6) =$	0.460538773120431

Table B.4: Coefficient Values of $H_a(z)$ in Design I of Example Two

$b_0 = -0.003380770212533$	$a_{11} = 0.539247695236686$
$b_1 = 0.004296394528310$	$a_{12} = 0.900031689957206$
$b_2 = -0.007704637092241$	$a_{21} = 0.178542192864560$
$b_3 = 0.009429935674593$	$a_{22} = 0.890180782296144$
$b_4 = -0.011046367856844$	$a_{31} = 0.114370808728947$
$b_5 = 0.012845989392493$	$a_{32} = 0.331849050024612$
$b_6 = -0.011403251714619$	$a_{41} = 0.168167637811082$
$b_7 = -0.003060812032810$	$a_{42} = 0.900031689720771$
$b_8 = -0.000825890333521$	
$b_9 = 0.016162800442781$	
$b_{10} = 0.148463979814737$	
$b_{11} = 0.162585230352644$	
$b_{12} = 0.312335688964153$	
$b_{13} = 0.196888917573409$	
$b_{14} = 0.207317174026574$	
$b_{15} = 0.063769643242859$	
$b_{16} = 0.040891105623264$	

Table B.5: Coefficient Values of $H_{ma}(z)$ and $H_{mc}(z)$ in Design I of Example Two

$h_{ma}(0) =$	0.057706892668878	$h_{mc}(0) =$	-0.005816573907431
$h_{ma}(1) =$	0.169371364415969	$h_{mc}(1) =$	0.001645917701875
$h_{ma}(2) =$	-0.137151128523582	$h_{mc}(2) =$	0.004996513260403
$h_{ma}(3) =$	-0.020748204943600	$h_{mc}(3) =$	-0.017266839460919
$h_{ma}(4) =$	0.199793606150968	$h_{mc}(4) =$	-0.019555989377291
$h_{ma}(5) =$	-0.099619319811184	$h_{mc}(5) =$	0.034592901422428
$h_{ma}(6) =$	-0.098001893037744	$h_{mc}(6) =$	-0.017984260916728
$h_{ma}(7) =$	0.254558273752464	$h_{mc}(7) =$	-0.036715351561883
$h_{ma}(8) =$	-0.085268958261785	$h_{mc}(8) =$	0.065329425448778
$h_{ma}(9) =$	-0.213514581038074	$h_{mc}(9) =$	-0.023209228933690
$h_{ma}(10) =$	0.313544301845012	$h_{mc}(10) =$	-0.119444140764621
$h_{ma}(11) =$	0.003004992952378	$h_{mc}(11) =$	0.281043931581274
$h_{ma}(12) =$	-0.343886366356723	$h_{mc}(12) =$	0.640720764368268
$h_{ma}(13) =$	0.305923071005216		
$h_{ma}(14) =$	0.205883119939136		
$h_{ma}(15) =$	-0.423878493167656		
$h_{ma}(16) =$	0.090950968469219		
$h_{ma}(17) =$	0.616611238587718		
$h_{ma}(18) =$	0.300019538708967		

Table B.6: Coefficient Values of $H_a(z)$ in Design II of Example Two

$b_0 = -0.002973507170150$	$a_{11} = 0.521311340054611$
$b_1 = 0.000507891659505$	$a_{12} = 0.900031689980024$
$b_2 = -0.004870823154343$	$a_{21} = 0.525204494455205v$
$b_3 = 0.001111599996539$	$a_{22} = 0.900031689978869$
$b_4 = -0.004469362231209$	$a_{31} = 0.222186503132099$
$b_5 = 0.001282705097131$	$a_{32} = 0.880674037869880$
$b_6 = -0.001841817004303$	$a_{41} = 0.165809746583077$
$b_7 = -0.007614792548834$	$a_{42} = 0.900031689941740$
$b_8 = -0.006587868449323$	
$b_9 = 0.011746951496139$	
$b_{10} = 0.126514141573441$	
$b_{11} = 0.222968461679072$	
$b_{12} = 0.391377144833263$	
$b_{13} = 0.361260699954724$	
$b_{14} = 0.340789439575977$	
$b_{15} = 0.166216355511043$	
$b_{16} = 0.081962495161488$	
$b_{17} = 0.006968925308761$	
$b_{18} = -0.004287734073378$	
$b_{19} = 0.000193879156107$	
$b_{20} = -0.000935860067054$	
$b_{21} = 0.001587963277971$	
$b_{22} = -0.000882256820788$	

