On the ranks of partitions modulo certain integers

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**NANYANG
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SINGAPORE

**ON THE RANKS OF PARTITIONS
MODULO CERTAIN INTEGERS**

Nankun Hong

SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

2020

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MODULO CERTAIN INTEGERS**

Nankun Hong

SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

A thesis submitted to the Nanyang Technological
University in partial fulfilment of the requirement for the
degree of Doctor of Philosophy

2020

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research done by me except where otherwise stated in this thesis. The thesis work has not been submitted for a degree or professional qualification to any other university or institution. I declare that this thesis is written by myself and is free of plagiarism and of sufficient grammatical clarity to be examined. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

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Nankun Hong

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Authorship Attribution Statement

This thesis contains material from one paper published in the following peer-reviewed journal in which I am listed as an author.

Chapter 4 is published as Z. S. Aygin and N. Hong. Ramanujan's convolution sum twisted by Dirichlet characters. *International Journal of Number Theory*, 15(01) (2019): 137-152. DOI: <https://doi.org/10.1142/S1793042119500027>

The contributions of co-authors are as follows:

- Dr. Aygin provided the initial project direction.
- I worked out values of $\eta(z)$, $\eta(2z)$, $\eta(4z)$, $\eta(8z)$ and $\eta(16z)$ at different cusps and gave results in Theorem 3.1 and Theorem 6.1.
- Dr. Aygin proved all the other theorems and wrote the manuscript.

13-Aug.-2019
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Date

洪颖坤
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Nankun Hong

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To Siyu Hong

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Abstract

This thesis focuses on the rank of partition functions, identities related to generating functions of ranks modulo different integers and Ramanujan's convolution sum. Most results in Chapters 2 and 4 are reproduced from [14] and [10], respectively.

Ramanujan had three famous congruences for the partition function modulo 5, 7 and 11. F. J. Dyson defined the rank of partitions and conjectured that ranks could provide combinatorial explanations for the cases of 5 and 7. A. O. L. Atkin and H. P. F. Swinnerton-Dyer proved his conjecture using generating functions for the rank difference modulo 5 and 7. From Theorem 8.16 in F. G. Garvan's paper [20], we know that results on dissections of the rank modulo m are equivalent to results on rank difference results modulo m , which inspired us to find a 3-dissection of ranks modulo 9 in Chapter 2. We also give an identity involving generating functions of ranks modulo 3 and 9.

Ramanujan recorded several entries which are related to generating functions of the rank modulo different integers. Finding analogous identities is the motivation of Chapter 3. We give some identities, some of which are obtained by using Ramanujan's entries.

In Ramanujan's paper [36], he proved a formula for convolutions of sum of divisors functions. In Chapter 4, we find formulas for convolutions of the sum of divisor functions twisted by the Dirichlet character, which are analogous to Ramanujan's.

Chapter 1

Introduction

In this thesis, we discuss two topics in number theory. Both Chapters 2 and 3 are on the same topic, ranks of partitions modulo certain integers. In Chapter 2, we prove a new identity related to the rank of partitions modulo 3 and 9. Then in Chapter 3, we provide a way to generate identities involving ranks of partitions modulo different integers. As applications, we apply our method to several identities of Ramanujan to get new ones. Chapter 4 is on Ramanujan's convolution sum. We find formulas for convolution sums of divisor functions twisted by the Dirichlet character $(\frac{-4}{*})$. Chapter 2 and Chapter 4 are reproduced from [14] and [10], respectively.

In this chapter, we describe a brief history of our research topics and introduce some important definitions.

A *partition* of a positive integer n is a sequence of non-increasing positive integers which add up to n . We use the partition function $p(n)$ to count the number of partitions of n , and we agree that $p(0) = 1$. For example, $p(4) = 5$, i.e., there are 5 partitions of 4:

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

The study on partitions can be traced back to Euler, who discovered the generating function of $p(n)$ as

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k} = \frac{1}{(q; q)_{\infty}}.$$

Here and for the rest of this thesis, we use the standard notations

$$\begin{aligned} (x)_0 &:= (x; q)_0 := 1, \\ (x)_n &:= (x; q)_n := \prod_{k=0}^{n-1} (1 - xq^k), \quad n \geq 1 \\ (x)_{\infty} &:= (x; q)_{\infty} := \prod_{k=0}^{\infty} (1 - xq^k), \\ (x_1, \dots, x_m)_n &:= (x_1, \dots, x_m; q)_n := (x_1; q)_n \dots (x_m; q)_n, \\ (x_1, \dots, x_m)_{\infty} &:= (x_1, \dots, x_m; q)_{\infty} := (x_1; q)_{\infty} \dots (x_m; q)_{\infty}, \\ J_a &:= (q^a; q^a)_{\infty}, \\ J_{a,b} &:= (q^a, q^{b-a}, q^b; q^b)_{\infty}, \\ j(x; q) &:= (x, q/x, q; q)_{\infty}, \\ [x; q]_n &:= (x; q)_n (q/x; q)_n, \quad n \geq 0, a \neq 0, \\ [x_1, x_2, \dots, x_m; q]_n &:= [x_1; q]_n [x_2; q]_n \dots [x_m; q]_n, \quad n \geq 0, \\ [x]_{\infty} &:= [x; q]_{\infty} := \prod_{k=0}^{\infty} (1 - xq^k)(1 - q^k/x), \\ [a_1, a_2, \dots, a_m; q]_{\infty} &:= [a_1; q]_{\infty} [a_2; q]_{\infty} \dots [a_m; q]_{\infty}, \end{aligned}$$

and we assume $|q| < 1$ for convergence.

In 1919, Ramanujan [37] discovered three simple yet remarkable congruences for $p(n)$:

$$p(5n + 4) \equiv 0 \pmod{5}, \tag{1.1}$$

$$p(7n + 5) \equiv 0 \pmod{7}, \tag{1.2}$$

and

$$p(11n + 6) \equiv 0 \pmod{11}. \quad (1.3)$$

Ramanujan provided a generating function for (1.1), namely,

$$\sum_{n=0}^{\infty} p(5n + 4)q^n = 5 \frac{(q^5; q^5)_{\infty}^5}{(q; q)_{\infty}^6}, \quad (1.4)$$

and he stated without proof the following generating function of (1.2)

$$\sum_{n=0}^{\infty} p(7n + 5)q^n = 7 \frac{(q^7; q^7)_{\infty}^3}{(q; q)_{\infty}^4} + 49q \frac{(q^7; q^7)_{\infty}^7}{(q; q)_{\infty}^8}. \quad (1.5)$$

One can deduce (1.1) and (1.2) from (1.4) and (1.5) directly. The first elementary proof of (1.3) was published by L. Winquist [43]. B. C. Berndt, S. H. Chan, Z.-G. Liu and H. Yesilyurt gave a different proof by an identity for $(q; q)_{\infty}^{10}$ in [12].

In 1944, F. J. Dyson [19] defined the *rank of a partition* as the largest part minus the number of parts. He tried to use the rank to separate the partitions into classes of equal size. Let $N(m, n)$ denote the number of partitions of n with rank m and let $N(k, l, n)$ denote the number of partitions of n with rank congruent to k modulo l . Then in the same paper, Dyson raised the following conjectures:

$$\begin{aligned} N(k, 5, 5n + 4) &= \frac{p(5n + 4)}{5}, \quad 0 \leq k \leq 4, \\ N(k, 7, 7n + 5) &= \frac{p(7n + 5)}{7}, \quad 0 \leq k \leq 6. \end{aligned}$$

Indeed, these give combinatorial interpretations for (1.1) and (1.2) since $p(5n + 4) = \sum_{k=0}^4 N(k, 5, 5n + 4)$ and $p(7n + 5) = \sum_{k=0}^6 N(k, 7, 7n + 5)$. Dyson also showed that the generating function for $N(m, n)$ is

$$G(z, q) := \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N(m, n) q^n z^m = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (q/z)_n}. \quad (1.6)$$

These conjectures were proved by A. O. L. Atkin and P. Swinnerton-Dyer in [7]. For every rank difference $N(b, l, ln + d) - N(c, l, ln + d)$ with $l = 5$ or 7 and $0 \leq b, c, d < l$, they gave the generating functions.

Dyson's rank failed to explain the third congruence (1.3). This led him to conjecture the existence of another partition statistic that could interpret all the three congruences (1.1), (1.2) and (1.3) combinatorially. Dyson named this statistic the *crank of a partition*.

In [20], F. G. Garvan defined *vector partitions* and gave the definition of the crank via vector partitions. Then in 1988, G. E. Andrews and Garvan found the crank of a partition in [4]. The crank of a partition is defined as following:

For a partition λ , let $\ell(\lambda)$ denote the largest part of λ , $\omega(\lambda)$ denote the number of 1's in λ , and $\mu(\lambda)$ denote the number of parts of λ larger than $\omega(\lambda)$. The crank $c(\lambda)$ is given by

$$c(\lambda) = \begin{cases} \ell(\lambda) & \text{if } \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0. \end{cases}$$

If we use $M(m, n)$ to denote the number of partitions of n with crank equal to m , then the generating function of $M(m, n)$ for $n \geq 2$ is given by Andrews and Garvan as

$$C(z, q) := \sum_{n=0}^{\infty} M(m, n) z^m q^n = \frac{(q)_{\infty}}{(zq, q/z)_{\infty}}.$$

In Ramanujan's *Lost Notebook* [38, p. 2 & p. 17], there are four identities involving the generating function of rank. As mentioned above, the rank is defined by Dyson in 1944. It is not likely that Ramanujan used the terminology *rank*.

To state these four identities, we start with Ramanujan's *general theta function* which is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \quad (1.7)$$

One particular case is, in Ramanujan's notation,

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}. \quad (1.8)$$

Note that the last equality follows from the famous Jacobi triple product identity [11, p.35, Entry 19].

Then we define:

$$f_a(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1 + aq + q^2)(1 + aq^2 + q^4) \cdots (1 + aq^n + q^{2n})}, \quad (1.9)$$

where a is a real number. Recall $G(z, q)$ is the generating function of rank defined in (1.6). We observe that

$$f_a(q) = G\left(\frac{-a \pm \sqrt{a^2 - 4}}{2}, q\right).$$

Now we list the four identities of Ramanujan:

Entry 1.1. [38, p. 2] *Suppose that a and b are real numbers such that $a^2 + b^2 = 4$. Then*

$$\begin{aligned} & \frac{b-a+2}{4} f_a(-q) + \frac{b+a+2}{4} f_{-a}(-q) - \frac{b}{2} f_b(q) \\ &= \frac{(q^4; q^4)_{\infty}}{(-q; q^2)_{\infty}} \prod_{n=1}^{\infty} \frac{1 - bq^n + q^{2n}}{1 + (a^2 b^2 - 2) q^{4n} + q^{8n}}. \end{aligned} \quad (1.10)$$

Entry 1.2. [38, p. 2] *Suppose that a and b are real numbers such that $a^2 + ab + b^2 = 3$. Then*

$$\begin{aligned} & (a+1)f_{-a}(q) + (b+1)f_{-b}(q) - (a+b-1)f_{a+b}(q) \\ &= 3 \frac{(q^3; q^3)_\infty^2}{(q; q)_\infty} \prod_{n=1}^{\infty} \frac{1}{1 + ab(a+b)q^{3n} + q^{6n}}. \end{aligned} \quad (1.11)$$

Entry 1.3. [38, p. 17]

$$\begin{aligned} & \frac{1+\sqrt{3}}{2}f_{-1}(-q) + \frac{3+\sqrt{3}}{6}f_1(-q) \\ &= f_{\sqrt{3}}(q) + \frac{2}{\sqrt{3}}\psi(-q) \frac{(q^4; q^4)_\infty}{(q^6; q^6)_\infty} \prod_{n=1}^{\infty} \frac{1}{1 + \sqrt{3}q^n + q^{2n}}. \end{aligned} \quad (1.12)$$

Entry 1.4. [38, p. 17]

$$\begin{aligned} & \frac{1}{2}(1 + e^{\pi i/4})f_0(iq) + \frac{1}{2}(1 + e^{-\pi i/4})f_0(-iq) \\ &= f_{\sqrt{2}}(q) + \frac{1}{\sqrt{2}}\psi(-q)(-q^2; q^4)_\infty \prod_{n=1}^{\infty} \frac{1}{1 + \sqrt{2}q^n + q^{2n}}. \end{aligned} \quad (1.13)$$

The above four identities were first proved by Yesilyurt [44]. His proofs depend upon a lemma of Atkin and Swinnerton-Dyer [7, Lemma 2]. In 2018, Andrews, Berndt, Chan, S. Kim and A. Malik [3] offered new proofs of the four identities. In [33], E. T. Mortenson used the rank deviations to provide a different proof of Entry 1.4.

The new proof of Entry 1.4 in [3] uses the 2-dissections of the rank generating function modulo 4 and 8, which are proved for the first time in [3]. Definition of dissection is given below:

Definition 1.1. Let $P(q)$ denote any power series in q . Then the t -dissection of $P(q)$ is given by

$$P(q) =: \sum_{k=0}^{t-1} q^k P_k(q^t).$$

Now we state the 2-dissections of ranks modulo 4 and 8:

Theorem 1.2 ([3], Thm. 1.6). The 2-dissection of the rank function $G(i, q)$ is given by

$$\begin{aligned} G(i, q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(iq)_n (q/i)_n} \\ &= \frac{2}{(q^{16}; q^{16})_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{24n^2+8n}}{1 + q^{16n+2}} - \frac{[q^4; q^{16}]_{\infty}^2 (q^{16}; q^{16})_{\infty}}{[-q^2, -q^2, -q^6; q^{16}]_{\infty}} \\ &\quad + \frac{2}{(q^{16}; q^{16})_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{24n^2+24n+5}}{1 + q^{16n+6}} + q \frac{[q^4; q^{16}]_{\infty}^2 (q^{16}; q^{16})_{\infty}}{[-q^2, -q^6, -q^6; q^{16}]_{\infty}}. \end{aligned}$$

Theorem 1.3 ([3], Thm. 1.7). Let ω_8 be a primitive eighth root of unity. Then

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\omega_8 q)_n (q/\omega_8)_n} \\ &= \frac{2 - \omega_8 - 1/\omega_8}{(q^{16}; q^{16})_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{24n^2+8n}}{1 - q^{16n+2}} \\ &\quad + \frac{(\omega_8 + 1/\omega_8 - 1) [q^4; q^{16}]_{\infty}^2 (q^{16}; q^{16})_{\infty}}{[q^2, q^2, q^6; q^{16}]_{\infty}} \\ &\quad + q \frac{[q^4; q^{16}]_{\infty}^2 (q^{16}; q^{16})_{\infty}}{[q^2, q^6, q^6; q^{16}]_{\infty}} + \frac{\omega_8 + 1/\omega_8}{(q^{16}; q^{16})_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{24n^2+24n+5}}{1 - q^{16n+6}}. \end{aligned} \tag{1.14}$$

The results about generating functions for rank differences modulo 5 and 7 are provided by Atkin and Swinnerton-Dyer in [7]. Years later, Atkin with S. M. Hussain [6] established the rank differences modulo 11. From Theorem 8.16 in Garvan's paper [20], we know that results on dissections of the rank modulo m are equivalent to results on rank difference results modulo m .

From the above discussion we can see that a dissection of generating function of rank implies properties of the rank of partition. Chan and R. Mao worked out

the 3-dissections of the rank modulo 3 in [16],

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\omega q, q/\omega)_n} \\
&= -\frac{q^3 J_{27}^3}{J_9 J_{12,27}} + \frac{J_{12,27} J_{27}^3}{J_9 J_{3,27} J_{6,27}} - \frac{3}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+9}}{1 - q^{27n+21}} \\
&\quad + \frac{q J_{27}^3}{J_9 J_{3,27}} + \frac{q^4 J_{3,27} J_{27}^3}{J_9 J_{6,27} J_{12,27}} + \frac{3}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+7}}{1 - q^{27n+12}} \\
&\quad + \frac{q^2 J_{27}^3}{J_9 J_{6,27}} + \frac{q^2 J_{6,27} J_{27}^3}{J_9 J_{3,27} J_{12,27}} - \frac{3}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+2}}{1 - q^{27n+3}},
\end{aligned}$$

where ω is a third root of unity. This gives the rank modulo 3. Mao [30] also gave a 5-dissection of the rank modulo 10. The previous results concentrate on identities on the rank differences, and there are also many research papers discussing about inequalities on the rank differences, see [15, 16, 24]. One can also consider the rank differences for other types of partitions, for example overpartitions, see [29].

Except for a 3-dissection of the rank modulo 3 and a 5-dissection of the rank modulo 10 mentioned above, there is no further result on dissections of rank. This inspired us to find dissections of rank modulo different integers. In fact, in Chapter 2, we obtained a 3-dissection of the rank modulo 9.

If we focus on Entries 1.1–1.4 themselves, we can observe that all of them link generating functions of rank modulo different integers with an infinite product. A natural question that arises is whether there are other analogous identities. From Theorem 8.16 in Garvan [20], we know that a dissection is a good starting point to explore.

Combining the 3-dissection of the rank modulo 9 with the 3-dissection of the rank modulo 3 [16], we get our main result in Chapter 2, which involves generating functions of rank and crank and an infinite product.

In Chapter 3, we continue to explore this topic and try to find similar identities. We exhibit our general method in Chapter 3, which gives identities connecting

generating functions of ranks modulo certain integers. Moreover, we apply it to Entries 1.2, 1.3 and 1.4, respectively, to obtain new identities.

The second topic of this thesis is Ramanujan's convolution sum. To begin, we define some notations. Let $\mathbb{N}_0$ denote non-negative integers and $\mathbb{N}$ for positive integers. Let χ and ψ be primitive Dirichlet characters with conductors L and M , respectively. Let $k \in \mathbb{N}$ and we assume $z \in \mathbb{H}$ and $q = e^{2\pi iz}$. The generalized Bernoulli numbers $B_{k,\chi}$ attached to χ and the Dirichlet L-function of χ are defined by

$$\sum_{k=0}^{\infty} \frac{B_{k,\chi} x^k}{k!} = \sum_{a=1}^L \frac{\chi(a) x e^{ax}}{e^{Lx} - 1} \quad \text{and} \quad L(k, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^k},$$

respectively. For each quadratic discriminant t , we put $\chi_t(n) = \left(\frac{t}{n}\right)$, where $\left(\frac{t}{n}\right)$ is the Kronecker symbol (see [32, p.296], and the definition is given in Chapter 4).

For $k, n \in \mathbb{N}_0$ we define $\sigma_{(k,\chi,\psi)}(n)$ by

$$\sigma_{(k,\chi,\psi)}(n) = \begin{cases} 0 & \text{if } n = 0, \psi \neq \chi_1 \text{ and } k \in \mathbb{N}, \\ 0 & \text{if } n = 0, \chi \neq \chi_1, \psi \neq \chi_1 \text{ and } k = 0, \\ \frac{-B_{1,\psi}}{2} & \text{if } n = 0, \chi = \chi_1 \text{ and } k = 0, \\ \frac{-B_{k+1,\chi}}{2(k+1)} & \text{if } n = 0, \psi = \chi_1 \text{ and } k \in \mathbb{N}_0, \\ \sum_{1 \leq d|n} \chi(d) \psi(n/d) d^k & \text{if } n \geq 1. \end{cases} \quad (1.15)$$

If $n \notin \mathbb{N}_0$ we set $\sigma_{(k,\chi,\psi)}(n) = 0$. For convenience, we use the following notations

$$\begin{aligned} \sigma_k(n) &:= \sigma_{(k,\chi_1,\chi_1)}(n), \quad \hat{\sigma}_k(n) := \sigma_{(k,\chi_1,\chi_{-4})}(n), \\ \tilde{\sigma}_k(n) &:= \sigma_{(k,\chi_{-4},\chi_1)}(n), \quad \bar{\sigma}_k(n) := \sigma_{(k,\chi_{-4},\chi_{-4})}(n). \end{aligned}$$

Let

$$R(l, m, k) := -\frac{B_{2l}}{2l} \frac{B_{2m}}{2m} \frac{k}{B_{2k}}.$$

In [36] Ramanujan proved the following formula for convolutions of sum of divisors functions.

