Nonlinear, non-Hermitian and high-dimensional classical topological systems

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**NANYANG
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SINGAPORE

**NONLINEAR, NON-HERMITIAN AND
HIGH-DIMENSIONAL CLASSICAL
TOPOLOGICAL SYSTEMS**

WANG YOU

SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

2019

NONLINEAR, NON-HERMITIAN AND HIGH-DIMENSIONAL CLASSICAL TOPOLOGICAL SYSTEMS

WANG YOU

SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

A thesis submitted to the Nanyang Technological
University in partial fulfilment of the requirement for
the degree of Doctor of Philosophy

2019

Statement of Originality

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Authorship Attribution Statement

This thesis contains material from 3 papers published in the following peer-reviewed journals in which I am listed as an author.

Chapter 2 is published as Xin Zhou, You Wang, Daniel Leykam, and Y.D. Chong. Optical isolation with nonlinear topological photonics. *New Journal of Physics* **19**, 095002 (2017).

The contributions of the co-authors are as follows:

- Yidong Chong conceived the idea.
- I performed the theoretical investigations and simulations in coupled ring resonators section. Xin Zhou performed the theoretical investigations and simulations in 1D SSH model and 2D Haldane model sections. Daniel Leykam and Yidong Chong supervised the project.
- I prepared the manuscript drafts for coupled ring resonator section. Xin Zhou prepared the manuscript drafts for 1D SSH model and 2D Haldane model sections. Daniel Leykam and Yidong Chong revised the manuscript.

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The contributions of the co-authors are as follows:

- I conducted the simulation, designed the printed circuit board and performed the measurements.
- I and Li-Jun Lang analyzed the data.
- Baile Zhang and Y.D. Chong supervised the project.
- I, Li-Jun Lang, Ching Hua Lee, Baile Zhang, and Y.D. Chong participated in the design and interpretation of the experiment.
- I prepared the manuscript draft, Y.D. Chong revised it.

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The contributions of the co-authors are as follows:

- Li-Jun Lang and Y.D. Chong conceived the idea.
- I and Li-Jun Lang performed the simulation. Li-Jun Lang performed theoretical analysis.
- I, Li-Jun Lang, Hailong Wang and Y.D. Chong participated in the analysis and interpretation of the simulation result.
- Li-Jun Lang prepared the manuscript draft, Y.D. Chong revised it.

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For my family and friends

Abstract

This thesis concerns classical topological systems with properties that are difficult to explore in solid state topological systems, namely nonlinearity, non-Hermiticity and high-dimensionality.

In the first part, we study topological photonic structures with nonlinearity, with a particular focus on their application as optical isolators. We study three representative designs that respectively implement a nonlinear 1D Su-Schrieffer-Heeger (SSH) model, a nonlinear 2D Haldane model, and a 2D lattice of coupled-ring waveguides. We show that power thresholds and discontinuities due to nonlinearity induced topological phase transition can lead to isolator functionalities either with the emergence of self-induced topological solitons or with discontinuities in the transmittance curve.

In the second part, we experimentally demonstrate an application of topological edge modes in a left-handed nonlinear circuit transmission line (NLTL), which is a nonlinear analogue of a 1D SSH lattice. We find that strong higher-harmonic signals that propagate into the lattice are produced by a topological electric edge mode at the first harmonic frequency and act as an effectively nonlocal cross-phase nonlinearity. This process leads to strongly enhanced third-harmonic generation that outperforms a comparable left-handed NLTL of conventional design. This advances our fundamental understanding of the effects of nonlinearities on topological states and can be utilized to design compact electronic frequency generators.

In the third part, we turn to a non-Hermitian (rather than nonlinear) version of the SSH model. We analyze the sublattice/time-reversal symmetry breaking for large gain/loss magnitudes, and show that a pair of defect states appears, which corresponds to nontopologically protected defect states of the SSH model at Hermitian limit. This work advances our theoretical understanding of the interaction between topology and non-Hermiticity.

In the fourth part, I present an experimental realization of a new type of four-dimensional(4D) topological system in a circuit lattice, based on a recently theoretically proposed 4D quantum Hall model. Surface states that correspond to the second Chern number can be directly observed, due to absence of nontrivial first Chern number in this model. This work shows that circuit lattices can be a convenient platform to experimentally explore topological phenomena in high-dimensional lattices.

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Contents

Abstract	i
Acknowledgements	ii
List of Figures	v
1 Introduction	15
1.1 Overview of classical topological systems	15
1.2 Nonlinear topological insulators	19
1.3 Non-Hermitian topological insulators	21
1.4 High dimensional topological insulators	23
2 Optical isolation with nonlinear topological photonics	27
2.1 Introduction	27
2.2 Nonlinear coupled waveguide arrays: SSH model	33
2.3 Nonlinear coupled waveguide arrays: Haldane model	38
2.4 Nonlinear coupled ring lattices	42
2.5 Discussion and conclusions	49
3 Topologically Enhanced Harmonic Generation in a Nonlinear Transmission Line Metamaterial	52
3.1 Introduction	52

3.2	Sample design and fabrication	54
3.2.1	Circuit design	54
3.2.2	Circuit fabrication	59
3.3	Results	60
3.3.1	Experimental results	60
3.4	Discussion	69
4	Effects of Non-Hermiticity on Su-Schrieffer-Heeger Defect States	73
4.1	Complex Su-Schrieffer-Heeger (cSSH) model	76
4.2	Symmetries of the cSSH model	76
4.3	Defect states in the cSSH model	81
4.4	$\mathcal{ST}$ -breaking of defect states	91
4.5	Summary	94
5	Circuit realization of a four-dimensional topological lattice	96
5.1	Circuit model of a 4D quantum Hall lattice	99
5.2	Topological phase transition in the 4D quantum Hall circuit	101
5.3	Circuit simulation and experiment	105
6	Conclusion	109
	Bibliography	111

List of Figures

1.1	An example of diode pair capacitance C versus bias voltage V , calculated from varactor manufacturer data. Inset: varactor pair schematic symbol.	21
1.2	By presence or absence of time-reversal T , particle-hole P and chiral C symmetry, topological insulators can be classified into ten symmetry classes. Combined with spatial dimension, different classes of topological systems show a periodic pattern. Here ± 1 and 0 indicates whether the symmetry operator squares to ± 1 or if the symmetry is absent. This table is excerpted from [1].	24
2.1	(a) Schematic of SSH model realized by an array of coupled waveguides. (b,c) Mode intensity distributions in a linear SSH lattice of 29 sites, generated by an excitation on the left-most site (waveguide 1). Results are computed numerically using the z -evolution operator. (b) For a “trivial” lattice with $\kappa_1 = 1$ and $\kappa_2 = 0.2$, the excitation diffracts into the bulk. (c) For a “nontrivial” lattice with $\kappa_1 = 1$ and $\kappa_2 = 5$, the excitation is confined to the edge. (d) Transmittance on waveguide 1 (blue line) versus κ_2/κ_1 . The intensity of the edge eigenmode on waveguide 1, $ a_1 ^2$, is also shown (green dashes). In (b)–(d), the input/output couplings η_A and η_B are set to unity.	32

- 2.2 Behavior of the nonlinear SSH lattice as an optical isolator. (a) Nonlinear transmittance along waveguide 0, $\mathcal{T}$, versus input power I . The lattice has 29 sites, with $\eta_A = 1$, $\eta_B = 0.5$, $\kappa_1 = 1$, $\kappa_0 = 0.5$, and $\alpha = 1$; the output is at $Z = 20$. Results are computed numerically using evolution operators with step size $\delta z = 10^{-3}$. (b) Averaged values of κ_2/κ_1 versus input power I , where the nonlinear κ_2 is averaged over $18 \leq z \leq 22$ (i.e., around the output Z) and over a few sites (4, 8, or 12) closest to the left lattice edge. The topological transition point of the linear SSH model, $\kappa_2 = \kappa_1$, is indicated by the horizontal dashes. (c) Isolation ratio (IR), forward transmittance T_f , and backward transmittance T_b , versus input power I . Horizontal dashes show the approximate upper bounds $T_{f,b}^{\max} = \eta_A^2 \eta_B^2$ and $\text{IR}^{\max} \approx 1/T(0)$ 35
- 2.3 (a) Intensities after evolution through a nonlinear Haldane lattice, with parameters $t_1 = 1$, $t_2 = 1/3$, $\phi = \pi/2$, and $m_0 = 2$. A corner site (A) is initially excited with intensity I , and colors indicate site intensities $|a_\mu|^2$ and $|b_\nu|^2$ (with arbitrary normalization) after propagation through $Z = 19.2$. Two cases are shown: $I = 1$ and $I = 150$. The input/output couplings η_A and η_B are set to unity. (b) Nonlinear transmittance $\mathcal{T}(I)$ from one corner (A) to the opposite corner (B), versus input power I . (c) Mean values of $\langle m_A \rangle - \langle m_B \rangle$, versus I . The averages are taken over sites up to n nearest-neighbor hops from site B, for $n \in \{1, 2, 3, 4\}$. (d) Isolation ratio (IR), forward transmittance T_f , and backward transmittance T_b , versus I 39

- 2.4 (a) Schematic of a lattice of coupled rings, consisting of “site rings” (large circles) arranged in a square lattice, separated by auxiliary “coupler rings” (small circles). Light is coupled in/out of the lattice on the left and right, with coupling parameters θ_A and θ_B . (b) Close-up of an inter-site coupling, showing the definitions of the complex wave amplitudes $\{a, b, c, d\}$ (in the site rings) and $\{\alpha, \beta, \gamma, \delta\}$ (in the coupler ring). 43
- 2.5 (a)–(b) Band diagrams for a semi-infinite lattice of coupled rings, with width 10 unit cells in the y direction. For the coupler ring parameters, we choose $\theta_0 = \pi/6$ and two different choices of the phase shift ξ ; the resulting bandstructures are (a) topologically trivial for $\xi_1 = \xi_2 = \xi = 0.55\pi$, and (b) topologically nontrivial for $\xi = 0.85\pi$. (c) Numerically-obtained phase diagram of the system for $\theta_0 = \pi/6$. Grey regions show the ungapped parts of the bandstructure, white regions show trivial (T) bandgaps, and pink regions show nontrivial (NT) bandgaps. For fixed $\phi = \pi/4$, the system can move from a trivial to a nontrivial gap by varying $\xi_{1,2}$ as indicated by the arrow. The gaps depend only on ξ_1 and ξ_2 through the combination $\xi_1 + \xi_2$ 44

- 2.6 (a) Transmittance T through a linear lattice with phase shift ξ in all arms of all coupler rings. The input/output couplings are taken to be perfect (the coupling matrices have the form of Eq. (2.17) with $\theta_A = \theta_B = \pi/2$), and the lattice parameters are $\phi = \pi/4$, and $\theta_0 = \pi/6$. For $\xi \approx \pi/2$, ϕ lies in a trivial gap, and T is close to zero; with increasing $\Delta\xi$, ϕ enters a band (grey region) and T exhibits bulk transmission resonances; for still larger ξ , ϕ enters a nontrivial gap (pink region) and T approaches unity. Vertical dashes indicate the boundaries between the band and gap regions shown in Fig. 2.5(c). (b) Transmittance through the nonlinear lattice versus input power I , using nonlinearity parameters $\xi_0 = \pi/2$ and $\kappa = 1$, with all other parameters kept the same as in (a). This and subsequent results are obtained by solving the nonlinear transfer matrix relations self-consistently via a numerical nonlinear solver. (c) Normalized field intensity distribution at $I = 3.21$, just above the large discontinuity in (b). For clarity, the coupler rings are omitted and only the site ring intensities are shown. (d) Mean phase shifts $\langle \xi \rangle = \langle (\xi_1 + \xi_2)/2 \rangle$ versus I , with averages taken over both arms of the eight coupler rings on the upper lattice edge [see Fig. 2.4(a)]. Horizontal dashes indicate the boundaries from Fig. 2.5(c). 47
- 2.7 Isolation ratio (IR), forward transmittance T_f , and backward transmittance T_b , versus input power I , in disordered nonlinear coupled ring lattices. Thin lines show results for 100 different disorder realizations, and thick black lines show the mean values. The disorder consists of shifts in individual coupler ring phase shifts ξ_j , normally distributed with mean 0 and standard deviation 0.01π . Input/output loss is applied at port B by multiplying amplitudes by $e^{0.1}$ (with no loss at port A); all other parameters are the same as in Fig. 2.6. 48

- 3.1 Design and implementation of SSH-like left-handed transmission line.
- a** Schematic of the transmission line circuit with alternating capacitances: linear capacitors with capacitance C_a , and back-to-back varactors with nonlinear capacitance C_b . The capacitances act like hoppings in a nonlinear Su-Schrieffer-Heeger (SSH) model. An input voltage is applied at points A or B to probe the topologically trivial or nontrivial lattice. **b** Photograph of the printed circuit board. **c** Capacitance C_b versus bias voltage. Dots are calculated from varactor manufacturer data, and the solid curve is the fit based on Eq. (3.17). **d, e** Calculated eigenfrequencies of a finite closed linear circuit, versus the capacitance ratio $\alpha = C_a/C_b$. The characteristic frequency is $f_a = \omega_a/2\pi \approx 19$ MHz, and the lattice has 40 sites. Two cases are shown: **d** C_a -type capacitors at the edge, for which the $\alpha > 1$ gap is topologically nontrivial; **e** C_b -type capacitors at the edge, for which the $\alpha > 1$ gap is trivial. Red dotted curves indicate the band-edge frequencies $f_a/\sqrt{2}$ and $\sqrt{\alpha/2} f_a$. Blue dashes indicate the operating regime of the nonlinear circuit, with $\alpha \approx 1.3$ in the linear (low-voltage) limit and α effectively increasing with voltage amplitude. 56
- 3.2 **a** Schematic of the left-handed transmission line circuit. **b** Band diagram for the infinite lattice. Results are shown for $\alpha = 0.5$ (solid curves) and 3.0 (dashes). 57

- 3.3 Experimental observation of harmonic generation in the nonlinear circuit. **a, b** Magnitude of the first- and third-harmonic voltage signals measured at different lattice sites, for **(a)** the nontrivial lattice, which has an SSH-like edge state in the linear limit, and **(b)** the trivial lattice, which has no edge state in the linear limit. The sinusoidal input signal, applied at the lattice edge (site 0), has frequency $f_{\text{in}} = 16$ MHz and amplitude $V_{\text{in}} = 2.5$ V. **c** Measured spectrum at site 3 for the nontrivial lattice corresponding to subfigure **a**. **d, e** Plot of the inverse participation ratio (IPR) versus input frequency f_{in} and input voltage amplitude V_{in} , calculated from experimental measurements of the first-harmonic signal in the **(d)** nontrivial lattice and **(e)** trivial lattice. Here, f_{in} is measured in steps of 0.2 MHz, and V_{in} in steps of 0.1 V. 61
- 3.4 Simulated on-site voltage amplitudes for the first-harmonic signal (orange circles) and third-harmonic signal (blue squares). The frequency of the input is $f_{\text{in}} = 16.4$ MHz. **a, b** Nontrivial and trivial lattices with $V_{\text{in}} = 2.5$ V. **c, d** Nontrivial and trivial lattices with $V_{\text{in}} = 3.4$ V. 62
- 3.5 Simulated squared bias voltage amplitudes on the nonlinear capacitors, for the first-harmonic signal (orange circles) and higher-harmonic signals (purple squares). The frequency of the input is $f_{\text{in}} = 16.4$ MHz. The higher-harmonic data is obtained by summing over the third, fifth, and seventh harmonic data (further harmonics are negligible). **a, b** Nontrivial and trivial lattices with $V_{\text{in}} = 2.5$ V. **c, d** Nontrivial and trivial lattices with $V_{\text{in}} = 3.4$ V. 63

- 3.6 Response of the linear circuit to an external voltage, calculated numerically from the circuit equations with the parameters $f_a = 19$ MHz and $C_a = 47$ pF. **a–b** Resonant response of **(a)** the amplitude of the voltage on the first site (to the right of the leftmost capacitor), and **(b)** the amplitude of the stored charge on the first (leftmost) capacitor, versus source frequency $f_{\text{in}} = \omega/2\pi$, for $\alpha = 1.5$. The resonance frequency predicted by equation (3.15) is indicated by the horizontal dashed line. **c** Spatial distribution of the voltage amplitudes for $f_{\text{in}} = 14.75$ MHz and $\alpha = 1.5$. **d** Intensity of the nontrivial lattice's resonant mode, as given by the value of $\sum_k |v_k|^2$ (where $|v_k|$ is the voltage amplitude at site k) at resonance, versus the α parameter. For each value of α , the input voltage has amplitude 1 V and frequency given by equation (3.15). 66
- 3.7 Plot of the normalized third-harmonic intensity χ versus input frequency f_{in} and input voltage V_{in} , for **(a)** the SSH-like lattice in its topologically nontrivial configuration, and **(b)** a conventional left-handed nonlinear transmission line (NLTL) with identical nonlinear capacitors. The figure of merit χ is defined as the mean squared third-harmonic amplitude on the first 10 sites relative to the squared input voltage amplitude. In the SSH case, we obtain $\chi \gtrsim 2.4$, which is comparable to the experimental result of $\chi \approx 2.5$. In the conventional NLTL, we observe $\chi < 0.47$ throughout the parameter range. 67

- 3.8 Measured third-harmonic signal intensities in the nonlinear circuit. **a**, **b** Normalized third-harmonic signal intensity χ versus input frequency f_{in} and input voltage V_{in} , for the **(a)** nontrivial lattice and **(b)** trivial lattice. Here, χ is derived from experimental data using the definition (3.19). **c** Ratio of the trivial and nontrivial intensities, $\chi_{\text{nontrivial}}/\chi_{\text{trivial}}$, within the region indicated by boxes in **a** and **b**. **d** Measured third-harmonic intensities (normalized to the input signal) at different sites, for the three sets of input parameters indicated by stars in **c**: $V_{\text{in}} = 0.5 \text{ V}$, 1.5 V , and 2.5 V , with fixed $f_{\text{in}} = 16 \text{ MHz}$ 68
- 3.9 Voltage amplitude profiles for the first and third harmonic signals, from SPICE simulation on a 40-site NLTL with **(a)** site 20 excited, and **(b)** site 21 excited. The choice of excitation site partitions the lattice into topologically trivial and nontrivial sections. The input signal has voltage amplitude $V_{\text{in}} = 3 \text{ V}$, frequency $f_{\text{in}} = 16 \text{ MHz}$, and input impedance 1Ω . All NLTL simulation parameters are the same as previous simulation where edge site is excited. 71

- 4.1 (a) The bulk cSSH chain. Sites with gain ($i\gamma$) and loss ($-i\gamma$) are respectively indicated by red and blue circles, and couplings $1 + \delta$ and $1 - \delta$ are indicated by double and single horizontal lines. (b) Phase diagram of the bulk cSSH chain, which contains a $\mathcal{PT}$ -unbroken phase where all Bloch state energies are real (yellow), an intermediate phase with both real and imaginary energies (white), and an $\mathcal{ST}$ -unbroken phase with purely imaginary energies (blue). (c)–(e) Complex bulk band structures for the cSSH chain, for the parameters indicated in (b) by the points labelled c , d , and e respectively: $\delta = 0.5$ and (c) $\gamma = 0$, (d) $\gamma = 1.5$, and (e) $\gamma = 2.5$. Solid (dashed) curves show the real (imaginary) part of the eigenenergy E . (f)–(h) cSSH chains with different lattice defects (vertical dashes). (f) $\mathcal{P}$ -preserving defect. (g) $\mathcal{PT}$ -preserving and $\mathcal{P}$ -breaking defect. (h) A defect that breaks both $\mathcal{P}$ and $\mathcal{PT}$, while reversing γ and δ across the interface. 77

- 4.2 (a) Phase diagram for defect states of the cSSH lattice shown in Fig. 4.1(h). In the white regions (I), the lattice has one defect state, in the green regions (III) it has three defect states, and in the pink (II) and orange regions (IIA) it has two defect states. The defect states in II are $\mathcal{ST}$ -broken, and the defect states in IIA are $\mathcal{ST}$ -symmetric. The points labelled b , c , d , and e indicate the parameters for the spectra plotted in (b)–(e). The gray dashes are the phase boundaries of the bulk lattice bandstructure, corresponding to Fig. 4.1(b). The blue dots indicate the critical line segment ($\delta = 0$ and $0 \leq \gamma \leq 1$) over which there are no localized defect states. (b)–(e) Complex eigenenergy spectra, calculated numerically for a finite lattice with 75 unit cells on each side of the defect, for (b) $\gamma = 0.5$ and $\delta = 2$, (c) $\gamma = 0.5$ and $\delta = 0.5$, (d) $\gamma = 1.5$ and $\delta = 0.5$, and (e) $\gamma = 2.5$ and $\delta = 0$. Defect states continuable to SSH mid-gap states are shown as triangles, defect states continuable to non-mid-gap states as squares, bulk states as black circles, and edge states (due to the finite lattice size) as grey circles. 84
- 4.3 Variation of $|\lambda|$ with γ for (a) $\delta = 0.5$ and (b) $\delta = 2$ 85
- 4.4 Mapping between (a) a pair of coupled uniform-coupling ($\delta = 0$) cSSH chains and (c) the two-chain model of Ref. [2], where the purple (white) sites represent counterclockwise (clockwise) resonator modes, and the red and blue arrows represent asymmetric internal scattering processes. 90