Table B.7: Coefficient Values of $H_{ma}(z)$ and $H_{mc}(z)$ in Design II of Example Two

$h_{ma}(0) =$	0.121718502640177	$h_{mc}(0) =$	-0.006509027597090
$h_{ma}(1) =$	-0.111284210905154	$h_{mc}(1) =$	-0.010551933335253
$h_{ma}(2) =$	0.066023803919310	$h_{mc}(2) =$	0.018021196837003
$h_{ma}(3) =$	0.177894991216151	$h_{mc}(3) =$	-0.016569820658683
$h_{ma}(4) =$	-0.104685934219106	$h_{mc}(4) =$	-0.016753939149578
$h_{ma}(5) =$	-0.093998424371916	$h_{mc}(5) =$	0.029292207699387
$h_{ma}(6) =$	0.327419456088910	$h_{mc}(6) =$	-0.012556811869399
$h_{ma}(7) =$	-0.115362237932963	$h_{mc}(7) =$	-0.045803234790992
$h_{ma}(8) =$	-0.198231135204272	$h_{mc}(8) =$	0.065585339531538
$h_{ma}(9) =$	0.345305010015101	$h_{mc}(9) =$	-0.015046976317745
$h_{ma}(10) =$	-0.043721458193987	$h_{mc}(10) =$	-0.130931807598059
$h_{ma}(11) =$	-0.346064863771246	$h_{mc}(11) =$	0.281691442609170
$h_{ma}(12) =$	0.332108137518078	$h_{mc}(12) =$	0.649964851213081
$h_{ma}(13) =$	0.107822725276900		
$h_{ma}(14) =$	-0.425435813700961		
$h_{ma}(15) =$	0.267849554679374		
$h_{ma}(16) =$	0.334936406510266		
$h_{ma}(17) =$	-0.483506400705310		
$h_{ma}(18) =$	0.225372278979851		
$h_{ma}(19) =$	0.498318242917677		
$h_{ma}(20) =$	0.341315216070618		

Appendix C

Coefficient Values of the Filters Designed in Chapter 5

Table C.1: Coefficient Values of the FRM-Based IIR Filter in the Design Example of Section 5.3.4

	$A_1(z)$	$A_2(z)$
a_{11}	-0.213771065059320	-0.438211030548531
a_{12}	0.000000000000000	0.189994548651190
a_{21}	-0.456912876428144	-0.446117751739313
a_{22}	0.524941801016586	0.857823513932283
	$A_{ma}(z)$	$A_{mc}(z)$
a_{11}	-0.302937266930349	-0.259399747645791
a_{12}	-0.217929107935141	-0.325206473590511
a_{21}	0.482747446633444	0.669503047462860
a_{22}	0.159187797069588	0.218361743988525
a_{31}	-0.325350085980321	0.048579666524114
a_{32}	0.734193951069146	0.271313375436403
a_{41}		-0.982750209390104
a_{42}		0.795680402718069

Table C.2: Coefficient Values of the FRM-Based IIR Filter in the Numerical Example of Section 5.5

	$A_1(z)$	$A_2(z)$
a_{11}	-0.198057789685961	-0.067092340101191
a_{12}	0.167975664819287	0.000000000000000
a_{21}	-0.421682648446497	-0.301891560566884
a_{22}	0.734118706838430	0.468034945321432
a_{21}	-0.599019943971560	-0.537906704592143
a_{22}	0.964622029660135	0.900169759337457
a_{21}		-0.621898852102840
a_{22}		0.991197906733658
	$A_{ma}(z)$	$A_{mc}(z)$
a_{11}	0.445203349131871	0.005080646830229
a_{12}	0.000000000000000	-0.389579296032324
a_{21}	-0.797554511031337	0.241523731689715
a_{22}	0.231069081359599	0.538461359104737
a_{31}	0.884422702979764	-0.963927906708942
a_{32}	0.714023439336566	0.442443164667437
a_{41}		1.561512114467170
a_{42}		0.830743360723876