Theorem 1.4 (Ramanujan). For $k = l + m$ and $n \in \mathbb{N}_0$,

$$\begin{aligned} & \sum_{0 \leq j \leq n} \sigma_{2l-1}(j) \sigma_{2m-1}(n-j) \\ &= R(l, m, k) \sigma_{2k-1}(n) + \frac{\zeta(2-2l) + \zeta(2-2m)}{2k-2} n \sigma_{2k-3}(n) + O(n^{\frac{2}{3}(2l+2m-1)}). \end{aligned}$$

In the literature, some formulas are given for these convolution sums of fixed weight divisor sums where certain arithmetic progressions on j are considered. For instance, K. S. Williams [42] worked out the case of $\sum_{m < n/8} \sigma(m) \sigma(n-8m)$. In [35], B. Ramakrishnan and B. Sahu evaluated the convolution sums $\sum_{l+15m=n} \sigma(l) \sigma(m)$ and $\sum_{3l+5m=n} \sigma(l) \sigma(m)$. S. Cooper and D. Ye evaluated $\sum_{l+20m=n} \sigma(l) \sigma(m)$, $\sum_{4l+5m=n} \sigma(l) \sigma(m)$ and $\sum_{2l+5m=n} \sigma(l) \sigma(m)$ in [18]. Z. S. Aygin [9] computed Fourier series expansions of weight 2 and weight 4 Eisenstein series at various cusps (elements from $\mathbb{Q} \cup \infty$), which could be used to give formulas for the convolution sums $\sum_{a+pb=n} \sigma(a) \sigma(b)$, $\sum_{p_1 a + p_2 b = n} \sigma(a) \sigma(b)$ and $\sum_{a+p_1 p_2 b = n} \sigma(a) \sigma(b)$ where p, p_1, p_2 are primes.

Some mathematicians worked on certain twists with Dirichlet characters with fixed weights. For example, H. Hahn obtained relations like $8 \sum_{m < n/2} \bar{\sigma}(m) \bar{\sigma}(n-2m) = -\tilde{\sigma}_3(n/2) + (n-1)\tilde{\sigma}(n) + (2n-1)\bar{\sigma}(n/2)$ in [21]. D. Kim computed values for various convolution sums in some special case in [25]. For example, he provided values of $\sum_{k=1}^{N-1} \sigma_1(dk) \sigma_1(N-mk)$ where d and m are powers of 2. P. C. Toh established $48 \sum_{m < n} \hat{\sigma}_1(n) \hat{\sigma}_1(n-m) = -4\hat{\sigma}_1(n) + 5\sigma_3(n) - \tilde{\sigma}_3(n)$ in [41].

In Chapter 4, we consider convolution sums of divisors functions twisted by χ_{-4} . Ramanujan's formula given by Theorem 1.4 covers the cases when powers of divisors are odd, while our results are the cases when powers of divisors are even. In this sense, our results can be viewed as complementary to Ramanujan's formula.

Chapter 2

An Identity Related To The Partition Rank Modulo 3 and 9

2.1 Introduction

We restate several definitions and identities in this chapter to enhance the coherence.

Recall (1.13) is

Entry 1.4. [38, p. 17]

$$\begin{aligned} & \frac{1}{2}(1 + e^{\pi i/4})f_0(iq) + \frac{1}{2}(1 + e^{-\pi i/4})f_0(-iq) \\ &= f_{\sqrt{2}}(q) + \frac{1}{\sqrt{2}}\psi(-q)(-q^2; q^4)_\infty \prod_{n=1}^{\infty} \frac{1}{1 + \sqrt{2}q^n + q^{2n}}, \end{aligned} \quad (2.1.1)$$

where for $a \in \mathbb{R}$

$$f_a(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1 + aq + q^2)(1 + aq^2 + q^4) \cdots (1 + aq^n + q^{2n})} \quad (2.1.2)$$

and

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}. \quad (2.1.3)$$

The q -series occurring in (2.1.1) and (2.1.2) are related to the generating function for partition ranks. Recall that Dyson [19] introduced and defined the *rank of a partition* as the largest part minus the number of parts. Let $N(m, n)$ denote the number of partitions of n with rank m . Then Dyson showed that the generating function for $N(m, n)$ is

$$G(z, q) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N(m, n) q^n z^m = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (q/z)_n},$$

and we see that f_a is a specialization of this generating function,

$$f_a(q) = G\left(\frac{-a \pm \sqrt{a^2 - 4}}{2}, q\right).$$

If $|a| \leq 2$ then $(-a \pm \sqrt{a^2 - 4})/2$ is a root of unity, and so $f_a(q)$ is a mock theta function for $-2 < a \leq 2$ by [45, Thm. 7.1]. Several of Ramanujan's third order mock theta functions are instances of $f_a(q)$. Our results involve third order mock theta functions, and there are many papers related to mock theta functions of third order, see [5, 17].

Attempts to find other dissections of $G(z; q)$ that lead to analogous identities bring us to the following result.

Theorem 2.1.1. Let ζ be a primitive ninth root of unity and let $\omega = \zeta^3$, i.e., ω is a primitive third root of unity. Define

$$C(z, q) := \frac{(q)_{\infty}}{(zq, q/z)_{\infty}}, \quad (2.1.4)$$

and recall that by (2.1.2),

$$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\omega q, q/\omega)_n} \quad \text{and} \quad f_{-2\Re(\zeta)}(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta q, q/\zeta)_n}.$$

Then

$$\begin{aligned} & \frac{1}{3}(1 - \zeta) (f_1(\omega q) - \omega^2 C(\omega, \omega q)) + \frac{1}{3}(1 - \zeta^{-1}) (f_1(\omega^2 q) - \omega C(\omega, \omega^2 q)) \\ &= f_{-2\Re(\zeta)}(q) + \frac{1}{3}(\zeta^2 + \zeta^{-2} - \zeta - \zeta^{-1}) \frac{J_3^3}{J_1 J_9}. \end{aligned} \quad (2.1.5)$$

The proof of identity (2.1.5) relies on properties of Appell-Lerch series [22, Eq. (16)] along with the 3-dissection of the rank modulo 3, which first appeared in [16]. (See (2.2.1) and (2.2.2)).

Observe that (2.1.5), unlike (2.1.1), contains extra infinite products on the left-hand side. These products are specializations of $C(z, q)$, which is the generating function for the partition crank. The crank is a partition statistic that was predicted by Dyson [19] and finally discovered by Andrews and Garvan [4] in 1988. To define it, let $w(\lambda)$ denote the number of ones in a partition λ and let $v(\lambda)$ denote the number of parts greater than $w(\lambda)$. Then the crank is equal to the largest part, if $w(\lambda) = 0$, and $v(\lambda) - w(\lambda)$ otherwise. Andrews and Garvan showed that if $M(m, n)$ is defined via

$$C(z, q) = \sum_{n=0}^{\infty} M(m, n) z^m q^n,$$

then for $n \geq 2$, $M(m, n)$ is the number of partitions of n with crank equal to m .

A consequence of (2.1.5) is the 3-dissection of the rank generating function modulo 9.

Theorem 2.1.2. Let ζ be a primitive ninth root of unity. Then

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta q, q/\zeta)_n} \\ &= (\zeta^2 + \zeta^{-2}) \frac{q^3 J_{27}^3}{J_9 J_{12,27}} + \frac{J_{12,27} J_{27}^3}{J_9 J_{3,27} J_{6,27}} + \frac{\zeta + \zeta^{-1} - 2}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+9}}{1 - q^{27n+21}} \\ &+ \frac{q J_{27}^3}{J_9 J_{3,27}} + (\zeta + \zeta^{-1} - 1) \frac{q^4 J_{3,27} J_{27}^3}{J_9 J_{6,27} J_{12,27}} + \frac{\zeta + \zeta^{-1} + \zeta^2 + \zeta^{-2} - 1}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+7}}{1 - q^{27n+12}} \\ &+ (\zeta + \zeta^{-1} - 1) \frac{q^2 J_{27}^3}{J_9 J_{6,27}} - (\zeta^2 + \zeta^{-2}) \frac{q^2 J_{6,27} J_{27}^3}{J_9 J_{3,27} J_{12,27}} + \frac{1 + \zeta^2 + \zeta^{-2}}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+2}}{1 - q^{27n+3}}. \end{aligned}$$

Dissections like this can be used to obtain results on $N(k, m, n)$, the number of partitions of n where the rank is congruent to k modulo m . Dyson [19] was the first to observe identities like

$$N(1, 5, 5n + 1) = N(2, 5, 5n + 1),$$

$$N(i, 5, 5n + 4) = N(j, 5, 5n + 4) \text{ for } 0 \leq i, j \leq 4,$$

$$N(i, 7, 7n + 5) = N(j, 7, 7n + 5) \text{ for } 0 \leq i, j \leq 6,$$

$$N(0, 7, 7n + 6) + N(1, 7, 7n + 6) = N(2, 7, 7n + 6) + N(3, 7, 7n + 6).$$

Dyson's observations were confirmed by Atkin and Swinnerton-Dyer [7].

By comparing Theorem 2.1.2 with the 3-dissection of the rank modulo 3 given in (2.2.1), we see that the same series and products appear in the dissections. This leads to the following rank identities, which were conjectured by R. P. Lewis [28] and first proved by Santa-Gadea in his thesis [39]. This phenomenon is also presented in [3], where dissections of the rank modulo 4 and 8 imply relations on ranks modulo 8.

Corollary 2.1.3. For all $n \geq 0$ we have

$$N(3, 9, 3n) = N(4, 9, 3n) \quad (2.1.6)$$

$$N(1, 9, 3n+1) + N(2, 9, 3n+1) = N(3, 9, 3n+1) + N(4, 9, 3n+1) \quad (2.1.7)$$

$$N(0, 9, 3n+2) = N(4, 9, 3n+2). \quad (2.1.8)$$

In Section 2.2, we prove Theorems 2.1.1 and 2.1.2 as well as Corollary 2.1.3. In Section 2.3 we give a different proof of Theorem 2.1.2.

2.2 Proofs of Theorems 2.1.1 and 2.1.2

First, we record some results from [16]. Let ω be a primitive third root of unity. Theorem 1.1 and Corollary 1.2 of [16] give two versions of the 3-dissections of the rank modulo 3,

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\omega q, q/\omega)_n} \\ &= -\frac{q^3 J_{27}^3}{J_9 J_{12,27}} + \frac{J_{12,27} J_{27}^3}{J_9 J_{3,27} J_{6,27}} - \frac{3}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+9}}{1 - q^{27n+21}} \\ &+ \frac{q J_{27}^3}{J_9 J_{3,27}} + \frac{q^4 J_{3,27} J_{27}^3}{J_9 J_{6,27} J_{12,27}} + \frac{3}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+7}}{1 - q^{27n+12}} \\ &+ \frac{q^2 J_{27}^3}{J_9 J_{6,27}} + \frac{q^2 J_{6,27} J_{27}^3}{J_9 J_{3,27} J_{12,27}} - \frac{3}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+2}}{1 - q^{27n+3}} \end{aligned} \quad (2.2.1)$$

$$\begin{aligned} &= \frac{1}{J_9} \sum_{n=-\infty}^{\infty} \left\{ \frac{(-1)^n q^{(27n^2+9n)/2}}{1 - q^{27n+3}} - \frac{(-1)^n q^{(27n^2+27n)/2+3}}{1 - q^{27n+12}} - 2 \frac{(-1)^n q^{(27n^2+45n)/2+9}}{1 - q^{27n+21}} \right\} \\ &+ \frac{q}{J_9} \sum_{n=-\infty}^{\infty} \left\{ \frac{(-1)^n q^{(27n^2+27n)/2}}{1 - q^{27n+3}} + 2 \frac{(-1)^n q^{(27n^2+45n)/2+6}}{1 - q^{27n+12}} + \frac{(-1)^n q^{(27n^2+63n)/2+15}}{1 - q^{27n+21}} \right\} \\ &+ \frac{q^2}{J_9} \sum_{n=-\infty}^{\infty} \left\{ -2 \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+3}} - \frac{(-1)^n q^{(27n^2+63n)/2+9}}{1 - q^{27n+12}} + \frac{(-1)^n q^{(27n^2+81n)/2+21}}{1 - q^{27n+21}} \right\}. \end{aligned} \quad (2.2.2)$$

Similarly, from Theorem 1.3 and Corollary 1.4 of [16], we have respectively,

$$\begin{aligned} & \frac{(q)_\infty}{(\omega q, q/\omega)_\infty} \\ &= 2 \frac{q^3 J_{27}^3}{J_9 J_{12,27}} + \frac{J_{12,27} J_{27}^3}{J_9 J_{3,27} J_{6,27}} - 2 \frac{q J_{27}^3}{J_9 J_{3,27}} + \frac{q^4 J_{3,27} J_{27}^3}{J_9 J_{6,27} J_{12,27}} - 2 \frac{q^2 J_{27}^3}{J_9 J_{6,27}} + \frac{q^2 J_{6,27} J_{27}^3}{J_9 J_{3,27} J_{12,27}} \end{aligned} \quad (2.2.3)$$

$$\begin{aligned} &= \frac{1}{J_9} \sum_{n=-\infty}^{\infty} \left\{ \frac{(-1)^n q^{(27n^2+9n)/2}}{1 - q^{27n+3}} + 2 \frac{(-1)^n q^{(27n^2+27n)/2+3}}{1 - q^{27n+12}} + \frac{(-1)^n q^{(27n^2+45n)/2+9}}{1 - q^{27n+21}} \right\} \\ &+ \frac{q}{J_9} \sum_{n=-\infty}^{\infty} \left\{ -2 \frac{(-1)^n q^{(27n^2+27n)/2}}{1 - q^{27n+3}} - \frac{(-1)^n q^{(27n^2+45n)/2+6}}{1 - q^{27n+12}} + \frac{(-1)^n q^{(27n^2+63n)/2+15}}{1 - q^{27n+21}} \right\} \\ &+ \frac{q^2}{J_9} \sum_{n=-\infty}^{\infty} \left\{ \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+3}} - \frac{(-1)^n q^{(27n^2+63n)/2+9}}{1 - q^{27n+12}} - 2 \frac{(-1)^n q^{(27n^2+81n)/2+21}}{1 - q^{27n+21}} \right\}. \end{aligned} \quad (2.2.4)$$

Next, by Lemma 2.2 of [16], we see that

$$\begin{aligned} m(q, q^3, \omega) &= \frac{1}{(1-\omega)J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+9n)/2}}{1 - q^{27n+3}} (-\omega - \omega^2 q^{9n+1} - q^{18n+2}) \\ &+ \frac{1}{(1-\omega)J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+27n)/2+3}}{1 - q^{27n+12}} (\omega^2 + q^{9n+4} + \omega q^{18n+8}) \\ &+ \frac{1}{(1-\omega)J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+9}}{1 - q^{27n+21}} (-1 - \omega q^{9n+7} - \omega^2 q^{18n+14}) \end{aligned} \quad (2.2.5)$$

and

$$\begin{aligned} \omega^2 m(q, q^3, \omega^2) &= \frac{1}{(1-\omega^2)J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+9n)/2}}{1 - q^{27n+3}} \{-\omega - q^{9n+1} - \omega^2 q^{18n+2}\} \\ &+ \frac{1}{(1-\omega^2)J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+27n)/2+3}}{1 - q^{27n+12}} \{1 + \omega^2 q^{9n+4} + \omega q^{18n+8}\} \\ &+ \frac{1}{(1-\omega^2)J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+9}}{1 - q^{27n+21}} \{-\omega^2 - \omega q^{9n+7} - q^{18n+14}\}, \end{aligned} \quad (2.2.6)$$

where

$$m(x, q, z) := \frac{1}{(z, q/z, q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n-1)/2} z^n}{1 - q^{n-1} x z} \quad (2.2.7)$$

is the Appell-Lerch series.

Using the above results, we derive the following lemma.

Lemma 2.2.1.

$$m(q, q^3, \omega) = \frac{1}{3} (f_1(q) - \omega C(\omega, q)), \quad (2.2.8)$$

$$m(q, q^3, \omega^2) = \frac{1}{3} (f_1(q) - \omega^2 C(\omega, q)). \quad (2.2.9)$$

Proof. Identity (2.2.8) follows from applying (2.2.2) and (2.2.4) to the right side of (2.2.5). Similarly, (2.2.9) follows from applying (2.2.2) and (2.2.4) to (2.2.6). $\square$

Next we need an expression for the rank generating function in terms of the Appell-Lerch series.

Lemma 2.2.2.

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(xq, q/x)_n} &= (1-x)m(x^3q, q^3, x^{-3}z^{-1}) + (1-x^{-1})m(x^{-3}q, q^3, x^3z) \\ &\quad + \frac{x(1-x)J_1^2j(xz; q)j(z; q^3)}{j(x; q)j(z; q)j(x^3z; q^3)}. \end{aligned}$$

Proof. From [22, Eq. (16)], we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(xq, q/x)_n} &= x(1-x) \left(-\frac{1}{x}m(x^{-3}q^2, q^3, x^3z) - \frac{1}{x^2}m(x^{-3}q, q^3, x^3z) \right. \\ &\quad \left. + \frac{J_1^2j(xz; q)j(z; q^3)}{j(x; q)j(z; q)j(x^3z; q^3)} \right) + (1-x) \\ &= (1-x)(1-m(x^{-3}q^2, q^3, x^3z)) + (1-1/x)m(x^{-3}q, q^3, x^3z) \\ &\quad + \frac{x(1-x)J_1^2j(xz; q)j(z; q^3)}{j(x; q)j(z; q)j(x^3z; q^3)}. \end{aligned} \quad (2.2.10)$$

Using [22, Prop. (21b)(21c)], we get

$$\begin{aligned} 1 - m(x^{-3}q^2, q^3, x^3z) &= 1 - x^3q^{-2}m(x^3q^{-2}, q^3, x^{-3}z^{-1}) \\ &= m(x^3q, q^3, x^{-3}z^{-1}). \end{aligned} \quad (2.2.11)$$

Substituting identity (2.2.11) into (2.2.10), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(xq, q/x)_n} &= (1-x)m(x^3q, q^3, x^{-3}z^{-1}) + (1-x^{-1})m(x^{-3}q, q^3, x^3z) \\ &\quad + \frac{x(1-x)J_1^2j(xz; q)j(z; q^3)}{j(x; q)j(z; q)j(x^3z; q^3)}, \end{aligned} \quad (2.2.12)$$

and this completes the proof of the lemma. $\square$

Set $x = \zeta$, and let z tend to 1 in Lemma 2.2.2. Noting that $j(\omega, q) = (1-\omega)J_9$, we obtain the following corollary.