- 4.5 (a) Close-up view of the defect state phase diagram near the $\mathcal{ST}$ -breaking line. The region labels have the same meanings as in Fig. 4.2(a). The points labelled b , c , and d indicate the parameters for the spectra plotted in (b)–(d). (b)–(d) Complex eigenenergy spectra, calculated numerically for a finite lattice with 500 unit cells on each side of the defect, for (b) $\gamma = 1.9$ and $\delta = 1.5$, (c) $\gamma = 2.1$ and $\delta = 1.625$, and (d) $\gamma = 2.1$ and $\delta = \sqrt{2}$. The $\mathcal{ST}$ -symmetric defect states are indicated by triangles; the $\mathcal{ST}$ -broken defect states are indicated by squares. The arrows in (c) and (d) indicate the direction of motion of the defect state eigenvalues as δ decreases from point c to point d in (a). 92
- 4.6 Magnitudes and phases of the cSSH defect state wavefunctions, before and after an $\mathcal{ST}$ -breaking transition. In (a) and (b), the lattice parameters are $\gamma = 2.1$ and $\delta = 1.625$, the same as in Fig. 4.5(c); both eigenstates are $\mathcal{ST}$ -symmetric. The up-pointing triangles show the state that evolved from the SSH mid-gap defect state; the down-pointing triangles show the additional $\mathcal{ST}$ -symmetric defect state which emerged from the continuum. In (c) and (d), the lattice parameters are $\gamma = 2.1$ and $\delta = \sqrt{2}$, the same as in Fig. 4.5(d); both eigenstates are $\mathcal{ST}$ -broken, and are related by $\mathcal{ST}$ operation. The filled squares show the state with $\text{Re}(E) > 0$ and the hollow squares show the state with $\text{Re}(E) < 0$. Red (blue) symbols indicate gain (loss) sites. The gauge is fixed by setting the phase of the first site to the right of the domain wall to zero. 93

- 5.1 (a) A 4D lattice with four sites per unit cell. Solid and dotted lines denote hoppings with positive and negative amplitudes J and $-J$ respectively. (b) Example of longer-range hoppings (Eq. 5.3) that can make the lattice topologically nontrivial. (c) Doubly-degenerate energy bands (Eq. 5.2) for $k_y = k_w = 0$, with the 4D Dirac points labelled. (d)(e) Energy dispersion at the topological phase transitions for $k_y = k_w = 0$, $J' = J/2$ and $J'' = 0$, corresponding to (d) $m = -J$ and (e) $m = J/2$, showing that there is only one pair of Dirac points at each transition. Figures reproduced from Ref. [3]. 98
- 5.2 Two slices of the 4D band diagram through the Brillouin zone origin, with $k_y = k_w = 0$ and in two directions: (a) $k_x = k_z = k$, and (b) $k_x = -k_z = k$. Calculated from a circuit lattice model with parameters used in Fig. 5.1(e). 100
- 5.3 A phase diagram for the unbounded lattice showing bandgap closing and reopening with varying mass term m . The blue curve indicates the upper band minimum and orange curve indicates lower band maximum. 101
- 5.4 Phase diagram of a $6 \times 6 \times 6 \times 6$ lattice. With open boundary condition in x, z directions and periodic boundary in y, w directions. The lowest 256 eigenenergies are presented for varying m value. 102
- 5.5 The bandgap closing m value shifts toward $m = 6$ with increasing size of the lattice. We fix the number of unit cells in y and w direction which has periodic boundary condition to 1, and change the unit cell number of x and z direction from 3 up to 25 and plot only the band edge (two edge energies closest to $E = 0$). As we increase the lattice size we can see that the band edge approaches the infinite model, and gap-closing m value approaches 6, verifying the finite-size effect. 103

5.6	Lowest energy eigenwave for a lattice with $J = 1, J' = -J'' = 2, atm = 0$. Each 3D plot presents a slice of the 4D wave distribution in W direction for $W = 1, 2, 3, 4, 5, 6$	104
5.7	Lowest energy eigenwave for a lattice with $J = 1, J' = -J'' = 2$, at $m = 8$. Each 3D plot presents a slice of the 4D wave distribution in W direction for $W = 1, 2, 3, 4, 5, 6$	105
5.8	Averaged bulk intensity as defined by Eq. 5.5, calculated from simulation result. A lower averaged bulk intensity indicates that the wave distribution is more localized to the surface.	106
5.9	Phase diagram and averaged bulk intensity similar to Fig. 5.4 and Fig. 5.8 for a smaller lattice with only two sites in y and w direction. .	107
5.10	(a) Simulation of an averaged surface mode. Parameters applied to the simulation are $C = 1\text{nF}, C' = C'' = 2\text{nF}, L = 2\text{mH}, L' = L'' = 1\text{mH}, f_{input} = 112.54\text{kHz}, C_m = 0$, extra inductors are added as groundings to balance on-site impedance and tune the pseudo-energy to $E = 0$, their values are calculated by site. Two 3D plots show slices of the 4D wave for $W = 0$ and $W = 1$ respectively. The simulation is an averaged result of multiple measurements applying input source to each site. (b) Simulation of a bulk mode. Same parameters as in (a) except that $C_m = 16\text{nF}$ and values of extra inductors are calculated to raise pseudo energy to $E = 5$. (c) A photo of the fabricated 4D quantum Hall lattice in a circuit. (d) Schematic of the circuit's node to node couplings. . . .	108

Chapter 1

Introduction

1.1 Overview of classical topological systems

From the crystallization of ice to the demagnetization of magnets, people have long been curious about how phase transitions of matter happen. Landau developed a theory that explains phase transitions to be a result of symmetry changes in how the component particles are ordered, as controlled by order parameters. This theory has been very successful [4]. However, researchers in the past few decades have shown that Landau's approach is not adequate to cover all kinds of phase transitions that matter can exhibit. For example, in the quantum Hall effect [5, 6], a two dimensional electron gas can go through a topological phase transition in which topological properties of the band structure change but the underlying symmetry is unchanged.

Topological phases of matter are new kinds of phases that go beyond Landau's phase transition theory. In 1980, the integer quantum Hall effect was discovered [5, 6], where it was shown that the Hall conductance of two-dimensional electron gas takes values that are multiples of e^2/h when the strength of the perpendicular magnetic field is tuned. This phenomenon was soon related to the Chern number, an integer topological invariant describing the system's momentum space structure [7, 8].

The interface between two materials (or between a material and the vacuum) with different topological invariants can host localized edge states in the energy gap of the surrounding bulk materials, due to the bulk-edge correspondence principle [9–12]: the band gap must close at the interface because the topological invariants have discrete values. It was also shown that edge currents supported by the interface modes of topologically nontrivial quantum Hall matter and topologically trivial vacuum can propagate unidirectionally along the boundary, and are immune to back scattering [13–15], due to broken time reversal symmetry.

Quantum Hall matter is not the only kind of topological matter. In 2005 the quantum spin-Hall effect was predicted and discovered [16–20]. This is a different topological effect from the quantum Hall effect; in quantum spin-Hall systems, the Chern number is zero, and the system’s topology is characterized by a $\mathbb{Z}_2$ topological invariant that can take on nontrivial values without breaking time reversal symmetry. Since then, people have started to explore other possible topological phases of matter in condensed-matter physics [21–23].

Topological phases are not restricted to quantum electron waves; they can be realized by other kinds of waves, including classical waves, in periodic media. Topological electromagnetic waves were first proposed by Haldane and Raghu in their work on two-dimensional photonic crystals [24, 25], where the photonic bands acquire nontrivial topological invariants due to magneto-optical elements that break time-reversal symmetry, giving rise to robust unidirectional chiral edge modes that mimic unidirectional edge currents in the quantum Hall effect. The experimental realization of this idea was first proposed [26] and implemented [27] by Wang *et al* with a two-dimensional microwave magneto-optical photonic crystal. After that there were several implementations that overcame the need for strong magneto-optical response, which is unavailable at optical frequencies [28–37], leading to intense research in the field of topological photonics [38–43].

Researchers have also shown that topological phases can be implemented using other classical systems, such as electric circuits [44–46], and mechanical [47–49] and acoustic [50–54] metamaterials.

There are several reasons for recent increasing research interest in topological classical systems. First of all, topology as a brand new feature may lead to new behaviors as well as novel applications in classical wave systems which have previously been thought to be thoroughly understood. Secondly, some classical systems can serve as a much more flexible and convenient platform for exploring topological phases, compared with condensed matter systems. For instance, topological acoustic lattices can be prototyped quickly with 3D printing technology [55], and topological electric circuits can be realized using basic laboratory equipment [44]. Thirdly, in classical systems, we can explore the interplay of topological phases with some other phenomena that are inaccessible in solid-state systems such as nonlinearity and non-Hermiticity, as well as system dimensions of four and higher. Such systems will be discussed in this thesis.

In the rest of this chapter, I will briefly review non-linear, non-Hermitian, and high-dimensional classical systems.

In Chapter 2, I show that power thresholds and discontinuities due to nonlinearity induced topological phase transition can lead to isolator functionalities either with the emergence of self-induced topological solitons or with discontinuities in the transmittance curve, through study of three representative theoretical photonic models. They respectively implement a nonlinear 1D Su-Schrieffer-Heeger (SSH) model, a 2D Haldane model and a 2D lattice of coupled-ring waveguides.

Chapter 3 presents an experimental study of nonlinearity-induced harmonic generation behavior in a left-handed nonlinear transmission line (NLTL) circuit, which is a nonlinear analogue of a 1D topological model known as the SSH lattice. We find that frequency-mixing processes in an SSH NLTL have important effects that

were previously unknown. The presence of a topological edge mode at the first harmonic can produce strong higher-harmonic signals that propagate into the lattice, acting as an effectively nonlocal cross-phase nonlinearity. We observe maximum third-harmonic signal intensities that are 5 times that of a comparable left-handed NLTL of a conventional design, and a 250-fold intensity contrast between the topologically nontrivial and trivial lattice configurations. This work may have applications for compact electronic frequency generators, as well as for advancing the fundamental understanding of the effects of nonlinearities on topological states.

In Chapter 4, we turn to a complex SSH (cSSH) model that is a non-Hermitian rather than nonlinear version of SSH model. A non-Hermitian defect state that is localized to the interface between two cSSH domains has been studied previously. This defect state becomes the topologically protected defect state of the Hermitian SSH model at Hermitian limit. We show that when the magnitudes of gain/loss are large, the lattice spontaneously breaks the sublattice and time-reversal symmetry. This gives rise to a pair of defect states that corresponds to nontopologically protected defect states of the SSH model at Hermitian limit. This work advances the theoretical understanding of how topological phenomena interacts with the effects of non-Hermiticity.

In Chapter 5, I present an experimental realization of a new type of 4D topological system in a circuit lattice, based on a recently proposed 4D topological model [3]. Unlike previously proposed 2D and 4D quantum Hall models, this model does not require particle spin or broken time-reversal symmetry, which allows it to be mapped to a circuit model. We can directly realize a 4D circuit instead of using synthetic dimensions [56–63], as circuit lattice topology is defined by its connections. The 4D circuit model will be a good platform to explore the topological features of high dimensional lattices.

1.2 Nonlinear topological insulators

In some systems such as photonics and electric circuits, nonlinearity can be conveniently introduced and controlled with nonlinear optical materials or nonlinear elements such as varactors. This makes topological photonics and “topoelectrical” circuits good platforms to look into the interplay of nonlinear effects and band topology.

Weak photonic nonlinearities can be treated semi-classically as nonlinear terms in the dependence of the dielectric polarization on the electric field:

$$P = \chi^{(1)}E + \chi^{(2)}E^2 + \chi^{(3)}E^3 + \dots \quad (1.1)$$

In this thesis we focus on effects induced by the third order susceptibility $\chi^{(3)}$.

For monochromatic waves of a certain frequency, $E = \text{Re}[E_0 e^{i\omega t}]$, the first-harmonic term proportional to $e^{i\omega t}$ expanded from the third-order susceptibility can be treated as intensity-induced refractive index change, or Kerr nonlinearity:

$$P_{\omega}^{(3)} \sim \chi^{(3)} \text{Re}[3E_0 |E_0|^2 e^{i\omega t}]. \quad (1.2)$$

At the same time, the third-harmonic term proportional to $e^{3i\omega t}$ leads to third harmonic generation (THG):

$$P_{3\omega}^{(3)} \sim \chi^{(3)} \text{Re}[E_0^3 e^{i3\omega t}]. \quad (1.3)$$

In electric circuits, a Kerr type susceptibility can be mimicked by introducing back-to-back varactors pairs as shown in Fig. 1.1.

A varactor is a special kind of p-n junction diode. A p-n junction diode cannot conduct DC currents when reverse-biased, because a depletion region forms between its p-type and n-type regions. At the same time, it acts like a capacitor and can conduct AC currents, as its depletion region has high resistance for electrons and holes,

acting like the insulating material in a capacitor, while its p-type and n-type regions have low resistance for carriers and act like the electrodes of a capacitor. When the reverse-biased voltage is increased, some holes from p-type region and some electrons from n-type region move away, making the depletion region wider and effectively the diode's capacitance smaller. So reverse-biased diodes act like voltage-dependent capacitors.

A varactor is a diode with this voltage-dependent capacitance exploited to work as a nonlinear capacitor in reversed-biased states. Its capacitance-voltage dependence can be described by [64]

$$C_T(V_R) = C_J(V_R) + C_P = \frac{C_{J0}}{(1 + \frac{V_R}{V_J})^M} + C_P, \quad (1.4)$$

where C_T, C_J, C_P respectively denote the varactor's total capacitance, varactor junction capacitance and the package capacitance. V_R is the reverse-biased voltage and C_{J0}, M, V_J are constants related to the varactor's geometry, junction material and doping density.

For a pair of back-to-back connected varactors, shown in Fig. 1.1, one of the two varactors is always reverse-biased and the pair demonstrates an even $C - V$ characteristic curve. This induces only odd-order voltage nonlinearity in a transmission line [65], due to the symmetric configuration.

The interplay of nonlinearity with topology can induce many novel effects. For example, an intensity-dependent refractive index can lead to self-induced topological phase transitions. This was first demonstrated theoretically in a Floquet photonic topological lattice where a soliton can induce topological edge states on a inner lattice boundary and localize itself in the lattice bulk [66]. It was also shown that a topologically nontrivial soliton can be nonlinearly induced in a topologically trivial lattice [67]. Chapter 2 discusses possible application of nonlinearly induced topological

Figure 1.1: An example of diode pair capacitance C versus bias voltage V , calculated from varactor manufacturer data. Inset: varactor pair schematic symbol.

phase transitions for optical isolation [68].

Nonlinear topological models have also been experimentally studied, such as non-linearity induced phase transitions in a nonlinear SSH circuit [69] following previous theoretical analysis [70], enhanced harmonic generation in a nonlinear SSH circuit [71], which will be discussed in Chapter 3, as well as topologically enhanced harmonic generation in nanophotonic structures [72].

1.3 Non-Hermitian topological insulators

Another interesting feature that is hard to implement in topological condensed matter is non-Hermiticity. Non-hermitian effects, in the form of gain and loss, are common in photonic systems and electric circuits. Optical gain media and operational amplifiers can be used to introduce gain in photonic systems and electric circuits, while loss comes in the form of intrinsic (eg. Ohmic) dissipation or scattering losses.

In quantum mechanics, Hamiltonians are assumed to be Hermitian (or self-adjoint) operators, which ensures that their eigenvalues are real. However, Hermiticity is too strong a condition to ensure that a Hamiltonian's eigenvalues are real [73].

For instance, a 2×2 complex matrix H can be written as $H = \sum d_i \sigma_i$ where $\sigma_0 = I$, $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices and d_i are complex coefficients. H being Hermitian is equivalent to d_i being all real: $\text{Im}d_1 = \text{Im}d_2 = \text{Im}d_3 = \text{Im}d_4 = 0$. This is, however, not the only way for a Hamiltonian to have real eigenvalues.

A PT symmetric Hamiltonian is one that commutes with a parity-time operator, $[H, \hat{P}\hat{T}] = 0$ [73]. Here the time reversal operator $\hat{T}$ is (in the simplest case) the complex conjugation operator and the parity operator $\hat{P}$ flips the sign of position (as well as momentum).

Consider the time-independent Schrödinger equation

$$H\psi = E\psi. \quad (1.5)$$

The PT symmetry condition implies that:

$$H(\hat{P}\hat{T}\psi) = \hat{P}\hat{T}H\psi = \hat{P}\hat{T}E\psi = E^*(\hat{P}\hat{T}\psi). \quad (1.6)$$

Hence, for a PT symmetric Hamiltonian, if there is an eigenvalue E and corresponding eigenvector ψ , there must also be an eigenvalue E^* and corresponding eigenvector $\hat{P}\hat{T}\psi$. Each of the eigenvalue is either real, or related pairwise by complex conjugation. This is equivalent to the fact that the characteristic function $\det[H - \lambda I] = 0$ is a real polynomial.

For the 2×2 matrix Hamiltonian, this requires:

$$\begin{vmatrix} d_0 + d_3 - \lambda & d_1 - id_2 \\ d_1 + id_2 & d_0 - d_3 - \lambda \end{vmatrix} = 0 \quad (1.7)$$

to be a real polynomial. This implies $\text{Im}d_0 = \text{Im}(d_1^2 + d_2^2 + d_3^2) = 0$.

When $\text{Im}(d_1^2 + d_2^2 + d_3^2) > 0$, the Hamiltonian has two different real eigenvalues. When $\text{Im}(d_1^2 + d_2^2 + d_3^2) < 0$, the Hamiltonian has a pair of conjugated complex eigenvalues, with d_0 as its real component. The transition point $\text{Im}(d_1^2 + d_2^2 + d_3^2) = 0$ is called an exceptional point [74–77], where the Hamiltonian has one degenerate eigenstate (i.e., its eigenvectors do not span the 2D space).

The existence of PT symmetric Hamiltonians encourages people to explore physics beyond Hermitian systems. Many results in quantum mechanics as well as in condensed matter theory, including topological band theory, assume Hermiticity. Therefore there is interest in phenomena associated with non-Hermiticity, especially in combination with band topology.

The simplest type of non-Hermitian system with topological features is the 1D complex Su-Schrieffer–Heeger (cSSH) model [78], which is the SSH model [79] with periodically balanced gain and loss on coupling links.

It has been shown that certain topological interface sites can be selectively enhanced by gain [78] in a cSSH model. Microwave[80] and optics experiments [81] have verified this. These interface states are non-Hermitian and continuable to the real SSH nontrivial edge states, while some non-Hermitian lattices support topological defect states are reported to have no evident Hermitian counterpart[82, 2, 83–85]. In Chapter 4, we demonstrate certain inherently non-Hermitian phenomena that have no counterpart in the Hermitian SSH model.

1.4 High dimensional topological insulators

The last topic in this thesis is an exploration of topological phases in four or higher spatial dimensions. As we live in a three dimensional world, high-dimensional topological phases are obviously inaccessible in condensed matter physics. However we

class	$\mathcal{C}$	$\mathcal{P}$	$\mathcal{T}$	$d = 0$	1	2	3	4	5	6	7
A				$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	4D QH	$\mathbb{Z}$	
AIII	1				$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
AI			1	$\mathbb{Z}$				$2\mathbb{Z}$	New	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\mathbb{Z}$		$\mathbb{Z}_2$
D		1		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\mathbb{Z}$	
DIII	1	1	-1		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\mathbb{Z}$
AII			-1	$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	4D QSH		
CII	1	-1	-1		$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
C		-1				$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
CI	1	-1	1				$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Figure 1.2: By presence or absence of time-reversal $\mathcal{T}$, particle-hole $\mathcal{P}$ and chiral $\mathcal{C}$ symmetry, topological insulators can be classified into ten symmetry classes. Combined with spatial dimension, different classes of topological systems show a periodic pattern. Here ± 1 and 0 indicates whether the symmetry operator squares to ± 1 or if the symmetry is absent. This table is excerpted from [1].

can not only theoretically but also experimentally explore higher dimensional physics using classical wave systems. Basically there are two experimental approaches: the first is by interpreting certain internal degrees of freedom as additional spatial dimensions, or “synthetic dimensions” [56–63]. The second approach is to increase lattice connectivity using long distance coupling, so that some dimensions can be compressed into a lower dimensional space [86–89].

In this thesis, we focus on a four-dimensional quantum Hall system. The topology of any system can be classified by its symmetry and dimensionality [21]. For example, in 2D there are two kinds of quantum Hall systems; the integer quantum Hall system is in class A and describes a spinless particle with broken time reversal symmetry, while the quantum spin Hall system is in class AII and describes time reversal symmetry preserved spinful particles, as shown in Fig. 1.2.

A 2D quantum Hall model Hamiltonian can be simplified to a two-band momentum-

space Hamiltonian [90]:

$$H(\mathbf{k}) = \varepsilon(\mathbf{k})\hat{I} + \mathbf{d}(\mathbf{k}) \cdot \boldsymbol{\sigma}, \quad (1.8)$$

where $\boldsymbol{\sigma}$ is a vector of Pauli matrices. Solving the eigenvalue problem gives two bands: $E_{\pm} = \varepsilon(\mathbf{k}) \pm \sqrt{\mathbf{d}(\mathbf{k}) \cdot \mathbf{d}(\mathbf{k})}$, where $\mathbf{d}(\mathbf{k})$ is a three-component vector and $\varepsilon(\mathbf{k})$ is an overall energy shift. When the two bands are gapped, the Chern number can be calculated as [90]:

$$\nu_-^{(1)} = \frac{1}{2\pi} \int_{\text{BZ}} \Omega_- = \frac{1}{4\pi} \int_{\text{BZ}} d^2\mathbf{k} \epsilon^{abc} \hat{d}_a \partial_{k_x} \hat{d}_b \partial_{k_y} \hat{d}_c, \quad (1.9)$$

where the integrals run over the 2D Brillouin zone, ϵ^{abc} is the 3D Levi-Civita symbol, $\hat{\mathbf{d}} = \mathbf{d}/|\mathbf{d}|$ and Ω_- is the lower band Berry curvature. The Chern number can be interpreted as the “winding number” that counts how many times $\hat{\mathbf{d}}(\mathbf{k})$ covers the unit Bloch sphere, S^2 , across the 2D Brillouin Zone [90].