Corollary 2.2.3.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta q, q/\zeta)_n} = (1-\zeta)m(\omega q, q^3, \omega^2) + (1-1/\zeta)m(\omega^2 q, q^3, \omega) + \frac{\zeta(1-\zeta)}{1-\omega} \frac{J_3^3}{J_1 J_9}. \quad (2.2.13)$$

We need one more lemma for our proof of Theorems 2.1.1 and 2.1.2.

Lemma 2.2.4.

$$\frac{1}{J_1} = \frac{J_9}{J_3^4} \{ (J_{12,27}^2 - q^3 J_{3,27} J_{6,27}) + (q J_{6,27} J_{12,27} + q^4 J_{3,27}^2) + (q^2 J_{6,27}^2 + q^2 J_{3,27} J_{12,27}) \} \quad (2.2.14)$$

Proof. First note that a simple consequence of the Jacobi triple product identity [11, p. 35, Entry 19] and [11, p. 48, Entry 31] is

$$J_1 = J_{12,27} - q J_{6,27} - q^2 J_{3,27}. \quad (2.2.15)$$

Substituting (2.2.15) into (2.2.14), cross multiplying and simplifying, we see that it suffices to prove

$$\frac{J_9}{J_3^4} (J_{12,27}^3 - q^3 J_{6,27}^3 - q^6 J_{3,27}^3 - 3q^3 J_{3,27} J_{6,27} J_{12,27}) = 1. \quad (2.2.16)$$

By applying the second and third identities from the bottom of page 351 of [11], we see that (2.2.16) is equivalent to

$$\frac{J_9}{J_3^4} (J_1^3 + 3qJ_9^3 - 9q^3 J_{3,27} J_{6,27} J_{12,27}) = 1. \quad (2.2.17)$$

Multiplying both sides by J_3^3 and rearranging, we see that (2.2.17) is equivalent to

$$\frac{J_9}{J_3} (J_1^3 + 3qJ_9^3) = J_3^3 + 9q^3 J_{27}^3. \quad (2.2.18)$$

This follows from Entry 1(iv) on page 345 of [11], and so we complete the proof of Lemma 2.2.4. $\square$

Theorems 2.1.1 and 2.1.2 now follow readily from the work above.

Proof of Theorem 2.1.1. First, using

$$1 + \zeta^3 + \zeta^6 = 0, \quad (2.2.19)$$

it is easy to verify that $\zeta(1 - \zeta)/(1 - \omega) = \frac{1}{3}(\zeta + \zeta^{-1} - \zeta^2 - \zeta^{-2})$. To complete the proof, it suffices to apply Lemma 2.2.1 to Corollary 2.2.3 and rearrange terms. $\square$

Proof of Theorem 2.1.2. Apply (2.2.1) and (2.2.3) on the left hand side of (2.1.5) and Lemma 2.2.4 on the right hand side and rearrange terms. $\square$

We end this section with a proof of Corollary 2.1.3.

Proof of Corollary 2.1.3. We use the notation

$$N_a = \sum_{n \geq 0} N(a, 9, n) q^n,$$

$$N_a(d) = \sum_{n \geq 0} N(a, 9, 3n + d) q^{3n+d},$$

and note that since the rank generating function is invariant under $z \leftrightarrow 1/z$, we have $N_a = N_{9-a}$ and $N_a(d) = N_{9-a}(d)$. Together with (2.2.19) and the fact that

$$\zeta + \zeta^2 + \zeta^4 + \zeta^5 + \zeta^7 + \zeta^8 = 0, \quad (2.2.20)$$

we find that

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta q, q/\zeta)_n} = N_0 - N_3 + (\zeta + \zeta^{-1})(N_1 - N_4) + (\zeta^2 + \zeta^{-2})(N_2 - N_4) \quad (2.2.21)$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\omega q, q/\omega)_n} = N_0 - N_1 - N_2 + 2N_3 - N_4. \quad (2.2.22)$$

Comparing coefficients of $1, \zeta + \zeta^{-1}$, and $\zeta^2 + \zeta^{-2}$ in (2.2.21) and the first line of Theorem 2.1.2, we find

$$N_0(0) - N_3(0) = \frac{J_{12,27} J_{27}^3}{J_9 J_{3,27} J_{6,27}} - \frac{2}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+9}}{1 - q^{27n+21}},$$

$$N_1(0) - N_4(0) = \frac{1}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+9}}{1 - q^{27n+21}},$$

$$N_2(0) - N_4(0) = \frac{q^3 J_{27}^3}{J_9 J_{12,27}}.$$

Comparing this with (2.2.22) and (2.2.1) we deduce that

$$N_0(0) - N_3(0) - N_1(0) + N_4(0) - N_2(0) + N_4(0) = N_0(0) + 2N_3(0) - N_1(0) - N_2(0) - N_4(0).$$

Simplifying, we obtain (2.1.6). The proofs of equations (2.1.7) and (2.1.8) are similar. $\square$

2.3 A Different Proof of Theorem 2.1.2

In this section, we provide a different proof of Theorem 2.1.2 using a result from [13].

Letting $q \rightarrow q^{27}$ in [13, Eq. (4.4)] leads us to following lemma.

Lemma 2.3.1.

$$\begin{aligned} \frac{[a; q^{27}]_{\infty} (q^{27}; q^{27})_{\infty}^2}{[b; q^{27}]_{\infty} [c; q^{27}]_{\infty}} &= \frac{[a/b; q^{27}]_{\infty}}{[c/b; q^{27}]_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+27n)/2} (a/c)^n}{1 - bq^{27n}} \\ &\quad + \frac{[a/c; q^{27}]_{\infty}}{[b/c; q^{27}]_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+27n)/2} (a/b)^n}{1 - cq^{27n}}. \end{aligned} \quad (2.3.1)$$

Proof of Theorem 2.1.2. For $m(\omega q, q^3, \omega^2)$, we have

$$\begin{aligned} &m(\omega q, q^3, \omega^2) \\ &= -\frac{\omega^2}{j(\omega^2; q^3)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2} \omega^{2n}}{1 - q^{3n+1} \omega^3} \\ &= -\frac{\omega^2}{j(\omega^2; q^3)} \left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n(n+1)/2}}{1 - q^{9n+1}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{3(3n+1)(3n+2)/2} \omega^2}{1 - q^{9n+4}} \right. \\ &\quad \left. + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3(3n+2)(3n+3)/2} \omega}{1 - q^{9n+7}} \right) \\ &= -\frac{\omega^2}{j(\omega^2; q^3)} \left\{ \left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+9n)/2}}{1 - q^{27n+3}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+27n)/2+1}}{1 - q^{27n+3}} \right. \right. \\ &\quad \left. + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+2}}{1 - q^{27n+3}} \right) + \omega^2 \left(\sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{27n^2+27n/2+3}}{1 - q^{27n+12}} \right. \\ &\quad \left. + \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{(27n^2+45n)/2+7}}{1 - q^{27n+12}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{(27n^2+63n)/2+11}}{1 - q^{27n+12}} \right) \\ &\quad \left. + \omega \left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+9}}{1 - q^{27n+21}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+63n)/2+16}}{1 - q^{27n+21}} \right. \right. \\ &\quad \left. \left. + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+81n)/2+23}}{1 - q^{27n+21}} \right) \right\}, \end{aligned} \quad (2.3.2)$$

where $m(x, q, z)$ is defined in (2.2.7). Note that in the second equality, we split the sum by separating n into $3n, 3n + 1$ and $3n + 2$, and for the third one, we used the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ to transform all the denominators $1 - q^{9n+1}, 1 - q^{9n+4}$ and $1 - q^{9n+7}$ into the forms $1 - q^{27n+1}, 1 - q^{27n+12}$ and $1 - q^{27n+21}$, respectively. For the product part in (2.3.2), we have

$$j(\omega^2; q^3) = (\omega^2; q^3)_\infty (q^3/\omega^2; q^3)_\infty (q^3; q^3)_\infty = (1 - \omega^2)J_9 = \frac{3J_9}{1 - \omega}. \quad (2.3.3)$$

Similarly, for $m(\omega^2 q, q^3, \omega)$,

$$\begin{aligned} & m(\omega^2 q, q^3, \omega) \\ &= -\frac{\omega}{j(\omega; q^3)} \left\{ \left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+9n)/2}}{1 - q^{27n+3}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+27n)/2+1}}{1 - q^{27n+3}} \right. \right. \\ & \quad \left. \left. + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+2}}{1 - q^{27n+3}} \right) + \omega \left(\sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{(27n^2+27n)/2+3}}{1 - q^{27n+12}} \right. \right. \\ & \quad \left. \left. + \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{(27n^2+45n)/2+7}}{1 - q^{27n+12}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{(27n^2+63n)/2+11}}{1 - q^{27n+12}} \right) \right. \\ & \quad \left. + \omega^2 \left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+9}}{1 - q^{27n+21}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+63n)/2+16}}{1 - q^{27n+21}} \right. \right. \\ & \quad \left. \left. + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+81n)/2+23}}{1 - q^{27n+21}} \right) \right\}. \end{aligned} \quad (2.3.4)$$

Here for the product part in (2.3.4), we have

$$j(\omega; q^3) = (\omega; q^3)_\infty (q^3/\omega; q^3)_\infty (q^3; q^3)_\infty = (1 - \omega)J_9 = \frac{3J_9}{1 - \omega^2}. \quad (2.3.5)$$

Taking $(a, b, c) = (q^{12}, q^3, q^{21}), (q^{12}, q^3, q^{12}), (q^{21}, q^{12}, q^{21}), (q^{21}, q^{12}, q^3), (q^{30}, q^{21}, q^{12}), (q^{30}, q^{21}, q^3)$, respectively, in (2.3.1), then we get:

$$\frac{[q^{12}; q^{27}]_\infty J_{27}^2}{[q^3; q^{27}]_\infty [q^{21}; q^{27}]_\infty} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+9n)/2}}{1 - q^{27n+3}} + \frac{[q^{-9}; q^{27}]_\infty}{[q^{-18}; q^{27}]_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+21}}, \quad (2.3.6)$$

$$\frac{J_{27}^2}{[q^3; q^{27}]_\infty} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+27n)/2}}{1 - q^{27n+3}}, \quad (2.3.7)$$

$$\frac{J_{27}^2}{[q^{12}; q^{27}]_\infty} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+27n)/2}}{1 - q^{27n+12}}, \quad (2.3.8)$$

$$\frac{[q^{21}; q^{27}]_\infty J_{27}^2}{[q^3; q^{27}]_\infty [q^{12}; q^{27}]_\infty} = \frac{[q^9; q^{27}]_\infty}{[q^{-9}; q^{27}]_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+63n)/2}}{1 - q^{27n+12}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+3}}, \quad (2.3.9)$$

$$\frac{[q^{30}; q^{27}]_\infty J_{27}^2}{[q^{21}; q^{27}]_\infty [q^{12}; q^{27}]_\infty} = \frac{[q^9; q^{27}]_\infty}{[q^{-9}; q^{27}]_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+63n)/2}}{1 - q^{27n+21}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+12}}, \quad (2.3.10)$$

$$\frac{[q^{30}; q^{27}]_\infty J_{27}^2}{[q^{21}; q^{27}]_\infty [q^3; q^{27}]_\infty} = \frac{[q^9; q^{27}]_\infty}{[q^{-18}; q^{27}]_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+81n)/2}}{1 - q^{27n+21}}, \quad (2.3.11)$$

Substituting (2.3.6)–(2.3.11) and (2.3.3) into (2.3.2) gives us

$$\begin{aligned} & m(\omega q, q^3, \omega^2) \\ &= -\frac{\omega^2(1-\omega)}{3J_9} \left(\frac{[q^{12}; q^{27}]_\infty J_{27}^2}{[q^3; q^{27}]_\infty [q^{21}; q^{27}]_\infty} - q^9 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+21}} + q \frac{J_{27}^2}{[q^3; q^{27}]_\infty} \right. \\ & \quad + q^2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+3}} - \omega^2 q^3 \frac{J_{27}^2}{[q^{12}; q^{27}]_\infty} - \omega^2 q^7 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+12}} \\ & \quad + \omega^2 q^2 \frac{[q^{21}; q^{27}]_\infty J_{27}^2}{[q^3; q^{27}]_\infty [q^{12}; q^{27}]_\infty} - \omega^2 q^2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+3}} + \omega q^9 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+21}} \\ & \quad \left. - \omega q^7 \frac{[q^{30}; q^{27}]_\infty J_{27}^2}{[q^{21}; q^{27}]_\infty [q^{12}; q^{27}]_\infty} + \omega q^7 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+12}} - \omega q^5 \frac{[q^{30}; q^{27}]_\infty J_{27}^2}{[q^{21}; q^{27}]_\infty [q^3; q^{27}]_\infty} \right) \\ &= \frac{1-\omega^2}{3J_9} \left(\frac{J_{12,27} J_{27}^3}{J_{3,27} J_{6,27}} - q^9 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+21}} + q \frac{J_{27}^3}{J_{3,27}} + q^2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+3}} \right. \\ & \quad - \omega^2 q^3 \frac{J_{27}^3}{J_{12,27}} - \omega^2 q^7 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+12}} + \omega^2 q^2 \frac{J_{6,27} J_{27}^3}{J_{3,27} J_{12,27}} \\ & \quad - \omega^2 q^2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+3}} + \omega q^9 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+21}} \\ & \quad \left. + \omega q^4 \frac{J_{3,27} J_{27}^3}{J_{6,27} J_{12,27}} + \omega q^7 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+12}} + \omega q^2 \frac{J_{27}^3}{J_{6,27}} \right). \quad (2.3.12) \end{aligned}$$

Substituting (2.3.6)–(2.3.11) and (2.3.5) into (2.3.4) yields

$$\begin{aligned}
& m(\omega^2 q, q^3, \omega) \\
&= -\frac{\omega(1-\omega^2)}{3J_9} \left(\frac{[q^{12}; q^{27}]_\infty J_{27}^2}{[q^3; q^{27}]_\infty [q^{21}; q^{27}]_\infty} - q^9 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1-q^{27n+21}} + q \frac{J_{27}^2}{[q^3; q^{27}]_\infty} \right. \\
&+ q^2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1-q^{27n+3}} - \omega q^3 \frac{J_{27}^2}{[q^{12}; q^{27}]_\infty} - \omega q^7 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1-q^{27n+12}} \\
&+ \omega q^2 \frac{[q^{21}; q^{27}]_\infty J_{27}^2}{[q^3; q^{27}]_\infty [q^{12}; q^{27}]_\infty} - \omega q^2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1-q^{27n+3}} + \omega^2 q^9 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1-q^{27n+21}} \\
&- \omega^2 q^7 \frac{[q^{30}; q^{27}]_\infty J_{27}^2}{[q^{21}; q^{27}]_\infty [q^{12}; q^{27}]_\infty} + \omega^2 q^7 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1-q^{27n+12}} - \omega^2 q^5 \frac{[q^{30}; q^{27}]_\infty J_{27}^2}{[q^{21}; q^{27}]_\infty [q^3; q^{27}]_\infty} \Bigg) \\
&= \frac{1-\omega}{3J_9} \left(\frac{J_{12,27} J_{27}^3}{J_{3,27} J_{6,27}} - q^9 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1-q^{27n+21}} + q \frac{J_{27}^3}{J_{3,27}} \right. \\
&+ q^2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1-q^{27n+3}} - \omega q^3 \frac{J_{27}^3}{J_{12,27}} - \omega q^7 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1-q^{27n+12}} \\
&+ \omega q^2 \frac{J_{6,27} J_{27}^3}{J_{3,27} J_{12,27}} - \omega q^2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1-q^{27n+3}} + \omega^2 q^9 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1-q^{27n+21}} \\
&+ \omega^2 q^4 \frac{J_{3,27} J_{27}^3}{J_{6,27} J_{12,27}} + \omega^2 q^7 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1-q^{27n+12}} + \omega^2 q^2 \frac{J_{27}^3}{J_{6,27}} \Bigg). \tag{2.3.13}
\end{aligned}$$

Substituting (2.3.12) and (2.3.13) into the two sums on the right hand side of (2.2.13), we find that

$$\begin{aligned}
& (1-\zeta)m(\omega q, q^3, \omega^2) + (1-1/\zeta)m(\omega^2 q, q^3, \omega) \\
&= \frac{1}{3J_9} \left\{ (3-\zeta-1/\zeta+\zeta^2+1/\zeta^2) \left(\frac{qJ_{27}^3}{J_{3,27}} + \frac{J_{12,27}J_{27}^3}{J_{3,27}J_{6,27}} \right) \right. \\
&+ (1+\zeta^2+1/\zeta^2)3q^2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1-q^{27n+3}} \\
&+ (-3+2\zeta+2/\zeta+\zeta^2+1/\zeta^2) \left(\frac{q^2J_{27}^3}{J_{6,27}} + \frac{q^4J_{3,27}J_{27}^3}{J_{6,27}J_{12,27}} \right) \\
&+ (-1+\zeta+1/\zeta+\zeta^2+1/\zeta^2)3q^7 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1-q^{27n+12}} \Bigg\}
\end{aligned}$$

$$\begin{aligned}
& + (\zeta + 1/\zeta + 2\zeta^2 + 2/\zeta^2) \left(\frac{q^3 J_{27}^3}{J_{12,27}} - \frac{q^2 J_{6,27} J_{27}^3}{J_{3,27} J_{12,27}} \right) + (-2 + \zeta + 1/\zeta) 3q^9 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+21}} \Big\} \\
& = \frac{1}{3J_9} \left\{ (3 - \zeta - 1/\zeta + \zeta^2 + 1/\zeta^2) \left(\frac{q J_{27}^3}{J_{3,27}} + \frac{J_{12,27} J_{27}^3}{J_{3,27} J_{6,27}} \right) \right. \\
& \quad + (-3 + 2\zeta + 2/\zeta + \zeta^2 + 1/\zeta^2) \left(\frac{q^2 J_{27}^3}{J_{6,27}} + \frac{q^4 J_{3,27} J_{27}^3}{J_{6,27} J_{12,27}} \right) \\
& \quad \left. + (\zeta + 1/\zeta + 2\zeta^2 + 2/\zeta^2) \left(\frac{q^3 J_{27}^3}{J_{12,27}} - \frac{q^2 J_{6,27} J_{27}^3}{J_{3,27} J_{12,27}} \right) \right\} \\
& \quad + \frac{(1 + \zeta^2 + 1/\zeta^2)}{J_9} q^2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+3}} \\
& \quad + \frac{(-1 + \zeta + 1/\zeta + \zeta^2 + 1/\zeta^2)}{J_9} q^7 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+12}} \\
& \quad + \frac{(-2 + \zeta + 1/\zeta)}{J_9} q^9 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+21}}. \tag{2.3.14}
\end{aligned}$$

Applying (2.2.14) into the product on right-hand side of (2.2.13) leads us to:

$$\begin{aligned}
& \frac{\zeta(1 - \zeta)}{1 - \omega} \frac{J_3^3}{J_1 J_9} \\
& = (\zeta - \zeta^2)(1 - \omega^2) \frac{1}{3J_3} \left\{ (J_{12,27}^2 - q^3 J_{3,27} J_{6,27}) \right. \\
& \quad \left. + (q J_{6,27} J_{12,27} + q^4 J_{3,27}^3) + (q^2 J_{6,27}^2 + q^2 J_{3,27} J_{12,27}) \right\} \\
& = (\zeta + 1/\zeta - \zeta^2 - 1/\zeta^2) \frac{1}{3J_9} \left(\frac{J_{12,27} J_{27}^3}{J_{3,27} J_{6,27}} - q^3 \frac{J_{27}^3}{J_{12,27}} \right. \\
& \quad \left. + q \frac{J_{27}^3}{J_{3,27}} + q^4 \frac{J_{3,27} J_{27}^3}{J_{6,27} J_{12,27}} + q^2 \frac{J_{6,27} J_{27}^3}{J_{3,27} J_{12,27}} + q^2 \frac{J_{27}^3}{J_{6,27}} \right), \tag{2.3.15}
\end{aligned}$$

where in the second equality, we used the identity

$$\frac{1}{J_3} = \frac{1}{J_{3,9}} = \frac{J_{27}^3}{J_9 J_{3,27} J_{6,27} J_{12,27}}.$$

Finally, we obtain the 3-dissection of rank modulo 9 by combining (2.3.14) and (2.3.15):

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta q, q/\zeta)_n}$$

$$\begin{aligned}
& = (1 - \zeta)m(\omega q, q^3, \omega^2) + (1 - 1/\zeta)m(\omega^2 q, q^3, \omega) + \frac{\zeta(1 - \zeta)}{1 - \omega} \frac{J_3^3}{J_1 J_9} \\
& = \frac{1}{3J_9} \left\{ 3 \left(\frac{qJ_{27}^3}{J_{3,27}} + \frac{J_{12,27}J_{27}^3}{J_{3,27}J_{6,27}} \right) + 3(-1 + \zeta + 1/\zeta) \left(\frac{q^2 J_{27}^3}{J_{6,27}} + \frac{q^4 J_{3,27}J_{27}^3}{J_{6,27}J_{12,27}} \right) \right. \\
& \quad \left. + 3(\zeta^2 + 1/\zeta^2) \left(\frac{q^3 J_{27}^3}{J_{12,27}} - \frac{q^2 J_{6,27}J_{27}^3}{J_{3,27}J_{12,27}} \right) \right\} + \frac{(1 + \zeta^2 + 1/\zeta^2)}{J_9} q^2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+3}} \\
& \quad + \frac{(-1 + \zeta + 1/\zeta + \zeta^2 + 1/\zeta^2)}{J_9} q^7 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+12}} \\
& \quad + \frac{(-2 + \zeta + 1/\zeta)}{J_9} q^9 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2}}{1 - q^{27n+21}} \\
& = (\zeta^2 + \zeta^{-2}) \frac{q^3 J_{27}^3}{J_9 J_{12,27}} + \frac{J_{12,27}J_{27}^3}{J_9 J_{3,27}J_{6,27}} + \frac{\zeta + \zeta^{-1} - 2}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+9}}{1 - q^{27n+21}} \\
& \quad + \frac{qJ_{27}^3}{J_9 J_{3,27}} + (\zeta + \zeta^{-1} - 1) \frac{q^4 J_{3,27}J_{27}^3}{J_9 J_{6,27}J_{12,27}} + \frac{\zeta + \zeta^{-1} + \zeta^2 + \zeta^{-2} - 1}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+7}}{1 - q^{27n+12}} \\
& \quad + (\zeta + \zeta^{-1} - 1) \frac{q^2 J_{27}^3}{J_9 J_{6,27}} - (\zeta^2 + \zeta^{-2}) \frac{q^2 J_{6,27}J_{27}^3}{J_9 J_{3,27}J_{12,27}} + \frac{1 + \zeta^2 + \zeta^{-2}}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+2}}{1 - q^{27n+3}}.
\end{aligned}$$

□

Chapter 3

On Ranks Of Partitions Modulo Certain Integers

3.1 Introduction

This chapter continues the work of Chapter 2. In Chapter 2, an identity related to the rank modulo 3 and 9 was given after getting a 3-dissection of the rank modulo 9 and comparing it with a 3-dissection of the rank modulo 3. In this chapter, similar identities are obtained and a general way to find such identities modulo certain integers is presented. The second aim of this chapter is to apply our method to Ramanujan's Entries 1.1, 1.3 and 1.4, to obtain new identities. These entries were first proved by Yesilyurt [44]. They were reproved by Andrews, Berndt, Chan, S. Kim and Malik [3], and Mortenson [33] gave a different proof of Entry 1.4. These two papers provided new proofs using different methods. Our results show a possible way to give a new proof.

Recall that Dyson defined *the rank of partition* by the largest part minus the number of parts. Let $N(m, n)$ denote the number of partitions of the positive

integer n with rank m . Dyson gave the generating function for $N(m, n)$ as

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N(m, n) q^n z^m = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (q/z)_n} =: G(z, q), \quad |q| < 1. \quad (3.1.1)$$

We restate three identities of Ramanujan:

Entry 1.1. [38, p. 2] *Suppose that a and b are real numbers such that $a^2 + b^2 = 4$.*

Then

$$\begin{aligned} & \frac{b-a+2}{4} f_a(-q) + \frac{b+a+2}{4} f_{-a}(-q) - \frac{b}{2} f_b(q) \\ &= \frac{(q^4; q^4)_{\infty}}{(-q; q^2)_{\infty}} \prod_{n=1}^{\infty} \frac{1 - bq^n + q^{2n}}{1 + (a^2 b^2 - 2) q^{4n} + q^{8n}}. \end{aligned} \quad (3.1.2)$$

Entry 1.3. [38, p. 17]

$$\begin{aligned} & \frac{1+\sqrt{3}}{2} f_{-1}(-q) + \frac{3+\sqrt{3}}{6} f_1(-q) \\ &= f_{\sqrt{3}}(q) + \frac{2J_2 J_4}{\sqrt{3}(-q; q^2)_{\infty} J_6} \prod_{n=1}^{\infty} \frac{1}{1 + \sqrt{3} q^n + q^{2n}}. \end{aligned} \quad (3.1.3)$$

Entry 1.4. [38, p. 17]

$$\begin{aligned} & \frac{1}{2} (1 + e^{\pi i/4}) f_0(iq) + \frac{1}{2} (1 + e^{-\pi i/4}) f_0(-iq) \\ &= f_{\sqrt{2}}(q) + \frac{J_2(-q^2; q^4)_{\infty}}{\sqrt{2}(-q; q^2)_{\infty}} \prod_{n=1}^{\infty} \frac{1}{1 + \sqrt{2} q^n + q^{2n}}. \end{aligned} \quad (3.1.4)$$

Here, in the above, $f_a(q)$ is defined as

$$f_a(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1 + aq + q^2)(1 + aq^2 + q^4) \cdots (1 + aq^n + q^{2n})}, \quad (3.1.5)$$

where a is a real number and we can observe that

$$f_a(q) = G\left(\frac{-a \pm \sqrt{a^2 - 4}}{2}, q\right). \quad (3.1.6)$$

It is natural to try to find similar relations between ranks modulo different integers. For example, we prove a theorem which involves ranks modulo 2 and 4:

Theorem 3.1.1. For generating functions of ranks of partitions modulo 2 and 4, we have the following equation:

$$\frac{1}{2}f_2(-q) - f_0(q) = \frac{1}{2}p_2(-q) - p_0(q). \quad (3.1.7)$$

Here, we have used $f_a(q)$ defined by (3.1.5) and

$$\omega(a) := \frac{-a + \sqrt{a^2 - 4}}{2} \quad (3.1.8)$$

$$p_a(q) := (1 - \omega(a)) \frac{J_2 J_{2,4}^2}{j(\omega(a); q) j(-q\omega(a)^2; q^2)}. \quad (3.1.9)$$

In Section 3.2 we will see that the above definitions come from Lemma 3.2.1, and $p_a(q)$ will remain invariant if we use the conjugate of $\omega(a)$ instead of $\omega(a)$ in (3.1.9).

In Sections 3.3 and 3.4, we work with Ramanujan's Entries 1.1, 1.3 and 1.4. For instance, we have the following theorem and corollary by utilizing Entry 1.4:

Theorem 3.1.2. For generating functions of ranks of partitions modulo 4 and 8, we have the equation

$$\begin{aligned} & \frac{1}{2}(1 + e^{\pi i/4})f_0(iq) + \frac{1}{2}(1 + e^{-\pi i/4})f_0(-iq) - f_{\sqrt{2}}(q) \\ &= \frac{1}{2}(1 + e^{\pi i/4})p_0(iq) + \frac{1}{2}(1 + e^{-\pi i/4})p_0(-iq) - p_{\sqrt{2}}(q), \end{aligned} \quad (3.1.10)$$

where $p_a(q)$ is defined by (3.1.9).

Corollary 3.1.3. Let $p_a(q)$ be defined by (3.1.9). Then

$$\frac{J_2(-q^2; q^4)_\infty}{\sqrt{2}(-q; q^2)_\infty} \prod_{n=1}^{\infty} \frac{1}{1 + \sqrt{2}q^n + q^{2n}} = \frac{1}{2}(1 + e^{\pi i/4})p_0(iq) + \frac{1}{2}(1 + e^{-\pi i/4})p_0(-iq) - p_{\sqrt{2}}(q).$$

3.2 Ranks modulo 2, 4 and 16

We first state a lemma.

Lemma 3.2.1. For the generating function of the rank function, we have following identity:

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{q^{n^2}}{(xq)_n(q/x)_n} \\
&= (2 - x - x^{-1}) + (x - 1) \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1 + x^{-2} q^{4n+1}} \\
&+ (x^{-1} - 1) \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1 + x^2 q^{4n+1}} + (1 - x) \frac{J_2 J_{2,4}^2}{j(x; q) j(-qx^2; q^2)}. \quad (3.2.1)
\end{aligned}$$

To prove the above lemma we first recall an identity from Ramanujan's Lost Notebook, Part I [2, (12.2.3)]:

Lemma 3.2.2.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(c)_{n+1}(q/c)_n} = \frac{1}{(q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1 - cq^n}. \quad (3.2.2)$$

Proof of Lemma 3.2.1. The identity in Entry 12.5.3 in [2] is reexpressed by Mortenson [33, Eq. (2.7)] as

$$g(x; q) = -x^{-1} + qx^{-3}g(-qx^{-2}; q^4) - qg(-qx^2; q^4) + \frac{J_2 J_{2,4}^2}{xj(x; q)j(-qx^2; q^2)},$$

where

$$g(x; q) := x^{-1} \left(-1 + \sum_{n=0}^{\infty} \frac{q^{n^2}}{(x)_{n+1}(q/x)_n} \right).$$

In other words,

$$\begin{aligned}
& x^{-1} \left(-1 + \sum_{n=0}^{\infty} \frac{q^{n^2}}{(x)_{n+1}(q/x)_n} \right) \\
&= -x^{-1} - x^{-1} \left(-1 + \sum_{n=0}^{\infty} \frac{q^{4n^2}}{(-qx^{-2}; q^4)_{n+1}(-q^3x^2; q^4)_n} \right) \\
&+ x^{-2} \left(-1 + \sum_{n=0}^{\infty} \frac{q^{4n^2}}{(-qx^2; q^4)_{n+1}(-q^3x^{-2}; q^4)_n} \right) + \frac{J_2 J_{2,4}^2}{xj(x; q)j(-qx^2; q^2)}.
\end{aligned}$$

After rearrangement, we get:

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{q^{n^2}}{(xq)_n(q/x)_n} \\
&= (1-x) \left(1 - \sum_{n=0}^{\infty} \frac{q^{4n^2}}{(-qx^{-2}; q^4)_{n+1}(-q^3x^2; q^4)_n} \right) \\
&+ \frac{1-x}{x} \left(-1 + \sum_{n=0}^{\infty} \frac{q^{4n^2}}{(-qx^2; q^4)_{n+1}(-q^3x^{-2}; q^4)_n} \right) + (1-x) \frac{J_2 J_{2,4}^2}{j(x; q)j(-qx^2; q^2)}.
\end{aligned}$$

Applying (3.2.2) in the above leads us to the desired result. $\square$

3.2.1 Ranks Modulo 2 and 4

For ranks modulo 2 and 4, we observe that the generating functions of them are exactly $f_2(q)$ and $f_0(q)$, respectively.

Proof of Theorem 3.1.1. Taking $x = -1$ and $x = i$ into (3.2.1) gives us

$$f_2(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n(-q)_n} = 4 - \frac{4}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1+q^{4n+1}} + p_2(q), \quad (3.2.3)$$

and

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(iq)_n(q/i)_n} = 2 - \frac{2}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1-q^{4n+1}} + p_0(q), \quad (3.2.4)$$

respectively.

Letting $q \rightarrow -q$ in (3.2.3) and multiplying 1/2 on both sides give us

$$\frac{1}{2}f_2(q) = 2 - \frac{2}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1 - q^{4n+1}} + p_2(-q). \quad (3.2.5)$$

Then simplifying (3.2.5) minus (3.2.4) out leads us to the desired result:

$$\frac{1}{2}f_2(-q) - f_0(q) = \frac{1}{2}p_2(-q) - p_0(q).$$

□

3.2.2 Ranks Modulo 2 and 16

We use ω_{16} to denote the primitive 16th root of unity and let $r := \omega_{16} + \omega_{16}^{-1}$, $s := \omega_{16}^2 + \omega_{16}^{-2}$, $t := \omega_{16}^3 + \omega_{16}^{-3}$. We have the following theorem:

Theorem 3.2.3. For generating functions of ranks of partitions modulo 2 and 16, we have the following equation

$$\begin{aligned} & f_{-r}(q) - p_{-r}(q) \\ &= \frac{1-r+s-t}{2}(f_0(q) - p_0(q)) + \frac{1-s}{2}(f_0(-q) - p_0(-q)) \\ & \quad + \frac{2-(1-i)(r-t)}{4}(f_0(iq) - p_0(iq)) + \frac{2-(1+i)(r-t)}{4}(f_0(-iq) - p_0(-iq)). \end{aligned}$$

Proof. Take $x = \omega_{16}$ in (3.2.1) to get:

$$\begin{aligned} f_{-r}(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\omega_{16}q)_n(q/\omega_{16})_n} \\ &= (2-r) + \frac{\omega_{16}-1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1 + \omega_{16}^{-2} q^{4n+1}} + \frac{\omega_{16}^{-1}-1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1 + \omega_{16}^2 q^{4n+1}} \\ & \quad + (1-\omega_{16}) \frac{J_2 J_{2,4}^2}{j(\omega_{16}; q) j(-q\omega_{16}^2; q^2)}. \end{aligned} \quad (3.2.6)$$

Let $F_q(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ denote the following series:

$$\begin{aligned}
& \sum_{n=-\infty}^{\infty} \frac{\alpha_1 q^{24n^2+4n} - \alpha_2 q^{24n^2+12n+1} + \alpha_3 q^{24n^2+20n+2} + \alpha_4 q^{24n^2+28n+3}}{1 - q^{64n+8}} \\
& + \sum_{n=-\infty}^{\infty} \frac{\alpha_1 q^{24n^2+36n+4} - \alpha_2 q^{24n^2+44n+5} + \alpha_3 q^{24n^2+52n+6} - \alpha_4 q^{24n^2+60n+7}}{1 - q^{64n+8}} \\
& - \sum_{n=-\infty}^{\infty} \frac{\alpha_1 q^{24n^2+28n+8} - \alpha_2 q^{24n^2+36n+9} + \alpha_3 q^{24n^2+44n+10} + \alpha_4 q^{24n^2+52n+23}}{1 - q^{64n+40}} \\
& - \sum_{n=-\infty}^{\infty} \frac{\alpha_1 q^{24n^2+60n+28} - \alpha_2 q^{24n^2+68n+33} + \alpha_3 q^{24n^2+76n+38} - \alpha_4 q^{24n^2+84n+43}}{1 - q^{64n+40}}.
\end{aligned} \tag{3.2.7}$$

Then we rewrite (3.2.6) as:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\omega_{16}q)_n (q/\omega_{16})_n} = (2-r) + \frac{4}{J_4} F_q(r-2, s-r, t, s-t) + p_r(q). \tag{3.2.8}$$

For ranks modulo 2, we also rewrite (3.2.3) as:

$$f_2(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n (-q)_n} = 4 - \frac{4}{J_4} F_q(1, 1, 1, 1) + \frac{J_1^3}{J_2^2}. \tag{3.2.9}$$

For the process of transforming Lambert series in (3.2.6) and (3.2.3) to F_q in (3.2.8) and (3.2.9) above, please refer to Appendix B.

Here, $F_q(r-2, s-r, t, s-t)$ can be presented as a linear combination of $F_q(1, 1, 1, 1)$, $F_{-q}(1, 1, 1, 1)$, $F_{iq}(1, 1, 1, 1)$ and $F_{-iq}(1, 1, 1, 1)$ (we use $G(q)$ to denote $F_q(1, 1, 1, 1)$ for simplicity of the following identity):

$$\begin{aligned}
& F_q(r-2, s-r, t, s-t) \\
& = \frac{r-2}{4} (G(q) + G(-q) + G(iq) + G(-iq)) + \frac{s-r}{4} \left(-G(q) + G(-q) - \frac{G(iq)}{i} + \frac{G(-iq)}{i} \right) \\
& \quad + \frac{t}{4} (G(q) + G(-q) - G(iq) - G(-iq)) + \frac{s-t}{4} \left(-G(q) + G(-q) + \frac{G(iq)}{i} - \frac{G(-iq)}{i} \right) \\
& = \frac{r+t-s-1}{2} G(q) + \frac{s-1}{2} G(-q) + \frac{(1-i)(r-t)-2}{4} G(iq) + \frac{(1+i)(r-t)-2}{4} G(-iq).
\end{aligned} \tag{3.2.10}$$

Replacing $G(q), G(-q), G(iq)$ and $G(-iq)$ on the right hand side of (3.2.10) with the corresponding series in (3.2.7) and simplifying all the series will give $F_q(r-2, s-r, t, s-t)$. Then (3.2.8), (3.2.9) and (3.2.10) imply our theorem. $\square$

Remark 3.2.4. In fact, using the above method we can get a similar relation between modulo 2 and k , where $k = 2^l$ for $l > 1, l \in \mathbb{Z}$.

3.3 Applications to Entry 1.3 and Entry 1.4

3.3.1 Application to Entry 1.4, Ranks Modulo 4 and 8

Entry 1.4 in fact is related to ranks modulo 4 and 8. We have stated the results in Section 3.1 as Theorem 3.1.2 and Corollary 3.1.3.