The integer quantum Hall and quantum spin Hall systems have 4D counterparts in their respective symmetry classes. The 4D integer quantum Hall model in a 2D quasicrystal was proposed first as a toy model [91] then in ultracold atomic gases [92] and in photonics [62]. It was recently realized [93] in photonics, where the 4D quantum Hall edge states were probed with topological pumping, as well as in an ultracold atomic system [94] where the quantized second Chern number were measured. The 4D quantum spin Hall model has also been proposed [95]. Unlike the 2D case, the simplest 4D quantum Hall system is described by a doubly degenerate four-band model:

$$H(\mathbf{k}) = \varepsilon(\mathbf{k})\Gamma_0 + \mathbf{d}(\mathbf{k}) \cdot \boldsymbol{\Gamma}, \quad (1.10)$$

where Γ_0 is the 4×4 identity; $\varepsilon(\mathbf{k})$ is an overall energy shift and $\mathbf{\Gamma}$ is a vector of 4×4 Dirac matrices, chosen as $\Gamma_1 = \sigma_1 \otimes \sigma_3$, $\Gamma_2 = \sigma_2 \otimes \hat{1}$, $\Gamma_3 = \sigma_1 \otimes \sigma_1$, $\Gamma_4 = \sigma_1 \otimes \sigma_2$, $\Gamma_5 = \sigma_3 \otimes \hat{1}$. The energy bands are $E_{\pm} = \varepsilon(\mathbf{k}) \pm \sqrt{\mathbf{d}(\mathbf{k}) \cdot \mathbf{d}(\mathbf{k})}$. The second Chern number for the lower pair of bands is [95]:

$$\begin{aligned} \nu_-^{(2)} &= \frac{1}{8\pi^2} \int_{\text{BZ}} \text{tr}(\Omega_- \wedge \Omega_-), \\ &= \frac{3}{8\pi^2} \int_{\text{BZ}} d^4\mathbf{k} \epsilon^{abcde} \hat{d}_a \partial_{k_x} \hat{d}_b \partial_{k_y} \hat{d}_c \partial_{k_z} \hat{d}_d \partial_{k_w} \hat{d}_e. \end{aligned} \quad (1.11)$$

This can be seen as a 4D analogue of the 2D Chern number, where ϵ^{abcde} is the 5D Levi-Civita symbol and the integral runs over the 4D Brillouin zone. The second Chern number is a “winding number” counting how many times $\hat{\mathbf{d}}(\mathbf{k})$ covers the unit sphere, S^4 , across the 4D Brillouin zone.

Besides the 4D version of quantum Hall and quantum spin Hall models, there is a quantum Hall system in class AI which has no 2D counterpart and describes a time reversal symmetry preserved spinless particle. A tight-binding lattice model for this class has been proposed [3]. This model is interesting for it has no nontrivial first Chern number, which ensures that its topological surface states are solely attributable to the nontrivial second Chern number. As this new class of quantum Hall lattice requires no spin or time reversal symmetry breaking, it can be implemented in an electric circuit. In Chapter 5, I will discuss my work on realizing such a circuit.

Chapter 2

Optical isolation with nonlinear topological photonics

Some of the contents of this chapter were previously published in the journal New Journal of Physics [68]. Wherever figures and materials from other sources have been taken, the authors have been given due credit.

2.1 Introduction

Nonreciprocal light transmission plays a key role in modern optical technologies. Optical isolators are devices that allow light to pass in one direction (e.g., along a waveguide), while blocking transmission in the other direction, thus acting as the analogues of diodes in electronic circuits. To realize an optical isolator, the reciprocity principle of ordinary electromagnetism must be broken [96].

The reciprocity principle can be derived by considering a volume containing two sets of excitation of fields $(\mathbf{E}_1, \mathbf{H}_1)$ and $(\mathbf{E}_2, \mathbf{H}_2)$ under same frequency ω . Inserting

sourceless Maxwell's curl equations into the quantity

$$\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = \quad (2.1)$$

$$(\nabla \times \mathbf{E}_1) \cdot \mathbf{H}_2 - (\nabla \times \mathbf{H}_2) \cdot \mathbf{E}_1 - (\nabla \times \mathbf{E}_2) \cdot \mathbf{H}_1 + (\nabla \times \mathbf{H}_1) \cdot \mathbf{E}_2 \quad (2.2)$$

gives

$$\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = \quad (2.3)$$

$$-i\omega \mathbf{H}_1 \cdot \boldsymbol{\mu} \mathbf{H}_2 - i\omega \mathbf{E}_2 \cdot \boldsymbol{\epsilon} \mathbf{E}_1 + i\omega \mathbf{H}_2 \cdot \boldsymbol{\mu} \mathbf{H}_1 + i\omega \mathbf{E}_1 \cdot \boldsymbol{\epsilon} \mathbf{E}_2. \quad (2.4)$$

When $\boldsymbol{\epsilon}, \boldsymbol{\mu}$ are constant symmetric tensors, the right-hand side of the equation cancels and we have

$$\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = 0. \quad (2.5)$$

When this principle is followed by optical waveguide systems which can be generally described by a scattering matrix $\mathbf{S}$ as $\mathbf{B} = \mathbf{S}\mathbf{A}$, where $\mathbf{A}$ and $\mathbf{B}$ are respectively column vectors of forward and backward propagating waves' complex mode amplitudes, it implies that [96]

$$\mathbf{S}^T = \mathbf{S}. \quad (2.6)$$

This means the system is reciprocal. To make the system non reciprocal, we need $\boldsymbol{\epsilon}, \boldsymbol{\mu}$ be not both constant symmetric tensors, so that

$$\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \neq 0. \quad (2.7)$$

This can be accomplished in three distinct ways: using magneto-optic effects, temporal

modulation of the electromagnetic medium, or optical nonlinearity. Magneto-optic isolators are the most widely used in current technology, but are challenging to incorporate into on-chip optical circuits [97–99]. For this reason, there has been a great deal of research into isolator designs based on spatio-temporal modulation [100, 101] and nonlinear materials [97, 102–121].

This chapter explores the possibility of realizing optical isolators using nonlinearity-induced topological phase transitions. As discussed in previous chapter 1, topological photonic structures can exhibit topological edge states that are robust against perturbations when tuned into a “topologically nontrivial” phase. Although these topological edge states have mostly been studied in the linear regime, there have been several recent papers exploring how they are affected by optical nonlinearities [66, 122, 70, 67, 120, 123]. It appears that photonic topological transitions—transitions from a conventional or topologically trivial phase to a topologically nontrivial phase—can be “driven” by nonlinearities, so that the light intensity itself determines whether the light can propagate via an edge state. We will study how this behavior might be exploited in nonlinear optical isolators. There have also been a number of papers seeking to implement optical isolators using magneto-optic topological photonics [98, 99], but such schemes lie outside the scope of the present discussion.

We will analyze three representative photonic designs that (i) are known to exhibit topological transitions in the linear regime, and (ii) can feasibly operate in the optical frequency range, where optical isolation is a particularly pressing problem due to the absence of strong magneto-optic effects [97–99]. Our goal is to obtain a conceptual understanding of the features and limitations of these novel isolation schemes; as such, we will make use of simplified models, based on the coupled-mode theory and transfer matrix frameworks, capturing just the essentials of nonlinearity and bandstructure topology. In particular, we will not attempt to study the actual device geometries and

material nonlinearities needed to achieve the nonlinear lattice parameters appearing in our models, nor to optimize our designs to maximize their performance.

The first type of structure we will study is an array of coupled optical waveguides, where light is guided (“evolves”) either forward or backward along each waveguide, and can hop between adjacent waveguides via evanescent coupling. We begin with an exemplary waveguide array corresponding to a 1D SSH model [90], with Kerr-like nonlinearities added to the inter-waveguide coupling strengths. Hadad, Khanikaev, and Alù have shown that such a model can exhibit a self-induced topological transition [70], in which the nonlinearity drives a local region of the lattice into a different topological phase, giving rise to self-trapped soliton-like edge states. These nonlinear edge states allow a high-intensity signal injected in a edge waveguide to resist diffraction into the rest of the lattice.

We show that when such a lattice has asymmetric input and output coupling losses, it can function as an efficient optical isolator. Light is injected into one port of an edge waveguide, evolves through a fixed distance, and leaves at the other end of the same waveguide. With appropriately-chosen system parameters, the forward transmittance (via a self-induced topological edge state) is of order unity, while the backward transmittance (without an edge state) is suppressed by several orders of magnitude.

The isolator relies on the fact that the self-induced topological transition has a power threshold—i.e., the soliton-like edge state appears only above a certain power. The asymmetric input and output couplings ensure that the edge state exists under forward transmission, but not backward transmission. However, the existence of a threshold is not unique to the nonlinear SSH model. It has previously been shown that nonlinear lattices with a conventional design can support edge solitons with a power threshold [124–126], distinct from bulk solitons which normally bifurcate from zero power [127]. In those studies [124–126], the edge soliton thresholds had no apparent

connection to the topology of the underlying bandstructure. In the present case, the threshold—and hence the operating power of the isolator—is set by the topological phase transition of the SSH model.

Next, we generalize these results to a structure exhibiting a 2D topological phase [36]. Unlike the 1D SSH model, the 2D lattice has edge solitons that are mobile, and can travel around defects such as corners [67]. In this case, to achieve strong optical isolation, the input and output must correspond to different (spatially separated) waveguides, chosen according to the nonzero velocity of the edge soliton. Similar to the 1D case, we find that the soliton has a power threshold determined by the underlying linear model’s topological phase transition—in this case, a transition from conventional insulator to Chern insulator. This shows that the soliton-based optical isolation scheme can be extended to more complex 2D lattices. Unlike in 1D topological phases, topologically nontrivial behavior in 2D can occur without requiring any special lattice symmetries, and is thus more robust.

The final type of nonlinear isolator that we will study is based on a periodic 2D array of coupled ring-like waveguides [31, 31, 128–133]. Unlike the waveguide array case, which had a separate “evolution” axis z (with light injected at $z = 0$, and undergoing nonlinear evolution up to $z = Z$), here the system is entirely on-chip. Light is in-coupled and out-coupled at two different positions on the lattice edge, and the steady-state solution within the nonlinear lattice is determined self-consistently. Roughly speaking, this is like solving a nonlinear steady-state scattering problem at a fixed “energy”, rather than a nonlinear evolution problem over a fixed “time interval”.

We show that the coupled-ring lattice can also function as an efficient optical isolator, but with substantially different characteristics due to its steady-state nature. The isolation is again based on a topological phase transition in the lattice bandstructure [129, 131], driven by a nonlinearity-induced variation in the effective coupling between rings. Unlike in the waveguide array, the transition manifests as a discontinuity in

Figure 2.1: (a) Schematic of SSH model realized by an array of coupled waveguides. (b,c) Mode intensity distributions in a linear SSH lattice of 29 sites, generated by an excitation on the left-most site (waveguide 1). Results are computed numerically using the z -evolution operator. (b) For a “trivial” lattice with $\kappa_1 = 1$ and $\kappa_2 = 0.2$, the excitation diffracts into the bulk. (c) For a “nontrivial” lattice with $\kappa_1 = 1$ and $\kappa_2 = 5$, the excitation is confined to the edge. (d) Transmittance on waveguide 1 (blue line) versus κ_2/κ_1 . The intensity of the edge eigenmode on waveguide 1, $|a_1|^2$, is also shown (green dashes). In (b)–(d), the input/output couplings η_A and η_B are set to unity.

the nonlinear transmittance: above a critical input power, light propagation switches abruptly from very low transmittance (via bulk states) to very high transmittance (via edge states). Thus, with varying input power, the structure is able to exhibit a discontinuous jump from very low to very high isolation ratios.

2.2 Nonlinear coupled waveguide arrays: SSH model

We begin our study with a nonlinear version of the 1D SSH model [70], which is the simplest model to exhibit topological modes. As shown schematically in Fig. 2.1(a), such a model can be implemented with a 1D array of waveguides with nonlinear couplings. Each unit cell consists of two waveguides, with identical wave-guiding characteristics. We take the tight-binding (coupled-mode) approximation, which applies to guided modes moving in one direction along the axial direction z , without backscattering. We let $a_n(z)$, $b_n(z)$ denote the complex wave amplitudes in the two waveguides of unit cell n . With an appropriate gauge choice, these obey the coupled-mode equations [127]

$$i \frac{da_n}{dz} = \kappa_1^n b_n + \kappa_2^{n-1} b_{n-1} \quad (2.8)$$

$$i \frac{db_n}{dz} = \kappa_1^n a_n + \kappa_2^n a_{n+1}, \quad (2.9)$$

where $\kappa_1^n, \kappa_2^n \in \mathbb{R}^+$ are intra-cell and inter-cell coupling coefficients. In the linear regime, the κ parameters are constants independent of n .

The linear SSH model has been extensively investigated in photonics, including in femtosecond-laser-written waveguide arrays [134, 135] and plasmonic waveguide arrays [136]. It has a phase transition at $\kappa_1 = \kappa_2$; when $\kappa_1 < \kappa_2$, there is an edge state with zero eigenvalue localized to the left edge of the lattice. We can observe this by exciting the leftmost site (waveguide 0) with input power $I = |a_0(0)|^2$, and letting the state evolve up to a fixed distance Z . As shown in Fig. 2.1(b)–(c), the excitation diffracts into a superposition of bulk modes for $\kappa_2/\kappa_1 < 1$, but remains confined to the edge for $\kappa_2/\kappa_1 > 1$. In Fig. 2.1(d), we plot the transmittance on the edge waveguide, defined as $\mathcal{T} = |a_0(Z)|^2/I$. This is seen to closely track the edge intensity of the exact edge eigenstate (the normalized eigenstate of the linear Hamiltonian having eigenvalue zero).

We now introduce nonlinearity into the model. In accordance with the goals of this study, we will choose a nonlinearity that is conceptually simple and easy to model, leaving aside the question of how best to physically implement it. The inter-cell coupling coefficient is made dependent on the local intensity, as follows [70]:

$$\kappa_2^n(z) = \kappa_0 + \alpha \left(|a_{n+1}(z)|^2 + |b_n(z)|^2 \right), \quad (2.10)$$

where κ_0 stands for static inter-cell coupling, and α is a Kerr-like coefficient multiplying the sum of the intensities in the two coupled sites. We will take $\kappa_1 = 1.0$, $\kappa_0 = 0.5$, and $\alpha > 0$, i.e. the inter-cell coupling becomes stronger at higher intensities. Thus, the bandstructure is topologically trivial in the linear (zero-intensity) limit, but increasing the intensity will (roughly speaking) drive it into the nontrivial phase. Without loss of generality, we set $\alpha = 1$ (other values are equivalent up to a rescaling of intensities).

The nonlinearity allows for the possibility of a self-induced topological transition [70]. Suppose we prepare an initial state by exciting just the left edge waveguide with input power $I = |a_0(0)|^2$. The light undergoes nonlinear evolution for distance Z , and we compute the transmittance $\mathcal{T}(I) = |a_0(Z)|^2/I$. The results are shown in Fig. 2.2(a). For small I , $\mathcal{T}(I)$ is close to zero (similar to the linear case, the light mostly diffracts into the lattice bulk); but for $I \gtrsim 9$, $\mathcal{T}(I)$ abruptly (but continuously) increases towards unity.

This abrupt change in the nonlinear transmittance is related to a self-induced topological transition. To see this, in Fig. 2.2(b) we plot the averaged values of κ_2/κ_1 as a function of I (since κ_2 varies between sites and also with z , the averages shown here are taken over different numbers of unit cells near the left lattice edge, and over a fixed range of z). We find that the increase in $\mathcal{T}(I)$ starts to occur when $\kappa_2/\kappa_1 \approx 1$, which is precisely the topological transition point of the SSH model. Note that correspondence is apparent even when we average κ_2 over 12 sites, a relatively

Figure 2.2: Behavior of the nonlinear SSH lattice as an optical isolator. (a) Nonlinear transmittance along waveguide 0, $\mathcal{T}$, versus input power I . The lattice has 29 sites, with $\eta_A = 1$, $\eta_B = 0.5$, $\kappa_1 = 1$, $\kappa_0 = 0.5$, and $\alpha = 1$; the output is at $Z = 20$. Results are computed numerically using evolution operators with step size $\delta z = 10^{-3}$. (b) Averaged values of κ_2/κ_1 versus input power I , where the nonlinear κ_2 is averaged over $18 \leq z \leq 22$ (i.e., around the output Z) and over a few sites (4, 8, or 12) closest to the left lattice edge. The topological transition point of the linear SSH model, $\kappa_2 = \kappa_1$, is indicated by the horizontal dashes. (c) Isolation ratio (IR), forward transmittance T_f , and backward transmittance T_b , versus input power I . Horizontal dashes show the approximate upper bounds $T_{f,b}^{\max} = \eta_A^2 \eta_B^2$ and $\text{IR}^{\max} \approx 1/T(0)$.

far distance from the edge.

The regime of high nonlinear transmittance is due to the self-induced edge soliton described by Hadad *et al.* [70]. This is a nonlinear mode that inherits some properties of the linear SSH model’s edge state, such as leaving the b_n sites unexcited. Its onset also closely matches the topological transition of the SSH model. However, it differs in other ways: its eigenvalue is not pinned to zero (as the SSH model’s “particle-hole” symmetry is broken by the nonlinearity), and the intensity goes to a small constant in the bulk rather than decaying exponentially with distance from the edge [70].

A nonlinear photonic structure with strongly intensity-dependent transmittance can serve as the basis for an optical isolator [105, 111, 115, 116, 118, 119]. To accomplish this, we introduce couplings η_A and η_B , as shown schematically in Fig. 2.1(a). In forward-transmission mode, light is coupled into port A on an edge waveguide, is guided in the $+z$ direction, and is out-coupled at port B of the same edge waveguide. In backward-transmission mode, light enters at B , is guided in the $-z$ direction, and is out-coupled at port A . (Note that this is distinct from “asymmetric light transmission” schemes, such as those studied in Ref. [101], where light propagates either left-to-right or right-to-left along the lattice, while being guided in a single axial direction $+z$; those schemes have no direct bearing on the problem of optical isolation, as they do not swap physical input and output ports.)

In forward-transmission mode, the input light has intensity I , and the intensity coupled into the edge waveguide is $\eta_A^2 I$; the intensity at the end of the waveguide is $\mathcal{T}(\eta_A^2 I) \eta_B^2 I$, where $\mathcal{T}$ is the nonlinear transmittance of the lattice itself. The overall forward transmittance is thus

$$T_f = \eta_A^2 \eta_B^2 \mathcal{T}(\eta_A^2 I). \quad (2.11)$$

Similarly, the backward transmittance (in the $-z$ direction) is

$$T_b = \eta_A^2 \eta_B^2 \mathcal{T}(\eta_B^2 I). \quad (2.12)$$

Note that $T_f = T_b$ in the linear regime $I = 0$, in accordance with the reciprocity principle. The isolation ratio is defined by

$$\text{IR} \equiv T_f/T_b = \frac{\mathcal{T}(\eta_A I)}{\mathcal{T}(\eta_B I)}. \quad (2.13)$$

Fig. 2.2(c) shows numerical results for the forward and backward transmission, and the isolation ratio, for $\eta_A = 1$ and $\eta_B = 0.5$. We can understand these results qualitatively by using Eqs. (2.11)–(2.12) and the features of the nonlinear waveguide transmittance $\mathcal{T}(I)$. First, note that the upper bound for the transmittance (in either direction) is $T_{f,b}^{\max} \leq \eta_A^2 \eta_B^2$, which occurs when the light is transmitted predominantly along the edge, with losses only at the input and output ports. As I is increased from zero, the transmittances increase exponentially from the very low value of $\eta_A^2 \eta_B^2 \mathcal{T}(0)$, before saturating near the upper bound. Since $\eta_A \neq \eta_B$, the initial increase of T_f and T_b occur with different exponential factors, and as a result the IR increases exponentially with the input power I . It reaches a maximum of $\text{IR}^{\max} \approx 1/\mathcal{T}(0)$, where $\mathcal{T}(0)$ is the waveguide transmittance in the linear regime. This corresponds to the case where T_f has saturated but not T_b . The results in Fig. 2.2(c) show good agreement with these approximate bounds.

We can compare these results to optical isolation schemes based on “non-topological” edge solitons, such as solitons induced by on-site Kerr nonlinearity [124–126]. Such solitons also exhibit a power threshold proportional to the coupling κ . Optimizing the isolation ratio in a device of fixed length thus requires a trade-off between minimizing the linear transmittance (larger κ), or minimizing the threshold power (smaller κ). In contrast, in the nonlinear SSH model one can reduce the linear transmittance by

increasing $\kappa_{0,1}$ without substantially effecting the threshold power, determined by $\kappa_1 - \kappa_0$. Moreover, the staggered profile of the topological edge soliton enables further optimization of the performance, for example by incorporating lossy elements onto the unexcited b_n sites to further reduce the backward transmission [99].

2.3 Nonlinear coupled waveguide arrays: Haldane model

The phenomenon of optical isolation aided by a topological transition can also be observed in 2D lattices. There are two important difference between 1D and 2D. Firstly, topological edge states in 2D can exhibit unidirectional propagation along the edge, so we choose the input and output ports to be different waveguides. Secondly, whereas topological protection in the SSH model requires a specific sublattice symmetry, topological protection in 2D generally does not; hence, 2D lattices could provide robust isolation under a wider range of fabrication imperfections or nonlinearities.

In the linear regime, it has previously been shown that a 2D optical waveguide array can be made to act as a 2D topological insulator with broken “time-reversal” symmetry (a Chern insulator) by adding a helical twist to the waveguides [36]. A variant design has been shown to support tunable topological phase transitions [137]. Here, we leave implementation details to one side, and focus instead on the Haldane model [138], the simplest and most well-known 2D model with a topological phase transition between a conventional insulator phase and a Chern insulator phase [138].