Proof of Theorem 3.1.2 and Corollary 3.1.3. Considering

$$f_{\sqrt{2}}(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(e^{3\pi i/4}q)_n (e^{5\pi i/4}q)_n},$$

we take $x = e^{3\pi i/4}$ into (3.2.1) and get (here we use ω_8 to denote $e^{3\pi i/4}$):

$$\begin{aligned} f_{\sqrt{2}}(q) = & (1 - \omega_8) \left(1 - \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1 + iq^{4n+1}} \right) \\ & + \frac{1 - \omega_8}{\omega_8} \left(-1 + \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1 - iq^{4n+1}} \right) + p_{\sqrt{2}}(q). \end{aligned} \quad (3.3.1)$$

For the two Lambert series above, we transform them as following:

$$\begin{aligned}
& \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1+iq^{4n+1}} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n} (1-iq^{4n+1})(1-q^{8n+2})}{1-q^{16n+4}} \\
&= \sum_{n=-\infty}^{\infty} \frac{q^{24n^2+4n} (1-iq^{8n+1}-q^{16n+2}+iq^{24n+3})}{1-q^{32n+4}} \\
&\quad - \sum_{n=-\infty}^{\infty} \frac{q^{24n^2+28n+8} (1-iq^{8n+5}-q^{16n+10}+iq^{24n+15})}{1-q^{32n+20}}, \tag{3.3.2}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1-iq^{4n+1}} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n} (1+iq^{4n+1})(1-q^{8n+2})}{1-q^{16n+4}} \\
&= \sum_{n=-\infty}^{\infty} \frac{q^{24n^2+4n} (1+iq^{8n+1}-q^{16n+2}-iq^{24n+3})}{1-q^{32n+4}} \\
&\quad - \sum_{n=-\infty}^{\infty} \frac{q^{24n^2+28n+8} (1+iq^{8n+5}-q^{16n+10}-iq^{24n+15})}{1-q^{32n+20}}. \tag{3.3.3}
\end{aligned}$$

Substituting (3.3.2) and (3.3.3) into (3.3.1), we get:

$$\begin{aligned}
& f_{\sqrt{2}}(q) \\
&= (2+\sqrt{2}) + \frac{1}{J_4} \left(\sum_{n=-\infty}^{\infty} \frac{q^{24n^2+4n} ((-2-\sqrt{2}) + \sqrt{2}q^{8n+1} + (2+\sqrt{2})q^{16n+2} - \sqrt{2}q^{24n+3})}{1-q^{32n+4}} \right. \\
&\quad \left. + \sum_{n=-\infty}^{\infty} \frac{q^{24n^2+28n+8} ((2+\sqrt{2}) - \sqrt{2}q^{8n+5} - (2+\sqrt{2})q^{16n+10} + \sqrt{2}q^{24n+15})}{1-q^{32n+20}} \right) + p_{\sqrt{2}}(q). \tag{3.3.4}
\end{aligned}$$

Similarly, we have the following identity for $f_0(q)$:

$$\begin{aligned}
f_0(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(iq)_n (q/i)_n} = 2 - \frac{2}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1 - q^{4n+1}} + p_0(q) \\
&= 2 - \frac{2}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n} (1 + q^{4n+1} + q^{8n+2} + q^{12n+3})}{1 - q^{16n+4}} + p_0(q) \\
&= 2 - \frac{2}{J_4} \left(\sum_{n=-\infty}^{\infty} \frac{q^{24n^2+4n} (1 + q^{8n+1} + q^{16n+2} + q^{24n+3})}{1 - q^{32n+4}} \right. \\
&\quad \left. - \sum_{n=-\infty}^{\infty} \frac{q^{24n^2+28n+8} (1 + q^{8n+5} + q^{16n+10} + q^{24n+15})}{1 - q^{32n+20}} \right) + p_0(q). \tag{3.3.5}
\end{aligned}$$

Taking $q \rightarrow iq$ and $q \rightarrow -iq$ in (3.3.5) gives

$$\begin{aligned}
f_0(iq) &= 2 - \frac{2}{J_4} \left(\sum_{n=-\infty}^{\infty} \frac{q^{24n^2+4n} (1 + iq^{8n+1} - q^{16n+2} - iq^{24n+3})}{1 - q^{32n+4}} \right. \\
&\quad \left. - \sum_{n=-\infty}^{\infty} \frac{q^{24n^2+28n+8} (1 + iq^{8n+5} - q^{16n+10} - iq^{24n+15})}{1 - q^{32n+20}} \right) + p_0(iq) \tag{3.3.6}
\end{aligned}$$

and

$$\begin{aligned}
f_0(-iq) &= 2 - \frac{2}{J_4} \left(\sum_{n=-\infty}^{\infty} \frac{q^{24n^2+4n} (1 - iq^{8n+1} - q^{16n+2} + iq^{24n+3})}{1 - q^{32n+4}} \right. \\
&\quad \left. - \sum_{n=-\infty}^{\infty} \frac{q^{24n^2+28n+8} (1 - iq^{8n+5} - q^{16n+10} + iq^{24n+15})}{1 - q^{32n+20}} \right) + p_0(-iq), \tag{3.3.7}
\end{aligned}$$

respectively. We replace $f_0(iq)$, $f_0(-iq)$ and $f_{\sqrt{2}}(q)$ on the left hand side of (3.1.10) with (3.3.6), (3.3.7) and (3.3.4), respectively, which will give us Theorem 3.1.2 after simplification. Comparing Theorem 3.1.2 with Entry 1.4, we get Corollary 3.1.3 directly. $\square$

3.3.2 Application to Entry 1.3, Ranks Modulo 3, 6 and 12

Entry 1.3 involves ranks modulo 3, 6 and 12. We have the following theorem and corollary.

Theorem 3.3.1. For the generating functions of ranks of partitions modulo 3, 6 and 12, we have

$$\frac{3 + \sqrt{3}}{6} f_1(-q) + \frac{1 + \sqrt{3}}{2} f_{-1}(-q) - f_{\sqrt{3}}(q) = \frac{3 + \sqrt{3}}{6} p_1(-q) + \frac{1 + \sqrt{3}}{2} p_{-1}(-q) - p_{\sqrt{3}}(q). \quad (3.3.8)$$

Corollary 3.3.2. We have

$$\frac{2J_2J_4}{\sqrt{3}(-q; q^2)_\infty J_6} \prod_{n=1}^{\infty} \frac{1}{1 + \sqrt{3}q^n + q^{2n}} = \frac{3 + \sqrt{3}}{6} p_1(-q) + \frac{1 + \sqrt{3}}{2} p_{-1}(-q) - p_{\sqrt{3}}(q). \quad (3.3.9)$$

Proof. Let $\omega_{12} = e^{\pi i/6}$ and notice:

$$\begin{aligned} f_1(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\omega_{12}^4 q)_n (q \omega_{12}^{-4})_n}, \\ f_{-1}(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\omega_{12}^2 q)_n (q \omega_{12}^{-2})_n}, \\ f_{\sqrt{3}}(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\omega_{12}^5 q)_n (q \omega_{12}^{-5})_n}. \end{aligned}$$

Similar to the previous subsection, we have:

$$\begin{aligned} f_1(q) &:= 3 + \frac{(\omega_{12}^4 - 1)}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n} (1 + \omega_{12}^{-2} q^{4n+1} + \omega_{12}^{-4} q^{8n+2})}{1 + q^{12n+3}} \\ &\quad + \frac{(\omega_{12}^{-4} - 1)}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n} (1 + \omega_{12}^2 q^{4n+1} + \omega_{12}^4 q^{8n+2})}{1 + q^{12n+3}} + p_1(q), \end{aligned} \quad (3.3.10)$$

$$\begin{aligned}
f_{-1}(q) := & 1 + \frac{(\omega_{12}^2 - 1)}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n} (1 - \omega_{12}^{-4} q^{4n+1} + \omega_{12}^{-8} q^{8n+2})}{1 + q^{12n+3}} \\
& + \frac{(\omega_{12}^{-2} - 1)}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n} (1 - \omega_{12}^4 q^{4n+1} + \omega_{12}^8 q^{8n+2})}{1 + q^{12n+3}} + p_{-1}(q),
\end{aligned} \tag{3.3.11}$$

and

$$\begin{aligned}
f_{\sqrt{3}}(q) := & (2 + \sqrt{3}) + \frac{(\omega_{12}^5 - 1)}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n} (1 - \omega_{12}^2 q^{4n+1} + \omega_{12}^4 q^{8n+2})}{1 - q^{12n+3}} \\
& + \frac{(\omega_{12}^{-5} - 1)}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n} (1 - \omega_{12}^{-2} q^{4n+1} + \omega_{12}^{-4} q^{8n+2})}{1 - q^{12n+3}} + p_{\sqrt{3}}(q).
\end{aligned} \tag{3.3.12}$$

Take $q \rightarrow -q$ in both (3.3.10) and (3.3.11) and work out $\frac{3+\sqrt{3}}{6}f_1(-q) + \frac{1+\sqrt{3}}{2}f_{-1}(-q) - f_{\sqrt{3}}(q)$ using (3.3.12). We find that all the constants and series will be canceled off and what remains is exactly Theorem 3.3.1. Comparing Theorem 3.3.1 with Entry 1.3 will lead us to Corollary 3.3.2 directly. $\square$

3.4 Application to Entry 1.1

Theorem 3.4.1. Let a and b be real numbers with $a^2 + b^2 = 4$. Then we have

$$\begin{aligned}
& \frac{b-a+2}{4} f_a(-q) + \frac{b+a+2}{4} f_{-a}(-q) - \frac{b}{2} f_b(q) \\
& = \frac{b-a+2}{4} p_a(-q) + \frac{b+a+2}{4} p_{-a}(-q) - \frac{b}{2} p_b(q).
\end{aligned}$$

Corollary 3.4.2. Let a and b be real numbers with $a^2 + b^2 = 4$. Then we have

$$\frac{(q^4; q^4)_{\infty}}{(-q; q^2)_{\infty}} \prod_{n=1}^{\infty} \frac{1 - bq^n + q^{2n}}{1 + (a^2 b^2 - 2) q^{4n} + q^{8n}} = \frac{b-a+2}{4} p_a(-q) + \frac{b+a+2}{4} p_{-a}(-q) - \frac{b}{2} p_b(q).$$

Proof. Firstly, we observe that if $ab = 0$, using Lemma 3.2.1 will lead us to Theorem 3.4.1 directly. So we assume $ab \neq 0$ in following proof.

From (3.1.6), we have

$$f_a(-q) = G\left(\frac{-a + \sqrt{a^2 - 4}}{2}, -q\right)$$

and

$$f_{-a}(-q) = G\left(\frac{a - \sqrt{a^2 - 4}}{2}, -q\right).$$

Let α denote $\frac{-a + \sqrt{a^2 - 4}}{2}$ and

$$\beta := \begin{cases} \frac{-b + \sqrt{b^2 - 4}}{2} & \text{if } ab > 0, \\ \frac{-b - \sqrt{b^2 - 4}}{2} & \text{if } ab < 0. \end{cases}$$

Now we prove the theorem for all a with $\alpha^m = 1$ for some nonzero integer m , i.e., α is a root of unity. Then we have $\alpha\beta = i$ if $ab > 0$ and $\alpha\beta = -i$ if $ab < 0$.

Case $\alpha\beta = i$: From Lemma 3.2.1, $f_a(q)$ is equal to:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\alpha q)_n (q/\alpha)_n} &= (2 - \alpha - \alpha^{-1}) + (\alpha - 1) \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1 + \alpha^{-2} q^{4n+1}} \\ &\quad + (\alpha^{-1} - 1) \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1 + \alpha^2 q^{4n+1}} + p(a). \end{aligned} \quad (3.4.1)$$

Since $\alpha^m = 1$, we have the following transformation for the Lambert series in (3.4.1):

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1 + \alpha^{-2} q^{4n+1}} \\ &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1 - q^{(4n+1)2m}} (1 - \alpha^{4m-2} q^{4n+1} + \alpha^{4m-4} q^{8n+2} - \dots - \alpha^2 q^{(4n+1)(2m-1)}), \end{aligned} \quad (3.4.2)$$

and

$$\begin{aligned}
& \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1 + \alpha^2 q^{4n+1}} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1 - q^{(4n+1)2m}} (1 - \alpha^{-4m+2} q^{4n+1} + \alpha^{-4m+4} q^{8n+2} - \dots - \alpha^{-2} q^{(4n+1)(2m-1)}).
\end{aligned} \tag{3.4.3}$$

Substituting (3.4.2), (3.4.3) into (3.4.1), we find that

$$\begin{aligned}
f_a(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\alpha q)_n (q/\alpha)_n} \\
&= (2 - \alpha - \alpha^{-1}) + \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (-2 + \alpha + \alpha^{-1}) q^{6n^2+2n}}{1 - q^{(4n+1)2m}} \\
&\quad - \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (-\alpha^{4m-2} - \alpha^{-4m+2} + \alpha^{4m-1} + \alpha^{-4m+1}) q^{6n^2+6n+1}}{1 - q^{(4n+1)2m}} \\
&\quad + \dots \\
&\quad - \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (-\alpha^2 - \alpha^{-2} + \alpha^3 + \alpha^{-3}) q^{6n^2+(8m-2)n+2m-1}}{1 - q^{(4n+1)2m}} \\
&\quad + p(a).
\end{aligned} \tag{3.4.4}$$

Therefore, the coefficients of

$$\frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n+k(4n+1)}}{1 - q^{(4n+1)2m}},$$

where k is a non-negative integer less than $2m$, in $f_a(-q)$, $f_{-a}(-q)$ and $f_b(q)$ should be

$$\begin{aligned}
F_1 &:= \alpha^{-2k+1} + \alpha^{2k-1} - \alpha^{-2k} - \alpha^{2k}, \\
F_2 &:= -\alpha^{-2k+1} - \alpha^{2k-1} - \alpha^{-2k} - \alpha^{2k}
\end{aligned}$$

and

$$F_3 := (-1)^k (\beta^{-2k+1} + \beta^{2k-1} - \beta^{-2k} - \beta^{2k}),$$

respectively.

Then we work on following equation:

$$\begin{aligned} & \frac{b-a+2}{4}F_1 + \frac{b+a+2}{4}F_2 - \frac{b}{2}F_3 \\ &= \frac{1}{4} \left\{ 2(\alpha^{-2k+1} + \alpha^{2k-1})(\alpha + \alpha^{-1}) - (\alpha^{-2k} + \alpha^{2k})(4 - 2(\beta + \beta^{-1})) \right. \\ & \quad \left. + 2(-1)^k(\beta + \beta^{-1})(\beta^{-2k+1} + \beta^{2k-1} - \beta^{-2k} - \beta^{2k}) \right\}. \end{aligned}$$

Use $\alpha\beta = i$ in the above and we see that all of the terms are canceled off, in other word, the right hand side equals 0. This implies that all the Lambert series in $\frac{b-a+2}{4}f_a(-q) + \frac{b+a+2}{4}f_{-a}(-q) - \frac{b}{2}f_b(q)$ will disappear. At the same time, the constant part is also canceled off. Consequently, only $p_a(q), p_{-a}(-q)$ and $p_b(q)$ remain.

For the case $\alpha\beta = -i$, the proof is similar. Therefore, we have proved the theorem for all the roots of unity. We conclude that the proof of Theorem 3.4.1 is complete by utilizing the fact that the roots of unity are dense in the unit circle.

Corollary 3.4.2 is obtained by comparing Entry 1.1 and Theorem 3.4.1 immediately. $\square$

Remark 3.4.3. Corollaries 3.1.3, 3.3.2 and 3.4.2 are obtained by comparing Theorems 3.1.2, 3.3.1 and 3.4.1 and Entries 1.1, 1.3 and 1.4, respectively. If one can prove all the corollaries directly without using our theorems, then combining with corresponding theorems lead us to those entries. One would then obtain new proofs of these entries.

Chapter 4

Ramanujan's Convolution Sum Twisted By Dirichlet Characters

4.1 Introduction

In this chapter we consider convolution sums of divisors functions twisted by χ_{-4} . We recall some definitions first.

Let $k \in \mathbb{N}$. Let χ and ψ be primitive Dirichlet characters with conductors L and M , respectively. Throughout this chapter we assume $z \in \mathbb{H}$ and $q = e^{2\pi iz}$. The generalized Bernoulli numbers $B_{k,\chi}$ attached to χ and the Dirichlet L-function of χ are defined by

$$\sum_{k=0}^{\infty} \frac{B_{k,\chi} x^k}{k!} = \sum_{a=1}^L \frac{\chi(a) x e^{ax}}{e^{Lx} - 1} \quad \text{and} \quad L(k, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^k},$$

respectively. For each quadratic discriminant t , we put $\chi_t(n) = \left(\frac{t}{n}\right)$, where $\left(\frac{t}{n}\right)$ is the Kronecker symbol. To define Kronecker symbol, we have to define Legendre symbol:

The Legendre symbol is a function of integer a and odd prime p defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod{p}, \\ -1 & \text{if } a \text{ is a non-quadratic residue modulo } p, \\ 0 & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

Then the Kronecker symbol $\left(\frac{a}{n}\right)$ equals

$$\left(\frac{a}{n}\right) = \left(\frac{a}{u}\right) \prod_{i=1}^k \left(\frac{a}{p_i}\right)^{e_i},$$

where $n = u \cdot p_1^{e_1} \cdots p_k^{e_k}$ is the prime factorization for non-zero integer n . For $p_i = 2$,

$$\left(\frac{a}{2}\right) = \begin{cases} 0 & \text{if } a \text{ is even,} \\ 1 & \text{if } a \equiv \pm 1 \pmod{8}, \\ -1 & \text{if } a \equiv \pm 3 \pmod{8}. \end{cases}$$

For $k, n \in \mathbb{N}_0$ we define $\sigma_{(k, \chi, \psi)}(n)$ by

$$\sigma_{(k, \chi, \psi)}(n) = \begin{cases} 0 & \text{if } n = 0, \psi \neq \chi_1 \text{ and } k \in \mathbb{N}, \\ 0 & \text{if } n = 0, \chi \neq \chi_1, \psi \neq \chi_1 \text{ and } k = 0, \\ \frac{-B_{1, \psi}}{2} & \text{if } n = 0, \chi = \chi_1 \text{ and } k = 0, \\ \frac{-B_{k+1, \chi}}{2(k+1)} & \text{if } n = 0, \psi = \chi_1 \text{ and } k \in \mathbb{N}_0, \\ \sum_{1 \leq d|n} \chi(d) \psi(n/d) d^k & \text{if } n \geq 1. \end{cases} \quad (4.1.1)$$

If $n \notin \mathbb{N}_0$ we set $\sigma_{(k, \chi, \psi)}(n) = 0$. For convenience, we use the following notations

$$\begin{aligned} \sigma_k(n) &:= \sigma_{(k, \chi_1, \chi_1)}(n), \quad \hat{\sigma}_k(n) := \sigma_{(k, \chi_1, \chi_{-4})}(n), \\ \tilde{\sigma}_k(n) &:= \sigma_{(k, \chi_{-4}, \chi_1)}(n), \quad \bar{\sigma}_k(n) := \sigma_{(k, \chi_{-4}, \chi_{-4})}(n). \end{aligned}$$

Let

$$R(l, m, k) := -\frac{B_{2l}}{2l} \frac{B_{2m}}{2m} \frac{k}{B_{2k}}.$$

In [36] Ramanujan proved the following formula for convolutions of sum of divisors functions.

Theorem 4.1.1 (Ramanujan). Let $k = l + m$ and $n \in \mathbb{N}_0$,

$$\begin{aligned} & \sum_{0 \leq j \leq n} \sigma_{2l-1}(j) \sigma_{2m-1}(n-j) \\ &= R(l, m, k) \sigma_{2k-1}(n) + \frac{\zeta(2-2l) + \zeta(2-2m)}{2k-2} n \sigma_{2k-3}(n) + O(n^{\frac{2}{3}(2l+2m-1)}). \end{aligned}$$

In this chapter we consider convolution sums of divisors functions twisted by χ_{-4} . Ramanujan's formula given by Theorem 4.1.1 covers the cases when powers of divisors are odd, our results (4.1.2)–(4.1.4) below are the cases when powers of divisors are even. In this sense, our results can be viewed as complementary to Ramanujan's formula.