The tight-binding model describes a 2D honeycomb lattice with broken time-reversal symmetry; the honeycomb lattice is divided into two sub-lattices, A and B , with on-site mass terms $m_A = m_0$ and $m_B = -m_0$. The other model parameters are the nearest-neighbor hopping $t_1 \in \mathbb{R}$, the next-nearest-neighbor hopping amplitude $t_2 \in \mathbb{R}$, and an Aharonov-Bohm phase $\phi \in [-\pi, \pi]$ which determines the magnetic

Figure 2.3: (a) Intensities after evolution through a nonlinear Haldane lattice, with parameters $t_1 = 1$, $t_2 = 1/3$, $\phi = \pi/2$, and $m_0 = 2$. A corner site (A) is initially excited with intensity I , and colors indicate site intensities $|a_\mu|^2$ and $|b_\nu|^2$ (with arbitrary normalization) after propagation through $Z = 19.2$. Two cases are shown: $I = 1$ and $I = 150$. The input/output couplings η_A and η_B are set to unity. (b) Nonlinear transmittance $\mathcal{T}(I)$ from one corner (A) to the opposite corner (B), versus input power I . (c) Mean values of $\langle m_A \rangle - \langle m_B \rangle$, versus I . The averages are taken over sites up to n nearest-neighbor hops from site B, for $n \in \{1, 2, 3, 4\}$. (d) Isolation ratio (IR), forward transmittance T_f , and backward transmittance T_b , versus I .

flux penetrating sub-regions of each unit cell [138]. The bandstructure is in a Chern insulator phase when

$$|m_0/t_2| < 3\sqrt{3}|\sin \phi|, \quad (2.14)$$

with Chern numbers $C = \text{sgn}(\phi)$. The other phase is a conventional insulator with Chern number $C = 0$.

We now introduce a nonlinearity designed to drive the system through a topological transition. We make the on-site mass terms on the A and B sublattices nonlinear, depending on the local intensity, as follows:

$$\begin{aligned} m_A^\mu &= \frac{m_0}{1 + |a_\mu|^2} \\ m_B^\nu &= \frac{-m_0}{1 + |b_\nu|^2}, \end{aligned} \quad (2.15)$$

where $m_0 \in \mathbb{R}$ is the Haldane model's mass parameter in the linear limit; μ, ν are site indices on the A and B sublattices respectively; and $a_\mu, b_\nu \in \mathbb{C}$ denote the optical wave amplitude (wavefunction) at those sites. Similar saturable nonlinearities have previously been studied in the context of non-topological photonic lattices with mobile discrete solitons [139]. We will take $\phi = \pi/2$, $t_2 = 1/3$, and $m_0 = 2$, so that according to Eq. (2.14) the linear system is in the trivial insulator phase. With increasing intensity, the saturable nonlinearity decreases the on-site mass parameters, “driving” the system towards the Chern insulator phase.

The resulting nonlinear propagation is shown in Fig. 2.3(a). Input light is injected onto a single site at a corner of the lattice, with intensity $I = |a_0|^2$, and undergoes nonlinear evolution over a fixed distance Z ; as discussed in Section 2.3, the z axis plays the role of time. For small I , the light diffracts into the bulk of the lattice, in accordance with the fact that the bandstructure of the linear system is topologically trivial. For large I , the light remains tightly confined to the edge, and propagates in one direction along the edge, including around corners, in accordance with the

existence of a unidirectional edge state in the Chern insulator phase.

The nonlinear transmittance $\mathcal{T}(I)$ is plotted in Fig. 2.3(b), and it exhibits a power threshold similar to what we observed in the nonlinear SSH model. Here, $\mathcal{T}(I)$ is defined as the transmittance from the input site (A) at $z = 0$, to the output site (B) on the opposite lattice corner at $z = Z$. Again, we can demonstrate a close correspondence between the power threshold and the topological transition of the linear lattice. In Fig. 2.3(c), we plot $\langle m_A \rangle - \langle m_B \rangle$ versus I , where $\langle m_A \rangle$ and $\langle m_B \rangle$ are the nonlinear on-site mass terms averaged over sites closest to the output waveguide at $z = Z$. Based on Eq. (2.14), the linear lattice exhibits a topological transition at $m_A - m_B = |6\sqrt{3}t_2 \sin \phi| = 2\sqrt{3}$; in Fig. 2.3(c), we indeed observe that the nonlinear transmittance's threshold power occurs as $\langle m_A \rangle - \langle m_B \rangle$ drops below this value. In Fig. 2.3(d), we plot the isolation ratio and forward and backward transmittances. The behavior is similar to the nonlinear SSH results in Fig. 2.2(c). One difference is that T_f and T_b are dissimilar even though the input/output couplings are symmetric (here we set $\eta_A = \eta_B = 1$); this is because the inputs and outputs are on different sites, so the lattice itself provides the asymmetry.

Optical isolation based on traveling edge solitons has important qualitative differences compared to schemes based on immobile solitons, such as the nonlinear SSH model discussed in Section 2.2. The device length Z and/or the choice of output waveguide must be matched to the edge soliton velocity, so as to ensure a high forward transmittance. This kind of traveling discrete soliton is not easily achievable with “conventional” nonlinear lattice designs not tied to a topological transition; in those cases, traveling solitons require excitation of several waveguides [139], and discrete solitons are typically immobile and/or suffer from strong radiative losses [140].

2.4 Nonlinear coupled ring lattices

We now turn our attentions to a quite different type of photonic structure: a lattice of nonlinear coupled rings. The structure is shown schematically in Fig. 2.4(a), and consists of ring-shaped waveguides arranged in a 2D square lattice, with adjacent “site rings” connected by auxiliary “coupler rings”. The structure is assumed to be engineered so that there is negligible back-scattering at the coupling regions where neighboring waveguides approach one another; in other words, the circulation of light in the site rings—clockwise or anti-clockwise—is preserved under inter-site hopping [31, 31, 128].

Suppose the entire lattice, including the auxiliary rings, is periodic. In this case, the lattice’s bandstructure is known to exhibit a topological transition [129]: as we increase the effective coupling between site rings, the bandstructure goes from a conventional phase to a topologically nontrivial phase. In the latter, there exists (for each circulation) a family of topological edge states that move unidirectionally along the lattice edge [129–133].

For the waveguide arrays discussed in Section 2.2–2.3, the “forward” and “backward” modes of the optical isolator corresponded to $+z$ and $-z$ propagation. By contrast, the present coupled-ring lattice structure is “on-chip”, i.e. purely 2D. In forward-transmission mode, light is coupled into one ring, propagates through the lattice in a given circulation direction (say, clockwise), and is subsequently out-coupled. In backward-transmission mode, the input and output ports are switched, and hence the in-lattice propagation takes place via the *opposite* circulation (anti-clockwise). It is important to note that this switches the directionality of the topological edge states. For instance, in Fig. 2.4(a) we show a right-moving edge state on the upper edge, with clockwise circulation; the reciprocal partner is a left-moving edge state on the upper edge, with anti-clockwise circulation.

Figure 2.4: (a) Schematic of a lattice of coupled rings, consisting of “site rings” (large circles) arranged in a square lattice, separated by auxiliary “coupler rings” (small circles). Light is coupled in/out of the lattice on the left and right, with coupling parameters θ_A and θ_B . (b) Close-up of an inter-site coupling, showing the definitions of the complex wave amplitudes $\{a, b, c, d\}$ (in the site rings) and $\{\alpha, \beta, \gamma, \delta\}$ (in the coupler ring).

The transmission of light through the lattice can be calculated using transfer matrices [31, 129]. Details are given in the Appendix chapter. Fig. 2.4(b) shows a coupler ring joining two site rings; the complex wave amplitudes labeled in this figure are related by

$$\begin{bmatrix} c \\ \gamma \end{bmatrix} = S_c(\theta_0) \begin{bmatrix} a \\ \alpha \end{bmatrix}, \quad \begin{bmatrix} d \\ \delta \end{bmatrix} = S_c(\theta_0) \begin{bmatrix} b \\ \beta \end{bmatrix}, \quad (2.16)$$

where

$$S_c(\theta_0) = \begin{bmatrix} \sin \theta_0 & i \cos \theta_0 \\ i \cos \theta_0 & \sin \theta_0 \end{bmatrix} \quad (2.17)$$

is a unitary 2×2 scattering matrix describing evanescent coupling with energy conservation and negligible back-scattering, as well as 180-degree rotational symmetry [31, 129, 130]. The strength of the evanescent coupling is described by θ_0 . Moreover,

Figure 2.5: (a)–(b) Band diagrams for a semi-infinite lattice of coupled rings, with width 10 unit cells in the y direction. For the coupler ring parameters, we choose $\theta_0 = \pi/6$ and two different choices of the phase shift ξ ; the resulting bandstructures are (a) topologically trivial for $\xi_1 = \xi_2 = \xi = 0.55\pi$, and (b) topologically nontrivial for $\xi = 0.85\pi$. (c) Numerically-obtained phase diagram of the system for $\theta_0 = \pi/6$. Grey regions show the ungapped parts of the bandstructure, white regions show trivial (T) bandgaps, and pink regions show nontrivial (NT) bandgaps. For fixed $\phi = \pi/4$, the system can move from a trivial to a nontrivial gap by varying $\xi_{1,2}$ as indicated by the arrow. The gaps depend only on ξ_1 and ξ_2 through the combination $\xi_1 + \xi_2$.

wave amplitudes acquire a phase $\xi_{1,2}$ on traversing each arm of the coupler ring:

$$\alpha = e^{i\xi_1} \delta, \quad \beta = e^{i\xi_2} \gamma. \quad (2.18)$$

The effective coupling between adjacent site rings can be determined [129–131] from the parameters θ_0 and $\xi_{1,2}$ (which depend, in turn, on the waveguide geometry and operating frequency). We will assume that the couplings in the x and y direction are identical.

The bandstructure is defined in terms of the phase shift ϕ over each quarter of a site ring, as indicated in Fig. 2.4(b). This plays the role of a “quasi-energy” [129, 131], whose value is fixed by the waveguide geometry and operating frequency. We can regard ϕ as being analogous to the chemical potential in a band insulator. (Note that the other model parameters, θ_0 and $\xi_{1,2}$, will also simultaneously depend on the operating frequency; in designing a real device, these model parameters must be mapped onto physical quantities including the frequency and geometrical parameters.) In Fig. 2.5(a)–(b), we plot band diagrams of ϕ versus k for a semi-infinite lattice with transverse width of 10 unit cells; here, k is the usual Bloch wavenumber, defined as the phase shift produced by translating one unit cell along the strip [129]. We fix $\theta_0 = \pi/6$, and show results for two choices of $\xi = \xi_1 = \xi_2$. Fig. 2.5(a) shows a conventional bandstructure, while Fig. 2.5(b) shows a nontrivial bandstructure with gaps spanned by topological edge states. Evidently, we can switch between these two distinct cases by tuning only ξ , with all other model parameters kept constant. Fig. 2.5(c) shows a phase diagram indicating the parameter choices for observing trivial and nontrivial gaps.

Fig. 2.6(a) shows the transmittance across the linear lattice for fixed $\phi = \pi/4$ and varying coupler ring phase shifts $\xi = \xi_1 = \xi_2$. The transmittance is close to zero for $\xi \approx \pi/2$, when the system is in a trivial gap, and approaches unity when ξ increases and the system enters a topologically nontrivial gap. For intermediate ξ , the system lies in a band, and the transmittance exhibits numerous resonances.

We now introduce a nonlinearity designed to drive the system through the topological transition. Let the phase shift parameter on each arm of a coupler ring be intensity-dependent:

$$\xi = \xi_0 + \kappa \mathcal{I}, \quad (2.19)$$

where ξ_0 is the phase shift in the linear limit, κ is a Kerr coefficient, and $\mathcal{I}$ is the

local intensity in the arm of the coupler ring, defined as $\mathcal{I} = |\psi|^2$ where ψ is the local complex amplitude. Physically, this may be accomplished by fabricating the coupler rings out of a nonlinear material (note that the intensity within the coupler rings will be strongly enhanced if they are close to resonance with the operating frequency [130]). We choose $\phi = \pi/4$, $\theta_0 = \pi/6$ (both assumed to be intensity-independent), $\xi_0 = \pi/2$, and $\kappa = 1$. Referring to the phase diagram in Fig. 2.5(c), we see that in the linear limit the bands are topologically trivial, with ϕ lying within a trivial gap. Increasing $\mathcal{I}$, and hence ξ , drives the system (locally) into a nontrivial gap. In interpreting this phase diagram, note that although the nonlinearity changes ξ independently in the two arms of a coupler ring, the band gaps of the linear system depend only on the *sum* of ξ in the two arms.

Fig. 2.6(b) plots the transmittance across the lattice, $\mathcal{T}(I)$, versus input power I . Here, we assume the input and output couplings to be perfect (the coupling matrices have the form of Eq. (2.17), with coupling angle $\pi/2$). These results are obtained by using a numerical nonlinear solver to find self-consistent solutions to the entire set of transfer matrix relations within the lattice, including the nonlinear phase shifts described by Eq. (2.19). The solution to the numerical problem is discussed in the Appendix chapter.

From Fig. 2.6(b), we observe that the transmittance in the linear limit, $\mathcal{T}(0)$, is negligible, in accordance with the fact that the system is in a trivial gap. Moreover, with increasing I , $\mathcal{T}(I)$ initially remains low, but at a critical intensity it jumps discontinuously to $\mathcal{T} \approx 0.8$. Above this discontinuity, transmission takes place along the upper lattice edge, as shown in Fig. 2.6(c).

In Fig. 2.6(d), we plot $\langle \xi \rangle$ versus I , averaging over the coupler rings on the upper lattice edge. This shows that the large discontinuity in $\mathcal{T}(I)$ found in Fig. 2.6(b) occurs at values of $\langle \xi \rangle$ corresponding to the boundary between the trivial gap and the band, and between the band and the nontrivial gap. By comparison with the

Figure 2.6: (a) Transmittance T through a linear lattice with phase shift ξ in all arms of all coupler rings. The input/output couplings are taken to be perfect (the coupling matrices have the form of Eq. (2.17) with $\theta_A = \theta_B = \pi/2$), and the lattice parameters are $\phi = \pi/4$, and $\theta_0 = \pi/6$. For $\xi \approx \pi/2$, ϕ lies in a trivial gap, and T is close to zero; with increasing $\Delta\xi$, ϕ enters a band (grey region) and T exhibits bulk transmission resonances; for still larger ξ , ϕ enters a nontrivial gap (pink region) and T approaches unity. Vertical dashes indicate the boundaries between the band and gap regions shown in Fig. 2.5(c). (b) Transmittance through the nonlinear lattice versus input power I , using nonlinearity parameters $\xi_0 = \pi/2$ and $\kappa = 1$, with all other parameters kept the same as in (a). This and subsequent results are obtained by solving the nonlinear transfer matrix relations self-consistently via a numerical nonlinear solver. (c) Normalized field intensity distribution at $I = 3.21$, just above the large discontinuity in (b). For clarity, the coupler rings are omitted and only the site ring intensities are shown. (d) Mean phase shifts $\langle \xi \rangle = \langle (\xi_1 + \xi_2)/2 \rangle$ versus I , with averages taken over both arms of the eight coupler rings on the upper lattice edge [see Fig. 2.4(a)]. Horizontal dashes indicate the boundaries from Fig. 2.5(c).

Figure 2.7: Isolation ratio (IR), forward transmittance T_f , and backward transmittance T_b , versus input power I , in disordered nonlinear coupled ring lattices. Thin lines show results for 100 different disorder realizations, and thick black lines show the mean values. The disorder consists of shifts in individual coupler ring phase shifts ξ_j , normally distributed with mean 0 and standard deviation 0.01π . Input/output loss is applied at port B by multiplying amplitudes by $e^{0.1}$ (with no loss at port A); all other parameters are the same as in Fig. 2.6.

linear system’s phase diagram from Fig. 2.5(c), it appears that the nonlinear lattice is “jumping” past the in-band regime.

Fig. 2.6(b) and (d) also shows a “secondary” discontinuity, where $\langle \xi \rangle$ jumps to a larger value while remaining in the topological gap region. This seems to be triggered when the nonlinearity causes the value of ξ in a few individual coupler rings to exceed the upper boundary of the topological gap region, entering into another in-band region at $\xi \sim 1.1\pi$. This destabilizes the solution, and the system compensates by jumping to a different field distribution, with larger $\langle \xi \rangle$, that keeps all coupling rings in the topological gap region. On both sides of this secondary discontinuity, the field distribution remains confined to the upper lattice edge.

Unlike the nonlinear Haldane model studied in Section 2.3, there is no “time-of-flight” limitation on the propagation of light from the input to the output port, due

to the steady-state nature of the model. As such, optical isolation in the nonlinear coupled-ring lattice can exhibit topological protection against lattice defects. To test this idea, we apply an input/output asymmetry by adding a small amount of loss (coupling factor of $e^{-0.1}$) to one of the ports (B). We then compute the self-consistent forward transmittance (port A to B), backward transmittance (B to A), and isolation ratio. Disorder is introduced by adding normally distributed shifts to each ξ_i parameter (with standard deviation 0.01π). The results are shown in Fig. 2.7. For a range of input powers near $I \gtrsim 3$, there is a sharp jump in the mean forward transmittance $\langle T_f \rangle$, and correspondingly a jump in the mean isolation ratio. These jumps come from ensemble averages of transmittance discontinuities shifted by the disorder; note that the secondary discontinuities, being much smaller in magnitude, are “smeared out” and are thus not visible in the plotted mean values.

2.5 Discussion and conclusions

In this chapter, we have studied how optical isolation can be accomplished in three different models of topological photonics. A few basic ingredients are common to all three models. Firstly, the model must be nonlinear, so as to break optical reciprocity [96]. Secondly, the structure must contain an asymmetry (e.g., asymmetric input/output couplings) that distinguishes between “forward” and “backward” directions. The third ingredient is the use of a topological phase transition to associate forward transmission and backward transmission with different topological phases, whose physical properties are qualitatively different from each other. This last design principle is reminiscent of recently-proposed optical isolation schemes based on parity/time-reversal (PT) symmetric structures, which rely on a non-Hermitian transition (between “PT symmetric” and “PT broken” phases) rather than a topological phase transition [115, 116, 118, 121].

In the presence of nonlinearity, topological phase transitions manifest as “self-induced” topological solitons: local regions of the lattice where the optical field self-sustains its own edge state-like behavior, even when the lattice is topologically trivial in the zero-intensity limit. Such solitons were previously discovered in the 1D nonlinear SSH model [70], as well as 2D continuum “Floquet topological insulator” models [66, 67]. Here, we have shown that the nonlinear SSH soliton is useful for optical isolation, and moreover that topological solitons also occur in two new 2D models—nonlinear versions of the Haldane model [24] and the coupled-ring lattice [31, 31, 128, 129, 131]. The soliton in the coupled-ring lattice model is notable for being a static (steady-state) solution, whereas the solitons in the nonlinear Haldane model and other previously-studied 2D lattices [66, 67] are dynamical. As we have seen, this gives rise to the distinctive feature: a discontinuity (not just a threshold or kink) in the power-dependent optical transmittance.

To realize a self-induced topological soliton, nonlinearity must be applied to a model in a non-arbitrary way: as exemplified by Eqs. (2.10), (2.15), and (2.19), the nonlinearity needs to act on parameters that drive the system towards a topological phase transition. Interestingly, homogeneous nonlinearities such as local Kerr effects, which are the most commonly-studied nonlinearities in lattice models [124–127], may *not* be suited to inducing topological phase transitions. Roughly speaking, such uniform nonlinearities play the role of altering the scalar potential, which is an inefficient way to induce topological band inversions. In real experiments, nonlinearities may be inhomogeneous and present in both on-site terms and “off-diagonal” (inter-site coupling) terms [141]. Our results demonstrate that the latter, though frequently ignored, can lead to soliton behaviors that are both distinctive and useful. It should also be noted that this study has omitted the temporal effects of optical nonlinearity, such as frequency generation and pulse dispersion. Such effects are important in certain materials, or for ultrashort pulses [142], and we shall see an example in the

next chapter.

In future design studies, it would be interesting to introduce additional features to the models that could further improve their performance as optical isolators. For instance, loss can be selectively added to the lattice to suppress the transmittance from bulk modes and/or diffusive non-topological edge modes, while leaving the topological edge modes relatively unaffected. It would also be interesting to make a comparison with various non-topological nonlinear isolator designs that are fine-tuned to achieve high isolation ratios and/or high transmittance [110–113]; one might be able to show that isolation schemes based on topological solitons can achieve similarly high performance, while being less sensitive to random defects due to the intrinsic robustness of the topological edge modes.

Dynamical effects in the coupled-ring lattice are another avenue for further study. For example, the transmittance jumps in Fig. 2.6(b) may form hysteresis loops under an additional slow modulation of the input intensity. Such designs may also be useful for limiting unwanted dynamical reciprocity [100], as the nonlinear isolation is provided by topological edge states localized in both frequency and space. Small amplitude signals with sufficiently large frequency detuning from the input could propagate via the qualitatively different bulk modes, and might thus be efficiently filtered or suppressed.

This chapter used simplified theoretical models to obtain a conceptual understanding of the features and limitations of novel topology-based isolation schemes. In future work it would be interesting to explore experimental realizations of these various models [143]. In the next chapter, a nonlinear SSH model is experimentally realized, and we look into the temporal effects of nonlinearity, that were neglected in the steady-state analysis of this chapter.

Chapter 3

Topologically Enhanced Harmonic Generation in a Nonlinear Transmission Line Metamaterial

Some of the contents of this chapter were previously published on the journal Nature Communications [71]. Wherever figures and materials from other sources have been taken, the authors have been given due credit.

3.1 Introduction

Recently, electronic LC circuits have emerged as a highly promising method of realizing topologically nontrivial lattices [144, 145, 44, 45, 146, 147, 69, 148]. Compared to other classical platforms like photonics [27, 40, 38, 43], acoustics [51, 149, 150], and mechanical lattices [47, 151, 152], which have also been used to realize topologically nontrivial bandstructures and topological edge states, electronic circuits have several compelling advantages: extreme ease of experimental analysis; the ability to fabricate complicated structures via printed circuit board (PCB) technology; and the intriguing

prospect of introducing nonlinear and/or amplifying circuit elements to easily study how topological edge states behave in novel physical regimes. Notably, circuits have been used to study the Su-Schrieffer-Heeger (SSH) chain (the simplest one-dimensional topologically-nontrivial lattice) [79, 44], nonlinear SSH chains supporting solitonic edge states [69], two-dimensional topological insulator lattices [144], and the corner states of high-order topological insulators [45, 148].

One of the most interesting questions raised by the emergence of topologically non-trivial classical lattices is how topological edge states interact with nonlinear media. Previous studies, including the work presented in the previous chapter, have focused on nonlinearity-induced local self-interactions in the fundamental harmonic. Such effects can give rise to solitons with anomalous plateau-like decay profiles in nonlinear SSH chains [70, 69], or chiral solitons in two-dimensional lattices [66, 122, 67, 153–155]. Apart from the applications for optical isolation studied in the previous chapter, it has also been suggested that topological edge states in nonlinear lattices could be used for robust traveling-wave parametric amplification [156], and other useful functionalities [157–160].

In this chapter, we report on the implementation of a nonlinear SSH chain based on a left-handed nonlinear transmission line (NLTL) [161–165, 65, 166, 167], in which the topological edge state induces highly efficient harmonic generation. Although previous studies have emphasized the role of local self-interactions, including in a previous demonstration of a nonlinear SSH circuit based on weakly-coupled LC resonators [69], an important feature of our circuit is the decisive role of higher-harmonic signals in modulating the first-harmonic modes: they can drive the entire lattice, not just the edge, deeper into the nontrivial regime at the first harmonic frequencies. This behaviour is aided by the fact that the left-handed NLTL has an unbounded dispersion curve supporting traveling-wave higher-harmonic modes [168, 165, 65, 166, 167].

Our measurements on the nonlinear circuit reveal a first harmonic mode that is

localized to the lattice edge, similar to a linear topological edge state, as well as higher-harmonic waves that propagate into the lattice bulk and have voltage amplitudes reaching over an order of magnitude larger than the first harmonic signal. The intensity of the generated third harmonic signal has a maximum of ≈ 2.5 times that of the input first-harmonic signal, compared to < 0.5 for a comparable conventional left-handed NLTL without a topological edge state. The important role played by the topological edge state is further demonstrated by the fact that the third-harmonic intensity is 250 times larger than in a trivial circuit, which has equivalent parameters but lacks a topological edge state in the linear limit, using the same input parameters.

3.2 Sample design and fabrication

3.2.1 Circuit design

The transmission line circuit is shown schematically in Fig. 3.1(a). It contains inductors of inductance L and capacitors of alternating (dimerized) capacitances C_a and C_b . We will shortly treat the case where the C_b capacitors are nonlinear (the L and C_a elements are always linear). First, consider the linear limit where C_b is a constant. We define the characteristic angular frequency $\omega_a = (LC_a)^{-1/2}$, and the capacitance ratio

$$\alpha = C_a/C_b. \quad (3.1)$$

The case of $\alpha = 1$ corresponds to a standard (non-dimerized) left-handed transmission line. This type of transmission line is characterized by having sites separated by capacitors, and connected to ground by inductors, rather than vice versa. Left-handed NLTLs have been shown to be useful for parametric amplification and pulse generation [165, 65, 166, 167].

Let us treat the points adjacent to the capacitors as lattice sites, indexed by an

integer k , and close the circuit by grounding the edges [the left edge is the site labelled A in Fig. 3.1(a)].

Consider the circuit shown in Fig. 3.2a below. This is the same as Fig. 3.1(a), but with the complex voltage variables relabeled for convenience. The band diagram, plotted in Fig. 3.2(b), shows that the upper band exhibits negative dispersion, regardless of the value of α . On unit cell n , the voltages on the two sites are v_n^a (to the right of the C_a capacitor) and v_n^b (to the right of the C_b capacitor). Also, we let i_n^a (i_n^b) denote the current through the inductor to the right of the C_a (C_b) capacitor.

Let C_a , C_b , and L be constants, and take a harmonic mode with angular frequency ω . For $n > 1$, we apply Kirchhoff's laws to the inductors and capacitors, with the $\exp(i\omega t)$ phasor convention, and obtain

$$i\omega L i_n^a + v_n^a = 0 \quad (3.2)$$

$$i\omega L i_n^b + v_n^b = 0 \quad (3.3)$$

$$-i\omega C_a (v_n^a - v_{n-1}^b) + i_n^a + i\omega C_b (v_n^b - v_n^a) = 0 \quad (3.4)$$

$$-i\omega C_b (v_n^b - v_n^a) + i_n^b + i\omega C_a (v_{n+1}^a - v_n^b) = 0. \quad (3.5)$$

Combining these to eliminate i_n^a and i_n^b yields the following pair of coupled equations:

$$v_{n-1}^b - \frac{1}{\alpha} v_n^a + \frac{1}{\alpha} v_n^b = (1 - \Omega^2) v_n^a \quad (3.6)$$

$$\frac{1}{\alpha} v_n^a - \frac{1}{\alpha} v_n^b + v_{n+1}^a = (1 - \Omega^2) v_n^b, \quad (3.7)$$

where $\Omega^2 \equiv \omega_a^2/\omega^2$, $\omega_a \equiv 1/\sqrt{LC_a}$ and $\alpha \equiv C_a/C_b$.

Suppose we close the circuit by grounding the leftmost and rightmost sites. Consider the left edge (the right edge is handled similarly). There, the Kirchhoff equations

Figure 3.1: Design and implementation of SSH-like left-handed transmission line. **a** Schematic of the transmission line circuit with alternating capacitances: linear capacitors with capacitance C_a , and back-to-back varactors with nonlinear capacitance C_b . The capacitances act like hoppings in a nonlinear Su-Schrieffer-Heeger (SSH) model. An input voltage is applied at points A or B to probe the topologically trivial or nontrivial lattice. **b** Photograph of the printed circuit board. **c** Capacitance C_b versus bias voltage. Dots are calculated from varactor manufacturer data, and the solid curve is the fit based on Eq. (3.17). **d, e** Calculated eigenfrequencies of a finite closed linear circuit, versus the capacitance ratio $\alpha = C_a/C_b$. The characteristic frequency is $f_a = \omega_a/2\pi \approx 19$ MHz, and the lattice has 40 sites. Two cases are shown: **d** C_a -type capacitors at the edge, for which the $\alpha > 1$ gap is topologically nontrivial; **e** C_b -type capacitors at the edge, for which the $\alpha > 1$ gap is trivial. Red dotted curves indicate the band-edge frequencies $f_a/\sqrt{2}$ and $\sqrt{\alpha/2} f_a$. Blue dashes indicate the operating regime of the nonlinear circuit, with $\alpha \approx 1.3$ in the linear (low-voltage) limit and α effectively increasing with voltage amplitude.

simplify to

$$i\omega Li_1^a + v_n^a = 0 \quad (3.8)$$

$$-i\omega C_a v_1^a + i_1^a + i\omega C_b (v_1^b - v_1^a) = 0, \quad (3.9)$$

resulting in the boundary equation

$$-\frac{1}{\alpha} v_1^a + \frac{1}{\alpha} v_1^b = (1 - \Omega^2) v_1^a. \quad (3.10)$$

Figure 3.2: **a** Schematic of the left-handed transmission line circuit. **b** Band diagram for the infinite lattice. Results are shown for $\alpha = 0.5$ (solid curves) and 3.0 (dashes).

Hence, we arrive at the modified SSH problem:

$$\begin{pmatrix} -\frac{1}{\alpha} & \frac{1}{\alpha} & & & \\ \frac{1}{\alpha} & -\frac{1}{\alpha} & 1 & & \\ & 1 & -\frac{1}{\alpha} & \frac{1}{\alpha} & \\ & & \frac{1}{\alpha} & -\frac{1}{\alpha} & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} v_1^a \\ v_1^b \\ v_2^a \\ v_2^b \\ \vdots \end{pmatrix} = (1 - \Omega^2) \begin{pmatrix} v_1^a \\ v_1^b \\ v_2^a \\ v_2^b \\ \vdots \end{pmatrix}, \quad (3.11)$$

or with labeling in Fig. 3.1(a):

$$\left(\mathcal{H} - \frac{1}{\alpha} \right) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{pmatrix} = \left(1 - \frac{\omega_a^2}{\omega^2} \right) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{pmatrix}, \quad (3.12)$$

where v_k denotes the complex voltage on site k . The matrix $\mathcal{H}$ has the form of the

SSH Hamiltonian:

$$\mathcal{H} = \begin{pmatrix} 0 & \frac{1}{\alpha} & & & \\ \frac{1}{\alpha} & 0 & 1 & & \\ & 1 & 0 & \frac{1}{\alpha} & \\ & & \frac{1}{\alpha} & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \quad (3.13)$$

Thus, the eigenfrequency modes of the circuit have a one-to-one correspondence with the SSH eigenstates.

This is the configuration referred to as the “nontrivial lattice”. The “trivial lattice” can be described by removing the first row and column of the matrix. In either case, the edges of the band gap are

$$\omega_{\pm} = \left\{ \sqrt{\frac{\alpha}{2}} \omega_a \quad \text{or} \quad \frac{\omega_a}{\sqrt{2}} \right\}, \quad (3.14)$$

and the angular frequency of the edge state is

$$\omega_{\text{es}} = \sqrt{\alpha/(1+\alpha)} \omega_a. \quad (3.15)$$

The band diagram for the linear closed circuit is shown in Fig. 3.1(d). The lack of an upper cutoff frequency is a characteristic of left-handed transmission lines [166]. There is a bandgap in the range $\omega_a/\sqrt{2} < \omega < \sqrt{\alpha/2} \omega_a$. For $\alpha > 1$, the bandgap contains edge states, which are zero-eigenvalue eigenstates of $\mathcal{H}$ that can be characterized via a topological invariant derived from the Zak phase [90]. The edge state’s angular frequency is

$$\omega_{\text{es}} = \sqrt{\alpha/(1+\alpha)} \omega_a. \quad (3.16)$$

Note that the edge state are not at zero frequency, nor do they lie at precisely the

middle of the bandgap; this is due to the aforementioned mapping from the circuit equations to the SSH model—specifically, the fact that ω is not the eigenvalue in Eq. (3.12).

For $\alpha < 1$, there is a finite bandgap below $\omega_a/\sqrt{2}$, which is topologically trivial and contains no edge states. If we swap the two types of capacitors, so that the C_b -type capacitors are the ones at the edge, then the $\alpha > 1$ bandgap is trivial and the $\alpha < 1$ bandgap nontrivial, as shown in Fig. 3.1(e).

Next, consider a nonlinear circuit with each C_b capacitor consisting of a pair of back-to-back varactors. The nonlinear capacitance C_b decreases with the magnitude of the bias voltage (the voltage between the end-points of the capacitor), as shown in Fig. 3.1(c). For theoretical analyses, it is convenient to model this nonlinearity by

$$\alpha_{\text{nl}}(t) \approx A + B [\Delta V(t)]^2, \quad (3.17)$$

where $\alpha_{\text{nl}}(t) \equiv C_a/C_b(t)$, and $\Delta V(t)$ is the bias voltage. The key feature of the nonlinearity is that at higher voltages, the effective value of α increases. Depending on the chosen boundary conditions, this drives the circuit deeper into the topologically trivial or nontrivial regime.

3.2.2 Circuit fabrication

The nonlinear transmission line was implemented on a PCB (Seeed Tech. Co), with each nonlinear capacitor consisting of a pair of back-to-back varactors (Skyworks Solutions, SMV1253-004LF). The transmission line, as fabricated, is topologically nontrivial, as shown in Fig. 3.1(a). To probe the trivial circuit, we use a switch to add one sublattice unit cell at the rightmost end of the transmission line, and disconnect the leftmost C_a and L in Fig. 3.1(a). This yields a nontrivial circuit of same length, with the C_a and C_b capacitors swapped.

A function generator (Tektronix AFG3102C) supplies the continuous-wave sinusoidal input voltage, and the voltages on successive lattice sites, $k \geq 1$, are measured by an oscilloscope (Rohde & Schwarz RTE1024) in high-impedance mode. Numerical results were obtained using the SPICE circuit simulator.

3.3 Results

3.3.1 Experimental results

The implemented nonlinear transmission line, shown Fig. 3.1(b), contains a total of 40 sites, or 20 unit cells. The linear circuit elements have $L = 1.5 \mu\text{H}$ and $C_a = 47 \text{ pF}$, so that $\omega_a/2\pi \approx 19 \text{ MHz}$. By fitting Eq. (3.17) to manufacturer data for the varactors at low bias voltages, we obtain $A = 1.32$ and $B = 0.51 \text{ V}^{-2}$ (thus, in the linear limit, $\alpha \approx 1.3 > 1$). The fitted capacitance-voltage relation is shown in Fig. 3.1(c).

We supply a continuous-wave sinusoidal input voltage signal, with tunable frequency f_{in} and amplitude V_{in} , to either of the points labelled A and B in Fig. 3.1(a). This allows us to study the cases corresponding to Fig. 3.1(d),(e), which we refer to as the “nontrivial” and “trivial” lattices respectively (see Methods). In both cases, the input site is denoted as $k = 0$.

A typical set of measurement results is shown in Fig. 3.3(a)–(c), for $f_{\text{in}} = 16 \text{ MHz}$ and $V_{\text{in}} = 2.5 \text{ V}$. On each site k , the spectrum of the voltage signal is shown in Fig. 3.3(c), with prominent peaks at odd harmonics (f_{in} , $3f_{\text{in}}$, $5f_{\text{in}}$, etc.); even harmonics are suppressed due to the symmetry of the capacitance-voltage relation [65]. Focusing on the first and third harmonics, we define the respective peak values as $|v_k^f|$ and $|v_k^{3f}|$, and use these to plot Fig. 3.3(a)–(b).

These experimental data agree well with results obtained from a SPICE circuit simulator. The simulations reproduce the basic features of the experimental results, though the results are not in exact agreement, probably due to imperfections in the

Figure 3.3: Experimental observation of harmonic generation in the nonlinear circuit. **a, b** Magnitude of the first- and third-harmonic voltage signals measured at different lattice sites, for **(a)** the nontrivial lattice, which has an SSH-like edge state in the linear limit, and **(b)** the trivial lattice, which has no edge state in the linear limit. The sinusoidal input signal, applied at the lattice edge (site 0), has frequency $f_{\text{in}} = 16$ MHz and amplitude $V_{\text{in}} = 2.5$ V. **c** Measured spectrum at site 3 for the nontrivial lattice corresponding to subfigure **a**. **d, e** Plot of the inverse participation ratio (IPR) versus input frequency f_{in} and input voltage amplitude V_{in} , calculated from experimental measurements of the first-harmonic signal in the **(d)** nontrivial lattice and **(e)** trivial lattice. Here, f_{in} is measured in steps of 0.2 MHz, and V_{in} in steps of 0.1 V.

circuit components. Fig. 3.4 shows the simulated on-site voltage amplitudes for the

Figure 3.4: Simulated on-site voltage amplitudes for the first-harmonic signal (orange circles) and third-harmonic signal (blue squares). The frequency of the input is $f_{\text{in}} = 16.4$ MHz. **a**, **b** Nontrivial and trivial lattices with $V_{\text{in}} = 2.5$ V. **c**, **d** Nontrivial and trivial lattices with $V_{\text{in}} = 3.4$ V.

first- and third-harmonic signals. These are extracted from the simulation results in a manner similar to the experiment: after the simulation reaches steady-state, we take a time-dependent sample, Fourier transform, and extract the peak heights. To obtain simulation results matching the experimental results, we find that it is necessary to apply a higher input voltage amplitude than in the experiment, $V_{\text{in}} \approx 3.4$ V. The results are shown in Fig. 3.4(c)–(d). Similar to the experiment, the first-harmonic mode in the nontrivial lattice decays away from the edge, reaching values much lower than in the trivial lattice.

Fig. 3.5 shows the simulated bias voltage amplitudes on the nonlinear capacitors, which were not measured in the experiment. To obtain this data from the simulations, we extract the time-dependent bias voltage samples (i.e., the time-dependent voltages

Figure 3.5: Simulated squared bias voltage amplitudes on the nonlinear capacitors, for the first-harmonic signal (orange circles) and higher-harmonic signals (purple squares). The frequency of the input is $f_{\text{in}} = 16.4$ MHz. The higher-harmonic data is obtained by summing over the third, fifth, and seventh harmonic data (further harmonics are negligible). **a, b** Nontrivial and trivial lattices with $V_{\text{in}} = 2.5$ V. **c, d** Nontrivial and trivial lattices with $V_{\text{in}} = 3.4$ V.

between the ports of the nonlinear capacitors), Fourier transform, and extract the peak heights. Fig. 3.5 shows a comparison between the first-harmonic contribution (orange circles) and the higher-harmonic contributions (purple squares). In particular, in Fig. 3.5(c), which correspond to the voltage plot of Fig. 3.4c, the higher-harmonic signals are found to increasingly dominate the nonlinearity as we go deeper into the lattice.

From Fig. 3.3(a)–(b), we see that the nontrivial and trivial lattices exhibit very different behaviours for both the first- and third-harmonic signals. First, consider the first-harmonic signal. In both lattices, there is an exponential decay away from the edge, but the decay is sharper in the nontrivial lattice, which may be attributed to the enhanced intensity arising from the coupling of the input signal to the topological

edge state. As a quantitative measure of the localization of the first-harmonic signal, Fig. 3.3(d),(e) shows the inverse participation ratio (IPR) $\sum_k |v_k^f|^4 / (\sum_k |v_k^f|^2)^2$; a larger IPR corresponds to a more localized profile [169]. We see that the IPR is substantially larger in the nontrivial lattice than in the trivial lattice, over a broad range of f_{in} and V_{in} . The strong difference in localization is a key signature of nonlinearity: in the linear regime, a driving voltage on the edges of the nontrivial and trivial lattices would produce different overall amplitudes, but the same exponential decay profile.

This can be shown by considering the response of the circuit to a harmonic voltage source. Instead of grounding the left edge, we apply an input voltage of amplitude V_{in} and frequency ω . Then equation (3.10) is replaced by

$$V_{\text{in}} - \frac{1}{\alpha} v_1^a + \frac{1}{\alpha} v_1^b = (1 - \Omega^2) v_1^a, \quad (3.18)$$

and the eigenvalue equation (3.11) is replaced by an inhomogenous equation.

Numerical solutions for this problem are shown in Fig. 3.6. Fig. 3.6(a)–(b) shows that the voltage amplitude in the nontrivial lattice is resonantly enhanced when ω matches ω_{es} . The trivial lattice, on the other hand, does not exhibit a resonant enhancement. However, when we plot the voltage distributions, they have the same decay constant, as shown in Fig. 3.6(c). This is due to the fact that they have the same *bulk* Hamiltonians.

Fig. 3.6(d) shows the modal intensity (the sum of squared voltage amplitudes over the lattice) at resonance, versus the α parameter, for a nontrivial lattice. In this plot, the input frequency for each value of α is adjusted to the edge state frequency given by equation (3.15). With increasing α , the bandgap becomes larger (i.e., the lattice moves deeper into the topologically nontrivial phase); accordingly, the edge state is more strongly confined, and responds more strongly to the resonant excitation. The

modal intensity scales exponentially with the bandgap size.

It is interesting to note that the region of enhanced IPR, shown in Fig. 3.3(d), closely resembles the nontrivial bandgap in Fig. 3.1(d).

We can also see from Fig. 3.3(a),(c) that strong higher-harmonic signals are present in the nontrivial lattice. Moreover, Fig. 3.3(a) indicates that the third-harmonic signal is extended, not localized to the edge. To understand this in more detail, we define

$$\chi = \left\langle \left| v_k^{3f} \right|^2 \right\rangle / V_{\text{in}}^2, \quad (3.19)$$

which quantifies the intensity of the third-harmonic signal relative to the input intensity at the first harmonic. Here, $\langle \dots \rangle$ denoting an average over the first ten lattice sites. Fig. 3.8(a),(b) plots the variation of χ with f_{in} and V_{in} . In the nontrivial circuit, the maximum value of the normalized intensity is $\chi \approx 2.5$ for $f_{\text{in}} \sim 16$ MHz and $1 \text{ V} \lesssim V_{\text{in}} \lesssim 4 \text{ V}$. The fact that χ peaks over a relatively narrow frequency range, as shown in Fig. 3.8(a), may be a finite-size effect: the high-frequency modes of the lattice form discrete sub-bands due to the finite lattice size [see Fig. 3.1(d),(e)].

In computer simulations, we obtained a similar maximum value of $\chi \approx 2.4$ for the nontrivial lattice, whereas a comparable left-handed NLTL of the usual design (containing only identical nonlinear capacitances) has maximum $\chi \approx 0.47$. In the conventional left-handed NLTL, all the linear C_a capacitors are replaced with nonlinear C_b capacitors (i.e., the lattice is no longer dimerized). In the linear limit, the conventional left-handed NLTL with $C_b = 35 \text{ pF}$ is a high-pass filter with a Bragg cutoff frequency of 11 MHz.

Fig. 3.7 plots the simulation results for the normalized third-harmonic intensity χ , versus the input parameters f_{in} and input voltage V_{in} . The simulation results for the SSH-like lattice, shown in Fig. 3.7(a), are similar to the experimental results; in particular, the maximum value of χ is $\gtrsim 2.4$, comparable to the experimentally-

Figure 3.6: Response of the linear circuit to an external voltage, calculated numerically from the circuit equations with the parameters $f_a = 19$ MHz and $C_a = 47$ pF. **a–b** Resonant response of (a) the amplitude of the voltage on the first site (to the right of the leftmost capacitor), and (b) the amplitude of the stored charge on the first (leftmost) capacitor, versus source frequency $f_{\text{in}} = \omega/2\pi$, for $\alpha = 1.5$. The resonance frequency predicted by equation (3.15) is indicated by the horizontal dashed line. **c** Spatial distribution of the voltage amplitudes for $f_{\text{in}} = 14.75$ MHz and $\alpha = 1.5$. **d** Intensity of the nontrivial lattice's resonant mode, as given by the value of $\sum_k |v_k|^2$ (where $|v_k|$ is the voltage amplitude at site k) at resonance, versus the α parameter. For each value of α , the input voltage has amplitude 1 V and frequency given by equation (3.15).