Before stating the main theorem we define the constant

$$S(l, m, k) := \frac{B_{2l+1, \chi_{-4}}}{2l+1} \frac{B_{2m+1, \chi_{-4}}}{2m+1} \frac{k}{B_{2k}},$$

and note that $C_{2k,i}(z)$ defined by (4.2.4) is cusp form for $1 \leq i \leq 4k-5$.

Theorem 4.1.2 (Main Theorem). Let $l, m \in \mathbb{N}_0$, $l', m' \in \mathbb{N}$, $1 < l + m + 1 = k$ or, $l' + m' + 1 = k$, and $C_{2k,i}(z)$ be the eta quotient given by (4.2.4), then for all

$n \in \mathbb{N}_0$ there exist $c_i \in \mathbb{C}$ such that

$$\sum_{0 \leq j \leq n} \tilde{\sigma}_{2l}(j) \tilde{\sigma}_{2m}(n-j) = S(l, m, k) \frac{\sigma_{2k-1}(n/2) - 2^{2k} \sigma_{2k-1}(n/4)}{(2^{2k} - 1)} + [n] \sum_{i=1}^{4k-5} c_i C_{2k,i}(z), \quad (4.1.2)$$

$$\sum_{0 \leq j \leq n} \hat{\sigma}_{2l}(j) \hat{\sigma}_{2m}(n-j) = S(l, m, k) \frac{4(\sigma_{2k-1}(n) - \sigma_{2k-1}(n/2))}{2^{2k}(2^{2k} - 1)} + [n] \sum_{i=1}^{4k-5} c_i C_{2k,i}(z), \quad (4.1.3)$$

$$\sum_{0 \leq j \leq n} \tilde{\sigma}_{2l'}(j) \hat{\sigma}_{2m'}(n-j) = [n] \sum_{i=1}^{4k-5} c_i C_{2k,i}(z). \quad (4.1.4)$$

Let $l, l', m \in \mathbb{N}$, $l' > 1$, $k = l(\text{or } l') + m$ and $C_{2k,i}(z)$ be the eta quotient given by (4.2.4), then for all $n \in \mathbb{N}_0$ there exist $c_i \in \mathbb{C}$ such that

$$\begin{aligned} \sum_{0 \leq j \leq n} \bar{\sigma}_{2l-1}(j) \bar{\sigma}_{2m-1}(n-j) = \\ R(l, m, k) \frac{(2^{2l} - 1)(2^{2m} - 1)}{4(1 - 1/2^{2k})} (\sigma_{2k-1}(n/2) - (2^{2k} + 1)\sigma_{2k-1}(n/4) + 2^{2k}\sigma_{2k-1}(n/8)) \\ + [n] \sum_{i=1}^{4k-5} c_i C_{2k,i}(z), \end{aligned} \quad (4.1.5)$$

$$\begin{aligned} \sum_{0 \leq j \leq n} \sigma_{2l'-1}(j) \bar{\sigma}_{2m-1}(n-j) = R(l', m, k) \frac{2^{2m}(2^{2m} - 1)}{2^{2k}(2^{2k} - 1)} \bar{\sigma}_{2k-1}(n) \\ + [n] \sum_{i=1}^{4k-5} c_i C_{2k,i}(z). \end{aligned} \quad (4.1.6)$$

Below we note some cases when formulas can be given with vanishing cusp parts.

Corollary 4.1.3. For all $n \in \mathbb{N}_0$ we have

$$\begin{aligned}
\sum_{0 \leq j \leq n} \tilde{\sigma}_0(j) \tilde{\sigma}_2(n-j) &= \sigma_3(n/2) - 16\sigma_3(n/4), \\
\sum_{0 \leq j \leq 2n} \tilde{\sigma}_0(j) \tilde{\sigma}_4(2n-j) &= \frac{5}{2}(\sigma_5(n) - 64\sigma_5(n/2)), \\
\sum_{0 \leq j \leq 2n} \tilde{\sigma}_2(j) \tilde{\sigma}_2(2n-j) &= \frac{1}{2}(\sigma_5(n) - 64\sigma_5(n/2)), \\
\sum_{0 \leq j \leq n} \hat{\sigma}_0(j) \hat{\sigma}_4(2n-j) &= \frac{5}{32}(\sigma_5(2n) - \sigma_5(n)), \\
\sum_{0 \leq j \leq 2n} \hat{\sigma}_2(j) \hat{\sigma}_2(2n-j) &= \frac{1}{32}(\sigma_5(2n) - \sigma_5(n)), \\
\sum_{0 \leq j \leq 2n} \tilde{\sigma}_2(j) \hat{\sigma}_2(2n-j) &= 0, \\
\sum_{0 \leq j \leq n} \bar{\sigma}_1(j) \bar{\sigma}_1(n-j) &= \sigma_3(n/2) - 17\sigma_3(n/4) + 16\sigma_3(n/8), \\
\sum_{0 \leq j \leq n} \bar{\sigma}_1(j) \bar{\sigma}_3(n-j) &= \sigma_5(n/2) - 65\sigma_5(n/4) + 64\sigma_5(n/8).
\end{aligned}$$

In the next section we give a basis for $M_{2k}(\Gamma_0(16))$ (the space of modular forms of weight $2k$ for $\Gamma_0(16)$), and prove some results that will be useful in Section 4.3, where we state and prove Theorem 4.3.1. This theorem gives the main terms of any modular form in $M_{2k}(\Gamma_0(16))$ explicitly. In Section 4.4 we use Theorem 4.3.1 to prove the main theorem. In Section 4.5, we first give the proof of Corollary 4.1.3. Then we give an example of computing the coefficients c_i of cusp forms. In Section 4.6, we apply Theorem 4.3.1 to find formulas for representations of positive integers by certain diagonal quadratic forms with coefficients 1, 2 and 4 and compare our results with those from [27].

4.2 Further Notations and Preliminary Calculations

We use the theory of modular forms to prove our results. The modular form spaces are known to have the following property (see [40, p. 83])

$$M_{2k}(\Gamma_0(16)) = E_{2k}(\Gamma_0(16)) \oplus S_{2k}(\Gamma_0(16)), \quad (4.2.1)$$

where $E_{2k}(\Gamma_0(16))$ and $S_{2k}(\Gamma_0(16))$ denote the subspaces of Eisenstein series and cusp forms of $M_{2k}(\Gamma_0(16))$, respectively. The main contribution to the coefficients of expansions at infinity comes from the Eisenstein part. In [36], Ramanujan refers to the cusp part as the error term. In this section we give a basis of $E_{2k}(\Gamma_0(16))$ in terms of Eisenstein series and a basis of $S_{2k}(\Gamma_0(16))$ in terms of eta quotients.

Let ϵ and ψ are primitive Dirichlet characters with conductors L and M , respectively; such that $\epsilon(-1)\psi(-1) = (-1)^k$. Following Miyake's notation in [31] we define the Eisenstein series

$$E_k(Mz; \epsilon, \psi) = \sum_{\substack{m, n = -\infty \\ m, n \neq 0}}^{\infty} \frac{\epsilon(m)\psi(n)}{(Mmz + n)^k}. \quad (4.2.2)$$

When $k \geq 3$, this series is convergent and the Fourier series expansion is given by [31, Theorem 7.1.3]

$$E_k(Mz; \epsilon, \psi) = \frac{2(-2\pi i)^k W(\epsilon)}{M^k(k-1)!} \sum_{n \geq 0} \sigma_{(k-1, \psi, \epsilon)}(n) q^n, \quad (4.2.3)$$

where $W(\epsilon) = \sum_{a=0}^{L-1} \epsilon(a) e^{2\pi i a/L}$ is the Gauss sum. The expansions of weight 1 and weight 2 Eisenstein series will be discussed in Section 4.4. For the future use we compute the following values of Gauss sums: $W(\chi_1) = 1$ and $W(\chi_{-4}) = 2i$.

The Dedekind eta function $\eta(z)$ is the holomorphic function defined on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ by the product formula

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}).$$

Let $N \in \mathbb{N}$ and $r_\delta \in \mathbb{Z}$ for all $1 \leq \delta \mid N$, where the r_δ 's are not all zero. Let $z \in \mathbb{H}$. We define an eta quotient by the product formula

$$f(z) = \prod_{\delta \mid N} \eta^{r_\delta}(\delta z).$$

We define the following family of eta quotients to give a basis for $S_{2k}(\Gamma_0(16))$:

$$C_{2k,i}(z) := \frac{\eta^{8k-2i-10}(z)\eta^6(4z)\eta^{1-i}(8z)}{\eta^{4k-i-5}(2z)\eta^{2-2i}(16z)}. \quad (4.2.4)$$

Below we give a basis for $E_{2k}(\Gamma_0(16))$ in terms of Eisenstein series, and a basis for $S_{2k}(\Gamma_0(16))$ in terms of eta quotients.

Lemma 4.2.1. Let $k > 1$. The set of Eisenstein series

$$\{E_{2k}(z; \chi_1, \chi_1), E_{2k}(2z; \chi_1, \chi_1), E_{2k}(4z; \chi_1, \chi_1), \\ E_{2k}(8z; \chi_1, \chi_1), E_{2k}(16z; \chi_1, \chi_1), E_{2k}(4z; \chi_{-4}, \chi_{-4})\}$$

is a basis for $E_{2k}(\Gamma_0(16))$, and the set of eta quotients

$$\{C_{2k,i}(z) \mid 1 \leq i \leq 4k - 5\}$$

is a basis for $S_{2k}(\Gamma_0(16))$.

Proof. The basis for $E_{2k}(\Gamma_0(16))$ follows from the Algorithm 5.11 in [40].

We have $\dim(S_{2k}(\Gamma_0(16))) = 4k - 5$. It is straightforward to check

$$C_{2k,i}(z) \in S_{2k}(\Gamma_0(16)), \text{ for all } 1 \leq i \leq 4k - 5.$$

The linear independence follows from the fact that $v_{1/16}(C_{2k,i}(z)) = i$, which gives the system of coefficient matrix of these eta quotients a lower triangular shape. $\square$

Let

$$A_r = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}), \quad (4.2.5)$$

such that $-d/c = r$. Then the Fourier series expansion of $f(z) \in M_k(\Gamma_0(N))$ at the cusp $\frac{-d}{c}$ is given by the Fourier series expansion of $(cz - a)^{-k} f(A_r^{-1}z)$ at the cusp ∞ , see [26, page 35]. Let the Fourier series expansion of $f(z)$ at the cusp r be given by the infinite sum

$$(cz - a)^{-k} f(A_r^{-1}z) = \sum_{n \geq 0} a_r(n) e^{2\pi i(n+\kappa)z/h},$$

where h is the width of $\Gamma_0(N)$ at the cusp r , and κ is the cusp parameter, see [26, (1.17)]. Then we use the notation $[n + \kappa]_r f(z)$ to denote $a_r(n)$. We write $[n]$, instead of $[n]_\infty$, at the cusp ∞ . If we say Fourier series expansion (or Fourier coefficients) without specifying the cusp, we mean the expansion (or coefficients) at the cusp ∞ . We define $v_r(f)$, the order of $f(z)$ at r , to be the smallest $n + \kappa$ such that $[n + \kappa]_r f(z) \neq 0$. For modular forms, ‘the constant term of the Fourier expansion of $f(z)$ at cusp r ’ refers to the term $[0]_r f(z)$. Note that if $\kappa \neq 0$, or f is a cusp form then the constant term is 0. Next we compute the constant terms of the Eisenstein series at the cusps in $R(16)$. A set of inequivalent cusps of $\Gamma_0(16)$ is

$$R(16) = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{3}{4} \right\},$$

whose corresponding matrices such that $A_r^{-1}(\infty) = r$ can be given by

$$\begin{aligned} A_{1/1} &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad A_{1/2} = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}, \quad A_{1/4} = \begin{bmatrix} -1 & 0 \\ 4 & -1 \end{bmatrix}, \\ A_{1/8} &= \begin{bmatrix} -1 & 0 \\ 8 & -1 \end{bmatrix}, \quad A_{1/16} = \begin{bmatrix} -1 & 0 \\ 16 & -1 \end{bmatrix}, \quad A_{3/4} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}, \end{aligned}$$

respectively.

Lemma 4.2.2. Let $k > 1$ and $r = \frac{-d}{c} \in R(16)$, then we have

$$[0]_r E_{2k}(tz; \chi_1, \chi_1) = \left(\frac{\gcd(c, t)}{t} \right)^{2k} 2\zeta(2k), \quad (4.2.6)$$

$$[0]_r E_{2k}(4z; \chi_{-4}, \chi_{-4}) = \begin{cases} -(1 - \frac{1}{2^{2k}})2\zeta(2k) & \text{if } r = 1/4, \\ (1 - \frac{1}{2^{2k}})2\zeta(2k) & \text{if } r = 3/4, \\ 0 & \text{otherwise.} \end{cases} \quad (4.2.7)$$

Proof. Let $k > 1$. Then $E_{2k}(4z; \chi_{-4}, \chi_{-4})$ is convergent and we have

$$\begin{aligned} E_{2k}(4A_{1/1}^{-1}(z); \chi_{-4}, \chi_{-4}) &= \sum_{\substack{m, n=-\infty, \\ m, n \neq 0}}^{\infty} \frac{\chi_{-4}(m)\chi_{-4}(n)}{(4mA_{1/1}^{-1}(z) + n)^{2k}} \\ &= (z+1)^{2k} \sum_{\substack{m, n=-\infty, \\ m, n \neq 0}}^{\infty} \frac{\chi_{-4}(m)\chi_{-4}(n)}{(-(4m+n)z - n)^{2k}}. \end{aligned}$$

That is, we obtain

$$\begin{aligned} [0]_{1/1} E_{2k}(4z; \chi_{-4}, \chi_{-4}) &= \sum_{\substack{m, n=-\infty, \\ m, n \neq 0, \\ (4m+n)=0}}^{\infty} \frac{\chi_{-4}(m)\chi_{-4}(n)}{(-(4m+n)z - n)^{2k}} = \sum_{\substack{m=-\infty, \\ m \neq 0}}^{\infty} \frac{\chi_{-4}(m)\chi_{-4}(-4m)}{(4m)^{2k}} \\ &= 0. \end{aligned}$$

Similarly, we have

$$\begin{aligned}
[0]_{1/4}E_{2k}(4z; \chi_{-4}, \chi_{-4}) &= \sum_{\substack{m,n=-\infty, \\ m,n \neq 0, \\ (m+n)=0}}^{\infty} \frac{\chi_{-4}(m)\chi_{-4}(n)}{(-(4m+4n)z-n)^{2k}} = \sum_{\substack{m,n=-\infty, \\ m \neq 0}}^{\infty} \frac{\chi_{-4}(m)\chi_{-4}(-m)}{(m)^{2k}} \\
&= - \sum_{m=-\infty}^{\infty} \frac{1}{(2m+1)^{2k}} = -2 \sum_{m=1}^{\infty} \frac{1}{(m)^{2k}} + 2 \sum_{m=1}^{\infty} \frac{1}{(2m)^{2k}} \\
&= - \left(1 - \frac{1}{2^{2k}}\right) 2\zeta(2k).
\end{aligned}$$

The rest of the proofs can be done similarly. $\square$

4.3 The Space $M_{2k}(\Gamma_0(16))$

Theorem 4.3.1 below gives a formula for any modular form in $M_{2k}(\Gamma_0(16))$ in terms of Eisenstein series and eta quotients. For notational convenience let us define

$$A(k) = \frac{1}{4\zeta(k)(2^k - 1)}.$$

Theorem 4.3.1. Let $k > 1$ and $f(z) \in M_{2k}(\Gamma_0(16))$. Then there exist constants $c_i \in \mathbb{C}$, such that

$$\begin{aligned}
f(z) &= A(2k) \Big((2^{2k+1}[0]_{1/1}f - 2[0]_{1/2}f) E_{2k}(z; \chi_1, \chi_1) \\
&\quad + (-2^{2k+1}[0]_{1/1}f + (2^{2k+1} + 2)[0]_{1/2}f - ([0]_{1/4}f + [0]_{3/4}f)) E_{2k}(2z; \chi_1, \chi_1) \\
&\quad + (-2^{2k+1}[0]_{1/2}f + (2^{2k} + 1)([0]_{1/4}f + [0]_{3/4}f) - 2[0]_{1/8}f) E_{2k}(4z; \chi_1, \chi_1) \\
&\quad + (-2^{2k}([0]_{1/4}f + [0]_{3/4}f) + (2^{2k+1} + 2)[0]_{1/8}f - 2[0]_{1/16}f) E_{2k}(8z; \chi_1, \chi_1) \\
&\quad + (-2^{2k+1}[0]_{1/8}f + 2^{2k+1}[0]_{1/16}f) E_{2k}(16z; \chi_1, \chi_1) \\
&\quad + (-2^{2k}([0]_{1/4}f - [0]_{3/4}f)) E_{2k}(4z; \chi_{-4}, \chi_{-4}) \Big) \\
&\quad + \sum_{i=1}^{4k-5} c_i C_{2k,i}(z).
\end{aligned}$$

Proof. Let $k > 1$ and $f(z) \in M_{2k}(\Gamma_0(16))$. Then by Lemma 4.2.1 we have

$$f(z) = \sum_{t|16} a_t E_{2k}(tz; \chi_1, \chi_1) + b_1 E_{2k}(4z; \chi_{-4}, \chi_{-4}) + \sum_{i=1}^{4k-5} c_i C_{2k,i}(z) \quad (4.3.1)$$

for some $a_t, b_1, c_i \in \mathbb{C}$. Noting that $[0]_r \sum_{i=1}^{4k-5} c_i C_{2k,i}(z) = 0$ for all $r \in R(16)$, we compare first coefficients of both sides of (4.3.1) at cusps in $R(16)$ to obtain the following set of linear equations:

$$\begin{aligned} [0]_{1/1} f &= \sum_{t|16} a_t \left(\frac{\gcd(1, t)}{t} \right)^{2k} 2\zeta(2k), \\ [0]_{1/2} f &= \sum_{t|16} a_t \left(\frac{\gcd(2, t)}{t} \right)^{2k} 2\zeta(2k), \\ [0]_{1/4} f &= \sum_{t|16} a_t \left(\frac{\gcd(4, t)}{t} \right)^{2k} 2\zeta(2k) + b_1 \left(-\left(1 - \frac{1}{2^{2k}}\right) 2\zeta(2k) \right), \\ [0]_{1/8} f &= \sum_{t|16} a_t \left(\frac{\gcd(8, t)}{t} \right)^{2k} 2\zeta(2k), \\ [0]_{1/16} f &= \sum_{t|16} a_t \left(\frac{\gcd(16, t)}{t} \right)^{2k} 2\zeta(2k), \\ [0]_{3/4} f &= \sum_{t|16} a_t \left(\frac{\gcd(4, t)}{t} \right)^{2k} 2\zeta(2k) + b_1 \left(\left(1 - \frac{1}{2^{2k}}\right) 2\zeta(2k) \right). \end{aligned}$$

The theorem follows from solving these equations for a_t and b_1 . □

4.4 Proof of Theorem 4.1.2

To prove Theorem 4.1.2 we need the first coefficients of certain Eisenstein series.

We then apply Theorem 4.3.1 to obtain Theorem 4.1.2.