Figure 3.7: Plot of the normalized third-harmonic intensity χ versus input frequency f_{in} and input voltage V_{in} , for (a) the SSH-like lattice in its topologically nontrivial configuration, and (b) a conventional left-handed nonlinear transmission line (NLTL) with identical nonlinear capacitors. The figure of merit χ is defined as the mean squared third-harmonic amplitude on the first 10 sites relative to the squared input voltage amplitude. In the SSH case, we obtain $\chi \gtrsim 2.4$, which is comparable to the experimental result of $\chi \approx 2.5$. In the conventional NLTL, we observe $\chi < 0.47$ throughout the parameter range.

obtained maximum value $\chi \approx 2.5$. By contrast, Fig. 3.7(b) shows that the conventional NLTL exhibits no comparable enhancement of the third-harmonic signal intensity, with $\chi < 0.47$ throughout the entire parameter regime we investigated. Hence, the introduction of the topological edge mode has contributed to a five-fold increase in the intensity of the generated third-harmonic signal.

The trivial lattice exhibits a much weaker third-harmonic signal. As indicated in Fig. 3.8(c), for certain choices of f_{in} and V_{in} , the value of χ in the nontrivial lattice is 200 times that in the trivial lattice. Fig. 3.8(d) plots the normalized third-harmonic signal intensities versus the site index k , showing that they do not decay exponentially away from the edge. In the nontrivial lattice, the normalized third-harmonic signal increases with V_{in} (i.e., stronger nonlinearity).

Figure 3.8: Measured third-harmonic signal intensities in the nonlinear circuit. **a**, **b** Normalized third-harmonic signal intensity χ versus input frequency f_{in} and input voltage V_{in} , for the (a) nontrivial lattice and (b) trivial lattice. Here, χ is derived from experimental data using the definition (3.19). **c** Ratio of the trivial and nontrivial intensities, $\chi_{\text{nontrivial}}/\chi_{\text{trivial}}$, within the region indicated by boxes in **a** and **b**. **d** Measured third-harmonic intensities (normalized to the input signal) at different sites, for the three sets of input parameters indicated by stars in **c**: $V_{\text{in}} = 0.5$ V, 1.5 V, and 2.5 V, with fixed $f_{\text{in}} = 16$ MHz.

3.4 Discussion

The results presented in the previous section point to a complex interplay between the topological edge state and higher-harmonic modes in the SSH-like NLTL. When a topological edge state exists in the linear lattice, it can be excited by an input signal at frequencies matching the bandgap of the linear lattice. The importance of the edge state is evident from the comparisons between the topologically trivial and nontrivial lattices (Fig. 3.3 and Fig. 3.8). Note also that when the excitation frequency lies outside the linear bandgap, the two lattices behave similarly and the harmonic generation is relatively weak.

In the topologically nontrivial lattice, the resonant excitation generates third- and higher-harmonic signals that penetrate deep into the lattice, unlike the first-harmonic mode which is localized to the edge. Away from the edge, the higher-harmonic signals become stronger than the first harmonic, and hence dominate the effective value of the nonlinear α parameter. In the linear lattice, α is the parameter that drives the topological transition, and increasing α leads to a larger bandgap and hence a more confined edge state. In the nonlinear regime, Fig. 3.3 shows an order-of-magnitude increase in the third-harmonic signal amplitude in the nontrivial lattice, relative to the trivial lattice; this implies an effective increase in α , and indeed we see that the first-harmonic mode profile is more strongly localized. A more localized edge state, in turn, produces a stronger response to an input signal.

We are able to show that propagating waves can be self-consistently realized for higher ($m \geq 3$) harmonics in the presence of non-linearity, even if the fundamental ($m = 1$) mode only has decaying solutions. The first-harmonic mode is localized to the edge, with localization length decreasing with $\langle \alpha_n \rangle$ in a manner similar to the linear SSH-like lattice. The generation of the higher-harmonic signals occurs mainly near the edge of the lattice, where the first-harmonic mode is largest. The nonlinearity-

induced harmonic generation is aided by the well-known fact that the SSH edge state changes sign in each unit cell, corresponding to the fact that the gap closing in the SSH model takes place at the corner of the Brillouin zone [79]. This feature increases the bias voltages across the nonlinear capacitors, which can thus exceed the values of the voltages at individual sites.

The input signal can also be applied to the middle of the lattice. In this context, it is interesting to note that when we choose to excite a single site in the bulk of an SSH-like lattice, the sections to either side of the excitation have different topological phases: either trivial on the left and nontrivial on the right, or vice versa, depending on the two possible choices of excitation site. If the source impedance is sufficiently low, the effect is similar to exciting independent chains to the left and right; thus, the enhanced higher-harmonic signal is preferentially emitted toward the topologically nontrivial side.

First, consider a low-impedance voltage source. In this case, the voltage on the input site is rigidly determined by the voltage source, so this is similar to exciting two independent transmission lines.

If lattice is uniform (i.e., defect free), once we pick an excitation site, the lattice sections to the left and right of the excitation site necessarily have different topological phases. For instance, for $\alpha > 1$, if there is a C_a capacitor to the right of the excitation site, then the section on the right is nontrivial and the section on the left is trivial. Under such circumstances, reasoning from the behavior of the circuit under edge excitation, we expect the higher-harmonic signal to be emitted asymmetrically: a strong higher-harmonic signal should propagate to the topologically nontrivial side, with a weak higher-harmonic signal on the trivial side. This prediction is verified by the SPICE simulation results plotted in Fig. 3.9.

When the input impedance of the voltage source is high, the behavior is less clear-cut, as both the first- and higher-harmonic signals can easily cross the excitation

Figure 3.9: Voltage amplitude profiles for the first and third harmonic signals, from SPICE simulation on a 40-site NLTL with (a) site 20 excited, and (b) site 21 excited. The choice of excitation site partitions the lattice into topologically trivial and nontrivial sections. The input signal has voltage amplitude $V_{\text{in}} = 3$ V, frequency $f_{\text{in}} = 16$ MHz, and input impedance 1Ω . All NLTL simulation parameters are the same as previous simulation where edge site is excited.

site. In both the topologically trivial and nontrivial lattices, the higher-harmonic modes are in-band; thus, any higher-harmonic signal that is generated can propagate to either the trivial or nontrivial side. At each frequency, the dominant direction of higher-harmonic emission will depend on the availability of circuit modes, based on finite-size effects, in each lattice section.

The presence of higher-harmonic signals distinguishes the present system from previous studies of nonlinear topological edge states, which were based on nonlinear self-modulation at a single harmonic. For instance, in a nonlinear SSH lattice where the coupling depends on the local intensity of a single mode, soliton-like edge states with anomalous mode profiles were predicted [70], and subsequently verified using a NLTL-like circuit [69]. That circuit, unlike ours, had narrow frequency bands and thus did not support propagating higher-harmonic modes. Topological solitons based on nonlinear self-modulation are also predicted to exist in higher-dimensional lattices [66, 122, 67, 153–155]. In our case, the effective value of α away from the edge is dominated by the higher-harmonic signals; from the point of view of the first-harmonic

mode, these act as a nonlocal nonlinearity, driving the entire lattice deeper into the topologically nontrivial regime, not just the sites with large first-harmonic intensity.

This work opens the door to the application of topological edge states for enhancing harmonic generation, not just in transmission line circuits, but also a variety of other interesting systems. These include two-dimensional electronic lattices, where topological edge states have already been observed in the linear regime [144], and the unidirectional nature of the edge states may be even more beneficial for frequency-mixing [156]. Higher dimensional circuit lattices may possess different thresholds for bulk propagation in different directions, with an extreme generalization being that of a corner mode circuit constructed in Ref. [44]. Electronic circuits incorporating amplifiers and resistances may also be able to explore behaviours analogous to topological lasers [170–174], combining topological states with both nonlinearity and non-Hermiticity. Finally, circuits containing varactors that are explicitly time-modulated may be suitable for generating synthetic dimensions to realize topological features in higher dimensions [56, 57, 175, 62, 92, 176].

Chapter 4

Effects of Non-Hermiticity on Su-Schrieffer-Heeger Defect States

Some of the contents of this chapter were previously published in the journal Physical Review B [177]. Wherever figures and materials from other sources have been taken, the authors have been given due credit.

As discussed in chapter 2 and 3, the SSH model is one of the simplest topological models, and a useful platform for studying the interplay between topological phases and other physical phenomena. Unlike the nonlinear SSH model discussed in the last chapter, the present chapter examines a different class of physical phenomena, associated with the effects of non-Hermiticity or energy non-conservation.

The complex Su-Schrieffer-Heeger (cSSH) model [78] is a non-Hermitian extension of the Su-Schrieffer-Heeger (SSH) model [79]. This model has been the subject of recent interest in experiments [80, 135, 81, 178, 179] as a simple testing-ground for the interaction of topological states with *non-Hermiticity*—i.e., the presence of loss and/or gain in the underlying medium [73, 180]. The standard topological invariants used to characterize topological states of matter [1, 90], including the SSH model, assume Hermiticity; for instance, Hermiticity guarantees the existence of a well-defined inner product, which is used to calculate the Zak phases [181] for characterizing the SSH

model. Non-Hermitian generalizations of topological concepts, such as the bulk-edge correspondence principle, are thus of significant theoretical interest [171, 182, 82, 183–188].

Moreover, non-Hermitian variants of topological states may have applications in photonics, where topological protection can be implemented by lattice engineering [40, 38, 43, 189, 190], and non-Hermiticity can be introduced by introducing optical loss and/or gain to the optical medium [191, 192]. The robustness of topological modes may be usefully exploited in amplifiers [129, 156], lasers [78, 174, 193, 193, 178, 179, 171, 194], and other non-Hermitian photonic devices.

Previous studies of the cSSH model, starting with the work of Schomerus [78], have focused on the existence of a defect state that is exponentially localized to an interface between different cSSH domains. In the Hermitian limit (no gain or loss), this defect state is explicitly continuable to the well-known topological mid-gap defect state of the SSH model [79]. In the non-Hermitian case (gain and loss on alternating lattice sites), the energy of the defect state can acquire a nonzero imaginary part, but the real part remains pinned to the mid-gap value. If the defect configuration is chosen appropriately, the defect state can have a larger amplification rate than any of the bulk states [78]; this has been demonstrated using microwave resonators [80] and lasers [179, 178]. Alternatively, if the gain and loss are distributed in a parity/time-reversal ($\mathcal{PT}$) symmetric pattern [73], the bulk and defect state energies can be purely real [135, 81].

These studies did not, however, look into whether SSH defect states always have a counterpart in the cSSH model, and, conversely, whether the defect states of the cSSH model are always SSH-like. This is a noteworthy omission because non-Hermitian states are known to be able to exhibit behaviors that have no Hermitian analogue. $\mathcal{PT}$ -symmetric dimer eigenstates, for example, can exhibit spontaneous $\mathcal{PT}$ symmetry breaking [195], while some non-Hermitian lattices have been shown to support defect

states that seem to be topological but have no evident Hermitian counterpart [2, 84, 83, 85, 82].

In this chapter, we analyze the effect of non-Hermiticity on cSSH defect states. We find that two interesting things can happen to the SSH-like defect state as the gain and loss magnitude is increased. First, the defect state can disappear via a divergence in its localization length, which corresponds to the merging of the defect state energy into the complex continuum of bulk energies. Second, the SSH-like defect state can interact with a second defect state that emerges from the continuum. Both of these states satisfy a composite sublattice/time-reversal ($\mathcal{ST}$) symmetry, which pins the real parts of their energies to zero, and is the non-Hermitian counterpart of the $\mathcal{S}$ symmetry that pins the energy of the SSH mid-gap defect state to zero. The two states can coalesce in a spontaneous $\mathcal{ST}$ -breaking transition (an exceptional point [180]), breaking apart into two $\mathcal{ST}$ -broken defect states. The latter are continuable to the *non*-topologically-protected defect states of the SSH model, which have hitherto been ignored but can exist as well in the cSSH model. Both methods of destabilizing the SSH-like “mid-gap” defect state require $\mathcal{PT}$ symmetry to be spontaneously broken in the bulk bandstructure.

We thus find that the cSSH chain has two anomalous defect state phases not present in the SSH model: (i) a phase with two “mid-gap” states localized to the domain wall, rather than one, and (ii) a phase with two “non-mid-gap” states but no “mid-gap” state. Phase (ii) includes the special case where the inter-site couplings are uniform (so that the cSSH lattice reduces to a lattice of gain/loss dimers with a defect in the gain/loss pattern); in this limit, the two “non-mid-gap” states appear abruptly when the gain/loss magnitude is increased above a certain nonzero threshold, similar to the “intrinsically non-Hermitian” defect states that have previously been seen in other non-Hermitian lattice models [2, 84, 85, 82].

4.1 Complex Su-Schrieffer-Heeger (cSSH) model

The bulk cSSH lattice, depicted in Fig. 4.1(a), consists of a chain of dimers with alternating coupling strengths $t \pm \delta$ between adjacent a and b sites, and alternating on-site gain/loss represented by imaginary on-site potentials $\pm i\gamma$. The gain/loss averages to zero over the lattice. The bulk Hamiltonian is

$$\begin{aligned} \mathcal{H}_{\text{bulk}} = & \sum_n \left[(t + \delta) |a_n\rangle \langle b_n| + (t - \delta) |a_n\rangle \langle b_{n-1}| + \text{h.c.} \right] \\ & + \sum_n \left[i\gamma |a_n\rangle \langle a_n| - i\gamma |b_n\rangle \langle b_n| \right], \end{aligned} \quad (4.1)$$

where $|a_n\rangle$ and $|b_n\rangle$ denotes the state on site a and b , respectively, in the n -th unit cell. The parameters t , δ , and γ are all real; we set $t = 1$ as the energy unit. When $\gamma = 0$, $\mathcal{H}_{\text{bulk}}$ reduces to the SSH Hamiltonian [79]. Performing a Fourier decomposition yields the reduced Hamiltonian

$$H_k = \begin{pmatrix} i\gamma & W_k \\ W_{-k} & -i\gamma \end{pmatrix}, \quad (4.2)$$

where $W_k = (1 + \delta) + (1 - \delta)e^{-ik}$ and k is the crystal momentum, with the unit of length chosen so that the lattice constant is unity. The resulting eigenvalue spectrum,

$$E_{k,\pm} = \pm \sqrt{2(1 + \delta^2) + 2(1 - \delta^2) \cos k - \gamma^2}, \quad (4.3)$$

is shown in Fig. 4.1(c)–(e) for different values of γ .

4.2 Symmetries of the cSSH model

Consider a one-dimensional discrete lattice with N_s sites in each unit cell. Let ψ_n be a column vector consisting of the N_s annihilation operators in unit cell n . In this general context, we can define the time-reversal operator ($\mathcal{T}$), charge-conjugation

Figure 4.1: (a) The bulk cSSH chain. Sites with gain ($i\gamma$) and loss ($-i\gamma$) are respectively indicated by red and blue circles, and couplings $1 + \delta$ and $1 - \delta$ are indicated by double and single horizontal lines. (b) Phase diagram of the bulk cSSH chain, which contains a $\mathcal{PT}$ -unbroken phase where all Bloch state energies are real (yellow), an intermediate phase with both real and imaginary energies (white), and an $\mathcal{ST}$ -unbroken phase with purely imaginary energies (blue). (c)–(e) Complex bulk band structures for the cSSH chain, for the parameters indicated in (b) by the points labelled c , d , and e respectively: $\delta = 0.5$ and (c) $\gamma = 0$, (d) $\gamma = 1.5$, and (e) $\gamma = 2.5$. Solid (dashed) curves show the real (imaginary) part of the eigenenergy E . (f)–(h) cSSH chains with different lattice defects (vertical dashes). (f) $\mathcal{P}$ -preserving defect. (g) $\mathcal{PT}$ -preserving and $\mathcal{P}$ -breaking defect. (h) A defect that breaks both $\mathcal{P}$ and $\mathcal{PT}$, while reversing γ and δ across the interface.

operator ($\mathcal{C}$, also called the particle-hole operator), sublattice operator ($\mathcal{S}$, also called the chiral operator), and parity operator ($\mathcal{P}$) as follows [1]:

$$\begin{aligned}\mathcal{T}\psi_n\mathcal{T}^{-1} &= U_{\mathcal{T}}\psi_n, & \mathcal{C}\psi_n\mathcal{C}^{-1} &= U_{\mathcal{C}}^*\psi_n^\dagger, \\ \mathcal{S}\psi_n\mathcal{S}^{-1} &= U_{\mathcal{S}}\psi_n, & \mathcal{P}\psi_n\mathcal{P}^{-1} &= U_{\mathcal{P}}\psi_{-n}.\end{aligned}\tag{4.4}$$

Here, $U_{\mathcal{T},\mathcal{C},\mathcal{S},\mathcal{P}}$ are unitary matrices, and the parity operation is taken around the origin.

A system is said to be $\mathcal{T}$, $\mathcal{P}$, and $\mathcal{S}$ symmetric if its lattice Hamiltonian $\mathcal{H}$ satisfies, respectively,

$$[\mathcal{T}, \mathcal{H}] = 0, \quad [\mathcal{P}, \mathcal{H}] = 0, \quad \{\mathcal{S}, \mathcal{H}\} = 0.\tag{4.5}$$

These definitions apply to translationally invariant lattices as well as lattices with defects.

The bulk SSH model given by the first line of Eq. (4.1) satisfies all three symmetries, with the matrix representations

$$U_{\mathcal{T}} = I, \quad U_{\mathcal{P}} = \sigma_x, \quad U_{\mathcal{S}} = \sigma_z.\tag{4.6}$$

Using these same matrix representations, the bulk cSSH model breaks $\mathcal{T}$, $\mathcal{S}$, and $\mathcal{P}$ individually, but preserves $\mathcal{PT}$ and $\mathcal{ST}$.

For any infinite translationally invariant one-dimensional lattice, the Hamiltonian has the form

$$\mathcal{H} = \sum_k (\psi_k^\dagger)^T H_k \psi_k,\tag{4.7}$$

where $\psi_k = N^{-1/2} \sum_n \exp(-iknd) \psi_n$. For simplicity, we set the lattice constant d to unity. In terms of these Bloch-state operators, the above symmetry operators have

the following form:

$$\begin{aligned}\mathcal{T}\psi_k\mathcal{T}^{-1} &= U_{\mathcal{T}}\psi_{-k}, & \mathcal{C}\psi_k\mathcal{C}^{-1} &= U_{\mathcal{C}}^*\psi_{-k}^\dagger, \\ \mathcal{S}\psi_k\mathcal{S}^{-1} &= U_{\mathcal{S}}\psi_k, & \mathcal{P}\psi_k\mathcal{P}^{-1} &= U_{\mathcal{P}}\psi_{-k}.\end{aligned}\tag{4.8}$$

Thus,

$$\mathcal{T}\mathcal{H}\mathcal{T}^{-1} = \sum_k (\psi_k^\dagger)^T (U_{\mathcal{T}}^\dagger H_{-k}^* U_{\mathcal{T}}) \psi_k, \tag{4.9}$$

$$\mathcal{C}\mathcal{H}\mathcal{C}^{-1} = \text{Tr}(\mathcal{H}) + \sum_k (\psi_k^\dagger)^T (\pm U_{\mathcal{C}}^\dagger H_{-k}^T U_{\mathcal{C}}) \psi_k, \tag{4.10}$$

$$\mathcal{S}\mathcal{H}\mathcal{S}^{-1} = \sum_k (\psi_k^\dagger)^T (U_{\mathcal{S}}^\dagger H_k U_{\mathcal{S}}) \psi_k, \tag{4.11}$$

$$\mathcal{P}\mathcal{H}\mathcal{P}^{-1} = \sum_k (\psi_k^\dagger)^T (U_{\mathcal{P}}^\dagger H_{-k} U_{\mathcal{P}}) \psi_k. \tag{4.12}$$

In Eq. (4.10), $\pm$ hold for bosons and fermions respectively. Notably, the relation $\mathcal{C}\mathcal{H}\mathcal{C}^{-1} = \text{Tr}(\mathcal{H}) \pm \mathcal{H}$ is always satisfied with $U_{\mathcal{C}} = I$ (identity matrix); hence, $H_{-k}^T = H_k$, which implies the general eigenenergy pair (E_k, E_{-k}) at each crystal momentum k , i.e., the band structure is mirror-symmetric with respect to $k = 0$.

For the bulk SSH lattice, Eqs. (4.9)–(4.12) lead respectively to the relations

$$H_{-k}^* = H_k \tag{4.13}$$

$$\sigma_x H_{-k} \sigma_x = H_k \tag{4.14}$$

$$\sigma_z H_k \sigma_z = -H_k. \tag{4.15}$$

Hence, the eigenenergies appear in pairs, (E_k, E_{-k}) and $(E_k, -E_k)$, for each k . The band diagram is mirror-symmetric around both $k = 0$ and $E = 0$.

In the bulk cSSH lattice given by the full Eq. (4.1), the $\mathcal{PT}$ and $\mathcal{ST}$ symmetries

respectively imply that

$$\sigma_x H_k^* \sigma_x = H_k \quad (4.16)$$

$$\sigma_z H_{-k}^* \sigma_z = -H_k. \quad (4.17)$$

As a consequence, if there is a bulk state of energy E_k , there must exist a bulk state of energy E_k^* (due to $\mathcal{PT}$), and a bulk state of energy $-E_{-k}^*$ (due to $\mathcal{ST}$). The former ensures that in the $\mathcal{PT}$ -unbroken phase, the bands are purely real. The latter can be combined with the definition of $\mathcal{C}$ to yield $\{\mathcal{H}, \mathcal{STC}\} = \mp 1$, and hence

$$\sigma_z H_k^\dagger \sigma_z = -H_k. \quad (4.18)$$

This ensures that if there is a bulk state of energy E_k , there must exist a bulk state of energy $-E_k^*$. In the $\mathcal{ST}$ -unbroken phase, the bands are purely imaginary [179].

In summary, for $\gamma \neq 0$, the cSSH model breaks $\mathcal{P}$, $\mathcal{S}$, and $\mathcal{T}$ individually, but retains two composite symmetries. First, it is $\mathcal{ST}$ symmetric [78, 179, 135], which implies that if E_k is an eigenvalue, $-E_{-k}^*$ is also an eigenvalue [83, 84]; as discussed above, this symmetry is responsible for the flatness of the real part of the spectrum in the $\mathcal{ST}$ -unbroken regime [83, 84, 196]. Second, it is $\mathcal{PT}$ symmetric, which implies that if E_k is an eigenvalue, E_k^* is also an eigenvalue [73, 191, 192].

The phase diagram for the cSSH chain's bulk bandstructure is shown in Fig. 4.1(b). It is divided into three parts: (i) a $\mathcal{PT}$ -unbroken phase where the Bloch states have real energies for all k , (ii) an intermediate phase where the energies are real for some ranges of k and imaginary elsewhere, and (iii) an $\mathcal{ST}$ -unbroken phase where the energies are purely imaginary for all k .

4.3 Defect states in the cSSH model

It is well known that when the SSH model ($\gamma = 0$) is gapped ($\delta \neq 0$), connecting two domains with opposite signs of δ leads to the emergence of a topological mid-gap defect state localized at the domain wall. The energy of the defect state is pinned to exactly zero by the $\mathcal{S}$ symmetry. In a similar vein, we can consider putting domain walls or defects in the cSSH model ($\gamma \neq 0$). However, due to the presence of both alternating coupling strengths and alternating gain/loss in the cSSH model, there is some leeway in how the defect is defined. Schomerus' original study of the cSSH model [78] used the configuration shown in Fig. 4.1(f), with gain applied to the defect site. In this case, the defect site can also be regarded as a domain wall, and the lattice is symmetric under a $\mathcal{P}$ operation across the defect site, whereas $\mathcal{PT}$ is broken [78, 80, 179]. (Henceforth, we will let the bulk lattices on the two sides of the defect have the same values of δ and γ .) This configuration has the notable feature that the bulk lattices on the two sides are “incompatible”: they have different patterns of alternating δ and γ , and are related to each other by a swap of either δ or γ . Subsequent studies [135, 81, 178] have also considered the configuration shown in Fig. 4.1(g); without gain or loss at the defect site, the overall lattice preserves $\mathcal{PT}$, although $\mathcal{P}$ is broken. The lattices on either side of the defect are “compatible”, in the sense that this configuration could be generated by inserting an additional site and link into a uniform cSSH lattice (similar to the original SSH case).

We will consider a third defect configuration, shown in Fig. 4.1(h). This has also been employed in a recent study by Yuce [197]. Unlike the previous two cases, the domain wall can be regarded as lying between two lattice sites; this defect configuration can be generated by inserting an additional gain site and link with coupling strength 1 into a uniform cSSH lattice, similar to the defect of Fig. 4.1(g) or the SSH model.

The motivation for studying this configuration is that in the $\delta \rightarrow 0$ limit, the lattice takes the form of a gain/loss dimer lattice [83] with a missing-site defect, which can also be regarded as a domain wall lying across a link. Such a lattice supports “non-Hermiticity-induced” defect states, the implications of which will be discussed later. As indicated in Fig. 4.1(h), we label the unit cells by $n = 1, 2, \dots$ to the right of the defect, and $n = -1, -2, \dots$ to the left. The cSSH sublattices on the two sides are related by a simultaneous swap of δ and γ . The defect breaks both $\mathcal{P}$ and $\mathcal{PT}$, but the $\mathcal{ST}$ symmetry of the underlying cSSH lattice is preserved [using the representations (4.6)]. Hence, defect states must either be $\mathcal{ST}$ -unbroken (and hence have purely imaginary eigenenergies), or appear in pairs with eigenenergies $\{E_D, -E_D^*\}$ and eigenstates related to each other by $\mathcal{ST}$:

$$\begin{aligned} \mathcal{H}(\mathcal{ST}|\psi_D\rangle) &= -(\mathcal{ST})\mathcal{H}|\psi_D\rangle \\ &= -E_D^*(\mathcal{ST}|\psi_D\rangle). \end{aligned} \tag{4.19}$$

In the following, we focus on the case of $\gamma \geq 0$; the $\gamma \leq 0$ case is just the time-reversed counterpart, with complex conjugated eigenenergies.

We look for states that are exponentially localized to the defect, having the form

$$|\psi_D\rangle = \sum_{\pm} \sum_{n>0} \lambda^n \left(\alpha_{\pm} |a_{\pm n}\rangle + \beta_{\pm} |b_{\pm n}\rangle \right), \tag{4.20}$$

with undetermined complex constants λ , $\alpha_{\pm}$, and $\beta_{\pm}$, constrained by $|\lambda| < 1$. The lattice is assumed to be infinite, so the sum over n extends to infinity. The energy of the defect state is related to λ by

$$E_D^2 + \gamma^2 = 2(1 + \delta^2) + (1 - \delta^2)(\lambda + \lambda^{-1}). \tag{4.21}$$

This ansatz is applied to the lattice configuration shown in Fig. 4.1(h), with coupling

strength 1 on the defect link.

We find defect state solutions by substituting the ansatz (4.20) into the time-independent Schrödinger equation $(\mathcal{H}_+ + \mathcal{H}_-) |\psi_D\rangle = E_D |\psi_D\rangle$, where H_+ and H_- are the Hamiltonians for semi-infinite lattices to the right ($n > 0$) and left ($n < 0$) of the domain wall. This yields

$$H_{\pm} \psi_{\pm} = E_D \psi_{\pm}, \quad (4.22)$$

$$(1 - \delta)\alpha_- + \alpha_+ = (E_D - i\gamma)\beta_-, \quad (4.23)$$

$$(1 + \delta)\beta_+ + \beta_- = (E_D - i\gamma)\alpha_+, \quad (4.24)$$

where

$$H_{\pm} = \begin{pmatrix} \pm i\gamma & W_{\pm\delta}(\lambda^{\mp 1}) \\ W_{\pm\delta}(\lambda^{\pm 1}) & \mp i\gamma \end{pmatrix} \quad (4.25)$$

$$W_{\delta}(\lambda) = (1 + \delta) + (1 - \delta)\lambda, \quad (4.26)$$

$$\psi_{\pm} = (\alpha_{\pm}, \beta_{\pm})^T. \quad (4.27)$$

From Eq. (4.22), we derive Eq. (4.21). Without loss of generality, we can take

$$\psi_- = \begin{pmatrix} E_D - i\gamma \\ W_{-\delta}(\lambda^{-1}) \end{pmatrix}, \quad \psi_+ = \eta \begin{pmatrix} E_D + i\gamma \\ W_{\delta}(\lambda) \end{pmatrix}. \quad (4.28)$$

Eqs. (4.23) and (4.24) describe the wave-matching conditions around the domain wall.

Substituting Eq. (4.28) into them, and using Eq. (4.21), gives

$$\begin{aligned} E_D^2 + \gamma^2 &= 2(1 + \delta^2) + (1 - \delta^2)(\lambda + \lambda^{-1}), \\ \lambda &= \frac{(E_D - i\gamma)(1 - \delta^2)}{E_D + i\gamma}, \quad \eta = \frac{1}{1 - \delta}. \end{aligned} \quad (4.29)$$

Figure 4.2: (a) Phase diagram for defect states of the cSSH lattice shown in Fig. 4.1(h). In the white regions (I), the lattice has one defect state, in the green regions (III) it has three defect states, and in the pink (II) and orange regions (IIA) it has two defect states. The defect states in II are $\mathcal{ST}$ -broken, and the defect states in IIA are $\mathcal{ST}$ -symmetric. The points labelled b , c , d , and e indicate the parameters for the spectra plotted in (b)–(e). The gray dashes are the phase boundaries of the bulk lattice bandstructure, corresponding to Fig. 4.1(b). The blue dots indicate the critical line segment ($\delta = 0$ and $0 \leq \gamma \leq 1$) over which there are no localized defect states. (b)–(e) Complex eigenenergy spectra, calculated numerically for a finite lattice with 75 unit cells on each side of the defect, for (b) $\gamma = 0.5$ and $\delta = 2$, (c) $\gamma = 0.5$ and $\delta = 0.5$, (d) $\gamma = 1.5$ and $\delta = 0.5$, and (e) $\gamma = 2.5$ and $\delta = 0$. Defect states continuable to SSH mid-gap states are shown as triangles, defect states continuable to non-mid-gap states as squares, bulk states as black circles, and edge states (due to the finite lattice size) as grey circles.

Figure 4.3: Variation of $|\lambda|$ with γ for (a) $\delta = 0.5$ and (b) $\delta = 2$.

We then search analytically or numerically for solutions satisfying $|\lambda| < 1$, corresponding to exponentially localized defect states. From (4.29), we can see that if (E_D, λ) is a solution, $(-E_D^*, \lambda^*)$ is also a solution. Specifically, the $\mathcal{ST}$ counterpart satisfies

$$U_{\mathcal{S}} \mathcal{K}(\psi_{\pm} \lambda^n) = \sigma_z (\psi_{\pm} \lambda^n)^*, \quad (4.30)$$

where $\mathcal{K}$ is the complex conjugation operator. This means that if the $\mathcal{ST}$ -paired defect states are two distinct solutions, they have the same intensity profiles and the phases are symmetric with respect to 0 ($\pi/2$) for the left (right) sites in each unit cell, which is precisely the behavior observed in Fig. 4.6.

The number of defect state solutions is determined by the number of solutions with $|\lambda| < 1$. Eliminating E_D in (4.29) yields a fourth-order polynomial in λ . In Fig. 4.3, we plot $|\lambda|$ versus γ for two different values of δ . If a root crosses the $|\lambda| = 1$ line, it corresponds to a defect state appearing out of, or disappearing into, the continuum.

Analytic solutions can be found for some special cases. First, in the Hermitian

limit ($\gamma = 0$), there is a mid-gap defect state of the form

$$\psi_- = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \psi_+ = \begin{pmatrix} 1 - \delta \\ 0 \end{pmatrix}, \quad \lambda = -\frac{1 + \delta}{1 - \delta} \quad (4.31)$$

for $\delta < 0$, and

$$\psi_- = \begin{pmatrix} 0 \\ 1 + \delta \end{pmatrix}, \quad \psi_+ = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \lambda = -\frac{1 - \delta}{1 + \delta} \quad (4.32)$$

for $\delta > 0$. The energy is pinned to $E_D = 0$ by the $\mathcal{S}$ symmetry [79, 90], and the existence of this defect state is tied to the topologically distinct configurations on both sides of the domain wall as characterized by a π difference in Zak phases [181]. Additionally, when $0 < |\delta| < \sqrt{2}$, there are two non-mid-gap defect state solutions:

$$\begin{aligned} \psi_- &= \begin{pmatrix} \pm(1 - \delta)\sqrt{4 + \delta^4} \\ 2 - 2\delta + \delta^2 \end{pmatrix}, \quad \psi_+ = \begin{pmatrix} \pm\sqrt{4 + \delta^4} \\ 2 - \delta^2 + \delta^3 \end{pmatrix} \\ \lambda &= 1 - \delta^2, \quad E_D = \pm\sqrt{4 + \delta^4}. \end{aligned} \quad (4.33)$$

Another analytic solution can be obtained when the inter-site couplings are uniform ($\delta = 0$). In this case, there can be a pair of defect state solutions of form

$$\begin{aligned} \psi_- &= \begin{pmatrix} \pm\lambda \\ 1 \end{pmatrix}, \quad \psi_+ = \begin{pmatrix} 1 \\ \pm\lambda \end{pmatrix}, \\ \lambda &= \pm(\sqrt{1 - \gamma^2} - i\gamma), \quad E_D = \sqrt{1 - \gamma^2} \pm 1. \end{aligned} \quad (4.34)$$

This is valid only for $\gamma > 1$; for $0 < \gamma < 1$, the defect state is not localized since $|\lambda| = 1$. The two eigenenergies have the same imaginary part and opposite real parts, due to the $\mathcal{ST}$ antisymmetry.

Finally, for $\delta = \pm 1$, there is an isolated trimer at the defect, and we can determine

the three eigenvalues

$$(E_D + i\gamma)(E_D - i\gamma)^2 = 5E_D - 3i\gamma. \quad (4.35)$$

In the Hermitian case, the roots are $E_D = \{0, \pm\sqrt{5}\}$.

The resulting phase diagram for the defect states is shown in Fig. 4.2(a). (As previously discussed, we consider only $\gamma \geq 0$.) The left edge of this phase diagram ($\gamma = 0$) corresponds to an SSH model with a domain wall through a link, and topologically distinct configurations on either side. This supports *three* distinct defect state solutions. One of them is the mid-gap defect state, which exists for all $\delta \neq 0$ and is pinned to energy $E_D^0 = 0$ by the $\mathcal{S}$ symmetry [79, 90] and topologically protected by a π difference in the Zak phases calculated for two bulk lattices [181]. The other two defect states, which exist for $0 < |\delta| < \sqrt{2}$, have eigenenergies

$$E_D^\pm = \pm\sqrt{4 + \delta^4}, \quad (4.36)$$

which respectively lie above and below the bulk energy bands. These defect states do not lie in a band gap, and are usually regarded as being topologically trivial in the Hermitian sense, because the Zak phases related to the non-mid-gap states have no difference for the two configurations on both sides of the domain wall. In the limit $\delta = \pm 1$, the three defect states reduce to the eigenstates of a trimer with coupling strengths of 1 and 2 on its two links, whose eigenenergies are 0 and $\pm\sqrt{5}$.

Suppose $0 < |\delta| < \sqrt{2}$, for which the SSH chain has three defect states. As we gradually increase γ from zero, all three defect states evolve continuously into exponentially-localized defect states of the non-Hermitian lattice, and their energies $\{E_D^0, E_D^\pm\}$ become complex. The topological mid-gap state becomes an unpaired $\mathcal{ST}$ -symmetric defect state with imaginary E_D^0 . The two non-mid-gap states become a pair of defect states that individually break the $\mathcal{ST}$ symmetry, and map to each other under

$\mathcal{ST}$, satisfying $E_D^+ = -(E_D^-)^*$. Within the regions labelled III in Fig. 4.2(a), the lattice supports three distinct and well-defined defect states. Note that the boundary of these regions lie close to, but outside, the phase boundaries of the bulk bandstructure’s $\mathcal{PT}$ -symmetric phase [Fig. 4.1(b)]. Fig. 4.2(c) shows a typical complex energy spectrum in domain III, calculated for a large but finite lattice. (We emphasize, however, that the phase boundaries in Fig. 4.2(a) were derived for infinite lattices, with and without the domain wall.) Note that the two $\mathcal{ST}$ -broken defect states (blue squares) have values of $\text{Re}(E_D^\pm)$ overlapping with the real bulk energy bands, but are nonetheless exponentially localized to the defect. This is reminiscent of the phenomenon of “bound states in the continuum” in Hermitian systems [198, 199], whose realization in non-Hermitian systems has recently been discussed by several authors [200–202].

As γ is further increased, the unpaired $\mathcal{ST}$ -symmetric defect state abruptly disappears. In the domain labelled II in Fig. 4.2(a), the system contains only two defect states (the $\mathcal{ST}$ -broken pair). The disappearance of the $\mathcal{ST}$ -symmetric defect state occurs via a divergence in its exponential decay constant (i.e., $|\lambda| \geq 1$ in the solution to the Schrödinger equation with the ansatz (4.20)), which coincides with the merging of E_D^0 into the continuum of bulk state eigenenergies. Since E_D^0 is imaginary, this can only happen outside the $\mathcal{PT}$ -symmetric phase of the bulk bandstructure, where the bulk spectrum is at least partially imaginary—hence the relationship between the defect state phase boundary and bulk phase boundary in Fig. 4.2(a).

Fig. 4.2(d) and (e) shows the complex energy spectra at two points in domain II. In Fig. 4.2(d), the bulk is in the intermediate phase, and its energies lie partly on the real line and partly on the imaginary line; the defect state eigenenergies $E_D^\pm$ stand apart from the bulk energies in the complex plane, but their real parts can be embedded in the $\text{Re}(E)$ continuum. In Fig. 4.2(e), the bulk is in the $\mathcal{ST}$ -symmetric phase and all of its energies are imaginary, whereas $\text{Re}(E_D^\pm) \neq 0$.

The case of $\delta = 0$ deserves special attention. For $\gamma = 0$, this is just an undimerized

chain, with no defect states. More interestingly, there are no defect states over the finite range $0 \leq \gamma \leq 1$. Only for $\gamma > 1$ do the pair of $\mathcal{ST}$ -broken defect states appear, described by

$$\alpha_- = \beta_+ = \pm\lambda, \quad \alpha_+ = \beta_- = 1, \quad (4.37)$$

$$\lambda = \pm \left(\sqrt{1 - \gamma^2} - i\gamma \right). \quad (4.38)$$

The corresponding eigenenergies are $E_D^\pm = \sqrt{1 - \gamma^2} \pm 1$. Within the range $0 \leq \gamma \leq 1$, Eq. (4.38) states that $|\lambda| = 1$, so the states are not localized. This results in a critical line segment in the phase diagram, indicated by the blue dots in Fig. 4.2(a), on which no localized defect states exist. The end of the line segment (at $\delta = 0$, $\gamma = 1$) is an exceptional point for Eq. (4.38).

Malzard, Poli, and Schomerus [2] have recently drawn attention to a class of intrinsically non-Hermitian defect states that (i) are not present in the Hermitian limit, and (ii) appear when a non-Hermiticity parameter exceeds a certain nonzero magnitude. They argued that such defect states may be considered “topologically protected”, in the sense that they are associated with non-Hermitian spectral phases bounded by exceptional points related to $\mathcal{PT}$ symmetry breaking. The defect states of our $\delta = 0$ lattice behave similarly.

To make the relation explicit, let us couple a $\delta = 0$ cSSH lattice to its time-reversed counterpart, transversely and site-by-site, to produce a two-chain lattice, as shown in Fig. 4.4(a). The inter-chain coupling is a new parameter denoted by h . Next, we exchange the positions of even (odd) sites between two chains, as shown in Fig. 4.4(b). This causes the unit cell to shrink to a single column. We then perform the following

Figure 4.4: Mapping between (a) a pair of coupled uniform-coupling ($\delta = 0$) cSSH chains and (c) the two-chain model of Ref. [2], where the purple (white) sites represent counterclockwise (clockwise) resonator modes, and the red and blue arrows represent asymmetric internal scattering processes.

pseudo-rotation in each unit cell:

$$\begin{pmatrix} |u'_n\rangle \\ |v'_n\rangle \end{pmatrix} = \mathcal{U} \begin{pmatrix} |u_n\rangle \\ |v_n\rangle \end{pmatrix}, \quad (4.39)$$

where $\mathcal{U} = e^{i\frac{\pi}{4}\sigma_x} = \frac{1}{\sqrt{2}}(1 + i\sigma_x)$, and $|u_n\rangle$ and $|v_n\rangle$ are states localized to sites in the upper and lower chains in the n -th unit cell. In this new basis, the model is identical to that of Ref. [2], as shown in Fig. 4.4(c), with the parameter correspondence

$$A = h + \gamma, \quad B = h - \gamma, \quad W = t, \quad (4.40)$$

where $\{A, B, W\}$ are the parameters defined in Ref. [2].

Our uniform-coupling model corresponds to $h = 0$, which is the diagonal line in the phase diagram Fig. 2(c) in Ref. [2] (i.e., $A = -B$).

Thus the $\delta = 0$ lattice is a particular limit of the model in Ref. [2]. However, the present analysis reveals qualifications to regarding these as topological defect states. In the $0 \leq \gamma \leq 1$ range, the defect states are indeed absent for $\delta = 0$, but

instantly re-appear when an infinitesimal δ is introduced (which causes $|\lambda|$ to drop below 1). Moreover, the defect states themselves are continuable to the non-mid-gap defect states of the SSH lattice, which are not topologically protected in the Hermitian sense.

4.4 $\mathcal{ST}$ -breaking of defect states

Fig. 4.5(a) shows a close-up view of the phase diagram for $\delta \sim \sqrt{2}$. For $|\delta| > \sqrt{2}$ and $\gamma = 0$, the SSH lattice has a single defect state (the mid-gap defect state). As we increase γ from zero, keeping δ fixed, the eigenvalue E_D^0 moves up the imaginary axis, as shown in Fig. 4.5(b).

As the system enters the region labelled IIA, another $\mathcal{ST}$ -symmetric defect state emerges from the continuum, as shown in the complex spectrum plotted in Fig. 4.5(c). Hence, in this region there are *two* $\mathcal{ST}$ -symmetric states that are localized to the defect, a phenomenon with no counterpart in the Hermitian SSH model.

As the system moves from region IIA to region II, the two imaginary energies approach each other, meet, then move off the imaginary axis to either side, as shown in Fig. 4.5(d). The transition line, shown as a thick brown line in Fig. 4.5(a), signifies a $\mathcal{ST}$ -breaking transition. In region II, as we have previously discussed, the system has two $\mathcal{ST}$ -broken defect states, which are continuable to topologically trivial defect states of the SSH chain.

Thus, there appear to be two distinct ways for the SSH mid-gap defect state to disappear from the non-Hermitian lattice. The first is to merge into the continuum; the second is to undergo a $\mathcal{ST}$ -breaking transition with another $\mathcal{ST}$ -symmetric defect state emerging from the continuum.

Fig. 4.6 shows the eigenstate magnitudes and phases of the defect states on either side of the $\mathcal{ST}$ -breaking transition. On one side of the transition, the two $\mathcal{ST}$ -

Figure 4.5: (a) Close-up view of the defect state phase diagram near the $\mathcal{ST}$ -breaking line. The region labels have the same meanings as in Fig. 4.2(a). The points labelled b , c , and d indicate the parameters for the spectra plotted in (b)–(d). (b)–(d) Complex eigenenergy spectra, calculated numerically for a finite lattice with 500 unit cells on each side of the defect, for (b) $\gamma = 1.9$ and $\delta = 1.5$, (c) $\gamma = 2.1$ and $\delta = 1.625$, and (d) $\gamma = 2.1$ and $\delta = \sqrt{2}$. The $\mathcal{ST}$ -symmetric defect states are indicated by triangles; the $\mathcal{ST}$ -broken defect states are indicated by squares. The arrows in (c) and (d) indicate the direction of motion of the defect state eigenvalues as δ decreases from point c to point d in (a).