Lemma 4.4.1. Let $r \in R(16)$, for $l, m \in \mathbb{N}$ we have

$$[0]_r E_1(4z; \chi_1, \chi_{-4}) = \begin{cases} \frac{-i\pi}{4} = -iL(1, \chi_{-4}) & \text{if } r = 1/1, \\ 0 & \text{if } r = 1/2, \\ \frac{\pi}{2} = 2L(1, \chi_{-4}) & \text{otherwise,} \end{cases} \quad (4.4.1)$$

$$[0]_r E_2(4z; \chi_{-4}, \chi_{-4}) = \begin{cases} \frac{-\pi^2}{4} = -\left(1 - \frac{1}{2^2}\right) 2\zeta(2) & \text{if } r = 1/4, \\ \frac{\pi^2}{4} = \left(1 - \frac{1}{2^2}\right) 2\zeta(2) & \text{if } r = 3/4, \\ 0 & \text{otherwise,} \end{cases} \quad (4.4.2)$$

$$[0]_r E_{2l+1}(4z; \chi_1, \chi_{-4}) = \begin{cases} 2L(2l+1, \chi_{-4}) & \text{if } r = 1/4, 1/8, 1/16, \\ 2L(2l+1, \chi_{-4}) & \text{if } r = 3/4, \\ 0 & \text{otherwise,} \end{cases} \quad (4.4.3)$$

$$[0]_r E_{2l+1}(z; \chi_{-4}, \chi_1) = \begin{cases} -2L(2l+1, \chi_{-4}) & \text{if } r = 1/1, \\ 0 & \text{otherwise.} \end{cases} \quad (4.4.4)$$

Proof. $E_{2l+1}(4z; \chi_1, \chi_{-4})$ and $E_{2l+1}(z; \chi_{-4}, \chi_1)$ are convergent for $l, m \in \mathbb{N}$, thus we use arguments similar to the proof of Lemma 4.2.2 to obtain (4.4.3)–(4.4.4). The Fourier series expansions

$$E_1(4z; \chi_1, \chi_{-4}) = \frac{\pi}{2} + 2\pi q + 2\pi q^2 + 2\pi q^4 + 4\pi q^5 + O(q^8) \text{ and}$$

$$E_2(4z; \chi_{-4}, \chi_{-4}) = -i\pi^2 q + i\pi^2 4q^3 - i\pi^2 6q^5 + i\pi^2 8q^7 - i\pi^2 13q^9 + O(q^{11})$$

can be calculated by [31, Theorems 7.2.12, 7.2.13]. Using this and Sturm Theorem ([23, Theorem 3.13]), we obtain

$$E_1(4z; \chi_1, \chi_{-4}) = \frac{\pi}{2} \frac{\eta^{10}(2z)}{\eta^4(z)\eta^4(4z)}, \quad E_2(4z; \chi_{-4}, \chi_{-4}) = -i\pi^2 \frac{\eta^4(2z)\eta^4(8z)}{\eta^4(4z)}.$$

We then use [26, Proposition 2.1] to obtain

$$\begin{aligned} & \frac{\eta^{10}(2A_{1/1}^{-1}(z))}{\eta^4(A_{1/1}^{-1}(z))\eta^4(4A_{1/1}^{-1}(z))} \\ &= \frac{(-z-1) \left(e^{5\pi i/12} \sum_{n \geq 1} \left(\frac{12}{n} \right) e^{2\pi i(n^2(z+1)/48)} \right)^{10}}{2 \left(e^{\pi i/3} \sum_{n \geq 1} \left(\frac{12}{n} \right) e^{2\pi i(n^2(z+1)/24)} \right)^4 \left(e^{7\pi i/12} \sum_{n \geq 1} \left(\frac{12}{n} \right) e^{2\pi i(n^2(z+1)/96)} \right)^4} \end{aligned} \quad (4.4.5)$$

$$= -(z+1) \left(\frac{i}{2} + \sum_{n > 0} a_n e^{2\pi i n/16} \right). \quad (4.4.6)$$

Thus we have $[0]_{1/1} \frac{\eta^{10}(2z)}{\eta^4(z)\eta^4(4z)} = \frac{-i}{2}$. Similarly we have

$$[0]_r \frac{\eta^{10}(2z)}{\eta^4(z)\eta^4(4z)} = \begin{cases} \frac{-i}{2} & \text{if } r = 1/1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$[0]_r \frac{\eta^4(2z)\eta^4(8z)}{\eta^4(4z)} = \begin{cases} \frac{-i}{4} & \text{if } r = 1/4, \\ \frac{i}{4} & \text{if } r = 3/4, \\ 0 & \text{otherwise.} \end{cases}$$

This proves (4.4.1) and (4.4.2). For the details of (4.4.5) and above two results, please see Appendix B.1 □

We have

$$\begin{aligned} E_{2l+1}(4z; \chi_1, \chi_{-4}) E_{2m+1}(4z; \chi_1, \chi_{-4}) &\in M_{2(l+m)+2}(\Gamma_0(16)) && \text{for all } l, m \in \mathbb{N}_0, \\ E_{2l+1}(4z; \chi_1, \chi_{-4}) E_{2m+1}(z; \chi_{-4}, \chi_1) &\in M_{2(l+m)+2}(\Gamma_0(16)) && \text{for all } l, m \in \mathbb{N}_0, \\ E_{2l+1}(z; \chi_{-4}, \chi_1) E_{2m+1}(z; \chi_{-4}, \chi_1) &\in M_{2(l+m)+2}(\Gamma_0(16)) && \text{for all } l, m \in \mathbb{N}_0, \\ E_{2l}(4z; \chi_{-4}, \chi_{-4}) E_{2m}(4z; \chi_{-4}, \chi_{-4}) &\in M_{2(l+m)}(\Gamma_0(16)) && \text{for all } l, m \in \mathbb{N}. \end{aligned}$$

Then the proof of Theorem 4.1.2 follows from Theorem 4.3.1 using the constant terms given by Lemma 4.4.1, and comparing coefficients of q^n in resulting equations.

Below we give the details of the proof of (4.1.3), the rest can be done similarly.

Letting $l, m \in \mathbb{N}$ by Theorem 4.3.1 and (4.4.4) we obtain

$$\begin{aligned} E_{2l+1}(z; \chi_{-4}, \chi_1) E_{2m+1}(z; \chi_{-4}, \chi_1) = \\ A(2k) 2^{2k+1} 4L(2l+1, \chi_{-4}) L(2m+1, \chi_{-4}) (E_{2k}(z; \chi_1, \chi_1) - E_{2k}(2z; \chi_1, \chi_1)) \\ + \sum_{i=1}^{4k-5} c_i C_{2k,i}(z). \end{aligned} \quad (4.4.7)$$

Noting that by [31, Theorem 3.3.4] we have

$$\begin{aligned} L(2l+1, \chi_{-4}) &= (-1)^{l+1} \left(\frac{2\pi}{4} \right)^{2l+1} \frac{B_{2l+1, \chi_{-4}}}{(2l+1)!}, \\ \zeta(2k) &= (-1)^{k+1} \frac{(2\pi)^{2k}}{2} \frac{B_{2k}}{(2k)!}, \end{aligned}$$

and using the Fourier series expansion given by (4.2.3) we obtain

$$\frac{E_{2l+1}(z; \chi_{-4}, \chi_1)}{L(2l+1, \chi_{-4})} = i \frac{4^{2l+1} (2l+1)}{B_{2l+1, \chi_{-4}}} \sum_{n \geq 0} \hat{\sigma}_{2l}(n) q^n, \quad (4.4.8)$$

$$A(2k) E_{2k}(z; \chi_1, \chi_1) = \frac{-4k}{2(2^{2k} - 1) B_{2k}} \sum_{n \geq 0} \sigma_{2k-1}(n) q^n. \quad (4.4.9)$$

Putting (4.4.8)–(4.4.9) in (4.4.7) we have

$$\begin{aligned} \sum_{n \geq 0} \hat{\sigma}_{2l}(n) q^n \sum_{n \geq 0} \hat{\sigma}_{2m}(n) q^n \\ = \frac{B_{2l+1, \chi_{-4}} B_{2m+1, \chi_{-4}}}{4(2l+1)(2m+1)} \frac{4k}{B_{2k}} \frac{4}{2^{2k}(2^{2k} - 1)} \left(\sum_{n \geq 0} \sigma_{2k-1}(n) q^n - \sum_{n \geq 0} \sigma_{2k-1}(n/2) q^n \right) \\ + \sum_{i=1}^{4k-5} c_i C_{2k,i}(z). \end{aligned} \quad (4.4.10)$$

For $l, m > 0$, (4.1.3) follows from comparing coefficients of q^n on both sides of (4.4.10). Note that we have

$$(-2i)E_1(4z; \chi_1, \chi_{-4}) = E_1(z; \chi_{-4}, \chi_1), \quad (4.4.11)$$

$$E_1(z; \chi_{-4}, \chi_1) = -4\pi i \sum_{n \geq 0} \hat{\sigma}_0(n) q^n. \quad (4.4.12)$$

We prove the cases when $l = 0$, or $m = 0$ similarly by employing (4.4.1) and (4.4.4) in Theorem 4.3.1 and using the Fourier expansion (4.4.12). We finish the section with the following equations that are needed in the proofs of other cases.

$$\begin{aligned} \frac{E_{2l+1}(4z, \chi_1, \chi_{-4})}{2L(2l+1, \chi_{-4})} &= \frac{-2(2l+1)}{B_{2l+1, \chi_{-4}}} \sum_{n \geq 0} \tilde{\sigma}_{2l}(n) q^n, \\ A(2k)E_{2k}(4z, \chi_{-4}, \chi_{-4}) &= i \frac{-4k}{4^{2k}(2^{2k}-1)B_{2k}} \sum_{n \geq 0} \bar{\sigma}_{2k-1}(n) q^n. \end{aligned}$$

4.5 Proof of Corollary 4.1.3 and on Cusp Parts

Equalities in Corollary 4.1.3 can be shown using the Sturm Theorem, see [23, Theorem 3.13]. Let

$$a_n := \sum_{0 \leq j \leq 2n} \tilde{\sigma}_0(j) \tilde{\sigma}_4(2n-j) - \frac{5}{2}(\sigma_5(n) - 64\sigma_5(n/2)).$$

Then $\sum_{n \geq 0} a_n q^n$ is a modular form in $M_6(\Gamma_0(16))$, see [34, Proposition 2.22]. We use MAPLE to compute $a_n = 0$ for $0 \leq n \leq 13$. Thus by Sturm Theorem, $a_n = 0$ for all $n \in \mathbb{N}_0$. This proves the second equality in Corollary 4.1.3. The rest of the equalities in Corollary 4.1.3 can be proven similarly.

The rest of the formulas appear to have non-vanishing cusps. For example for (4.1.6) when $(l', m) = (2, 1)$ we have

$$[n] \sum_{i=1}^7 c_i C_{6,i}(z) = \sum_{0 \leq j \leq n} \sigma_3(j) \bar{\sigma}_1(n-j) - \frac{1}{3840} \bar{\sigma}_5(n). \quad (4.5.1)$$

We compare LHS and RHS of (4.5.1) for $0 \leq n \leq 7$ to obtain the following matrix corresponding to this system of equations

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{256} \\ -12 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 60 & -10 & 1 & 0 & 0 & 0 & 0 & \frac{579}{64} \\ -160 & 40 & -8 & 1 & 0 & 0 & 0 & 24 \\ 246 & -80 & 24 & -6 & 1 & 0 & 0 & \frac{4635}{128} \\ -240 & 84 & -32 & 12 & -4 & 1 & 0 & 20 \\ 184 & -52 & 18 & -8 & 4 & -2 & 1 & \frac{587}{32} \end{bmatrix}.$$

We solve the above matrix to obtain

$$c_1 = \frac{1}{256}, c_2 = \frac{67}{64}, c_3 = \frac{617}{32}, c_4 = 137, c_5 = \frac{1913}{4}, c_6 = 819, c_7 = 546.$$

As noted at the end of proof of Lemma 4.2.1, coefficients of $C_{2k,i}$ will always have a lower triangular shape, which allows us to compute c_i recursively.

Additionally, in general the Fourier coefficients of cusp forms in $M_4(\Gamma_0(16))$ and $M_6(\Gamma_0(16))$ are vanishing on certain arithmetic progressions.

Lemma 4.5.1. For all $n \in \mathbb{N}_0$ we have

$$[4n]C_{4,i}(z) = 0, \text{ for } 1 \leq i \leq 3, \quad (4.5.2)$$

$$[8n]C_{6,i}(z) = 0, \text{ for } 1 \leq i \leq 7. \quad (4.5.3)$$

Proof. (4.5.2) and (4.5.3) follow from [1, Theorem 1.1] and [1, Theorem 1.2], respectively. $\square$

Thus letting $k \in \{2, 3\}$, for any modular form $f(z) \in M_{2k}(\Gamma_0(16))$ and $n \in \mathbb{N}_0$, we have

$$\begin{aligned}
[(4k-4)n]f(z) = & A(2k)[(4k-4)n] \Big((2^{2k+1}[0]_{1/1}f - 2[0]_{1/2}f) E_{2k}(z; \chi_1, \chi_1) \\
& + (-2^{2k+1}[0]_{1/1}f + (2^{2k+1} + 2)[0]_{1/2}f - ([0]_{1/4}f + [0]_{3/4}f)) E_{2k}(2z; \chi_1, \chi_1) \\
& + (-2^{2k+1}[0]_{1/2}f + (2^{2k} + 1)([0]_{1/4}f + [0]_{3/4}f) - 2[0]_{1/8}f) E_{2k}(4z; \chi_1, \chi_1) \\
& + (-2^{2k}([0]_{1/4}f + [0]_{3/4}f) + (2^{2k+1} + 2)[0]_{1/8}f - 2[0]_{1/16}f) E_{2k}(8z; \chi_1, \chi_1) \\
& + (-2^{2k+1}[0]_{1/8}f + 2^{2k+1}[0]_{1/16}f) E_{2k}(16z; \chi_1, \chi_1) \\
& + (-2^{2k}([0]_{1/4}f - [0]_{3/4}f)) E_{2k}(4z; \chi_{-4}, \chi_{-4}) \Big).
\end{aligned}$$

4.6 Application to Representations by Quadratic Forms

In this section we use Theorem 4.3.1 to derive formulas for representations by certain quadratic forms. Let

$$\begin{aligned}
N(r, s, t; n) = & \#\{(x_1, \dots, x_{r+s+t}) \in \mathbb{N}^{r+s+t} \mid (x_1^2 + \dots + x_r^2) + 2(x_{r+1}^2 + \dots + x_{r+s}^2) \\
& + 4(x_{r+s+1}^2 + \dots + x_{r+s+t}^2) = n\}.
\end{aligned}$$

Lemire in his thesis [27], has proven formulas for $N(r, s, t; n)$ under certain restrictions on r, s, t . His methods to prove these formulas are different from ours. Below we provide an another proof of all his results by using Theorem 4.3.1.

Theorem 4.6.1. Let $r, s, t \in \mathbb{N}$ such that s is even and $r + s + t = 4k$ with $1 < k \in \mathbb{N}$. Then if $r, s, t \neq 0$ we have

$$\begin{aligned} N(r, s, t; n) = & \frac{-4k}{(2^{2k} - 1)B_{2k}} \frac{(-1)^k (\sigma_{2k-1}(n) - \sigma_{2k-1}(n/2))}{2^{\frac{s+2t}{2}}} \\ & - \frac{-4k}{(2^{2k} - 1)B_{2k}} (\sigma_{2k-1}(n/8) - 2^{2k} \sigma_{2k-1}(n/16)) \\ & + [n] \sum_{i=1}^{4k-5} c_i C_{2k,i}(z), \end{aligned} \quad (4.6.1)$$

$$\begin{aligned} N(r, s, 0; n) = & \frac{-4k}{(2^{2k} - 1)B_{2k}} \frac{(-1)^k (\sigma_{2k-1}(n) - \sigma_{2k-1}(n/2))}{2^{\frac{s}{2}}} \\ & + \frac{-4k}{(2^{2k} - 1)B_{2k}} (-\sigma_{2k-1}(n/4) + 2^{2k} \sigma_{2k-1}(n/8)) \\ & + [n] \sum_{i=1}^{4k-5} c_i C_{2k,i}(z). \end{aligned} \quad (4.6.2)$$

$$\begin{aligned} N(r, 0, t; n) = & (-1)^k \frac{-4k}{2^t (2^{2k} - 1)B_{2k}} \sigma_{2k-1}(n) \\ & - (-1)^k \frac{-4k}{(2^{2k} - 1)B_{2k}} \left(2^{-t} + \frac{\cos(\pi r/4)}{2^{\frac{t}{2}}} \right) \sigma_{2k-1}(n/2) \\ & + (-1)^k \frac{-4k}{(2^{2k} - 1)B_{2k}} \left(\frac{(2^{2k} + 1) \cos(\pi r/4)}{2^{\frac{t}{2}}} \right) \sigma_{2k-1}(n/4) \\ & - (-1)^k \frac{-4k}{(2^{2k} - 1)B_{2k}} \left(\frac{\cos(\pi r/4)}{2^{\frac{t}{2}-2k}} + (-1)^k \right) \sigma_{2k-1}(n/8) \\ & + \frac{-4k}{(2^{2k} - 1)B_{2k}} 2^{2k} \sigma_{2k-1}(n/16) \\ & + (-1)^k \frac{-4k}{(2^{2k} - 1)B_{2k}} \left(\frac{\sin(\pi r/4)}{2^{\frac{t}{2}+2k-1}} \right) \bar{\sigma}_{2k-1}(n) \\ & + [n] \sum_{i=1}^{4k-5} c_i C_{2k,i}(z). \end{aligned} \quad (4.6.3)$$

Proof. The generating function of $N(r, s, t; n)$ is given by the eta quotient

$$f(r, s, t; z) := \frac{\eta^{5r-2s}(2z) \eta^{5s-2r-2t}(4z) \eta^{5t-2s}(8z)}{\eta^{2r}(z) \eta^{2t}(16z)},$$

see [8] for details. By [8, Theorem 2.1], if s is even then we have

$$f(r, s, t; z) \in M_{2k}(\Gamma_0(16)).$$

On the other hand, if $r, s, t \neq 0$ by [26, Proposition 2.1], proceeding similar to (4.4.6), we compute

$$f(r, s, t; A_{1/1}^{-1}(z)) = 2^{\frac{-r-2s-3t}{2}} e^{\frac{\pi i(r+s+t)}{4}} (-z-1)^{\frac{r+s+t}{2}} \left(1 + \sum_{n \geq 1} a_n e^{2\pi i z n / 16} \right).$$

From this we obtain $[0]_{1/1} f(r, s, t; z) = (-1)^k 2^{-2k - \frac{s+2t}{2}}$. Similar calculations will yield to

$$[0]_r f(r, s, t; z) = \begin{cases} (-1)^k 2^{-2k - \frac{s+2t}{2}} & \text{if } r = 1/1, \\ 1 & \text{if } r = 1/16, \\ 0 & \text{otherwise.} \end{cases}$$

Then (4.6.1) follows from putting these values in Theorem 4.3.1 and comparing coefficients of q^n on both sides of the resulting equation. When $r, s \neq 0, t = 0$ and $r, t \neq 0, s = 0$ we have

$$[0]_r f(r, s, 0; z) = \begin{cases} (-1)^k 2^{-2k - \frac{s}{2}} & \text{if } r = 1/1, \\ 1 & \text{if } r = 1/8, \\ 1 & \text{if } r = 1/16, \\ 0 & \text{otherwise,} \end{cases}$$

$$[0]_r f(r, 0, t; z) = \begin{cases} (-1)^k 2^{-2k+t} & \text{if } r = 1/1, \\ (-1)^k 2^{-\frac{t}{2}} e^{\frac{\pi i r}{4}} & \text{if } r = 1/4, \\ 1 & \text{if } r = 1/16, \\ (-1)^k 2^{-\frac{t}{2}} e^{-\frac{\pi i r}{4}} & \text{if } r = 3/4, \\ 0 & \text{otherwise,} \end{cases}$$

from which (4.6.2) and (4.6.3) can be proven similarly. $\square$

The Eisenstein parts of (4.6.1)–(4.6.3) agree with Lemire’s results [27, Theorems 6.1.3–6.3.3], cusp parts look different due to the choice of cusp basis. In (4.6.3) we reveal an additional Eisenstein term $(-1)^k \frac{-4k}{(2^{2k}-1)B_{2k}} \left(\frac{\sin(\pi r/4)}{2^{\frac{t}{2}+2k-1}} \right) \bar{\sigma}_{2k-1}(n)$ which is hiding inside the eta quotients of [27, Theorem 6.3.3].