Figure 4.6: Magnitudes and phases of the cSSH defect state wavefunctions, before and after an $\mathcal{ST}$ -breaking transition. In (a) and (b), the lattice parameters are $\gamma = 2.1$ and $\delta = 1.625$, the same as in Fig. 4.5(c); both eigenstates are $\mathcal{ST}$ -symmetric. The up-pointing triangles show the state that evolved from the SSH mid-gap defect state; the down-pointing triangles show the additional $\mathcal{ST}$ -symmetric defect state which emerged from the continuum. In (c) and (d), the lattice parameters are $\gamma = 2.1$ and $\delta = \sqrt{2}$, the same as in Fig. 4.5(d); both eigenstates are $\mathcal{ST}$ -broken, and are related by $\mathcal{ST}$ operation. The filled squares show the state with $\text{Re}(E) > 0$ and the hollow squares show the state with $\text{Re}(E) < 0$. Red (blue) symbols indicate gain (loss) sites. The gauge is fixed by setting the phase of the first site to the right of the domain wall to zero.

symmetric states have different intensity profiles with different localization lengths; moreover, the phases in the gain and loss sites differ by $\pi/2$, which is a characteristic feature of unbroken $\mathcal{ST}$ symmetry. On the other side of the transition, the two $\mathcal{ST}$ -broken eigenstates have identical intensity profiles, and the phases on the right (left) side of the domain wall are symmetric with respect to 0 ($\pi/2$) for the gain sites and $-\pi/2$ (0) for the loss sites; these features arise from the fact that the two states are related by the $\mathcal{ST}$ operation.

4.5 Summary

The cSSH model may be regarded as the simplest one-dimensional non-Hermitian model with a clear link to Hermitian concepts of band topology. In this chapter, we have examined the conditions under which a cSSH lattice supports exponentially localized defect states. Previous works on the subject have focused on the simplest case of a single non-Hermitian defect state that is $\mathcal{ST}$ -symmetric, whose energy has exactly zero real part. Such a defect state has a clear connection to the physics of topological states: it is continuable, in the Hermitian limit, to the SSH model's well-known topological defect state [78, 80, 135, 179, 194, 81, 178, 193].

Our study has revealed a richer variety of behaviors. In particular, the cSSH model has defect states that are $\mathcal{ST}$ -broken, with energies having non-zero real parts. Although these are continuable to the SSH model's "trivial" defect states, they play an interesting role in the cSSH model. In some parameter regimes, a pair of $\mathcal{ST}$ -broken defect states can co-exist with an $\mathcal{ST}$ -symmetric defect state. Alternatively, an $\mathcal{ST}$ -symmetric defect state can coalesce with another $\mathcal{ST}$ -symmetric defect state emerging out of the continuum, turning into an $\mathcal{ST}$ -broken pair. This is an inherently non-Hermitian phenomenon that lacks any analogue in the SSH model.

We have focused on the specific defect configuration of Fig. 4.1(h), with coupling

strength 1 on the defect link. If the coupling strength is not 1, the phase diagram for the defect states is qualitatively similar, though the positions of the phase boundaries are shifted, and the critical line segment at $\delta = 0$ is not present. For the alternative configurations shown in Figs. 4.1(f)-(g), there exist similar combinations of $\mathcal{ST}$ -broken and $\mathcal{ST}$ -unbroken defect states, but the phase diagrams are different. For the configuration of Fig. 4.1(f), a single $\mathcal{ST}$ -unbroken defect state exists for $\delta > 0$, whereas for $\delta > 0$ there are three defect states (one $\mathcal{ST}$ -unbroken and two $\mathcal{ST}$ -broken, or three $\mathcal{ST}$ -unbroken). For the configuration of Fig. 4.1(g), there is a phase with no defect states, similar to the critical line in Fig. 4.2(a); since this configuration is also $\mathcal{PT}$ symmetric, its $\mathcal{ST}$ -broken defect state pairs have real energies.

Chapter 5

Circuit realization of a four-dimensional topological lattice

This chapter presents an experimental realization of a class AI 4D quantum Hall system (see section 1.4) with an electric circuit. The circuit design is based on the tight-binding model proposed in [3]. In the tight-binding model, each unit cell consists of four sites denoted by (A,B,C,D).

The nearest-neighbor hoppings of the 4D lattice are constructed as shown in Fig. 5.1(a). The corresponding real-space tight-binding Hamiltonian for nearest-neighbor hopping is given by:

$$\begin{aligned} H &= H_x + H_y + H_z + H_w + H_{\text{on-site}}, \\ H_x &= J \sum_{m,n,j,l} (c_{m,n,j,l}^\dagger a_{m,n,j,l} + a_{m+1,n+1,j,l}^\dagger c_{m,n,j,l} - b_{m,n,j,l}^\dagger d_{m,n,j,l} - d_{m+1,n+1,j,l}^\dagger b_{m,n,j,l} + \text{h.c.}) \\ H_y &= J \sum_{m,n,j,l} (c_{m-1,n,j,l}^\dagger a_{m,n,j,l} - b_{m-1,n,j,l}^\dagger d_{m,n,j,l} + \text{h.c.}) \\ H_z &= J \sum_{m,n,j,l} (d_{m,n,j,l}^\dagger a_{m,n,j,l} + a_{m,n,j+1,l+1}^\dagger d_{m,n,j,l} + b_{m,n,j,l}^\dagger c_{m,n,j,l} + c_{m,n,j+1,l+1}^\dagger b_{m,n,j,l} + \text{h.c.}) \\ H_w &= J \sum_{m,n,j,l} (d_{m,n,j-1,l}^\dagger a_{m,n,j,l} + b_{m,n,j-1,l}^\dagger c_{m,n,j,l} + \text{h.c.}) \end{aligned} \tag{5.1}$$

$$H_{\text{on-site}} = m \sum_{m,n,j,l} (a_{m,n,j,l}^\dagger a_{m,n,j,l} + b_{m,n,j,l}^\dagger b_{m,n,j,l} - c_{m,n,j,l}^\dagger c_{m,n,j,l} - d_{m,n,j,l}^\dagger d_{m,n,j,l})$$

where J is the hopping amplitude, m is an energy offset between the A , B and C , D sites. The Hamiltonian splits into hopping terms along each direction. The index (m, n, j, l) indicates a particular unit cell with respect to four-dimensional lattice vectors $(\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_4)$, defined as $\mathbf{R}_1 = (1, -1, 0, 0)$, $\mathbf{R}_2 = (1, 1, 0, 0)$, $\mathbf{R}_3 = (0, 0, 1, -1)$, $\mathbf{R}_4 = (0, 0, 1, 1)$, with $a = 1$ being the distance between any two nearest-neighbour lattice sites. We have also introduced the operators $\alpha_{m,n,j,l}$ ($\alpha_{m,n,j,l}^\dagger$) which annihilate (create) a particle on an α -site in the (m, n, j, l) unit cell.

By Fourier-transforming the real-space Hamiltonian, we get the momentum-space Hamiltonian:

$$H(\mathbf{k}) = J [(2 \cos k_x + \cos k_y) \Gamma_1 + \sin k_y \Gamma_2 + (2 \cos k_z + \cos k_w) \Gamma_3 + \sin k_w \Gamma_4 + m \Gamma_5], \quad (5.2)$$

with the Γ_i defined in section 1.4.

The longer-range hoppings are shown in Fig. 5.1(b). Similar to nearest-neighbor hoppings, the momentum-space Hamiltonian with the longer-range hopping configuration shown in Fig. 5.1(b) is:

$$H'(\mathbf{k}) = [2J' \cos(2k_x + 2k_z) + 2J'' \cos(2k_x - 2k_z)] \Gamma_5. \quad (5.3)$$

When the longer-range hopping is absent ($J' = J'' = 0$) and the mass term is zero ($m = 0$), the model has four 4D Dirac points in the Brillouin zone, as shown in Fig. 5.1(c). Two pairs of Dirac cones near $\mathbf{K}_{1,2} = (\mp 2\pi/3, 0, \mp 2\pi/3, 0)$ and $\mathbf{K}_{3,4} = (\pm 2\pi/3, 0, \mp 2\pi/3, 0)$ contribute oppositely in sign to the second Chern number in Eq. (1.11). Their contributions cancel, so the model is topologically trivial.

When longer-range hoppings are present, the two types of Dirac points can be

Figure 5.1: (a) A 4D lattice with four sites per unit cell. Solid and dotted lines denote hoppings with positive and negative amplitudes J and $-J$ respectively. (b) Example of longer-range hoppings (Eq. 5.3) that can make the lattice topologically nontrivial. (c) Doubly-degenerate energy bands (Eq. 5.2) for $k_y = k_w = 0$, with the 4D Dirac points labelled. (d)(e) Energy dispersion at the topological phase transitions for $k_y = k_w = 0$, $J' = J/2$ and $J'' = 0$, corresponding to (d) $m = -J$ and (e) $m = J/2$, showing that there is only one pair of Dirac points at each transition. Figures reproduced from Ref. [3].

separated, making the second Chern number non-trivial. For example, let $J'' = 0$ and $J' = J/2$; then the second Chern number is -2 for $-J < m < J/2$ and 0 otherwise (see Fig. 5.1(d),(e)). Since this model preserves time-reversal symmetry, the first Chern number is always zero, thus the 3D surface states are purely related to non-zero second Chern numbers.

5.1 Circuit model of a 4D quantum Hall lattice

Such a 4D quantum Hall lattice can be realized in an LC electric circuit. Consider a general case of an electric circuit lattice where nodes of the lattice are marked as N_i . For any node N_i connected to any other node N_j via certain electrical components, we denote the combined complex admittance of the components linking node N_i and N_j as iD_{ij} , where D_{ij} is real if loss is ignored. And if any node N_i is grounded via certain components, we denote their combined complex admittance as iD_i , where D_i is real.

By Kirchhoff's law,

$$-(D_i + \sum_j D_{ij})V_i + \sum_j D_{ij}V_j = 0, \quad (5.4)$$

where V_i is the i th node's complex voltage. This maps to a linear matrix equation $H_{ij}V_i = 0$, where the diagonal terms are $H_{ii} = -(D_i + \sum_j D_{ij})$ and off-diagonal terms are $H_{ij} = D_{ij}$. Note that $D_{ij} = D_{ji}$. A real symmetric Hamiltonian can thus be mapped to an LC circuit, by properly assigning admittances to the components.

The real-space tight-binding Hamiltonian of our 4D quantum Hall model is real and symmetric. With fixed input frequency ω_d , we can map each positive hopping J between two nearest-neighbor nodes to a capacitor with capacitance C , and each negative hopping $-J$ to an inductor with inductance $L = 1/(\omega_d^2 C)$. Longer-range

hopping terms $\pm J', \pm J''$ and on-site grounding terms $\pm m$ can be similarly mapped to $C' = C(J'/J), C'' = C(J''/J), L_m = L(J/m)$ for positive hoppings and $L' = L(J/J'), L'' = L(J/J''), C_m = C(m/J)$ for negative hoppings. Note that grounding terms have a minus sign in Eq. (5.4). Also note that off-diagonal hopping terms appear in the diagonal terms as well, as shown in Eq. (5.4), and can be cancelled by grounding a copy of the hopping components in parallel, for each node of the circuit.

This procedure maps the circuit lattice to the 4D quantum Hall model for only one single frequency $\omega_d = 1/\sqrt{LC}$. At other frequencies, the band degeneracy is broken. However, this is adequate to probe the 4D quantum Hall model's band topology.

For example, with the parameters in Fig. 5.1(e), set $C = 1$ and $\omega_d = 1$. The lattice's band-diagram is calculated, and a slice of the 4D band diagram with $k_y = k_w = 0$ and along two directions, $k_x = k_z = k$ (Fig. 5.2(a)) and $k_x = -k_z = k$ (Fig. 5.2(b)), respectively. The results replicate the Dirac cone corresponding to $\mathbf{K}_{1,2}$ and the opened gap corresponding to $\mathbf{K}_{3,4}$ near $\omega = \omega_d = 1$.

Figure 5.2: Two slices of the 4D band diagram through the Brillouin zone origin, with $k_y = k_w = 0$ and in two directions: (a) $k_x = k_z = k$, and (b) $k_x = -k_z = k$. Calculated from a circuit lattice model with parameters used in Fig. 5.1(e).

5.2 Topological phase transition in the 4D quantum Hall circuit

The simplest way to demonstrate topological behavior of the 4D quantum Hall circuit is by directly showing how its topological 3D surface modes (which are solely related to the lattice's second Chern number, as the first Chern number is zero) appear and disappear during a topological phase transition, by altering the mass term m .

Before we look into a lattice with a boundary (and the associated surface states), we first observe the band diagram for an infinite lattice extended along all coordinates. For convenience, we set $J = 1, J' = -J'' = 2$ in Eq. (5.3). As shown in Fig. 5.3, the

Figure 5.3: A phase diagram for the unbounded lattice showing bandgap closing and reopening with varying mass term m . The blue curve indicates the upper band minimum and orange curve indicates lower band maximum.

bulk bandgap closes at $m = \pm 6$, as expected. The lattice is topologically nontrivial

for $-6 < m < 6$ and is topologically trivial otherwise.

We expect topological 3D surface states to appear in the non-trivial gap for a lattice with open boundaries. This is verified in a lattice of size $6 \times 6 \times 6 \times 6$, with periodic boundary condition in y and w directions and open boundary condition in x and z directions. The phase diagram for this lattice, shown in Fig. 5.4, can be obtained by numerically solving for the eigenenergies of the lattice's real-space Hamiltonian. Indeed, there are states lying between the two gap-closing m values, where topological 3D surface states are theoretically expected. However, the gap-closing m values are not ± 6 as in the infinite unbounded model. This is a finite size effect, as can be verified by varying the lattice's size (see Fig. 5.5). A demonstration

Figure 5.4: Phase diagram of a $6 \times 6 \times 6 \times 6$ lattice. With open boundary condition in x, z directions and periodic boundary in y, w directions. The lowest 256 eigenenergies are presented for varying m value.

of a surface state eigenwave, which corresponds to the lowest eigenenergy at $m = 0$

Figure 5.5: The bandgap closing m value shifts toward $m = 6$ with increasing size of the lattice. We fix the number of unit cells in y and w direction which has periodic boundary condition to 1, and change the unit cell number of x and z direction from 3 up to 25 and plot only the band edge (two edge energies closest to $E = 0$). As we increase the lattice size we can see that the band edge approaches the infinite model, and gap-closing m value approaches 6, verifying the finite-size effect.

in the nontrivial bandgap in the phase diagram of Fig. 5.4, is shown in Fig. 5.6. The surface wave circulates around the open boundary of the $x - z$ plane.

As a comparison, Fig. 5.7 shows the lowest energy eigenstate at $m = 8$, which exhibits the features of a trivial bulk mode.

To quantify the phase transition with the eigenwave distribution, we define a quantity called the averaged bulk intensity (ABI):

$$\text{ABI} = \frac{\sum_{i=1}^6 \sum_{y,w \neq 0,1,4,5} |\psi_{x,y,z,w}^i|^2 / n_{\text{bulk}}}{\sum_{i=1}^6 \sum |\psi_{x,y,z,w}^i|^2 / n_{\text{all}}} \quad (5.5)$$

Figure 5.6: Lowest energy eigenwave for a lattice with $J = 1$, $J' = -J'' = 2$, $atm = 0$. Each 3D plot presents a slice of the 4D wave distribution in W direction for $W = 1, 2, 3, 4, 5, 6$.

This is defined as the ratio of averaged wave intensity of bulk sites to averaged intensity of all sites, both accumulated for six eigenwaves whose eigenenergies have the lowest absolute values. From Fig. 5.8 we can clearly see that the eigenwaves are strongly localized to the surface when the mass term lies between the two gap-closing m values. They become delocalized upon exceeding the two gap-closing m values, indicating the existence of a topological phase transition.

A lattice of size $6 \times 6 \times 6 \times 6$ has 1296 sites and is unrealistically large for a circuit experiment. We find that the number of sites in the y and w directions can be reduced to 2, with the lattice retaining similar phase diagram and averaged-bulk-intensity curve (see Fig. 5.9). Such a lattice has 144 sites and is experimentally accessible.

Figure 5.7: Lowest energy eigenwave for a lattice with $J = 1, J' = -J'' = 2$, at $m = 8$. Each 3D plot presents a slice of the 4D wave distribution in W direction for $W = 1, 2, 3, 4, 5, 6$.

5.3 Circuit simulation and experiment

In a circuit experiment, surface state eigenmodes are resonant and can be excited by sources with frequencies corresponding to its eigenenergies. Before fabricating the electric circuits and experimentally measuring them, we test it using Ngspice, an electric and electronic circuits simulator. Fig. 5.10(a) and Fig. 5.10(b) clearly demonstrate an excited surface mode and a bulk mode of the circuit respectively. In this simulation we set $C = 1\text{nF}, C' = C'' = 2\text{nF}, L = 2\text{mH}, L' = L'' = 1\text{mH}, f_{\text{input}} = 112.54\text{kHz}$, and $C_m = 0$ for Fig. 5.10(a) and $C_m = 16\text{nF}$ for Fig. 5.10(b). Extra grounding inductors with values calculated node by node are added to balance the on-site impedance. This circuit maps to the tight-binding model with $J = 1, J' = -J'' = 2$, and $m = 0, E = 0$ for the former case and $m = 8, E = 5$ for the latter case. During the simulation, a source with fixed input frequency $f_{\text{input}} = 112.5\text{kHz}$ is connected to one node each time, while absolute voltage on other nodes are recorded.

Figure 5.8: Averaged bulk intensity as defined by Eq. 5.5, calculated from simulation result. A lower averaged bulk intensity indicates that the wave distribution is more localized to the surface.

Fig. 5.10 is then produced by calculating an average of 144 measurements with source connected to every nodes of the lattice.

Guided by the circuit design supported by above simulation results, the circuit lattice has been designed and fabricated (see Fig. 5.10(c)). As of this writing, the measurements and detailed analysis are still in progress. The results will be presented in a separate work.

Figure 5.9: Phase diagram and averaged bulk intensity similar to Fig. 5.4 and Fig. 5.8 for a smaller lattice with only two sites in y and w direction.

Figure 5.10: (a) Simulation of an averaged surface mode. Parameters applied to the simulation are $C = 1\text{nF}$, $C' = C'' = 2\text{nF}$, $L = 2\text{mH}$, $L' = L'' = 1\text{mH}$, $f_{\text{input}} = 112.54\text{kHz}$, $C_m = 0$, extra inductors are added as groundings to balance on-site impedance and tune the pseudo-energy to $E = 0$, their values are calculated by site. Two 3D plots show slices of the 4D wave for $W = 0$ and $W = 1$ respectively. The simulation is an averaged result of multiple measurements applying input source to each site. (b) Simulation of a bulk mode. Same parameters as in (a) except that $C_m = 16\text{nF}$ and values of extra inductors are calculated to raise pseudo energy to $E = 5$. (c) A photo of the fabricated 4D quantum Hall lattice in a circuit. (d) Schematic of the circuit's node to node couplings.

Chapter 6

Conclusion

In this thesis, we have examined several classical topological systems and explored a variety of phenomena inaccessible in solid state topological systems, namely the interplay of topology with nonlinearity, the interplay of topology with non-Hermiticity, as well as topology in high-dimensional lattices.

Nonlinear topological systems were discussed in first two chapters. In the first chapter, we showed theoretically that when nonlinearity is applied on parameters that can drive a system through a topological phase transition, and asymmetry is properly introduced in the system, optical isolation with topological robustness can be accomplished. In the second chapter, we experimentally realized an SSH-like nonlinear transmission line (NLTL). We examined the complex interplay between the topological edge states and higher-harmonic modes: localized topological edge modes at the resonant excitation frequency (within the linear bandgap) generate third- and higher-harmonic signals that penetrate into the lattice, acting as a cross-phase modulation on the lattice parameter that leads to a larger bandgap and hence a more confined edge state. A more localized edge state, in turn, produces a stronger harmonic signal. Further studies may attempt to obtain a more detailed understanding of these self- and cross-phase modulation processes, and explore their applications for frequency-

conversion technologies.

A non-Hermitian topological tight-binding model was discussed in the third chapter. Here, we extended previous theoretical research that showed that the cSSH lattice domain wall supports a non-Hermitian defect state with purely imaginary energy. Our study revealed a richer variety of behaviors of the defect states, including the fact that non-Hermitian extensions of Hermitian topological systems can support inherently non-Hermitian phenomena that have no analogue in their Hermitian counterparts. The extension of this principle to other types of lattices, including 2D and 3D lattices, is an active area of current research.

A circuit realization of a 4D quantum Hall lattice was presented in the fourth chapter. Topological surface states that correspond to nontrivial second Chern numbers were directly measured. While most previous experiments of high dimensional topological systems rely on synthetic dimensions, our work shows that as the dimensionality of a circuit is defined by its connectivity, extra dimensions can be easily implemented. Circuit lattices are a convenient platform to study high-dimensional topological models, and one can envision exploring other entries in the “periodic table” of topological insulators that had previously been assumed to be only theoretical.

In the future, I plan to introduce some of the phenomena studied in this thesis to nanophotonics settings. This includes topologically robust optical isolation along the lines proposed in chapter 2; a nanophotonic version of the harmonic generation enhancement studied in chapter 3; and possible applications in laser technology based on the non-Hermitian cSSH model behavior uncovered in chapter 4. Also, the high-dimensional circuit realized in chapter 5 is a good platform to continue experimental studies on nonlinear phenomena in high dimensions.

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