Appendix A

In this appendix, we prove the following theorem, which shows that we may express generating functions of ranks modulo two different integers using the same Lambert series and similar products.

Theorem A.1. Let k_1 and k_2 be two different positive integers. Let a_1 and a_2 denote primitive k_1 -th and k_2 -th roots of unity, respectively. We use m to denote the lcm of k_1 and k_2 and let $M = m$ if m is even and $M = 2m$ if m is odd. Define d_l and d'_l as

$$d_l = \begin{cases} (-1)^l (a_1^{2m-2l+2} + a_1^{-2m+2l-2} - a_1^{2m-2l+3} - a_1^{-2m+2l-3}), & \text{if } m \text{ is even,} \\ (-1)^l (a_1^{4m-2l+2} + a_1^{-4m+2l-2} - a_1^{4m-2l+3} - a_1^{-4m+2l-3}), & \text{if } m \text{ is odd,} \end{cases}$$

and

$$d'_l = \begin{cases} (-1)^l (a_2^{2m-2l+2} + a_2^{-2m+2l-2} - a_2^{2m-2l+3} - a_2^{-2m+2l-3}), & \text{if } m \text{ is even,} \\ (-1)^l (a_2^{4m-2l+2} + a_2^{-4m+2l-2} - a_2^{4m-2l+3} - a_2^{-4m+2l-3}), & \text{if } m \text{ is odd.} \end{cases}$$

Recall that $p_a(q)$ is defined by (3.1.9) in Chapter 3 and we use $\Re(z)$ to denote the real part of a complex number z . Then the following identities for generating functions of ranks modulo k_1 and k_2 are true.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(a_1 q)_n (q/a_1)_n} = -d_1 + \sum_{l=1}^M d_l L_l(q) + p_{-2\Re(a_1)}(q)$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(a_2 q)_n (q/a_2)_n} = -d'_1 + \sum_{l=1}^M d'_l L_l(q) + p_{-2\Re(a_2)}(q),$$

where

$$L_l(q) = \begin{cases} \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n+(4n+1)(l-1)}}{1-q^{(4n+1)m}}, & \text{if } m \text{ is even,} \\ \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n+(4n+1)(l-1)}}{1-q^{(4n+1)2m}}, & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Take $x = a_1$ in (3.2.1) and we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{q^{n^2}}{(a_1 q)_n (q/a_1)_n} \\ &= (1 - a_1) \left(1 - \sum_{n=0}^{\infty} \frac{q^{4n^2}}{(-qa_1^{-2}; q^4)_{n+1} (-q^3 a_1^2; q^4)_n} \right) \\ & \quad + (a_1^{-1} - 1) \left(-1 + \sum_{n=0}^{\infty} \frac{q^{4n^2}}{(-qa_1^2; q^4)_{n+1} (-q^3/a_1^2; q^4)_n} \right) \\ & \quad + (1 - a_1) \frac{J_2 J_{2,4}^2}{j(a_1; q) j(-qa_1^2; q^2)}. \end{aligned} \tag{A.1}$$

If the lcm of k_1 and k_2 , m , is even, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{q^{4n^2}}{(-qa_1^{-2}; q^4)_{n+1} (-q^3 a_1^2; q^4)_n} \\ &= \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1 + a_1^{-2} q^{4n+1}} \quad (\text{by (3.2.2)}) \\ &= \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (q^{6n^2+2n} - a_1^{2m-2} q^{6n^2+6n+1} + a_1^{2m-4} q^{6n^2+10n+2} - \dots - a_1^2 q^{6n^2+(4m-2)n+m-1})}{1 - q^{(4n+1)m}}, \end{aligned} \tag{A.2}$$

and

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{q^{4n^2}}{(-qa_1^2; q^4)_{n+1}(-q^3/a_1^2; q^4)_n} \\
&= \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1+a_1^2 q^{4n+1}} \quad (\text{by (3.2.2)}) \\
&= \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (q^{6n^2+2n} - a_1^{-2m+2} q^{6n^2+6n+1} + a_1^{-2m+4} q^{6n^2+10n+2} - \dots - a_1^{-2} q^{6n^2+(4m-2)n+m-1})}{1 - q^{(4n+1)m}}.
\end{aligned} \tag{A.3}$$

Substituting (A.2) and (A.3) into (A.1), we find that

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{q^{n^2}}{(a_1 q)_n (q/a_1)_n} \\
&= (2 - a_1 - a_1^{-1}) + \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (-2 + a_1 + a_1^{-1}) q^{6n^2+2n}}{1 - q^{(4n+1)m}} \\
&+ \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a_1^{2m-2} + a_1^{-2m+2} - a_1^{2m-1} - a_1^{-2m+1}) q^{6n^2+6n+1}}{1 - q^{(4n+1)m}} \\
&+ \dots \\
&+ \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a_1^2 + a_1^{-2} - a_1^3 - a_1^{-3}) q^{6n^2+(4m-2)n+m-1}}{1 - q^{(4n+1)m}} \\
&+ (1 - a_1) \frac{J_2 J_{2,4}^2}{j(a_1; q) j(-qa_1^2; q^2)}.
\end{aligned} \tag{A.4}$$

For a_2 we can use same method and obtain a similar result with same summation parts. In other words,

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{q^{n^2}}{(a_2 q)_n (q/a_2)_n} \\
&= (2 - a_2 - a_2^{-1}) + \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (-2 + a_2 + a_2^{-1}) q^{6n^2+2n}}{1 - q^{(4n+1)m}} \\
&+ \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a_2^{2m-2} + a_2^{-2m+2} - a_2^{2m-1} - a_2^{-2m+1}) q^{6n^2+6n+1}}{1 - q^{(4n+1)m}} \\
&+ \dots \\
&+ \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a_2^2 + a_2^{-2} - a_2^3 - a_2^{-3}) q^{6n^2+(4m-2)n+m-1}}{1 - q^{(4n+1)m}} \\
&+ (1 - a_2) \frac{J_2 J_{2,4}^2}{j(a_2; q) j(-q a_2^2; q^2)}. \tag{A.5}
\end{aligned}$$

We observe that (A.4) and (A.5) are in the same form as shown in Theorem A.1.

If m is odd, we have

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{q^{4n^2}}{(-q a_1^{-2}; q^4)_{n+1} (-q^3 a_1^2; q^4)_n} \\
&= \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+2n}}{1 + a_1^{-2} q^{4n+1}} \quad (\text{by (3.2.2)}) \\
&= \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (q^{6n^2+2n} - a_1^{4m-2} q^{6n^2+6n+1} + a_1^{4m-4} q^{6n^2+10n+2} - \dots - a_1^2 q^{6n^2+(8m-2)n+2m-1})}{1 - q^{(4n+1)2m}}. \tag{A.6}
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{q^{n^2}}{(a_1 q)_n (q/a_1)_n} \\
&= (2 - a_1 - a_1^{-1}) + \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (-2 + a_1 + a_1^{-1}) q^{6n^2+2n}}{1 - q^{(4n+1)2m}} \\
&+ \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a_1^{4m-2} + a_1^{-4m+2} - a_1^{4m-1} - a_1^{-4m+1}) q^{6n^2+6n+1}}{1 - q^{(4n+1)2m}} \\
&+ \dots \\
&+ \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a_1^2 + a_1^{-2} - a_1^3 - a_1^{-3}) q^{6n^2+(8m-2)n+2m-1}}{1 - q^{(4n+1)2m}} \\
&+ (1 - a_1) \frac{J_2 J_{2,4}^2}{j(a_1; q) j(-q a_1^2; q^2)}. \tag{A.7}
\end{aligned}$$

Using the same method for a_2 , we find that

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{q^{n^2}}{(a_2 q)_n (q/a_2)_n} \\
&= (2 - a_2 - a_2^{-1}) + \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (-2 + a_2 + a_2^{-1}) q^{6n^2+2n}}{1 - q^{(4n+1)2m}} \\
&+ \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a_2^{4m-2} + a_2^{-4m+2} - a_2^{4m-1} - a_2^{-4m+1}) q^{6n^2+6n+1}}{1 - q^{(4n+1)2m}} \\
&+ \dots \\
&+ \frac{1}{J_4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a_2^2 + a_2^{-2} - a_2^3 - a_2^{-3}) q^{6n^2+(8m-2)n+2m-1}}{1 - q^{(4n+1)2m}} \\
&+ (1 - a_2) \frac{J_2 J_{2,4}^2}{j(a_2; q) j(-q a_2^2; q^2)}. \tag{A.8}
\end{aligned}$$

Now we again see that (A.7) and (A.8) are in the same form as shown in Theorem A.1, which complete the proof. $\square$

Remark A.2. From the proof we observe that, getting a 2-dissection of the product part, $(1 - a) \frac{J_2 J_{2,4}^2}{j(a; q) j(-q a^2; q^2)}$, is sufficient to give a 2-dissection of the generating function of the rank.

Appendix B

In this appendix, we provide some calculation details for Chapter [4](#).

B.1 Fouries Expansion of $\eta(z), \eta(2z), \eta(4z), \eta(8z)$ and $\eta(16z)$

In this section, we use $\eta_i(z)$ to denote $\eta(iz)$ for $i = 1, 2, 4, 8$ and 16 .

B.1.1 $\eta(z), \eta(2z), \eta(4z), \eta(8z)$ and $\eta(16z)$ at cusp 1

Let $A_1 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \in SL_2(\mathbb{Z})$, so $A_1^{-1} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$.

For $\eta_1(z), \eta_2(z), \eta_4(z), \eta_8(z)$ and $\eta_{16}(z)$, we choose $L_1 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}, L_4 = \begin{pmatrix} -4 & 1 \\ -1 & 0 \end{pmatrix}, L_8 = \begin{pmatrix} -8 & 1 \\ -1 & 0 \end{pmatrix}$ and $L_{16} = \begin{pmatrix} -16 & 1 \\ -1 & 0 \end{pmatrix}$, respectively.

From [26, Proposition 2.1] , we get

$$\begin{aligned}
\eta_1(A_1^{-1}z) &= e^{\pi i/3}(-z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(z+1)\right), \\
\eta_2(A_1^{-1}z) &= \frac{e^{5\pi i/12}}{2^{1/2}}(-z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{48}(z+1)\right), \\
\eta_4(A_1^{-1}z) &= \frac{e^{7\pi i/12}}{2}(-z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{96}(z+1)\right), \\
\eta_8(A_1^{-1}z) &= \frac{e^{11\pi i/12}}{8^{1/2}}(-z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{192}(z+1)\right), \\
\eta_{16}(A_1^{-1}z) &= \frac{e^{19\pi i/12}}{4}(-z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{384}(z+1)\right).
\end{aligned}$$

The above results lead us to (4.4.5), and the two results following (4.4.5) are obtained from the following subsections.

B.1.2 $\eta(z), \eta(2z), \eta(4z), \eta(8z)$ and $\eta(16z)$ at cusp $1/2$

Let $A_2 = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} \in SL_2(\mathbb{Z})$, so $A_2^{-1} = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$. For $\eta_1(z), \eta_2(z), \eta_4(z), \eta_8(z)$

and $\eta_{16}(z)$, we choose $L_1 = \begin{pmatrix} -1 & -1 \\ -2 & -3 \end{pmatrix}, L_2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, L_4 = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}, L_8 =$

$\begin{pmatrix} -4 & 1 \\ -1 & 0 \end{pmatrix}$ and $L_{16} = \begin{pmatrix} -8 & 1 \\ -1 & 0 \end{pmatrix}$, respectively. We get

$$\begin{aligned}\eta_1(A_2^{-1}z) &= e^{5\pi i/12}(-2z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(z-1)\right), \\ \eta_2(A_2^{-1}z) &= e^{\pi i/3}(-2z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(2z+1)\right), \\ \eta_4(A_2^{-1}z) &= \frac{e^{5\pi i/12}}{2^{1/2}}(-2z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{48}(2z+1)\right), \\ \eta_8(A_2^{-1}z) &= \frac{e^{7\pi i/12}}{2}(-2z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{96}(2z+1)\right), \\ \eta_{16}(A_2^{-1}z) &= \frac{e^{11\pi i/12}}{8^{1/2}}(-2z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{192}(2z+1)\right).\end{aligned}$$

B.1.3 $\eta(z), \eta(2z), \eta(4z), \eta(8z)$ and $\eta(16z)$ at cusp $1/4$

Here we set $A_3 = \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix} \in SL_2(\mathbb{Z})$, so $A_3^{-1} = \begin{pmatrix} -1 & 0 \\ -4 & -1 \end{pmatrix}$. For $\eta_1(z), \eta_2(z), \eta_4(z), \eta_8(z)$ and $\eta_{16}(z)$, we choose $L_1 = \begin{pmatrix} -1 & -1 \\ -4 & -5 \end{pmatrix}$, $L_2 = \begin{pmatrix} -1 & -1 \\ -2 & -3 \end{pmatrix}$, $L_4 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $L_8 = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$ and $L_{16} = \begin{pmatrix} -4 & 1 \\ -1 & 0 \end{pmatrix}$, respectively. We get

$$\begin{aligned}\eta_1(A_3^{-1}z) &= e^{\pi i/4}(-4z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(z-1)\right), \\ \eta_2(A_3^{-1}z) &= e^{5\pi i/12}(-4z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(2z-1)\right), \\ \eta_4(A_3^{-1}z) &= e^{\pi i/3}(-4z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(4z+1)\right), \\ \eta_8(A_3^{-1}z) &= \frac{e^{5\pi i/12}}{2^{1/2}}(-4z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{48}(4z+1)\right), \\ \eta_{16}(A_3^{-1}z) &= \frac{e^{7\pi i/12}}{2}(-4z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{96}(4z+1)\right).\end{aligned}$$

B.1.4 $\eta(z), \eta(2z), \eta(4z), \eta(8z)$ and $\eta(16z)$ at cusp $1/8$

Let $A_4 = \begin{pmatrix} -1 & 0 \\ 8 & -1 \end{pmatrix} \in SL_2(\mathbb{Z})$, so $A_4^{-1} = \begin{pmatrix} -1 & 0 \\ -8 & -1 \end{pmatrix}$. For $\eta_1(z), \eta_2(z), \eta_4(z), \eta_8(z)$ and $\eta_{16}(z)$: We choose $L_1 = \begin{pmatrix} -1 & -1 \\ -8 & -9 \end{pmatrix}, L_2 = \begin{pmatrix} -1 & -1 \\ -4 & -5 \end{pmatrix}, L_4 = \begin{pmatrix} -1 & -1 \\ -2 & -3 \end{pmatrix}, L_8 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ and $L_{16} = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$, respectively. We get

$$\begin{aligned} \eta_1(A_4^{-1}z) &= e^{-\pi i/12}(-8z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(z-1)\right), \\ \eta_2(A_4^{-1}z) &= e^{\pi i/4}(-8z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(2z-1)\right), \\ \eta_4(A_4^{-1}z) &= e^{5\pi i/12}(-8z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(4z-1)\right), \\ \eta_8(A_4^{-1}z) &= e^{\pi i/3}(-8z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(8z+1)\right), \\ \eta_{16}(A_4^{-1}z) &= \frac{e^{5\pi i/12}}{2^{1/2}}(-8z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{48}(8z+1)\right). \end{aligned}$$

B.1.5 $\eta(z), \eta(2z), \eta(4z), \eta(8z)$ and $\eta(16z)$ at cusp $1/16$

Let $A_5 = \begin{pmatrix} -1 & 0 \\ 16 & -1 \end{pmatrix} \in SL_2(\mathbb{Z})$, so $A_5^{-1} = \begin{pmatrix} -1 & 0 \\ -16 & -1 \end{pmatrix}$. For $\eta_1(z), \eta_2(z), \eta_4(z), \eta_8(z)$ and $\eta_{16}(z)$, we choose $L_1 = \begin{pmatrix} -1 & -1 \\ -16 & -17 \end{pmatrix}, L_2 = \begin{pmatrix} -1 & -1 \\ -8 & -9 \end{pmatrix}, L_4 = \begin{pmatrix} -1 & -1 \\ -4 & -5 \end{pmatrix}, L_8 =$

$\begin{pmatrix} -1 & -1 \\ -2 & -3 \end{pmatrix}$ and $L_{16} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, respectively. We get

$$\begin{aligned}\eta_1(A_5^{-1}z) &= e^{-3\pi i/4}(-16z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(z-1)\right), \\ \eta_2(A_5^{-1}z) &= e^{-\pi i/12}(-16z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(2z-1)\right), \\ \eta_4(A_5^{-1}z) &= e^{\pi i/4}(-16z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(4z-1)\right), \\ \eta_8(A_5^{-1}z) &= e^{5\pi i/12}(-16z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(8z-1)\right), \\ \eta_{16}(A_5^{-1}z) &= e^{\pi i/3}(-16z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(16z+1)\right).\end{aligned}$$

B.1.6 $\eta(z), \eta(2z), \eta(4z), \eta(8z)$ and $\eta(16z)$ at cusp $3/4$

Let $A_6 = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} \in SL_2(\mathbb{Z})$, so $A_6^{-1} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$. For $\eta_1(z), \eta_2(z), \eta_4(z), \eta_8(z)$ and $\eta_{16}(z)$, we choose $L_1 = \begin{pmatrix} 3 & -4 \\ 4 & -5 \end{pmatrix}, L_2 = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, L_4 = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}, L_8 = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$ and $L_{16} = \begin{pmatrix} 12 & -1 \\ 1 & 0 \end{pmatrix}$, respectively. We get

$$\begin{aligned}\eta_1(A_5^{-1}z) &= e^{-\pi i/6}(4z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(z+1)\right), \\ \eta_2(A_5^{-1}z) &= (4z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{12}(z-1)\right), \\ \eta_4(A_5^{-1}z) &= (4z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24}(4z-1)\right), \\ \eta_8(A_5^{-1}z) &= \frac{e^{\pi i/4}}{2^{1/2}}(4z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{48}(4z-1)\right), \\ \eta_{16}(A_5^{-1}z) &= \frac{e^{3\pi i/4}}{2}(4z-1)^{1/2} \sum_{n \geq 1} \left(\frac{12}{n}\right) e\left(\frac{n^2}{96}(4z-1)\right).\end{aligned}$$

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